Problems and Solutions (Generalized and FEM) Related to Rapid and Impulsive Changes for Incompressible Flows

P.M. Gresho
R.L. Sani

This paper was prepared for submittal to the
10th International Conference on Finite Elements in Fluids
Tucson, AZ
January 5-8, 1998

August 1997

This is a preprint of a paper intended for publication in a journal or proceedings. Since changes may be made before publication, this preprint is made available with the understanding that it will not be cited or reproduced without the permission of the author.
Problems and solutions (generalized and FEM) related to rapid and impulsive changes for incompressible flows

by
P.M. Gresho
(Lawrence Livermore National Laboratory)
and
R.L. Sani
(University of Colorado)

1. INTRODUCTION

We shall be interested in examining so-called 'impulsive' changes in normal boundary conditions for the incompressible Navier-Stokes equations, both theoretically and numerically—via the Galerkin finite element method (GFEM) and several time-marching methods. We begin by stating two facts:

1. In the 'strictest' sense, impulsive (instantaneous/discontinuous) changes in the normal component of the velocity are illegal in that they cause violations of incompressibility, as well as concomitant unbounded pressure—briefly.

2. For the very common case in the fluid dynamics literature, the misnomer "impulsive start from rest" is confusingly employed. It is a misnomer because the true initial condition for most of these analyses and/or simulations is potential flow—far from a fluid at rest.

To help clarify the above issues, we shall also study rapid changes, via driving functions for normal velocity like \( 1 - e^{-\lambda t} \) for 'large' \( \lambda \)—eventually permitting \( \lambda \) to become unbounded. We focus on but a single simple and common example: flow past a circular cylinder. The startup—both rapid and impulsive—was treated extensively in Gresho and Sani (1997) (hereafter referred to as GS). In this paper, we concentrate mostly on the opposite case: sudden shutdown (rapid and impulsive) of the flow past the same cylinder from an IC that was generated in the startup phase.

For a discussion of some of the previous work in the startup case, see GS. For the shutdown case, we cite the few with which we are familiar: Gresho (1991a) Wang and Dalton (1991), Chang and Maxey (1995), who state, "An impulsively stopped free stream... is much different than an impulsively started flow," and Mei and Lawrence (1996), who state, "When it is suddenly brought to rest, the wake behind the body will continue to move to the right with the wake origin traveling as \( x \sim t \)."

In the remainder of this paper, we shall 'analyze' the problems associated with rapid/impulsive changes—both in the continuum and in the finite \( h \), finite \( \Delta t \) discrete world in which we are forced to do our computations. For the latter case, we shall demonstrate the performance of two popular 'elements' \( (Q_2P_1 \text{ and } Q_2Q_0) \) and the manner in which they (or any other approximate method) behave; viz., dismal failure at small time—especially for the impulsive case—with recovery fortunately occurring once the mesh near the cylinder has had time to 'recognize' the situation and respond to the severe challenge. Another problem, somewhat surprising, is the discovery that the 'stable'-for-Stokes-flow element \( (Q_2P_1) \) is less stable (bigger wiggles) for potential flow than the \( Q_2Q_0 \) element that is deemed by some to be unstable (again for Stokes flow).

2. THEORY

We shall 'sneak up' on impulsive changes as follows: Find \( \bar{u} \) and \( P \) from

\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} + \nabla P = \nu \nabla^2 \bar{u}
\]

and

\[
\nabla \cdot \bar{u} = 0 \quad \text{in} \quad \Omega,
\]

\[ u = w_0 e^{-\lambda t} + w_1(1-e^{-\lambda t}) \quad \text{on} \quad \Gamma, \]

with

\[ \bar{u} = u_0 \quad \text{at} \quad t = 0 \]
that satisfies both
\[ \nabla \cdot u_0 = 0 \quad \text{in } \Omega \]  
and
\[ n \cdot u_0 = n \cdot w_0 \quad \text{on } \Gamma; \]  
where \( \lambda \) is a parameter that will be allowed to become arbitrarily large—-with \( \lambda \to \infty \) defining our impulsive change. (Actually, the Dirichlet BC of (2) will only be applied on a portion of \( \Gamma \); the remaining BC's will be presented later.)

The pressure Poisson equation (PPE) implied by (1),
\[ \nabla^2 p = \nabla \cdot (\nabla^2 u - u \cdot \nabla u) \quad \text{in } \Omega \]  
and, from (1) and (2), it's implied (Neumann) BC (see, e.g., Gresho and Sani [1987], GS),
\[ \partial p / \partial n = n \cdot \left[ \nabla^2 u - u \cdot \nabla u - \partial u / \partial t \right] = n \cdot \left( \nabla^2 u - u \cdot \nabla u \right) - n \cdot (w_1 - w_0) \lambda e^{-\lambda t} \quad \text{on } \Gamma, \]  
are especially important to understand. Hopefully it is obvious (at least after due reflection) that for \( \lambda \) sufficiently large, and \( t \) sufficiently small, the pressure satisfies
\[ \nabla^2 p = 0 \quad \text{in } \Omega \]  
with
\[ \partial p / \partial n \equiv -n \cdot (w_1 - w_0) \lambda e^{-\lambda t} \quad \text{on } \Gamma; \]  
i.e., the acceleration BC dominates the problem.

Remarks:
1) For impulsive (or rapid) starts from rest, \( u_0 = 0 \) and \( w_0 = 0 \).
2) For impulsive (or rapid) stops, \( w_1 = 0 \).

Noting that (8) and (9) are 'solved' by
\[ p = -\phi \lambda e^{-\lambda t}, \]  
where \( \phi \) is the potential function satisfying
\[ \nabla^2 \phi = 0 \quad \text{in } \Omega \]  
and
\[ \partial \phi / \partial n = n \cdot (w_1 - w_0) \quad \text{on } \Gamma, \]  
permits the realization that an impulsive change via Dirichlet BC changes in the normal velocity is actually nothing more than a potential flow 'adjustment' + vortex sheet; i.e. for \( \lambda \to \infty \) our 'new' IBVP is simply this: Find \( u \) and \( P \) with the modified IC and BC,
\[ u = u_0 - \nabla \phi \quad \text{in } \Omega \quad \text{and} \quad u = w_1 \quad \text{on } \Gamma; \]  
and we note the presence of a vortex sheet on \( \Gamma \) since \( \nabla \cdot u = \nabla \cdot w_1 \) on \( \Gamma \) but \( \nabla \cdot u = \nabla \cdot w_0 - \nabla \cdot \phi \neq \nabla \cdot w_1 \) just off of the surface. For \( \lambda \) large-but-finite, this is of course only an approximation (still for small time) to the true IBVP given by (1) through (5).

It is also worthwhile pointing out the following projection connection for \( \lambda \to \infty \): The solution of (11)-(13) can also be derived as follows: (1) make a step change in the BC from \( n \cdot u = n \cdot w_0 \) to \( n \cdot u = n \cdot w_1 \); (2) call the resulting non-solenoidal velocity \( \tilde{\mathbf{u}} \) (i.e. \( \tilde{\mathbf{u}} = u_0 \) in \( \Omega \) with \( n \cdot \tilde{\mathbf{u}} = n \cdot w_1 \) \( \neq n \cdot u_0 \)) on \( \Gamma \); (3) this is realized via (i) solve \( \nabla^2 \phi = \nabla \cdot \tilde{\mathbf{u}} \) in \( \Omega \) with \( \partial \phi / \partial n = n \cdot (\tilde{\mathbf{u}} - u) = 0 \) on \( \Gamma \), where \( \nabla \cdot \tilde{\mathbf{u}} \) is to be interpreted as a Dirac delta function on \( \Gamma \); (4) compute \( u = \tilde{\mathbf{u}} - \nabla \phi \), which will give the same result as Eqn. (13). (Admittedly, the 'continuous' approach, with \( \lambda \to \infty \) at the end, is somewhat more intuitively acceptable—but the equivalence is important.)

Note from (9) that \( \lambda \to \infty \) puts a version of the Dirac delta function, a so-called generalized function, on the boundary:
\[ \delta(t) \equiv \lambda e^{-\lambda t}, \]  
which satisfies \( \int_0^{\infty} \delta(t) dt = 1 \) and \( \int_0^t f(t) \delta(t) dt = f(0) \) for \( \lambda \to \infty \), and causes \( \partial p / \partial n \) to become unbounded.

Note too that this same limit causes both an infinite pressure and a violation of incompressibility—both briefly—the latter a consequence of a step change discontinuity in the normal velocity at the boundary \( (n \cdot u = n \cdot w_0 \quad \text{at } t = 0 \quad \text{but } n \cdot u = n \cdot w_1 \quad \text{for } t = 0^+). \) It is this violation that has caused us to state that impulsive changes are "illegal" for incompressible flow (cf. Gresho 1991b, GS). Arbitrarily large \( \lambda \)'s are quite permissible, of course. Here, however, we shall 'soften up' and take the broader view that impulsive changes are just as legitimate as are generalized solutions.

Consider now the application of the above theory to a particular case: rapid startup of flow past a circular cylinder (radius \( a \)) in an unbounded domain. We first treat the inviscid (potential flow) case, as the resulting analytical solutions provide some useful insight.
For this case, the following solution (in polar coordinates) is easily obtained:

\[ u_t = (1 - e^{-\lambda t}) \nabla \phi, \quad (15) \]

where

\[ \phi = w_1 (r + a^2/r) \cos \theta. \quad (16) \]

Also,

\[ P_l = -\phi \lambda e^{-\lambda t} + P_{pot} \left(1 - e^{-\lambda t}\right)^2 \quad (17) \]

gives the concomitant pressure, where the potential pressure is

\[ P_{pot} = \frac{w_1^2}{2} \frac{a^2}{r^2} \left(2 \cos 2\theta - \frac{a^2}{r^2}\right). \quad (18) \]

For the limiting case of an impulsive start, \( \lambda \to \infty \) and we get

\[ u_l = H_1(t) \nabla \phi \quad (19) \]

and

\[ P_l = -\phi \delta(t) + H_2(t) P_{pot}, \quad (20) \]

where we 'interpret' \( H_1(t) \) as Heaviside step functions—\( H_1(0) = 0 \) and \( H_1(t) = 1 \) for \( t > 0 \). Note that \( \delta(t) \) generates the potential flow and \( H_2(t) \) maintains it, which is our interpretation of an (ideal) impulsive start from rest.

For the viscous (no-slip) case, we do not know the exact solution, but in GS we have developed (with much help from A.C. Hindmarsh, to whom we remain indebted) the following useful approximation (model), valid within the boundary layer (BL) and close to the cylinder; i.e., for \( r - a \ll \sqrt{4\nu t} \) and \( r - a \ll a : \)

\[ u_\theta \equiv -4w_1 \sin \theta \cdot \frac{r - a}{\sqrt{\pi \nu \tau_{ac}}} \ D\left(\sqrt{\lambda t}\right), \quad (21) \]

where \( \tau_{ac} \equiv 1/\lambda \) is the acceleration time constant and \( D() \) is Dawson’s integral, which behaves like so: \( D(y) \equiv y \) for \( y \ll 1, \ D(y) \approx \sqrt{\pi} y \) for \( y \gg 1 \), with \( D(y) \) attaining its only maximum of \( D(-0.92) \approx 0.54 \). The corresponding pressure is given by

\[ P = P_l - \frac{4w_1}{r} \cos \theta \cdot \frac{a}{\sqrt{\pi \nu \tau_{ac}}} \ D\left(\sqrt{\lambda t}\right). \quad (22) \]

For both the rapid start and the impulsive start, we emphasize that \( D(0) = 0 \)—and also note that \( D(\sqrt{\lambda t})/\sqrt{\tau_{ac}} \rightarrow 1/\sqrt{2\lambda t} \) for \( \lambda \to \infty \) (and \( t > 0 \)), which latter result makes our model agree with previous impulsive start theoretical analyses [e.g., Wang (1968), Collins and Dennis (1973), Bar-Lev and Yang (1975)] in that both viscous and pressure contributions to the drag coefficient, for small time, can be shown to be (see GS for details)

\[ C_D^p = C_D^p \equiv \frac{4}{w_1} \sqrt{\frac{\pi \nu}{\tau_{ac}}} D\left(\sqrt{\lambda t}\right) \quad (23) \]

giving, for \( \lambda t \ll 1 \),

\[ C_D^p = C_D^p \equiv 4\lambda \sqrt{\pi \nu t}/w_1, \quad (24) \]

which corresponds to the acceleration phase of flow within the BL, while for \( \lambda t \gg 1 \) we get

\[ C_D^p = C_D^p \equiv \frac{2}{w_1} \sqrt{\frac{\pi \nu}{t}}, \quad (25) \]

which accounts for the deceleration within the BL—and is the result referred to above in the impulsive start literature (none of whom treat the accelerating phase), and we emphasize that even for the limiting case (\( \lambda \to \infty \)), (25) only applies for \( t > 0 \); \( C_D^p = C_D^p = 0 \) at \( t = 0 \). The final contribution to the drag coefficient comes from the acceleration portion of the transient pressure field and can be derived as

\[ C_D^p = \frac{2\pi a}{w_1} \lambda e^{-\lambda t}, \quad (26) \]

using (17) and (22).

The total drag coefficient is, of course,

\[ C_D = C_D^p + C_D^p + C_D^p, \quad (27) \]

and we note that \( P_{pot} \) in (17) contributes naught to \( C_D \), per d’Alembert. Thus, for \( \lambda \) arbitrarily large, \( C_D \) starts at an arbitrarily large value owing to \( C_D^p \) (the Dirac function, in the limit) and for \( t > 0 \) but small, goes like \( \sqrt{\lambda t} \) in which both the viscous component and the pressure component are ‘caused’ by the no-slip BC, the pressure part needing to utilize \( \nabla : u = 0 \) to clearly see its viscous ‘roots’; see GS for details.

Finally we note, for \( \lambda \to \infty \), that there exists a near-vortex sheet on the cylinder at \( t = 0^+ \). For a ‘classic’ impulsive start, \( (\lambda = \infty) \) if such a thing exists, the IC is potential flow for \( r > a \) with a true vortex
sheet on \( r = a \) caused by suddenly imposing the no-slip BC. We are in fact somewhat ‘bothered’ by such a definition because it is quite misleading; the flow is not at rest at \( t = 0 \).

3. NUMERICS

So much for theory. How does a CFD code deal with the above issues, which clearly must lead to some serious numerical challenges for large \( \lambda \) and, as we shall show, to total failure at small \( t \) for \( \lambda = \infty \)? We begin by returning to the analytical pressure solution, (22), and note its satisfaction of the following BC on the cylinder (for \( \lambda \) sufficiently large that the contribution from \( \partial P/\partial r \) is negligible):

\[
\frac{\partial P}{\partial r} = \frac{4\nu \cos \theta}{a} \sqrt{\pi \nu \tau_{ac}} D(\sqrt{\lambda r}),
\]

which behaves like (and causes the viscous part of the pressure to also behave like) \( O(\lambda^{-1}) \) for small \( t/\lambda \ll 1 \) and like \( O(1/\sqrt{t}) \) for ‘large’ \( t/\lambda \gg 1 \). But the ‘numerical’ version of this Neumann BC, derived in GS, is rather different:

\[
\frac{\partial P}{\partial r} = \frac{\nu \cos \theta}{a} \left( a/h - 4 \right) \left( 1 - e^{-\lambda t} \right),
\]

where the second term might be argued to at least approximate the physics (accelerating potential flow) outside the BL. (\( h \) is the distance from the cylinder to the first node point in the fluid.) But the first and spurious term, which necessarily dominates for all cases of interest (\( h \ll a \)) does not—and the physics within the BL is completely lost. Not only does this ‘numerical BC’ fail to describe reality, it clearly diverges and leads to unbounded pressure for \( h \to 0 \)!

By requiring that the far field PPE BC, \( \partial P/\partial r = -W_1 \cos \theta \cdot e^{-\lambda t} \), dominate the spurious one in (29) the following (approximate) “window of non-believability,” can be derived (see GS for details):

\[
\tau_{ac} \ln(ah/\sqrt{\nu \tau_{ac}}) < t < \tau_{MTB},
\]

where

\[
\tau_{MTB} = h^2/4\nu.
\]

is called the Minimum Time of Believability—and actually came about while analyzing the limiting case \( \lambda = \infty \), the impulsive start. For \( \lambda = \infty \), the numerical solution can not be believed for \( t < \tau_{MTB} \); whereas for the rapid startup (finite \( \lambda \)) the numerical solution is mostly believable for \( t < \tau_{ac} \ln(ah/\nu \tau_{ac}) \) because the acceleration-dominated phase is not so difficult to compute successfully. It is not believable when (30) applies, and both types of startup are believable/useful when \( t > \tau_{MTB} \), because the mesh can now ‘follow’ the viscous diffusion process because the BL now contains at least 1 node point. Of course, for the finite \( \lambda \) case, a good mesh design would preclude the left inequality in (30) via

\[
\tau_{MTB} = h^2/4\nu = \tau_{ac} \ln(ah/\nu \tau_{ac})
\]

which, for given \( \nu \) and \( \tau_{ac} \), can be solved for \( h \)—and we point out the bad news that \( \tau_{ac} \to 0 \Rightarrow h \to 0 \). Even when (32) is satisfied, however, the details of the solution within the BL cannot be captured by the finite mesh for \( t < \tau_{MTB} \); e.g., in GS is shown the following for the ‘numerical version’ of \( C^D \):

\[
C^D = 2\pi\nu(\lambda/\omega_1 h)
\]

for \( t < \tau_{MTB} \), vis-a-vis the correct result, given in (23)–(25).

Enough on the ‘theory’ of the spatial numerics. Now we address the fact that, just as we cannot compute with \( h = 0 \), so too can we not time-integrate our resulting differential-algebraic equations (DAE’s) exactly—we must also deal with finite \( \Delta t \). The GFEM approximation to (1)–(5) is (cf., e.g., GS for details)

\[
M\dot{u} + N(u)u + CP = -Ku + f(t)
\]

and

\[
C^T u = g(t)
\]

with, when well-posed (in the ‘strict’ sense), an IC \( u_0 \) that satisfies

\[
C^T u_0 = g(0) \equiv g_0.
\]

These equations also imply the following discrete PPE—complete with built in BC’s:

\[
(CTM^{-1}C)\dot{P} = CTM^{-1}[f - Ku - N(u)u] - \dot{g},
\]

and we briefly discuss how two simple ODE methods behave when applied to either the index 2 system of DAE’s given by (34), or the index 1 system ‘defined’ by (34a) and (36); i.e., the PPE replaces (34b) in the index 1 formulation. Implicit time integration methods are ‘natural for the index 2 formulation whereas the index 1 formulation is—after the mass is lumped (which is not always possible) to convert \( M \) (and thus \( M^{-1} \)) to a
diagonal matrix—a menable to explicit methods; see GS for details on these issues.

But since we will also show some results using the penalty method, which method reduces the index on the DAE’s to zero (i.e., to ODE’s), we first briefly summarize the method: the fluid’s incompressibility is ‘relaxed’ via the ‘penalty’ equation that relates pressure to incompressibility violation,

\[ QP = \lambda(C^T u - g), \quad (37) \]

where \( \lambda \) is the “penalty parameter” and is very large (typically \( 10^6 - 10^{10} \)), and \( Q \) is the pressure mass matrix (see, e.g., GS). Inserting (37) into (34a) yields the (very stiff) system of penalty ODE’s:

\[
\begin{align*}
M\ddot{u} + N(u)u &= -(K + \lambda CQ^{-1}C^T)u + f(t) + \lambda CQ^{-1}g(t). \\
&= -(K + \lambda CQ^{-1}C^T)u + f(t), \quad (38)
\end{align*}
\]

See Engelman et al. (1982) for discussion related to the “penalty matrix,” \( B = CQ^{-1}C^T \), and see GS for a discussion of the spurious penalty transient—whose behavior we shall later demonstrate. Note the implied ODE for the pressure [from (37) and (38)]:

\[
\frac{1}{\lambda} Q \ddot{p} + (C^T M^{-1} C) \dot{p} = C^T M^{-1} [f - (K + N(u))u] - \dot{g}, \quad (39)
\]

vis-a-vis (36), to which (39) ‘returns’ when \( \lambda t >> 1 \). Finally, we apologize for the introduction of a different parameter with the same symbol (\( \lambda \))—but ‘excuse ourselves’ by stating that we will not (but could have) apply the penalty method to the \( e^{-\lambda t} \) case.

We now turn to the simplest implicit ODE method; backward Euler applied to the index 2 DAE’s gives

\[
\begin{align*}
M(u_{n+1} - u_n) + N(u_{n+1})u_{n+1} &= -(K + \lambda CQ^{-1})u_{n+1} + f_{n+1} \\
\Delta t \\
+ C\dot{p}_{n+1} &= -Ku_{n+1} + f_{n+1} \\
\therefore \quad C^T u_{n+1} &= \phi_{n+1}, \quad (40)
\end{align*}
\]

and

\[
C^T u_{n+1} = \phi_{n+1}, \quad (41)
\]

whereas the simplest explicit method (forward Euler, FE) applied to the index 1 DAE’s gives

\[
\begin{align*}
\frac{M(u_{n+1} - u_n)}{\Delta t} + N(u_n)u_n &= -(Ku_n + f_n) \\
\therefore \quad C^T u_n &= \phi_n, \quad (42)
\end{align*}
\]

wherein \( \phi_n \) is first computed from

\[
\begin{align*}
(C^T M^{-1} C) \phi_n = C^T M^{-1} f_n \\
-\frac{Ku_n - N(u_n)u_n}{\Delta t} + \frac{g_{n+1} - g_n}{\Delta t} \quad (43)
\end{align*}
\]

We presented these two well-known methods in detail only because we need to address the question of how they ‘perform’ for a step change in velocity that violates discrete mass conservation; i.e., for our impulsive changes, we are employing IC’s that do not satisfy \( C^T u_0 = g_0 \)—and we are facing an ill-posed system of DAE’s, and one for which an honest/rigorous trapezoid rule (TR) for time integration would ‘announce’ the ill-posedness via WIGGLES (ringing via \( 2\Delta t \) oscillations; see GS for details).

Thus, starting with BE for \( n = 0 \) we obtain from (40) and (41) for the first time step, with \( C^T u_1 = g_1 \) but \( C^T u_0 \neq g_0 \),

\[
\begin{align*}
(C^T M^{-1} C) \phi_1 &= C^T u_0 - g_0 \\
+ \frac{C^T M^{-1} [f_1 - Ku_1 - N(u_1)u_1] - \frac{g_1 - g_0}{\Delta t}}{\Delta t} \quad (44)
\end{align*}
\]

for the pressure—where in our case \( g_1 = g_0 \), for the impulsive start (or stop). Clearly, when \( C^T u_0 \neq g_0 \), it gives \( \phi_1 \to \infty \) as \( \Delta t \to 0 \), thus reflecting the ill-posedness of impulsive changes. It turns out, however, that we can turn this apparent ‘problem’ into a solution simply by noting that for sufficiently small \( \Delta t \) (44) becomes

\[
\begin{align*}
(C^T M^{-1} C) \phi \equiv C^T u_0 - g_0, \quad (45)
\end{align*}
\]

where \( \phi \equiv \Delta t \phi_1 \), which remains finite for all \( \Delta t \), corresponds to the “potential flow adjustment” presented earlier for the continuous case. Thus, for sufficiently small \( \Delta t \), a single BE step performs the \( L^2 \)-projection to the div-free subspace that is associated with impulsive changes in normal velocity—a fact that we shall put to good use later.

Switching now to the explicit Euler method on the lowered-index DAE’s, we examine the first step of FE applied to (42) and (43) by setting \( n = 0 \) there. Inserting \( \phi_0 \) from (43) into (42) and operating on the result with \( C^T M^{-1} \) gives
\[ C^T u_1 - g_1 = C^T u_0 - g_0, \]  
(46)
a 'general' result in the sense that any ODE method applied to the index 1 DAE's will 'preserve' the divergence (see Gresho 1991a, GS, for details). Thus, the 'PPE method' can not be used to obtain useful results when applied to impulsive changes.

Later, we shall demonstrate the above; i.e., we will show useful results from BE and discuss the useless ones from FE. We will also show good results for rapid, not impulsive, changes via the \( e^{-\lambda t} \) BC. In GS we spent lots of time on both rapid and impulsive startups, both theoretical and via the GFEM and two time-marching methods, BE and TR, both in the 'smart' mode (variable \( \Delta t \) based on physics). Here we shall summarize some of that work, to set the stage for what is new herein—viz.,: (1) impulsive and fast shutdowns, (2) the behavior of several time integration methods applied to the 'illegal' impulsive case, and (3) the solution to the sudden stop case via the penalty method. Thus we shall consider this paper to be an extension of the impulsive (and rapid) start case that is Section 3.19 in GS.

4. NUMERICAL RESULTS FOR STARTUPS

In this section we summarize and paraphrase some of the results in Section 3.19 of GS—partly because they are rather interesting and merit a second visit, but mainly to set the stage for the next section—rapid shutdowns. Fig. 1 shows both the domain and the mesh employed. The origin of the \( x-y \) coordinate system is at the center of the unit radius cylinder, and the domain covers \(-3 < x < 3\) and \(0 < y < 2\).

![Figure 1: The mesh of 4290 Q2P-1(9/3) elements; 16,965 nodes.](image)

The IC is \( u_0 = 0 \) and the BC's at the inlet \((x = -3)\) are \( u = w_1(1 - e^{-\lambda t}) \), \( \nu = 0 \) where \( w_1 = 0.1 \) and \( \lambda = 100 \) or \( \infty \)—the latter case (impulsive) being 'solved' by the technique discussed above of taking one very small BE time step. Homogeneous natural boundary conditions (NBC's) were used at \( x = 3 \) (\( \nu \partial \nu / \partial x = 0 \)) as outflow conditions; \( u = 0 \) on the cylinder, and symmetry BC's were used elsewhere. We chose a Reynolds number, \( Re = 2aw_1/\nu \), of 1000 which translates to \( \nu = 0.0002 \). The mesh shown in Fig 1(a) has 4290 9-node elements, with 16,965 elements.
nodes—and we used an equivalent mesh (same number of nodes) for the 4-node element.

To 'set the stage', we show first the pressure field for two steady-state results, in Figs. 2 and 3: potential flow and Stokes flow. Fig. 4 shows the velocity potential, which will be important for both startups and shutdowns. The potential flow pressure in Fig. 2, while looking quite good, is actually associated with a somewhat bad velocity—revealed by the wiggles in the line plots of Fig. 5.

The less 'stable' (for Stokes flow) element, \( Q_4(0) \) (4-node velocity, piecewise-constant pressure), was run on (virtually) the same mesh with the less wiggly results shown on the right side of the same figure. We do not understand the cause of these wiggles—which vanish later in time when viscosity has had a chance to 'help', and which also decrease with mesh refinement thus not precluding convergence—and we implore again (as we did in GS) the FEM mathematicians to study this second-order elliptic problem \( (\mathbf{u} = \nabla \phi \text{ and } \nabla \cdot \mathbf{u} = 0 \text{ with mixed methods}) \) using 'Stokes elements'.

**Figure 2:** Potential flow pressure; \( P_{\text{max}} = 0.00505, P_{\text{min}} = -0.0284 \) \((\Delta P = 0.00167)\)

**Figure 3:** Stokes flow pressure; \( P_{\text{max}} = 3.73 \times 10^{-4}, P_{\text{min}} = -0.5 \times 10^{-4} \) \((\Delta P = 2.14 \times 10^{-5})\).

**Figure 4:** Velocity potential; \( \phi_{\text{max}} = 0.80, \phi_{\text{min}} = 0 \) \((\Delta \phi = 0.04)\)
Moving on to the transient behavior, we point out that using either the potential flow plus no slip IC (realized via $u_0 = 0$, $u = w = 0.1$ at inlet, and the $L_2$-projection via the small $\Delta t$ BE 'trick' described earlier), or the $w_1(1-e^{-\lambda t})$ inlet BC, gives the 'same' numerical results for $\Delta t > 10$ (see GS for details, and for the differences when $\Delta t < 10$); we display here merely a sample of our ('long time')computed results—
in Figs. 6 through 8, just to give the reader a 'feel' for the startup results, and to provide an IC, that in Fig. 8, for the shutdown simulations to follow. Here, and in the sequel, all results are obtained using the $Q_2P_1$ element (also called 9/3). (The 4/1 element would deliver virtually the same results.) Flow separation, plus advection, began somewhat prior to $t = 5$, and visible eddy growth was present by $t = 5$. 

---

**Figure 5**: Potential flow: 9/3 results on the left, 4/1 on the right.
Figure 6: Solution at $t = 10$. 

(a) Pressure; $P_{\text{max}} = 0.0091$, $P_{\text{min}} = -0.0232$ ($\Delta P = 0.00162$).

(b) Vorticity; $\omega_{\text{max}} = 3.20$, $\omega_{\text{min}} = -7.20$ ($\Delta \omega = 0.52$).
(a) Stream function

(b) Pressure; $P_{\text{max}} = 0.014, P_{\text{min}} = -0.034$ ($\Delta P = 0.0032$).

(c) Vorticity; $\omega_{\text{max}} = 4.62, \omega_{\text{min}} = -7.12$ ($\Delta \omega = 0.78$).

Figure 7: Solution at $t = 25$. 
(b) Pressure; $P_{\text{max}} = 0.014$, $P_{\text{min}} = -0.040$ ($\Delta P = 0.0027$).

(c) Vorticity; $\omega_{\text{max}} = 2.09$, $\omega_{\text{min}} = -7.01$ ($\Delta \omega = 0.35$).

Figure 8: Solution at $t = 40$.

For further details regarding the startup cases, as well as drag coefficients and the demonstrated bad early time results—such as pressure and vorticity $\sim O(1/h)$ for $t < \tau_{MTB}$, et al.—see GS.
5. NUMERICAL RESULTS FOR SHUTDOWNS

We now switch gears and, starting from the solution in Fig. 8, present and discuss some shutdown results—performed both impulsively and quickly, the latter via \( w_0 = 0.1 \) and \( \lambda = 100 \) at the inlet in (2), with \( w_1 = 0 \). We shall also investigate the behavior of the penalty method. In all cases, we employed slippery BC’s \( (f_x = 0, \text{ a zero pseudo-shear stress—see GS for details}) \) except on the cylinder, which had \( u = 0 \).

We begin by noting the somewhat remarkable fact that there are rather many seemingly different/disparate ways of getting to (virtually) the same place—a result that may surprise some. (We hope so!) The “place” they get to, starting from the IC’s discussed above, is this: the vorticity-preserving potential flow adjustment plus a new vortex sheet on the cylinder. Here is the list of those that we have discovered thus far—all of which are virtually independent of Reynolds number, and most of which are discussed in GS:

1. An \( L^2 \)-projection.
2. One very small backward Euler time step.
3. Exponential decay BC\( (e^{-\lambda t}) \) for large \( \lambda \).
4. The penalty method with accurate time integration through the spurious penalty transient.
5. The penalty method with inaccurate time integration.
6. One very small FE step on the index 2 DAE’s.
7. The exact (analytical) generalized (and discontinuous) solution to the (ostensibly ill-posed) index 2 DAE’s.

Another way to perhaps state these results is this: We have found six different ways to closely approximate the \( L^2 \)-projection—some which we demonstrate below.

Also interesting—and somewhat surprising—is that each of these techniques can generate nearly the same result (at least for small \( t \)) for 3 different sets of BC’s:

1. Close both ends (slam the door at inlet and outlet).
2. Close only the right end.
3. Close only the left (inlet) end.

In the latter two cases, the “standard” homogeneous (“do nothing”) NBC’s \( \nu \frac{\partial u}{\partial x} - p = 0 \) \( = \nu \frac{\partial v}{\partial x} \), are applied at the open end.

Also noteworthy is the general observation that, since the impulsive shutdown is obtained (effectively if not directly) by subtracting the appropriate potential flow from the given flow, the closer the original flow is to a potential flow, the closer will be the sudden stop to a slow and ‘boring’ flow—with the limit being this: any potential flow IC will be stopped dead; \( u = 0 \) everywhere. Another way to state this is this: only a potential flow can be impulsively stopped. [In the general case, (only) the vortical portion of the flow remains after the ‘attempted’ impulsive stop.] This fact, in fact, leads to a potentially useful way to debug portions of a code; viz (1) solve a (slippery) potential flow problem with the code, (2) slam the doors on this IC and take one small BE time step (still permitting slip, of course). Any non-zero flow that your code computes should be close to zero—and vary like \( O(\Delta t) \). We in fact begin our shutdown presentation with such a flow; Fig. 9 shows the residual flow from the code used in this paper (FIDAP)—in which only the left end is closed off (the small inflow/outflow at the right in Fig. 9(a) would vanish if both ends had been closed). The upper left corner “noise” results from the use of large, distorted elements, and is further discussed in GS. But the main points are these: (1) the flow is nearly zero,
and (2) the error does scale with $\Delta t$ (here $\Delta t = 5 \times 10^{-6}$) thus verifying both theory and code.

We now present some shutdown simulations starting from the $Re = 1000$ solution in Fig. 8—and, unless stated to the contrary, the results to follow are also at $Re = 1000$. Fig. 10 shows the impulsive stop stream function after one small BE step ($\Delta t = 10^{-5}$) for each of the 3 BC's discussed above—and Fig. 11 shows the corresponding vorticity and pressure for (virtually) all 3.

**Figure 9:** Residual flow upon closing the left end on a potential flow.
(a) Both ends closed (\(\psi_{\text{min}} = -0.12293, \ \psi_{\text{max}} = 0.00224\))

(b) Right end closed, left end open (\(\psi_{\text{min}} = -0.12293, \ \psi_{\text{max}} = 0.00356\))

(c) Left end closed, right end open (\(\psi_{\text{min}} = -0.12589, \ \psi_{\text{max}} = 0.00224\))

Figure 10: Flow field (stream function) after 1 small time step via backward Euler.
It is obvious that the differences in $\psi$ are quite small, showing once more the utility of the homogeneous NBC's as OBC's. Noteworthy also are the following: (1) The trapped eddy is spinning clockwise, as is the large eddy in Fig. 8, and the flow "upstream" is near-zero—both consequences of subtracting the potential flow from the viscous flow—and its intensity in the eddy is now greater ($\psi \equiv -0.125$ at the center of the eddy, vs. $-0.073$ prior to the "projection"); (2) The vortex "sheet" has a peak vorticity of $+23.5$ on the cylinder (a rather far cry from infinity!), which corroborates almost perfectly with that from the impulsive start case reported in GS: $-23.6$; (3) The vorticity away from the vortex sheet is virtually the same as that in Fig. 8—another consequence of the potential flow adjustment: conservation of vorticity; (4) The pressure is that after 2 time steps, since the 1st step is a potential field—per Fig. 4 and Eqn. (45).

Next, in Figs. 12 through 15 for the left-end-closed BC, we trace the evolution of the trapped flow—and point out that the only "brakes" on the system (besides viscous dissipation) are those from the no-slip BC on the cylinder; all other BC's are frictionless.
Figure 12: Solution at $t = 5$ after impulsive stop; left end closed.

(a) Stream function ($\psi_{\text{min}} = -0.12364$, $\psi_{\text{max}} = 0.00859$)

(b) Vorticity ($\omega_{\text{min}} = -1.335$, $\omega_{\text{max}} = 10.406$)

Figure 13: As in Figure 12 except $t = 10$.

(a) Stream function ($\psi_{\text{min}} = -0.12076$, $\psi_{\text{max}} = 0.01038$)

(b) Vorticity ($\omega_{\text{min}} = -4.5260$, $\omega_{\text{max}} = 11.571$)
(a) Stream function (ψ_{min} = -0.11109, ψ_{max} = 0.03460)

(b) Vorticity (ω_{min} = -1.9212, ω_{max} = 6.2933)

**Figure 14:** As in Figure 12 except t = 20.

---

(a) Stream function (ψ_{min} = 0 - 0.09867, ψ_{max} = 0.040937)

(b) Vorticity (ω_{min} = -1.333, ω_{max} = 1.308)

(c) Pressure (P_{min} = -0.01383, P_{max} = 0.014525)

**Figure 15:** As in Figure 12 except t = 40, and pressure is shown.
The combination of the initial large vortex and the vortex sheet evolves into a separated flow that spawns another eddy (Fig. 13) and yet another (Fig. 14), while the large eddy's center performs another sort of clockwise rotation on the right side of the cylinder—and these figures can also be 'interpreted' as an upstream motion of the original 'wake'. Although we went beyond $t=40$ (Fig. 15) in our simulations, we show no more because—in fact—the OBC at the right finally did cause some poor, nonphysical behavior. We could have closed both ends, but chose not to because we wanted to compare the sudden stop with the $e^{-\lambda t}$ shutdown, which BC would be pretty awkward to apply at the right end. Thus, this and all remaining simulations are for the case of closing off only the left (inlet) boundary.

Figs. 16 through 19 show the analogous results for the somewhat unreal case of Stokes flow—starting from the same IC—and were generated just to see how the flow would evolve with advection absent (no wake "washing" over the cylinder).

(a) Stream function ($\psi_{\text{min}} = -0.12583$, $\psi_{\text{max}} = 0.002259$)

(b) Vorticity ($\omega_{\text{min}} = -6.8677$, $\omega_{\text{max}} = 20.619$)

(c) Pressure ($P_{\text{min}} = -0.007235$, $P_{\text{max}} = 0.002236$)

Figure 16: Stokes flow solution at $t = 0.1$ after impulsive stop; left end closed.
Figure 17: As in Figure 16 except $t = 1$.

Figure 18: As in Figure 16 except $t = 10$. 
Whereas the stream function decay is rather boring, the vorticity evolution via diffusion is interesting. Recall that vorticity flux through the cylinder surface is given by $\nu \frac{\partial \omega}{\partial t} = \partial \psi / \partial r$ (see Gresho 1991a); the flux is large when the tangential pressure gradient is. Finally we point out that the decay toward zero flow occurs on a very long time scale; $\tau_D = D^2 / \nu = 20,000$.

Now we turn to the “rapid” shutdown, via the $e^{-\lambda t}$ decay of the inlet BC. And we only show the part that is interesting and different from the impulsive stop—namely that for $t \leq 10 \tau_{ac} = 10 / \lambda = 0.1$, because for times greater than this the two results are very close. Thus, Figs. 20 through 26 show the interesting evolution when one “shuts down the pump” via an exponential spindown.
Figure 20: Stream function at $t = 0.001$ (inflow = 90.5% of initial) for $e^{-\lambda t}$ shutdown; ($\psi_{\text{min}} = -0.0773$, $\psi_{\text{max}} = 0.18097$).

Figure 21: As in Figure 20 except $t = 0.01$ (inflow = 36.8% of initial); $\psi_{\text{min}} = -0.10485$, $\psi_{\text{max}} = 0.07390$.

Figure 22: As in Figure 20 except $t = 0.03$ (inlet = 4.98% of initial); $\psi_{\text{min}} = -0.12283$, $\psi_{\text{max}} = 0.01022$.

Figure 23: As in Figure 20 except pressure at $t = 0.04$ ($P_{\text{min}} = -1.5455$, $P_{\text{max}} = 0.00011$).
Figure 24: As in Figure 23 except $t = 0.06$ ($P_{\text{min}} = -0.2306$, $P_{\text{max}} = 0.00011$).

Figure 25: As in Figure 23 except $t = 0.08$ ($P_{\text{min}} = 0.04248$, $P_{\text{max}} = 0.00240$).

Figure 26: As in Figure 23 except $t = 0.10$ ($P_{\text{min}} = -0.03733$, $P_{\text{max}} = 0.00918$).

The flow field seems to "evolve" somewhat more quickly than the pressure in that it has undergone most of its transition by $t = 3\tau_{ac}$ (Fig. 22) and the new, opposite-signed, vortex sheet is already nearly at full strength (not shown)—yet the pressure is still "dominated" by the deceleration transient (i.e., it's still mostly a potential field decaying like $e^{-\lambda t}$); see Fig. 23. But from $5\tau_{ac}$ to $10\tau_{ac}$ (Figs. 24 through 26), the pressure undergoes a rapid transition from acceleration-dominated to one reacting to the advective source term $-\nabla \cdot (u \cdot \nabla u)$ and the viscous Neumann BC; see (6) and (7). Beyond $t \approx 0.10$, this simulation closely mimics that in Figs. 12 through 15.

Turning now to the penalty method, we solved the impulsive stop problem (up to a time corresponding to the attainment of the equivalent $L^2$ - projection) in two ways: (1) using an initially very small time step and integrating the penalty ODE’s with a smart, variable-step time-accurate integrator (see GS for how to do this) and (2) via a "large" $\Delta t$, 2-time-step procedure that bypasses the entire penalty transient yet, perhaps surprisingly, ends up at the same place. We chose the penalty parameter, $\lambda$, to be $10^7$, and began our time-accurate simulation with $\Delta t_0 = 10^{-10}$. Conversely, for the 2-step run, we set $\Delta t_0 = 10^{-2}$ and took just two BE steps. Defining the penalty time constant as $\tau_p \equiv D^2/\lambda = 4 \times 10^{-3}$, gives $\Delta t_0/\tau_p = 0.00025$ for the time-accurate case, whereas $\Delta t/\tau_p = 25000$ for the 2 step case—rather a large range of step sizes.

Figs. 27 through 30 show the penalty pressure [from (39); actually, of course, from (37)] for the impulsive stop at the left end (slam the left door, leave the right one open).
Figure 27: Pressure at $t = 0$ for penalty method (left end closed); 
\[ P_{\text{min}} = -1.69 \times 10^7, \quad P_{\text{max}} = 1.33 \times 10^6. \]

Figure 28: Penalty pressure at $t = 10^{-9}$ \[ (P_{\text{fin}} = -5.44 \times 10^6, \quad P_{\text{min}} = 4.48 \times 10^4) \]

Figure 29: Penalty pressure at $t = 10^{-8}$ \[ (P_{\text{fin}} = -1.68 \times 10^6, \quad P_{\text{max}} = 7.53 \times 10^{-3}) \]

Figure 30: Penalty pressure at $t = 10^{-7}$ \[ (P_{\text{min}} = -7.26 \times 10^5, \quad P_{\text{max}} = 1.97) \]
During this phase of the penalty transient (the spurious, pressure 'shock wave,' portion), the bulk of the velocity field remains virtually unchanged from the IC; i.e., both $\psi$ and $\omega$ look much like Fig. 8 and the magnitudes change little until $t$ reaches $10^{-7}(= l/\lambda)$ or so. But they change *rapidly* for $t > 10^{-7}$ and are 'finished' by $-t = 10^{-5}$. Fig. 31(a) shows the stream function at $t = 10^{-6}$—for which the pressure, in Fig. 31(b) still looks like a potential—which is clearly tending toward the desired result.

(a) Stream function ($\psi_{\text{min}} = -0.09075$, $\psi_{\text{max}} = 0.13054$)

(b) Pressure ($P_{\text{min}} = -2.16 \times 10^5$, $P_{\text{max}} = 0.034$)

**Figure 31:** Penalty solution at $t = 10^{-6}$.

(a) Stream function ($\psi_{\text{min}} = -0.12546$, $\psi_{\text{max}} = 0.00269$)

(b) Pressure ($P_{\text{min}} = -872.1$, $P_{\text{max}} = 0.00022$)

**Figure 32:** Penalty solution at $t = 10^{-5}$
The penalty transient for the velocity is over by $t = 10^{-5}$ [see Fig. 32(a), both $\psi$ and $\omega$ now agree very closely with the projected solution], but certainly not for the pressure (Fig. 32(b), which is still more like a potential function—of the previous, $e^{-\mu}$, case. Even 10-fold farther in time ($t = 10^{-4} = 250\tau_p$) still shows a potential-like pressure—see Fig. 33. Finally, after one more decade, $t = 10^{-3}$, we see the pressure transient completed; the pressure field in Fig. 34 agrees well with the “projected” pressure field corresponding to Fig. 11b.

![Figure 33: Penalty pressure at $t = 10^{-4}$](image)

$(P_{\text{min}} = -0.03673, P_{\text{min}} = 2.26 \times 10^{-5}, P_{\text{max}} = 13.38)$

![Figure 34: Penalty pressure at $t = 10^{-3}$](image)

$(P_{\text{min}} = -0.03673, P_{\text{max}} = 0.01002)$

Incidentally, the entire penalty transient was accurately tracked via the smart (variable-step) BE method described in GS; $\Delta t$ grew to about $10^{-3}$ by a time of $10^{-3}$, the entire simulation needing only 80 steps.

Having shown that a time-accurate penalty transient finds the correct “common” result at the conclusion of the penalty transient, we performed a much more important/relevant simulation that overlooks the spurious penalty transient. After all, if it were necessary to track this transient in order to “recover” properly, the penalty method would be much less attractive—especially for fixed-$\Delta t$ integrators! Fortunately, this is not the case. With no need to show the results, we simply state that the 2-step run using $\Delta t = 10^{-2}$ (or $10^{-4}$, in fact) achieved the proper result for the velocity after a single BE time step (as for the index 2 DAE formulation), while a ‘proper’ result for $P$ was not obtained until step 2 because—as discussed in GS—the first ‘large $\Delta t$’ penalty pressure is, like BE on the index 2 problem, really $\phi/\Delta t$. So we have shown again that the penalty method is quite viable when $\Delta t$ is large, causing the spurious penalty transient to be gracefully by-passed via BE. (Do not try this trick using the more accurate trapezoid rule!)

To conclude this portion of the presentation, we present in Table 1 a summary of results. Although mostly self-explanatory, we offer a few additional remarks:

1. The near-invariance of $\psi$ and $\omega$ between steps 1 and 2 of the 2-step runs ‘justifies’ our $\Delta t$ selection.
2. The solutions at $t = 0.10$ have suffered some reduction in the vorticity; perhaps a larger $\lambda$ is needed.
3. These mathematical simulations naturally preclude the two ‘real-world’ phenomena of cavitation and water hammer.

The final time integration method applied to the sudden shutdown was explicit Euler—applied to the index 1/$PPE$ version of the DAE’s. But the results are hardly worth presenting, let alone showing. The “div-preservation” equation, (46), holds true and causes the following behavior: (1) the only mass imbalance at
\( t = 0 \) occurs in the first column of elements at the inlet—via the change in BC from \( u = w_0 = 0.1 \) to \( u = 0 \); (2) the requirement to preserve the div necessarily causes the u-velocity at the first row of nodes to the right of the inlet nodes to hold a value very close to \( w_0 \); (3) the result of this is this: the flow is not shut-down—rather, it's much like discarding the first column of elements and applying the BC \( u = w_0 \) at the second column. And this is just what occurred: FE produced a continued integration of the startup run!!

To finish the presentation, we show in Fig. 35 the drag coefficients for three of the shutdown cases presented earlier.

**Figure 35:** Drag coefficients for three stopped flows.
The heavy lines are the total drag coefficients, the lighter solid lines are the pressure contributions, and the dotted lines are the viscous contributions. In all three cases, the initial value of \( C_D \) (\( t = 40 \) for the startup case) was about 3; see GS. Figs. 35(a) and (b) show the \( Re = 1000 \) shutdowns, about which we make four remarks:

1. Negative \( C_D \Rightarrow \) the fluid force on the cylinder is in the \(-x\)-direction.
2. \( C_D(t) \) is truncated/clipped at \(-1.5\) for the exponential case for plotting purposes; \( C_D(0) \) is actually about \(-8000\), and approximates the analytical result given by \( C_D(t) = -2na\lambda e^{-\lambda t}/\omega_0 \) from the decelerating potential flow; see (26).
3. \( C_D(0) = -0.3 \) for the impulsive stop is a spurious result caused by our finite mesh; \( C_D(0^+) \) should be \(-\infty \) owing to the vortex sheet.
4. The small difference (for \( t > \tau_{MTB} \equiv 0.6 \)) between Figs. 35 (a) & (b), which are not even visible for the analogous and simpler startup case (see GS), are a reflection of the extra ‘dynamics’; i.e., there is enough difference in the two cases for \( t < \tau_{MTB} \) to be ‘noticed,’ even on our finite mesh.

Finally, Fig. 35(c) shows the impulsive stop drag coefficient for Stokes flow. Again the value of \( C_D(0) \) is quite spurious; it is in fact not too far from the (negative of the) empirically-determined mesh-dependent result reported in GS—\( C_D \equiv 38\nu/\omega_s \lambda \equiv 3.45 \) for the startup case, both representing the finite \( h \) spurious representation of a vortex sheet. Finally, we opine that all of our \( C_D(t) \) results are reasonably accurate for \( t > \tau_{MTB} \equiv 0.6 \), partly because 9/3 and 4/1 'agree'.

6. CONCLUDING COMMENTS

It might seem that our shutdown simulations are too much influenced by our tightly-bounded domain and—especially for those familiar with the problem of stopping a moving cylinder (or sphere, or other object)—perhaps not even ‘correct’ in that the original wake/eddy simply tends to mostly spin in place. While we plead guilty to a rather small domain (designed originally for studying only small-time results near the cylinder for fast startups), we believe and assert that a larger one (even much larger) would have only ‘secondary’ effects on the resulting shutdown flow (and M. Maxey agrees with us; personal communication); i.e., the eddy motion and behavior would be much the same as presented here. Evidence for this assertion can be seen in Fig. 13, and related discussion, in Chang and Maxey (1995).

Finally, in addition to the obvious conclusions from the results presented, we offer the following conclusion, based on our experience with both startups (see GS for details) and shutdowns (summarized herein): our \( e^{-\lambda t} \) method is both more realistic than the ‘impulse-method’ and it sheds better understanding via analysis of the large \( \lambda \) situation (ultimately for \( \lambda \to \infty \)).

ACKNOWLEDGMENTS

We have profited considerably from discussions with Profs. R. Mei, M. Maxey, and J. Brady—and Dr. P. Lovalenti. Assistance from Drs. A. Hindmarsh and D. Veyret are also gratefully acknowledged, as is the expert document preparation by A. Henke. This work was sponsored by the U.S. Department of Energy Environmental Sciences Division and performed by the Lawrence Livermore National Laboratory under Contract No. W-7405-Eng-48.

REFERENCES


<table>
<thead>
<tr>
<th></th>
<th>IC(1)</th>
<th>Both closed; step1/step2(2)</th>
<th>Right closed; step1/step2(2)</th>
<th>Left closed; step1/step2(2)</th>
<th>$e^{-x}$ at $t = 0.10^{(3)}$</th>
<th>Impulsive at $t = 0.10^{(3)}$</th>
<th>Penalty(4) at $t = 10^{-3}$</th>
<th>Penalty at $\Delta t = .01$; step1/step2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\psi_{min}$</td>
<td>0.07325</td>
<td>0.12293/0.12293</td>
<td>0.12293/0.12293</td>
<td>0.12589/0.12589</td>
<td>0.12579</td>
<td>0.12577</td>
<td>0.12589</td>
<td>0.12587/0.12587</td>
</tr>
<tr>
<td>$\psi_{max}$</td>
<td>0.20000</td>
<td>0.00224/0.00224</td>
<td>0.00356/0.00356</td>
<td>0.00224/0.00224</td>
<td>0.00225</td>
<td>0.00225</td>
<td>0.00224</td>
<td>0.00256/0.00244</td>
</tr>
<tr>
<td>$-P_{min}$</td>
<td>0.03997</td>
<td>82070$^{(5)}/0.04595$</td>
<td>0.0837$^{(5)}/0.03560$</td>
<td>80356$^{(5)}/0.03672$</td>
<td>0.03733</td>
<td>0.03682</td>
<td>0.03673</td>
<td>4.4x10$^{-5}$/0.03443</td>
</tr>
<tr>
<td>$P_{max}$</td>
<td>0.01431</td>
<td>$0^{(6)}/0^{(6)}$</td>
<td>81739$^{(5)}/0.01027$</td>
<td>0.0577$^{(5)}/0.01000$</td>
<td>0.00918</td>
<td>0.00986</td>
<td>0.01002</td>
<td>80.319$^{(5)}/0.01302$</td>
</tr>
</tbody>
</table>

(1) $t = 40$ from startup run
(2) BE 'projections' (2 steps at $\Delta t = 10^{-5}$)
(3) Left end closed
(4) Time-accurate
(5) The "pressure" is actually $\phi/\Delta t$
(6) Pressure ‘pegged’ at right edge (1 node) to set hydrostatic level

**TABLE 1. Summary of Results**