LS-179
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THE EFFECT OF SMALL ELLIPSOIDAL MATERIAL ON THE RESONANT FREQUENCY OF A CAVITY

We assume that the medium inside the cavity has no losses with $\epsilon = \epsilon_o = \frac{1}{36\pi} 10^{-9}$ farads/meter and $\mu = \mu_o = 4\pi \times 10^{-7}$ henrys/meter. Excited at resonance, the fields inside the cavity can be written in the form

$$E(x,y,z,t) = E_o(x,y,z)e^{j\omega t}$$
$$H(x,y,z,t) = H_o(x,y,z)e^{j\omega t}.$$

The electric and magnetic fields are 90^o out of phase. In other words, if E_o is real H_o is imaginary and vice versa. Insertion of a small piece of material with $\epsilon \neq \epsilon_o$ and $\mu \neq \mu_o$ will change the field values and the resonant frequency

$$E = (E_o + E_1)e^{j(\omega + \delta\omega)t}$$
$$H = (H_o + H_1)e^{j(\omega + \delta\omega)t}. \tag{1}$$

Note that $\delta\omega$ will be a complex quantity if the inserted material is lossy. Substitution of equations (1) in Maxwell's equations

$$\text{curl } E = -\frac{\partial B}{\partial t}$$

$$\text{curl } H = \frac{\partial D}{\partial t}$$

give

$$\mathrm{curl}\,(E_o + E_1) = -j(\omega + \delta\omega)\,(B_o + B_1)$$

$$\mathrm{curl}\,(H_o + H_1) = j(\omega + \delta\omega)\,(D_o + D_1)\ .$$

Noting that the fields E_o, D_o, H_o and B_o satisfy the same Maxwell's equations we obtain

$$\mathrm{curl}\,E_1 = -j\,[\,\delta\omega B_o + (\omega + \delta\omega)B_1\,] \tag{2a}$$

$$\mathrm{curl}\,H_1 = j\,[\,\delta\omega D_o + (\omega + \delta\omega)D_1\,]\ . \tag{2b}$$

Multiplication of equation (2a) by H_o and equation (2b) by E_o and addition give

$$H_o \cdot \mathrm{curl}\,E_1 + E_o \cdot \mathrm{curl}\,H_1 = j(\omega + \delta\omega)\,(E_o \cdot D_1 - H_o \cdot B_1) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) \tag{3}$$

Using the vector relation

$$\mathrm{div}\,(A \times B) = B \cdot \mathrm{curl}\,A - A \cdot \mathrm{curl}\,B$$

we can rewrite the L.H.S. of equation (3) in the form

$$H_o \cdot \text{curl } E_1 + H_o \cdot \text{curl } H_1$$
$$= E_1 \cdot \text{curl } H_o + H_1 \cdot \text{curl } E_o$$
$$- \text{div } (H_o \times E_1 + E_o \times H_1),$$

or

$$H_o \cdot \text{curl } E_1 + E_o \cdot \text{curl } H_1$$
$$= j\omega(E_1 \cdot D_o - H_1 \cdot B_o)$$
$$- \text{div } (H_o \times E_1 + E_o \times H_1). \qquad (4)$$

Comparison of equations (3) and (4) gives

$$j(\omega + \delta\omega)\,(E_o \cdot D_1 - H_o \cdot B_1) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o)$$
$$= j\omega(E_1 \cdot D_o - H_1 \cdot B_o) - \text{div } (H_o \times E_1 + E_o \times H_1)$$

In practice $\omega >> \delta\omega$ so that in the expression $(\omega + \delta\omega)\,(E_o \cdot D_1 - H_o \cdot B_1)$ we can neglect $\delta\omega$ with respect to ω and obtain

$$j\omega(E_o \cdot D_1 - H_o \cdot B_1 - E_1 \cdot D_o + H_1 \cdot B_o) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o)$$
$$= - \text{div } (H_o \times E_1 + E_o \times H_1) \quad . \qquad (5)$$

Outside the ellipsoid we have

$$D_0 = \epsilon_0 E_0, \quad D_1 = \epsilon_0 E_1, \quad B_0 = \mu_0 H_0, \quad B_1 = \mu_0 H_1,$$

and equation (5) reduces to

$$j\delta\omega(E_0 \cdot D_0 - H_0 \cdot B_0) = - \operatorname{div} (H_0 \times E_1 + E_0 \times H_1) .$$

We integrate this equation over the volume bounded by the cavity wall and the surface of the ellipsoid

$$j\delta\omega \int_{V-\Delta V} (E_0 \cdot D_0 - H_0 \cdot B_0)\, dV = - \int_{V-\Delta V} \operatorname{div} (H_0 \times E_1 + E_0 \times H_1)\, dV \tag{6}$$

where V = volume of the cavity

and ΔV = volume of the ellipsoid.

Using the divergence theorem, the R.H.S. of equation (6) can be written as a surface integral

$$\int_{V-\Delta V} \operatorname{div} (H_0 \times E_1 + E_0 \times H_1)\, dV = \int_{s+\Delta s} (H_0 \times E_1 + E_0 \times H_1) \cdot ds$$

where $s + \Delta s$ is the surface bounding the volume V-ΔV. Since the cavity is assumed to be a good conductor, E_o and E_1 will be practically perpendicular to the cavity surface and the contribution of the cavity wall to the surface integral can be neglected. In this case we find

$$j\delta\omega \int_{V-\Delta V} (E_o \cdot D_o - H_o \cdot B_o)\, dV = \int_{\Delta s} (H_o \times E_1 + E_o \times H_1) \cdot ds \qquad (7)$$

where Δs = surface of the ellipsoid. Note that ds is in the direction of the now outward normal to the ellipsoid surface. Inside the ellipsoid we have

$$D_1 = \epsilon_o E_1 + P \quad \text{and} \quad B_1 = \mu_o H_1 + M \quad .$$

where P = polarization or electric dipole moment per unit volume, and

M = magnetization or magnetic dipole moment per unit volume.

Substitution in equation (5) gives

$$j\omega(E_o \cdot P - H_o M) + j\delta\omega(E_o \cdot D_o - H_o \cdot B_o) = -\operatorname{div}(H_o \times E_1 + E_o \times H_1).$$

Integrating over the volume of the ellipsoid and using the divergence theorem we obtain

$$j\omega \int_{\Delta V} (E_0 \cdot P - H_0 \cdot M)\, \Delta V + j\delta\omega \int_{\Delta V} (E_0 \cdot D_0 - H_0 \cdot B_0) \Delta V$$
$$= - \int_{\Delta s} (H_0 \times E_1 + E_0 \times H_1) \cdot ds. \tag{8}$$

Comparison of equations (7) and (8) and some manipulation gives

$$\frac{\delta\omega}{\omega} = - \frac{\int_{\Delta V} (E_0 \cdot P - H_0 \cdot M)\, dV}{\int_V (E_0 \cdot D_0 - H_0 \cdot B_0)\, dV}$$

or, since H_0 and B_0 are imaginary if E_0 and D_0 are real, we can write

$$\frac{\delta\omega}{\omega} = - \frac{1}{4U} \int_{\Delta V} (E_0 \cdot P + H_0 \cdot M)\, dV \tag{9}$$

where U = cavity stored energy and E_0, P, H_0 and M are now all real quantities. For the small region in and around the ellipsoid, E_0 and H_0 are practically uniform. For an ellipsoid of semi-axis a, b and c with a field <u>parallel to the a-axis</u>, P and M are given by (see M. Mason and W. Weaver, Dover Publications, § 36).

$$P = \frac{\epsilon_r - 1}{L(\epsilon_r - 1) + 1} \epsilon_0 E_0$$

$$M = \frac{\mu_r - 1}{L(\mu_r - 1) + 1} \mu_0 H_0$$

where ϵ_r = relative permittivity, μ_r = relative permeability, and

$$L = \frac{abc}{2}\int_0^\infty \frac{du}{(a^2 + u) \sqrt{(a^2 + u) (b^2 + u) (c^2 + u)}}$$

Axial Symmetrical Ellipsoid

For axial symmetry about the a-axis we have b = c, and this reduces the integral to an elementary one,

$$L = \frac{ab^2}{2}\int_0^\infty \frac{du}{(a^2 + u)^{3/2} (b^2 + u)}$$

Performing the integration we find for a oblate spheroid (a < b)

$$L = \frac{1 + e^2}{e^3} (e - \arctan e), \qquad e = \frac{1}{a} \sqrt{b^2 - a^2}$$

and for a prolate spheroid (a > b)

$$L = \frac{1 - e^2}{e^3} \left(\frac{1}{2} \ln \frac{1 + e}{1 - e} - e\right), \qquad e = \frac{1}{a} \sqrt{a^2 - b^2}$$

For both cases the spheroid reduces to a sphere for $e \to 0$ and $L = 1/3$. For the limit as $a \to 0$ $(e \to \infty)$, the oblate spheroid becomes a circular disk of radius b and $L = 1$. On the other hand, as $b \to 0$ the prolate spheroid becomes a very thin rod of length 2a and $L = 0$.

If the field is perpendicular to the axis of revolution we have

$$L = \frac{ab^2}{2} \int_0^\infty \frac{du}{(b^2 + u)^2 \sqrt{a^2 + u}} .$$

For a prolate spheroid (a > b) we find

$$L = \frac{1 - e^2}{4e^3} \left(\frac{2e}{1-e^2} - \ln \frac{1 + e}{1 - e} \right), \qquad e = \frac{1}{a} \sqrt{a^2 - b^2} .$$

For an oblate spheroid (a < b)) we find

$$L = \frac{1 + e^2}{8e^3} \left(\arctan e - \frac{e(1 - e^2)}{(1 + e^2)^2} \right), \quad e = \frac{1}{a} \sqrt{b^2 - a^2} .$$

In both cases for $e \to 0$ (sphere), $L = 1/3$. For the prolate spheroid for $e = 1$ (rod), $L = 1/2$. For the oblate spheroid for $e = \infty$ (disk), $L = 0$.

EXAMPLES

(1) Dielectric Sphere:

Radius R, $\mu_r = 1$, $P = 3\frac{\epsilon_r - 1}{\epsilon_r + 2}\epsilon_o E_o$, $M = o$

$$\frac{\delta f}{f} = -\frac{\pi R^3}{U}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\epsilon_o E_o^{\cdot}$$

(2) Metal Sphere:

Radius R, $\mu_r = 0$, $\epsilon_r = \infty$, $P = 3\epsilon_o E_o$, $M = -3/2\ \mu_o H_o$

$$\frac{\delta f}{f} = -\frac{\pi R^3}{U}\left(\epsilon_o E_o^2 - \frac{1}{2}\mu_o H_o^2\right) .$$

(3) Dielectric Needle:

Parallel to E_o, volume ΔV, $P = (\epsilon_r - 1)\ \epsilon_o E_o$, $M = 0$

$$\frac{\delta f}{f} = -\frac{\Delta V}{4U}(\epsilon_r - 1)\ \epsilon_o E_o^2 .$$

(4) Metal Needle:

Perpendicular to E_o and parallel to H_o, volume ΔV, $P = 2\ \epsilon_o E_o$, $M = -\mu_o H_o$

$$\frac{\delta f}{f} = -\frac{\Delta V}{4U}(2\ \epsilon_o E_o^2 - \mu_o H_o^2) .$$

(5) Dielectric Disk:

Perpendicular to E_o, volume ΔV, $P = \frac{\epsilon_r - 1}{e_r} \epsilon_o E_o$, $M = 0$

$$\frac{\delta f}{f} = - \frac{\Delta V}{4U} \frac{\epsilon_r - 1}{\epsilon_r} \epsilon_o E_o^2 \quad .$$

(6) Metal Needle Parallel to E_o and Metal Disk Parallel or Perpendicular to E_o

For the metal needle parallel to E_o, the approximation of $e = \frac{1}{a} \sqrt{a^2 - b^2} = 1$ is not valid since for $L = 0$, $P \rightarrow \infty$. For the metal disk, the approximation $e = \frac{1}{a} \sqrt{b^2 - a^2} = \infty$ is not valid since for $L = 0$, $P \rightarrow \infty$ and for $L = 1$, $M \rightarrow \infty$. In these cases, the actual value of L must be calculated and substituted in $P = \frac{1}{L} \epsilon_o E_o$ and $M = \frac{1}{L-1} \mu_o H_o$.