SLAC-PUB-95-7045
October 1995

# NUMERICAL CONSTRUCTION OF THE POINCARÉ MAP, WITH APPLICATION TO ACCELERATORS*

ROBERT L. WARNOCK AND J. SCOTT BERG
*Stanford Linear Accelerator Center*
*Stanford University, Stanford, CA 94309*

**Abstract.** We show how to construct a symplectic approximation to the Poincaré map, using data from a symplectic integrator. We illustrate by producing a full-turn map for a realistic model of the Large Hadron Collider. Mapping of one turn is typically faster by a factor of 60 than direct integration. This allows one to follow orbits over times comparable to the required storage time of the beam, on a workstation computer. Fast mapping also allows the construction of quasi-invariant actions, which aid in estimates of long-term stability.

## 1. Introduction

The question of stability of a Hamiltonian system is simplified, at least in a conceptual sense, by considering the Poincaré map rather than the full Hamiltonian flow. In numerical studies this theoretical simplification might carry as well a practical advantage if it were possible to make a good approximate evaluation of the map in much less time than is required for direct integration of the flow over one return to the Poincaré section. We explore this possible advantage, having in mind complicated systems for which accurate modelling is required and direct integration is expensive. Our starting point is a numerical algorithm to compute the flow.

Let us denote the Poincaré map defined directly by the given numerical flow as $T_0$. We seek an approximation $T_1 \approx T_0$ in a region $U$ of phase space. To meet the symplectic condition we construct the mixed-variable, canonical generator $G$ of $T_1$, which defines $T_1$ implicitly [1, 2, 3]. Attempts at explicit representations of $T_1$ have shown either excessive violation of symplecticity or difficulties in controlling accuracy; see Ref. [2] for a survey.

In problems of accelerator physics, $T_0$ is defined by a *tracking code* which is based on a *symplectic integrator* [4] that follows the orbit in small steps through the fields of the various magnets that guide the beam. For large storage rings the cost of tracking is so great that one cannot follow orbits for the desired storage time of the beam. Fortunately, the generator of an adequate approximation $T_1 \approx T_0$ can be obtained from relatively few evaluations of $T_0$, and $T_1$ can be iterated so quickly as to follow orbits over times approaching the storage time.

*Work supported by Department of Energy contract DE-AC03-76SF00515.

*Presented at the NATO Advanced Study Institute on Hamiltonian Systems with Three or More Degrees of Freedom, Catalunya, Spain, June 19-30, 1995.*





## 2. Construction of the generator

We describe the map $T_0$ in angle-action coordinates $(I, \phi) \in R_+^n \times T^n$, where $R_+$ is the positive real line. If $T_0 : (I, \phi) \mapsto (I', \phi')$ we write

$$I' = I + R(I, \phi) , \qquad \phi' = \phi + \Theta(I, \phi) \ . \tag{1}$$

The same transformation is defined implicitly through the generator $G : R_+^n \times T^n \to R$ by the following equations:

$$I' = I + G_{\phi'}(I, \phi') , \qquad \phi = \phi' + G_I(I, \phi') \ , \tag{2}$$

where subscripts denote partial derivatives. To determine $G$ we must find a solution of the partial differential equations

$$G_{\phi'}(I, \phi') = R(I, \phi) , \qquad G_I(I, \phi') = -\Theta(I, \phi) \ , \tag{3}$$

where $\phi = \phi(I, \phi')$ is a solution of the second equation in (1).

The equations (3) are solved by Fourier analysis of $G$:

$$G(I, \phi') = \sum_{m \in Z^n} g_m(I) e^{i\langle m, \phi' \rangle} \ . \tag{4}$$

Now express the Fourier transform of $G_{\phi'}$ in terms of $R$ through (3), and change the integration variable from $\phi'$ to $\phi$:

$$\begin{aligned} img_m(I) &= \frac{1}{(2\pi)^n} \int_{T^n} e^{-i\langle m, \phi' \rangle} R(I, \phi) d\phi' \\ &= \frac{1}{(2\pi)^n} \int_{T^n} e^{-i\langle m, \phi + \Theta(I, \phi) \rangle} R(I, \phi) \det(1 + \Theta_\phi(I, \phi)) d\phi \ . \end{aligned} \tag{5}$$

This is valid if the Jacobian matrix $1 + \Theta_\phi$ is nonsingular at all $\phi$. Equation (5) solves our problem by giving $g_m$ when at least one component of $m$ is non-zero. If $m = 0$, we apply a similar method to $G_I$ and get $\partial g_0(I)/\partial I$ as an integral involving $\Theta$. We then integrate on $I$ to get $g_0$ itself; a constant of integration is inconsequential.

In practice we calculate the integral (5) by the trapezoid rule over a uniform mesh in $\phi$. The calculation is repeated for each $I$ on some finite mesh $\{I_i\}$ (not necessarily uniform). The $g_m(I_i)$ are then interpolated by spline functions to define $g_m(I)$ as a smooth function of $I$. Our program allows splines of arbitrary degree, thus continuous derivatives to an arbitrary order.

## 3. Iteration of the Map

To calculate $T_1(I, \phi) = (I', \phi')$ from $G$, we solve the second equation of (2) for $\phi'$ by Newton's method, then substitute in the first equation to get $I'$. To maximize the speed of this process we take the following steps: (1) We throw away small terms in (4). (Among all Fourier modes with $|m_i| \le M$, a great many are found to be negligible, even with a mode cutoff as small as $M = 8$.) (2) We do the interpolation in $I$ with a Kronecker-product B-spline basis. (This saves a lot of time, since at any point only a few of the basis functions are nonzero.) (3) We start the Newton iteration with a good guess for $\phi'$ obtained from

a rough *explicit* formula for $T_1$. (The formula is obtained by direct Fourier analysis and $I-$interpolation of $T_0$, retaining only a few modes.)

Typically, four or five Newton iterations are sufficient to solve (2) to computer precision. Note that the derivatives of $G$ are expressed analytically, so that an accurate solution of the equations will imply some corresponding accuracy of symplecticity.

## 4. The Large Hadron Collider as a Hamiltonian system

In the LHC to be built at CERN, two counter-circulating proton beams of energy 7.7 TeV will collide to produce an energy of 15.4 TeV in the center-of-mass system. The beams travel in two parallel storage rings of circumference 27 km, guided by 1792 superconducting dipole bending magnets with a very high field strength of 8 Tesla, and 392 quadrupole focussing magnets. In colliding mode at full energy, the beams are stored for 10 hours, or $4 \cdot 10^8$ turns, during which time they have about $10^{12}$ encounters with localized nonlinear fields. In the injection mode at 0.45 TeV, the beams must last for 10 minutes, or $7 \cdot 10^6$ turns. In this paper we study the injection mode, which in many respects is the more critical.

The protons radiate little electromagnetic energy and therefore are well described by Hamiltonian mechanics. The particles are subject to linear magnetic forces from dipoles and quadrupoles, and nonlinear forces from sextupoles (introduced to compensate the energy dependence of the focussing) and higher multipoles. The latter have a big effect and cannot be completely controlled, since they arise from sources such as errors in placement of superconducting coils. The beam is also subject to a localized, longitudinal r.f. electric field, which (in a storage mode) serves to bunch the beam longitudinally, and causes a small oscillation in energy of a single particle from turn to turn. Interparticle forces can be neglected in a first approximation, except for the beam-beam interaction at the collision point of the two beams. In the injection mode, the beams do not collide.

Position coordinates are referred to the *design orbit*, which is a periodic, linearly stable orbit. The time-like independent variable of Hamilton's equations will be $s$, the arc length along the design orbit. The Hamiltonian $H(P, Q, s)$ is periodic in $s$ with period $C$, the circumference of the design orbit. The "betatron" motion, oscillation transverse to the design orbit, is described by transverse displacements $x_i$ and conjugate momenta $p_i$, the latter being the slopes $dx_i/ds$ (up to a small correction). The "synchrotron" motion, oscillation in energy due to the r.f. field, is described by the coordinate $\tau = t - t_0$, where $t(s)$ is the time of arrival at $s$, and conjugate momentum $p_\tau = -(E - E_0)/p_0$, where $E$ is the energy. Here $t_0$ is the time of arrival for a particle with nominal energy $E_0$ and corresponding momentum $p_0$. We have a system in 3 1/2 degrees of freedom, with $P = (p_1, p_2, p_\tau), Q = (x_1, x_2, \tau)$. The motion consists essentially of three harmonic oscillators perturbed by nonlinear terms that are localized in $s$. Our Poincaré section in the 7-d extended phase space is just all of the 6-d mechanical phase space at $s = 0 \pmod C$, and the Poincaré map is the "full turn map" giving the evolution of $(P, Q)$ over one turn.

Although the synchrotron motion has an important effect on the betatron motion (since particles of different energy take different routes through nonlinear magnets), the latter has relatively little effect on the former. To a good approximation, the energy just oscillates harmonically from turn to turn, changing only once per turn within a short r.f. cavity. For a first investigation we can work with a model in which the particles travel through most of the ring at constant $p_\tau$, which at the $n-$th turn has the prescribed value

$$p_\tau = \hat{p} \sin 2\pi\nu_s n \ . \tag{6}$$

The constant $\nu_s$, the synchrotron tune, has the value 1/129.97 in the injection mode. Approximating by $\nu_s = 1/130$ and ignoring the coordinate $\tau$, we then have a system with 2 1/2 degrees of freedom, periodic in $s$ with period $130C$. We can study two-dimensional invariant tori and resonances on a Poincaré section $\mathcal{S}$ at $s = 0 \pmod{130C}$, by techniques that have already proved effective in the case of pure betatron motion [5].

## 5. Energy dependent four-dimensional map for the LHC

We report some results with this simplified model of synchrotron oscillations, but with a full, realistic model of the betatron motion at constant energy. We apply the construction of Section 2 in the betatron phase space ($n = 2$) taking $T_0$ to be the evolution of $(p_1, x_1, p_2, x_2)$ over one turn at constant $p_\tau$. The map $T_1$ is determined for values of $p_\tau$ on a mesh, and then interpolated by splines to provide every value that occurs in (6). Values of the Fourier coefficients $g_m(I_1, I_2, p_\tau)$ for all 130 values of $p_\tau$ are stored for later use in map iteration, so as to avoid duplicate evaluations of splines in $p_\tau$. The required map on the surface $\mathcal{S}$ at the synchrotron period is obtained by iterating the one-turn map 130 times. Direct construction of the 130 turn map is possible in principle, but probably not advisable.

Maps are constructed on rectangles in action space, $I_{ia} < I_i < I_{ib}$, with $I_{ia}$ not too small. A method to deal with the singularity of polar coordinates where one action vanishes is a topic for further work. For each $p_\tau$ the origin of coordinates is at the fixed point of the four-dimensional map; the coordinates are normal coordinates for the linear part of the map.

For the LHC we construct a map for a rectangle centered at about one half of the short-term dynamic aperture in $x_1, x_2$ (the roughly defined border of stability over a thousand turns or so). The domain of validity of the map allows for a 50% variation in initial actions $I_i(0)$. We allow mode numbers $|m_i| \leq 8$ in either dimension, and take 10 spline interpolation points for each $I_i$ and 6 for $p_\tau$. The map construction requires 264600 evaluations of $T_0$ at 68ms per evaluation (thus 5 hours) on an IBM RS6000-590. The resulting map $T_1$ can be iterated in 1.2ms, giving $10^7$ turns in 3.6 hours. This speed is sufficient to study many orbits over the required beam lifetime of the storage mode. It is also sufficient to compute a quasi-invariant torus on the section $\mathcal{S}$ in 20 minutes, using the method of Ref. [5] with 20 Fourier modes of the torus in each dimension.

Figure 1 shows a plot of $I_1$ on a long orbit of $T_1$ ($10^7$ turns). The orbit appears to be rather chaotic, and shows a slow drift in $I$ space. This "diffusion" is a typical phenomenon that accelerator builders strive to understand and control.

## 6. Validation of the map

An obvious test of validity of $T_1$ is to check its agreement with $T_0$ at many points $(I, \phi)$. For several hundred randomly chosen points we find a maximum deviation at one turn of about 1 part in $10^4$. The agreement can be increased essentially at will by increasing the number of Fourier modes and spline interpolation points, but we think it more useful to see how well the physical phenomena of $T_0$ are reproduced at this modest level of agreement. Since resonances form in some sense the "skeleton of phase space", an immediate requirement is that resonant structures be reproduced down to some fine scale. Another test is to

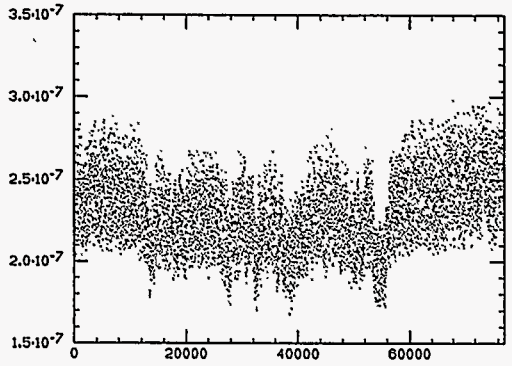


*Figure 1.* $I_1$ (in meters) plotted at every 8th synchrotron period over 77000 periods ($10^7$ turns). Initial conditions: $(I_1(0), I_2(0)) = (2.19, 1.41) \cdot 10^{-7} m, \phi(0) = (0, 0)$.

construct nearly invariant tori of $T_1$, and then check their invariance under $T_0$. In Ref. [6] we report such tests, and take the results as evidence that $T_1$ and $T_0$ represent closely similar physical systems.

## 7. Long-term bounds on the motion

A numerical method to set bounds on the motion for long but finite times was proposed in Ref. [5]. In the spirit of the Nekhoroshev theorem, one constructs globally defined quasi-invariants, and estimates their maximum change over $N_0$ turns for any initial condition in an open region $\Omega$. One can then set a bound on the maximum change during $N \gg N_0$ turns for any initial condition in an open subset $\Omega_0 \subset \Omega$. The techniques of Ref. [5], presented there for a simple model of betatron motion, can now be applied to the present realistic model of the LHC, thanks to our fast mapping. A brief account of initial work in this direction is to be found in Ref. [6].

## Acknowledgements

This work was made possible by the kind help of Frank Schmidt and Étienne Forest, who provided lattice data and a tracking code for the LHC.

## References

1. J. S. Berg and R. L. Warnock, Proc. 1991 IEEE Particle Accelerator Conf., San Francisco, May 6-9, 1991, pp. 1654-1656.
2. J. S. Berg, R. L. Warnock, R. D. Ruth, and É. Forest, *Phys. Rev. E* 49, pp. 722-739, (1994).
3. Y. Yan, P. Channell, and M. Syphers, SSC Lab. Report No. SSCL-Preprint-157, 1992 (unpublished).
4. R. D. Ruth, *IEEE Trans. Nucl. Sci.* 30, pp. 2669-2671, (1983); J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian Problems*, Chapman & Hall, London, 1994.
5. R. L. Warnock and R. D. Ruth, *Physica D* 56, pp. 188-215, (1992).
6. R. L. Warnock, J. S. Berg, and É. Forest, SLAC-PUB-95-6840, to be published in Proc. 1995 IEEE Particle Accelerator Conf., Dallas, Texas, May 1-5, 1995.