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Acoustic-Structure Interaction Problems

E. Love
R. L. Taylor

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**FINAL REPORT
LLNL B249873**

**STRUCTURAL ENGINEERING,
MECHANICS AND MATERIALS**

**ACOUSTIC-STRUCTURE
INTERACTION PROBLEMS**

**by
EDWARD LOVE
and
ROBERT L. TAYLOR**

DECEMBER 1993

**DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
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by

Edward Love

and

Robert L. Taylor

Department of Civil Engineering
University of California at Berkeley

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Chapter 1

Introduction

The purpose of this report is to compare and evaluate different numerical methods for solving problems of interaction between elastic solids and acoustic fluids. In particular, we concentrate our efforts on solution techniques involving the finite element method.

To that end, in Chapter 2 we discuss different options for analysis of infinite fluids. In particular, the method of mesh truncation and the use of radiation elements and the use of infinite elements are discussed. Also discussed is the analysis of scattering from rigid boundaries.

Chapter 3 is a brief discussion of finite element formulations for elastic solids. We review the development of two dimensional plane strain elements and one dimensional plate and shell elements.

In Chapter 4, there is a discussion of the method used to couple the solid and the fluid. We give examples for solution of scattering of pressure waves from thin elastic shell structures.

Chapter 5 is a brief conclusion of results and includes recommendations for the best methods of solution and additional research.

Chapter 2

Fluid Pressure Formulations

2.1 Bounded Total Pressure Problems

The pressure in an acoustic medium is governed by the wave equation, which is

$$\nabla^2 p = \frac{1}{c^2} \ddot{p} \quad (2.1)$$

where p is the time dependent pressure and c is the speed of sound in the acoustic medium. If equation 2.1 is multiplied by an arbitrary weighting function w and integrated over the domain Ω , the result is

$$\int_{\Omega} w \frac{1}{c^2} \ddot{p} d\Omega - \int_{\Omega} w \nabla^2 p d\Omega = 0 \quad (2.2)$$

The next step is to use the divergence theorem on the second term of equation 2.2. This will result in

$$\int_{\Omega} w \frac{1}{c^2} \ddot{p} d\Omega + \int_{\Omega} \frac{\partial w}{\partial x_i} \frac{\partial p}{\partial x_i} d\Omega - \int_{\Gamma} w \frac{\partial p}{\partial x_i} n_i d\Gamma = 0 \quad (2.3)$$

where Γ is that part of the boundary upon which the derivative of the pressure is prescribed, n_i is the outward normal to the domain in the x_i direction and indicial notation with summation convention has been used. It is important to note that the weighting function w is arbitrarily prescribed to be zero on boundaries where the pressure is known. Noting that $p_{,i} n_i = \nabla p \cdot \mathbf{n}$ we can write

$$\int_{\Omega} w \frac{1}{c^2} \ddot{p} d\Omega + \int_{\Omega} \frac{\partial w}{\partial x_i} \frac{\partial p}{\partial x_i} d\Omega - \int_{\Gamma} w \nabla p \cdot \mathbf{n} d\Gamma = 0 \quad (2.4)$$

A standard finite element discretization is used to approximate p within each element such that

$$p = \mathbf{N} \mathbf{p} = N_I p^I \quad (2.5)$$

where N_I and p^I are the standard finite element nodal shape functions and unknown pressures at the nodes, respectively. We also introduce

$$\frac{\partial p}{\partial \mathbf{x}} = \mathbf{B} \mathbf{p} = \mathbf{B}_I p^I \quad (2.6)$$

where

$$\mathbf{B}_I = \begin{bmatrix} \frac{\partial N_I}{\partial x_1} \\ \frac{\partial N_I}{\partial x_2} \end{bmatrix} \quad (2.7)$$

The same interpolations are used for the weighting function w and its derivatives.

Introducing the above interpolations into equation 2.4 results in

$$\sum_1^{numel} w^T \left(\frac{1}{c^2} \int_{\Omega} \mathbf{N}^T \mathbf{N} d\Omega \ddot{\mathbf{p}} + \int_{\Omega} \mathbf{B}^T \mathbf{B} d\Omega \mathbf{p} - \int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} d\Gamma \right) = 0 \quad (2.8)$$

Since the nodal parameters w are arbitrary, they can be cancelled from 2.8 resulting in

$$\sum_1^{numel} \left(\frac{1}{c^2} \int_{\Omega} \mathbf{N}^T \mathbf{N} d\Omega \ddot{\mathbf{p}} + \int_{\Omega} \mathbf{B}^T \mathbf{B} d\Omega \mathbf{p} = \int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} d\Gamma \right) \quad (2.9)$$

Finally, we can write

$$\mathbf{E} \ddot{\mathbf{p}} + \mathbf{H} \mathbf{p} = \mathbf{h} \quad (2.10)$$

where $\mathbf{H}$ is the 'stiffness' matrix, $\mathbf{E}$ is the 'mass' matrix and $\mathbf{h}$ is the 'load' vector formed by integrating the prescribed normal derivative of the pressure and the shape functions over the Neumann boundary. Boundaries where the pressure is described rather than the derivative can be handled by arranging the the vector of nodal pressures such that known boundary pressures can be easily multiplied by their stiffness and mass terms to become part of the forcing function.

In many applications of interest, we would like to solve frequency dependent problems. To do so, we will introduce the Fourier transform and inverse Fourier transform as

$$\mathbf{p}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{p}(t) e^{-i\omega t} dt \quad (2.11)$$

$$\mathbf{p}(t) = \int_{-\infty}^{\infty} \mathbf{p}(\omega) e^{i\omega t} d\omega \quad (2.12)$$

If equation 2.12 is substituted into equation 2.10 the result is

$$\int_{-\infty}^{\infty} (\mathbf{H}\mathbf{p}e^{i\omega t} - \omega^2 \mathbf{E}\mathbf{p}e^{i\omega t} - \mathbf{h}e^{i\omega t}) d\omega = 0 \quad (2.13)$$

Since the integral above must be zero for any time t , the integrand must be zero throughout the domain. Taking this fact into account and cancelling the $e^{i\omega t}$ we are left with

$$(\mathbf{H} - \omega^2 \mathbf{E}) \mathbf{p} = \mathbf{h} \quad (2.14)$$

where now the unknowns $\mathbf{p}$ and the forcing function $\mathbf{h}$ are frequency dependent rather than time dependent. To evaluate the time dependent response of a system, equation 2.14 should be solved for various frequencies and then the resulting $\mathbf{p}(\omega)$ inverted to the time domain using equation 2.12.

2.2 Unbounded Total Pressure Problems

In the previous section numerical solutions of the wave equation for bounded domains were considered. However, in many problems, such as radiation and scattering of sound from solid bodies, the fluid domain is effectively infinite. In such cases, the infinite field properties of the fluid must be taken into account.

A well known fact in acoustics is that radiated and scattered pressure waves must move away from their source. For infinite problems, this fact is known as the Sommerfeld radiation condition. In one dimension, the Sommerfeld radiation condition can be written as

$$\frac{\partial p}{\partial x} + \frac{1}{c} \frac{\partial p}{\partial t} = 0 \quad (2.15)$$

For two and three dimensional problems, the time dependent radiation condition is an infinite product of differential operators and will not be discussed here. However, the frequency dependent radiation condition can be written concisely as

$$\lim_{r \rightarrow \infty} r^{\frac{1}{2}(n-1)} \left(\frac{\partial p}{\partial r} + ikp \right) = 0 \quad (2.16)$$

where n is the dimension of the problem and k is the wavenumber, which is calculated as

$$k = \frac{\omega}{c} \quad (2.17)$$

Equation 2.16 is well known and can be found in almost all references on acoustics, including Bettess [4], Junger [12] and Morse [14].

If we have an unbounded fluid domain, the right hand side of equation 2.9 can be broken into two integrals, one over the finite part of the boundary, and one over the infinite part of the boundary. We can write

$$\int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma + \int_{\Gamma_{\infty}} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma \quad (2.18)$$

where Γ_I is the finite boundary of the domain and Γ_{∞} is the boundary at an infinite distance from the source of the radiation or scattering. Obviously, we cannot perform an integration over an infinite boundary. However, we can approximate the boundary at infinity by a finite boundary Γ_{rad} at a suitably large but finite distance from the source of the pressure waves. Equation 2.18 becomes

$$\int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma + \int_{\Gamma_{rad}} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma \quad (2.19)$$

Noting that for one dimensional problems $\nabla p \cdot \mathbf{n} = \frac{\partial p}{\partial x}$, by substituting this relationship into the integral over Γ_{rad} equation 2.19 can be written as

$$\int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma + \int_{\Gamma_{rad}} \mathbf{N}^T \frac{\partial p}{\partial x} \, d\Gamma \quad (2.20)$$

By substituting the one dimensional radiation condition of equation 2.15 into the second term on the right hand side of equation 2.20 the result is

$$\int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma - \frac{1}{c} \int_{\Gamma_{rad}} \mathbf{N}^T \dot{p} \, d\Gamma \quad (2.21)$$

If we recall the finite element discretization of the pressure is

$$\dot{p} = \mathbf{N} \dot{\mathbf{p}} \quad (2.22)$$

within each finite element and substitute this into equation 2.21 we obtain

$$\int_{\Gamma} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} \, d\Gamma - \frac{1}{c} \int_{\Gamma_{rad}} \mathbf{N}^T \mathbf{N} \, d\Gamma \, \dot{\mathbf{p}} \quad (2.23)$$

The second term on the right hand side of the above equation is the pressure velocities multiplied by the 'damping' matrix of what will be referred to as a Sommerfeld radiation element. The damping matrix of an individual element is given by

$$\mathbf{C} = \frac{1}{c} \int_{\Gamma_{rad}} \mathbf{N}^T \mathbf{N} d\Gamma \quad (2.24)$$

If the one dimensional relationship 2.23 is substituted back into equation 2.9 the result becomes

$$\begin{aligned} \sum_1^{numel} \left(\frac{1}{c^2} \int_{\Omega} \mathbf{N}^T \mathbf{N} d\Omega \ddot{\mathbf{p}} + \int_{\Omega} \mathbf{B}^T \mathbf{B} d\Omega \mathbf{p} \right) = \\ \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} d\Gamma - \frac{1}{c} \int_{\Gamma_{rad}} \mathbf{N}^T \mathbf{N} d\Gamma \dot{\mathbf{p}} \right) \end{aligned} \quad (2.25)$$

Bringing the 'damping' terms to the other side of the equation leads to the result

$$\mathbf{E} \ddot{\mathbf{p}} + \mathbf{C} \dot{\mathbf{p}} + \mathbf{H} \mathbf{p} = \mathbf{h} \quad (2.26)$$

where $\mathbf{h}$ is now calculated by integrating over only Γ_I and is

$$\mathbf{h} = \sum_1^{numel} \int_{\Gamma_I} \mathbf{N}^T \nabla p \cdot \mathbf{n} d\Gamma \quad (2.27)$$

Again, many problems are solved in the frequency domain. Introducing the inverse Fourier Transform relationship into equation 2.26 yields

$$(\mathbf{H} + i\omega\mathbf{C} - \omega^2\mathbf{E}) \mathbf{p} = \mathbf{h} \quad (2.28)$$

where now all quantities are functions of frequency. It should also be noted that now all quantities, including both the pressure and forcing function, are complex valued and have both real and imaginary components.

The Sommerfeld radiation element discussed above can be used for one, two or three dimensional problems. Of primary interest are two and dimensional domains, in which the element would be a line or surface, respectively. The element can be physically oriented arbitrarily in multi-dimensional problems. When placed at the outer 'edge' of a finite element mesh the element has the effect of approximately enforcing the one dimensional radiation condition at a finite distance from the disturbance causing the pressure waves.

The element effectively forces all pressure waves to be outgoing and travelling normal to the surface of the element. Obviously, at abrupt changes in the first derivatives of the radiation boundary, such as sharp corners, the results may not be reliable. However, if the element is placed at a sufficiently large distance from the disturbance, the error can be reduced to within an acceptable tolerance.

2.3 Infinite Elements

Another option for taking into account the effects of unbounded domains is to use infinite elements. They are applicable for use in static scalar field problems, static elasticity problems, diffusion problems and frequency dependent acoustic wave propagation problems. The last item is of course the primary concern of this report, but before discussing the elements used for wave problems, some basic concepts of infinite elements must be discussed first.

2.3.1 Basic Concepts

One very useful type of infinite element, and the one which will be used for our acoustic problems, is known as a mapped infinite element. This type of element involves a mapping, which is used to perform coordinate transformations between the real domain of the problem and the finite element natural coordinate domain.

To elaborate on the concept, consider a three node quadratic one dimensional finite element. The first node is placed at $x = x_1$, the second node at $x = x_2$ and the third node placed at $x = \infty$. We also assume that $0 \leq x_1 < x_2 < \infty$. Now, introduce the coordinate transformation

$$x = m_1(\xi) x_1 + m_2(\xi) x_2 \quad (2.29)$$

where ξ is the finite element natural coordinate and M_1 and M_2 are the

mapping functions defined as

$$\begin{aligned} m_1 &= \frac{-2\xi}{1-\xi} \\ m_2 &= \frac{1+\xi}{1-\xi} \end{aligned} \quad (2.30)$$

The mapping function for node 3, the node at infinity, is not needed. We note that

$$\begin{aligned} \text{At } \xi = -1, \quad x &= \frac{(-2)(-1)}{1+1}x_1 + \frac{1-1}{1+1}x_2 = x_1 \\ \text{At } \xi = 0, \quad x &= \frac{(-2)(0)}{1-0}x_1 + \frac{1+0}{1-0}x_2 = x_2 \\ \text{At } \xi = 1, \quad x &= \frac{(-2)(1)}{1-1}x_1 + \frac{1+1}{1-1}x_2 = \frac{2(x_2-x_1)}{1-1} = +\infty \end{aligned} \quad (2.31)$$

If we invert the mapping to obtain the equation for ξ in terms of x we obtain

$$\xi = \frac{x - x_2}{x_2 - 2x_1 + x} = 1 - \frac{2(x_2 - x_1)}{x_2 - 2x_1 + x} \quad (2.32)$$

An important fact to note about the mapping is that ξ varies in a manner inversely proportional to x . A general polynomial P in ξ will have the form

$$P = \alpha_0 + \frac{\alpha_1}{x} + \frac{\alpha_2}{x^2} + \frac{\alpha_3}{x^3} + \dots \quad (2.33)$$

where α_i are constants which can be determined using equation 2.33.

The above coordinate transformation is used when the stiffness, mass and residual terms are found by integration on the element level. The unknown variables are still interpolated for using standard Lagrangian finite element shape functions, which for a three node one dimensional element are

$$\begin{aligned} N_1 &= \frac{1}{2}\xi(1-\xi) \\ N_2 &= (1+\xi)(1-\xi) \\ N_3 &= \frac{1}{2}\xi(1+\xi) \end{aligned} \quad (2.34)$$

Then, for a simple one dimensional element the mass matrix takes the form

$$\mathbf{M} = \int_{\Omega} \mathbf{N}^T \mathbf{N} \, d\Omega \quad (2.35)$$

where $\mathbf{N}$ is formed from the shape functions in equation 2.34. This matrix, when integrated becomes

$$\mathbf{M} = \int_{-1}^1 \mathbf{N}^T \mathbf{N} j d\xi \quad (2.36)$$

where j is the determinant of the Jacobian coordinate transformation matrix $\mathbf{J}$ from ξ to x . For mapped infinite elements, the Jacobian is calculated using the mapping functions of equation 2.30 and the transformation 2.29. Similarly, the stiffness matrix is

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{B} d\Omega \quad (2.37)$$

which when integrated becomes

$$\mathbf{K} = \int_{-1}^1 \mathbf{B}^T \mathbf{B} j d\xi \quad (2.38)$$

The $\mathbf{B}$ matrix is determined by calculating the global shape function derivatives using the relationship

$$\frac{\partial N_I}{\partial x} = \mathbf{J}^{-1} \frac{\partial N_I}{\partial \xi} \quad (2.39)$$

The extension to two dimensional problems is now straightforward. In order to generate mapping functions for two dimensional elements with one infinite direction, (this is the type of element that will be used for our acoustic problems) we need only combine the mapping functions of equation 2.30 with the shape functions in equation 2.34. The element is assumed to be infinite in the ξ direction and finite in the η direction.

For a nine node infinite element with the following natural coordinates (ξ, η) :

$$\begin{aligned}
 \text{Node 1 : } & (-1, -1) \\
 \text{Node 2 : } & (0, -1) \\
 \text{Node 3 : } & (1, -1) \\
 \text{Node 4 : } & (-1, 0) \\
 \text{Node 5 : } & (0, 0) \\
 \text{Node 6 : } & (1, 0) \\
 \text{Node 7 : } & (-1, 1) \\
 \text{Node 8 : } & (0, 1) \\
 \text{Node 9 : } & (1, 1)
 \end{aligned}
 \tag{2.40}$$

where nodes 3, 6 and 9 have infinite coordinate(s) in the real domain the

mapping functions are

$$\begin{aligned}
 M_1(\xi, \eta) &= m_1(\xi)N_1(\eta) \\
 M_2(\xi, \eta) &= m_2(\xi)N_1(\eta) \\
 M_4(\xi, \eta) &= m_1(\xi)N_2(\eta) \\
 M_5(\xi, \eta) &= m_2(\xi)N_2(\eta) \\
 M_7(\xi, \eta) &= m_1(\xi)N_3(\eta) \\
 M_8(\xi, \eta) &= m_2(\xi)N_3(\eta)
 \end{aligned} \tag{2.41}$$

The coordinate transformation can then be written as

$$\begin{aligned}
 x &= M_1x_1 + M_2x_2 + M_4x_4 + M_5x_5 + M_7x_7 + M_8x_8 \\
 y &= M_1y_1 + M_2y_2 + M_4y_4 + M_5y_5 + M_7y_7 + M_8y_8
 \end{aligned} \tag{2.42}$$

The shape functions used to interpolate the unknown variables are simply constructed from the one dimensional shape functions of equation 2.34 in the manner typical for two dimensional finite elements. When the tangent matrices and residuals are integrated on the element level, the Jacobian matrix $\mathbf{J}$ and its' determinant j are calculated from the relationship 2.42.

2.3.2 The Helmholtz Equation

Before beginning the discussion of the type of element used for acoustic wave problems, the Helmholtz equation must be introduced. If the Fourier tranform relationships are introduced into the wave equation 2.1 the resulting frequency dependent equation is

$$\nabla^2 p + k^2 p = 0 \tag{2.43}$$

where k is the wavenumber introduced in equation 2.17. This equation is known as the reduced wave or Helmholtz Equation. This is exactly the equation which is being used when the finite element equation 2.14 is solved. The Green's function, or fundamental solution, to the three dimensional Helmholtz equation is

$$G(r) = \frac{1}{4\pi r} e^{ikr} \tag{2.44}$$

where r is the distance between the source point and the observation point in spherical coordinates. The two dimensional Green's function is

$$G(r) = \frac{1}{4i} H_0^{(1)}(kr) \quad (2.45)$$

where r is the distance between the source point and the observation point in cylindrical polar coordinates. $H_0^{(1)}$ is a Hankel function of the first kind of order zero. It is defined as

$$H_0^{(1)}(kr) = J_0(kr) + iY_0(kr) \quad (2.46)$$

where $J_0(kr)$ is a Bessel function of the first kind of order zero and $Y_0(kr)$ is a Bessel function of the second kind of order zero. For large r , the Hankel function of equation 2.46 varies approximately as

$$H_0^{(1)}(kr) \approx r^{-\frac{1}{2}} (\cos(kr) - i \sin(kr)) \quad (2.47)$$

2.3.3 Mapped Wave Infinite Elements

For the element now to be discussed, the mapping part is exactly as described in Section 2.3.1

Exterior acoustic problems in three dimensions have solutions which decay in proportion to $1/r$, which is apparent from the three dimensional Green's function for the Helmholtz Equation. We can account for this behavior by modifying the element shape functions to have the form

$$N(\xi, \eta) = C P(\xi, \eta) \exp(-iks) \quad (2.48)$$

where

$$s = \frac{A}{1 - \xi} \quad (2.49)$$

C a phase constant to be determined and A is a decay constant to be chosen. $P(\xi, \eta)$ are standard finite element interpolation functions, now called parent shape functions. The variable s goes to infinity as ξ approaches 1. The $1/r$ decay is accounted for since ξ has a $1/r$ behavior from the mapping. (see equations 2.32 and 2.33). Thus, the decay is contained in the parent shape functions. Note that the function $\exp(-iks)$ satisfies the Sommerfeld radiation condition of equation 2.16

For two dimensional plane problems, which are the primary concern of this report, the shape functions are

$$N(\xi, \eta) = C P(\xi, \eta) s^{1/2} \exp(-iks) \quad (2.50)$$

The $s^{1/2}$ is added to correctly model the $r^{-1/2}$ decay of the two dimensional Green's function. The $s^{1/2}$ term combined with the $1/r$ decay in the parent shape functions leads to the needed $r^{-1/2}$ decay. It still remains to determine the phase constant C .

The phase constant C can be determined by requiring C^0 continuity between finite and infinite elements. The value of C must be found such that the infinite element shape functions match the values of the shape functions of bordering finite elements when $\xi = -1$. This is the boundary with standard finite elements. Equation 2.50 can be written

$$N(\xi, \eta) = C P(\xi, \eta) \left[\frac{A}{1-\xi} \right]^{\frac{1}{2}} \exp \left[-\frac{ikA}{1-\xi} \right] \quad (2.51)$$

If $\xi = -1$ is substituted into the above the result is

$$N(-1, \eta) = C P(-1, \eta) \left[\frac{A}{2} \right]^{\frac{1}{2}} \exp \left[-\frac{ikA}{2} \right] = P(-1, \eta) \quad (2.52)$$

To insure continuity the constant C must be

$$C = \left[\frac{2}{A} \right]^{\frac{1}{2}} \exp \left[\frac{ikA}{2} \right] \quad (2.53)$$

Now that C is known, the derivatives $\frac{\partial N}{\partial \xi}$ and $\frac{\partial N}{\partial \eta}$ can be determined by straight forward differentiation.

The 'mass' and 'stiffness' matrices for the mapped wave infinite element take the same form as the matrices of section 1.1. They can be written as

$$\mathbf{H} = \int_{\Omega} \mathbf{B}^T \mathbf{B} d\Omega \quad (2.54)$$

and

$$\mathbf{E} = \frac{1}{c^2} \int_{\Omega} \mathbf{N}^T \mathbf{N} d\Omega \quad (2.55)$$

Standard Gaussian quadrature can be used to integrate these terms. However, given the oscillatory nature of the shape functions, more accurate numerical quadrature methods have been developed to perform the integration. The reader is encouraged to read Bettess [4] and Zienkiewicz [18] for detailed information about the integration procedures.

This element theoretically has nine nodes. However, the nodal pressures at infinity are prescribed to be zero. This can be motivated by the fact that a solution to a two dimensional problem can be written as combinations of Green's functions, which decay to zero as $r \rightarrow \infty$. Also, the acoustic medium is non - dissipative, so the amount of energy passing through an area A_1 of mean distance r_1 has to be the same amount of energy which passes through an area A_2 at a distance r_2 . If $r_2 > r_1$ and $A_2 > A_1$ then the pressure must decrease with increasing distance. Because we know they are zero, the three pressure variables at infinity are not assembled into the system of equations. The element is coded to have only six nodes.

The element just presented can be used only for frequency dependent problems. The stiffness and mass terms must be assembled into frequency dependent system of equations such as 2.14. The element was developed to solve frequency dependent acoustic problems, and cannot be used for transient time dependent problems. Given that the element is infinite, the forcing term is calculated from equation 2.27 since now only Γ_I , the finite part of the boundary, can cause the loading.

2.4 Scattering From Rigid Boundaries

In this section, we will consider scattering of plane waves from rigid boundaries. The primary variable will be the scattered pressure.

If we consider the wave equation 2.1 and note that

$$p = p^{in} + p^{sc} \quad (2.56)$$

where p^{in} is the incident pressure and p^{sc} is the scattered pressure the result is

$$\nabla^2(p^{in} + p^{sc}) - \frac{1}{c^2}(\ddot{p}^{in} + \ddot{p}^{sc}) = 0 \quad (2.57)$$

Since the incident pressure wave indentially satisfies the wave equation, we

are left with

$$\nabla^2(p^{sc}) - \frac{1}{c^2}(\ddot{p}^{sc}) = 0 \quad (2.58)$$

By performing the same steps as in previous sections we arrive at a finite element discretization for the scattered pressure. The global equation, assuming the use of Sommerfeld radiation elements, is

$$\mathbf{E}\ddot{\mathbf{p}}^{sc} + \mathbf{C}\dot{\mathbf{p}}^{sc} + \mathbf{H}\mathbf{p}^{sc} = \mathbf{h} \quad (2.59)$$

where $\mathbf{h}$ is

$$\mathbf{h} = \sum_1^{numel} \int_{\Gamma_I} \mathbf{N}^T \nabla p^{sc} \cdot \mathbf{n} d\Gamma \quad (2.60)$$

Making the substitution of 2.56 into the expression for $\mathbf{h}$ yields

$$\mathbf{h} = \sum_1^{numel} \int_{\Gamma_I} \mathbf{N}^T \nabla(p - p^{in}) \cdot \mathbf{n} d\Gamma \quad (2.61)$$

On rigid scattering boundaries, we have the boundary condition

$$\nabla p \cdot \mathbf{n} = 0 \quad (2.62)$$

Taking this fact into account gives the needed result for $\mathbf{h}$, which is

$$\mathbf{h} = \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}^T \nabla p^{in} \cdot \mathbf{n} d\Gamma \right) \quad (2.63)$$

2.5 Scattering from Rigid Cylinder

In this section, in order to test the Sommerfeld radiation element and the mapped wave infinite element, we consider the problem of scattering of plane waves from an infinite rigid solid cylinder. The incident waves have a harmonic time dependence of the form $P_0 \exp[i(-kx + \omega t)]$ and are incident from an angle π with the x - axis. P_0 , the magnitude of the incident wave, is set equal to 1. The cylinder has a radius $a = 1.0$. An analytical solution to this problem, in terms of Bessel functions, can be found in many texts, including Junger [12] and Morse [14].

Figure 1 shows the mesh used to test the Sommerfeld elements. We have used nine node quadrilateral fluid pressure elements and three node

Sommerfeld elements. The Sommerfeld elements have been placed at the outer edge of the mesh. The results for the scattered pressure on the surface of the cylinder vs. angle(in degrees) with the x - axis are plotted in Figure 2. The continuous solid line is the real part of the analytical solution and the continuous dashed line is the imaginary part. The squares and triangles are the finite element solution. Both real and imaginary parts of the solution are shown. The numerical results show good agreement with the formal solution.

Figure 3 shows the mesh used to test the mapped wave infinite elements. The decay parameter A was chosen as $A = 2.0\alpha k$. We chose $\alpha = 6.0$. The solution is plotted in Figure 4. Again, the numerical results are very close to the analytical solution.

Chapter 3

Solid/Structure Formulations

3.1 Two Dimensional Solid Element

The equation of linear momentum for a solid body is

$$\text{div } \mathbf{T} + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad (3.1)$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is the vector of body forces, ρ is the density of the material and $\ddot{\mathbf{u}}$ is the vector of accelerations. The coordinate form of this equation, assuming a rectangular Cartesian coordinate system, is

$$\frac{\partial t_{ij}}{\partial x_j} + b_i = \rho \ddot{u}_i \quad (3.2)$$

If equation 3.2 is multiplied by an arbitrary vector valued weighting function w and integrated over the solid domain Ω , the result is

$$\int_{\Omega} w_i \left(\rho \ddot{u}_i - \frac{\partial t_{ij}}{\partial x_j} - b_i \right) d\Omega = 0 \quad (3.3)$$

Using the divergence theorem on second term of equation 3.3 will result in

$$\int_{\Omega} w_i \rho \ddot{u}_i d\Omega + \int_{\Omega} \frac{\partial w_i}{\partial x_j} t_{ij} d\Omega - \int_{\Gamma_T} w_i t_{ij} n_j d\Gamma - \int_{\Omega} w_i b_i d\Omega = 0 \quad (3.4)$$

where Γ_T is the traction boundary and n_i is the outward normal to the domain in the x_i direction. The weighting function w is prescribed to be

zero on displacement boundaries. Equation 3.4 is the same as the principle of virtual displacements. We can view the weighting function $\mathbf{w}$ as a virtual displacement. If we use a matrix notation, the symmetric gradient of $\mathbf{w}$ is defined as

$$\nabla^s \mathbf{w} = \left[\frac{1}{2} \nabla \mathbf{w} + (\nabla \mathbf{w})^T \right] \quad (3.5)$$

Note that the symmetric gradient of $\mathbf{w}$ is the same as the virtual strain field. Equation 3.4 can be put into matrix form.

$$\int_{\Omega} \mathbf{w}^T \rho \ddot{\mathbf{u}} d\Omega + \int_{\Omega} (\nabla^s \mathbf{w})^T \mathbf{T} d\Omega = \int_{\Omega} \mathbf{w}^T \mathbf{b} d\Omega + \int_{\Gamma_T} \mathbf{w}^T \mathbf{t} d\Gamma \quad (3.6)$$

where the $\mathbf{t}$ is the traction vector and the relationship $t_i = t_{ij}n_j$ has been used. $\mathbf{T}$, $\delta \mathbf{u}$ and $\delta \epsilon$ are now written in vector form. The constitutive relationship between stress and strain is

$$\mathbf{T} = \mathbf{D} \epsilon \quad (3.7)$$

where the strain is the symmetric gradient of $\mathbf{u}$. For plane strain elasticity, of primary interest in this report, this relationship takes the form

$$\begin{bmatrix} t_{11} \\ t_{22} \\ t_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad (3.8)$$

Substituting the relationship 3.7 into equation 3.6 gives

$$\int_{\Omega} \mathbf{w}^T \rho \ddot{\mathbf{u}} d\Omega + \int_{\Omega} (\nabla^s \mathbf{w})^T \mathbf{D} (\nabla^s \mathbf{u}) d\Omega = \int_{\Omega} \mathbf{w}^T \mathbf{b} d\Omega + \int_{\Gamma_T} \mathbf{w}^T \mathbf{t} d\Gamma \quad (3.9)$$

A standard finite element discretization is used to approximate ϵ and $\mathbf{u}$ within each element such that

$$\mathbf{u} = \mathbf{N} \mathbf{u} = \mathbf{N}_I \mathbf{u}^I \quad (3.10)$$

The strains are then given by

$$\nabla^s \mathbf{u} = \mathbf{B} \mathbf{u} = \mathbf{B}_I \mathbf{u}^I \quad (3.11)$$

where $\mathbf{N}_I$, $\mathbf{B}_I$ and $\mathbf{u}^I$ are the shape function matrix, the strain-displacement matrix and unknown displacements at the nodes, respectively. For two dimensional problems, these matrices take the form

$$\mathbf{u}^I = \begin{bmatrix} u_1^I \\ u_2^I \end{bmatrix} \quad (3.12)$$

$$\mathbf{N}_I = \begin{bmatrix} N_I & 0 \\ 0 & N_I \end{bmatrix} \quad (3.13)$$

$$\mathbf{B}_I = \begin{bmatrix} \frac{\partial N_I}{\partial x_1} & 0 \\ 0 & \frac{\partial N_I}{\partial x_2} \\ \frac{\partial N_I}{\partial x_2} & \frac{\partial N_I}{\partial x_1} \end{bmatrix} \quad (3.14)$$

The same interpolations are used for the virtual displacement field $\mathbf{w}$.

Introducing the above interpolations into equation 3.9 results in

$$\sum_1^{numel} \mathbf{w}^T \left(\int_{\Omega} \mathbf{N}^T \rho \mathbf{N} d\Omega \ddot{\mathbf{u}} + \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \mathbf{u} \right) = \sum_1^{numel} \mathbf{w}^T \left(\int_{\Omega} \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma_T} \mathbf{N}^T \mathbf{t} d\Gamma \right) \quad (3.15)$$

Since the nodal parameters $\mathbf{w}$ are arbitrary, they can be eliminated from the above equation, thus leaving

$$\sum_1^{numel} \left(\int_{\Omega} \mathbf{N}^T \rho \mathbf{N} d\Omega \ddot{\mathbf{u}} + \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \mathbf{u} \right) = \int_{\Omega} \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma_T} \mathbf{N}^T \mathbf{t} d\Gamma \quad (3.16)$$

Lastly, we can write

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{f}_B + \mathbf{f}_T \quad (3.17)$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, $\mathbf{f}_B$ is the load vector caused by body forces and $\mathbf{f}_T$ is the load vector caused by surface tractions.

Again, many problems are solved in the frequency domain. Substituting the inverse Fourier Transform relationship of equation 2.12 into equation 3.17 and cancelling the $\exp(i\omega t)$ term leaves

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{u} = \mathbf{f}_B + \mathbf{f}_T \quad (3.18)$$

In anticipation of the coupled fluid - structure system, the load $\mathbf{f}_B$ will be neglected and the load $\mathbf{f}_T$ becomes

$$\mathbf{f}_T = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}^T \mathbf{t} \, d\Gamma \right) \quad (3.19)$$

where Γ_I is the boundary of the fluid structure interface.

3.2 One Dimensional Infinite Strip Plate Element

In this section, we consider the equations for a linear elastic infinite plate. The plate is flat, lying in the $x - z$ plane, and is infinite in the z - direction. It has a thickness h in the y - direction and a length L in the x - direction. A y - direction distributed load $q(x)$ is applied along the length of the plate. The plate has a mass density ρ .

Using the notation u_x for displacement in the x - direction and u_y for displacement in the y - direction, the kinematical assumptions are

$$\begin{aligned} u_x &= u - \theta y \\ u_y &= v \end{aligned} \quad (3.20)$$

where u , v , θ are the displacement and rotation of the middle surface. The strains are

$$\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} - y \frac{\partial \theta}{\partial x} \\ \gamma_{xy} &= -\theta + \frac{\partial v}{\partial x} \end{aligned} \quad (3.21)$$

A plane stress assumption is made so that $t_{yy} = 0$. Also, we assume plane strain for the z - direction so that $\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0$. The constitutive equation for this plate is

$$\begin{bmatrix} t_{xx} \\ 0 \\ t_{xy} \end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & 0.5 - \nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad (3.22)$$

Performing a solution of the above relationship gives the expected

$$t_{xx} = \frac{E}{1 - \nu^2} \epsilon_{xx} \quad (3.23)$$

$$t_{xy} = \frac{E(0.5 - \nu)}{(1 - \nu)(1 - 2\nu)} \gamma_{xy} = \mu \gamma_{xy} \quad (3.24)$$

The force resultants axial force(P), shear(V) and moment(M) can be calculated. They are

$$P = \int_{-h/2}^{h/2} t_{xx} dy = \left(\frac{Eh}{1 - \nu^2} \right) \frac{\partial u}{\partial x} \quad (3.25)$$

$$V = \int_{-h/2}^{h/2} t_{xy} dy = \kappa \mu h \left(-\theta + \frac{\partial v}{\partial x} \right) \quad (3.26)$$

where κ is a shear correction factor, commonly selected as 5/6.

$$M = - \int_{-h/2}^{h/2} t_{xy} y dy = \left(\frac{Eh^3}{12(1 - \nu^2)} \right) \frac{\partial \theta}{\partial x} \quad (3.27)$$

The principle of virtual displacements for this plate can be written

$$\begin{aligned} \int_{\Omega} \begin{bmatrix} \delta \epsilon_{xx} & \delta \gamma_{xy} \end{bmatrix} \begin{bmatrix} t_{xx} \\ t_{xy} \end{bmatrix} d\Omega + \int_0^L \begin{bmatrix} \delta u & \delta v & \delta \theta \end{bmatrix} \begin{bmatrix} \rho h \ddot{u} \\ \rho h \ddot{v} \\ \frac{\rho h^3}{12} \ddot{\theta} \end{bmatrix} dx = \\ \int_0^L \begin{bmatrix} \delta u & \delta v & \delta \theta \end{bmatrix} \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} dx \quad (3.28) \end{aligned}$$

Equation 3.28 can be rewritten, using the force resultants, and will take the form

$$\int_0^L \begin{bmatrix} \frac{\partial(\delta u)}{\partial x} & -\delta \theta + \frac{\partial(\delta v)}{\partial x} & \frac{\partial(\delta \theta)}{\partial x} \end{bmatrix} \begin{bmatrix} \left(\frac{Eh}{1 - \nu^2} \right) \frac{\partial u}{\partial x} \\ \kappa \mu h \left(-\theta + \frac{\partial v}{\partial x} \right) \\ \left(\frac{Eh^3}{12(1 - \nu^2)} \right) \frac{\partial \theta}{\partial x} \end{bmatrix} dx +$$

$$\int_0^L \begin{bmatrix} \delta u & \delta v & \delta \theta \end{bmatrix} \begin{bmatrix} \rho h \ddot{u} \\ \rho h \ddot{v} \\ \frac{\rho h^3}{12} \ddot{\theta} \end{bmatrix} dx = \int_0^L \begin{bmatrix} \delta u & \delta v & \delta \theta \end{bmatrix} \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} dx \quad (3.29)$$

If the following three matrices are introduced :

$$\mathbf{D} = \begin{bmatrix} \frac{Eh}{1-\nu^2} & 0 & 0 \\ 0 & \kappa\mu h & 0 \\ 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} \end{bmatrix} \quad (3.30)$$

$$\rho = \begin{bmatrix} \rho h & 0 & 0 \\ 0 & \rho h & 0 \\ 0 & 0 & \frac{\rho h^3}{12} \end{bmatrix} \quad (3.31)$$

$$\mathbf{q} = \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} \quad (3.32)$$

and used in equation 3.29 the result is

$$\int_0^L \begin{bmatrix} \frac{\partial(\delta u)}{\partial x} & -\delta\theta + \frac{\partial(\delta v)}{\partial x} & \frac{\partial(\delta\theta)}{\partial x} \end{bmatrix} \mathbf{D} \begin{bmatrix} \frac{\partial u}{\partial x} \\ -\theta + \frac{\partial v}{\partial x} \\ \frac{\partial\theta}{\partial x} \end{bmatrix} dx + \int_0^L \begin{bmatrix} \delta u & \delta v & \delta\theta \end{bmatrix} \rho \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{\theta} \end{bmatrix} dx = \int_0^L \begin{bmatrix} \delta u & \delta v & \delta\theta \end{bmatrix} \mathbf{q} dx \quad (3.33)$$

A standard finite element discretization is used to approximate u , v and θ within each element such that

$$\begin{bmatrix} u \\ v \\ \theta \end{bmatrix} = \mathbf{N}\mathbf{u} = \mathbf{N}_I \mathbf{u}^I \quad (3.34)$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ -\theta + \frac{\partial v}{\partial x} \\ \frac{\partial \theta}{\partial x} \end{bmatrix} = \mathbf{B}\mathbf{u} = \mathbf{B}_I \mathbf{u}^I \quad (3.35)$$

where $\mathbf{N}_I$, $\mathbf{B}_I$ and $\mathbf{u}^I$ are the shape function matrix, the strain-displacement matrix and unknown displacements at the nodes, respectively. These matrices take the form

$$\mathbf{u}^I = \begin{bmatrix} u^I \\ v^I \\ \theta^I \end{bmatrix} \quad (3.36)$$

$$\mathbf{N}_I = \begin{bmatrix} N_I & 0 & 0 \\ 0 & N_I & 0 \\ 0 & 0 & N_I \end{bmatrix} \quad (3.37)$$

$$\mathbf{B}_I = \begin{bmatrix} \frac{\partial N_I}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_I}{\partial x} & -N_I \\ 0 & 0 & \frac{\partial N_I}{\partial x} \end{bmatrix} \quad (3.38)$$

The same interpolations are used for the virtual displacement field $\delta \mathbf{u}$ and its' derivatives.

Introducing the above interpolations into equation 3.33 results in

$$\sum_1^{numel} \delta \mathbf{w}^T \left(\int_0^L \mathbf{N}^T \rho \mathbf{N} dx \ddot{\mathbf{u}} + \int_0^L \mathbf{B}^T \mathbf{D} \mathbf{B} dx \mathbf{u} \right) = \sum_1^{numel} \delta \mathbf{w}^T \left(\int_0^L \mathbf{N}^T \mathbf{q} dx \right) \quad (3.39)$$

Since the nodal parameters $\delta \mathbf{w}$ are arbitrary, they can be eliminated from the above equation, thus leaving

$$\sum_1^{numel} \left(\int_0^L \mathbf{N}^T \rho \mathbf{N} dx \ddot{\mathbf{u}} + \int_0^L \mathbf{B}^T \mathbf{D} \mathbf{B} dx \mathbf{u} \right) = \int_0^L \mathbf{N}^T \mathbf{q} dx \quad (3.40)$$

Lastly, we can write

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}_T \quad (3.41)$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the consistent mass matrix and $\mathbf{f}_T$ is the load vector caused by the distributed load. In actuality, we use a lumped mass matrix rather than a consistent mass matrix for our numerical examples. For a two node plate element, this matrix takes the form

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2}\rho h L & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\rho h L & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{24}\rho h^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\rho h L & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\rho h L & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24}\rho h^3 \end{bmatrix} \quad (3.42)$$

The stiffness matrix contains axial, shear and bending terms. Assuming a two node linear element, the axial and bending terms are exactly integrated with one point quadrature. The shear terms are exactly integrated with two point quadrature. However, this element is known to be too 'stiff' if two point integration is used on all of the stiffness terms. A reduced order one point integration is used to improve the performance of the element.

In order to make this element useful for shell analysis applications, the stiffness matrix is transformed using an orthogonal transformation matrix $\mathbf{A}$. Assuming the element is oriented such that the tangent to the element and the x - axis form an angle β , the transformation matrix takes the form

$$\mathbf{A} = \begin{bmatrix} \cos(\beta) & \sin(\beta) & 0 & 0 & 0 & 0 \\ -\sin(\beta) & \cos(\beta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\beta) & \sin(\beta) & 0 \\ 0 & 0 & 0 & -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.43)$$

The transformed stiffness matrix is

$$\hat{\mathbf{K}} = \mathbf{A}^T \mathbf{K} \mathbf{A} \quad (3.44)$$

The transformed stiffness is now applicable for use in shell problems. The mass matrix must be transformed, but for either mass matrix of this element, the consistent or the lumped, the transformed matrix is exactly the same as the untransformed matrix. The load vector and is also transformed, and takes a form such as

$$\hat{\mathbf{f}}_T = \mathbf{A}^T \mathbf{f}_T \quad (3.45)$$

Chapter 4

Fluid - Structure Interaction

In this chapter, we consider scattering of pressure waves for the coupled fluid - structure system. First, coupling of fluid finite elements with solid finite elements is discussed. Then, coupling of the fluid with shell elements is discussed.

4.1 Coupling with Solid Element

First, we consider the finite element equations for the fluid medium. Recall from chapter two these are,

$$\mathbf{E}\ddot{\mathbf{p}}^{sc} + \mathbf{C}\dot{\mathbf{p}}^{sc} + \mathbf{H}\mathbf{p}^{sc} = \sum_1^{numel} \int_{\Gamma_I} \mathbf{N}_f^T \nabla(p - p^{in}) \cdot \mathbf{n} d\Gamma \quad (4.1)$$

where $\mathbf{N}_f$ are the fluid finite element shape functions.

The necessary fluid - structure interface boundary condition can be derived from the continuum equations for the fluid. If the fluid is viewed as a general continuous medium, it must satisfy the local balance of linear momentum equation, which is

$$t_{ij,j}^f = \rho \ddot{u}_i \quad (4.2)$$

where t_{ij}^f are the fluid stresses, ρ is the fluid density and $\ddot{u}_i$ are the fluid particle accelerations. Body forces in the fluid have been neglected. The constitutive equation for the fluid is given by

$$t_{ij}^f = -p\delta_{ij} \quad (4.3)$$

where p is the fluid pressure. Substituting the constitutive equation into the linear momentum equation yields

$$-p_{,i} = \rho \ddot{u}_i \quad (4.4)$$

Multiplying both sides of the above equation by the normal vector to some surface in the fluid medium gives

$$-p_{,i} n_i = \rho \ddot{u}_i n_i \quad (4.5)$$

Noting that on the interface of the solid and the fluid the two mediums share the same particle positions, as well as accelerations, on the fluid - structure interface, we have the coupling condition

$$\nabla p \cdot \mathbf{n} = -\rho \mathbf{n} \cdot \ddot{\mathbf{u}} \quad (4.6)$$

which can be written

$$\nabla p \cdot \mathbf{n} = -\rho \mathbf{n}^T \ddot{\mathbf{u}} \quad (4.7)$$

where $\mathbf{u}$ is the vector of solid displacements.

Making a substitution of equation 4.7 into equation 4.1 gives

$$\begin{aligned} \mathbf{E} \ddot{\mathbf{p}}^{sc} + \mathbf{C} \dot{\mathbf{p}}^{sc} + \mathbf{H} \mathbf{p}^{sc} &= \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}_f^T \nabla p^{in} \cdot \mathbf{n} \, d\Gamma \right) \\ &+ \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}_f^T \rho \mathbf{n}^T \ddot{\mathbf{u}} \, d\Gamma \right) \end{aligned} \quad (4.8)$$

Rearranging yields

$$\begin{aligned} \mathbf{E} \ddot{\mathbf{p}}^{sc} + \mathbf{C} \dot{\mathbf{p}}^{sc} + \mathbf{H} \mathbf{p}^{sc} + \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_f^T \rho \mathbf{n}^T \ddot{\mathbf{u}} \, d\Gamma \right) &= \\ \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}_f^T \nabla p^{in} \cdot \mathbf{n} \, d\Gamma \right) \end{aligned} \quad (4.9)$$

Recalling that

$$\mathbf{u} = \mathbf{N}_s \mathbf{u} \quad (4.10)$$

where $\mathbf{N}_s$ are the solid finite element shape functions, and substituting into equation 4.9 gives

$$\mathbf{E}\ddot{\mathbf{p}}^{sc} + \mathbf{C}\dot{\mathbf{p}}^{sc} + \mathbf{H}\mathbf{p}^{sc} + \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_f^T \rho \mathbf{n}^T \mathbf{N}_s d\Gamma \right) \ddot{\mathbf{u}} = \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}_f^T \nabla p^{in} \cdot \mathbf{n} d\Gamma \right) \quad (4.11)$$

Setting

$$\mathbf{h} = \sum_1^{numel} \left(- \int_{\Gamma_I} \mathbf{N}_f^T \nabla p^{in} \cdot \mathbf{n} d\Gamma \right) \quad (4.12)$$

and

$$\rho \mathbf{Q}^T = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_f^T \rho \mathbf{n}^T \mathbf{N}_s d\Gamma \right) \quad (4.13)$$

equation 4.11 can be written

$$\mathbf{E}\ddot{\mathbf{p}}^{sc} + \mathbf{C}\dot{\mathbf{p}}^{sc} + \mathbf{H}\mathbf{p}^{sc} + \rho \mathbf{Q}^T \ddot{\mathbf{u}} = \mathbf{h} \quad (4.14)$$

We now turn our attention to the finite element equations for the solid medium. These equations are

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{t} d\Gamma \right) \quad (4.15)$$

On the fluid - structure interface,

$$\mathbf{t} = \mathbf{n}p \quad (4.16)$$

where $\mathbf{n}$ is the outward unit normal to the fluid medium and p is the total fluid pressure. Substituting equation 4.16 into equation 4.15 gives

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} p d\Gamma \right) \quad (4.17)$$

Since $p = p^{in} + p^{sc}$, equation 4.17 becomes

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} (p^{in} + p^{sc}) d\Gamma \right) \quad (4.18)$$

Upon rearranging terms, we get

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} p^{sc} d\Gamma \right) = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} p^{in} d\Gamma \right) \quad (4.19)$$

The finite element interpolation for the scattered pressure is

$$p = \mathbf{N}_f \mathbf{p}^{sc} \quad (4.20)$$

Substituting this into equation 4.19 yields the result

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} \mathbf{N}_f d\Gamma \right) \mathbf{p}^{sc} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} p^{in} d\Gamma \right) \quad (4.21)$$

Setting

$$\mathbf{f} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} p^{in} d\Gamma \right) \quad (4.22)$$

and

$$\mathbf{Q} = \sum_1^{numel} \left(\int_{\Gamma_I} \mathbf{N}_s^T \mathbf{n} \mathbf{N}_f d\Gamma \right) \quad (4.23)$$

equation 4.21 can be written

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{Q}\mathbf{p}^{sc} = \mathbf{f} \quad (4.24)$$

The combined fluid - structure system of equations is the combination of equations 4.14 and 4.24. The overall system of equations can be written

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \rho \mathbf{Q}^T & \mathbf{E} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{p}}^{sc} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}}^{sc} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{Q} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p}^{sc} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{h} \end{bmatrix} \quad (4.25)$$

The frequency dependent system of equations is

$$\begin{bmatrix} \mathbf{K} - \omega^2 \mathbf{M} & -\mathbf{Q} \\ -\omega^2 \rho \mathbf{Q}^T & \mathbf{H} + i\omega \mathbf{C} - \omega^2 \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p}^{sc} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{h} \end{bmatrix} \quad (4.26)$$

4.2 Coupling with Shell Element

Now that we have discussed the method to couple the fluid finite elements with solid finite elements, coupling of fluid elements with the shell element discussed in chapter 3 is not difficult.

As a result of the first order shear theory used to develop the shell element, distributed loads along the length of the element cause no θ loading. Therefore, the pressure exerted by the acoustic medium on the shell element causes no θ loading. In addition, due to the kinematic assumptions for the shell element, the θ rotations cause no transverse displacements of the shell, and thus do not cause pressure changes in the acoustic fluid.

The result of the facts stated above is that the θ degree of freedom remains uncoupled with the fluid pressure variables. The method used to couple the shell element with the fluid finite element is the same as discussed in the previous section of this chapter, except that the θ variables remain uncoupled. The coupling matrix $\mathbf{Q}$ has no coefficients corresponding to rotational degrees of freedom of the shell.

4.3 Scattering from Cylindrical Shell

In order to verify that the developments for the coupled fluid - structure are correct, we consider the problem of scattering of time harmonic plane waves from an infinite flexible cylindrical shell. An analytical solution to this problem can be found in a 1952 paper by Junger [11]. The incident waves have the same form as in Section 2.5. The shell has a radius $a = 1.0m$ and a thickness $h = 0.02m$. The shell is made of steel and the acoustic medium is sea water. See Appendix B for the properties of these materials.

The shell was first modelled using solid plane strain nine node quadrilateral elements. Two elements were used through the thickness of the shell. Nine node elements were used for the fluid and three node elements were used on the radiation boundary. The mesh used is shown in Figure 1. The solid elements are not shown in the figure because they are quite thin and do not plot well given the scale and resolution of the figure. The results for scattered pressure of the surface of the shell are plotted in Figure 5. Again, the solid line is the real part of the solution and the dotted line is the imaginary. The numerical results are in very good agreement with the formal solution.

The shell was then modelled using two node linear shell elements such as those discussed in Chapter 3. Four node quadrilateral elements were used for the fluid and two node linear elements were used for the Sommerfeld elements. Figure 6 is a plot of the mesh used. The results are shown in Figure 7. Again, the finite element solution and the analytical solution differ only very slightly.

Chapter 5

Conclusion

The main issue of this study was to compare and evaluate the different available numerical approaches for the solution of problems involving the interaction of acoustic fluids and elastic solids. To that end, we investigated various techniques of solution, with primary emphasis on the finite element method.

In Chapter 2, we investigated the properties of Sommerfeld radiation elements and mapped wave infinite elements. Both types of elements appear to perform well and we notice little difference between the results given by the two approaches. However, both these types of elements lack rigorous convergence criteria. In addition, the infinite element needs additional study with respect to placement of intermediate nodes and the optimal value to be chosen for the decay constant A . We do not know at this time how much affect the variation of these parameters will have on results given by the infinite element.

In Chapter 4, we investigated the method used to couple fluid finite elements with solid finite elements. The numerical implementation of the coupled problem is not difficult. The results show that both two dimensional solid finite elements and one dimensional shell elements can be coupled with the fluid finite elements, and both types of elements give acceptable results. However, to avoid any possible problems with accurate and stable convergence, we recommend the use of the shell element in the analysis of thin shell structures. The two dimensional solid element may not perform well if one of its' dimensions is much less than the other dimension.

Another option for solution of frequency dependent problems is the use of fluid boundary elements, an integral equation method. While the boundary

element method does have rigorous convergence criteria, it is difficult to implement. The Burton - Miller formulation (see references [1] and [9]) is especially difficult to implement, although it is more robust than traditional boundary element formulations. In addition, it is difficult to incorporate the boundary element method into existing finite element codes and difficult to couple the boundary elements with finite elements.

Lastly, for transient time dependent problems, the only available option is the use of the Sommerfeld elements. Both the infinite elements and boundary elements can only be used for frequency dependent problems. However, the Sommerfeld elements work well for frequency dependent problems, and should perform equally well for transient problems. With this in mind, and also given that the Sommerfeld element is easy to implement, this type of element appears to be the best available option overall. Infinite elements show promise for frequency dependent problems, but more research is needed on some aspects of their behavior.

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Appendix A

Operation Counts

Appendix A. Operation Counts

The solution of the scattering problem is expressed in terms of the structure displacements, $\mathbf{u}$, and the fluid pressures, $\mathbf{p}$. Fluid pressures are associated with the fluid island, which is modeled using conventional finite elements, and the radiation condition, which is represented by either a Sommerfeld type element or an infinite element. If we denote the number of unknowns for each part of the problem by n_s for the structure, n_f for the fluid island, and n_r for the radiation condition, the matrix for the solution of the scattering problem may be written as:

$$\begin{bmatrix} \mathbf{K} - \omega^2 \mathbf{M} & -\mathbf{Q} & \mathbf{0} \\ -\omega^2 \rho \mathbf{Q}^T & \mathbf{H}_{ff} - \omega^2 \mathbf{E}_{ff} & \mathbf{H}_{fr} - \omega^2 \mathbf{E}_{fr} \\ \mathbf{0} & \mathbf{H}_{rf} - \omega^2 \mathbf{E}_{rf} & \hat{\mathbf{H}}_{rr}^* \end{bmatrix} \begin{bmatrix} \mathbf{u}_s^* \\ \mathbf{p}_r^* \\ \mathbf{p}_r^* \end{bmatrix} = \begin{bmatrix} \mathbf{f}_s \\ \mathbf{h}_f \\ \mathbf{h}_r \end{bmatrix} \quad \text{A.1}$$

In the above a superscript * denotes a complex quantity. The $\hat{\mathbf{H}}^*$ term is defined as

$$\hat{\mathbf{H}}^* = \mathbf{H} + i\omega \mathbf{C} - \omega^2 \mathbf{E} = \mathbf{H}^* - \omega^2 \mathbf{E} \quad \text{A.2}$$

We note that the only complex part (provided the structure is undamped) occurs in the radiation part of the problem and in the solution parameters. All other parts could be expressed in real arithmetic. We also note that the problem is unsymmetric. For a problem in which the density is constant the symmetry may be removed by dividing the second and third rows of (A.1) by $\omega^2 \rho$. For variable density problems this is not possible. Consequently, in our implementations we treat the unsymmetric formulation. This requires double the storage and decomposition cost compared with a symmetric solution.

The solution of the complex unsymmetric equations is implemented using a standard compact Gaussian elimination type scheme. For any direct solution method the procedures may be easily modified by replacing all real operations by complex arithmetic equivalents. There are two types of replacements which need to be made:

(1) Accumulation involving an add and a multiply.

$$\mathbf{y}^* = \mathbf{x}^* + \mathbf{a}^* \cdot \mathbf{b}^* \quad \text{(A.3)}$$

Expanded into real arithmetic the operation is

$$y_R + i y_I = x_R + x_I + (a_R + i a_I)(b_R + i b_I) \quad \text{(A.4)}$$

$$y_R + i y_I = (x_R + a_R \cdot b_R - a_I \cdot b_I) + i(x_I + a_R \cdot b_I + a_I \cdot b_R)$$

Thus, this type of step requires 4 additions and 4 multiplications for each operation to compute the real and imaginary parts. This is 4 times the cost of the pure real arithmetic operation.

(2) The second type of operation is a division.

$$y^* = x^* + a^* / b^* \quad (\text{A.5})$$

The complex divide is performed by multiplying numerator and denominator by the complex conjugate of b . Accordingly,

$$y_R + i y_I = x_R + i x_I + \frac{(a_R + i a_I) \cdot (b_R - i b_I)}{(b_R + i b_I) \cdot (b_R - i b_I)}$$

resulting in

$$y_R + i y_I = x_R + i x_I + \frac{(a_R \cdot b_R + a_I \cdot b_I) + i(a_I \cdot b_R - a_R \cdot b_I)}{(b_R \cdot b_R + b_I \cdot b_I)}$$

Separating into real and imaginary parts we have:

$$y_R = x_R + \frac{(a_R \cdot b_R + a_I \cdot b_I)}{(b_R \cdot b_R + b_I \cdot b_I)}$$

and

$$y_I = x_I + \frac{(a_I \cdot b_R - a_R \cdot b_I)}{(b_R \cdot b_R + b_I \cdot b_I)}$$

Thus, after defining the denominator as a single term, a complex divide requires 5 addition/subtractions, six multiplications, and 2 divisions for each operation. The real equivalent requires only one add and one divide.

The Fortran 77 standard does not define complex double precision. Thus, the implementation used performs all operations using double precision with real arithmetic operations as described above. Estimates for solving large systems of algebraic equations may be obtained using results for equivalent real systems. A large size triangular decomposition is dominated by type (1) operations, consequently, the time to factor a real symmetric system is multiplied by a factor of 8 to obtain an estimate of the solution time (2 for the unsymmetric factor multiplied by 4 for complex arithmetic). A forward and backward solution requires only the complex factor 4 to make the conversion.

Appendix B

Material Properties

Properties of steel :

$$\rho_s = 7840 \text{ kg/m}^3$$

$$E_s = 210 \text{ GPa}$$

$$\nu_s = 0.30$$

Properties of Sea Water :

$$\rho_f = 1025 \text{ kg/m}^3$$

$$c_f = 1520 \text{ m/s}$$

Appendix C

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Appendix D

Plots and Graphs

FEAP

Scattering from Rigid Cylinder (ka=2); Sommerfeld Elements

Scattering from Rigid Cylinder (ka=2); Infinite Elements alone

Scattering from Flexible Shell ($ka=2$); Sommerfeld Elements; Solid Elem.

FEAP

Scattering from Flexible Shell ($ka=2$); Sommerfeld Elements; Shell Elem.