A PERIODICALLY-SWITCHED ODE MODEL FOR N-BUNCH BEAMLOADING IN A STORAGE RING*  

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Abstract  
A new baseband formulation of the coupled cavity/longitudinal-bunch ODEs is derived. Assuming linearity, a model of the form \( \dot{x}(t) = A(t)x(t) + B(t)u(t) \) arises, where \( A(t) \) and \( B(t) \) are piecewise constant, and periodic with the revolution period \( T_0 \). Such models, known in the control community as (periodic) switched systems, have known (in)stability criteria and control theoretic properties, which can be useful in the analysis and control of multiple bunch beamloading.

1 BASEBAND FORMULATION

The term beamloading is to imply here the dynamical interaction between a given cavity resonance and the \( N \)-bunches' longitudinal dynamics. Resonance implies band-limitedness (BL), and a standard tool for the analysis of bandlimited signals and systems (ODEs) is the IQ formalism [1]. The formalism has been applied to beamloading, especially w.r.t. the cavity ODE in [2], [3].

In the way of review and to establish notation: The Fourier transform of the resonance \( Z(j\omega) \), of order \( M \) in \( j\omega \), is assumed to be (effectively) zero for \( \omega \) outside its band. Denote the positive part of the band by \( \Omega \). Then using some carrier frequency \( \omega_c \in \Omega \), the impulse response kernel of \( Z(j\omega) \) is

\[
\tilde{Z}(t) = z_1(t) \cos \omega_c t - z_2(t) \sin \omega_c t.
\]

(1)

The utility of the IQ formalism lies in the fact that we need only consider the complex envelope, defined as \( \tilde{x}(t) = \tilde{z}(t) + j\tilde{z}(t) \), whose Fourier transform \( \tilde{Z}(j\omega) \) is also of order \( M \) in \( j\omega \). In particular, the cavity output signal \( u(t) \) to an (AM/PM) sinusoid \( f(t) \) is obtained via \( \tilde{u}(t) = \tilde{f}(t) * \tilde{z}(t) \).

1.1 Bunch Train Signal

Use of the IQ formalism presupposes AM/PM signals of the form (1). It is now shown that the beam current, modeled here as an impulse train, is seen by the resonance approximately as an AM/PM signal about the carrier \( \omega_c \).

The width of \( \Omega \) determines the minimum number of bunches that need be considered in a time domain analysis; arbitrary gaps in the beam current may make this determination difficult. Here, the number of representative bunches \( N \) is assumed known, chosen through modal analysis or made safely large.

Define \( N_{\text{b}} \) as the number of bunch current "segments": each nth beam current segment is of duration \( T_b = T_0/N_\text{b} \) and has a charge \( q_n \), \( n = 1, \ldots, N_\text{b} \). \( q_n \) may be identically zero if and only if the segment represents a gap; if there are no gaps \( N = N_{\text{b}} \). Henceforth the word bunch shall mean bunch segment.

Let \( \tau_{n,p} \) denote the nth bunch's deviation in arrival time at the cavity from the nominal, for the pth arrival. Of course if \( q_n = 0 \) then \( \tau_{n,p} \) is devoid of physical meaning; otherwise it is governed by the synchrotron ODE. However, the cavity sees the beam current as a signal, and that is the perspective of this section.

The time-infinite beam current is written, using Wilson's phasor convention [4], as

\[
i(t) = -\sum_{n=1}^{N_{\text{b}}} q_n \sum_{p=-\infty}^{\infty} \delta\left(t - \left[p + \frac{n - 1}{N_\text{b}}\right] T_0 - \tau_{n,p}\right)
\]

but, as proven in section 1.2, the following Proposition applies:

Proposition The beam current (2) is seen by an \( \Omega \)-BL resonance approximately as

\[
i(t) \approx -q(t) \frac{2}{T_b} \left[\cos \omega_c t + \omega_c \tau(t) \sin \omega_c t\right].
\]

(3)

In (3), \( q(t) = q(t + T_b) \) is a continuous-time interpolation (CTI) of \( q_n, n = 1, \ldots, N_\text{b} \), and \( \tau(t) \) is a CTI of \( \tau_{n,p}, n, p \), as depicted in Figures 1-2 and defined in the next section. Note that \( \tau(t) \) is of use only in discussing the beam current as a signal; when addressing the system aspect (section 1.3 and on), \( \tau(t) \) will be abandoned.

1.2 Proof of the Proposition

The Proposition is proved in three steps: interpolation, Taylor series approximation, and application of some Fourier properties.

Interpolation [1] The signals \( q(t) \) and \( \tau(t) \) are formally constructed via the interpolation kernel \( S_{T_b}^\star(t) = u(t + \tau - T_b/2) - u(t - T_1/2) \), where \( T_1 \) is some period, and \( u(t) = 1 \) for \( t \geq 0 \), and is zero otherwise. Define \( \bar{q}_{k} = q(-1 + k \mod N_{\text{b}}) \). Then formally,

\[
q(t) = \sum_{k=-\infty}^{\infty} \bar{q}_{k} S_{T_b}^\star(t - kT_b),
\]

(4)

\[
\tau(t) = \sum_{p=-\infty}^{\infty} \sum_{n=1}^{N_{\text{b}}} \tau_{n,p} S_{T_b}^\star \left(t - \frac{n - 1}{N_\text{b}} - pT_b\right).
\]

(5)

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Fourier Analysis Via Fourier series it is known that the signal's Taylor series expansion approximate the signal. Hence, the RHS of (6) can be rewritten as

\[ q(t) = \sum_{p=0}^{\infty} \cos \omega_b t + \sum_{p=0}^{\infty} p \sin \omega_b t \]  

Finally, aside from the considerations mentioned in section 1.1 regarding the choice of \( N \) and hence \( N_o \), here a further imposition on \( N_o \) is introduced: \( N_o \) is chosen s.t. \( k_o \omega_b \in \Omega \) for one and only one integer \( k_o \). Choose the carrier frequency according to \( \omega_b = k_o \omega_b \). Thus by design, no harmonics of \( \omega_b \) fall in the band \( \Omega \), and it can be shown (e.g., via convolution) that the \( \Omega \)-BL resonance sees only the harmonic \( p = k_o \) of (8), concluding the Proof.

Remark The derivation of (3) does not impose any assumptions on the bunch longitudinal motion other than it is of small amplitude, cf. the traditional derivations using modal analysis and Bessel functions [6]. In addition, it allows for gaps in the bunch train.

1.3 Beamloading ODEs

It is convenient henceforth to use deviation from nominal values for all variables. Then the cavity portion of the overall beamloading system at baseband can be written as

\[ \Delta v(t) = \Delta v_q(t) - j [\Delta v_q(t)] + j [\Delta v_q(t)] \]

\[ = \Delta v_q(t) + j [\Delta v_q(t)] \]

where \( I_o \) is the DC beam current.

From the perspective of the cavity resonance \( \tau(t) \) is a signal, but in truth it is determined over time by \( N \) synchrotron ODEs. These are incorporated in the overall system by now abandoning the variable \( \tau(t) \), and instead working with the new continuous state variables \( \tau_m(t), m = 1, \ldots, N \), which represent each \( m \)th bunch's arrival time deviation from nominal. The relationship between \( \tau_m(t) \) and (9) will be evident in the following development.

From the perspective of the \( m \)th bunch, the cavity voltage represents a forcing function that is nonzero only during the time intervals \( T_m \), during which the bunch couples to, i.e., passes through, the cavity. For example, bunch No. 1 of Figure 1 passes through the cavity during the intervals

\[ T_1 = \bigcup_{p=-\infty}^{\infty} \left[ pT_o - \frac{T_b}{2}, pT_o + \frac{T_b}{2} \right] \]

Otherwise, i.e., \( \forall t \in T_m \), where \( ^c \) denotes set complement, the bunch is not coupled to the cavity. The set of all time intervals that correspond to gaps is \( T_o = T \cap T_m^c \).

Defining \( \phi_m(t) = \omega_c \tau_m(t) \), the linearized synchrotron ODE can thus be written as

\[ \dot{\phi}_m + 2\alpha \phi + \omega_c^2 \phi = \begin{cases} f_c(t) & t \in T_m \\ 0 & t \in T_m^c \end{cases} \]
where $\alpha$ is the inverse damping time, and $\omega_s$ is the synchrotron frequency. It can be shown that given linearization, the RHS forcing function is given by \( f_c(t) = \frac{\omega_s^2}{V_c \sin \phi_s} [\Delta u_{II}(t) - \Delta u_{QQ}(t)] \), where $V_c$ is the peak cavity voltage, and $\phi_s$ is the synchronous phase.

Remark $\Delta u_{II}(t)$ is a function of $q(t)$, and hence is not a state but a stationary forcing function, not governed by an ODE. Therefore it does not have any bearing on (linearized) stability, see e.g., [7], Theorem 12.6. Thus, the quadrature impulse response of the cavity resonance $q(t)$ and not $y(t)$ determines linearized beamloading stability, aside from the beam ODEs. Compare the argument proposed in [3] to prove this. Note also that the result is particularly transparent here after having used the phasor reference plane of [4].

## 2 SWITCHED SYSTEM FORMULATION

By merging the cavity (9) and beam (11) system formulations developed above, one arrives at a single ODE of the form $\dot{x}(t) = A(t)x(t) + B(t)\Delta u_{II}(t)$: The $M$ cavity states $\Delta u_{QQ}^{-1}(t), \ldots, \Delta u_{QQ}(t)$ and the two bunch states $\phi_m(t), \mu_m(t)$ for each of the $N$ bunches are ordered in the column vector $x$, where the superscript now denotes derivative order. Thus $x$ has $2N + M$ elements. The ODE coefficients are corresponding elements of $A(t)$ and $B(t)$. More specifically, the resulting overall system is of the form

$$
\dot{x}(t) = \begin{cases} 
A_m x(t) + B_m \Delta u_{II}(t) & t \in T_m \\
A_o x(t) & t \in T_o
\end{cases}
$$

(12)

The top part of the RHS conveys that while the $m$th bunch passes through the cavity, the beam current (quadrature modulated by the state $\phi_m(t)$) perturbs the cavity. Meanwhile the bunch is perturbed by the cavity state $\Delta u_{QQ}(t)$, and by the forcing function $\Delta u_{II}(t)$. During the periods corresponding to gaps (bottom of RHS), the cavity and bunches are uncoupled. Thus, $A_o$ is a $(N+M)$ by $(N+M)$ block diagonal matrix, and $A_m$ contains the same block diagonal elements as well as additional diagonal coupling terms. $B_m$ is $(N+M)$ by 1, and contains only a single nonzero element.

## 3 APPLICATIONS

Given an initial condition $x_0$, say at $t = 0$, then (12) represents an initial value problem, $\dot{x}(t) = A(t)x(t), x(0) = x_0$. Two properties of the system readily lend themselves to application of ODE system theory, see e.g., [7]. First, $A(t)$ is piecewise constant, or a switched system, which means that the state transition matrix $\Phi(t, 0)$ for any $t$ can be computed as the product of matrix exponentials. For example, in the case of Figure 1, starting at $t = 0$, the state at $3.5T_b$ is given by $e^{\Delta t A_1} e^{\Delta t A_2} e^{\Delta t A_3} e^{0.5T_b A_4} x_0$. Thus, using the state transition matrix, the state $x(t)$ for any $t$ can be computed, see [8].

Second, since $A(t)$ is $T_o$ periodic, Floquet Theory can be applied to assess (in)stability:

**Theorem [Floquet]** The system (12) is stable (unstable) if and only if the magnitudes of the eigenvalues of $\Phi(T_o, 0)$ are s.t. all are less than unity (at least one is greater than unity).

A particularly useful application of this criterion is the identification of the cavity higher mode(s) that cause coupled bunch instabilities in a partially filled storage ring [9]: For each cavity mode, a new $\omega_s$ is determined, and then the eigenvalues of the corresponding $\Phi(T_o, 0)$ are checked for stability.

A final application (but originally the motivating application) is beamloading control. The authors of [10] note that the now classic optimal state space control theory does not readily apply to multiple bunch beamloading. The switched system formulation, along with some recent control-theoretic results relating to the control of such systems [11], are therefore of particular interest and are currently under study.

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## 5 REFERENCES