Null-Plane Invariance of Hamiltonian Null-Plane Dynamics

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Relativistic Hamiltonian few-body dynamics [1, 2] involves two unitary representations of the Poincaré group on the Hilbert space $\mathcal{H}$ of physical states, with and without interactions. These two representations, $U(\Lambda, a)$ and $U_0(\Lambda, a)$, coincide for a kinematic subgroup H . The “Hamiltonians” are the generators not in the Lie algebra of the kinematic subgroup. The kinematic subgroup of null-plane dynamics leaves the null-plane $z \cdot x \equiv x^0 + x_3 = 0$ invariant.

Few-body Hamiltonians satisfying the required commutation relations can be constructed as functions of a mass operator and kinematic quantities. For more than two particles there are nontrivial problems in satisfying cluster separability. [3] Consistency of electro-weak interactions with strong interactions also involves significant problems: Poincaré covariance of current operators requires the construction of appropriate interaction currents.

In formal Fock-space representations of local field theories the Poincaré generators are obtained by integrating the energy momentum tensor over the hypersurface that is invariant under the kinematic subgroup.[4] The elements of the Lie algebra so constructed are bilinear functionals over the Fock space, but they are not self-adjoint Fock-space operators. Thus there is no unitary Poincaré representation, $U(\Lambda, a)$, on the Fock space.

In the context sketched above it is useful to consider the manifold of null planes $n \cdot x \equiv x^0 + \hat{n} \cdot x = 0$ parameterized by the unit vectors $\hat{n}$, and treat the orientations $\hat{n}$ of the null planes as an additional unphysical degree of freedom. [5, 6]

The essential mathematical feature is the coset decomposition $\Lambda = \mathcal{R}_H \Lambda_H$, $\Lambda_H \in H$ for all Lorentz transformations Λ , where Λ_H leaves the null-plane $z \cdot x = 0$ invariant and the rotations $\mathcal{R}_H$ are parameterized by the components of the spatial unit vector $\hat{n}$, $n := \{-1, \hat{n}\} = \mathcal{R}(n)\{-1, 0, 0, 1\}$. The Poincaré transformations Λ_{H_n}, a_{H_n} that leave the null plane $n \cdot x = 0$ invariant are related to Λ_H, a_H by

$$\Lambda_{H_n} = \mathcal{R}(n) \Lambda_H \mathcal{R}^{-1}(n), \quad a_{H_n} = \mathcal{R}(n) a_H, \quad (1)$$

and the corresponding coset representatives are $\mathcal{R}_{H_n}(\Lambda) = \Lambda \mathcal{R}(n) \Lambda_H^{-1} \mathcal{R}^{-1}(n)$. It

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follows that

$$\mathcal{R}(\Lambda n) = \Lambda \mathcal{R}(n) \Lambda_H^{-1} = \mathcal{R}_H(\Lambda) \Lambda_H \mathcal{R}(n) \Lambda_H^{-1}. \quad (2)$$

It is easy to verify that the unitary operators $C(n) := U_0[\mathcal{R}(n)]U^\dagger[(\mathcal{R}(n))]$ satisfy the intertwining relations $U_0(\Lambda)C(n) = C(\Lambda n)U(\Lambda)$. [7] The representations $U_n(\Lambda, a) \equiv \exp(iP_n \cdot a)U_n(\Lambda) := C(n)U(\Lambda, a)C^\dagger(n)$ are identical to $U_0(\Lambda, a)$ for $\Lambda, a \in H_n$. The fact that the coset representatives form a compact manifold is a special feature of null-plane dynamics.

The treatment of the null-plane orientation $\hat{n}$ as a degree of freedom depends on the existence of an irreducible unitary representation, $\bar{U}(\Lambda)$, of $SO(1, 3)$ on a Hilbert space $\bar{\mathcal{H}}$ of functions $f(\hat{n})$, with the inner product $(f, g) = \int d^2\hat{n} f^*(\hat{n})g(\hat{n})$. Translations are represented by the identity. The extended Hilbert space of states is the tensor product $\bar{\mathcal{H}} := \mathcal{H} \otimes \bar{\mathcal{H}}$. On the extended Hilbert space we have the Poincaré representation

$$\bar{U}(\Lambda, a) = \exp(i\bar{P} \cdot a)[U_0(\Lambda) \otimes \bar{U}(\Lambda)] \quad (3)$$

where only the momentum operator, $\bar{P}^\mu := \int dE_n P_n^\mu$, depends on the dynamics.

From properties of the representation $U(\Lambda, a)$ follows the existence of generalized wave operators, $\bar{\Omega}_\pm$, which map the Hilbert space $\mathcal{H}$ onto the Hilbert space $\mathcal{H}_f$ of free physical particles. [3] The corresponding extended wave operators, $\bar{\Omega}_\pm := \int dE_n C(n)\bar{\Omega}_\pm$ map $\bar{\mathcal{H}}$ onto $\bar{\mathcal{H}}_f := \mathcal{H}_f \otimes \bar{\mathcal{H}}$ with the Poincaré representation

$$\bar{U}_f(\Lambda, a) := \exp(i\bar{P}_f \cdot a)[U_f(\Lambda) \otimes \bar{U}(\Lambda)]. \quad (4)$$

The operators $\bar{P}_f$ and $\bar{S} := \bar{\Omega}_+^\dagger \bar{\Omega}_-$ commute with $\bar{U}(\Lambda)$. Assuming existence and completeness of the wave operators, null-plane independence of the observables,

$$[\bar{U}(\Lambda), \bar{\Omega}^\dagger \bar{P}^\mu \bar{\Omega}] = 0, \quad \text{and} \quad [\bar{U}(\Lambda), \bar{S}] = 0, \quad (5)$$

is both necessary and sufficient for the existence of the unitary operator $C(n)$,

$$C(n)dE_n = \bar{\Omega}_+ dE_n U_f^\dagger[\mathcal{R}(n)]\bar{\Omega}_+ U_0[\mathcal{R}(n)] = \bar{\Omega}_- dE_n U_f^\dagger[\mathcal{R}(n)]\bar{\Omega}_- U_0[\mathcal{R}(n)]. \quad (6)$$

Attempts to construct Poincaré covariant Hamiltonian dynamics from the Fock-space representation of a local Lagrangean requires additional considerations. [8] Since for a local interaction Lagrangean $\int d^3\mathbf{x} \mathcal{L}(\mathbf{x})\bar{\Psi}$ is not in the Fock space, $\mathcal{F}$, for any $\bar{\Psi} \in \mathcal{F}$, there is no unitary Poincaré representation on the Fock space. There is, of course, the free-field unitary representation $U_0(\Lambda)$ and it is possible to regularize the interaction in such a manner that the four-momentum $\bar{P}$ is a covariant self-adjoint operator with commuting components. That much is sufficient for the existence of a unitary representation $\bar{U}(\Lambda, a)$ on $\mathcal{F} \otimes \bar{\mathcal{H}}$. For a renormalizable field theory it should be straightforward to verify the null-plane independence of perturbative results in the limit of infinite cutoff. The framework we have outlined may provide a systematic approach to non-perturbative approximations in suitably defined subspaces of the Fock space.

Null-plane dynamics has the unique feature that, for space-like momentum transfer the impulse approximation is kinematically consistent, provided the null

plane orientations are restricted to $n \cdot Q = 0$. The invariant $Q^2 = Q_\perp^2$ depends only on the kinematic components of the momentum operator, and the momenta of all contributing constituents are related by kinematic Lorentz transformations. The treatment of the null-plane orientation so restricted as an additional degree of freedom provides a convenient framework for the determination of “minimal” interaction currents required for full Lorentz covariance. The operator $N^\mu := n^\mu / (n \cdot P)$ transforms as a four vector and commutes with the Fourier transform $\tilde{I}^\mu(Q)$ of the current density operator. Since $N^2 = 0$, $N \cdot Q = 0$ and $N \cdot P = 1$ there are no $\hat{n}$ -dependent invariants. The projection, $\tilde{P}(Q)$, of the four-momentum operator into the hyperplane perpendicular to Q also commutes with $\tilde{I}^\mu(Q)$. Let the boosts $B(\tilde{P}, Q)$ be defined by

$$Q = B(\tilde{P}, Q)\{0, 0, 0, \sqrt{Q^2}\}, \quad \tilde{P}(Q) = B(\tilde{P}, Q)\{M(Q^2), 0, 0, 0\} \quad (7)$$

and define

$$\begin{aligned} \mathcal{N} &:= B^{-1}(\tilde{P}, Q)N\tilde{M}(Q^2) = \{-1, \mathcal{N}, 0\}, \quad \mathcal{N} = \{\cos \varphi, \sin \varphi\} \\ \mathcal{I} &:= B^{-1}(\tilde{P}, Q)\tilde{I}(Q) = \{\mathcal{I}^0, \mathcal{I}, 0\} \end{aligned} \quad (8)$$

Under Lorentz transformations the operator $\mathcal{I}(Q^2, \mathcal{N})$ undergoes $SO(2)$ Wigner rotations,

$$\mathcal{R}_W(A; \tilde{P}, Q) := B^{-1}(A\tilde{P}, AQ)AB(\tilde{P}, Q). \quad (9)$$

The operators $\mathcal{I}^\beta$, $\beta = \{0, \pm 1\}$ transform respectively as an $O(2)$ scalar and vector and their matrix elements are $O(2)$ Clebsch-Gordan coefficients multiplied by invariant form factors,

$$\langle \sigma', j' | \mathcal{I}^\beta(Q^2, \mathcal{N}) | j, \sigma \rangle = e^{im\varphi} \delta(m + \beta + \sigma - \sigma') F(Q^2; |\beta|, m, |\sigma|) \quad (10)$$

The null-plane dependent part of the current, $m \neq 0$ must be cancelled by “minimal” interaction currents. This construction is equally applicable to impulse currents and “model dependent” meson exchange currents.

Equivalent results obtain by consideration of $O(1, 3)$ covariant current kernels $(\rho', j', p' | I^\mu(0, N^\nu) | p, j, \rho)$ where ρ denotes generalized spinor indices. Such kernels are linear combinations of invariants multiplied by irreducible tensors which may involve monomials of N . Since there are no N -dependent invariants the null-plane independent part of the current is well defined.[6]

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Appendix

A. Unitary Representation of $SO(1,3)$ and Degenerate Representation of the Poincaré Group

Let $\hat{n}$ be a unit vector parameterized by an azimuthal angle φ and $n_z = \cos \theta$.

$$\hat{n}(n_z, \varphi) := \{\sqrt{1 - n_z^2} \cos \varphi, \sqrt{1 - n_z^2} \sin \varphi, n_z\} \quad (\text{A.1})$$

The Hilbert space $\tilde{\mathcal{H}}$ of square integrable functions has the inner product

$$(f, g) := \int_0^{2\pi} d\varphi \int_{-1}^{+1} dn_z f^*(n_z, \varphi) g(n_z, \varphi) = \int d^2 \hat{n} f^*(\hat{n}) g(\hat{n}), \quad \forall f, g \in \tilde{\mathcal{H}}. \quad (\text{A.2})$$

The Lorentz transformation $\Lambda \in O(1,3)$ acts on $\tilde{\mathcal{H}}$ according to

$$[\tilde{U}(\Lambda)f](\hat{n}) = s(\Lambda; \hat{n}) f(\hat{n}'), \quad \text{where } s(\Lambda, \hat{n})\{-1, \hat{n}'\} := \Lambda^{-1}\{-1, \hat{n}\} \quad (\text{A.3})$$

defines how the Lorentz transformation $\Lambda \in SO(1,3)$ changes $\hat{n} \rightarrow \hat{n}'$ and induces a scale factor $s(\Lambda, \hat{n})$. For space reflection and time reversal we have $[\mathcal{P}f](\hat{n}) = f(-\hat{n})$ and $[\mathcal{PT}f](\hat{n}) = f^*(\hat{n})$

The group property of the scale factors,

$$s(\Lambda_1 \Lambda_2, \hat{n}) = s(\Lambda_1, \hat{n}) s(\Lambda_2, \hat{n}') \quad (\text{A.4})$$

follows from their definition.

$$s(\Lambda_1 \Lambda_2, \hat{n})\{-1, \hat{n}''\} = \Lambda_2^{-1} s(\Lambda_1, \hat{n})\{-1, \hat{n}'\} = s(\Lambda_1, \hat{n}) s(\Lambda_2, \hat{n}')\{-1, \hat{n}''\} \quad (\text{A.5})$$

The Jacobians $\mathcal{J}(\Lambda, \hat{n})$,

$$\mathcal{J}(\Lambda, \hat{n}) := \frac{\partial(u', \varphi')}{\partial(u, \varphi)} \equiv \frac{d^2 \hat{n}'}{d^2 \hat{n}}. \quad (\text{A.6})$$

have the same group property,

$$\mathcal{J}(\Lambda_1 \Lambda_2, \hat{n}) = \mathcal{J}(\Lambda_1, \hat{n}) \mathcal{J}(\Lambda_2, \hat{n}') \quad (\text{A.7})$$

Thus

$$(\tilde{U}(\Lambda)f)(\hat{n}) = \sqrt{\mathcal{J}(\Lambda, \hat{n})} f(\hat{n}') \quad (\text{A.8})$$

defines a unitary, irreducible representation of the Lorentz group $\forall f \in \tilde{\mathcal{H}}$.

$$\|\tilde{U}(\Lambda)f\|^2 = \int d^2 \hat{n} \mathcal{J}(\Lambda, \hat{n}) |f(\hat{n}')|^2 = \int d^2 \hat{n}' |f(\hat{n}')|^2 = \|f\|^2 \quad (\text{A.9})$$

Improper basis states $|\hat{n}\rangle$, transform according to

$$\tilde{U}(\Lambda)|\hat{n}\rangle = |\hat{n}'\rangle s(\Lambda, \hat{n}) \quad (\text{A.10})$$

so that the spectral projector $dE_n := d^2 \hat{n} |\hat{n}\rangle \langle \hat{n}|$ is invariant.