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Minimum-Time Trajectory Tracking of an Under-Actuated System[#]

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Abstract: Minimum-time trajectory tracking of an under-actuated mechanical system called the Acrobot is presented. The success of the controller is demonstrated by the fact that the tracking error is reduced by more than an order of magnitude when compared to the open-loop system response. The control law is obtained by linearizing the system about the nominal trajectory and applying differential dynamic programming to the resulting linear time-varying system, while using a weighted sum of the state-deviation and input-deviation as the cost function.

1. Introduction

In controlling under-actuated systems, it is often necessary to determine the desired state and/or output trajectory that the plant under its control must follow. The determination of the desired trajectory is often performed in order to minimize a certain performance index while satisfying the required state and/or input constraints. A frequently used performance index is the "time" to perform a certain task.

Most of the work in minimum-time trajectory planning and tracking has been for rigid, non-redundant, completely actuated robotic manipulators. This includes the work of [Bobrow, et al, 1983, 1985] and [Shin and McKay, 1984, 1985, 1986]. There has been some work where there are no path constraints and only the terminal state is specified. This scenario allows for the use of switching times as optimization parameters (see [Meier and Bryson, 1990] and [Sadegh and

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Driessen, 1999)) if the state equation of the dynamic system is linear in the inputs and the inputs have explicit upper and lower bounds. (This can be proved with the Pontryagin Minimum Principle {see [Bryson and Ho, 1975] and [Lewis, 1986]}.) The Acrobot system considered in the present paper falls into this category and the optimal trajectories are of this bang-bang type, as expected.

Some of the previous work in tracking of minimum-time trajectories of under-actuated dynamic systems includes the work of [Schoenwald, et al, 1991], [Eisler, et al, 1990, 1993], and [Parker, et al, 1997]. In all of these works, an off-line trajectory is found by optimization methods, and then a controller is designed to attempt to keep the system close to the off-line optimal trajectory.

The present work presents and demonstrates a method of real-time tracking of minimum-time trajectories that is applicable to under-actuated as well as fully actuated systems. The method used is to first linearize the nonlinear discrete-time dynamic system about the nominal trajectory. Then, we apply differential dynamic programming (ddp) to the resulting time-varying linear system while treating our cost function as the weighted sum of the squares of the state deviations (state errors) and the input deviations. For the Acrobot problem considered herein, with an initial state error, the closed loop control law (obtained from the ddp) gives tracking performance that is more than an order of magnitude better than the open-loop tracking response.

2. Problem Statement

Given a nonlinear time invariant dynamic system whose state equation is:

$$\dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}) \quad (2.1)$$

with initial condition:

$$\bar{x}(0) = \bar{x}_0 \quad (2.2)$$

and required terminal state:

$$\bar{x}(t_f) = \bar{x}_{f,des} \quad (2.3)$$

and input bounds:

$$\bar{u}_{\min} \leq \bar{u} \leq \bar{u}_{\max} \quad (2.4)$$

the objective is to find $\bar{u}(t)$ (and $\bar{x}(t)$ that satisfies constraints (2.3) and (2.4), and minimizes the total trajectory execution time t_f .

By applying a Runge-Kutta formula to equation (2.1), we obtain:

$$\bar{x}_{k+1} = f(\bar{x}_k, \bar{u}_k, h), (k=1, \dots, N) \quad (2.5)$$

where h is the integration time step: $h = t_f / N$. Let a change in $\bar{u}_k$ be denoted by u_k (thus defining $u_k, (k=1, \dots, N)$).

Linearizing equation (2.5), we obtain

$$x_{k+1} = A_k x_k + B_k u_k, (k=1, \dots, N) \quad (2.6)$$

where of course,

$$A_k = \frac{\partial f(\bar{x}_k, \bar{u}_k)}{\partial \bar{x}_k}, B_k = \frac{\partial f(\bar{x}_k, \bar{u}_k)}{\partial \bar{u}_k} \quad (2.7)$$

3. Overview of the Tracking Method/Derivation of Feedback Gain Matrices

We suppose we have found an optimal (minimum-time) trajectory off-line and that we have left enough slack in the inequality constraints (2.4) to allow for closed-loop tracking with disturbances present, and we want to track this trajectory in the presence of initial errors and/or other disturbances. Obviously, to robustify the tracking performance to such disturbances, we must have feedback. Let us denote the off-line optimal trajectory by:

$$\bar{u}_i^*, \quad (i=1, \dots, N) \quad (3.1)$$

and

$$\bar{x}_i^*, \quad (i=1, \dots, N) \quad (3.2)$$

The equations (2.5) will be linearized about this trajectory. (So, in equation (2.7), we replace $\bar{u}_k$ and $\bar{x}_k$ by $\bar{u}_k^*$ and $\bar{x}_k^*$, respectively.) The ddp method provides us with a feedback law of the form:

$$u_k = G_k x_k, (k=1, \dots, N) \quad (3.3)$$

where again the u_k is the change in $\bar{u}_k$, or $u_k = \Delta \bar{u}_k$, and x_k is the change in $\bar{x}_k$, or $x_k = \Delta \bar{x}_k$, and where the G_k are constant matrices.

Obviously we have reduced (by first order approximation) the problem to one in which our system is the following linear, time-varying system:

$$x_{k+1} = A_k x_k + B_k u_k, (k=1, \dots, N) \quad (3.4)$$

and our cost function is:

$$\text{cost} = \frac{1}{2} \sum_{k=2}^{N+1} x_k^T x_k \quad (3.5)$$

In equation (3.4), however, N is not the same N as was used in the off-line optimization. We integrated the open loop torques (from the off-line optimization) forward at a finer integration step-size to obtain data (i.e., A_k and B_k) that are spaced at 0.7569 ms intervals.

The following schematic will be very useful in deriving the feedback gain matrices:

Figure 1. Schematic for finding the ddp gain matrices

We first want to solve for u_4 in terms of x_4 . We will denote immediate cost functions by v_k and entire future cost functions by w_k . For the last time stage in Figure 1, of course $w_5 = v_5$. We have:

$$w_5 = v_5 = \frac{1}{2} x_5^T x_5 = \frac{1}{2} (A_4 x_4 + B_4 u_4)^T (A_4 x_4 + B_4 u_4) \quad (3.6)$$

Expanding, we obtain:

$$w_5 = v_5 = \frac{1}{2} x_4^T A_4^T A_4 x_4 + x_4^T A_4^T B_4 u_4 + \frac{1}{2} u_4^T B_4^T B_4 u_4 \quad (3.7)$$

We can now solve explicitly for the value of u_4 that will minimize w_5 . It is:

$$u_4 = -(B_4^T B_4)^{-1} B_4^T A_4 x_4 \quad (3.8)$$

from which we recognize:

$$G_4 = -(B_4^T B_4)^{-1} B_4^T A_4 \quad (3.9)$$

We now want to find u_3 in terms of x_3 . We have:

$$w_4 = v_4 + w_5 \quad (3.10)$$

So, let us first express w_4 in terms of x_3 and u_3 . After substituting (3.8) into (3.7) and then

substituting $x_4 = A_3 x_3 + B_3 u_3$ into that result and simplifying, it can be shown that:

$$\begin{aligned} w_4 = & \frac{1}{2} u_3^T \left[B_3^T A_4^T A_4 B_3 - B_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 B_3 + B_3^T B_3 \right] u_3 + \\ & x_3^T \left(A_3^T A_4^T A_4 B_3 - A_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 B_3 + A_3^T B_3 \right) u_3 + \\ & \frac{1}{2} x_3^T \left(A_3^T A_4^T A_4 A_3 - A_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 A_3 + A_3^T A_3 \right) x_3 \end{aligned} \quad (3.11)$$

Then, solving for the value of u_3 that minimizes w_4 , we obtain:

$$u_3 = \left[B_3^T A_4^T A_4 B_3 - B_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 B_3 + B_3^T B_3 \right]^{-1} \left(B_3^T A_4^T A_4 A_3 - B_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 A_3 + B_3^T A_3 \right) x_3 \quad (3.12)$$

from which we recognize that:

$$G_3 = \left[B_3^T A_4^T A_4 B_3 - B_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 B_3 + B_3^T B_3 \right]^{-1} \left(B_3^T A_4^T A_4 A_3 - B_3^T A_4^T B_4 (B_4^T B_4)^{-1} B_4^T A_4 A_3 + B_3^T A_3 \right) \quad (3.13)$$

The recursive pattern here can be established as follows. If we let:

$$w_i = \frac{1}{2} u_{i-1}^T C_i u_{i-1} + x_{i-1}^T D_i^T u_{i-1} + \frac{1}{2} x_{i-1}^T P_i x_{i-1} \quad (3.14)$$

(thus defining C_i , D_i , and P_i), we have:

$$u_i = -C_i^{-1} D_i^T x_i \quad (3.15)$$

(and thus $G_i = -C_i^{-1} D_i$),

Mere inspection/comparison of equations (3.13) and (3.9) reveal that the following recursive relations hold:

$$C_{i-1} = \left[B_{i-1}^T P_i B_{i-1} - B_{i-1}^T D_i^T C_i^{-1} D_i B_{i-1} + B_{i-1}^T B_{i-1} \right] \quad (3.16)$$

$$D_{i-1}^T = [A_{i-1}^T P_i B_{i-1} - A_{i-1}^T D_i^T C_i^{-1} D_i B_{i-1} + A_{i-1}^T B_{i-1}] \quad (3.17)$$

$$P_{i-1} = [A_{i-1}^T P_i A_{i-1} - A_{i-1}^T D_i^T C_i^{-1} D_i A_{i-1} + A_{i-1}^T A_{i-1}] \quad (3.18)$$

Thus, we can find G_{N-1} from G_N , and G_{N-2} from G_{N-1} , and proceed recursively all the way to G_1 .

(In general, we could have modified our cost function to include some weighting on the inputs by simply adding the weighting matrices to the C_i .)

4. Simulation Case Study

This feedback law was applied to the following problem of the Acrobot, illustrated below in Figure 2.

Figure 2. Acrobot Schematic

The Acrobot is a single input system. The first joint is not actuated at all. The initial state of the system is the "straight down" position (i.e., $\theta_1 = 0$, $\theta_2 = 0$) at rest, and the desired terminal state is the "straight up" position (i.e., $\theta_1 = \pi$, $\theta_2 = 0$), also at rest.

We will denote the masses and centroidal moments of inertia of the two links by m_1 , m_2 , I_1 , I_2 (which take into account the motor inertia at the second joint), their center of mass positions by l_{1c} , l_{2c} , which are measured from the joint of each respective link, the link lengths by l_1 , and l_2 , and the gravitational acceleration by g . The system has its link mass distributed such that $l_{2c} = 0$. The complete set of nominal parameter values are: $m_1 = 1.4$, $m_2 = 1.4$, $l_1 = 1$, $l_2 = 1$, $g = 9.81$, $I_1 = 1/6$, $I_2 = 1/2$, $l_{1c} = 1/2$, $l_{2c} = 0$ (all units SI). The motor torque $\bar{u}$ at the second link is restricted as

follows: $-\bar{u}_{\max} \leq \bar{u} \leq \bar{u}_{\max}$, where $\bar{u}_{\max} = 3$. The equations of motion can be found in [DeJong and Spong, 1994]. The open-loop off-line trajectory is illustrated below.

Figure 3. Torques (Nm) versus time (sec.), Off-Line Minimum Time Trajectory, Acrobot Problem

Figure 4. Joint angles (rad) versus time (sec.), Off-Line Minimum Time Trajectory, Acrobot Problem

The closed loop simulation performed was one in which the system was given an initial state error. In particular, the initial state was set to: $x_1 = (0.2 \text{ rad}, 0.2 \text{ rad}, 0 \text{ rad/s}, 0 \text{ rad/s})$. The associated open loop response to the initial error is shown immediately below.

Figure 5. Joint angles (rad) versus time (sec.), open-loop response to initial state error, Acrobot Problem, Desired Trajectory (--), Actual Trajectory (-)

In particular, the terminal state, which is desired to be: (3.14159 rad, 0 rad, 0 rad/s, 0 rad/s), was actually: (3.83 rad, -0.28 rad, 2.2 rad/s, -2.2 rad/s). This will contrast significantly with the closed loop performance.

We will now show the closed loop tracking performance in the presence of the same initial state error. In these simulations, a 1.32 kHz sampling rate was assumed for the digital controller, and we had one G_k for each sampling period. The cost function used for the ddp was $\frac{1}{2} \sum_k x_k^T x_k + \frac{1}{2} \sum_k u_k^T u_k$. Closed loop tracking plots are immediately below.

Figure 6. Joint angles (rad) versus time (sec.), Closed Loop Tracking Performance, with initial state error, Acrobot Problem, Desired Angles (--), Actual Angles (-)

Of special note is the terminal state that was achieved. The terminal state achieved by the closed loop controller was: (3.16 rad, -0.0022 rad, 0.10 rad/s, -0.0055 rad/s). These terminal state errors are the following percentage of the open loop terminal state errors: (4%, 0.8%, 5%, 0.3%). Thus, with feedback control, we have a 20-fold decrease in the terminal state error.

Of course, we are also very interested in the closed loop torque history. This is plotted immediately below.

Figure 7. Torques (Nm) Versus time (sec.) Closed Loop tracking, with an initial state error, Acrobot Problem

We see from Figure 7 that the closed loop torque does not exceed the off-line torque by too much. The maximum torque in Figure 7 is about 3.5Nm, which is 15% above the open loop torque. Thus, our actual torque bounds would have had to have been 3.5Nm for us to be able to achieve the tracking performance shown in Figure 6.

5. Conclusions

We have demonstrated the ability of the ddp feedback law to provide tracking performance for a minimum-time trajectory of the Acrobot that is at least 20 times better than the open-loop response with the same initial state error. Furthermore, the closed loop torques were not terribly above the open loop torques, exceeding them only by 15%.

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