An Update on Projection Methods for Transient Incompressible Viscous Flow

P.M. Gresho and S.T. Chan
Regional Atmospheric Sciences Division
Lawrence Livermore National Laboratory, L-262
Livermore, CA 94551

This was prepared for submittal to the
6th International Symposium on Computational Fluid Dynamics
Lake Tahoe, Nevada
September 4-8, 1995

July 1995
DISCLAIMER

This document was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor the University of California nor any of their employees, makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or the University of California, and shall not be used for advertising or product endorsement purposes.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
An Update on Projection Methods for Transient Incompressible Viscous Flow

P.M. Gresho and S.T. Chan

Lawrence Livermore National Laboratory
P.O. Box 808, L-264
Livermore, California 94551 USA

ABSTRACT

In 1990 we introduced the biharmonic equation (for the pressure) and the concomitant biharmonic miracle when transient incompressible viscous flow is solved approximately by a projection method. Herein we introduce the biharmonic catastrophe that sometimes occurs with these same projection methods.

THEORY

The object of the exercise is the usual: solve the incompressible Navier-Stokes equations.

An increasingly popular way to time 'integrate' these equations is via so-called projection methods. Herein we focus on one of the most popular, "Semi-implicit Projection 2" (Gresho 1990), although most if not all of what we reveal below would apply to the other variants. We shall consider only the continuous-in-space case, partly for simplicity/generality, and partly because the 'problem' that we shall unveil is independent of spatial discretization. The semi-implicit projection method we discuss, analyze, and use is the following, using backward Euler for simplicity:

Step 1; intermediate velocity. Given \( P_n \) and \( u_n \) with \( \nabla \cdot u_n = 0 \), find \( u_{n+1} \) from

\[
\frac{u_{n+1} - u_n}{\Delta t} + u_n \cdot \nabla u_n + \nabla P_n = \nu \nabla^2 u_{n+1}. \tag{1}
\]

Step 2; the projection. Find the final velocity and the new pressure from

\[
\frac{u_{n+1} - u_n}{\Delta t} + \nabla \left( \frac{P_{n+1} - P_n}{2} \right) = 0 \quad \text{and} \quad \nabla \cdot u_n = 0, \tag{2a, b}
\]

which is realized by first solving the pressure Poisson equation (PPE) implied by (2).

This is the algorithm—rather simple, and often quite efficient. But the analysis of the algorithm is less simple, leading as it does to, among other things, a biharmonic equation (BHE) for the pressure [in addition to the PPE 'that defines' the pressure], which is derived by first inserting \( u_{n+1} = u_{n+1} + \Delta t \nabla (P_{n+1} - P_n)/2 \) from (2a) into (1):

\[
\frac{u_{n+1} - u_n}{\Delta t} + u_n \cdot \nabla u_n + \nabla \left( \frac{P_{n+1} + P_n}{2} \right) = \nu \nabla^2 u_{n+1} + \frac{\nu \Delta t}{2} \nabla^2 (P_{n+1} - P_n), \tag{3}
\]

a 'perturbed' momentum equation—perturbed to \( O(\Delta t^2) \) by the last term. The divergence of this equation leads to, using \( \nabla \cdot u = 0 \) and \( \nabla \cdot \nabla^2 u = \nabla \cdot (\nabla \nabla \cdot u - \nabla \times \nabla \times u) = 0 \),

\[
(I - \nu \Delta t \nabla^2) \nabla^2 P_{n+1} = -2 \nabla \cdot (u_n \cdot \nabla u_n) - \left( I + \nu \Delta t \nabla^2 \right) \nabla^2 P_n. \tag{4}
\]
a BHE that is satisfied by $P_{n+1}$. Note that it is also a singularly-perturbed PPE in that, for $\Delta \rightarrow 0$, the proper PPE is recovered.

In Gresho (1990), wherein this equation was first derived and discussed, the existence of a spurious numerical boundary layer (BL), $\delta = \sqrt{\nu \Delta t}$, was also discussed—as was the biharmonic miracle (BHM), which is this: whereas the pressure from a projection method is generally in error, by $O(\delta)$, near and at the wall, and its normal gradient is in serious error, by $O(1)$ in $\delta$, near and at the wall, the pressure, and its gradient, recovers at a distance $O(\delta)$ from the wall to the correct pressure from the Navier-Stokes equations. Whereas the mathematics in the above reference was as much heuristic as rigorous, it did seem to explain how a method that should be bad can be good—or at least, OK. It has also been strongly bolstered recently by Schwab (1995), a mathematician who understands well elliptic PDE's, and who more rigorously analyzed the BHE and associated spurious numerical BL's. He also properly restricts such analysis to problems with sufficient regularity—which is rather higher, naturally, than that required of the Navier-Stokes equations, or even the PPE.

Now that we know, not merely assert, that the projection method pressure also satisfies a BHE, we can address the major concern/issue of this paper: Since the BHE is much more prone to displaying geometrically-caused singularities (cf. fracture mechanics) than is the PPE (which itself is more "prone," in the pressure, than are the original NS equations), we ask: are there common situations in which the BHE of projection methods causes difficulty by polluting the pressure field with a 'biharmonic reaction' to sharp changes in geometry? That the answer is yes forms the basis for the rest of this paper.

RESULTS

Consider the flow in a diverging channel with sharp changes in angle. The angle of the sloped portion of $\Gamma$ in Fig. 1 is $14^\circ$ from the horizontal, the Reynolds number is zero, and the BC's are: $u = 0.2 \ln (100y + 1)$, $v = 0$ at the inlet, $u = v = 0$ on the bottom surface, $u = 1.0$, $v = 0$ on the top, and $\partial u/\partial n - \mu \partial P = 0$ at the outlet. The problem seems simple, even boring, until you seriously refine the mesh near the lower wall—when using a projection method. We were preparing for some analysis involving turbulent BL's, thus explaining $u(y)$, but got somewhat side-tracked by tripping over the biharmonic catastrophe (BHC). The mesh in Fig. 1 has the following properties: $\Delta x_{\min} \equiv 14 \times 10^{-5}$ (near $x = 1$ and $x = 3$, and growing at 15% per element) and $\Delta x_{\max} \equiv 0.13$ (at inlet and outlet), $\Delta y_{\min} \equiv 4.4 \times 10^{-5}$ (at the lower wall, and growing at 50% per element), and $\Delta y_{\max} = 0.5 - 0.6$ (at the upper wall). (Note that the aspect ratio is $\sim 3000$ near $x = y = 0$.) The suspicious-looking concentration of 'ink' near the lower wall at $x = 1$ and 3 is the manifestation of the BHC, the catastrophic nature of which behavior is only revealed by zooming. Hence, Figs. 2-4 focus on the details of the flow field near $x = 1$—wherein is displayed a rather anomalous eddy pattern that has no relationship to fluid mechanics. Though the spurious eddy is very small [its size is $O(0.02)$], the velocity magnitude is not ($\lvert u \rvert_{\max} \equiv 0.28$ at $x = 1$ and $y = 1.11 \times 10^{-4}$, the second node from the surface). Above all, small or not, it is clearly wrong—moving against the prevailing flow near the 'corner.' We have here, we believe, a manifestation of the BHE in that the very well-resolved numerical BL ($\delta = \sqrt{1 \times 0.005} \equiv 0.071$, in which reside $\sim 17$ nodes) is showing the results of the geometric singularity of the pressure BHE. Fig. 5 shows the pressure field at the first node above the surface and Fig. 6 shows a zoom near $x = 1$. This singular behavior must be caused by the BHE ... and it is noteworthy that the global picture is rather misleading in that it suggests that $P(x)$ first increases to $+\infty$ and then jumps to $-\infty$. The better resolution offered in Fig. 6 corrects this impression; there is a lot going on near the singularity.

Turning our attention now to the second (weaker?) singularity at $x = 3$ which, unlike that at $x = 1$, is not a so-called re-entrant singularity, we see another clockwise eddy—in Figs. 7 and 8. The flow is not weaker ($\lvert u \rvert_{\max} \equiv 0.38$ at $x = 3$ and 2 nodes up from the wall). Fig. 9 shows the corresponding pressure at 1 node up—a rather different singularity, at least in pressure, from that at $x = 1$ (cf. Fig. 6).

A small change in the inlet BC can make a very large change in the spurious eddy flow. Thus, when we changed the inlet velocity profile from logarithmic to parabolic—$u(y) = 2(y - y_0^2)$ which now peaks at $u(1.5) = 1.5$, giving a mean velocity equal to the peak velocity in the previous case—the first eddy is changed only quantitatively [the pressure extrema are rather larger ($\sim \pm 4500$)], but the second one is drastically different. Fig. 10 shows a counter clockwise eddy and Fig. 11 the corresponding pressure at 1 node from the wall (cf. Fig. 9). Strange indeed—and further manifestation of the spurious BHC.
To see the truth of the matter, we enlisted the help of two colleagues, each with a different code. Thus, R. Martin (LLNL) ran FIDAP (1993) and D. Gartling (SNL) ran NACHOS (Gartling 1987), each in the steady Stokes flow mode for the above parabolic inlet BC. (In our projection code, we must ‘time-march’ to the steady-state.) Whereas we and Martin ran the same mesh of $24 \times 200$ bilinear elements (piecewise-constant pressure), Gartling used about the same number ($25 \times 200$) of higher-order elements ($Q_2P_1$, 9-node velocity, 3 node linear, discontinuous, pressure)—arguably the best 2D element on the planet—but on a mesh that did not resolve the corner singularities in nearly as much details (Eq. $\Delta y_{\min} = 3.5 \times 10^{-5}$), although it is probably quite good enough for Stokes flow. Both codes showed the “boring” flow alluded to earlier—basically Couette flow near the lower wall with minor deviations near the singularities (not shown). Figure 12 shows the FIDAP pressure one node up over the entire domain (a result that agrees well with that from NACHOS) and Fig. 13 is a zoom near $x=1$ showing also the (miniscule) $x$- and $y$-velocities (plus a few little wiggles). In Fig. 14 are the NACHOS results for the vorticity along the lower wall which probably best displays the nature of the true velocity singularities (for the Laplacian operator, not the biharmonic)—even though $u$ and $v$ are $O(10^{-2})$ near these ‘corners,’ their derivatives are significant.

The interim conclusion from the above would seem to be a dire one: dump projection methods. That it is not quite that bad is shown in the next result, in which we return to the logarithmic inlet profile but reduce our time step 10-fold—reducing $\delta t$ to $0.022$, with now about 14 nodes in the spurious BL (The perturbation to the Laplacian operator, see (4), has dropped 10-fold—from $(1 - 0.005V_2^2)$ to $(1 - 0.0005V_2^2)$. Fig. (15) shows the velocity field near $x=1$ and is to be compared to Fig. 4. Now the spurious flow is very highly localized, of size $O(0.002)$ and is much weaker [$u_{\max} = 0.014$ at $x=1$ and 2 nodes up]; i.e. it is almost gone. Because we still have enough nodes to resolve the spurious BL, the new result is clearly caused by the 10-fold reduction in the perturbed operator. It is, however, interesting and surprising that the pressure field itself, Fig. 16, is only a little different than that in Fig. 6 (note the scale change in $x$)—at least with respect to the primary extrema. The ‘secondary’ extrema, however, are significantly reduced—from $\pm 250$ in Fig. 6 to $\pm 50$ in Fig. 16. Finally, Fig. 17 and 18 show the analogous results at the second singularity [$u_{\max} = 0.029$ at $x=3$ and 2 nodes up].

In another experiment, we coarsened the mesh significantly and virtually all bad behavior vanished. Specifically, we re-zoned using the same number of elements but with $\Delta y_{\min} = 0.2$ and $\Delta t = 0.22$, placing about 8 nodes within the spurious BL—even though $(1 - \delta V = 1 - 0.05V_2^2)$, a larger perturbation than before. The good news may be that the BHC is only present on ultra-fine meshes and can be vanquished by mesh coarsening and/or the use of a small $\Delta t$. The bad news is that good simulation of turbulent boundary layers (using modern models and no wall functions/numerical graffiti) may be more difficult to do well via projection methods.

**DISCUSSION**

In further ‘defense’ of projection methods, we briefly summarize the results of some time-dependent (Stokes flow) runs via NACHOS (by Gartling) and via FIDAP (by R. Sani at Univ. Col. and M. Engelman at FDI) on the same grids used for the steady Stokes flows. The IC was potential flow (as was ours) which means that the tangential velocity at the lower wall is very high (slip), relative to what it will ultimately be for steady Stokes flow. The initial pressure field [Euler velocity and Stokes pressure— a result of applying the no-slip BC at $t=0^+$] is also quite different than that for steady flow. I.e., we have defined a difficult transient Stokes flow on a very fine mesh. The ‘bottom line’ is this: when the two codes were run with the variable step trapezoid rule for time integration (see Gresho et al. 1996), both ‘crashed,’ i.e. behaved poorly, erratically, unbelievably—for reasons that remain unexplained. When a variable step backward Euler method was used, both codes behaved somewhat better—although neither code was used to pursue the matter in any depth.

**CONCLUDING REMARKS**

If a projection method is used in situations that involve sharp angular shape changes in polygonal domains and if fine spatial resolution is employed near ‘corners’ and if $\Delta t$ is sufficiently large that the spurious numerical boundary layer, of thickness $O(\sqrt{\nu \Delta t})$, contains many nodes (is well-resolved), then it may happen that small-scale spurious velocities that are associated with large-scale spurious pressures are the sad result. It is, however, often possible to obtain better results by coarsening the mesh (in the region of the spurious velocities) at the same $\Delta t$. It is of course always possible to obtain better results by reducing $\Delta t$. 
ACKNOWLEDGMENTS

We thank Dick Martin, Bob Sani, Dave Gartling, and Michael Engelman for helping us gauge the level of severity of the biharmonic catastrophe.

REFERENCES


*This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48.
Fig. 1. Domain with steady 'Stokes' flow.

Fig. 2. First vector zoom.

Fig. 3. Streamlines for figure 2.

Fig. 4. Second vector zoom.

Fig. 5. P(x) one node up.

Fig. 6. Zoom from figure 5.

Figures 1–6
Fig. 7. Vector zoom near x=3.

Fig. 8. Streamlines for figure 7.

Fig. 9. Zoom from figure 5.

Fig. 10. Vectors ala figure 7 for new BC.

Fig. 11. P(x) for new BC.

Fig. 12. P(x) one node up (FIDAP).

Figures 7-12
Fig. 13. P(x), u(x), v(x) from FIDAP.

Fig. 14. Vorticity on wall (from NACHOS)

Fig. 15. Vector zoom for smaller Δt.

Fig. 16. P(x) one node up.

Fig. 17. Vector zoom for smaller Δt.

Fig. 18. P(x) one node up.

Figures 13-18