UNIQUENESS RESULTS FOR THE INFINITE UNITARY, ORTHOGONAL AND ASSOCIATED GROUPS

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Let H be a separable infinite dimensional complex Hilbert space, let $U(H)$ be the Polish topological group of unitary operators on H, let G be a Polish topological group and $\varphi: G \rightarrow U(H)$ an algebraic isomorphism. Then φ is a topological isomorphism. The same theorem holds for the projective unitary group, for the group of *-automorphisms of $L(H)$ and for the complex isometry group. If H is a separable real Hilbert space with $\dim(H) \geq 3$, the theorem is also true for the orthogonal group $O(H)$, for the projective orthogonal group and for the real isometry group. The theorem fails for $U(H)$ if H is finite dimensional complex Hilbert space.

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CONTENTS

CHAPTER 1

INTRODUCTION

One of the general problems of topological algebra is to determine restrictions on the set of possible topological group topologies that are definable on a given abstract group G. This entails finding restrictions on the set of possible topologies on the abstract group G for which the group operations are continuous. There are many special known results related with this problem. Some of the most illustrious mathematicians of the twentieth century have been linked to this area. One of the first results belongs to Elie Cartan, who showed that if G is a compact semisimple Lie group, H is a Lie group and $\phi : G \to H$ is an abstract group homomorphism whose image is bounded, then ϕ is continuous [2]. Another important result is due to van der Waerden who proved that if a linear representation of a simple nonabelian compact Lie group is bounded around the identity, then it is continuous [27]. Hans Freudenthal proves a theorem similar to van der Waerden: he considers G to be a simple real Lie group (of dimension ≥ 3) which is absolutely simple, *i.e.* the complexification of its Lie algebra remains simple as a complex Lie algebra, and he shows that, under this assumption, any automorphism of G is continuous [4]. This result applies to $SL_2(\mathbb{R})$, but it is not true for $SL_2(\mathbb{C})$ as von Neumann noted that if ψ is a discontinuous automorphism of $\mathbb{C}$, the mapping

$$\tilde{\psi} : SL_2(\mathbb{C}) \to SL_2(\mathbb{C}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{pmatrix}$$

is not continuous. Furthermore, Borel and Tits extended the van der Waerden paper in a variety of ways to Lie groups over locally compact fields [1], [26]. Similar questions about metrizable topological groups arose naturally. One result is due to Robert Kallman, who answered a question posed by Ulam, Schreier and von Neumann. By combining ideas from algebra and descriptive set theory, he proved that if G is a complete separable metric group and if $\phi : G \to S_\infty$ is an algebraic isomorphism, then ϕ is a topological isomorphism [13]. This is perhaps a surprising result because, for example, it is false for the additive group $(\mathbb{R}, +)$. To see this, note that $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as vector spaces over $\mathbb{Q}$ and therefore are isomorphic as additive groups, but they are not homeomorphic in spite of the fact that both groups are Polish groups. Later, Kallman used similar methods to prove analogous theorems for large classes of groups, each of which requires unique special algebraic tricks: compact simple Lie groups [11]; compact connected metric groups with totally disconnected center [14]; the homeomorphism group of manifolds [17]; the diffeomorphism group of C^∞ manifolds [17]; the homeomorphism group of the Hilbert cube [17]; the homeomorphism group of pseudo-arc (unpublished); the p-adic integers [12]; the group of measure-preserving transforms of $[0, 1]$ [16]; the group of measurable, non-singular, invertible transforms of $[0, 1]$ (clarifying an example of Kakutani)(unpublished); semisimple Lie groups of second kind (unpublished); and the real $ax + b$ group [15].

The purpose of my dissertation is to add to this list by proving that $\mathcal{U}(\mathcal{H})$, the group of unitary operators acting on a separable infinite dimensional Hilbert space, admits a unique topology in which it is a complete separable metric group. The basic idea again is to combine algebraic techniques with descriptive set theoretical results and prove the following theorem "*Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then ϕ is a topological isomorphism*", Theorem 3.58. The same theorem holds for the projective unitary group $\mathcal{PU}(\mathcal{H})$ Theorem 4.18, for the group of *-automorphisms of $\mathcal{L}(\mathcal{H})$ Corollary 5.37 and for the complex isometry group Theorem 8.10. If $\mathcal{H}$ is a separable real Hilbert space with $\dim(\mathcal{H}) \geq 3$, the theorem is also true for the orthogonal group $\mathcal{O}(\mathcal{H})$ Theorem 6.40, for the projective orthogonal group $\mathcal{PO}(\mathcal{H})$ Theorem 7.13 and for the real isometry group Theorem 8.13. It is surprising that the theorem fails for $\mathcal{U}(n)$ if $\mathcal{H}$ is $n-$dimensional complex Hilbert space Corollary 3.64.

Some of the theorems and propositions in this project do not represent original work. They are reproduced here for the convenience of the reader, sometimes with slightly different than the original proofs. If a theorem is a well known result, the name of the author is listed, if it is just a general fact there is no name associated with it. Recommended references for the general facts are [20], [21], [25] and [18]. All of the original theorems are marked with $\bigstar$.

CHAPTER 2

BASICS OF HILBERT SPACES

2.1. Inner Products

DEFINITION 2.1. Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$. A norm on V is a function $p : V \to \mathbb{R}$ satisfying, for every $x, y \in V$ and every $a \in \mathbb{R}$ or $a \in \mathbb{C}$ the following:

1) $p(x + y) \leq p(x) + p(y)$;

2) $p(ax) = |a|p(x)$;

3) $p(x) > 0$ whenever $x \neq 0$.

The function p is usually denoted $\|\cdot\|$.

DEFINITION 2.2. A normed space is a pair $(V, \|\cdot\|)$, where V is a vector space over $\mathbb{C}$ or $\mathbb{R}$ and $\|\cdot\|$ is a norm on V.

DEFINITION 2.3. If $(V, \|\cdot\|)$ is a normed space, the closed unit ball is the set $\{x \in V \mid \|x\| \leq 1\}$ and is denoted by V_1.

DEFINITION 2.4. A bilinear functional on a complex vector space V is a complex-valued function ϕ on $V \times V$ such that $\phi(x, y)$ is linear in the first argument and it is complex conjugate linear in the second argument. A bilinear functional ϕ is positive if $\phi(x, x) \geq 0$ for every $x \in V$, and it is strictly positive if $\phi(x, x) > 0$, whenever $x \neq 0$. A bilinear functional ϕ is conjugate-symmetric if $\phi(x, y) = \overline{\phi(y, x)}$ for every $x, y \in V$. The quadratic form $\hat{\phi}$ induced by a bilinear functional ϕ on a complex vector space is the real-valued function defined for each $x \in V$ by $\hat{\phi}(x) = \phi(x, x)$.

A real bilinear functional on a real vector space is a real valued function defined in a similar way, except that the values $\phi(x, y)$ are required to be real and the conjugation no longer appear.

Definition 2.5. An inner product on a complex vector space V is a strictly positive, conjugate-symmetric, bilinear functional on V. An inner product space is a complex vector space V and a choice of inner product on V. The quadratic form $\langle x, x\rangle$ induced by the inner product is denoted by $\|x\|^2$. The positive square root $\|x\|$ of $\|x\|^2$ is a norm, called the norm of x.

A real inner product space is a real complex vector space and a strictly positive, symmetric, real bilinear functional on it.

Definition 2.6. We say that a bilinear functional ϕ is bounded if there is a real number c such that $|\phi(x, y)| \leq c\|x\|\|y\|$. When this is so, we denote by $\|\phi\|$ the least possible value of c, which is given by

$$\|\phi\| = \sup\{|\phi(x, y)| \mid \|x\| \leq 1, \|y\| \leq 1\}$$

Proposition 2.7 (Parallelogram Law). *If V is a complex or a real inner product space, then*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x^2\| + \|y\|^2)$$

for every $x, y \in V$.

Proof. $\|x+y\|^2 + \|x-y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle = 2\langle x, x\rangle + 2\langle y, y\rangle = 2(\|x^2\| + \|y\|^2)$. $\square$

Proposition 2.8 (Polarization identity). *If $\hat{\phi}$ is the quadratic form induced by a bilinear functional ϕ on a complex vector space V, then*

$$\phi(x, y) = \hat{\phi}(\frac{1}{2}(x + y)) - \hat{\phi}(\frac{1}{2}(x - y)) + i\hat{\phi}(\frac{1}{2}(x + iy)) - i\hat{\phi}(\frac{1}{2}(x - iy))$$

for every $x, y \in V$.

Proof. $\hat{\phi}(\frac{1}{2}(x + y)) - \hat{\phi}(\frac{1}{2}(x - y)) + i\hat{\phi}(\frac{1}{2}(x + iy)) - i\hat{\phi}(\frac{1}{2}(x - iy)) = \phi(\frac{1}{2}(x + y), \frac{1}{2}(x + y)) - \phi(\frac{1}{2}(x-y), \frac{1}{2}(x-y)) + i\phi(\frac{1}{2}(x+iy), \frac{1}{2}(x+iy)) - i\phi(\frac{1}{2}(x-iy), \frac{1}{2}(x-iy)) = \frac{1}{4}\phi(x, x) + \frac{1}{4}\phi(x, y) + \frac{1}{4}\phi(y, x) + \frac{1}{4}\phi(y, y) - \frac{1}{4}\phi(x, x) + \frac{1}{4}\phi(x, y) + \frac{1}{4}\phi(y, x) - \frac{1}{4}\phi(y, y) + \frac{1}{4}i\phi(x, x) + \frac{1}{4}\phi(x, y) - \frac{1}{4}\phi(y, x) + \frac{1}{4}i\phi(y, y) - \frac{1}{4}i\phi(x, x) + \frac{1}{4}\phi(x, y) - \frac{1}{4}\phi(y, x) - \frac{1}{4}i\phi(y, y) = \phi(x, y)$ $\square$

DEFINITION 2.9. Let V be a vector space over $\mathbb{C}$. Define the distance between two vectors x and y to be $\|x - y\|$. Then V is a metric space with respect to this distance function. A Hilbert space is an inner product space which, as a metric space, is complete. A Hilbert space is usually denoted by $\mathcal{H}$.

If V is a vector space over $\mathbb{R}$, a real Hilbert space is defined in a similar way. As regards elementary geometrical properties of Hilbert spaces, there is a little difference between the real and the complex cases. In the main we shall restrict attention to the complex case, making occasional comments on the modifications needed to deal with real spaces.

DEFINITION 2.10. A Hilbert space $\mathcal{H}$ is separable if there is $D \subset \mathcal{H}$ a countable dense subset. Throughout the Hilbert space $\mathcal{H}$ will be assumed to be separable.

DEFINITION 2.11. We define two topologies on a Hilbert space $\mathcal{H}$. The first topology is compatible with the metric induced by the norm and is called the strong topology. A base of neighborhoods for the strong topology at the point x_0 is the collection of all sets of the form

$$\{x \mid \|x - x_0\| < \epsilon\}$$

where $\epsilon > 0$. We say that the net x_j converges strongly to x if $\|x_j - x\| \to 0$ and we denote this by $x_j \xrightarrow{s} x$.

Another topology on a Hilbert space is called the weak topology. A base of neighborhoods for the weak topology at the point x_0 is the collection of all sets of the form

$$\{x \mid |\langle x - x_0, y_i \rangle| < \epsilon,\ 1 \leq i \leq k\}$$

where $y_1, y_2, ..., y_k \in \mathcal{H}$ and $\epsilon > 0$. We say that the net x_j converges weakly to x if $\langle x_j - x, y \rangle \to 0$ for every $y \in \mathcal{H}$ and we denote this by $x_j \xrightarrow{w} x$.

2.2. Linear Operators

DEFINITION 2.12. An operator is a linear transformation from $\mathcal{H}$ into $\mathcal{H}$. We say that the operator $T : \mathcal{H} \to \mathcal{H}$ is bounded if there exists $C \in \mathbb{R}$ such that $\|Tx\| \leq C\|x\|$ for every

$x \in \mathcal{H}$. The least such constant C is the norm of T. The collection of all bounded operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$.

LEMMA 2.13. *If $T \in \mathcal{L}(\mathcal{H})$, then $\|T\| = \sup\{|\langle Tx, y\rangle| \mid \|x\| \leq 1,\ \|y\| \leq 1\}$.*

Proof. If $\|Tx\| \neq 0$, let $y = \frac{Tx}{\|Tx\|}$. Then $\|y\| = 1$ and $\sup\{|\langle Tx, y\rangle| \mid \|x\| \leq 1,\ \|y\| \leq 1\} = \sup\left\{|\langle Tx, \frac{Tx}{\|Tx\|}\rangle| \mid \|x\| \leq 1\right\} = \sup\left\{\frac{\|Tx\|^2}{\|Tx\|} \mid \|x\| \leq 1\right\} = \sup\{\|Tx\| \mid \|x\| \leq 1\} = \sup\left\{\frac{\|Tx\|}{\|x\|} \mid x \neq 0\right\} = \inf\left\{C \mid \frac{\|Tx\|}{\|x\|} \leq C\right\} = \inf\{C \mid \|Tx\| \leq C\|x\|\} = \|T\|$ □

THEOREM 2.14 (Riesz's Representation Theorem). *If $\mathcal{H}$ is a Hilbert space and $y \in \mathcal{H}$, the equation $\phi_y(x) = \langle x, y\rangle$ defines a continuous linear functional ϕ_y on $\mathcal{H}$, and $\|\phi_y\| = \|y\|$. Each continuous linear functional on $\mathcal{H}$ arises in this way from a unique element y of $\mathcal{H}$.*

Proof. Since the inner product is linear in the first argument, it is clear that ϕ_y is linear. For every $y \in \mathcal{H}$ we have that $|\phi_y(x)| = |\langle x, y\rangle| \leq \|x\|\,\|y\|$ for every $x \in \mathcal{H} \Rightarrow \phi_y$ is bounded and hence continuous. If $x = y$ we have that $|\phi_y(x)| = \|x\|\,\|y\| \Rightarrow \|\phi_y\| = \|y\|$.

If $\phi \neq 0$ is a continuous linear functional on $\mathcal{H}$, let $Y = \phi^{-1}(0)$. Then, since $\phi \neq 0$ we have that $Y \neq \mathcal{H} \Rightarrow Y^\perp \neq \{0\}$. Let $u \in Y^\perp$ be such that $\|u\| = 1$. Note that $\phi(\phi(u)x - \phi(x)u) = \phi(u)\phi(x) - \phi(x)\phi(u) = 0$ for every $x \in \mathcal{H} \Rightarrow \phi(u)x - \phi(x)u \in Y$ and, since $u \in Y^\perp$ we have that $0 = \langle \phi(u)x - \phi(x)u, u\rangle = \phi(u)\langle x, u\rangle - \phi(x) \Rightarrow \phi(x) = \phi(u)\langle x, u\rangle = \langle x, \overline{\phi(u)}u\rangle$. Let $y = \overline{\phi(u)}u$. Then $\phi(x) = \phi_y(x) = \langle x, y\rangle$ for every $x \in \mathcal{H}$. If $\phi = 0$ then it is clear that $0 = \phi(x) = \phi_0(x) = \langle x, 0\rangle$ for every $x \in \mathcal{H}$. If also $\phi = \phi_z$, with $z \in \mathcal{H}$ then $\|y - z\| = \|\phi_{y-z}\| = \|\phi_y - \phi_z\| = \|\phi - \phi\| = 0 \Rightarrow y = z \Rightarrow$ the representation of ϕ is unique. □

THEOREM 2.15 (Banach-Alaoglu). *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ or $\mathbb{R}$. The weak topology on $\mathcal{H}_1 = \{x \in \mathcal{H} \mid \|x\| \leq 1\}$, the unit ball of $\mathcal{H}$, is compact Hausdorff.*

Proof. Here is the proof for the complex case only. The real case is similar.

For every $x \in \mathcal{H}$, let $D_x = \{z \in \mathbb{C} \mid |z| \leq \|x\|\}$ be the closed disc in $\mathbb{C}$. Let $D = \prod_{x \in \mathcal{H}} D_x$ equipped with the product topology. By Tychonoff's Theorem D is compact. For every $x \in$

$\mathcal{H}_1$ let $\delta(x) = \prod_{y\in\mathcal{H}}\langle x, y\rangle$. Since for every $x \in \mathcal{H}_1$ and every $y \in \mathcal{H}$, $|\langle x, y\rangle| \leq \|x\|\,\|y\| = \|y\|$, we have that $\delta(x) \in D$ and hence δ is a mapping from $\mathcal{H}_1$ into D.

If $x_1, x_2 \in \mathcal{H}_1$ such that $\delta(x_1) = \delta(x_2)$, then$\langle x_1, y\rangle = \langle x_2, y\rangle$ for every $y \in \mathcal{H} \Rightarrow x_1 = x_2 \Rightarrow \delta$ is one-to-one. If $x_j, x \subset \mathcal{H}_1$, then $x_j \xrightarrow{w} x \Leftrightarrow \langle x_j, y\rangle \to \langle x, y\rangle$ for every $y \in \mathcal{H} \Leftrightarrow \delta(x_j) \to \delta(x)$. Hence δ is an embedding of $\mathcal{H}_1$ with the weak topology into D with the product topology.

Let $x_1 \neq x_2 \in \mathcal{H}_1$. Then there exists $y_0 \in \mathcal{H}$ such that $\langle x_1, y_0\rangle \neq \langle x_2, y_0\rangle \in D_{y_0} \Rightarrow$ there exist $U_1, U_2 \subset D_{y_0}$ open, disjoint such that $\langle x_1, y_0\rangle \in U_1 \subset D_{y_0}$ and $\langle x_2, y_0\rangle \in U_2 \subset D_{y_0}$. Then $\delta^{-1}(U_1 \times \prod_{y\neq y_0} D_y)$ and $\delta^{-1}(U_2 \times \prod_{y\neq y_0} D_y)$ are disjoint weakly open sets and separate x_1 and x_2. Hence the weak topology on $\mathcal{H}_1$ is Hausdorff. We will show compactness by showing that the range of δ is closed in D, which can be viewed as the set of all complex valued functions acting on $\mathcal{H}$.

Let $f \in \mathrm{cl}_D(\delta(\mathcal{H}_1))$. Then $f : \mathcal{H} \to \mathbb{C}$ and there exists $x_j \subset \mathcal{H}_1$ such that $\delta(x_j) \to f$, which is that $\langle x_j, y\rangle \to f(y)$ for every $y \in \mathcal{H}$. Since $|\langle x_j, y\rangle| \leq \|y\|$, we have that $|f(y)| \leq \|y\|$ for every $y \in \mathcal{H} \Rightarrow \|f\| \leq 1$.

Let $\epsilon > 0$, $x_1, x_2 \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and let $x_3 = \alpha x_1 + \beta x_2$. Let $U = \{g \in D \mid |g(x_1) - f(x_1)| < \epsilon\ ,\ |g(x_2) - f(x_2)| < \epsilon\ ,\ |g(x_3) - f(x_3)| < \epsilon\}$. Then $U \subset D$ is open and contains $f \ \Rightarrow \delta(\mathcal{H}_1) \cap U \neq \emptyset \Rightarrow$ there exists $x_0 \in \mathcal{H}_1$ such that $|\langle x_0, x_1\rangle - f(x_1)| < \epsilon\ ,\ |\langle x_0, x_2\rangle - f(x_2)| < \epsilon\ ,\ |\langle x_0, x_3\rangle - f(x_3)| < \epsilon$. Then

$$|f(x_3) - \alpha f(x_1) - \overline{\beta} f(x_2)| =$$

$$|f(x_3) - \langle x_0, x_3\rangle + \langle x_0, x_3\rangle - \alpha f(x_1) - \overline{\beta} f(x_2)| =$$

$$|f(x_3) - \langle x_0, x_3\rangle + \alpha\langle x_0, x_1\rangle + \overline{\beta}\langle x_0, x_2\rangle - \alpha f(x_1) - \overline{\beta} f(x_2)| \leq$$

$$|f(x_3) - \langle x_0, x_3\rangle| + \alpha|f(x_1) - \langle x_0, x_1\rangle| + \overline{\beta}|f(x_2) - \langle x_0, x_2\rangle| <$$

$$\epsilon + \alpha\epsilon + \overline{\beta}\epsilon = \epsilon(1 + \alpha + \overline{\beta})$$

Since this is true for every ϵ, we have that f is linear. By Riesz's Representation Theorem we have that there exists $x \in \mathcal{H}_1$ such that $f(y) = \langle x, y\rangle$ for every $y \in \mathcal{H} \Rightarrow f \in \delta(\mathcal{H}_1) \Rightarrow \delta(\mathcal{H}_1)$ is closed in $D \Rightarrow \mathcal{H}_1$ is weakly compact. $\square$

THEOREM 2.16. *If $\mathcal{H}$ is separable, the weak topology on $\mathcal{H}_1$ is compact and metrizable. In this case, a metric compatible with the weak topology on $\mathcal{H}_1$ is*

$$d(x,y) = \sum_{l\geq 1} \frac{1}{2^l} |\langle x - y, e_l\rangle|$$

where $\{e_1, e_2, ..., e_l, ...\}$ is an orthonormal basis for $\mathcal{H}$.

Proof. We have shown in the Theorem 2.15 that the unit ball is compact. To show that the metric just defined is compatible with the weak topology, we have to show that if $(x_j) \subset \mathcal{H}_1$ is a net and $x \in \mathcal{H}_1$, then $x_j \xrightarrow{w} x \Leftrightarrow d(x_j, x) \to 0$.

If $(x_j) \subset \mathcal{H}_1$ is a net, $x \in \mathcal{H}_1$ and $x_j \xrightarrow{w} x$, then $\langle x_j - x, e_l\rangle \to 0$ for every $l \geq 1$. Let $\epsilon > 0$. Choose L so that $2^{L-1} > \frac{2}{\epsilon}$. Then $\frac{\epsilon}{2} > \frac{1}{2^{L-1}} = \frac{1}{2^{L-1}}(\sum_{l\geq 1} \frac{1}{2^l}) = \sum_{l\geq 1} \frac{1}{2^{L-1+l}} = \sum_{l>L} \frac{1}{2^{l-1}} \geq \sum_{l>L} \frac{1}{2^l} \|x_j - x\|\ \|e_l\| \geq \sum_{l>L} \frac{1}{2^l} |\langle x_j - x, e_l\rangle|$ for every j. For every $1 \leq l \leq L$ there is an J_l such that $\frac{1}{2^l}|\langle x_j - x, e_l\rangle| < \frac{\epsilon}{2L}$ for every $j \geq J_l$. Let $J \geq \{J_l \mid 1 \leq l \leq L\}$. Then $\sum_{1\leq l\leq L} \frac{1}{2^l}|\langle x_j - x, e_l\rangle| < \frac{\epsilon}{2}$ for every $j \geq J$. Hence, if $j \geq J$, then $\sum_{l\geq 1} \frac{1}{2^l}|\langle x_j - x, e_l\rangle| < \epsilon \Rightarrow d(x_j, x) \to 0$.

If $(x_j) \subset \mathcal{H}_1$ is a net, $x \in \mathcal{H}_1$ and $d(x_j, x) \to 0$, then $\sum_{l\geq 1} \frac{1}{2^l}|\langle x_j - x, e_l\rangle| \to 0$. This implies that $|\langle x_j - x, e_l\rangle| \to 0$ for every $l \geq 1 \Rightarrow |\langle x_j - x, v\rangle| \to 0$ for every $v = \sum_{l=1}^k a_l e_l$.

Let $\epsilon > 0$, and $y \in \mathcal{H}_1$. Choose $v = \sum_{l=1}^k a_l e_l$ be such that $\|y - v\| < \frac{\epsilon}{4}$. This can be done since finite linear combinations of e_l are dense. Then $|\langle x_j - x, y - v\rangle| \leq |\langle x_j, y - v\rangle| + |\langle x, y - v\rangle| \leq \|x_j\|\ \|y - v\| + \|x\|\ \|y - v\| \leq 2\|y - v\| < \frac{\epsilon}{2}$. Since $|\langle x_j - x, v\rangle| \to 0$ for every $v = \sum_{l=1}^k a_l e_l$, choose J such that $|\langle x_j - x, v\rangle| < \frac{\epsilon}{2}$ for every $j \geq J$. This implies that $|\langle x_j - x, y\rangle| \leq |\langle x_j - x, y - v\rangle| + |\langle x_j - x, v\rangle| < \epsilon$ for every $j \geq J \Rightarrow x_j \xrightarrow{w} x$. $\square$

THEOREM 2.17. *If $T \in \mathcal{L}(\mathcal{H})$ then the equation $b_T(x,y) = \langle Tx, y\rangle$ defines a bounded bilinear functional on $\mathcal{H} \times \mathcal{H}$ and $\|b_T\| = \|T\|$. Each bounded bilinear functional on $\mathcal{H} \times \mathcal{H}$ arises in this way from a unique element of $\mathcal{L}(\mathcal{H})$.*

Proof. Given $T \in \mathcal{L}(\mathcal{H})$ it is clear that b_T is a bilinear form on $\mathcal{H} \times \mathcal{H}$. Since $|b_T(x,y)| = |\langle Tx, y\rangle| \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\|$ we have that b_T is bounded and $\|b_T\| \leq \|T\|$. Since $\|Tx\|^2 = \langle Tx, Tx\rangle = b_T(x, Tx) \leq \|b_T\|\|x\|\|Tx\|$ we have that $\|Tx\| \leq \|b_T\|\|x\| \Rightarrow \|T\| \leq \|b_T\|$ and hence $\|T\| = \|b_T\|$.

Let $b : \mathcal{H} \times \mathcal{H}$ be a bounded bilinear form. For every $x \in \mathcal{H}$ let $(Rx)(y) = \overline{b(x,y)}$. Rx is a linear functional on $\mathcal{H}$ and, since $|(Rx)(y)| \leq \|b\|\|x\|\|y\|$, Rx is bounded and $\|Rx\| \leq \|b\|\|x\|$. Since b is linear in the first variable, the mapping R, $R(x) = Rx$ from $\mathcal{H}$ into the dual space of $\mathcal{H}$ is bounded, conjugate-linear. For every $y \in \mathcal{H}$ let $(Sy)(x) = \langle x, y\rangle$. It is clear that Sy is linear. Since $|(Sy)(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|$ with equality if $x = y \Rightarrow \|Sy\| = \|y\|$. Thus, S is a norm-preserving conjugate-linear from $\mathcal{H}$ into the dual of $\mathcal{H}$. Let $T = S^{-1}R$. Then $T : \mathcal{H} \to \mathcal{H}$ is linear and T is bounded by $\|b\|$. Moreover, $b_T(x,y) = \langle Tx, y\rangle = \langle S^{-1}Rx, y\rangle = \overline{\langle y, S^{-1}Rx\rangle} = \overline{(Rx)(y)} = b(x,y)$.

If also $b_U = b$ for some $U \in \mathcal{L}(\mathcal{H})$ then $\|T - U\| = \|b_{T-U}\| = \|b_T - b_U\| = \|b - b\| = 0$ and hence $T = U$. $\square$

PROPOSITION 2.18. *If $T \in \mathcal{L}(\mathcal{H})$ then*

$$4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle$$

for every $x, y \in \mathcal{H}$.

Proof. If $\phi(x,y) = \langle Tx, y\rangle$, then ϕ is a bilinear form on $\mathcal{H}$. It follows from Proposition 2.8 that $\langle Tx, y\rangle = \langle T(\frac{1}{2}(x+y)), \frac{1}{2}(x+y)\rangle - \langle T(\frac{1}{2}(x-y)), \frac{1}{2}(x-y)\rangle + i\langle T(\frac{1}{2}(x+iy)), \frac{1}{2}(x+iy)\rangle - i\langle T(\frac{1}{2}(x-iy)), \frac{1}{2}(x-iy)\rangle = \frac{1}{4}\langle T(x+y), x+y\rangle - \frac{1}{4}\langle T(x-y), x-y\rangle + \frac{1}{4}i\langle T(x+iy), x+iy\rangle - \frac{1}{4}i\langle T(x-iy), x-iy\rangle$ $\square$

PROPOSITION 2.19. *If S and T are bounded linear operators on a Hilbert space $\mathcal{H}$ and if $\langle Tx, x\rangle = \langle Sx, x\rangle$ for every $x \in \mathcal{H}$, then $S = T$.*

Proof. If $x, y \in \mathcal{H}$, using Proposition 2.18 we have that

$$4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle =$$

$$\langle S(x+y), x+y\rangle - \langle S(x-y), x-y\rangle + i\langle S(x+iy), x+iy\rangle - i\langle S(x-iy), x-iy\rangle = 4\langle Sx, y\rangle$$

$\Rightarrow S = T$. $\square$

THEOREM 2.20. *If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ then there is a unique element $T^* \in \mathcal{L}(\mathcal{H})$ such that*

$$\langle T^* x, y\rangle = \langle x, Ty\rangle$$

for every $x, y \in \mathcal{H}$. Moreover,

1) $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$

2) $(TS)^* = S^*T^*$

3) $(T^*)^* = T$

4) $\|T^*T\| = \|T\|^2$

5) $\|T^*\| = \|T\|$

for every $S, T \in \mathcal{L}(\mathcal{H})$ and every $a, b \in \mathbb{C}$.

Proof. The equation $b(x, Ty) = \langle x, Ty\rangle$ defines a bilinear functional b on $\mathcal{H} \times \mathcal{H}$. Since $|b(x, y)| = |\langle x, Ty\rangle| = |\langle Ty, x\rangle| = |b_T(y, x)|$, where b_T is the bilinear functional defined in Theorem 2.17, we have that b is bounded. By the same theorem that there exists a unique element $T^* \in \mathcal{L}(\mathcal{H})$ such that $\langle T^*x, y\rangle = b(x, y) = \langle x, Ty\rangle$ for every $x, y \in \mathcal{H}$ and $\|T^*\| = \|b\| = \|T\|$, which proves 5). If $x \in \mathcal{H}$ then $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\|\|x\|^2 \Rightarrow \|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$ and 4) follows.

Since $\langle(\overline{a}S^* + \overline{b}T^*)x, y\rangle = \overline{a}\langle S^*x, y\rangle + \overline{b}\langle T^*x, y\rangle = \overline{a}\langle x, Sy\rangle + \overline{b}\langle x, Ty\rangle = \langle x, (aS + bT)y\rangle = \langle(aS + bT)^*x, y\rangle$ for every $x, y \in \mathcal{H}$, we have that $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$.

Since $\langle S^*T^*x, y\rangle = \langle T^*x, Sy\rangle = \langle x, TSy\rangle = \langle(TS)^*x, y\rangle$ for every $x, y \in \mathcal{H}$, we have that $(TS)^* = S^*T^*$. Finally, since $\langle Ty, x\rangle = \overline{\langle x, Ty\rangle} = \overline{\langle T^*x, y\rangle} = \langle y, T^*x\rangle = \langle(T^*)^*y, x\rangle$ for every $x, y \in \mathcal{H}$, we have that $(T^*)^* = T$ and the theorem is proved. $\square$

DEFINITION 2.21. A bounded linear operator $T \in \mathcal{L}(\mathcal{H})$ is said to be self-adjoint or Hermitian if $T^* = T$.

Definition 2.22. The strong operator topology and the weak operator topology are topologies on the space of bounded linear operators on a Hilbert space. In the strong operator topology, an element T_0 has a base of neighborhoods consisting of all sets of the form

$$\{T \in \mathcal{L}(\mathcal{H}) \mid \|(T - T_0)x_i\| < \epsilon,\ 1 \leq i \leq k\}$$

where $x_1, x_2, ..., x_k \in \mathcal{H}$ and $\epsilon > 0$. We say that the net T_j converges to T in the strong operator topology if $\|(T_j - T)x\| \to 0$ for every $x \in \mathcal{H}$ and we denote this by $T_j \xrightarrow{so} T$.

A basic neighborhood at T_0 in the weak operator topology is the collection of all sets of the form

$$\{T \in \mathcal{L}(\mathcal{H}) \mid |\langle (T - T_0)x_i, y_i\rangle| < \epsilon,\ 1 \leq i \leq k\}$$

where $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k \in \mathcal{H}$ and $\epsilon > 0$. We say that the net T_j converges weakly to T in the weak operator topology if $\langle (T_j - T)x, y\rangle \to 0$ for every $x, y \in \mathcal{H}$ and we denote this by $T_j \xrightarrow{wo} T$.

Definition 2.23. Let X be a topological space. The set of all functions $f : X \to X$ such that f is bijective and f and f^{-1} are continuous is denoted $\mathcal{H}om(X)$. $\mathcal{H}om(X)$ together with the composition of functions is a group, called the homeomorphism group of X.

Theorem 2.24. *Let X be a separable compact metric space and let $\mathcal{H}om(X)$ be the homeomorphism group of X. Then $\mathcal{H}om(X)$ can be given a separable complete metric group topology. The metric compatible with this group topology is given by*

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x)) + \sup_{x \in X} d(f^{-1}(x), g^{-1}(x))$$

for every $f, g \in \mathcal{H}om(X)$, where d is the metric on X.

A condensed sketch of this proof is in [17]. □

Corollary 2.25. *Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$ or $\mathbb{R}$, $\mathcal{H}_1$ the unit ball and $\mathcal{H}om(\mathcal{H}_1)$ the homeomorphism group of the unit ball. Then*

$$\rho(f, g) = \sup_{x \in \mathcal{H}_1} d(f(x), g(x)) + \sup_{x \in \mathcal{H}_1} d(f^{-1}(x), g^{-1}(x))$$

where d is the metric on $\mathcal{H}_1$, defines a complete separable metric on $\mathcal{H}om(\mathcal{H}_1)$. $\mathcal{H}om(\mathcal{H}_1)$ is a topological group with respect to the corresponding topology. If $f_j \to f$ with respect to this topology, we will use the notation $f_j \xrightarrow{\rho} f$.

Proof. If $\mathcal{H}$ is separable, $\mathcal{H}_1$ is a separable compact metric space by Theorem 2.16. The conclusion follows from the Theorem 2.24. $\square$

2.3. Projections

Definition 2.26. An orthogonal projection on a subspace $\mathcal{M} \subset \mathcal{H}$ is the transformation $P : \mathcal{H} \to \mathcal{M}$ defined, for every $z = x + y \in \mathcal{H}$, with $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$, by $P(z) = x$.

Proposition 2.27. *The orthogonal projection P on a subspace $\mathcal{M}$ is an idempotent and Hermitian operator. If $\mathcal{M} \neq \mathcal{O}$, then $\|P\| = 1$. Conversely, if P is an idempotent Hermitian operator and if $\mathcal{M} = \{x \in \mathcal{H} \mid P(x) = x\}$, then P is the orthogonal projection on $\mathcal{M}$.*

Proof. It is clear that P is linear. If $z = x + y$ with $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$, then $\|P(z)\|^2 = \|x\|^2 \leq \|x\|^2 + \|y\|^2 = \|z\|^2$, and hence P is bounded and $\|P\| \leq 1$. Since $P^2(z) = P(x) = x = P(z)$, we have that P is idempotent. If $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$, where $x_1, x_2 \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{M}^\perp$, then $\langle P(z_1), z_2\rangle = \langle x_1, x_2 + y_2\rangle = \langle x_1, x_2\rangle + \langle x_1, y_2\rangle = \langle x_1, x_2\rangle = \langle x_1, x_2\rangle + \langle y_1, x_2\rangle = \langle x_1, P(z_2)\rangle + \langle y_1, P(z_2)\rangle = \langle z_1, P(z_2)\rangle$, and hence P is Hermitian. Also, if $\mathcal{M} \neq \mathcal{O}$, then $P(x) = x$ implies that $\|P\| = 1$.

Conversely, let P be an idempotent Hermitian operator, $\mathcal{M} = \{x \in \mathcal{H} \mid P(x) = x\}$ and let $z \in \mathcal{H}$. Since P is idempotent, $P(P(z)) = P(z)$ and hence $P(z) \in \mathcal{M}$. Since P is Hermitian, $\langle x, z - P(z)\rangle = \langle x, z\rangle - \langle x, P(z)\rangle = \langle x, z\rangle - \langle P(x), z\rangle = \langle x, z\rangle - \langle x, z\rangle = 0$ for every $x \in \mathcal{M}$, and hence $z - P(z) \in \mathcal{M}^\perp$. Since $z = P(z) + (z - P(z))$, the conclusion follows. $\square$

Definition 2.28. A partial isometry is an operator on a Hilbert space that is an isometry on the orthogonal complement of its kernel.

Proposition 2.29. *An operator U on a Hilbert space $\mathcal{H}$ is a partial isometry if and only if U^*U is an orthogonal projection.*

Proof. Let U be a partial isometry and P be the orthogonal projection on $\mathrm{Ker}(U)^\perp$. If $x \in \mathrm{Ker}(U)^\perp$, then $\langle U^*Ux, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2 = \|x\|^2 = \langle x, x\rangle$. Hence, if $z \in \mathcal{H}$ and $z = x + y$, where $x \in \mathrm{Ker}(U)^\perp$ and $y \in \mathrm{Ker}(U)$, then $\langle U^*Uz, z\rangle = \langle U^*Ux, x\rangle + \langle U^*Ux, y\rangle + \langle U^*Uy, x\rangle + \langle U^*Uy, y\rangle = \langle x, x\rangle = \langle x, x\rangle + \langle x, y\rangle = \langle x, z\rangle = \langle Pz, z\rangle$. By Proposition 2.19, $U^*U = P$ is the orthogonal projection on $\mathrm{Ker}(U)^\perp$.

Let U^*U be the orthogonal projection on $\mathcal{M}$. We will first show that $\mathcal{M} = \mathrm{Ker}(U)^\perp$. Let $x \in \mathcal{M}$ and $y \in \mathrm{Ker}(U)$. Then $\langle x, y\rangle = \langle U^*Ux, y\rangle = \langle Ux, Uy\rangle = \langle Ux, 0\rangle = 0$, and hence $\mathcal{M} \subset \mathrm{Ker}(U)^\perp$. Let $y \in \mathcal{M}^\perp$. Then $\|Uy\|^2 = \langle Uy, Uy\rangle = \langle U^*Uy, y\rangle = \langle 0, y\rangle = 0$. This implies that $y \in \mathrm{Ker}(U)$ and hence $\mathcal{M}^\perp \subset \mathrm{Ker}(U) \Rightarrow \mathrm{Ker}(U)^\perp \subset \mathcal{M}$.

It remains to show that U is an isometry on the orthogonal complement of its kernel. To this end, let $x \in \mathrm{Ker}(U)^\perp = \mathcal{M}$. Then, $\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle x, x\rangle = \|x\|^2$. $\square$

Lemma 2.30. *If P is the orthogonal projection on the subspace $\mathcal{M}$ and x is a vector such that $\|Px\| = \|x\|$, then $x \in \mathcal{M}$.*

Proof. Let x be any vector. Then $Px \in \mathcal{M}$ and, since $\langle x - Px, y\rangle = \langle x, y\rangle - \langle Px, y\rangle = \langle x, y\rangle - \langle x, Py\rangle = 0$ for every $y \in \mathcal{M}$, $x - Px \in \mathcal{M}^\perp$. Since $x = Px + (x - Px)$, we have that $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$ and, since $\|x\| = \|Px\|$, that $\|x - Px\| = 0$. Hence, $Px = x \Rightarrow x \in \mathcal{M}$. $\square$

Proposition 2.31. *Let P and Q be two orthogonal projections on subspaces $\mathcal{M}$ and $\mathcal{N}$ respectively. Then the following relations are equivalent.*

1) $P \leq Q$;

2) $\|Px\| \leq \|Qx\|$ *for every* x;

3) $\mathcal{M} \subset \mathcal{N}$;

4) $QP = P$;

5) $PQ = P$.

Proof. If $P \leq Q$, then $\|Px\|^2 = \langle Px, Px\rangle = \langle Px, P^*x\rangle = \langle P^2x, x\rangle = \langle Px, x\rangle \leq \langle Qx, x\rangle = \|Qx\|^2$ for every x.

If $\|Px\| \leq \|Qx\|$ for all x, let $x \in \mathcal{M}$, $x = y + z$, where $y \in \mathcal{N}$ and $z \in \mathcal{N}^{\perp}$. Then $\|x\|^2 = \|Px\|^2 \leq \|Qx\|^2 = \|y\|^2 \leq \|y\|^2 + \|z\|^2 = \|x\|^2 \Rightarrow \|x\| = \|Qx\|$. By Lemma 2.30 we have that $x \in \mathcal{N}$ and hence $\mathcal{M} \subset \mathcal{N}$.

If $\mathcal{M} \subset \mathcal{N}$, then $Px \in \mathcal{M} \subset \mathcal{N}$ for every x, and hence $QPx = Px$ for every x.

If $QP = P$, then $PQ = P^*Q^* = (QP)^* = P^* = P$.

If $PQ = P$, then $\langle Px, x\rangle = \|Px\|^2 = \|PQx\|^2 \leq \|Qx\|^2 = \langle Qx, x\rangle$ for every x, and therefore $P \leq Q$. $\square$

Proposition 2.32. *If P_1 and P_2 are two orthogonal projections on a Hilbert space $\mathcal{H}$, then $P_1 \geq P_2$ if and only if $P_1 - P_2$ is an orthogonal projection.*

Proof. If $P_1 \geq P_2$, then $P_2P_1 = P_1P_2 = P_2$. But then $(P_1 - P_2)^* = P_1^* - P_2^* = P_1 - P_2$ and $(P_1 - P_2)^2 = P_1^2 - P_1P_2 - P_2P_1 - P_2^2 = P_1 - P_2 - P_2 + P_2 = P_1 - P_2$. Hence, $P_1 - P_2$ is an orthogonal projection.

If $P_1 - P_2$ is an orthogonal projection, then $\langle P_1x, x\rangle - \langle P_2x, x\rangle = \langle (P_1 - P_2)x, x\rangle = \langle (P_1 - P_2)^2x, x\rangle = \langle (P_1 - P_2)x, (P_1 - P_2)x\rangle = \|(P_1 - P_2)x\|^2 \geq 0$ for every $x \in \mathcal{H}$. Hence, $P_1 \geq P_2$. $\square$

CHAPTER 3

THE UNITARY GROUP

Throughout this section $\mathcal{H}$ is considered to be a separable infinite dimensional complex Hilbert space.

3.1. Introduction

Definition 3.1. A bounded linear operator acting on a Hilbert space $\mathcal{H}$ is said to be unitary if it is a norm preserving mapping from $\mathcal{H}$ onto $\mathcal{H}$. We denote with $\mathcal{U}(\mathcal{H})$ the set of all unitary operators acting on the Hilbert space $\mathcal{H}$. If $\mathcal{H}$ is n-dimensional $\mathcal{U}(\mathcal{H})$ is sometimes denoted $\mathcal{U}(n)$.

Proposition 3.2. *A bounded linear operator U is unitary if and only if $U^*U = UU^* = I$.*

Proof. If U is unitary then, since $\|Ux_1 - Ux_2\| = \|U(x_1 - x_2)\| = \|x_1 - x_2\|$, U is one-to-one, onto by definition and hence invertible. Since $\langle U^*Ux, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2 = \|x\|^2 = \langle x, x\rangle$, by Proposition 2.19 we have that $U^*U = I$ and hence the inverse of U is the bounded operator U^*. Therefore $U^*U = UU^* = I$.

If $U^*U = UU^* = I$ then U is invertible and hence onto. Since $\|Ux\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle x, x\rangle = \|x\|^2$, then U preserves norms and hence U is unitary. $\square$

3.2. Topologies on $\mathcal{U}(\mathcal{H})$

Proposition 3.3. *The weak operator topology and the strong operator topology coincide on $\mathcal{U}(\mathcal{H})$.*

Proof. If $U_j \xrightarrow{so} U$ then, since $|\langle U_jx, y\rangle - \langle Ux, y\rangle| = |\langle (U_j - U)x, y\rangle| \le \|(U_j - U)x\|\ \|y\| \to 0$ for j large and for every $x, y \in \mathcal{H} \Rightarrow U_j \xrightarrow{wo} U$.

If $U_j \xrightarrow{wo} U$, then $\langle U_jx, y\rangle \to \langle Ux, y\rangle$ for every $x, y \in \mathcal{H}$. In particular, $\langle U_jx, Ux\rangle \to \langle Ux, Ux\rangle$ for every $x \in \mathcal{H}$. Then $\|(U_j - U)x\|^2 = \langle (U_j - U)x, (U_j - U)x\rangle = \langle U_jx, U_jx\rangle -$

$\langle U_jx, Ux\rangle - \langle Ux, U_jx\rangle + \langle Ux, Ux\rangle = \|x\|^2 - (\langle U_jx, Ux\rangle + \overline{\langle U_jx, Ux\rangle}) + \|x\|^2 = 2\|x\|^2 - 2\mathrm{Re}(\langle U_jx, Ux\rangle) \to 2\|x\|^2 - 2\mathrm{Re}(\langle Ux, Ux\rangle) = 2\|x\|^2 - 2\mathrm{Re}(\|x\|^2) = 0$ for every $x \in \mathcal{H}$. Hence $U_j \xrightarrow{so} U$. $\square$

Lemma 3.4. *If $(T_j), T \subset \mathcal{L}(\mathcal{H})$ are linear operators and if $T_j \xrightarrow{wo} T$, then $T_j^* \xrightarrow{wo} T^*$.*

Proof. If $T_j \xrightarrow{wo} T$, then $(T_j - T) \xrightarrow{wo} 0 \Rightarrow \langle (T_j - T)x, y\rangle \to 0$ for every $x, y \in \mathcal{H}$. This implies that $\langle x, (T_j - T)^* y\rangle \to 0 \Rightarrow T_j^* - T^* = (T_j - T)^* \xrightarrow{wo} 0 \Rightarrow T_j^* \xrightarrow{wo} T^*$. $\square$

Lemma 3.5. *If $\mathcal{H}$ is a separable complex Hilbert space and $f \in \mathcal{H}om(\mathcal{H}_1)$, then the mappings $f \mapsto \langle f(x), y\rangle$ and $f \mapsto \langle f^{-1}(x), y\rangle$, where $x \in \mathcal{H}_1$ and $y \in \mathcal{H}$, are continuous.*

Proof. The topology on $\mathcal{H}om(\mathcal{H}_1)$ is given by the metric

$$\rho(f, g) = \sup_{x \in \mathcal{H}_1} \sum_{l \geq 1} \frac{1}{2^l} |\langle f(x) - g(x), e_l\rangle| + \sup_{x \in \mathcal{H}_1} \sum_{l \geq 1} \frac{1}{2^l} |\langle f^{-1}(x) - g^{-1}(x), e_l\rangle|$$

where $\{e_l\}$ is an orthonormal basis for $\mathcal{H}$.

If $f_j, f \subset \mathcal{H}om(\mathcal{H}_1)$ such that $\rho(f_j, f) \to 0$, then $\sup_{x \in \mathcal{H}_1} \sum_{l \geq 1} \frac{1}{2^l} |\langle f_j(x) - f(x), e_l\rangle| \to 0$ and $\sup_{x \in \mathcal{H}_1} \sum_{l \geq 1} \frac{1}{2^l} |\langle f_j^{-1}(x) - f^{-1}(x), e_l\rangle| \to 0$. This implies that $|\langle f_j(x) - f(x), e_l\rangle| \to 0$ and $|\langle f_j^{-1}(x) - f^{-1}(x), e_l\rangle| \to 0$ for every $x \in \mathcal{H}_1$ and every $l \geq 1$ $\Rightarrow |\langle f_j(x) - f(x), v\rangle| \to 0$ and $|\langle f_j^{-1}(x) - f^{-1}(x), v\rangle| \to 0$ for every $x \in \mathcal{H}_1$ and every $v = \sum_{l=1}^k a_l e_l$.

Let $\epsilon > 0$, and $y \in \mathcal{H}$. Choose $v = \sum_{l=1}^k a_l e_l$ be such that $\|y - v\| < \frac{\epsilon}{4}$. This can be done since finite linear combinations of e_l are dense. Then, for every $x \in \mathcal{H}_1$ we have that $|\langle f_j(x) - f(x), y - v\rangle| \leq |\langle f_j(x), y - v\rangle| + |\langle f(x), y - v\rangle| \leq \|f_j(x)\|\ \|y - v\| + \|f(x)\|\ \|y - v\| \leq 2\|y - v\| < \frac{\epsilon}{2}$. Since $|\langle f_j(x) - f(x), v\rangle| \to 0$ for every $x \in \mathcal{H}_1$ and every $v = \sum_{l=1}^k a_l e_l$, choose J such that $|\langle f_j(x) - f(x), v\rangle| < \frac{\epsilon}{2}$ for every $j \geq J$. This implies that $|\langle f_j(x) - f(x), y\rangle| \leq |\langle f_j(x) - f(x), y - v\rangle| + |\langle f_j(x) - f(x), v\rangle| < \epsilon$ for every $j \geq J$. Hence, the mapping $f \mapsto \langle f(x), y\rangle$ is continuous. A similar argument shows that the mapping $f \mapsto \langle f^{-1}(x), y\rangle$ is continuous. $\square$

Proposition 3.6. ★ *If $\mathcal{H}$ is a separable complex Hilbert space, the weak operator topology on $\mathcal{U}(\mathcal{H})$ coincides with the relative topology on $\mathcal{U}(\mathcal{H})$ given by $\mathcal{H}om(\mathcal{H}_1)$.*

Proof. Let $(U_j) \subset \mathcal{U}(\mathcal{H})$ be a net and $U \in \mathcal{U}(\mathcal{H})$. We want to prove that $\rho(U_j, U) \to 0 \Leftrightarrow U_j \xrightarrow{wo} U$. If $\rho(U_j, U) \to 0$, since by Lemma 3.5 the mapping $f \mapsto \langle f(x), y\rangle$ is continuous, we have that $\langle (U_j - U)(x), y\rangle = \|x\| \langle (U_j - U)(\frac{x}{\|x\|}), y\rangle \to 0$ for every $x, y \in \mathcal{H}$. Hence $U_j \xrightarrow{wo} U$.

If $U_j \xrightarrow{wo} U$ then, by Lemma 3.4, we have that $U_j^* \xrightarrow{wo} U^*$ and then by Proposition 3.3 we have that $U_j^* \xrightarrow{so} U$. Since $|\langle U_j(x) - U(x), e_l\rangle| = |\langle (U_j - U)(x), e_l\rangle| = |\langle x, (U_j - U)^*(e_l)\rangle| \leq \|x\|\ \|(U_j^* - U^*)(e_l)\| \leq \|(U_j^* - U^*)(e_l)\| \to 0$, we have that $|\langle U_j(x) - U(x), e_l\rangle| \to 0$ uniformly for every $x \in \mathcal{H}_1$ and every $l \geq 1$.

Let $\epsilon > 0$. Choose L so that $2^{L-1} > \frac{2}{\epsilon}$. Then $\frac{\epsilon}{2} > \frac{1}{2^{L-1}} = \frac{1}{2^{L-1}}(\sum_{l\geq 1} \frac{1}{2^l}) = \sum_{l\geq 1} \frac{1}{2^{L-1+l}} = \sum_{l>L} \frac{1}{2^{l-1}} \geq \sum_{l>L} \frac{1}{2^l} \|U_j(x) - U(x)\|\ \|e_l\| \geq \sum_{l>L} \frac{1}{2^l} |\langle U_j(x) - U(x), e_l\rangle|$ for every $x \in \mathcal{H}_1$ and every j. Since $|\langle U_j(x) - U(x), e_l\rangle| \to 0$ uniformly for every $x \in \mathcal{H}_1$ and every $l \geq 1$, then for every $1 \leq l \leq L$ there is an J_l such that $\frac{1}{2^l}|\langle U_j(x) - U(x), e_l\rangle| < \frac{\epsilon}{2L}$ for every $x \in \mathcal{H}_1$ and every $j \geq J_l$. Let $J \geq \{J_l \mid 1 \leq l \leq L\}$. Then $\sum_{1\leq l\leq L} \frac{1}{2^l}|\langle U_j(x) - U(x), e_l\rangle| < \frac{\epsilon}{2}$ for every $x \in \mathcal{H}_1$ and every $j \geq J$. Hence, if $j \geq J$, then $\sum_l \frac{1}{2^l}|\langle U_j(x) - U(x), e_l\rangle| < \epsilon$ for every $x \in \mathcal{H}_1 \Rightarrow \sup_{x\in\mathcal{H}_1} \sum_l \frac{1}{2^l}|\langle U_j(x) - U(x), e_l\rangle| < \epsilon$ for every $j \geq J$.

A similar proof shows that $\sup_{x\in\mathcal{H}_1} \sum_l \frac{1}{2^l}|\langle U_j^{-1}(x) - U^{-1}(x), e_l\rangle| < \epsilon$ for every $j \geq J'$. Hence $\rho(U_j, U) \to 0$, and therefore the two topologies coincide. $\square$

THEOREM 3.7. ★ *If $\mathcal{H}$ is a complex separable Hilbert space, $\mathcal{U}(\mathcal{H})$ is a closed subgroup in $\mathcal{H}om(\mathcal{H}_1)$.*

Proof. If $U \in \mathcal{U}(\mathcal{H})$, then U is a bijection from $\mathcal{H}_1$ into $\mathcal{H}_1$. If $x_j, x \in \mathcal{H}_1$ such that $x_j \xrightarrow{w} x$, then for every $y \in \mathcal{H}$ we have that $\langle Ux_j, y\rangle = \langle x_j, U^*y\rangle \to \langle x, U^*y\rangle = \langle Ux, y\rangle \Rightarrow Ux_j \xrightarrow{w} Ux$, and hence U is weakly continuous. Since the inverse has the same properties U is a homeomorphism of $\mathcal{H}_1$ with the relative weak operator topology and hence $\mathcal{U}(\mathcal{H}) \subset \mathcal{H}om(\mathcal{H}_1)$. If $U, V \in \mathcal{U}(\mathcal{H}) \Rightarrow \|UVx\| = \|x\|$ and UV is onto $\Rightarrow UV \in \mathcal{U}(\mathcal{H})$. $I \in \mathcal{U}(\mathcal{H})$. If $U \in \mathcal{U}(\mathcal{H})$, then $\|U^*x\|^2 = \langle U^*x, U^*x\rangle = \langle UU^*x, x\rangle = \langle UU^{-1}x, x\rangle = \langle x, x\rangle = \|x\|^2$. This implies that $U^* \in \mathcal{U}(\mathcal{H})$, and hence that $\mathcal{U}(\mathcal{H}) \subset \mathcal{H}om(\mathcal{H}_1)$ is a subgroup.

Let $\{U_j\} \subset \mathcal{U}(\mathcal{H})$ be a net such that $U_j \xrightarrow{\rho} \phi \in \mathcal{H}om(\mathcal{H}_1)$. Since the inverse operation in a Polish group is continuous, we have that $U_j^* = U_j^{-1} \xrightarrow{\rho} \phi^{-1}$. According to Lemma 3.5 we have that $\langle U_j(x), y\rangle \to \langle \phi(x), y\rangle$ and $\langle U_j^*(x), y\rangle \to \langle \phi^{-1}(x), y\rangle$ for every $x \in \mathcal{H}_1$ and every $y \in \mathcal{H}$.

We will define $U : \mathcal{H} \to \mathcal{H}$ as follows. For every $x \in \mathcal{H}_1$ let $U(x) = \phi(x)$. If $x \in \mathcal{H}$, then there exists $\lambda > 0$ such that $\lambda x \in \mathcal{H}_1$, and let $U(x) = \frac{1}{\lambda}\phi(\lambda x)$. If $x \in \mathcal{H}$ and $\lambda_1, \lambda_2 > 0$ are such that $\lambda_1 x, \lambda_2 x \in \mathcal{H}_1$, then $\frac{1}{\lambda_1}\langle U_j(\lambda_1 x), y\rangle \to \frac{1}{\lambda_1}\langle \phi(\lambda_1 x), y\rangle = \langle \frac{1}{\lambda_1}\phi(\lambda_1 x), y\rangle$ and $\frac{1}{\lambda_1}\langle U_j(\lambda_1 x), y\rangle = \langle U_j(x), y\rangle = \frac{1}{\lambda_2}\langle U_j(\lambda_2 x), y\rangle \to \frac{1}{\lambda_2}\langle \phi(\lambda_2 x), y\rangle = \langle \frac{1}{\lambda_2}\phi(\lambda_2 x), y\rangle$ for every $x, y \in \mathcal{H}$. This implies that $\langle \frac{1}{\lambda_1}\phi(\lambda_1 x), y\rangle = \langle \frac{1}{\lambda_2}\phi(\lambda_2 x), y\rangle$ for every $x, y \in \mathcal{H}$, which implies that $\frac{1}{\lambda_1}\phi(\lambda_1 x) = \frac{1}{\lambda_2}\phi(\lambda_2 x)$ for every $x \in \mathcal{H}$. Hence, the definition of U is independent of λ.

If $x \in \mathcal{H}_1$ and $y \in \mathcal{H}$, then $\langle U_j(x), y\rangle \to \langle \phi(x), y\rangle = \langle U(x), y\rangle$. If $x, y \in \mathcal{H}$, let $\lambda > 0$ be such that $\lambda x \in \mathcal{H}_1$, and then $\langle U_j(x), y\rangle = \frac{1}{\lambda}\langle U_j(\lambda x), y\rangle \to \frac{1}{\lambda}\langle \phi(\lambda x), y\rangle = \langle \frac{1}{\lambda}\phi(\lambda x), y\rangle = \langle U(x), y\rangle$ and hence $\langle U_j(x), y\rangle \to \langle U(x), y\rangle$ for every $x, y \in \mathcal{H}$.

For every $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in \mathcal{H}$ we have $\alpha\langle U_j(x), z\rangle + \beta\langle U_j(y), z\rangle = \langle U_j(\alpha x + \beta y), z\rangle \to \langle U(\alpha x + \beta y), z\rangle$. Since $\langle U_j(x), z\rangle \to \langle U(x), z\rangle$ and $\langle U_j(y), z\rangle \to \langle U(y), z\rangle$, we have that $\alpha\langle U(x), z\rangle + \beta\langle U(y), z\rangle = \langle U(\alpha x + \beta y), z\rangle \Rightarrow U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)$ for every $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{H}$ and hence U is linear. Since $|\langle U(x), y\rangle| = \lim_j |\langle U_j(x), y\rangle| \leq \lim_j \|U_j(x)\|\ \|y\| \leq \|x\|\ \|y\|$, we have that $\|U(x)\| \leq \|x\| \Rightarrow \|U\| \leq 1$ and hence U is a linear operator. It remains to show that U is unitary.

Lemma 3.4 implies that $\langle U_j^*(x), y\rangle \to \langle U^*(x), y\rangle$ for every $x \in \mathcal{H}_1$ and every $y \in \mathcal{H}$. Hence $U^*(x) = \phi^{-1}(x)$ for every $x \in \mathcal{H}_1$. If $x \in \mathcal{H}_1$, then $\phi(x), \phi^{-1}(x) \in \mathcal{H}_1$ and then $U^*U(x) = U^*(\phi(x)) = \phi^{-1}(\phi(x)) = x$ and $UU^*(x) = U(\phi^{-1}(x)) = \phi(\phi^{-1}(x)) = x$. If $x \notin \mathcal{H}_1$, let $\lambda > 0$ be such that $\lambda x \in \mathcal{H}_1$. Then $U^*U(x) = U^*(\frac{1}{\lambda}\phi(\lambda x)) = \frac{1}{\lambda}U^*(\phi(\lambda x)) = \frac{1}{\lambda}\phi^{-1}(\phi(\lambda x)) = \frac{1}{\lambda}\lambda x = x$ and $UU^*(x) = \frac{1}{\lambda}UU^*(\lambda x) = \frac{1}{\lambda}U(\phi^{-1}(\lambda x)) = \frac{1}{\lambda}\phi(\phi^{-1}(\lambda x)) = \frac{1}{\lambda}\lambda x = x$. Hence $U^*U = UU^* = I$, and by Proposition 3.2 we have that U is unitary, and therefore $\mathcal{U}(\mathcal{H})$ is closed. □

Corollary 3.8. *$\mathcal{U}(\mathcal{H})$ is a complete separable metric topological group.*

Proof. From Corollary 2.25 we have that $\mathcal{H}om(\mathcal{H}_1)$ is a complete separable metric topological group. The conclusion follows from Theorem 3.7. □

LEMMA 3.9. *Let $D \subset \mathcal{H}$ be a dense subset of the Hilbert space $\mathcal{H}$ and let $U_0 \in \mathcal{U}(\mathcal{H})$ be unitary. Then the sets*

$$\cap_{1\leq i\leq k}\{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)d_i\| < \epsilon,\ d_i \in D\}$$

where $\epsilon > 0$ and $k \geq 1$, form a neighborhood base at U_0 in $\mathcal{U}(\mathcal{H})$ for the strong operator topology.

Proof. Let $\{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)x_i\| < \epsilon,\ 1 \leq i \leq k\}$ be a basic neighborhood of U_0, where $\epsilon > 0$ and $x_1, x_2, ..., x_k \in \mathcal{H}$. Since D is dense in $\mathcal{H}$, there exist $d_1, d_2, ..., d_k \in D$ such that $\|x_i - d_i\| < \frac{\epsilon}{3}$. If $U \in \{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)d_i\| < \frac{\epsilon}{3},\ 1 \leq i \leq k\}$ then $\|(U-U_0)x_i\| \leq \|Ux_i - Ud_i\| + \|Ud_i - U_0d_i\| + \|U_0d_i - U_0x_i\| = \|x_i - d_i\| + \|(U-U_0)d_i\| + \|x_i - d_i\| < \epsilon$ and hence $U \in \{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)x_i\| < \epsilon,\ 1 \leq i \leq k\}$. This implies that the sets $\{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)d_i\| < \epsilon,\ 1 \leq i \leq k\}$ form a neighborhood base at U_0 for the strong operator topology. □

LEMMA 3.10. *Let $\{e_l\}_{l\geq 1}$ be an orthonormal subset of a Hilbert space $\mathcal{H}$. Then finite linear combinations of e_l are dense in $\mathcal{H}$.*

Proof. Let $x = \sum_{l\geq 1} a_l e_l \in \mathcal{H}$ and let $\epsilon > 0$. Since $\|x\|^2 = \sum_{l\geq 1} |a_l|^2$ we have that there exists N such that $\sum_{l\geq N} |a_l|^2 < \epsilon$. Then $\|x - \sum_{1\leq l\leq N} a_l e_l\|^2 = \|\sum_{l>N} a_l e_l\|^2 = \sum_{l\geq N} |a_l|^2 < \epsilon$, and hence finite linear combinations of e_l are dense in $\mathcal{H}$. □

PROPOSITION 3.11. *Let $\{e_l\}_{l\geq 1}$ be an orthonormal subset of a Hilbert space $\mathcal{H}$ and let $U_0 \in \mathcal{U}(\mathcal{H})$ be unitary. Then the sets*

$$\cap_{1\leq l\leq k}\{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)e_l\| < \epsilon\}$$

where $\epsilon > 0$ and $k \geq 1$, form a neighborhood base at U_0 for the strong operator topology on $\mathcal{U}(\mathcal{H})$.

Proof. Let $\epsilon > 0$ and let $D = \{\sum_{1\leq l\leq N} a_l e_l \mid N \geq 1\}$. Then by Lemma 3.10 D is dense in $\mathcal{H}$ and thus by Lemma 3.9, $\mathcal{N} = \cap_{1\leq i\leq k}\{U \in \mathcal{U}(\mathcal{H}) \mid \|(U-U_0)d_i\| < \epsilon\}$, where $d_i = \sum_{1\leq l\leq N_i} a^i_l e_l$ for $1 \leq i \leq k$, is a basic open neighborhood at U_0 with respect to the strong operator topology. Let $N = \max_{1\le i\le k} N_i$ and $A = \max_{1\le i\le k,\, 1\le l\le N} |a_l^i|$. If $U \in \mathcal{U}(\mathcal{H})$ is such that $\|(U-U_0)e_l\| < \frac{\epsilon}{AN}$, then $\|(U-U_0)d_i\| \le \sum_{1\le l\le N_i} |a_l^i|\, \|(U-U_0)e_l\| \le \sum_{1\le l\le N_i} A\frac{\epsilon}{AN} < \epsilon$ for every $1 \le i \le k$ and hence $U \in \mathcal{N}$. □

3.3. The Subsets $\mathcal{U}(\mathcal{M})$ and $SU(\mathcal{M})$ of $\mathcal{U}(\mathcal{H})$

Definition 3.12. If $\mathcal{H}$ is a Hilbert space, we define $Z(\mathcal{U}(\mathcal{H})) = \{U \in \mathcal{U}(\mathcal{H}) \mid UV = VU,\ \forall V \in \mathcal{U}(\mathcal{H})\}$, the center of $\mathcal{U}(\mathcal{H})$.

Proposition 3.13. $Z(\mathcal{U}(\mathcal{H})) = \{\lambda I \mid |\lambda| = 1\}$

Proof. Let $U \in \mathcal{U}(\mathcal{H})$, let λ be such that $|\lambda| = 1$ and let $x \in \mathcal{H}$. Then $\lambda Ux = U\lambda x \Rightarrow (\lambda I)U = U(\lambda I) \Rightarrow \lambda I \in Z(\mathcal{U}(\mathcal{H}))$.

Let $W \in Z(\mathcal{U}(\mathcal{H}))$. Then $WA = AW$ for every $A \in \mathcal{L}(\mathcal{H})$ since A is a finite linear combination of unitary operators (Theorem 4.1.7., page 242, [10]). Let $\{e_l\}_{l\ge 1}$ be an orthonormal basis for $\mathcal{H}$ and let P_l be the orthogonal projection on the 1-dimensional subspace spanned by e_l. Then $W(e_l) = WP_l(e_l) = P_lW(e_l) = \lambda_l e_l$ for some scalar λ_l for every $l \ge 1$. If $i \ne j$ and if $U \in \mathcal{L}(\mathcal{H})$ is such that $Ue_i = e_j$, $Ue_j = e_i$ and $Ue_l = e_l$ for every $l \ne i, j$, then $\lambda_i e_i = We_i = WUe_j = UWe_j = U\lambda_j e_j = \lambda_j Ue_j = \lambda_j e_i \Rightarrow \lambda_i = \lambda_j$. Hence, there exists a scalar λ such that $\lambda_l = \lambda$ for every $l \ge 1$ and $We_l = \lambda e_l$. We also have that $1 = \|e_1\| = \|We_1\| = \|\lambda e_1\| = |\lambda|\, \|e_1\| = |\lambda|$. Hence $W = \lambda I$, with $|\lambda| = 1$. □

Proposition 3.14. *If $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{H}$ and if $\mathcal{U}_{\mathcal{M}} = \{U \in \mathcal{U}(\mathcal{H}) \mid U|_{\mathcal{M}^\perp} = I\}$, then $\mathcal{U}_{\mathcal{M}}$ is a closed subgroup of $\mathcal{U}(\mathcal{H})$ and the mapping $i : \mathcal{U}_{\mathcal{M}} \to \mathcal{U}(\mathcal{M})$, $i(U) = U|_{\mathcal{M}}$ is a well defined isomorphism of topological groups. Accordingly, $\mathcal{U}(\mathcal{M})$ may be identified with $\mathcal{U}_{\mathcal{M}}$, and we can consider $\mathcal{U}(\mathcal{M})$ to be a closed subgroup of $\mathcal{U}(\mathcal{H})$.*

Proof. If $U, V \in \mathcal{U}_{\mathcal{M}}$, then $U|_{\mathcal{M}^\perp} = I$ and $V|_{\mathcal{M}^\perp} = I \Rightarrow UV|_{\mathcal{M}^\perp} = I \Rightarrow UV \in \mathcal{U}_{\mathcal{M}}$. Let $U \in \mathcal{U}(\mathcal{M})$ and $x \in \mathcal{M}^\perp$. Then $x = Ux \Rightarrow U^*x = U^*Ux = x \Rightarrow U^*|_{\mathcal{M}^\perp} = I \Rightarrow U^* \in \mathcal{U}_{\mathcal{M}}$. This proves that $\mathcal{U}_{\mathcal{M}}$ is a subgroup of $\mathcal{U}(\mathcal{H})$.

Let $(U_n) \subset \mathcal{U}_{\mathcal{M}}$ be such that $U_n \to U \in \mathcal{U}(\mathcal{H})$. Since $U_n|_{\mathcal{M}^\perp} = I$ for every n, we have that $\langle x, y\rangle = \langle U_nx, y\rangle \to \langle Ux, y\rangle$ for every $x \in \mathcal{M}^\perp$ and every $y \in \mathcal{H} \Rightarrow Ux = x$ for every

$x \in \mathcal{M}^\perp \Rightarrow U \in \mathcal{U}_\mathcal{M} \Rightarrow \mathcal{U}_\mathcal{M}$ is closed in $\mathcal{U}(\mathcal{H})$. It remains to show that the mapping i is a topological isomorphism.

Let $U \in \mathcal{U}_\mathcal{M}$. Let $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$. Then $\langle i(U)x, y\rangle = \langle U|_\mathcal{M} x, y\rangle = \langle Ux, y\rangle = \langle x, U^* y\rangle = \langle x, y\rangle = 0 \Rightarrow i(U) : \mathcal{M} \to \mathcal{M}$. Since for every $x \in \mathcal{M}$ we have that $\|i(U)x\| = \|U|_\mathcal{M}(x)\| = \|Ux\| = \|x\| \Rightarrow i(U)$ is norm preserving. Let $y \in \mathcal{M}$. Since U is surjective, there exists $x \in \mathcal{H}$ such that $Ux = y$. If $x = x_1 + x_2$, with $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$ then $y = Ux_1 + x_2 \Rightarrow x_2 = y - Ux_1 \in \mathcal{M} \Rightarrow x_2 \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\} \Rightarrow y = Ux_1 = U|_\mathcal{M} x_1 = i(U)x_1 \Rightarrow i(U)$ is onto $\mathcal{M}$. Hence, if $U \in \mathcal{U}_\mathcal{M}$, then $i(U) : \mathcal{M} \to \mathcal{M}$ is a norm preserving surjection $\Rightarrow i(U) \in \mathcal{U}(\mathcal{M}) \Rightarrow i$ is well defined.

If $U_1, U_2 \in \mathcal{U}_\mathcal{M}$ are such that $i(U_1) = i(U_2)$ then $U_1|_\mathcal{M} = U_2|_\mathcal{M}$ and, since $U_1|_{\mathcal{M}^\perp} = U_2|_{\mathcal{M}^\perp} = I$ we have that $U_1 = U_2 \Rightarrow i$ is one-to-one. If $U \in \mathcal{U}(\mathcal{M})$ let $W : \mathcal{H} \to \mathcal{H}$ be defined as $Wx = UP_1x + P_2x$ for every $x \in \mathcal{H}$, where P_1 and P_2 are the orthogonal projections on $\mathcal{M}$ and $\mathcal{M}^\perp$, respective. Then $\|Wx\|^2 = \|UP_1x\|^2 + \|P_2x\|^2 = \|P_1x\|^2 + \|P_2x\|^2 = \|x\|^2 \Rightarrow W$ is norm preserving. Let $y \in \mathcal{H}$, then $P_1y \in \mathcal{M} \Rightarrow$ there exists $x' \in \mathcal{M}$ such that $Ux' = P_1y$. If $x = x' + P_2y$, then $Wx = UP_1x + P_2x = Ux' + P_2y = P_1y + P_2y = y \Rightarrow W$ is surjective $\Rightarrow W$ is unitary and, since $W|_{\mathcal{M}^\perp} = I$ we have that $W \in \mathcal{U}_\mathcal{M}$. Note that $i(W) = W|_\mathcal{M} = U$ and hence i is onto $\mathcal{U}(\mathcal{M})$.

Let $(U_n) \subset \mathcal{U}_\mathcal{M}$ be such that $U_n \to U \in \mathcal{U}_\mathcal{M}$. Then for every $x, y \in \mathcal{M}$ we have that $\langle i(U_n)x, y\rangle = \langle U_n|_\mathcal{M} x, y\rangle = \langle U_n x, y\rangle \to \langle Ux, y\rangle = \langle U|_\mathcal{M} x, y\rangle = \langle i(U)x, y\rangle \Rightarrow i$ is continuous.

Let $(U_n) \subset \mathcal{U}(\mathcal{M})$ be such that $U_n \to U \in \mathcal{U}(\mathcal{M})$. Then, since $i^{-1}(U_n)x = U_nP_1x + P_2x$ and $i^{-1}(U)x = UP_1x + P_2x$ for every $x \in \mathcal{H}$, we have that $\langle i^{-1}(U_n)x, y\rangle = \langle U_nP_1x + P_2x, y\rangle = \langle U_nP_1x, y\rangle + \langle P_2x, y\rangle \to \langle UP_1x, y\rangle + \langle P_2x, y\rangle = \langle UP_1x + P_2x, y\rangle = \langle i^{-1}(U)x, y\rangle \Rightarrow i^{-1}$ is continuous. □

Definition 3.15. If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{H}$ are two closed subspaces we define their sum to be $\mathcal{M}_1 + \mathcal{M}_2 = \{v_1 + v_2 \mid v_1 \in \mathcal{M}_1 \text{ and } v_2 \in \mathcal{M}_2\}$. $\mathcal{M}_1 + \mathcal{M}_2$ is a vector subspace.

Proposition 3.16. *If $\mathcal{A} \subset \mathcal{H}$ is a vector subspace, then $(\mathcal{A}^\perp)^\perp = cl(\mathcal{A})$.*

Proof. Let $x \in \mathcal{A}$ and $y \in \mathcal{A}^\perp$. Then $x \perp y$ and hence $x \in (\mathcal{A}^\perp)^\perp \Rightarrow \mathcal{A} \subset (\mathcal{A}^\perp)^\perp \Rightarrow \mathrm{cl}(\mathcal{A}) \subset (\mathcal{A}^\perp)^\perp$. If $\mathrm{cl}(\mathcal{A})$ were a proper subspace of $(\mathcal{A}^\perp)^\perp$, then $(\mathcal{A}^\perp)^\perp$ would have a non-zero vector x such that $x \perp \mathrm{cl}(\mathcal{A})$, *i.e.* there exists $0 \neq x \in (\mathcal{A}^\perp)^\perp \cap \mathcal{A}^\perp = \{0\}$, a contradiction. Thus $\mathrm{cl}(\mathcal{A}) = (\mathcal{A}^\perp)^\perp$. $\square$

LEMMA 3.17. *If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{H}$ are two closed subspaces, then $\mathcal{M}_1 \cap \mathcal{M}_2 = (\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp$.*

Proof. If $x \in \mathcal{M}_1 \cap \mathcal{M}_2$ then $\langle x, a\rangle = 0$ for every $a \in \mathcal{M}_1^\perp$ and $\langle x, b\rangle = 0$ for every $b \in \mathcal{M}_2^\perp \Rightarrow \langle x, a+b\rangle = 0$ for every $a+b \in \mathcal{M}_1^\perp + \mathcal{M}_2^\perp \Rightarrow x \in (\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp \Rightarrow \mathcal{M}_1 \cap \mathcal{M}_2 \subset (\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp$.

If $x \in (\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp \Rightarrow \langle x, a+b\rangle = 0$ for every $a \in \mathcal{M}_1^\perp$ and every $b \in \mathcal{M}_2^\perp \Rightarrow \langle x, a\rangle = 0$ for every $a \in \mathcal{M}_1^\perp$ and $\langle x, b\rangle = 0$ for every $b \in \mathcal{M}_2^\perp \Rightarrow x \in (\mathcal{M}_1^\perp)^\perp = \mathcal{M}_1$ and $x \in (\mathcal{M}_2^\perp)^\perp = \mathcal{M}_2 \Rightarrow x \in \mathcal{M}_1 \cap \mathcal{M}_2 \Rightarrow (\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp \subset \mathcal{M}_1 \cap \mathcal{M}_2$. $\square$

COROLLARY 3.18. *If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{H}$, are two closed subspaces, then $cl(\mathcal{M}_1^\perp + \mathcal{M}_2^\perp) = (\mathcal{M}_1 \cap \mathcal{M}_2)^\perp$.*

Proof. It follows from Proposition 3.16 and Lemma 3.17 that $(\mathcal{M}_1 \cap \mathcal{M}_2)^\perp = [(\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)^\perp]^\perp = \mathrm{cl}(\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)$. $\square$

PROPOSITION 3.19. *Let $\mathcal{M}_l \subset \mathcal{H}$, $l = 1, 2$ be two finite dimensional closed subspaces. If $U \in \mathcal{U}(\mathcal{M}_l)$ for $l = 1, 2$, then $U|_{\mathcal{M}_l} : \mathcal{M}_l \to \mathcal{M}_l$ is a linear mapping, the determinant $\det(U|_{\mathcal{M}_l})$ exists and $\det(U|_{\mathcal{M}_1}) = \det(U|_{\mathcal{M}_2})$.*

Proof. Since $U \in \mathcal{U}(\mathcal{M}_l)$ for $l = 1, 2$, we have that $U|_{\mathcal{M}_l^\perp} = I$ for $l = 1, 2 \Rightarrow U|_{\mathcal{M}_1^\perp + \mathcal{M}_2^\perp} = I \Rightarrow U|_{\mathrm{cl}(\mathcal{M}_1^\perp + \mathcal{M}_2^\perp)} = I$ and, using Corollary 3.18, we have that $U|_{(\mathcal{M}_1 \cap \mathcal{M}_2)^\perp} = I$. If $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ then, since $(\mathcal{M}_1 \cap \mathcal{M}_2)^\perp = \mathcal{H}$ we have that $U = I \Rightarrow \det(U|_{\mathcal{M}_1}) = \det(U|_{\mathcal{M}_2}) = 1$.

If $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \{0\}$, let $\{e_1, e_2, ..., e_k\}$ be an orthonormal basis for $\mathcal{M}_1 \cap \mathcal{M}_2$. Extend this to $\{e_1, ..., e_k, e_{k+1}, ..., e_n\}$, an orthonormal basis for $\mathcal{M}_1$ and denote $\mathcal{N} = span(\{e_{k+1}, ..., e_n\})$. Since $\mathcal{N} \subset (\mathcal{M}_1 \cap \mathcal{M}_2)^\perp$ it follows that $U|_{\mathcal{N}} = I$ and hence $\det(U|_{\mathcal{N}}) = 1$. This implies that $\det(U|_{\mathcal{M}_1}) = \det(U|_{\mathcal{M}_1 \cap \mathcal{M}_2}) \det(U|_{\mathcal{N}}) = \det(U|_{\mathcal{M}_1 \cap \mathcal{M}_2})$. Similarly, we have that $\det(U|_{\mathcal{M}_2}) = \det(U|_{\mathcal{M}_1 \cap \mathcal{M}_2})$ and hence $\det(U|_{\mathcal{M}_1}) = \det(U|_{\mathcal{M}_2})$. $\square$

Definition 3.20. Define the finite dimensional unitaries to be $\mathcal{U}_F(\mathcal{H}) = \cup\{\mathcal{U}(\mathcal{M}) \mid \mathcal{M} \subset \mathcal{H}, \mathcal{M}$ finite dimensional $\}$. For every $U \in \mathcal{U}_F(\mathcal{H})$, there exists $\mathcal{M} \subset \mathcal{H}$ finite dimensional such that $U|_{\mathcal{M}^\perp} = I$, and we define $\det(U) = \det(U|_{\mathcal{M}})$. According with Proposition 3.19 this definition is independent on the choice of $\mathcal{M}$ and hence $\det : \mathcal{U}_F(\mathcal{H}) \to \mathbb{C}$ is well defined. If $\mathcal{M} \subset \mathcal{H}$ is finite dimensional, we denote $SU(\mathcal{M})$ to be the set $SU(\mathcal{M}) = \{U \in \mathcal{U}(\mathcal{M}) \mid \det(U) = 1\}$, and $SU_F(\mathcal{H}) = \{U \in \mathcal{U}_F(\mathcal{H}) \mid \det(U) = 1\}$. $SU(\mathcal{M})$ is called the special unitary group and sometimes is denoted $SU(n)$, where n is the dimension of $\mathcal{M}$.

Proposition 3.21. *$SU(\mathcal{M}) \subset \mathcal{U}(\mathcal{M})$ is a subgroup.*

Proof. If $U, V \in SU(\mathcal{M})$, then $\det(U) = 1$ and $\det(V) = 1 \Rightarrow \det(UV^{-1}) = \det(U)\det(V^{-1}) = \det(U)\frac{1}{\det(V)} = 1 \Rightarrow UV^{-1} \in SU(\mathcal{M})$. □

Definition 3.22. If $\mathcal{M} \subset \mathcal{H}$ is a closed subspace, we denote with $Z(\mathcal{U}(\mathcal{M}))$ the center of $\mathcal{U}(\mathcal{M})$.

Remark 3.23. Note that $Z(\mathcal{U}(\mathcal{M}))$ is a closed subgroup of $\mathcal{U}(\mathcal{M})$ and, as an immediate consequence of Proposition 3.13, if $\emptyset \neq \mathcal{M} \subsetneq \mathcal{H}$, we have that $Z(\mathcal{U}(\mathcal{M})) = \{U \in \mathcal{U}(\mathcal{M}) \mid U|_{\mathcal{M}} = \lambda I,\ |\lambda| = 1 \text{ and } U|_{\mathcal{M}^\perp} = I\}$

Lemma 3.24. ★ *Let $\{e_l\}_{1\le l\le n}$ be an orthonormal subset of a Hilbert space $\mathcal{H}$ and let $U \in \mathcal{U}(\mathcal{H})$ a unitary operator acting on $\mathcal{H}$. Then there exists $\mathcal{M} \subset \mathcal{H}$ a subspace and $W \in \mathcal{U}(\mathcal{H})$ a unitary operator such that $We_l = Ue_l$ for every $1 \le l \le n$ and $W|_{\mathcal{M}^\perp} = I$.*

Proof. Let $\mathcal{M} = span(\{e_l, Ue_l\}_{1\le l\le n})$. Then $\mathcal{M}$ is a closed finite dimensional subspace of $\mathcal{H}$. Let $\{e_1, e_2, ..., e_n, f_1, ..., f_k\}$ be an orthonormal basis for $\mathcal{M}$ obtained by expanding the orthonormal system $\{e_l\}_{1\le l\le n}$. Since $\langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$, then $\{Ue_l\}_{1\le l\le n}$ is also an orthonormal system and expand this to $\{Ue_1, Ue_2, ..., Ue_n, g_1, ..., g_k\}$, another orthonormal basis for $\mathcal{M}$. Note that the two bases have the same cardinality. Define W to be $We_l = Ue_l$ for $1 \le l \le n$, $Wf_l = g_l$ for $1 \le l \le k$ and $W|_{\mathcal{M}^\perp} =$ I. We will show that W is unitary.

Let $y \in \mathcal{H}$. Then $y = y_1 + y_2$ with $y_1 \in \mathcal{M}$, $y_2 \in \mathcal{M}^\perp$ and $y_1 = \sum_{1\le l\le n} a_l Ue_l + \sum_{1\le l\le k} b_l g_l$. If $x = \sum_{1\le l\le n} a_l e_l + \sum_{1\le l\le k} b_l f_l + y_2$, then $Wx = y$ and hence W is onto.

If $x = x_1 + x_2$, where $x_1 = \sum_{1\le l\le n} a_l e_l + \sum_{1\le l\le k} b_l f_l \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$, then $\|Wx\|^2 = \|Wx_1\|^2 + \|Wx_2\|^2 = \|\sum_{1\le l\le n} a_l U e_l + \sum_{1\le l\le k} b_l g_l\|^2 + \|x_2\|^2 = \sqrt{a_l^2 + b_l^2} + \|x_2\|^2 = \|x_1\|^2 + \|x_2\|^2 = \|x\|^2$ and hence W is an isometry. $\square$

THEOREM 3.25. ★ *Let $\{e_l\}_{1\le l\le n}$ be an orthonormal subset of a Hilbert space $\mathcal{H}$ and let $U \in \mathcal{U}(\mathcal{H})$ be a unitary operator acting on $\mathcal{H}$. Then there exists $\mathcal{M} \subset \mathcal{H}$ a finite dimensional subspace,* $\dim(\mathcal{M}) = N \ge n$, *such that span$(\{e_l\}_{1\le l\le n}) \subset \mathcal{M}$, and there exists $V \in SU(\mathcal{M})$ such that $Ve_l = Ue_l$ for every $1 \le l \le n$.*

Proof. Let $\{e_l\}_{1\le l\le n}$ be an orthonormal subset of H and $U \in \mathcal{U}(\mathcal{H})$ a unitary operator acting on $\mathcal{H}$. According with Lemma 3.24 there exists $\mathcal{N} \subset \mathcal{H}$ a finite dimensional subspace of $\mathcal{H}$ and $W \in \mathcal{U}(\mathcal{N})$ a unitary operator such that $We_l = Ue_l$ for every $1 \le l \le n$. Note if $\lambda = \det(W)$, then $|\lambda| = 1$. Let $N = \dim(\mathcal{N}) + 1$, let $f_N \in \mathcal{N}^\perp$ be such that $\|f_N\| = 1$ and let $\mathcal{M} = span(\mathcal{N} \cup \{f_N\})$. Then $\dim(\mathcal{M}) = N \ge n$ and $span(\{e_l\}_{1\le l\le n}) \subset \mathcal{N} \subset \mathcal{M}$. Define $V : \mathcal{H} \to \mathcal{H}$ as $V|_{\mathcal{M}} = W$, $Vf_N = \frac{1}{\lambda} f_N$ and $V|_{\mathcal{M}^\perp} = I$. Obviously $V \in \mathcal{U}(\mathcal{M})$ and, since $\det(V) = \frac{1}{\lambda}\det(W) = 1$, it follows that $V \in SU(\mathcal{M})$. $\square$

3.4. $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is Closed

PROPOSITION 3.26. *If G is a Hausdorff topological group and $\emptyset \ne S \subset G$ then the set $\{g \in G \mid gs = sg\ \forall s \in S\}$ is closed in G.*

Proof. For every $s \in S$ let $C_s = \{g \in G \mid gs = sg\} = \{g \in G \mid gsg^{-1}s^{-1} = e\}$. Since G is Hausdorff, $\{e\}$ is closed in G, and since $\phi_s(g) = gsg^{-1}s^{-1}$ is continuous, $C_s = \phi_s^{-1}(\{e\})$ is closed in G. But then $\{g \in G \mid gs = sg\ \forall s \in S\} = \cap_{s\in S} C_s$ is closed in G. $\square$

LEMMA 3.27. *If $W \in \mathcal{U}(\mathcal{H})$ is such that $WV = VW$ for every $V \in \mathcal{U}(\mathcal{M}^\perp)$, then $W : \mathcal{M} \to \mathcal{M}$ is surjective and $W : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is surjective.*

Proof. Let $W \in \mathcal{U}(\mathcal{H})$ be such that $WV = VW$ for every $V \in \mathcal{U}(\mathcal{M}^\perp)$. Let $V : \mathcal{H} \to \mathcal{H}$ be defined as $Vx = x_1 - x_2$ for every $x = x_1 + x_2 \in \mathcal{H}$, where $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. It is clear that V is an isometry from $\mathcal{H}$ onto $\mathcal{H}$ and hence $V \in \mathcal{U}(\mathcal{H})$. Since $V|_{\mathcal{M}} = I$, we have that $V \in \mathcal{U}(\mathcal{M}^\perp)$ and hence $WV = VW$. Let $x_1 \in \mathcal{M}$ and let $Wx_1 = y_1 + y_2$, with $y_1 \in \mathcal{M}$ and

$y_2 \in \mathcal{M}^\perp$. Then $y_1 - y_2 = V(y_1 + y_2) = VWx_1 = WVx_1 = Wx_1 = y_1 + y_2 \Rightarrow y_2 = -y_2 \Rightarrow y_2 = 0 \Rightarrow Wx_1 = y_1 \in \mathcal{M} \Rightarrow W : \mathcal{M} \to \mathcal{M}$.

Let $x_2 \in \mathcal{M}^\perp$, and let $Wx_2 = y_1 + y_2$, with $y_1 \in \mathcal{M}$ and $y_2 \in \mathcal{M}^\perp$. Then $y_1 - y_2 = V(y_1 + y_2) = VWx_2 = WVx_2 = W(-x_2) = -Wx_2 = -y_1 - y_2 \Rightarrow y_1 = -y_1 \Rightarrow y_1 = 0 \Rightarrow Wx_2 = y_2 \in \mathcal{M}^\perp \Rightarrow W : \mathcal{M}^\perp \to \mathcal{M}^\perp$.

Let $y_1 \in \mathcal{M}$ and $y_2 \in \mathcal{M}^\perp$. Since W is onto $\mathcal{H}$, there exists $x = x_1 + x_2 \in \mathcal{H}$ and $z = z_1 + z_2 \in \mathcal{H}$ such that $Wx = y_1$ and $Wz = y_2$, where $x_1, z_1 \in \mathcal{M}$ and $x_2, z_2 \in \mathcal{M}^\perp$. Then $y_1 = Wx_1 + Wx_2 \Rightarrow Wx_2 = y_1 - Wx_1 \in \mathcal{M} \Rightarrow Wx_2 \in \mathcal{M} \cap W(\mathcal{M}^\perp) \subset \mathcal{M} \cap \mathcal{M}^\perp \Rightarrow Wx_2 = 0 \Rightarrow x_2 = 0 \Rightarrow y_1 = Wx_1 \Rightarrow W : \mathcal{M} \to \mathcal{M}$ is onto and $y_2 = Wz_1 + Wz_2 \Rightarrow Wz_1 = y_2 - Wz_2 \in \mathcal{M}^\perp \Rightarrow Wz_1 \in \mathcal{M}^\perp \cap W(\mathcal{M}) \subset \mathcal{M}^\perp \cap \mathcal{M} \Rightarrow Wz_1 = 0 \Rightarrow z_1 = 0 \Rightarrow y_2 = Wz_2 \Rightarrow W : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is onto. $\square$

THEOREM 3.28. ★ *Let G be a Polish topological group, $\mathcal{M}$ a closed subspace of $\mathcal{H}$ and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}[Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})]$ is closed in G.*

Proof. We will prove that $Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}) = \{W \in \mathcal{U}(\mathcal{H}) \mid WV = VW \ \forall V \in \mathcal{U}(\mathcal{M}^\perp)\}$. This will imply that $\phi^{-1}[Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})] = \phi^{-1}(\{W \in \mathcal{U}(\mathcal{H}) \mid WV = VW \ \forall V \in \mathcal{U}(\mathcal{M}^\perp)\}) = \{\phi^{-1}(W) \mid \phi^{-1}(W)\phi^{-1}(V) = \phi^{-1}(V)\phi^{-1}(W) \ \forall\, \phi^{-1}(V) \in \phi^{-1}(\mathcal{U}(\mathcal{M}^\perp))\}$ and then, according with the Proposition 3.26 we will have that $\phi^{-1}[Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})]$ is closed in G. Note that by Proposition 3.13 we have that $Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}) = \{\lambda U \mid U \in \mathcal{U}(\mathcal{M}),\ |\lambda| = 1\}$.

Let $U \in \mathcal{U}(\mathcal{M})$, let $V \in \mathcal{U}(\mathcal{M}^\perp)$ and let $x = x_1 + x_2 \in \mathcal{H}$, with $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. Then $Ux_2 = x_2$, $Vx_1 = x_1$ and, by Proposition 3.14, $Ux_1 \in \mathcal{M}$ and $Vx_2 \in \mathcal{M}^\perp$ and hence $VUx_1 = Ux_1$ and $UVx_2 = Vx_2$. It follows that $\lambda UVx = \lambda UV(x_1 + x_2) = \lambda(UVx_1 + UVx_2) = \lambda(Ux_1 + Vx_2) = \lambda(VUx_1 + VUx_2) = \lambda VUx = V\lambda Ux \Rightarrow \lambda UV = V\lambda U$ for every $V \in \mathcal{U}(\mathcal{M}^\perp) \Rightarrow Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}) \subset \{W \in \mathcal{U}(\mathcal{H}) \mid WV = VW \ \forall V \in \mathcal{U}(\mathcal{M}^\perp)\}$.

Let $W \in \mathcal{U}(\mathcal{H})$ be such that $WV = VW$ for every $V \in \mathcal{U}(\mathcal{M}^\perp)$. Let $U : \mathcal{M}^\perp \to \mathcal{M}^\perp$ be unitary, and let $V : \mathcal{H} \to \mathcal{H}$ be defined as $Vx = x_1 + Ux_2$ for every $x = x_1 + x_2 \in \mathcal{H}$, where $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. V is unitary since it is an isometry from $\mathcal{H}$ onto $\mathcal{H}$, and $V|_{\mathcal{M}} = I$. Thus $V \in \mathcal{U}(\mathcal{M}^\perp)$, and hence $VW = WV$. Let $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. Then, by Lemma 3.27 $Wx_1 \in \mathcal{M}$ and $Wx_2 \in \mathcal{M}^\perp$, and hence $Wx_1 + UWx_2 = VWx_1 + VWx_2 = VW(x_1 + x_2) = WV(x_1 + x_2) = W(x_1 + Ux_2) = Wx_1 + WUx_2 \Rightarrow UWx_2 = WUx_2$ for every $x_2 \in \mathcal{M}^\perp \Rightarrow UW|_{\mathcal{M}^\perp} = W|_{\mathcal{M}^\perp}U$. By Proposition 3.13 it follows that $W|_{\mathcal{M}^\perp} = \lambda I$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. But then $\overline{\lambda}W \in \mathcal{U}(\mathcal{H})$ and $\overline{\lambda}W|_{\mathcal{M}^\perp} = \overline{\lambda}\lambda I = I \Rightarrow \overline{\lambda}W \in \mathcal{U}(\mathcal{M}) \Rightarrow W = \lambda\overline{\lambda}W \in Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})$ and hence $\{W \in \mathcal{U}(\mathcal{H}) \mid WV = VW \ \forall V \in \mathcal{U}(\mathcal{M}^\perp)\} \subset Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})$. $\Box$

Proposition 3.29. ★ *Let G be a Polish topological group, $\mathcal{M} \subset \mathcal{H}$ an infinite dimensional closed subspace and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is an analytic subset of G.*

Proof. Let $[\cdot,\cdot] : G \times G \to G$ be defined as $[a, b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot,\cdot]$ is continuous. If $a, b \in \phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})) \subset G$ then $\phi(a), \phi(b) \in Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}) \Rightarrow$ there exist $U, V \in \mathcal{U}(\mathcal{M})$ and λ, μ scalars such that $\phi(a) = \lambda U$ and $\phi(b) = \mu V$. But then $[a, b] = \phi^{-1}(\lambda U)\phi^{-1}(\mu V)\phi^{-1}(\lambda^{-1}U^{-1})\phi^{-1}(\mu^{-1}V^{-1}) = \phi^{-1}(UVU^{-1}V^{-1}) \in \phi^{-1}(\mathcal{U}(\mathcal{M}))$. This proves that $[\cdot,\cdot]|_{\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))\times\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))}$ takes its values in $\phi^{-1}(\mathcal{U}(\mathcal{M}))$. Let $T \in \mathcal{U}(\mathcal{M})$ and denote $T|_{\mathcal{M}} = W$. Since $\mathcal{M}$ is infinite dimensional and since W is unitary on $\mathcal{M}$, we have by [7], page 134, problem 191, that there exist unitaries $U', V' : \mathcal{M} \to \mathcal{M}$ such that $W = U'V'U'^{-1}V'^{-1}$. If $U, V : \mathcal{H} \to \mathcal{H}$ are such that $U|_{\mathcal{M}} = U'$, $U|_{\mathcal{M}^\perp} = I$, $V|_{\mathcal{M}} = V'$ and $V|_{\mathcal{M}^\perp} = I$ then $U, V \in Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})$ and $[\phi^{-1}(U), \phi^{-1}(V)] = \phi^{-1}(UVU^{-1}V^{-1}) = \phi^{-1}(T)$ and hence $[\cdot,\cdot]|_{\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))\times\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))}$ is onto $\phi^{-1}(\mathcal{U}(\mathcal{M}))$. Since G is a Polish topological group, $G \times G$ is a Polish topological group and since $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))$ is closed in G by Theorem 3.28, we have that $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})) \times \phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))$ is closed in $G \times G$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is the continuous image of a closed subset of a Polish topological group, and therefore an analytic subset of G. $\Box$

Definition 3.30. Let X be a topological space. A set $A \subset X$ is said to be a set with the Baire property if there exists an open set $U \subset X$ such that $A \triangle U \equiv (U \setminus A) \cup (A \setminus U)$, the symmetric difference of A and U, is meager in X.

Remark 3.31. The collection of subsets of X which have the Baire property, $\mathcal{BP}(X)$, is a σ-algebra of subsets.(cf. [18], p.47)

Lemma 3.32 (D.E.Miller, [24]). *Let G be a Polish topological group and $H \subset G$ be a dense subgroup. Suppose $E \subset G$ is a subset with the Baire property which is right-invariant under H (i.e. $EH = E$). Then E is meager or comeager.*

Proof. This lemma and its proof are slightly different than the original of Miller, and is only valid in the separable case.

Since G is a separable metric space, it has a countable base for its topology. The relative topology on H is also second countable, and hence H is separable as a subspace of G. If $D \subset H$ is any countable dense subgroup of H, then D is dense in G and E is right-invariant under D. Thus, by replacing H with D we may assume that H is countable.

Since E is a set with the Baire property, there exists $U \subset G$ open, such that $E \triangle U$ is meager. If $a \in H$, then $(E \triangle U)a = Ea \triangle Ua = E \triangle Ua$ is meager $\Rightarrow E \triangle (\cup_{a\in H} Ua) = (\cup_{a\in H} Ea) \triangle (\cup_{a\in H} Ua) \subset \cup_{a\in H}(Ea \triangle Ua) = \cup_{a\in H}(E \triangle Ua)$ is meager. Let $V = \cup_{a\in H} Ua$. Then V is open, right-invariant under H, $E \triangle V$ is meager and, since H is dense in G, V is dense in G. If $V = \emptyset$, then $E = E \triangle V$ is meager.

If $V \neq \emptyset$, then $E^C \cap V \subset E \triangle V$ is meager and, since V is open and dense in G, $E^C \cap V^C \subset V^C$ is meager. This implies that $E^C = (E^C \cap V) \cup (E^C \cap V^C)$ is meager $\Rightarrow E$ is comeager. $\square$

Definition 3.33. If X is a topological space and $\mathcal{F}$ is a family of subsets of X, we say that $\mathcal{F}$ separates points in X, or is a separating family of points if given any two points $x, y \in X$ with $x \neq y$, there exists $E \in \mathcal{F}$ such that $x \in E$ and $y \notin E$. We say that $\mathcal{F}$ separates subsets of X if given any two disjoint subsets $A, B \subset X$ with $A \neq B$, there exists $E \in \mathcal{F}$ such that $A \subset E$ and $B \cap E = \emptyset$.

LEMMA 3.34. *Let G be a topological group, $H \subset G$ a dense subgroup and $\{E_i\}_{i\geq 1}$ a collection of subsets of G, right-invariant under H. Then $\{E_i\}_{i\geq 1}$ separates the H-cosets if and only if for every $g \in G$ we have that $gH = \cap\{E_i \mid g \in E_i\}$.*

Proof. Note that $g \in E_i \Leftrightarrow gH \subset E_i$ since E_i is right-invariant under H. Assume that for every $g \in G$ we have that $gH = \cap\{E_i \mid g \in E_i\}$ and suppose, for contradiction, that the collection $\{E_i\}_{i\geq 1}$ does not separate the H-cosets. Then there exist $a, b \in G$ such that $aH \neq bH$ and there is no E_l such that $aH \subset E_l$ and $bH \cap E_l = \emptyset$. Thus for every E_l if $aH \subset E_l$ then $bH \subset E_l \Rightarrow bH \subset \cap\{E_i \mid aH \subset E_i\} = \cap\{E_i \mid a \in E_i\} = aH \Rightarrow aH = bH$, a contradiction. Hence, the collection $\{E_i\}_{i\geq 1}$ separates the H-cosets.

Assume now that $\{E_i\}_{i\geq 1}$ separates the H-cosets and let $g \in G$. Since $g \in E_i \Leftrightarrow gH \subset E_i$, we have that $gH \subset \cap\{E_i \mid g \in E_i\}$. Let $x \in \cap\{E_i \mid g \in E_i\}$ and suppose that $x \notin gH$. Then $xH \neq gH$ and there exists E_l such that $gH \subset E_l$ and $xH \cap E_l = \emptyset \Rightarrow g \in E_l$ and $x \notin E_l$, a contradiction to $x \in \cap\{E_i \mid g \in E_i\}$. Thus $x \in gH \Rightarrow \cap\{E_i \mid g \in E_i\} \subset gH$. □

THEOREM 3.35 (D.E.Miller, [24]). *Let G be a Polish topological group, $H \subset G$ a subgroup and $\{E_i\}_{i\geq 1}$ a collection of subsets of G with the Baire property, right-invariant under H, which separates the H-cosets. Then H is closed in G.*

Proof. By replacing G with $\mathrm{cl}_G(H)$ and each E_i with $E_i \cap \mathrm{cl}_G(H)$, then each $E_i \cap \mathrm{cl}_G(H)$ has the Baire property is invariant under H and separate the H-cosets. Thus, we may assume that H is dense in G. It follows from Lemma 3.34 that for every $g \in G$, $gH = \cap\{E_i \mid g \in E_i\}$.

Suppose that H is meager, and let $g \in G$. Then $gH = \cap\{E_i \mid g \in E_i\}$ is meager. From Lemma 3.32 we have that each E_i is either meager or comeager. If each E_i, with $g \in E_i$ is comeager, then $G \setminus E_i$ is meager $\Rightarrow G \setminus gH = G \setminus \cap\{E_i \mid g \in E_i\} = \cup\{G \setminus E_i \mid g \in E_i\}$ is meager $\Rightarrow G = gH \cup (G \setminus gH)$ is meager, a contradiction with G being Polish. Hence there exists a meager E_i such that $g \in E_i$. Since $g \in G$ was arbitrary, this implies that $G \subset \cup\{E_i \mid E_i \text{ is meager }\} \Rightarrow G$ is meager, a contradiction. This implies that H is a nonmeager subset of G.

Since each E_i has the Baire property and the sets with the Baire property are closed under countable intersection and since $H = eH = \cap\{E_i \mid e \in E_i\}$, we have that H has the Baire property. Since it is also nonmeager, it follows from a theorem of Pettis (Theorem 9.9, page 61, [18]) that $H^{-1}H$ contains an open neighborhood of $e \in G$. Let $V \subset H$ be an open neighborhood of $e \in G$ and let $x \in G$. Then xV is an open neighborhood of x and, since H is dense, $xV \cap H \neq \emptyset$. This implies that $x \in HV^{-1} \subset H \Rightarrow G \subset H \Rightarrow H$ is closed. □

Corollary 3.36. ★ *Let G be a Polish topological group, $A \subset G$ an analytic subset and $H \subset G$ an analytic subgroup such that A intersects each H-coset in exactly one point and $G = AH$. Then H is closed in G.*

Proof. Since the topology on G is Polish, the relative topology on A is second countable, and there exist $\{C_i\}_{i\geq 1}$ a separating family of relatively open sets for the topology on A. Each C_i is the intersection of an open subset of G with an analytic subset of G and hence is analytic. Let $E_i = C_iH$ for every $i \geq 1$. Since each E_i is a product of two analytic sets, each E_i is analytic and hence has the Baire property. Since $E_iH = C_iHH = C_iH = E_i$ for every $i \geq 1$, we have that each E_i is right-invariant under H.

Let $a, b \in A$ be such that $aH \neq bH$. Then $a \neq b$, and there exists C_l such that $a \in C_l$ and $b \notin C_l$. We will show that $E_l = C_lH$ is such that $aH \subset E_l$ and $bH \cap E_l = \emptyset$. If $h \in H$, then $ah \in C_lH = E_l \Rightarrow aH \subset E_l$. Suppose that $bH \cap E_l \neq \emptyset$ and let $x \in bH \cap E_l = bH \cap C_lH$. Then there exist $c \in C_l$ and $h, k \in H$ such that $bh = ck \Rightarrow c = bhk^{-1} \in bH$. Since $c \in C_l \subset A \Rightarrow c \in A \cap bH$. Since $b \in A \cap bH$ and since A intersects the H-cosets in exactly one point, we have that $b = c \in C_l$, a contradiction. Hence, $bH \cap E_l = \emptyset$ and therefore $\{E_i\}_{i\geq 1}$ separates the H-cosets.

Since the hypothesis of the Theorem 3.35 is satisfied, it follows that H is closed in G. □

Definition 3.37. Let X be a set and E an equivalence relation on X. A selector for E is a map $s : X \to X$ such that $xEy \Rightarrow s(x) = s(y)$ and $s(y)Ex$. A transversal for E is a set $T \subset X$ that meets every equivalence class in exactly one point.

If X is a Borel subset of a Polish space and E an equivalence relation on X, a Borel selector for E is a selector for E which is also a Borel map and a Borel transversal for E is a transversal for E which is also a Borel subset of X.

LEMMA 3.38. *Let X be a Borel subset of a Polish space and let E be an equivalence relation on X. If $s : X \to X$ is a Borel selector for E, then $T = \{x \in X \mid x = s(x)\}$ is a Borel transversal for E.*

Proof. Let A be an equivalence class for E. Then $A \neq \emptyset$ and let $x \in A$. Since xEx we have that $s(x)Ex$ and $s(x) \in A \Rightarrow s(s(x)) = s(x) \Rightarrow s(x) \in T \Rightarrow s(x) \in A \cap T \Rightarrow A \cap T \neq \emptyset$. Let $x, y \in T \cap A$. Since $x, y \in A$ we have that $xEy \Rightarrow s(x) = s(y)$ and since $x, y \in T$ we have that $x = s(x)$ and $y = s(y)$. Thus $x = y$ and hence T is a transversal for E. It remains to show that T is a Borel subset of X.

Let $\phi : X \to X \times X$ be defined as $\phi(x) = (x, s(x))$. If $x \neq y \in X$ then $\phi(x) = (x, s(x)) \neq (y, s(y)) = \phi(y) \Rightarrow \phi$ is one-to-one. Let $A \subset X$ and $B \subset X$ be Borel subsets. Then $\phi^{-1}(A \times B) = \{x \in X \mid \phi(x) \in A \times B\} = \{x \in X \mid (x, s(x)) \in A \times B\} = \{x \in X \mid x \in A \text{ and } s(x) \in B\} = \{x \in X \mid x \in A \text{ and } x \in s^{-1}(B)\} = A \cap s^{-1}(B)$ is a Borel set, since A, B are Borel and s is a Borel map. This implies that ϕ is a Borel map. Using a well-known Theorem of Souslin (Corollary 15.2, page 89, [18]) we have that $\phi(X)$ is Borel. Let $\Delta = \{(x, x) \mid x \in X\}$ the diagonal of $X \times X$ and let $P : \Delta \to X$, $P(x, x) = x$ be the natural projection on the first coordinate. Then Δ is closed in $X \times X$ and since $\phi(X)$ is Borel, we have that $\phi(X) \cap \Delta$ is Borel. If $(x, x) \neq (y, y) \in \Delta$ then $P(x, x) = x \neq y = P(y, y)$ and hence P is one-to-one. If $(x_j, x_j) \to (x, x)$ then $P(x_j, x_j) = x_j \to x = P(x, x)$ and hence P is continuous. Using Souslin's Theorem again, we have that $P(\phi(X) \cap \Delta)$ is a Borel subset of X. But $P(\phi(X) \cap \Delta) = P(\{(x, s(x)) \mid x \in X\} \cap \{(x, x) \mid x \in X\}) = P(\{(x, s(x)) \mid x = s(x)\}) = \{x \mid x = s(x)\} = T$, and hence T is Borel. $\square$

COROLLARY 3.39. ★ *Let G be a Polish topological group, $A \subset G$ a closed subgroup and $H \subset G$ an analytic subgroup such that $A \cap H = C$ is closed in G and $G = AH$. Then H is closed in G.*

Proof. Since A is a closed subgroup of G, A is a Polish topological group. Since C is a closed subgroup of G and hence of A and using Theorem 12.17, page 78, [18], we have that there exists a Borel selector $s : A \to A$ for the equivalence relation whose classes are the C-cosets in A. Let $T = \{a \in A \mid s(a) = a\}$. By Lemma 3.38 we have that T intersects each C-coset in exactly one point and T is a Borel subset of A, thus an analytic subset of G. We will prove that $G = TH$ and that T intersects each H-coset in exactly one point. The conclusion will follow from Corollary 3.36.

Suppose for contradiction that there exists an H-coset aH such that $T \cap aH = \{x, y\}$ and $x \neq y$. Since $x, y \in aH$, we have that $y^{-1}x \in H$ and since $x, y \in T \subset A$ we have that $y^{-1}x \in A \Rightarrow y^{-1}x \in A \cap H = C \Rightarrow x$ and y belong to the same C-coset. But then T intersects a C-coset in two different points, a contradiction.

Let $g \in G = AH$. Then $g = ah$ with $a \in A$ and $h \in H$. Denote with E_C the equivalence relation whose classes are the C-cosets. Since $aE_Ca \Rightarrow s(a)E_Ca \Rightarrow a \in s(a)C \Rightarrow$ there exists $c \in C$ such that $a = s(a)c \Rightarrow g = s(a)ch$. Since $s(a)E_Ca$ we have that $s(s(a)) = s(a) \Rightarrow s(a) \in T$. Since $c \in C \subset H$ and $h \in H$ we have that $ch \in H$ and hence $g = s(a)ch \in TH \Rightarrow G \subset TH$. $\square$

COROLLARY 3.40. ★ *Let G be a Polish topological group, $\mathcal{M} \subset \mathcal{H}$ an infinite dimensional closed subspace of the Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G.*

Proof. If $\mathcal{M} = \mathcal{H}$ then $\mathcal{U}(\mathcal{M}) = \mathcal{U}(\mathcal{H}) \Rightarrow \phi^{-1}(\mathcal{U}(\mathcal{M})) = G$ is closed in G. Suppose $\mathcal{M} \neq \mathcal{H}$. By Theorem 3.28 we have that $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M})) = \phi^{-1}(Z(\mathcal{U}(\mathcal{H})))\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G and hence Polish. Since ϕ is an isomorphism we have that $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))) = Z(G)$, the center of G, is a closed subgroup of G and $\phi^{-1}(\mathcal{U}(\mathcal{M})) \subset G$ is analytic by Proposition 3.29. If $U \in Z(\mathcal{U}(\mathcal{H})) \cap \mathcal{U}(\mathcal{M})$, then $U = \lambda I$, with $|\lambda| = 1$, and, since $U|_{\mathcal{M}^\perp} = I$, we have that $\lambda = 1 \Rightarrow U = I \Rightarrow Z(\mathcal{U}(\mathcal{H})) \cap \mathcal{U}(\mathcal{M}) = \{I\} \Rightarrow \phi^{-1}(Z(\mathcal{U}(\mathcal{H}))) \cap \phi^{-1}(\mathcal{U}(\mathcal{M})) = \phi^{-1}(Z(\mathcal{U}(\mathcal{H})) \cap \mathcal{U}(\mathcal{M})) = \phi^{-1}(I) = \{e\}$ is closed in G. Using Corollary 3.39 we have that

$\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))$ and since $\phi^{-1}(Z(\mathcal{U}(\mathcal{H}))\mathcal{U}(\mathcal{M}))$ is closed in G it follows that $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G. $\square$

Corollary 3.41. ★ *Let G be a Polish topological group, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace of the infinite dimensional Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be a orthonormal basis for $\mathcal{M}$. Extend this to $\{e_1, ..., e_n, ..., e_{n+l}, ...\}$ an orthonormal basis for $\mathcal{H}$. For every $l \geq 1$, let $\mathcal{M}_l = span(\{e_i\}_{i\geq 1} \setminus \{e_{n+l}\})$. Each $\mathcal{M}_l$ is infinite dimensional. Hence, by Corollary 3.40, we have that $\phi^{-1}(\mathcal{U}(\mathcal{M}_l))$ is closed in G, for every $l \geq 1$.

Since $U \in \mathcal{U}(\mathcal{M}) \Leftrightarrow U|_{\mathcal{M}^\perp} = I \Leftrightarrow Ue_{n+l} = e_{n+l}$ for every $l \geq 1 \Leftrightarrow U \in \mathcal{U}(\mathcal{M}_l)$ for every $l \geq 1 \Leftrightarrow U \in \cap_{l\geq 1}\mathcal{U}(\mathcal{M}_l)$ we have that $\mathcal{U}(\mathcal{M}) = \cap_{l\geq 1}\mathcal{U}(\mathcal{M}_l) \Rightarrow \phi^{-1}(\mathcal{U}(\mathcal{M})) = \phi^{-1}(\cap_{l\geq 1}\mathcal{U}(\mathcal{M}_l)) = \cap_{l\geq 1}\phi^{-1}(\mathcal{U}(\mathcal{M}_l)) \Rightarrow \phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G. $\square$

Corollary 3.42. ★ *Let G be a Polish topological group, $\mathcal{M} \subset \mathcal{H}$ a closed subspace of the infinite dimensional Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G.*

Proof. Put together Corollary 3.40 and Corollary 3.41. $\square$

3.5. $\phi^{-1}(SU(\mathcal{M}))$ is Closed

Lemma 3.43.

$$If\ U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ then\ U \in SU(2)\ and\ U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^* = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Proof. Note that

$$U^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and then by a straight forward computation we have that $UU^* = U^*U = I$ and $\det(U) = 1$ and hence $U \in SU(2)$.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_2 \\ -\lambda_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

$\square$

LEMMA 3.44. *Let $\mathcal{M}$ be a finite dimensional Hilbert space with* $\dim(\mathcal{M}) = n$ *and let P, Q be two operators acting on $\mathcal{M}$. If*

$$P_k = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & 0 & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & 1 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \textit{ and } Q = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_k & & & \\ & & & \lambda_{k+1} & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$$

are the matrix representations of P_k, respective Q with respect to some basis in $\mathcal{M}$, then $P_k \in SU(\mathcal{M})$ and

$$P_k Q P_k^* = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_{k+1} & & & \\ & & & \lambda_k & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}$$

Proof. Note that P_k restricted to the appropiate two dimensional subspace equals the matrix U from Lemma 3.43 and outside that subspace is the identity. Lemma 3.43 implies that $P_k Q P_k^*$ is obtained from Q by interchanging the two entries of the diagonal λ_k and λ_{k+1}.

Straight forward computation shows that $P_kP_k^* = P_k^*P_k = I$ and that $\det(P_k) = 1$, and hence $P_k \in SU(\mathcal{M})$. $\square$

Lemma 3.45. *Let $\mathcal{M}$ be a finite dimensional Hilbert space and let $U \in SU(\mathcal{M})$. Then there exist $P, Q \in SU(\mathcal{M})$ such that $U = PQP^*Q^*$.*

Proof. This is a consequence of the main theorem in [5]. Here is a simple, direct proof.

If $U \in SU(\mathcal{M})$, then by the Spectral Theorem U is diagonalizable and U can be represented as

$$U = \begin{pmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_n} \end{pmatrix}$$

where $\alpha_1 + \alpha_2 + ... + \alpha_n = 0$.

Let $P = P_1P_2...P_{n-1}$, where P_k is defined in Lemma 3.44. Note that $P \in SU(\mathcal{M})$ and $P^* = P_{n-1}^*...P_1^*$. Let Q be defined as

$$Q = \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix}$$

where $\theta_n = \frac{(n-1)\alpha_1+(n-2)\alpha_2+...+\alpha_{n-1}}{n}$ and $\theta_l = \theta_n - (\alpha_1 + \alpha_2 + ... + \alpha_l)$ for every $1 \leq l \leq n-1$. Then $\theta_1 + ... + \theta_n = \theta_n - \alpha_1 + \theta_n - (\alpha_1 + \alpha_2) + ... + \theta_n - (\alpha_1 + \alpha_2 + ... + \alpha_{n-1}) + \theta_n = n\theta_n - (n-1)\alpha_1 - (n-2)\alpha_2 - ... - \alpha_{n-1} = 0 \Rightarrow \det(Q) = 1 \Rightarrow Q \in SU(\mathcal{M})$. Note that $\theta_n - \theta_1 = \theta_n - \theta_n + \alpha_1 = \alpha_1$ and $\theta_l - \theta_{l+1} = \theta_n - (\alpha_1 + ... + \alpha_l) - \theta_n + (\alpha_1 + ... + \alpha_{l+1}) = \alpha_{l+1}$. Using Lemma 3.44 we have that

$$PQP^* = \begin{pmatrix} e^{i\theta_n} & & & 0 \\ & e^{i\theta_1} & & \\ & & \ddots & \\ 0 & & & e^{i\theta_{n-1}} \end{pmatrix} \text{ and since } Q^* = \begin{pmatrix} e^{-i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{-i\theta_n} \end{pmatrix}$$

$$\Rightarrow PQP^*Q^* = \begin{pmatrix} e^{i(\theta_n-\theta_1)} & & & 0 \\ & e^{i(\theta_1-\theta_2)} & & \\ & & \ddots & \\ 0 & & & e^{i(\theta_{n-1}-\theta_n)} \end{pmatrix} = \begin{pmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_n} \end{pmatrix} = U$$

□

PROPOSITION 3.46. ★ *Let G be a Polish topological space, $\mathcal{H}$ infinite dimensional Hilbert space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(SU(\mathcal{M}))$ is an analytic subset of G.*

Proof. Since $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G by Corollary 3.42, $\phi^{-1}(\mathcal{U}(\mathcal{M})) \times \phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in $G \times G$. Let $[\cdot,\cdot] : \phi^{-1}(\mathcal{U}(\mathcal{M})) \times \phi^{-1}(\mathcal{U}(\mathcal{M})) \to G$ be defined as $[a,b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot,\cdot]$ is continuous. If $a, b \in \phi^{-1}(\mathcal{U}(\mathcal{M}))$ then $\phi(a), \phi(b) \in \mathcal{U}(\mathcal{M})$, $\phi([a,b]) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)(\phi(a))^{-1}(\phi(b))^{-1} \in \mathcal{U}(\mathcal{M})$ and $\det(\phi([a,b])) = \det(\phi(aba^{-1}b^{-1})) = \det(\phi(a))\det(\phi(b))(\det(\phi(a)))^{-1}(\det(\phi(b)))^{-1} = 1 \Rightarrow \phi([a,b]) \in SU(\mathcal{M}) \Rightarrow [a,b] \in \phi^{-1}(SU(\mathcal{M}))$. This proves that $[\cdot,\cdot]$ takes its values in $\phi^{-1}(SU(\mathcal{M}))$. Let $y \in \phi^{-1}(SU(\mathcal{M}))$. Then $\phi(y) = W \in SU(\mathcal{M})$. By Lemma 3.45 we have that there exist $U, V \in SU(\mathcal{M})$ such that $W = UVU^{-1}V^{-1}$. Let $a = \phi^{-1}(U) \in \phi^{-1}(\mathcal{U}(\mathcal{M}))$ and $b = \phi^{-1}(V) \in \phi^{-1}(\mathcal{U}(\mathcal{M}))$. Then $y = \phi^{-1}(W) = \phi^{-1}(UVU^{-1}V^{-1}) = \phi^{-1}(U)\phi^{-1}(V)(\phi^{-1}(U))^{-1}(\phi^{-1}(V))^{-1} = aba^{-1}b^{-1} = [a,b] \Rightarrow [\cdot,\cdot]$ is onto $\phi^{-1}(SU(\mathcal{M}))$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(SU(\mathcal{M}))$ is the continuous image of $\phi^{-1}(\mathcal{U}(\mathcal{M})) \times \phi^{-1}(\mathcal{U}(\mathcal{M}))$, a closed set of a Polish space, and therefore $\phi^{-1}(SU(\mathcal{M}))$ is an analytic subset of G. □

PROPOSITION 3.47. *If $\mathcal{M}$ is a finite dimensional Hilbert space, then $\mathcal{U}(\mathcal{M}) = Z(\mathcal{U}(\mathcal{M})) \cdot SU(\mathcal{M})$.*

Proof. Since both $Z(\mathcal{U}(\mathcal{M})), SU(\mathcal{M}) \subset \mathcal{U}(\mathcal{M})$ and since $\mathcal{U}(\mathcal{M})$ is a subgroup it follows that $Z(\mathcal{U}(\mathcal{M})) \cdot SU(\mathcal{M}) \subset \mathcal{U}(\mathcal{M})$.

Let $U \in \mathcal{U}(\mathcal{M})$ and let $\det(U) = \det(U|_{\mathcal{M}}) = \lambda$. Then $1 = \det(I) = \det(UU^*) = \det(U)\det(U^*) = \det(U)\overline{\det(U)} = \lambda\overline{\lambda} = |\lambda|^2 \Rightarrow |\lambda| = 1$. Choose θ such that $e^{in\theta} = \lambda$, where $n = \dim(\mathcal{M})$. Let V be defined as $V|_{\mathcal{M}} = e^{i\theta}I$, $V|_{\mathcal{M}^\perp} = I$ and W be defined as $W|_{\mathcal{M}} = e^{-i\theta}U|_{\mathcal{M}}$, $W|_{\mathcal{M}^\perp} = I$. Then $V \in Z(\mathcal{U}(\mathcal{M}))$ and, since $\det(W) = \det(e^{-i\theta}U|_{\mathcal{M}}) = (e^{-i\theta})^n \det(U|_{\mathcal{M}}) = \lambda^{-1}\lambda = 1$, we have that $W \in SU(\mathcal{M})$. Since $U|_{\mathcal{M}} = (e^{i\theta}I)(e^{-i\theta}U|_{\mathcal{M}}) = V|_{\mathcal{M}}W|_{\mathcal{M}}$ and since $U|_{\mathcal{M}^\perp} = I = V|_{\mathcal{M}^\perp}W|_{\mathcal{M}^\perp}$ we have that $U = VW \in Z(\mathcal{U}(\mathcal{M})) \cdot SU(\mathcal{M}) \Rightarrow \mathcal{U}(\mathcal{M}) \subset Z(\mathcal{U}(\mathcal{M})) \cdot SU(\mathcal{M})$. □

Corollary 3.48. ★ *Let G be a Polish topological space, $\mathcal{H}$ infinite dimensional Hilbert space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{U}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(SU(\mathcal{M}))$ is closed in G.*

Proof. From Corollary 3.42 we have that $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is closed in G and hence Polish. From Proposition 3.47 we have that $Z(\mathcal{U}(\mathcal{M}))SU(\mathcal{M}) = \mathcal{U}(\mathcal{M}) \Rightarrow \phi^{-1}(Z(\mathcal{U}(\mathcal{M})))\phi^{-1}(SU(\mathcal{M})) = \phi^{-1}(Z(\mathcal{U}(\mathcal{M}))SU(\mathcal{M})) = \phi^{-1}(\mathcal{U}(\mathcal{M}))$. $\phi^{-1}(Z(\mathcal{U}(\mathcal{M}))) = Z(\phi^{-1}(\mathcal{U}(\mathcal{M})))$, the center of $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ is a closed subgroup of $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ and $\phi^{-1}(SU(\mathcal{M}))$ is an analytic subgroup of G by Proposition 3.46, and hence analytic subgroup of $\phi^{-1}(\mathcal{U}(\mathcal{M}))$. Let $C = Z(\mathcal{U}(\mathcal{M})) \cap SU(\mathcal{M})$. Then $C = \{U \in \mathcal{U}(\mathcal{M}) \mid U|_{\mathcal{M}} = \lambda I,\ U|_{\mathcal{M}^\perp} = I \text{ and } \det(U) = \lambda^n = 1\}$, where $n = \dim(\mathcal{M}) \Rightarrow C$ is finite. Since ϕ is an isomorphism we have that $\phi^{-1}(Z(\mathcal{U}(\mathcal{M}))) \cap \phi^{-1}(SU(\mathcal{M})) = \phi^{-1}(C)$ is finite and hence closed in $\phi^{-1}(\mathcal{U}(\mathcal{M}))$. It follows from Corollary 3.39 that $\phi^{-1}(SU(\mathcal{M}))$ is closed in $\phi^{-1}(\mathcal{U}(\mathcal{M}))$ and hence closed in G. □

3.6. Main Result

Lemma 3.49. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let $\{e_l\}_{l\geq 1} \subset \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ and let P be the orthogonal projection on $span(\{e_1\})$. Then there exists $\mathcal{M}$ a three dimensional subspace of $\mathcal{H}$ such that for every $U \in \mathcal{U}(\mathcal{H})$ there exists $U_0 \in SU(\mathcal{M})$ such that $PU_0e_1 = PUe_1$.*

Proof. Let $\mathcal{M} = span(\{e_1, e_2, e_3\})$ be a three dimensional subspace of $\mathcal{H}$. Note that since P is the orthogonal projection on $span(\{e_1\})$, then $PUe_1 = \lambda e_1$ and since $|\lambda|^2 = |\lambda|^2\|e_1\|^2 = \|\lambda e_1\|^2 = \|PUe_1\|^2 \leq \|PUe_1\|^2 + \|(I-P)Ue_1\|^2 = \|Ue_1\|^2 = \|e_1\|^2 = 1$ we have that $|\lambda| \leq 1$.

If $|\lambda| = 0$ let

$$U_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the matrix representation of U_0 with respect to the basis $\{e_1, e_2, e_3\}$. Then

$$U_0^* = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by a straight forward computation we have that $U_0 U_0^* = U_0^* U_0 = I$ and $\det(U_0) = 1$ and hence $U_0 \in SU(\mathcal{M})$. Note that $U_0 e_1 = e_2$ and hence $PU_0 e_1 = 0 = \lambda e_1 = PUe_1$.

If $|\lambda| \neq 0$ let

$$U_0 = \begin{pmatrix} \lambda & -\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda & 0 \\ \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda & \lambda & 0 \\ 0 & 0 & |\lambda|^2\lambda^{-2} \end{pmatrix}$$

Then we have that

$$U_0^* = \begin{pmatrix} \overline{\lambda} & \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\overline{\lambda} & 0 \\ -\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\overline{\lambda} & \overline{\lambda} & 0 \\ 0 & 0 & |\lambda|^2\overline{\lambda}^{-2} \end{pmatrix}$$

and hence

$$U_0 U_0^* = \begin{pmatrix} \lambda & -\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda & 0 \\ \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda & \lambda & 0 \\ 0 & 0 & |\lambda|^2\lambda^{-2} \end{pmatrix} \begin{pmatrix} \overline{\lambda} & \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\overline{\lambda} & 0 \\ -\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\overline{\lambda} & \overline{\lambda} & 0 \\ 0 & 0 & |\lambda|^2\overline{\lambda}^{-2} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda\overline{\lambda} + \frac{1-|\lambda|^2}{|\lambda|^2}\lambda\overline{\lambda} & \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\overline{\lambda} - \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\overline{\lambda} & 0 \\ \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\overline{\lambda} - \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\overline{\lambda} & \frac{1-|\lambda|^2}{|\lambda|^2}\lambda\overline{\lambda} + \lambda\overline{\lambda} & 0 \\ 0 & 0 & |\lambda|^4(\lambda\overline{\lambda})^{-2} \end{pmatrix} =$$

$$= \begin{pmatrix} |\lambda|^2 + \frac{1-|\lambda|^2}{|\lambda|^2}|\lambda|^2 & & 0 \\ & \frac{1-|\lambda|^2}{|\lambda|^2}|\lambda|^2 + |\lambda|^2 & 0 \\ 0 & 0 & |\lambda|^4(|\lambda|^2)^{-2} \end{pmatrix} = I$$

and similarly $U_0^* U_0 = I$. We also have that $\det(U_0) = |\lambda|^2 - \left(\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\right)\left(-\frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda\right)|\lambda|^2\lambda^{-2} = |\lambda|^2 + (1-|\lambda|^2) = 1$ and hence $U_0 \in SU(\mathcal{M})$.

Since $U_0 e_1 = \lambda e_1 + \frac{\sqrt{1-|\lambda|^2}}{|\lambda|}\lambda e_2$ it follows that $PU_0 e_1 = \lambda e_1 = PUe_1$. $\square$

Lemma 3.50. *Let $\mathcal{H}$ be a Hilbert space, let $e \in \mathcal{H}$, let P be the orthogonal projection on span$(\{e\})$ and $Q = I - P$. If $W \in \mathcal{U}(\{e\}^\perp)$ then W commutes with P and with Q.*

Proof. Let $x \in \mathcal{H}$. Since $Px \in span(\{e\}$ and $W|_{span(\{e\})} = I$ we have that $WPx = Px$. Since $Qx \in \{e\}^\perp$ and $W|_{span(\{e\})} = I$ we have that $WQx \in \{e\}^\perp \Rightarrow PWQx = 0$. It follows that $PWx = PW(Px + Qx) = PWPx + PWQx = P^2x + 0 = Px = WPx$.

On the other hand we have that $WQx = W(x - Px) = Wx - WPx = PWx + QWx - WPx = QWx$. $\square$

Lemma 3.51. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let $e \in \mathcal{H}$ be such that $\|e\| = 1$ and let $\mathcal{S} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|e - Ue\| < \epsilon\}$. Then there exists $\mathcal{M} \subset \mathcal{H}$ a three dimensional subspace such that $\mathcal{S} = \mathcal{U}(\{e\}^\perp)\,[SU(\mathcal{M}) \cap \mathcal{S}]\,\mathcal{U}(\{e\}^\perp)$.*

Proof. Note that if $W \in \mathcal{U}(\{e\}^\perp)$ and if $U \in \mathcal{S}$ then $\|e - UWe\| = \|e - Ue\| < \epsilon \Rightarrow UW \in \mathcal{S} \Rightarrow \mathcal{S}\,\mathcal{U}(\{e\}^\perp) \subset \mathcal{S} \Rightarrow \mathcal{S}\,\mathcal{U}(\{e\}^\perp) = \mathcal{S}$ and $\|e - WUe\| = \|We - WUe\| = \|W(e - Ue)\| = \|e - Ue\| < \epsilon \Rightarrow WU \in \mathcal{S} \Rightarrow \mathcal{U}(\{e\}^\perp)\,\mathcal{S} \subset \mathcal{S} \Rightarrow \mathcal{U}(\{e\}^\perp)\,\mathcal{S} = \mathcal{S}$ and hence $\mathcal{U}(\{e\}^\perp)\,\mathcal{S}\,\mathcal{U}(\{e\}^\perp) = \mathcal{S}$.

Let $U \in \mathcal{S}$. Let P be the orthogonal projection on $span(\{e\})$ and let $Q = I - P$. By Lemma 3.49 we have that there exists $\mathcal{M}$ a three dimensional subspace and $U_0 \in SU(\mathcal{M})$ such that $PU_0e = PUe$. Since $\|PUe\|^2 + \|QUe\|^2 = \|Ue\|^2 = 1 = \|U_0e\|^2 = \|PU_0e\|^2 + \|QU_0e\|^2$ we have that $\|QUe\|^2 = \|QU_0e\|^2$. Since $QUe \in \{e\}^\perp$ and $QU_0e \in \{e\}^\perp$ there exists $W \in \mathcal{U}(\{e\}^\perp)$ such that $WQU_0e = QUe$. Since by Lemma 3.50 W commutes with P and with Q we have that $WU_0e = PWU_0e + QWU_0e = WPU_0e + WQU_0e = PU_0e + QUe =$

$PUe + QUe = Ue \Rightarrow U_0^* W^* Ue = e \Rightarrow U_0^* W^* U = V \in \mathcal{U}(\{e\}^\perp) \Rightarrow U = WU_0V$. We also have that $\|e - U_0 e\|^2 = \|e - PU_0 e\|^2 + \|QU_0 e\|^2 = \|e - PU_0 e\|^2 + \|WQU_0 e\|^2 = \|e - PUe\|^2 + \|QUe\|^2 = \|P(e - Ue)\|^2 + \|Q(e - Ue)\|^2 = \|e - Ue\|^2 < \epsilon^2 \Rightarrow U_0 \in \mathcal{S}$. Thus $U = WU_0V$, with $W, V \in \mathcal{U}(\{e\}^\perp)$ and $U_0 \in SU(\mathcal{M}) \cap \mathcal{S}$. This implies that $\mathcal{S} \subset \mathcal{U}(\{e\}^\perp)\ [SU(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{U}(\{e\}^\perp) \subset \mathcal{U}(\{e\}^\perp)\ \mathcal{S}\ \mathcal{U}(\{e\}^\perp) = \mathcal{S} \Rightarrow \mathcal{S} = \mathcal{U}(\{e\}^\perp)\ [SU(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{U}(\{e\}^\perp)$. □

Lemma 3.52. *The intersection of two analytic subsets of a Polish space is analytic.*

Proof. Let X be a Polish space and let $A_1, A_2 \subset X$ be analytic. Then there exist B_l Borel sets and $f_l : B_l \to A_l$ Borel mappings such that $f_l(B_l) = A_l$, for $l = 1, 2$. Let $F : B_1 \times B_2 \to X \times X$ be defined as $F(b_1, b_2) = (f_1(b_1), f_2(b_2))$. Then F is obviously a Borel mapping and hence if $D = \{(x, x) \mid x \in X\} \subset X \times X$ is the diagonal, then $F^{-1}(D) = \{(b_1, b_2) \mid b_l \in B_l,\ f_1(b_1) = f_2(b_2)\} \subset B_1 \times B_2$ is a Borel subset.

Let $y \in A_1 \cap A_2$. Then there exist $b_l \in B_l$ such that $y = f_l(b_l)$, for $l = 1, 2$ and $(b_1, b_2) \in F^{-1}(D)$. The mapping $\pi_1 \circ F : B_1 \times B_2 \to X$ is the composition between a continuous and a Borel mapping, and hence a Borel mapping and $(\pi_1 \circ F)(b_1, b_2) = y$. Hence $A_1 \cap A_2$ is the Borel image of the Borel subset $F^{-1}(D)$, and hence an analytic subset. □

Lemma 3.53. *The product of two analytic subsets of a Polish space is analytic.*

Proof. Let X be a Polish space and let $A_1, A_2 \subset X$ be analytic. Then there exist B_l Borel sets and $f_l : B_l \to A_l$ Borel mappings such that $f_l(B_l) = A_l$, for $l = 1, 2$. Let $F : B_1 \times B_2 \to X$ be defined as $F(b_1, b_2) = f_1(b_1) f_2(b_2)$. Since the multiplication is continuous, F is a composition between a continuous mapping and a Borel mapping and hence a Borel mapping. Since $B_1 \times B_2$ is Borel, it follows that $A_1 A_2 = F(B_1 \times B_2)$ is analytic. □

Lemma 3.54. ★ *Let G be a Polish topological group, let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $e \in \mathcal{H}$ be such that $\|e\| = 1$. Let $\mathcal{S} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|e - Ue\| < \epsilon\}$ and let $\phi : G \to \mathcal{U}(\mathcal{H})$ be an algebraic isomorphism. Then $\phi^{-1}(\mathcal{S})$ is analytic in G.*

Proof. Let $\mathcal{M}$ be as in Lemma 3.51 so that $\mathcal{S} = \mathcal{U}(\{e\}^\perp)\ [SU(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{U}(\{e\}^\perp)$. Since $SU(\mathcal{M})$ is a connected compact metric group with a totally disconnected center (Chapter I, Section

14, [19]), using the result from [14] we have that $\phi|_{\phi^{-1}(SU(\mathcal{M}))} : \phi^{-1}(SU(\mathcal{M})) \to SU(\mathcal{M})$ is a homeomorphism. $\mathcal{S} \cap SU(\mathcal{M})$ is a relatively open subset of $SU(\mathcal{M}) \Rightarrow \phi^{-1}(\mathcal{S} \cap SU(\mathcal{M}))$ is relatively open in $\phi^{-1}(SU(\mathcal{M}))$. Since $\phi^{-1}(SU(\mathcal{M}))$ is closed in G by Corollary 3.48, we have that $\phi^{-1}(\mathcal{S} \cap SU(\mathcal{M}))$ is a Borel subset of G. Since $\phi^{-1}(\mathcal{U}(\{e\}^{\perp}))$ is closed in G by Corollary 3.42, it follows from Lemma 3.53 that $\phi^{-1}(\mathcal{S}) = \phi^{-1}(\mathcal{U}(\{e\}^{\perp})\ [\mathcal{S} \cap SU(\mathcal{M})]\ \mathcal{U}(\{e\}^{\perp})) = \phi^{-1}(\mathcal{U}(\{e\}^{\perp}))\phi^{-1}(\mathcal{S} \cap SU(\mathcal{M}))\phi^{-1}(\mathcal{U}(\{e\}^{\perp}))$ is analytic. □

LEMMA 3.55. *The union of a sequence of analytic subsets of a Polish topological space is analytic.*

Proof. Let Y be a Polish topological space and let $\{A_l\}_{l \geq 1}$ be a sequence of analytic subsets of Y. Then there exist B_l Borel sets and $f_l : B_l \to A_l$ Borel mappings such that $f_l(B_l) = A_l$, for every $l \geq 1$. Without loss of generality we may assume that the B_l's are Borel subsets of the same Polish topological space X. Let $F : \mathbb{N} \times X \to Y$ be defined as $F((n, x)) = f_n(x)$. If we define $D : (\mathbb{N} \times X) \times (\mathbb{N} \times X) \to \mathbb{R}$ by $D((n, x), (n, y)) = d(x, y)$ and $D((n, x), (m, y)) = 1$ if $n \neq m$, then D is a complete metric on $\mathbb{N} \times X$ and hence $\mathbb{N} \times X$ becomes a Polish topological group. The mapping F is Borel, $\cup_{l \geq 1}\{l\} \times B_l$ is a Borel subset of $\mathbb{N} \times X$ and hence $\cup_{l \geq 1} A_l = F(\mathbb{N} \times \cup_{l \geq 1} B_l = F(\cup_{l \geq 1}\{l\} \times B_l)$ is analytic. □

LEMMA 3.56. *A translate of an analytic subset of a Polish topological group is analytic.*

Proof. Let X be a Polish topological group, let $x \in X$ and let $A \subset X$ be an analytic subset. Then there there exists B a Borel set and $f : B \to A$ a Borel mapping such that $f(B) = A$. Let $F : X \times B \to X$ be defined as $F((x, y)) = xf(y)$. Then $\{x\} \times B$ is a Borel set and since the multiplication is continuous, the mapping F is Borel. Hence $xA = F(\{x\} \times B)$ is analytic. □

LEMMA 3.57. *Let G and H be two Polish topological groups and let $\phi : G \to H$ be an algebraic isomorphism. If $\phi^{-1}(U)$ is a set with the Baire property for every U in a neighborhood basis $\mathcal{U}$ at e in H, then ϕ is a topological isomorphism.*

Proof. Let $U \subset H$ be open. Then $U = \cup_{n\geq 1} x_n V_n$, where $x_n \in U$ and $V_n \in \mathcal{U}$. Then $\phi^{-1}(x_n V_n) = \phi^{-1}(x_n)\phi^{-1}(V_n)$ is a set with the Baire property for every $n \geq 1 \Rightarrow \phi^{-1}(U) = \cup_{n\geq 1}\phi^{-1}(x_n V_n)$ is a set with the Baire property $\Rightarrow \phi$ is measurable with respect to the sets with the Baire property.

Since G is Baire and H is separable, it follows from a well-known theorem of Banach, Kuratowski and Pettis (Theorem 9.10, page 61, [18]) that ϕ is continuous. From Lusin-Souslin Theorem (page 89, [18]) we have that ϕ^{-1} is Borel measurable, and hence it is measurable with respect to the sets with the Baire property. From the same result of Banach-Kuratowski-Pettis it follows that ϕ^{-1} is continuous and hence ϕ is a topological isomorphism. □

THEOREM 3.58. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{U}(\mathcal{H})$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. Let $\{e_l\}_{l\geq 1}$ be an orthonormal basis for $\mathcal{H}$. Let $\mathcal{U}$ be a basic neighborhood of I in $\mathcal{U}(\mathcal{H})$. According with Proposition 3.11 $\mathcal{U}$ is of the form $\mathcal{U} = \cap_{1\leq l\leq n}\{U \in \mathcal{U}(\mathcal{H}) \mid \|Ue_l - e_l\| < \epsilon\}$ for some $\epsilon > 0$. $\phi^{-1}(\mathcal{U})$ is analytic by Lemma 3.54 and, since analytic sets have the Baire property, $\phi^{-1}(\mathcal{U})$ is a set with the Baire property. The conclusion follows from Lemma 3.57.

□

3.7. The Finite Dimensional Case

LEMMA 3.59. *Let G be a group, $A, B \subset G$ two subgroups such that $G = AB$ and $ab = ba$ for every $a \in A$ and $b \in B$. If $C = \{(c, c^{-1}) \mid c \in A \cap B\}$, then $(A \times B)/C$ is isomorphic to G.*

Proof. Let $\phi : A \times B \to G$ be defined as $\phi((a, b)) = ab$. Since $\phi(a_1, b_1)\phi(a_2, b_2) = a_1b_1a_2b_2 = a_1a_2b_1b_2 = \phi(a_1a_2, b_1b_2)$ we have that ϕ is a homomorphism. If $g \in G$ then $g = ab$, with $a \in A$ and $b \in B$ and $\phi(a, b) = g \Rightarrow \phi$ is onto G. Since $\ker(\phi) = \{(a, b) \mid \phi((a, b)) = e\} = \{(a, b) \mid ab = e\} = \{(a, b) \mid b = a^{-1} \in A \cap B\} = \{(a, a^{-1}) \mid a \in A \cap B\} = C$, it follows from the Isomorphism Theorem for groups that $(A \times B)/C$ is isomorphic to G. □

Lemma 3.60. *If A, B are two abstract groups, H is a normal subgroup of A and K is a normal subgroup of B then $H \times K$ is a normal subgroup of $A \times B$ and $(A \times B)/(H \times K) \simeq (A/H) \times (B/K)$.*

Proof. If $(a, b) \in A \times B$ and $(h, k) \in H \times K$ then $(a, b)(h, k)(a, b)^{-1} = (aha^{-1}, bkb^{-1}) \in H \times K$, we have that $H \times K$ is a normal subgroup of $A \times B$.

Let $\pi : A \times B \to (A/H) \times (B/K)$ be defined as $\pi(a, b) = (\pi_1(a), \pi_2(b))$, where π_1, π_2 are the natural quotient mappins $\pi_1 : A \to A/H$ and $\pi_2 : B \to B/K$. Since $\pi(a_1, b_1)\pi(a_2, b_2) = (\pi_1(a_1), \pi_2(b_1))(\pi_1(a_2), \pi_2(b_2)) = (\pi_1(a_1)\pi_1(a_2), \pi_2(b_1)\pi_2(b_2)) = (\pi_1(a_1a_2), \pi_2(b_1b_2)) = \pi(a_1a_2, b_1b_2)$ we have that π is a homomorphism. π is obviously onto since π_1 and π_2 are onto. Since $\pi(a, b) = (e, e) \in (A/H) \times (B/K) \Leftrightarrow \pi_1(a) = e \in A/H$ and $\pi_2(b) = e \in B/K \Leftrightarrow a \in H$ and $b \in K \Leftrightarrow \ker(\pi) = H \times K$ we have that $(A \times B)/(H \times K) \simeq (A/H) \times (B/K)$. $\square$

Lemma 3.61. *Let G be a group, let A, B be two subgroups such that $G = AB$, $A \cap B = \{e\}$ and $ab = ba$ for every $a \in A$ and $b \in B$. If N is a normal subgroup of B then N is a normal subgroup of G and $G/N \simeq A \times (B/N)$.*

Proof. Let $g = ab \in G$. If $c \in N$ then $gcg^{-1} = abcb^{-1}a^{-1} = baca^{-1}b^{-1} = bcaa^{-1}b^{-1} = bab^{-1} \in N \Rightarrow N$ is a normal subgroup of G. Let $\phi : A \times B \to G$ be the homomorphism defined in Lemma 3.59. Since $C = \{(c, c^{-1}) \mid c \in A \cap B\} = \{e\} \times \{e\}$, by the same Lemma we have that $G \simeq (A \times B)/C = A \times B$.

Let $\pi : G \to G/N$ be the natural quotient mapping. If $(\pi \circ \phi)(a, b) = \hat{e} \in G/N$ then $\phi(a, b) \in N \Rightarrow ab \in N \Rightarrow a \in Nb^{-1} \subset B \Rightarrow a = e \Rightarrow b \in N \Rightarrow \ker(\pi \circ \phi) = \{e\} \times N \Rightarrow (A \times B)/(\{e\} \times N) \simeq G/N$. From Lemma 3.60 it follows that $A \times (B/N) \simeq G/N$. $\square$

Lemma 3.62. *$\mathbb{R}/\mathbb{Z} \simeq \mathbb{R} \oplus \mathbb{R}/\mathbb{Z}$ as abstract groups.*

Proof. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Choose $\{1\} \cup \{r_\alpha \mid \alpha \in A\}$, a Hamel basis for $\mathbb{R}$. Then $\mathbb{R}$ is the weak direct sum of the vector spaces spanned by each element of the base, *i.e.* $\mathbb{R} = \mathbb{Q} \oplus (\oplus_{\alpha \in A} \mathbb{Q} r_\alpha)$. It follows from Lemma 3.61 that $\mathbb{R}/\mathbb{Z} \simeq \mathbb{Q}/\mathbb{Z} \oplus (\oplus_{\alpha \in A} \mathbb{Q} r_\alpha)$. Since $|A| = \mathfrak{c}$, there exist $B, C \subset A$ such that $B \cup C = A$, $B \cap C = \emptyset$, $|B| = |C| = \mathfrak{c}$ and

$\oplus_{\alpha\in A}\mathbb{Q}r_\alpha = (\oplus_{\beta\in B}\mathbb{Q}r_\beta)\oplus(\oplus_{\gamma\in C}\mathbb{Q}r_\gamma) \Rightarrow \mathbb{R} = \mathbb{Q}\oplus(\oplus_{\beta\in B}\mathbb{Q}r_\beta)\oplus(\oplus_{\gamma\in C}\mathbb{Q}r_\gamma)$. Using Lemma 3.61 again, we have that $\mathbb{R}/\mathbb{Z} \simeq (\mathbb{Q}/\mathbb{Z})\oplus(\oplus_{\beta\in B}\mathbb{Q}r_\beta)\oplus(\oplus_{\gamma\in C}\mathbb{Q}r_\gamma) \Rightarrow \mathbb{R}/\mathbb{Z} \simeq (\mathbb{R}/\mathbb{Z})\oplus\mathbb{R}$. $\square$

PROPOSITION 3.63. *If $\mathcal{H}$ is a n-dimensional Hilbert space, then $\mathcal{U}(\mathcal{H}) \simeq \mathbb{R}\times\mathcal{U}(\mathcal{H})$ as abstract groups.*

Proof. Let $T = \{\lambda I \mid |\lambda| = 1\}$. Then $T \simeq \mathbb{R}/\mathbb{Z}$, T and $SU(\mathcal{H})$ commute and $\mathcal{U}(\mathcal{H}) = T\cdot SU(\mathcal{H})$. Since $T\cap SU(\mathcal{H}) = \{\lambda I \mid \lambda^n = 1\} \simeq \mathbb{Z}_n$, using Lemma 3.59 we have that $\mathcal{U}(\mathcal{H}) \simeq (T\times SU(\mathcal{H}))/\mathbb{Z}_n \simeq ((\mathbb{R}/\mathbb{Z})\times SU(\mathcal{H}))/\mathbb{Z}_n$. Since $\mathbb{R}/\mathbb{Z} \simeq \mathbb{R}\times(\mathbb{R}/\mathbb{Z})$ by Lemma 3.62 and using Lemma 3.61 we have that $\mathcal{U}(\mathcal{H}) \simeq (\mathbb{R}\times(\mathbb{R}/\mathbb{Z})\times SU(\mathcal{H}))/\mathbb{Z}_n \simeq \mathbb{R}\times((\mathbb{R}/\mathbb{Z})\times SU(\mathcal{H}))/\mathbb{Z}_n \simeq \mathbb{R}\times\mathcal{U}(\mathcal{H})$. $\square$

COROLLARY 3.64. ★ *If $\mathcal{H}$ is an n-dimensional Hilbert space, there is no unique Polish topological group topology on $\mathcal{U}(\mathcal{H})$.*

Proof. According to Proposition 3.63, $\mathcal{U}(\mathcal{H})$ is algebraically isomorphic to $\mathbb{R}\times\mathcal{U}(\mathcal{H})$. If $\mathcal{T}$ is the standard Polish topological group topology on $\mathcal{U}(\mathcal{H})$ and $\mathbb{R}_{std}$ is the usual topology on $\mathbb{R}$, then the product topology on $\mathbb{R}\times\mathcal{U}(\mathcal{H})$ is a Polish topological group topology and it is different than $\mathcal{T}$ and hence $\mathcal{T}$ is not unique. $\square$

CHAPTER 4

THE PROJECTIVE GROUP

Throughout this section $\mathcal{H}$ is considered to be a separable infinite dimensional complex Hilbert space.

4.1. The Topology on $\mathcal{PU}(\mathcal{H})$

Definition 4.1. If H is a Hilbert space, the projective unitary group is the group $\mathcal{PU}(\mathcal{H}) = \mathcal{U}(\mathcal{H})/Z(\mathcal{U}(\mathcal{H}))$. If $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ is the natural quotient mapping and if $\mathcal{S} \subset \mathcal{U}(\mathcal{H})$ then $\pi(\mathcal{S}) = \{U \cdot Z(\mathcal{U}(\mathcal{H})) \mid U \in \mathcal{S}\}$ and $\pi^{-1}(\pi(\mathcal{S})) = \{\lambda U \mid |\lambda| = 1 \text{ and } U \in \mathcal{S}\}$.

Proposition 4.2. *If N is a normal subgroup of a topological group G, then G/N is a topological group.*

Proof. Let $aN, bN \in G/N$ and let $U \subset G/N$ be an open neighborhood of $aN \cdot bN = abN$. Then $\pi^{-1}(U) \subset G$ is open and contains ab. Let $a \in V \subset G$ and $b \in W \subset G$ be open and such that $V \cdot W \subset \pi^{-1}(U)$. Then $\pi(V)$ and $\pi(W)$ are open neighborhoods of aN and bN respectively, in G/N and $\pi(V)\pi(W) = \pi(VW) \subset \pi(\pi^{-1}(U)) = U \Rightarrow$ the multiplication in G/N is continuous. Let $U \subset G/N$ be open. Then $\pi^{-1}(U)$ is open in G and $(\pi^{-1}(U))^{-1}$ is open since inversion in G is continuous. Since $x \in (\pi^{-1}(U))^{-1} \Leftrightarrow x^{-1} \in \pi^{-1}(U) \Leftrightarrow \pi(x^{-1}) = (\pi(x))^{-1} \in U \Leftrightarrow \pi(x) \in U^{-1} \Leftrightarrow x \in \pi^{-1}(U^{-1})$ we have that $(\pi^{-1}(U))^{-1} = \pi^{-1}(U^{-1})$ and hence $\pi((\pi^{-1}(U))^{-1}) = \pi(\pi^{-1}(U^{-1})) = U^{-1}$ is open $\Rightarrow$ the inversion in G/N is continuous.
$\square$

Corollary 4.3. *$\mathcal{PU}(\mathcal{H})$ is a topological group.*

Proof. $Z(\mathcal{U}(\mathcal{H}))$ is a normal subgroup of $\mathcal{U}(\mathcal{H})$ and use Proposition 4.2. $\square$

Theorem 4.4. *Let G be a metrizable topological group and $H \subset G$ a closed subgroup. Then G/H is metrizable.*

Proof. Let d be a compatible right invariant metric on G and let $D(xH, yH) = \inf\{d(x, yh) \mid h \in H\}$. It is clear that $D(xH, yH) \geq 0$ for every $x, y \in G$. If $xH = yH$ then $y^{-1}x \in H \Rightarrow D(xH, yH) = \inf\{d(x, yh) \mid h \in H\} = d(x, y(y^{-1}x)) = 0$. If $D(xH, yH) = 0 \Rightarrow$ there exists a sequence $\{h_n\}_{n\geq 1} \subset H$ such that $yh_n \to x \Rightarrow h_n \to y^{-1}x \Rightarrow y^{-1}x \in H \Rightarrow xH = yH$. Hence $D(xH, yH) = 0 \Leftrightarrow xH = yH$. $D(xH, yH) = \inf\{d(x, yh) \mid h \in H\} = \inf\{d(xh^{-1}, y) \mid h \in H\} = \inf\{d(y, xh^{-1}) \mid h \in H\} = D(yH, xH)$. If $x, y, z \in G$ and $h_1, h_2 \in H$, then $D(xH, yH) \leq d(x, yh_2h_1^{-1}) = d(xh_1, yh_2) \leq d(z, xh_1) + d(z, yh_2) \Rightarrow D(xH, yH) \leq \inf\{d(z, xh_1) \mid h_1 \in H\} + \inf\{d(z, yh_2) \mid h_2 \in H\} = D(zH, xH) + D(zH, yH)$ and hence D is a metric.

To prove that the metric D is compatible with the topology on G/H it is enough to show that $\pi(B_d(a, \delta)) = B_D(\pi(a), \delta)$, where $\pi : G \to G/H$ is the natural quotient mapping, $a \in G$ and $\delta > 0$. Let $b \in B_d(a, \delta)$. Then $d(b, a) < \delta \Rightarrow D(aH, bH) = D(\pi(a), \pi(b)) < \delta \Rightarrow \pi(b) \in B_D(\pi(a), \delta)$ and so $\pi(B_d(a, \delta)) \subset B_D(\pi(a), \delta)$. Conversely, choose $b \in G$ such that $\pi(b) \in B_D(\pi(a), \delta)$. Then $D(\pi(b), \pi(a)) < \delta$ and hence there exists $h \in H$ such that $d(a, bh) < \delta \Rightarrow bh \in B_d(a, \delta) \Rightarrow \pi(bh) = \pi(b) \in \pi(B_d(a, \delta)) \Rightarrow B_D(\pi(a), \delta) \subset \pi(B_d(a, \delta))$. □

Proposition 4.5. *If G is a separable topological group and H a subgroup, the G/H is separable.*

Proof. Let $D \subset G$ be a countable dense subset. Then $\pi(D)$ is countable and, since π is continuous, we have that $G/H = \pi(G) = \pi(cl_G(D)) \subset cl_{G/H}(\pi(D)) \Rightarrow \pi(D)$ is dense in G.
□

Corollary 4.6. ★ *If $\mathcal{H}$ is separable, $\mathcal{PU}(\mathcal{H})$ is a Polish topological group.*

Proof. $\mathcal{PU}(\mathcal{H})$ is metrizable by Theorem 4.4. If $\mathcal{H}$ is separable, then $\mathcal{H}om(\mathcal{H}_1)$, the homeomorphism group of the unit ball, is completely metrizable by Corollary 2.25 and since $\mathcal{U}(\mathcal{H})$ is a closed subgroup of $\mathcal{H}om(\mathcal{H}_1)$ by Theorem 3.7, we have that $\mathcal{U}(\mathcal{H})$ is completely metrizable. Since the mapping π is continuous and onto, using a theorem of Hausdorff [8] we have that $\mathcal{PU}(\mathcal{H})$ is completely metrizable. $\mathcal{PU}(\mathcal{H})$ is separable by Proposition 4.5. □

THEOREM 4.7. ★ *Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$ and let $W \in \mathcal{U}(\mathcal{H})$ be such that $WUW^*U^* \in Z(\mathcal{U}(\mathcal{H}))$ for every $U \in \mathcal{U}(\mathcal{M})$. Then $WU = UW$ for every $U \in \mathcal{U}(\mathcal{M})$.*

Proof. Let $W \in \mathcal{U}(\mathcal{H})$ be such that $WUW^*U^* \in Z(\mathcal{U}(\mathcal{H}))$ for every $U \in \mathcal{U}(\mathcal{M})$. Then there exists $\lambda = \lambda(U)$, with $|\lambda| = 1$, such that $WU = \lambda(U)UW$. If $U_1, U_2 \in \mathcal{U}(\mathcal{M})$, then $\lambda(U_1U_2)U_1U_2W = WU_1U_2 = \lambda(U_1)U_1WU_2 = \lambda(U_1)\lambda(U_2)U_1U_2W \Rightarrow \lambda(U_1U_1) = \lambda(U_1)\lambda(U_2) \Rightarrow$ the mapping $\lambda : \mathcal{U}(\mathcal{M}) \to T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ is a homomorphism of groups. If $U \in \mathcal{U}(\mathcal{M})$ then $U^* \in \mathcal{U}(\mathcal{M})$ and $1 = \lambda(I) = \lambda(U^*U) = \lambda(U^*)\lambda(U) \Rightarrow \lambda(U^*) = (\lambda(U))^{-1} = \overline{\lambda(U)}$. If $\{U_j\}_{j\in J} \subset \mathcal{U}(\mathcal{M})$ and $U \in \mathcal{U}(\mathcal{M})$ are such that $U_j \xrightarrow{wo} U$, then $\lambda(U_j) = WU_jW^*U_j^* \xrightarrow{wo} WUW^*U^* = \lambda(U) \Rightarrow \lambda$ is continuous.

If $\mathcal{M}$ is infinite dimensional and if $U \in \mathcal{U}(\mathcal{M})$, according to [7], page 134, problem 191, there exist $P, Q \in \mathcal{U}(\mathcal{M})$ such that $U = PQP^*Q^*$ and then $\lambda(U) = \lambda(P)\lambda(Q)\lambda(P)^{-1}\lambda(Q)^{-1} = 1$ for every $U \in \mathcal{U}(\mathcal{M}) \Rightarrow WUW^*U^* = 1 \Rightarrow WU = UW$ for every $U \in \mathcal{U}(\mathcal{M})$.

Suppose first that $\mathcal{M}$ is one-dimensional, that $\mathcal{M} = \mathit{span}(\{e_1\})$ and that $\{e_l\}_{l\geq 1}$ is an orthonormal basis for $\mathcal{H}$. Note that in this case $\mathcal{U}(\mathcal{M}) = T$, the circle group, and hence $\mathcal{U}(\mathcal{M})$ is connected. Let $U \in \mathcal{U}(\mathcal{M})$. Then $Ue_1 = e^{i\alpha}e_1$, $Ue_l = e_l$ for every $l \geq 2$ and $U^*e_1 = e^{-i\alpha}e_1$ and $U^*e_l = e_l$ for every $l \geq 2$. If $\langle We_i, e_j\rangle \neq 0$ for some $i, j \geq 2$ then, since $WU = \lambda(U)UW$, we have that $\langle We_i, e_j\rangle = \langle WUe_i, e_j\rangle = \lambda(U)\langle UWe_i, ej\rangle = \lambda(U)\langle We_i, U^*e_j\rangle = \lambda(U)\langle We_i, e_j\rangle \Rightarrow \lambda(U) = 1$.

Otherwise, $\langle We_i, e_j\rangle = 0$ for every $i, j \geq 2$. In addition, if $\langle We_1, e_1\rangle \neq 0$ then $e^{i\alpha}\langle We_1, e_1\rangle = \langle WUe_1, e_1\rangle = \lambda(U)\langle UWe_1, e_1\rangle = \lambda(U)\langle We_1, U^*e_1\rangle = \lambda(U)e^{i\alpha}\langle We_1, e_1\rangle \Rightarrow \lambda(U) = 1$.

Otherwise, if $\langle We_1, e_1\rangle = 0$ and $\langle We_i, e_j\rangle = 0$ for all $i, j \geq 2$, then for every $l \geq 2$ we have that $\langle We_l, e_1\rangle = \langle WUe_l, e_1\rangle = \lambda(U)\langle UWe_l, e_1\rangle = \lambda(U)\langle We_l, U^*e_1\rangle = \lambda(U)e^{i\alpha}\langle We_l, e_1\rangle$. If $\langle We_l, e_1\rangle = 0$ for all $l \geq 2$ then $\langle We_l, e_1\rangle = 0$ for all $l \geq 1 \Rightarrow \langle Wx, e_1\rangle = 0$ for all $x \in \mathcal{H} \Rightarrow W^*e_1 = 0 \Rightarrow e_1 = WW^*e_1 = W(0) = 0$, a contradiction. Thus, there exists $l \geq 2$ such that $\langle We_l, e_1\rangle \neq 0 \Rightarrow \lambda(U)e^{i\alpha} = 1 \Rightarrow \lambda(U) = e^{-i\alpha}$. We also have that $e^{i\alpha}\langle We_1, e_l\rangle =$

$\langle WUe_1, e_l\rangle = \lambda(U)\langle UWe_1, e_l\rangle = \lambda(U)\langle We_1, U^*e_l\rangle = \lambda(U)\langle We_1, e_l\rangle$ for $l \geq 2$. If $\langle We_1, e_l\rangle = 0$ for all $l \geq 2$ then $\langle We_1, e_l\rangle = 0$ for all $l \geq 1 \Rightarrow \langle We_1, x\rangle = 0$ for all $x \in \mathcal{H} \Rightarrow We_1 = 0 \Rightarrow e_1 = W^*We_1 = W^*(0) = 0$, a contradiction. Thus, there exists $l \geq 2$ such that $\langle We_1, e_l\rangle \neq 0 \Rightarrow e^{i\alpha} = \lambda(U) \Rightarrow \lambda(U)^2 = 1 \Rightarrow \lambda(U) = \pm 1$. Since $\mathcal{U}(\mathcal{M})$ is connected, λ is continuous and $\lambda(I) = 1 \Rightarrow \lambda(U) = 1 \Rightarrow WU = UW$ for every $U \in \mathcal{U}(\mathcal{M})$.

Suppose now that $\mathcal{M} = span(\{e_1, ..., e_n\})$ is n-dimensional where $\{e_l\}_{l\geq 1}$ is an orthonormal basis for $\mathcal{H}$. If $U \in \mathcal{U}(\mathcal{M})$ then, according with the spectral theorem, we have that there exists $V \in \mathcal{U}(\mathcal{M})$ such that $VUV^*e_l = e^{i\alpha_l}e_l$ for every $1 \leq l \leq n$ and $VUV^*e_l = e_l$ for every $l > n$. If for every $1 \leq l \leq n$ we define $U_l|_{span(\{e_l\})}e_l = e^{i\alpha_l}e_l$ and $U_l|_{(span(\{e_l\}))^\perp} = I$ then $VUV^* = U_1U_2...U_n$ and hence $U = V^*U_1U_2...U_nV$. If we denote $\mathcal{M}_l = span(\{e_l\})$, then each $\mathcal{M}_l$ is one-dimensional, each $U_l \in \mathcal{U}(\mathcal{M}_l)$ and $\mathcal{U}(\mathcal{M}_l) \subset \mathcal{U}(\mathcal{M})$. Thus $WU_l = \lambda(U_l)U_lW$ for every $l \geq 1$ and by the previous paragraph we have that $\lambda(U_l) = 1$ for every $1 \leq l \leq n \Rightarrow \lambda(U) = \lambda(V^*)\lambda(U_1)\lambda(U_2)...\lambda(U_n)\lambda(V) = \overline{\lambda(V)}\lambda(V) = 1$ and hence $WU = UW$ for every $U \in \mathcal{U}(\mathcal{M})$. $\square$

Theorem 4.8. ★ *Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$, G a Polish topological group and $\phi : G \to \mathcal{PU}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ is closed in G, where $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ is the natural quotient mapping.*

Proof. We will prove that $\pi(\mathcal{U}(\mathcal{M})) = \{\hat{W} \in \mathcal{PU}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W}$ for all $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}^\perp))\}$. This will imply that $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M}))) = \{\phi^{-1}(\hat{W}) \mid \phi^{-1}(\hat{W})\phi^{-1}(\hat{V}) = \phi^{-1}(\hat{V})\phi^{-1}(\hat{W})\ \forall\ \phi^{-1}(\hat{V}) \in \phi^{-1}(\pi(\mathcal{U}(\mathcal{M}^\perp)))\}$ and then, according with the Proposition 3.26 we will have that $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ is closed in G. Note that if $\mathcal{S} \subset \mathcal{U}(\mathcal{H})$ and $\hat{U} \in \pi(\mathcal{S})$ then there exists $U \in \mathcal{S}$ such that $\pi(U) = \hat{U}$.

Let $\hat{U} \in \pi(\mathcal{U}(\mathcal{M}))$ and $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}^\perp))$. Let $U \in \mathcal{U}(\mathcal{M})$ be such that $\pi(U) = \hat{U}$ and $V \in \mathcal{U}(\mathcal{M}^\perp)$ be such that $\pi(V) = \hat{V}$. According with Theorem 3.28 we have that $UV = VU \Rightarrow \pi(U)\pi(V) = \pi(V)\pi(U) \Rightarrow \hat{U}\hat{V} = \hat{V}\hat{U} \Rightarrow \pi(\mathcal{U}(\mathcal{M}))\pi(\mathcal{U}(\mathcal{M}^\perp)) = \pi(\mathcal{U}(\mathcal{M}^\perp))\pi(\mathcal{U}(\mathcal{M})) \Rightarrow \pi(\mathcal{U}(\mathcal{M})) \subset \{\hat{W} \in \mathcal{PU}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W}$ for all $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}^\perp))\}$.

Let $\hat{W} \in \mathcal{PU}(\mathcal{H})$ be such that $\hat{W}\hat{V} = \hat{V}\hat{W}$ for all $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}^{\perp}))$. Let $W \in \mathcal{U}(\mathcal{H})$ be such that $\pi(W) = \hat{W}$ and, for every $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}^{\perp}))$, let $V \in \mathcal{U}(\mathcal{M}^{\perp})$ be such that $\pi(V) = \hat{V}$. Then $\pi(W)\pi(V) = \pi(V)\pi(W) \Rightarrow \pi(WV) = \pi(VW) \Rightarrow WVW^*V^* \in Z(\mathcal{U}(\mathcal{H})) \Rightarrow WV = VW$ by Theorem 4.7. Using Theorem 3.28 we have that $W \in Z(\mathcal{U}(\mathcal{H}))\cdot\mathcal{U}(\mathcal{M}) \Rightarrow$ there exist λ with $|\lambda| = 1$ and $U \in \mathcal{U}(\mathcal{M})$ such that $W = \lambda U \Rightarrow \pi(W) = \pi(U) \Rightarrow \hat{W} \in \pi(\mathcal{U}(\mathcal{M})) \Rightarrow \{\hat{W} \in \mathcal{PU}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W} \text{ for all } \hat{V} \in \pi(\mathcal{U}(\mathcal{M}^{\perp}))\} \subset \pi(\mathcal{U}(\mathcal{M}))$. □

Proposition 4.9. *If $\mathcal{M} \subset \mathcal{H}$ is a finite dimensional subspace, then*

$$\pi(\mathcal{U}(\mathcal{M})) = \pi(Z(\mathcal{U}(\mathcal{M})))\pi(SU(\mathcal{M}))$$

Proof. Since $Z(\mathcal{U}(\mathcal{M})), SU(\mathcal{M}) \subset \mathcal{U}(\mathcal{M})$ and $\mathcal{U}(\mathcal{M})$ is a subgroup we have that $Z(\mathcal{U}(\mathcal{M}))SU(\mathcal{M}) \subset \mathcal{U}(\mathcal{M}) \Rightarrow \pi(Z(\mathcal{U}(\mathcal{M})))\pi(SU(\mathcal{M})) \subset \pi(\mathcal{U}(\mathcal{M}))$.

Let $\hat{U} \in \pi(\mathcal{U}(\mathcal{M}))$. Then there exists $U \in \mathcal{U}(\mathcal{M})$ such that $\pi(U) = \hat{U}$ and by Proposition 3.47 we have that there exist $V \in Z(\mathcal{U}(\mathcal{M}))$ and $W \in SU(\mathcal{M})$ such that $U = VW \Rightarrow \pi(U) = \pi(VW) = \pi(V)\pi(W) \subset \pi(Z(\mathcal{U}(\mathcal{M})))\pi(SU(\mathcal{M})) \Rightarrow \pi(\mathcal{M}) \subset \pi(Z(\mathcal{U}(\mathcal{M})))\pi(SU(\mathcal{M}))$. □

Proposition 4.10. ★ *Let G be a Polish topological space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{PU}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(SU(\mathcal{M})))$ is an analytic subset of G.*

Proof. Since $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ is closed in G by Theorem 4.8, $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M}))) \times \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ is closed in $G \times G$. Let $[\cdot,\cdot] : \phi^{-1}(\pi(\mathcal{U}(\mathcal{M}))) \times \phi^{-1}(\pi(\mathcal{U}(\mathcal{M}))) \to G$ be defined as $[a, b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot,\cdot]$ is continuous. If $a, b \in \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ then $\phi(a), \phi(b) \in \pi(\mathcal{U}(\mathcal{M})) \Rightarrow$ there exist $U, V \in \mathcal{U}(\mathcal{M})$ such that $\phi(a) = \pi(U)$, $\phi(b) = \pi(V)$ and $(\phi(a))^{-1} = (\pi(U))^{-1} = \pi(U^*)$ and similarly $(\phi(b))^{-1} = \pi(V^*)$. Since $\phi([a, b]) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)(\phi(a))^{-1}(\phi(b))^{-1} = \pi(U)\pi(V)\pi(U^*)\pi(V^*) = \pi(UVU^*V^*) \in \pi(\mathcal{U}(\mathcal{M}))$ and since $\det(UVU^*V^*) = \det(U)\det(V)\overline{\det(U)}\ \overline{\det(V)} = 1$, we have that $\phi([a, b]) \in \pi(SU(\mathcal{M})) \Rightarrow [a, b] \in \phi^{-1}(\pi(SU(\mathcal{M})))$. This proves that $[\cdot,\cdot]$ takes its values in $\phi^{-1}(\pi(SU(\mathcal{M})))$.

Let $y \in \phi^{-1}(\pi(SU(\mathcal{M})))$. Then $\phi(y) \in \pi(SU(\mathcal{M})) \Rightarrow$ there exists $W \in SU(\mathcal{M})$ such that $\phi(y) = \pi(W)$. By Lemma 3.45 we have that there exist $U, V \in SU(\mathcal{M})$ such that $W =$

UVU^*V^*. Let $a = \phi^{-1}(\pi(U)) \in \phi^{-1}(\pi(SU(\mathcal{M}))) \subset \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ and $b = \phi^{-1}(\pi(V)) \in \phi^{-1}(\pi(SU(\mathcal{M}))) \subset \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$. Then $y = \phi^{-1}(\pi(W)) = \phi^{-1}(\pi(UVU^*V^*)) = \phi^{-1}(\pi(U))$ $\phi^{-1}(\pi(V))(\phi^{-1}(\pi(U)))^{-1}(\phi^{-1}(\pi(V)))^{-1} = aba^{-1}b^{-1} = [a,b] \Rightarrow [\cdot,\cdot]$ is onto $\phi^{-1}(\pi(SU(\mathcal{M})))$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(\pi(SU(\mathcal{M})))$ is the continuous image of $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))\times \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$, a closed subset of a Polish space, and therefore $\phi^{-1}(\pi(SU(\mathcal{M})))$ is an analytic subset of G. □

Lemma 4.11. ★ *If $\mathcal{M} \subset \mathcal{H}$ is a finite dimensional subspace, then $\pi(Z(\mathcal{U}(\mathcal{M}))) = Z(\pi(\mathcal{U}(\mathcal{M})))$.*

Proof. Let $\hat{U} \in \pi(Z(\mathcal{U}(\mathcal{M})))$. Then there exists $U \in Z(\mathcal{U}(\mathcal{M}))$ such that $\pi(U) = \hat{U}$. Let $\hat{V} \in \pi(\mathcal{U}(\mathcal{M}))$ and $V \in \mathcal{U}(\mathcal{M})$ be such that $\pi(V) = \hat{V}$. Then, since U and V commute, we have that $\hat{U}\hat{V} = \pi(U)\pi(V) = \pi(UV) = \pi(VU) = \pi(V)\pi(U) = \hat{V}\hat{U} \Rightarrow \hat{U} \in Z(\pi(\mathcal{U}(\mathcal{M}))) \Rightarrow \pi(Z(\mathcal{U}(\mathcal{M}))) \subset Z(\pi(\mathcal{U}(\mathcal{M})))$.

Let $\hat{U} \in Z(\pi(\mathcal{U}(\mathcal{M})))$ and let $U \in \mathcal{U}(\mathcal{H})$ be such that $\pi(U) = \hat{U}$. We will show that $U \in Z(\mathcal{U}(\mathcal{M}))$. This will imply that $\hat{U} \in \pi(Z(\mathcal{U}(\mathcal{M})))$ and therefore that $Z(\pi(\mathcal{U}(\mathcal{M}))) \subset \pi(Z(\mathcal{U}(\mathcal{M})))$. Let $V \in \mathcal{U}(\mathcal{M})$. Then $\pi(V) \in \pi(\mathcal{U}(\mathcal{M}))$ and hence $\hat{U}\pi(V) = \pi(V)\hat{U} \Rightarrow \pi(U)\pi(V) = \pi(V)\pi(U) \Rightarrow \pi(UVU^*V^*) = Id \in \mathcal{PU}(\mathcal{H}) \Rightarrow UVU^*V^* \in Z(\mathcal{U}(\mathcal{H})) \Rightarrow$ from Theorem 4.7 that $UV = VU \Rightarrow U \in Z(\mathcal{U}(\mathcal{M}))$. □

Corollary 4.12. ★ *Let G be a Polish topological space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{PU}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(SU(\mathcal{M})))$ is closed in G.*

Proof. From Corollary 4.8 we have that $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ is closed in G and hence Polish. From Proposition 4.9 we have that $\phi^{-1}(\pi(Z(\mathcal{U}(\mathcal{M}))))\phi^{-1}(\pi(SU(\mathcal{M}))) = \phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$. By Lemma 4.11 we have that $\pi(Z(\mathcal{U}(\mathcal{M}))) = Z(\pi(\mathcal{U}(\mathcal{M})))$ and, since ϕ is an isomorphism, it follows that $\phi^{-1}(\pi(Z(\mathcal{U}(\mathcal{M}))))$ is the center of $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ and therefore $\phi^{-1}(\pi(Z(\mathcal{U}(\mathcal{M}))))$ is closed in $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$. $\phi^{-1}(\pi(SU(\mathcal{M})))$ is an analytic subgroup of G by Proposition 4.10, and hence analytic subgroup of $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$. Let $C = \pi(Z(\mathcal{U}(\mathcal{M}))) \cap \pi(SU(\mathcal{M}))$ and let $\hat{U} \in C$. Then there exist $U \in Z(\mathcal{U}(\mathcal{M}))$ and $V \in SU(\mathcal{M})$ such that $\pi(U) = \hat{U} = \pi(V) \Rightarrow \pi(UV^*) = Id \in \mathcal{PU}(\mathcal{H}) \Rightarrow UV^* \in Z(\mathcal{U}(\mathcal{H})) \Rightarrow UV^* = \lambda I \Rightarrow U = \lambda V$. Since $U|_{\mathcal{M}^\perp} = I$

and $V|_{\mathcal{M}^\perp} = I$ we have that $\lambda = 1 \Rightarrow U = V \Rightarrow C = \{\pi(U) \mid U \in Z(\mathcal{U}(\mathcal{M})) \cap SU(\mathcal{M})\} = \{\pi(U) \mid U|_{\mathcal{M}} = \mu I,\ U|_{\mathcal{M}^\perp} = I,\ \mu^n = 1\}$, where $n = \dim(\mathcal{M}) \Rightarrow C$ is finite. Since ϕ is an isomorphism we have that $\phi^{-1}(C)$ is finite and hence closed in $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$. It follows from Corollary 3.39 that $\phi^{-1}(\pi(SU(\mathcal{M})))$ is closed in $\phi^{-1}(\pi(\mathcal{U}(\mathcal{M})))$ and hence closed in G. □

PROPOSITION 4.13. ★ *Let G be a Polish topological group, let $\mathcal{H}$ be a separable Hilbert space and let $e \in \mathcal{H}$ be such that $\|e\| = 1$. Let $\mathcal{S} = \{U \in \mathcal{U}(\mathcal{H})) \mid \|e - Ue\| < \epsilon\} \subset \mathcal{U}(\mathcal{H})$ and let $\phi : G \to \mathcal{PU}(\mathcal{H})$ be an algebraic isomorphism. Then $\phi^{-1}(\pi(\mathcal{S}))$ is analytic in G.*

Proof. Note first that the quotient mapping $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ is open and continuous. Let $\mathcal{M} \subset \mathcal{H}$ be a three dimensional subspace as in Lemma 3.51 so that $\mathcal{S} = \mathcal{U}(\{e\}^\perp) \cdot [SU(\mathcal{M}) \cap \mathcal{S}] \cdot \mathcal{U}(\{e\}^\perp)$. Then $\pi(\mathcal{S}) = \pi(\mathcal{U}(\{e\}^\perp))\pi[SU(\mathcal{M}) \cap \mathcal{S}]\pi(\mathcal{U}(\{e\}^\perp))$. Since $SU(\mathcal{M})$ is a connected compact metric group with a totally disconnected center (Chapter I, Section 14, [19]), then $\pi(SU(\mathcal{M}))$ is a connected compact metric group. A proof similar to the proof of Proposition 4.11 shows that $Z(\pi(SU(\mathcal{M}))) = \pi(Z(SU(\mathcal{M})))$ and hence the center of $\pi(SU(\mathcal{M}))$ is finite. Using the result from [14] we have that $\phi|_{\phi^{-1}(\pi(SU(\mathcal{M})))} : \phi^{-1}(\pi(SU(\mathcal{M}))) \to \pi(SU(\mathcal{M}))$ is a homeomorphism. $SU(\mathcal{M}) \cap \mathcal{S}$ is a relatively open subset of $SU(\mathcal{M})$ and hence Borel $\Rightarrow \pi[SU(\mathcal{M}) \cap \mathcal{S}]$ is analytic in $\pi(SU(\mathcal{M})) \Rightarrow \phi^{-1}(\pi[SU(\mathcal{M}) \cap \mathcal{S}])$ is analytic in $\phi^{-1}(\pi(SU(\mathcal{M})))$. Since $\phi^{-1}(\pi(\mathcal{U}(\{e\}^\perp)))$ is closed in G by Theorem 4.8 and therefore analytic, it follows from Lemma 3.53 that $\phi^{-1}(\pi(\mathcal{S})) = \phi^{-1}(\pi(\mathcal{U}(\{e\}^\perp))\pi[SU(\mathcal{M}) \cap \mathcal{S}]\pi(\mathcal{U}(\{e\}^\perp))) = \phi^{-1}(\pi(\mathcal{U}(\{e\}^\perp)))\phi^{-1}(\pi[SU(\mathcal{M}) \cap \mathcal{S}])\phi^{-1}(\pi(\mathcal{U}(\{e\}^\perp)))$ is analytic. □

4.3. Main Result

PROPOSITION 4.14. ★ *Let $\{e_m\}_{m \geq 1}$ be an orthonormal basis for the separable infinite dimensional Hilbert space $\mathcal{H}$. For every $m, n \geq 1$ let $\mathcal{U}_{m,n} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|e_m - Ue_m\| < \frac{1}{n}\}$. Let $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ be the natural quotient mapping. Then*

$$\bigcap_{m,n \geq 1} \pi^{-1}(\pi(\mathcal{U}_{m,n})) = \{W \in \mathcal{U}(\mathcal{H}) \mid We_m = \lambda_m e_m \textit{ for every } m \geq 1 \textit{ with } |\lambda_m| = 1\}$$

Proof. Note first that $\pi^{-1}(\pi(\mathcal{U}_{m,n})) = Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{m,n}$ for every $m, n \geq 1$. Let $W \in \mathcal{U}(\mathcal{H})$ be such that $We_m = \lambda_m e_m$ for every $m \geq 1$ and $|\lambda_m| = 1$. Then $(\overline{\lambda_1}W)e_1 = \overline{\lambda_1}\lambda_1 e_1 = e_1 \Rightarrow \|e_1 - (\overline{\lambda_1}W)e_1\| = 0 < \frac{1}{n}$ for every $n \geq 1 \Rightarrow \overline{\lambda_1}W \in \mathcal{U}_{1,n}$ for every $n \geq 1 \Rightarrow W \in Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{1,n}$ for every $n \geq 1$. Similarly we have that $W \in Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{m,n}$ for every $m, n \geq 1 \Rightarrow W \in \cap_{m,n\geq 1} Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{m,n} = \cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{U}_{m,n}))$.

Let $W \in \cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{U}_{m,n})) = \cap_{m,n\geq 1} Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{m,n}$. Then for every $m, n \geq 1$ there exist $\lambda_{m,n}$ with $|\lambda_{m,n}| = 1$ and $W_{m,n} \in \mathcal{U}_{m,n}$ such that $W = \lambda_{m,n} W_{m,n}$ and $\|e_m - W_{m,n}e_m\| < \frac{1}{n}$ for every $m, n \geq 1$. Fix m and let $p, q \geq 1$. Then $|\lambda_{m,p} - \lambda_{m,q}| = \|\lambda_{m,p}e_m - \lambda_{m,q}e_m\| \leq \|\lambda_{m,p}e_m - \lambda_{m,p}W_{m,p}e_m\| + \|\lambda_{m,p}W_{m,p}e_m - \lambda_{m,q}W_{m,q}e_m\| + \|\lambda_{m,q}W_{m,q}e_m - \lambda_{m,q}e_m\| = \|e_m - W_{m,p}e_m\| + \|We_m - We_m\| + \|e_m - W_{m,q}e_m\| < \frac{1}{p} + \frac{1}{q} \to 0$ as $p, q \to \infty \Rightarrow \{\lambda_{m,n}\}_{n\geq 1}$ is Cauchy $\Rightarrow \lambda_{m,n} \to \lambda_m$ as $n \to \infty$, with $|\lambda_m| = 1$. Thus $\|We_m - \lambda_m e_m\| = \|\lambda_{m,n}W_{m,n}e_m - \lambda_m e_m\| \leq \|\lambda_{m,n}W_{m,n}e_m - \lambda_m W_{m,n}e_m\| + \|\lambda_m W_{m,n}e_m - \lambda_m e_m\| = |\lambda_{m,n} - \lambda_m| \cdot \|W_{m,n}e_m\| + |\lambda_m| \cdot \|W_{m,n}e_m - e_m\| < |\lambda_{m,n} - \lambda_m| + \frac{1}{n} \to 0$ as $n \to \infty \Rightarrow We_m = \lambda_m e_m$. $\square$

COROLLARY 4.15. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ be the natural quotient mapping. Then there exists $\{\mathcal{S}_l\}_{l\geq 1} \subset \mathcal{U}(\mathcal{H})$ a sequence of subbasic open neighborhoods of I such that $\cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) = Z(\mathcal{U}(\mathcal{H}))$.*

Proof. Let $\{e_m\}_{m\geq 1}$ be an orthonormal basis for $\mathcal{H}$. Let $f_1 = \frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}$. Then $\|f_1\|^2 = \frac{6}{\pi^2}\sum_{m\geq 1}\frac{1}{m^2} = 1$ and expand $\{f_1\}$ to an orthonormal basis $\{f_m\}_{m\geq 1}$. Let $\mathcal{U}_{m,n} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|e_m - Ue_m\| < \frac{1}{n}\}$ and let $\mathcal{V}_{m,n} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|f_m - Uf_m\| < \frac{1}{n}\}$. Let $\{\mathcal{S}_l\}_{l\geq 1} = \{\mathcal{U}_{m,n}, \mathcal{V}_{m,n} \mid m, n \geq 1\}$. According with the Proposition 3.11 $\{\mathcal{S}_l\}_{l\geq 1}$ is a sequence of subbasic open neighborhoods of I in $\mathcal{U}(\mathcal{H})$.

Let $W \in \cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) = [\cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{U}_{m,n}))] \cap [\cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{V}_{m,n}))]$. Then, according with the Proposition 4.14 we have that $We_m = \lambda_m e_m$ and $Wf_m = \mu_m f_m$, with $|\lambda_m| = |\mu_m| = 1$ for every $m \geq 1$. But $Wf_1 = W\left(\frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}\right) = \frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{We_m}{m} = \frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{\lambda_m e_m}{m}$ and also $Wf_1 = \mu_1 f_1 = \mu_1\left(\frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}\right) = \left(\frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{\mu_1 e_m}{m}\right) \Rightarrow \lambda_m = \mu_1$ for every $m \geq 1 \Rightarrow We_m = \mu_1 e_m$ for every $m \geq 1 \Rightarrow W = \mu_1 I \in Z(\mathcal{U}(\mathcal{H})) \Rightarrow \cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) \subset Z(\mathcal{U}(\mathcal{H}))$.

If $W \in Z(\mathcal{U}(\mathcal{H}))$ then $W = \lambda I$ for some $|\lambda| = 1$ and since $I \in \mathcal{U}_{m,n}$ and $I \in \mathcal{V}_{m,n}$ for every $m, n \geq 1 \Rightarrow W \in Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{U}_{m,n} = \pi^{-1}(\pi(\mathcal{U}_{m,n}))$ and $W \in Z(\mathcal{U}(\mathcal{H})) \cdot \mathcal{V}_{m,n} = \pi^{-1}(\pi(\mathcal{V}_{m,n}))$ for every $m, n \geq 1 \Rightarrow W \in \cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l))$. □

THEOREM 4.16. ★ *Let G and H be two Polish topological groups and $\phi : G \to H$ an algebraic isomorphism. Suppose that there exists a sequence of open subsets of H, $\{U_n\}_{n\geq 1}$, such that $\cap_{n\geq 1}U_n = \{e\}$, $U_n = U_n^{-1}$ for every $n \geq 1$, for every n_0 there exists n_1 such that $U_{n_1}^2 \subset U_{n_0}$ and $\phi^{-1}(U_n)$ is a set with the Baire property in G for every $n \geq 1$. Then ϕ is a topological isomorphism.*

Proof. Let $\{a_m\}_{m\geq 1}$ be a countable dense subset of H. We will prove that the sequence $\{a_m U_n\}_{m\geq 1,\, n\geq 1}$ separate points in H. Then, according to a theorem of Mackey (Theorem 3.3, [22]) we have that $\{a_m U_n\}_{m\geq 1,\, n\geq 1}$ generates the Borel structure of H. Since $\phi^{-1}(U_n)$ is a set with the Baire property and since the sets with the Baire property are invariant under left translations, we have that $\phi^{-1}(a_m U_n) = \phi^{-1}(a_m)\phi^{-1}(U_n)$ is a set with the Baire property in G. Since $\{a_m U_n\}_{m\geq 1,\, n\geq 1}$ generates the Borel structure of H we have that $\phi^{-1}(B)$ is a set with the Baire property in G for every B Borel subset of H and hence ϕ is measurable with respect to the sets with the Baire property. Then, since G is Baire and $\mathcal{H}$ is separable, it follows from a well-known theorem of Banach, Kuratowski and Pettis (Theorem 9.10, page 61, [18]) that ϕ is continuous. From Lusin-Souslin Theorem (page 89, [18]) we have that ϕ^{-1} is Borel measurable, and hence it is measurable with respect to the sets with the Baire property. From the same result of Banach-Kuratowski-Pettis it follows that ϕ^{-1} is continuous and hence ϕ is a topological isomorphism.

To show that $\{a_m U_n\}_{m\geq 1,\, n\geq 1}$ separates points in H, let $x, y \in H$ be such that $x \neq y$. Then $x^{-1}y \neq e \Rightarrow x^{-1}y \notin \cap_{n\geq 1}U_n \Rightarrow$ there exists n_0 such that $x^{-1}y \notin U_{n_0}$. Let n_1 be such that $U_{n_1}^2 \subset U_{n_0}$. Then $x^{-1}y \notin U_{n_1}^2$. The set xU_{n_1} is open and since $\{a_m\}_{m\geq 1}$ is dense, there exists m_0 such that $a_{m_0} \in xU_{n_1} \Rightarrow x^{-1}a_{m_0} \in U_{n_1} \Rightarrow x^{-1} \in U_{n_1}a_{m_0}^{-1} \Rightarrow x \in a_{m_0}U_{n_1}^{-1} = a_{m_0}U_{n_1}$. If $y \in a_{m_0}U_{n_1}$ then $a_{m_0}^{-1}y \in U_{n_1}$ and since $x^{-1}a_{m_0} \in U_{n_1}$ we have that

$x^{-1}y = (x^{-1}a_{m_0})(a_{m_0}^{-1}y) \in U_{n_1}^2 \subset U_{n_0}$, a contradiction. Thus $y \notin a_{m_0}U_{n_1}$ and $x \in a_{m_0}U_{n_1} \Rightarrow$ the collection $\{a_mU_n\}_{m\geq 1,\, n\geq 1}$ separates points in H. $\square$

Lemma 4.17. *Let $f : X \to Y$ be onto and let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a collection of subsets of Y. Then $f(\cap_{\gamma\in\Gamma}f^{-1}(A_\gamma)) = \cap_{\gamma\in\Gamma}A_\gamma$.*

Proof. Let $y \in f(\cap_{\gamma\in\Gamma}f^{-1}(A_\gamma))$. Then there exists $x \in \cap_{\gamma\in\Gamma}f^{-1}(A_\gamma)$ such that $y = f(x) \Rightarrow x \in f^{-1}(A_\gamma)$ for every $\gamma \in \Gamma \Rightarrow f(x) \in A_\gamma$ for every $\gamma \in \Gamma \Rightarrow y = f(x) \in \cap_{\gamma\in\Gamma}A_\gamma \Rightarrow f(\cap_{\gamma\in\Gamma}f^{-1}(A_\gamma)) \subset \cap_{\gamma\in\Gamma}A_\gamma$.

Let $y \in \cap_{\gamma\in\Gamma}A_\gamma$. Then there exists $x \in X$ such that $f(x) = y \Rightarrow f(x) \in A_\gamma$ for every $\gamma \in \Gamma \Rightarrow x \in f^{-1}(A_\gamma)$ for every $\gamma \in \Gamma \Rightarrow x \in \cap_{\gamma\in\Gamma}f^{-1}(A_\gamma) \Rightarrow y = f(x) \in f(\cap_{\gamma\in\Gamma}f^{-1}(A_\gamma)) \Rightarrow \cap_{\gamma\in\Gamma}A_\gamma \subset f(\cap_{\gamma\in\Gamma}f^{-1}(A_\gamma))$. $\square$

Theorem 4.18. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{PU}(\mathcal{H})$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. Let $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{PU}(\mathcal{H})$ be the natural quotient mapping. Let $\{\mathcal{S}_l\}_{l\geq 1}$ be the sequence defined in Proposition 4.15, $\{\mathcal{S}_l\}_{l\geq 1} = \{\mathcal{U}_{m,n}, \mathcal{V}_{m,n} \mid m, n \geq 1\}$, where $\mathcal{U}_{m,n} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|e_m - Ue_m\| < \frac{1}{n}\}$, $\mathcal{V}_{m,n} = \{U \in \mathcal{U}(\mathcal{H}) \mid \|f_m - Uf_m\| < \frac{1}{n}\}$ and $\{e_m\}_{m\geq 1}$, $\{f_m\}_{m\geq 1}$ are two orthonormal bases for $\mathcal{H}$. We will prove that the sequence $\{\pi(\mathcal{S}_l)\}_{l\geq 1}$ of subsets of $\mathcal{PU}(\mathcal{H})$ satisfy the hypothesis of Theorem 4.16 and the conclusion will follow from the same theorem. Since the projection mapping is open we have that $\pi(\mathcal{S}_l)$ is open for every $l \geq 1$. Also note that each $\phi^{-1}(\pi(\mathcal{S}_l))$ is analytic in G by Proposition 4.13 and hence each $\phi^{-1}(\pi(\mathcal{S}_l))$ is a set with the Baire property.

Since $\|e_m - U^*e_m\| = \|U^*(Ue_m - e_m)\| = \|Ue_m - e_m\|$ we have that $U^* \in \mathcal{U}_{m,n}$ whenever $U \in \mathcal{U}_{m,n}$. Let $\hat{U} \in \pi(\mathcal{U}_{m,n})$ and $U \in \mathcal{U}_{m,n}$ be such that $\pi(U) = \hat{U}$. Then $U^* \in \mathcal{U}_{m,n} \Rightarrow \hat{U}^{-1} = (\pi(U))^{-1} = \pi(U^*) \in \pi(\mathcal{U}_{m,n}) \Rightarrow (\pi(\mathcal{U}_{m,n}))^{-1} \subset \pi(\mathcal{U}_{m,n})$. By replacing $\mathcal{U}_{m,n}$ with $\mathcal{U}_{m,n}^{-1}$ we have that $(\pi(\mathcal{U}_{m,n}^{-1}))^{-1} \subset \pi(\mathcal{U}_{m,n}^{-1}) \Rightarrow \pi(\mathcal{U}_{m,n}) \subset (\pi(\mathcal{U}_{m,n}))^{-1} \Rightarrow (\pi(\mathcal{U}_{m,n}))^{-1} = \pi(\mathcal{U}_{m,n})$ for every $m, n \geq 1$. Similarly $(\pi(\mathcal{V}_{m,n}))^{-1} = \pi(\mathcal{V}_{m,n})$ for every $m, n \geq 1 \Rightarrow (\pi(\mathcal{S}_l))^{-1} = \pi(\mathcal{S}_l)$ for every $l \geq 1$.

Let $U, V \in \mathcal{U}_{m,2n}$. Then $\|e_m - Ue_m\| < \frac{1}{2n}$ and $\|e_m - Ve_m\| < \frac{1}{2n}$ and hence $\|e_m - UVe_m\| \leq \|e_m - Ue_m\| + \|Ue_m - UVe_m\| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow UV \in \mathcal{U}_{m,n} \Rightarrow \mathcal{U}_{m,2n}^2 \subset \mathcal{U}_{m,n} \Rightarrow (\pi(\mathcal{U}_{m,2n}))^2 = \pi(\mathcal{U}_{m,2n}^2) \subset \pi(\mathcal{U}_{m,n})$ and hence for every $m_0, n_0 \geq 1$ there exists $m_1 = m_0$ and $n_1 = 2n_0$ such that $(\pi(\mathcal{U}_{m_1,n_1}))^2 \subset \pi(\mathcal{U}_{m_0,n_0})$. Similarly for every $m_0, n_0 \geq 1$ there exists $m_1 = m_0$ and $n_1 = 2n_0$ such that $(\pi(\mathcal{V}_{m_1,n_1}))^2 \subset \pi(\mathcal{V}_{m_0,n_0})$ and therefore for every $l_0 \geq 1$ there exists l_1 such that $(\pi(\mathcal{S}_{l_1})^2 \subset \pi(\mathcal{S}_{l_0})$.

From Corollary 4.15 we have that $\cap_{l \geq 1} \pi^{-1}(\pi(\mathcal{S}_l)) = Z(\mathcal{U}(\mathcal{H}))$. From Lemma 4.17 we have that $\pi(\cap_{l \geq 1} \pi^{-1}(\pi(\mathcal{S}_l))) = \cap_{l \geq 1} \pi(\pi^{-1}(\pi(\mathcal{S}_l))) = \cap_{l \geq 1} \pi(\mathcal{S}_l) \Rightarrow \cap_{l \geq 1} \pi(\mathcal{S}_l) = \pi(Z(\mathcal{U}(\mathcal{H}))) = Z(\mathcal{U}(\mathcal{H}))$ and hence $\cap_{l \geq 1} \pi(\mathcal{S}_l)$ is the identity in $\mathcal{PU}(\mathcal{H})$. $\square$

CHAPTER 5

THE GROUP OF $*-$AUTOMORPHISMS

Throughout this section $\mathcal{H}$ is considered to be a separable complex Hilbert space.

5.1. The Topology on $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$

THEOREM 5.1. *Let $\mathcal{H}$ be a separable Hilbert space and $\{e_l\}_{l\geq 1}$ be a maximal orthonormal subset. Then*

$$d(S,T) = \sum_{m,n\geq 1} \frac{1}{2^{m+n}} |\langle (S-T)e_m, e_n\rangle|$$

is a metric on $\mathcal{L}(\mathcal{H})_1$ compatible with the weak operator topology.

Proof. Since $|\langle (S-T)e_m, e_n\rangle| \leq \|S-T\|$, the series $\sum_{m,n\geq 1} \frac{1}{2^{m+n}} |\langle (S-T)e_m, e_n\rangle|$ converges. Clearly $d(S,T) \geq 0$, $d(S,T) = d(T,S)$ and $d(S,S) = 0$. If $d(S,T) = 0$ then $\langle (S-T)e_m, e_n\rangle = 0$ for all $m,n \geq 1$. Since $(S-T)e_n = \sum_{m\geq 1} \langle (S-T)e_n, e_m\rangle e_m$ for every $n \geq 1$ we have that $\|(S-T)e_n\|^2 = \sum_{m\geq 1} |\langle (S-T)e_n, e_m\rangle|^2 = 0 \Rightarrow (S-T)e_n = 0$ for all $n \geq 1 \Rightarrow S = T$. Finally, $d(S,T) = \sum_{m,n\geq 1} \frac{1}{2^{m+n}} |\langle (S-R+R-T)e_m, e_n\rangle| \leq \sum_{m,n\geq 1} \frac{1}{2^{m+n}} |\langle (S-R)e_m, e_n\rangle| + \sum_{m,n\geq 1} \frac{1}{2^{m+n}} |\langle (R-T)e_m, e_n\rangle| = d(S,R) + d(R,T)$ and hence d is a metric.

Let $\mathcal{U} \subset \mathcal{L}(\mathcal{H})_1$ be an open set with respect to the topology compatible with the metric d. Let $S_0 \in \mathcal{U}$ and let $\epsilon > 0$ so that $B_d(S_0, \epsilon) \subset \mathcal{U}$. Choose k such that $\frac{1}{k} + \frac{1}{2^{k-2}} < \epsilon$. Let $\mathcal{V} = \{S \in \mathcal{L}(\mathcal{H})_1 \mid |\langle (S-S_0)e_m, e_n\rangle| < \frac{1}{k},\ 1 \leq m,n \leq k\}$ be a basic weak operator open neighborhood of S_0. If $S \in \mathcal{V}$ then

$$d(S,S_0) = \sum_{m,n=1}^{k} \frac{1}{2^{m+n}} |\langle (S-S_0)e_m, e_n\rangle| + 2\sum_{m\geq k+1}\sum_{n\geq 1} \frac{1}{2^{m+n}} |\langle (S-S_0)e_m, e_n\rangle| \leq$$

$$\sum_{m,n=1}^{k} \frac{1}{2^{m+n}}\frac{1}{k} + 2\sum_{m\geq k+1}\sum_{n\geq 1} \frac{1}{2^{m+n}} (\|S\| + \|S_0\|) \leq \frac{1}{k}\left(\sum_{m=1}^{k}\frac{1}{2^m}\right)\left(\sum_{n=1}^{k}\frac{1}{2^n}\right) + 2\sum_{m\geq k+1}\sum_{n\geq 1}\frac{2}{2^{m+n}} \leq$$

$$\frac{1}{k} + 2\sum_{m\geq k+1}\frac{2}{2^m}\sum_{n\geq 1}\frac{1}{2^n} = \frac{1}{k} + 2\sum_{m\geq k}\frac{1}{2^m} = \frac{1}{k} + 2\frac{1}{2^{k-1}} = \frac{1}{k} + \frac{1}{2^{k-2}} < \epsilon$$

This implies that $\mathcal{V} \subset B_d(S_0, \epsilon) \subset \mathcal{U}$ and hence the metric topology is weaker than the weak operator topology.

Let $\mathcal{V} \subset \mathcal{L}(\mathcal{H})$ be an open set with respect to the weak operator topology and let $S_0 \in \mathcal{V}$. Let $\epsilon > 0$ and $k \geq 1$ so that $\{S \in \mathcal{L}(\mathcal{H}) \mid |\langle (S - S_0)e_m, e_n\rangle| < \epsilon,\ 1 \leq m, n \leq k\} \subset \mathcal{V}$. Let $\mathcal{U} = \{S \in \mathcal{L}(\mathcal{H}) \mid d(S, S_0) < \frac{\epsilon}{2^{2k}}\}$. If $S \in \mathcal{U}$ then for every $1 \leq m, n \leq k$ we have that

$$|\langle (S - S_0)e_m, e_n\rangle| \leq 2^{2k}\sum_{m,n=1}^{k}\frac{1}{2^{m+n}}|\langle (S - S_0)e_m, e_n\rangle| \leq$$

$$2^{2k}\sum_{m,n\geq 1}\frac{1}{2^{m+n}}|\langle (S - S_0)e_m, e_n\rangle| = 2^{2k}d(S, S_0) < 2^{2k}\frac{\epsilon}{2^{2k}} = \epsilon$$

This implies that $\mathcal{U} \subset \mathcal{V}$ and hence the weak operator topology is weaker than the metric topology on $\mathcal{L}(\mathcal{H})_1$. □

Corollary 5.2.

$$\rho(f, g) = \sup_{T\in\mathcal{L}(\mathcal{H})_1} d(f(T), g(T)) + \sup_{T\in\mathcal{L}(\mathcal{H})_1} d(f^{-1}(T), g^{-1}(T))$$

where d is the metric on $\mathcal{L}(\mathcal{H})_1$ defined in Theorem 5.1, defines a metric on $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$. $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ is a complete separable metric topological group with the topology compatible with this metric.

Proof. $\mathcal{L}(\mathcal{H})_1$ is weak operator compact by Theorem 5.1.3, page 306, [10]. From Theorem 5.1 we have that $\mathcal{L}(\mathcal{H})_1$ is a metric space. The conclusion follows from Theorem 2.24. □

5.2. The Subgroup $\mathcal{S}$

Definition 5.3. We say that $T \in \mathcal{L}(\mathcal{H})$ is positive if $\langle Tx, x\rangle \geq 0$ for every $x \in \mathcal{H}$. If $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{M}^+$ will denote the set of all positive elements of $\mathcal{M}$. If T, S are two self-adjoint operators, we say that $S \leq T$ if $T - S \in \mathcal{L}(\mathcal{H})^+$.

Proposition 5.4. *If $T \in \mathcal{L}(\mathcal{H})$ is a bounded linear operator, then T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for each $x \in \mathcal{H}$. In particular, positive operators are self-adjoint.*

Proof. For every $x \in \mathcal{H}$ we have that $\langle Tx, x\rangle - \langle T^*x, x\rangle = \langle Tx, x\rangle - \langle x, Tx\rangle = \langle Tx, x\rangle - \overline{\langle Tx, x\rangle} = 2i\mathrm{Im}(\langle Tx, x\rangle)$. Hence $\langle Tx, x\rangle$ is real if and only if $\langle Tx, x\rangle = \langle T^*x, x\rangle$ for every $x \in \mathcal{H}$. It follows from Proposition 2.19 that T is real if and only if $T^* = T$. $\square$

REMARK 5.5. According to the Proposition 5.4, if $T \in \mathcal{L}(\mathcal{H})^+$ then T is self-adjoint. If $S \leq T$ and $T \leq S$ then $T - S \in \mathcal{L}(\mathcal{H})^+$ and $-(T-S) \in \mathcal{L}(\mathcal{H})^+ \Rightarrow \langle (T-S)x, x\rangle = 0$ for every $x \in \mathcal{H} \Rightarrow T - S = 0$ by Proposition 2.19. This implies that $T = S$ and hence $\leq$ is a partial order on the set of self-adjoint operators.

LEMMA 5.6. *If $T \in \mathcal{L}(\mathcal{H})$ is a self-adjoint, bounded linear operator then*

$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}$$

In particular, if $T \in \mathcal{L}(\mathcal{H})^+$, then $\|T\| = \sup\{\langle Tx, x\rangle \mid \|x\| = 1\}$.

Proof. Let $a = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}$. Since $\{|\langle Tx, x\rangle| \mid \|x\| = 1\} \subset \{|\langle Tx, y\rangle| \mid \|x\| \leq 1,\ \|y\| \leq 1\}$ we have that $a = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\} \leq \sup\{|\langle Tx, y\rangle| \mid \|x\| \leq 1,\ \|y\| \leq 1\} = \|T\|$.

From Proposition 2.18 we have that $\langle Tx, y\rangle = \frac{1}{4}\langle T(x+y), x+y\rangle - \frac{1}{4}\langle T(x-y), x-y\rangle + \frac{1}{4}i\langle T(x+iy), x+iy\rangle - \frac{1}{4}i\langle T(x-iy), x-iy\rangle$ and, since by Proposition 5.4 $\langle Tx, x\rangle$ is real for each $x \in \mathcal{H}$, it follows that $\mathrm{Re}\langle Tx, y\rangle = \frac{1}{4}\langle T(x+y), x+y\rangle - \frac{1}{4}\langle T(x-y), x-y\rangle \Rightarrow$

$$|\mathrm{Re}\langle Tx, y\rangle| \leq \frac{1}{4}|\langle T(x+y), x+y\rangle| + \frac{1}{4}|\langle T(x-y), x-y\rangle| =$$

$$\frac{1}{4}\|x+y\|^2 \left|\langle T\frac{x+y}{\|x+y\|}, \frac{x+y}{\|x+y\|}\rangle\right| + \frac{1}{4}\|x-y\|^2 \left|\langle T\frac{x-y}{\|x-y\|}, \frac{x-y}{\|x-y\|}\rangle\right| \leq$$

$$\frac{1}{4}a(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{4}a(2\|x\|^2 + 2\|y\|^2) \leq a$$

for every $x, y \in \mathcal{H}$ with $\|x\| \leq 1,\ \|y\| \leq 1$. Here we are also using the Paralelogram Law, Proposition 2.7.

Let $x, y \in \mathcal{H}$ such that $\|x\| = \|y\| = 1$ and let $c = \frac{\mathrm{Re}\langle Tx,y\rangle - i\mathrm{Im}\langle Tx,y\rangle}{|\langle Tx,y\rangle|} \in \mathbb{C}$. Then $|c| = \sqrt{\frac{(\mathrm{Re}\langle Tx,y\rangle)^2}{|\langle Tx,y\rangle|^2} + \frac{(\mathrm{Im}\langle Tx,y\rangle)^2}{|\langle Tx,y\rangle|^2}} = 1 \Rightarrow \|cx\| = |c|\,\|x\| = 1$ and $\langle T(cx), y\rangle = c\langle Tx, y\rangle = \frac{\mathrm{Re}\langle Tx,y\rangle - i\mathrm{Im}\langle Tx,y\rangle}{|\langle Tx,y\rangle|}(\mathrm{Re}\langle Tx, y\rangle + i\mathrm{Im}\langle Tx, y\rangle) = \frac{(\mathrm{Re}\langle Tx,y\rangle)^2 + (\mathrm{Im}\langle Tx,y\rangle)^2}{|\langle Tx,y\rangle|} = |\langle Tx, y\rangle|$. It follows that

$\langle T(cx), y\rangle$ is real and positive and, using the previous inequality, we have that $|\langle Tx, y\rangle| = \langle T(cx), y\rangle = |\mathrm{Re}\langle T(cx), y\rangle| \leq a$ for every $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$. This implies that $\|T\| = \sup\{|\langle Tx, y\rangle| \mid \|x\| \leq 1,\ \|y\| \leq 1\} \leq a = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}$ and hence $\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}$. □

Corollary 5.7. *If $S, T \in \mathcal{L}(\mathcal{H})$ and $S - T \geq 0$, then $\|S\| \geq \|T\|$.*

Proof. $S \geq T \geq 0 \Rightarrow \langle Sx, x\rangle \geq \langle Tx, x\rangle$ for every $x \in \mathcal{H}$. It follows from Lemma 5.6 that $\|S\| = \sup\{\langle Sx, x\rangle \mid \|x\| = 1\} \geq \|T\| = \sup\{\langle Tx, x\rangle \mid \|x\| = 1\}$. □

Definition 5.8. If $(T_j)_{j\in J}$ is a net of self-adjoint operators, we say that $(T_j)_{j\in J}$ is bounded above if there exists S a self-adjoint operator such that $T_j \leq S$ for every $j \in J$. The least such S, if exists, is denoted $\sup_{j\in J}\{T_j\}$.

Definition 5.9. A $*-$subalgebra of $\mathcal{L}(\mathcal{H})$ is a subalgebra of $\mathcal{L}(\mathcal{H})$ which is stable with respect to the adjoint operation.

Definition 5.10. Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$. The commutant $\mathcal{M}'$ of $\mathcal{M}$ is the set the set defined as $\mathcal{M}' = \{T \in \mathcal{L}(\mathcal{H}) \mid TS = ST \text{ for every } S \in \mathcal{L}(\mathcal{H})\}$. The bicommutant $\mathcal{M}''$ of $\mathcal{M}$ is $\mathcal{M}'' = (\mathcal{M}')'$.

Definition 5.11. A von Neumann algebra in $\mathcal{H}$ is a $*-$subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}''$. The algebra $\mathcal{L}(\mathcal{H})$ is a von Neumann algebra.

Definition 5.12. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras. A linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ is said to be positive if $\phi(\mathcal{A}^+) \subset \mathcal{B}^+$. We say that ϕ is normal positive if, further, for every increasing net $\{T_j\}_{j\in J} \subset \mathcal{A}^+$ with supremum $T \in \mathcal{A}^+$, the net $\{\phi(T_j)\}_{j\in J}$ has supremum $\phi(T)$.

Proposition 5.13. *Let $\mathcal{A}$ be a von Neumann algebra and $T \in \mathcal{A}$. Then $T \in \mathcal{A}^+$ if and only if $T = S^*S$ for some $S \in \mathcal{A}$.*

Proof. If $T = S^*S$, then T is self adjoint and $\langle Tx, x\rangle = \langle S^*Sx, x\rangle = \langle Sx, Sx\rangle = \|Sx\| \geq 0 \Rightarrow T \geq 0$.

If $T \in \mathcal{A}^+$, then $T^{\frac{1}{2}} \in \mathcal{A}^+$ and $T = T^{\frac{1}{2}}T^{\frac{1}{2}} = (T^{\frac{1}{2}})^*T^{\frac{1}{2}}$. $\square$

DEFINITION 5.14. A $*-$automorphism acting on $\mathcal{L}(\mathcal{H})$ is a bijective mapping $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ satisfying, for every $S, T \in \mathcal{L}(\mathcal{H})$ and every $\lambda \in \mathbb{C}$ the following:

1) $\varphi(ST) = \varphi(S)\varphi(T)$;

2) $\varphi(S + T) = \varphi(S) + \varphi(T)$;

3) $\varphi(\lambda T) = \lambda\varphi(T)$;

4) $\varphi(T^*) = (\varphi(T))^*$.

We denote with $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ the set of all $*-$automorphisms acting on $\mathcal{L}(\mathcal{H})$.

A $*-$anti-automorphism on $\mathcal{L}(\mathcal{H})$ is a bijective mapping $\varphi' : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ satisfying, for every $S, T \in \mathcal{L}(\mathcal{H})$ and every $\lambda \in \mathbb{C}$, $\varphi'(ST) = \varphi'(T)\varphi'(S)$ and the conditions 2)-4) above.

REMARK 5.15. $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is a group under composition.

PROPOSITION 5.16. *If $\mathcal{S}$ is the group generated by the $*-$automorphisms and the $*-$anti-automorphisms and if φ' is any fixed $*-$anti-automorphism on $\mathcal{L}(\mathcal{H})$ then $\mathcal{S} = \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \cup \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$.*

Proof. If $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ then $\varphi'\varphi$ is a $*-$anti-automorphism and hence $\mathcal{A}ut(\mathcal{L}(\mathcal{H})) \cup \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H})) \subset \mathcal{S}$.

If ψ is any $*-$anti-automorphism, let $\varphi = \varphi'^{-1}\psi$. Then φ is linear, $\varphi(T^*) = (\varphi(T))^*$ for every $T \in \mathcal{L}(\mathcal{H})$ and since $\varphi(ST) = (\varphi'^{-1}\psi)(ST) = \varphi'^{-1}(\psi(T)\psi(S)) = \varphi'^{-1}(\psi(S))\varphi'^{-1}(\psi(T)) = \varphi(S)\varphi(T) \Rightarrow \varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \Rightarrow \psi = \varphi'\varphi \in \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ and hence $\mathcal{S} \subset \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \cup \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$. $\square$

PROPOSITION 5.17. *Let $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$. If $S, T \in \mathcal{L}(\mathcal{H})$ are self-adjoint such that $S \leq T$ then $\varphi(S), \varphi(T)$ are self-adjoint and $\varphi(S) \leq \varphi(T)$.*

Proof. Let $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ and let $S, T \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then $S^* = S \Rightarrow (\varphi(S))^* = \varphi(S^*) = \varphi(S)$. Similarly $(\varphi(T))^* = \varphi(T)$.

If $S \leq T$ then $T - S \geq 0 \Rightarrow$ there exists $R \in \mathcal{L}(\mathcal{H})$ such that $T - S = R^*R \Rightarrow \varphi(T-S) = \varphi(R^*)\varphi(R) = (\varphi(R))^*\varphi(R) \geq 0 \Rightarrow \varphi(T) \geq \varphi(S)$. $\square$

PROPOSITION 5.18. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then every element of $\mathcal{A}$ is a linear combination of unitary elements of $\mathcal{A}$.*

Proof. Since every $T \in \mathcal{A}$ can be uniquely expressed in the form $T = T_1 + iT_2$, where $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{i}{2}(T^* - T)$ are self-adjoint elements of $\mathcal{A}$, it is enough to consider the case of a self-adjoint operator $T \in \mathcal{A}$. We may also assume that $\|T\| \leq 1$ by replacing T with $\frac{T}{\|T\|}$. But then $\|Tx\| \leq \|x\| \Rightarrow \langle Tx, Tx\rangle \leq \langle x, x\rangle \Rightarrow I - T^2 \geq 0 \Rightarrow (I - T^2)^{\frac{1}{2}}$ exists and it's positive. Let $U = T + i(I - T^2)^{\frac{1}{2}}$. Then $U \in \mathcal{A}$ and $U^* = T - i(I - T^2)^{\frac{1}{2}}$. Since $I - T^2$ commutes with T, $(I - T^2)^{\frac{1}{2}}$ commutes with T and hence $U^*U = UU^* = (T - i(I - T^2)^{\frac{1}{2}})(T + i(I - T^2)^{\frac{1}{2}}) = T^2 + I - T^2 = I$. Moreover, $T = \frac{1}{2}(U + U^*)$. $\square$

COROLLARY 5.19. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $T \in \mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{A}$ if and only if $UT = TU$ for every unitary operator $U \in \mathcal{A}'$.*

Proof. If $T \in \mathcal{A} = \mathcal{A}''$ then T commutes with every operator of $\mathcal{A}'$, hence with every unitary operator $U \in \mathcal{A}'$.

If $UT = TU$ for every unitary operator $U \in \mathcal{A}'$ then, since by Proposition 5.18 every operator $S \in \mathcal{A}'$ is a linear combination of unitary operators of $\mathcal{A}'$, we have that T commutes with every operator of $\mathcal{A}$, and hence $T \in \mathcal{A}'' = \mathcal{A}$. $\square$

THEOREM 5.20. *If $(T_j)_{j\in J}$ is a net of self-adjoint operators on a Hilbert space $\mathcal{H}$, which is increasing and bounded above, then there exists $T \in \mathcal{L}(\mathcal{H})$ self-adjoint, such that $T_j \xrightarrow{so} T$. Moreover, $T = \sup_{j\in J}\{T_j\}$.*

Proof. Let $(T_j)_{j\in J}$ be an increasing, bounded above net of self-adjoint operators acting on the Hilbert space $\mathcal{H}$. By assumption, there exists S a self-adjoint operator such that $S \geq T_j$ for every $j \in J$. We may assume that $T_j \in \mathcal{L}(\mathcal{H})^+$, by considering the net $T_j - T_{j_0}$ for

$j \geq j_0$ if necessary, where T_{j_0} is some fixed element of the original net. If $M = \|S\|$ then by Corollary 5.7 we have that $\|T_j\| \leq M$ for all $j \in J$. This implies that $|\langle T_j x, x\rangle| \leq \|T_j x\| \; \|x\| \leq \|T_j\| \; \|x\|^2 \leq M\|x\|^2 \Rightarrow \langle T_j x, x\rangle$ is an increasing net, bounded above, and hence convergent. It follows from the polarization identity (Corollary 2.18) that $\langle T_j x, y\rangle$ is convergent for all $x, y \in \mathcal{H}$. If $u : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is defined as $u(x,y) = \lim_j \langle T_j x, y\rangle$ then, since $u(x,y) = \lim_j \langle T_j x, y\rangle = \lim_j \overline{\langle T_j y, x\rangle} = \overline{u(y,x)}$, u is a bilinear form on $\mathcal{H} \times \mathcal{H}$. Since $|u(x,y)| = \lim_j |\langle T_j x, y\rangle| \leq M \; \|x\| \; \|y\|$, we have that u is bounded. Hence, there exists $T \in \mathcal{L}(\mathcal{H})$ such that $u(x,y) = \langle Tx, y\rangle$. Since $\langle Tx, y\rangle = u(x,y) = \overline{u(y,x)} = \overline{\langle Ty, x\rangle} = \langle x, Ty\rangle$, we have that T is self-adjoint. Clearly $\langle Tx, x\rangle = u(x,x) \geq \langle T_j x, x\rangle \Rightarrow T \geq T_j$ for every $j \in J$ and $\|T\| = \sup_{\|x\| \leq 1, \; \|y\| \leq 1} \langle Tx, y\rangle = \sup_{\|x\| \leq 1, \; \|y\| \leq 1} |u(x,y)| \leq M$. Since $\|(T - T_j)x\|^2 = \|(T - T_j)^{\frac{1}{2}} (T - T_j)^{\frac{1}{2}} x\|^2 \leq \|T - T_j\| \; \|(T - T_j)^{\frac{1}{2}} x\|^2 \leq 2M \langle (T - T_j)x, x\rangle = 2M(\langle Tx, x\rangle - \langle T_j x, x\rangle) = 2M(u(x,x) - \langle T_j x, x\rangle) \to 0$, it follows that $T_j \xrightarrow{so} T$.

Let S be self-adjoint and such that $T_j \leq S$ for every $j \in J$. Then $\langle T_j x, x\rangle \leq \langle Sx, x\rangle$ for every $x \in \mathcal{H}$. Since $T_j \xrightarrow{so} T$ we have that $T_j \xrightarrow{wo} T$ and hence $\langle Tx, x\rangle \leq \langle Sx, x\rangle$ for every $x \in \mathcal{H} \Rightarrow \langle (S - T)x, x\rangle \geq 0$ for every $x \in \mathcal{H} \Rightarrow S - T \geq 0 \Rightarrow S \geq T$ and hence $T = \sup_{j \in J}\{T_j\}$. $\square$

Corollary 5.21. *If $\{A_j\} \subset \mathcal{A}^+$ is an increasing net, bounded above with supremum A, then $A \in \mathcal{A}^+$.*

Proof. Let $U \in \mathcal{A}'$ be unitary. Then $UAU^* = \sup_j\{UA_jU^*\} = \sup_j\{A_j\} = A$, and hence A commutes with every unitary operator in $\mathcal{A}'$. According with the Corollary 5.19, $A \in \mathcal{A}$. Since A is the supremum of positive operators, A is also positive. $\square$

Corollary 5.22. *Every $*-$automorphism acting on $\mathcal{L}(\mathcal{H})$ is a normal positive mapping.*

Proof. Let $\varphi \in Aut(\mathcal{L}(\mathcal{H}))$. By Lemma 5.17 we have that φ preserves order and hence $\varphi(\mathcal{L}(\mathcal{H})^+) \subset \mathcal{L}(\mathcal{H})^+$. Let $\{T_j\}_{j \in J} \subset \mathcal{L}(\mathcal{H})^+$ be a net with $T = \sup_{j \in J}\{T_j\} \in \mathcal{L}(\mathcal{H})^+$. Since φ preserves order we have that $\{\varphi(T_j)\}_{j \in J}$ is increasing and bounded above by $\varphi(T)$. Let $S = \sup_{j \in J} \varphi(T_j)$. Then $\varphi(T_j) \leq S \leq \varphi(T)$ for every $j \in J \Rightarrow T_j \leq \varphi^{-1}(S) \leq T$ for every $j \in J \Rightarrow \varphi^{-1}(S) = T \Rightarrow S = \varphi(T)$. $\square$

Proposition 5.23. *If φ is a $*-$automorphism acting on $\mathcal{L}(\mathcal{H})$ then $\varphi(T) \in \mathcal{L}(\mathcal{H})_1$ for every $T \in \mathcal{L}(\mathcal{H})_1$.*

Proof. If $T \in \mathcal{L}(\mathcal{H})$ and $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ then $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle \leq \|T\|^2\langle x, x\rangle \Rightarrow T^*T \leq \|T\|^2 I \Rightarrow (\varphi(T))^*\varphi(T) = \varphi(T^*)\varphi(T) = \varphi(T^*T) \leq \|T\|^2\varphi(I) = \|T\|^2 I \Rightarrow \|\varphi(T)\|^2 \leq \|T\|^2 \Rightarrow$ if $\|T\| \leq 1$ then $\|\varphi(T)\| \leq 1$ and hence $\varphi(T) \in \mathcal{L}(\mathcal{H})_1$ for every $T \in \mathcal{L}(\mathcal{H})_1$. □

Proposition 5.24. ★ *If $\mathcal{S}$ is the group defined in Proposition 5.16 then $\mathcal{S} \subset \mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$.*

Proof. If $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ then $\varphi|_{\mathcal{L}(\mathcal{H})_1} : \mathcal{L}(\mathcal{H})_1 \to \mathcal{L}(\mathcal{H})_1$ by Proposition 5.23 and it is normal by Corollary 5.22. According to Theorem 2, page 59 [3] we have that $\varphi|_{\mathcal{L}(\mathcal{H})_1}$ is continuous with respect to the weak operator topology. Similarly $\varphi^{-1}|_{\mathcal{L}(\mathcal{H})_1}$ is weak operator continuous and hence $\mathcal{A}ut(\mathcal{L}(\mathcal{H})) \subset \mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$. Since $\mathcal{S} = \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \cup \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ where φ' is any fixed $*-$anti-automorphism, it remains to show that there exists φ' a $*-$anti-automorphism such that $\varphi'|_{\mathcal{L}(\mathcal{H})_1}$ is continuous with respect to the weak operator topology.

Let $\{e_l\}_{l\geq 1}$ be an orthonormal basis for $\mathcal{H}$. If $x = \sum_{l\geq 1} a_l e_l$, let $Vx = \sum_{l\geq 1} \overline{a_l} e_l$. Then $V : \mathcal{H} \to \mathcal{H}$, $V(\lambda x + \mu y) = \overline{\lambda} Vx + \overline{\mu} Vy$ for every $x, y \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$ and, if $x = \sum_{l\geq 1} a_l e_l \in \mathcal{H}$ and $y = \sum_{l\geq 1} b_l e_l \in \mathcal{H}$, then $\langle Vx, Vy\rangle = \langle \sum_{l\geq 1} \overline{a_l} e_l, \sum_{l\geq 1} \overline{b_l} e_l\rangle = \sum_{l\geq 1} \overline{a_l} b_l = \langle y, x\rangle$. Also note that $V^2 = I$ and hence $V^{-1} = V$ and that $\|Vx\|^2 = |\langle Vx, Vx\rangle| = |\langle x, x\rangle| = \|x\|^2$.

Let $\varphi' : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be defined as $\varphi'(T) = VT^*V^{-1}$. Let $T \in \mathcal{L}(\mathcal{H})$, $x, y \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$. Then $\varphi(T)(\lambda x + \mu y) = VTV^{-1}(\lambda x + \mu y) = VT(\overline{\lambda} V^{-1} x + \overline{\mu} V^{-1} y) = V(\overline{\lambda} TV^{-1} x + \overline{\mu} TV^{-1} y) = \lambda VTV^{-1} x + \mu VTV^{-1} y = \lambda\varphi'(T)x + \mu\varphi'(T)y \Rightarrow \varphi'(T)$ is linear. Since $\|\varphi'(T)x\| = \|VTV^{-1}x\| = \|TV^{-1}x\| \leq \|T\| \cdot \|V^{-1}x\| = \|T\| \cdot \|x\|$ we have that $\varphi'(T)$ is bounded. Thus $\varphi'(T) \in \mathcal{L}(\mathcal{H})$ for every $T \in \mathcal{L}(\mathcal{H})$. We will show that φ' is a $*-$anti-automorphism and that $\varphi'|_{\mathcal{L}(\mathcal{H})_1}$ is continuous with respect to the weak operator topology.

If $S, T \in \mathcal{L}(\mathcal{H})$ and if $\lambda \in \mathbb{C}$ we have that $\varphi'(S + T) = V(S + T)^*V^{-1} = VS^*V^{-1} + VT^*V^{-1} = \varphi'(S) + \varphi'(T)$; $\varphi'(\lambda T) = V(\lambda T)^*V^{-1} = V(\overline{\lambda} T^*)V^{-1} = \lambda VT^*V^{-1} = \lambda\varphi'(T)$ and $\varphi'(ST) = V(ST)^*V^{-1} = VT^*S^*V^{-1} = VT^*V^{-1}VS^*V^{-1} = \varphi'(T)\varphi'(S)$. If $T \in \mathcal{L}(\mathcal{H})$, since $\langle \varphi'(T)^*x, y\rangle = \langle x, \varphi'(T)y\rangle = \langle x, VT^*V^{-1}y\rangle = \langle T^*V^{-1}y, V^{-1}x\rangle = \langle V^{-1}y, TV^{-1}x\rangle = \langle VTV^{-1}x, y\rangle = \langle \varphi(T^*)x, y\rangle$ for every $x, y \in \mathcal{H}$, we have that $\varphi'(T)^* = \varphi'(T^*)$.

Let $\psi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be defined as $\psi(T) = V^{-1}T^*V$. Same arguments as before shows that $\psi(T) \in \mathcal{L}(\mathcal{H})$ and that $\psi(T^*) = \psi(T)^*$. Since $\varphi'(\psi(T)) = V\psi(T)^*V^{-1} = V\psi(T^*)V^{-1} = VV^{-1}(T^*)^*VV^{-1} = T$ and since $\psi(\varphi'(T)) = V^{-1}\varphi'(T)^*V = V^{-1}\varphi'(T^*)V = V^{-1}V(T^*)^*V^{-1}V = T$ for every $T \in \mathcal{L}(\mathcal{H}) \Rightarrow \varphi'$ and ψ are inverses of each other and hence bijections.

To show continuity, let $\{T_j\}_{j\in J} \subset \mathcal{L}(\mathcal{H})$ be such that $T_j \xrightarrow{wo} T \in \mathcal{L}(\mathcal{H})$. Then $T_j^* \xrightarrow{wo} T^* \Rightarrow \langle T_j^*x, y\rangle \to \langle T^*x, y\rangle$ for every $x, y \in \mathcal{H}$. In particular, if we replace x with $V^{-1}x$ and y with $V^{-1}y$, then $\langle T_j^*V^{-1}x, V^{-1}y\rangle \to \langle T^*V^{-1}x, V^{-1}y\rangle \Rightarrow \langle y, VT_j^*V^{-1}x\rangle \to \langle y, VT^*V^{-1}x\rangle \Rightarrow \langle y, \varphi'(T_j)x\rangle \to \langle y, \varphi'(T)x\rangle \Rightarrow \varphi'(T_j) \xrightarrow{wo} \varphi'(T)$ and hence φ' is continuous with respect to the weak operator topology. $\square$

Definition 5.25. If $\rho : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a linear bijection and $\|\rho(T)\| = \|T\|$ for every $T \in \mathcal{L}(\mathcal{H})$ we say that ρ is a linear bijective isometry. We denote with LBIG the set of all linear bijective isometries on $\mathcal{L}(\mathcal{H})$.

Proposition 5.26. *LBIG is a group under composition.*

Proof. Let ρ, $\eta \in$ LBIG and let $T \in \mathcal{L}(\mathcal{H})$. Obviously $\rho\eta$ is linear, bijective and $\|\rho\eta(T)\| = \|\eta(T)\| = \|T\|$ and hence $\rho\eta \in$ LBIG. The identity mapping $\mathfrak{id} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is the identity element of the group LBIG.

If $\rho \in$ LBIG then ρ^{-1} is bijective. If $S, T \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ then $\rho^{-1}(\alpha T + S) = \rho^{-1}(\alpha\rho[\rho^{-1}(T)] + \rho[\rho^{-1}(S)]) = \rho^{-1}(\rho[\alpha\rho^{-1}(T) + \rho^{-1}(S)]) = \alpha\rho^{-1}(T) + \rho^{-1}(S)$ and hence ρ^{-1} is linear. Since $\|T\| = \|\rho(\rho^{-1}(T))\| = \|\rho^{-1}(T)\|$ we have that ρ^{-1} is an isometry $\Rightarrow \rho^{-1} \in$ LBIG and hence LBIG is a group. $\square$

Theorem 5.27. ★ *If $\mathcal{S}$ is the group defined in Proposition 5.16 then $cl_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H}))) \subset \mathcal{S}$. Here, the topology on $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ is the topology compatible with the metric ρ defined in Corollary 5.2.*

Proof. Let $f \in \mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H})))$. Let $\{\varphi_j\}_{j\in J} \subset \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ be such that $\varphi_j|_{\mathcal{L}(\mathcal{H})_1} \xrightarrow{\rho} f \in \mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$. Since $\rho(\varphi_j, f) = \sup_{T\in\mathcal{L}(\mathcal{H})_1} d(\varphi_j(T), f(T)) + \sup_{T\in\mathcal{L}(\mathcal{H})_1} d(\varphi_j^{-1}(T), f^{-1}(T))$ we have that $d(\varphi_j(T), f(T)) \to 0$ and $d(\varphi_j^{-1}(T), f^{-1}(T)) \to 0$ for every $T \in \mathcal{L}(\mathcal{H})_1$ and, since the weak operator topology and the d-metric topology on $\mathcal{L}(\mathcal{H})_1$ are equivalent, we have that $\langle \varphi_j(T)x, y\rangle \to \langle f(T)x, y\rangle$ and $\langle \varphi_j^{-1}(T)x, y\rangle \to \langle f^{-1}(T)x, y\rangle$ for every $T \in \mathcal{L}(\mathcal{H})_1$ and every $x, y \in \mathcal{H}$.

Define $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as $\varphi(T) = \|T\| f\left(\frac{T}{\|T\|}\right)$ if $T \neq 0$ and $\varphi(0) = 0$. Note that since $0 = \langle 0x, y\rangle = \langle \varphi_j(0)x, y\rangle \to \langle f(0)x, y\rangle$ for every $x, y \in \mathcal{H}$ we have that $f(0) = 0 = \varphi(0)$. If $0 \neq T \in \mathcal{L}(\mathcal{H})_1$, then $\langle \varphi_j(T)x, y\rangle = \|T\|\langle \varphi_j\left(\frac{T}{\|T\|}\right)x, y\rangle \to \|T\|\langle f\left(\frac{T}{\|T\|}\right)x, y\rangle = \langle \varphi(T)x, y\rangle$ for every $x, y \in \mathcal{H}$ and since $\langle \varphi_j(T)x, y\rangle \to \langle f(T)x, y\rangle$ for every $x, y \in \mathcal{H}$ we have that $\varphi(T) = f(T)$ for every $T \in \mathcal{L}(\mathcal{H})_1$ and hence $\varphi|_{\mathcal{L}(\mathcal{H})_1} = f$. We also have that $\langle x, y\rangle = \langle \varphi_j(I)x, y\rangle \to \langle f(I)x, y\rangle = \langle \varphi(I)x, y\rangle \Rightarrow \varphi(I) = f(I) = I$.

Let $S \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. If $S = 0$ or $\lambda = 0$ then $\lambda S = 0 \Rightarrow \varphi(\lambda S) = 0 = \lambda\varphi(S)$. If $S \neq 0$ and $\lambda \neq 0$ then $\langle \varphi_j(\lambda S)x, y\rangle = \lambda\|S\|\langle \varphi_j\left(\frac{S}{\|S\|}\right)x, y\rangle \to \lambda\|S\|\langle f\left(\frac{S}{\|S\|}\right)x, y\rangle = \langle \lambda\varphi(S)x, y\rangle$ and $\langle \varphi_j(\lambda S)x, y\rangle = \|\lambda S\|\langle \varphi_j\left(\frac{\lambda S}{\|\lambda S\|}\right)x, y\rangle \to \|\lambda S\|\langle f\left(\frac{\lambda S}{\|\lambda S\|}\right)x, y\rangle = \langle \varphi(\lambda S)x, y\rangle$ for every $x, y \in \mathcal{H}$ and hence $\lambda\varphi(S) = \varphi(\lambda S)$.

Let $S, T \in \mathcal{L}(\mathcal{H})$. If $S = 0$ then $\varphi(S + T) = \varphi(T) = \varphi(S) + \varphi(T)$. Similarly if $T = 0$. If $S + T = 0$ then $-S = T \Rightarrow \varphi(S + T) = 0 = \varphi(S) - \varphi(S) = \varphi(S) + \varphi(-S) = \varphi(S) + \varphi(T)$. If $S \neq 0$, $T \neq 0$ and $S + T \neq 0$ then $\langle \varphi_j(S)x, y\rangle + \langle \varphi_j(T)x, y\rangle = \langle \varphi_j(S + T)x, y\rangle = \|S + T\|\langle \varphi_j\left(\frac{S+T}{\|S+T\|}\right)x, y\rangle \to \|S + T\|\langle f\left(\frac{S+T}{\|S+T\|}\right)x, y\rangle = \langle \varphi(S + T)x, y\rangle$ for every $x, y \in \mathcal{H}$. Similarly $\langle \varphi_j(S)x, y\rangle \to \langle \varphi(S)x, y\rangle$ and $\langle \varphi_j(T)x, y\rangle \to \langle \varphi(T)x, y\rangle$ for every $x, y \in \mathcal{H}$. Hence $\langle \varphi(S + T)x, y\rangle = \langle \varphi(S)x, y\rangle + \langle \varphi(T)x, y\rangle$ for every $x, y \in \mathcal{H} \Rightarrow \varphi(S + T) = \varphi(S) + \varphi(T)$.

Define $\psi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ as $\psi(T) = \|T\| f^{-1}\left(\frac{T}{\|T\|}\right)$ if $T \neq 0$ and $\psi(0) = 0$. By the same reasoning as before we have that $\psi|_{\mathcal{L}(\mathcal{H})_1} = f^{-1}$ and ψ is linear. If $0 \neq T \in \mathcal{L}(\mathcal{H})_1$ then $\varphi(\psi(T)) = f(f^{-1}(T)) = T$ and $\psi(\varphi(T)) = f^{-1}(f(T)) = T$. If $0 \neq T \in \mathcal{L}(\mathcal{H})$, let $\lambda > 0$ be such that $\|\lambda T\| \leq 1$. Then $\varphi(\psi(T)) = \frac{1}{\lambda}\varphi(\psi(\lambda T)) = \frac{1}{\lambda} f(f^{-1}(\lambda T)) = \frac{1}{\lambda}\lambda T = T$ and similarly

$\psi(\varphi(T)) = T$. If $T = 0$ then $\varphi(\psi(0)) = 0$ and $\psi(\varphi(0)) = 0$. Thus φ and ψ are inverses of each other and hence φ is a bijection and $\varphi^{-1} = \psi$.

Let $T \in \mathcal{L}(\mathcal{H})$. Since $\langle x, \varphi_j(T)y\rangle = \langle(\varphi_j(T))^* x, y\rangle = \langle\varphi_j(T^*)x, y\rangle \to \langle\varphi(T^*)x, y\rangle$ for every $x, y \in \mathcal{H}$ and since $\langle x, \varphi_j(T)y\rangle \to \langle x, \varphi(T)y\rangle$ for every $x, y \in \mathcal{H}$, we have that $\langle\varphi(T^*)x, y\rangle = \langle x, \varphi(T)y\rangle$ for every $x, y \in \mathcal{H} \Rightarrow \varphi(T^*) = (\varphi(T))^*$ for every $T \in \mathcal{L}(\mathcal{H})$.

If $T \in \mathcal{L}(\mathcal{H})_1$ then $\|\varphi(T)\| = \|f(T)\| \leq 1 \Rightarrow \|\varphi\| = \sup_{T\in\mathcal{L}(\mathcal{H})_1} \|\varphi(T)\| \leq 1$. Let $T \in \mathcal{L}(\mathcal{H})$. Then $\|\varphi(T)\| \leq \|\varphi\| \cdot \|T\| \leq \|T\|$. Similarly $\|\varphi^{-1}\| = \sup_{T\in\mathcal{L}(\mathcal{H})_1} \|\varphi^{-1}(T)\| \leq 1$ and hence $\|\varphi^{-1}(T)\| \leq \|T\|$. Replace T with $\varphi(T)$ in the last inequality and get $\|T\| = \|\varphi^{-1}(\varphi(T))\| \leq \|\varphi(T)\|$ and hence $\|\varphi(T)\| = \|T\|$.

Thus $\varphi \in$ LBIG. Since $\varphi(I) = I$, according to Theorem 7 and Corollary 11 of [9] we have that φ is either a $*-$automorphism or a $*-$anti-automorphism. It follows from the definition of $\mathcal{S}$ that $\varphi \in \mathcal{S}$ and hence $\mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H}))) \subset \mathcal{S}$. $\square$

Corollary 5.28. ★ *$\mathcal{S}$ is a closed subgroup of $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$.*

Proof. $\mathcal{S} \subset \mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ by Proposition 5.24. Let φ' be any $*-$anti-automorphism of $\mathcal{L}(\mathcal{H})$. Since $\mathcal{S} = \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \cup \varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ by Proposition 5.16 and since $\mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H}))) \subset \mathcal{S}$ by Theorem 5.27, we have that $\mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{S}) = \mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H}))\cup\varphi'\mathcal{A}ut(\mathcal{L}(\mathcal{H}))) = \mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H})))\cup\varphi'\mathrm{cl}_{\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)}(\mathcal{A}ut(\mathcal{L}(\mathcal{H}))) \subset \mathcal{S}\cup\varphi'\mathcal{S} = \mathcal{S}\cup\mathcal{S} = \mathcal{S} \Rightarrow \mathcal{S}$ is a closed subgroup of $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1$. $\square$

5.3. The Surjection

Definition 5.29. Let $\mathcal{H}$ be a Hilbert space of dimension n. A family $(U_{i,j})_{1\leq i,j\leq n}$ of operators in $\mathcal{L}(\mathcal{H})$ is called a self-adjoint system of $n \times n$ matrix units if $U_{i,j}U_{k,l} = 0$ for $j \neq k$, $U_{i,j}U_{j,k} = U_{i,k}$, $\sum_{1\leq i\leq n} U_{i,i} = I$ and $U^*_{i,j} = U_{j,i}$.

If $\mathcal{H}$ is infinite dimensional, a family $(U_{i,j})_{1\leq i,j\leq n}$ of operators in $\mathcal{L}(\mathcal{H})$ is called a self-adjoint system of operator units if $U_{i,j}U_{k,l} = 0$ for $j \neq k$, $U_{i,j}U_{j,k} = U_{i,k}$, $U^*_{i,j} = U_{j,i}$ and $\sum_{i\geq 1} U_{i,i} = I$, with convergence of $\sum_{i\geq 1} U_{i,i}$ in the strong operator topology.

PROPOSITION 5.30. *The system of $n \times n$ matrix units in finite dimensional Hilbert space and the system of operator units in infinite dimensional Hilbert space as in Definition 5.29 exist.*

Proof. In finite dimensional case $U_{i,j}$ corresponds to the matrix with all entries 0 except in position (i, j), where the entry is 1.

In the infinite dimensional case, let $\{e_l\}_{l \in L}$ be an orthonormal basis for $\mathcal{H}$, and define $U_{i,j}$ for every e_l as

$$U_{i,j}(e_l) = \begin{cases} 0 & \text{if } j \neq l \\ e_i & \text{if } j = l \end{cases}$$

It is obvious that $U_{i,j}$'s are linear operators. We need to show that $U_{i,j}U_{k,l} = 0$ if $j \neq k$, $U_{i,j}U_{j,k} = U_{i,k}$, $\sum_{i \geq 1} U_{i,i} = I$ and $U_{i,j}^* = U_{j,i}$. Let $x = \sum_{l \in L} a_l e_l \in \mathcal{H}$.

If $j \neq k$ then $U_{i,j}U_{k,m}(x) = U_{i,j}U_{k,m}(\sum_{l \in L} a_l e_l) = U_{i,j}(\sum_{l \in L} a_l U_{k,m}(e_l)) = U_{i,j}(a_m U_{k,m}(e_m)) = a_m U_{i,j}(e_k) = 0$.

$U_{i,j}U_{j,k}(x) = U_{i,j}U_{j,k}(\sum_{l \in L} a_l e_l) = U_{i,j}(\sum_{l \in L} a_l U_{j,k}(e_l)) = U_{i,j}(a_k U_{j,k}(e_k)) = a_k U_{i,j}(e_j) = a_k e_i$. On the other hand, $U_{i,k}(x) = U_{i,k}(\sum_{l \in L} a_l e_l) = \sum_{l \in L} a_l U_{i,k}(e_l) = a_k U_{i,k}(e_k) = a_k e_i$, and hence $U_{i,j}U_{j,k} = U_{i,k}$.

If $y = \sum_{l \in L} b_l e_l \in \mathcal{H}$, then

$$\langle U_{i,j}(x), y \rangle = \langle U_{i,j}(\sum_{l \in L} a_l e_l), \sum_{l \in L} b_l e_l \rangle = \langle \sum_{l \in L} a_l U_{i,j}(e_l), \sum_{l \in L} b_l e_l \rangle = \langle a_j U_{i,j}(e_j), \sum_{l \in L} b_l e_l \rangle =$$

$$= a_j \langle e_i, \sum_{l \in L} b_l e_l \rangle = a_j \overline{b_i}$$

On the other hand,

$$\langle x, U_{j,i}(y) \rangle = \langle \sum_{l \in L} a_l e_l, U_{j,i}(\sum_{l \in L} b_l e_l) \rangle = \langle \sum_{l \in L} a_l e_l, \sum_{l \in L} b_l U_{j,i}(e_l) \rangle = \langle \sum_{l \in L} a_l e_l, b_i U_{j,i}(e_i) \rangle =$$

$$= \overline{b_i} \langle \sum_{l \in L} a_l e_l, e_j \rangle = \overline{b_i} a_j$$

and hence $U_{i,j}^* = U_{j,i}$.

Since $\| \sum_{1\leq i\leq n} U_{i,i}(x) - x\|^2 = \| \sum_{i>n} a_i e_i\|^2 = \sum_{i>n} |a_i|^2 \to 0$ as $n \to \infty$ for every $x \in \mathcal{H}$ we have that $\sum_{i\geq 1} U_{i,i}$ converges in the strong operator topology to I. $\square$

PROPOSITION 5.31. *If $\mathcal{H}$ is finite dimensional, then for every $*-$automorphism φ acting on $\mathcal{L}(\mathcal{H})$ there is an unitary operator W such that $\varphi(T) = WTW^*$ for every $T \in \mathcal{L}(\mathcal{H})$.*

Proof. Let $n = \dim(\mathcal{H})$ and let φ be a $*-$automorphism on $\mathcal{L}(\mathcal{H})$. If P is an orthogonal projection, then $(\varphi(P))^2 = \varphi(P)\varphi(P) = \varphi(P^2) = \varphi(P)$ and $(\varphi(P))^* = \varphi(P^*) = \varphi(P)$, and hence $\varphi(P)$ is an orthogonal projection. If P_1 and P_2 are two orthogonal projections such that $P_1 \geq P_2$, then $P_1 - P_2$ is an orthogonal projection, and $\varphi(P_1 - P_2) = \varphi(P_1) - \varphi(P_2)$ is an orthogonal projection, and then $\varphi(P_1) \geq \varphi(P_2)$. Hence φ preserves the order of projections and sends minimal nonzero projections into minimal nonzero projections. If U is a partial isometry, then $(\varphi(U))^*\varphi(U) = \varphi(U^*)\varphi(U) = \varphi(U^*U)$ is an orthogonal projection, since U^*U is, and hence $\varphi(U)$ is a partial isometry.

Let $(U_{i,j})_{1\leq i,j\leq n}$ be a self-adjoint system of $n \times n$ matrix units as in Definition 5.29. Note that since $U_{i,i}^2 = U_{i,i}$, $U_{i,i}^* = U_{i,i}$ and $U_{i,i}U_{j,j} = 0$ for $i \neq j$, then $U_{i,i}$ is a family of nonzero orthogonal projections with sum I. Also note that since $U_{i,j}U_{i,j}^* = U_{i,j}U_{j,i} = U_{i,i}$ is an orthogonal projection, then each $U_{i,j}$ is a partial isometry. Since $U_{i,i}$ is a minimal nonzero projection, we have that $U_{i,i}(\mathcal{H})$ is 1-dimensional for every $1 \leq i \leq n$. Since $\varphi(U_{i,i})$ is also a minimal nonzero projection, we have that $\varphi(U_{i,i})(\mathcal{H})$ is 1-dimensional.

Let $e_1 \in U_{1,1}(\mathcal{H})$ and $f_1 \in \varphi(U_{1,1})(\mathcal{H})$ be such that $\|e_1\| = 1$ and $\|f_1\| = 1$. For every $l \geq 1$ let $e_l = U_{l,1}(e_1)$ and $f_l = \varphi(U_{l,1})(f_1)$. If $i \neq j$, then $\langle e_i, e_j\rangle = \langle U_{i,1}(e_1), U_{j,1}(e_1)\rangle = \langle U_{1,j}U_{i,1}(e_1), e_1\rangle = \langle 0(e_1), e_1\rangle = 0$ and $\langle e_i, e_i\rangle = \langle U_{i,1}(e_1), U_{i,1}(e_1)\rangle = \langle e_1, U_{1,i}U_{i,1}(e_1)\rangle = \langle e_1, U_{1,1}(e_1)\rangle = \langle e_1, e_1\rangle = 1$. Hence, $\{e_i\}_{1\leq i\leq n}$ is orthonormal and therefore an orthonormal basis since any orthonormal set is independent and its size equals the dimension of the space. A similar argument shows that $\{f_i\}_{1\leq i\leq n}$ is also orthonormal basis.

Define $W : \mathcal{H} \to \mathcal{H}$ by $W(e_l) = f_l$ for every $1 \leq l \leq n$. It is clear that W is an invertible operator. If $x = \sum a_i e_i$, then $\|W(x)\|^2 = \|W(\sum a_i e_i)\|^2 = \|\sum a_i f_i\|^2 = \sum |a_i|^2 = \|x\|^2$.

Hence, W is an isometry and, since it is surjective, W is unitary. Next we will show that $\varphi(U_{i,j}) = WU_{i,j}W^*$.

Note first that $WU_{l,1}(e_1) = W(e_l) = f_l = \varphi(U_{l,1})(f_1) = \varphi(U_{1,l})W(e_1)$. If $l \neq 1$, then $WU_{l,1}(e_l) = WU_{l,1}U_{l,1}(e_1) = W0(e_1) = 0$ and $\varphi(U_{l,1})W(e_l) = \varphi(U_{l,1})(f_l) = \varphi(U_{l,1})\varphi(U_{l,1})(f_1) = \varphi(U_{l,1}U_{l,1})(f_1) = \varphi(0)(f_1) = 0(f_1) = 0$. Since $\{e_i\}_{1\leq i\leq n}$ and $\{f_i\}_{1\leq i\leq n}$ are orthonormal bases, we have that $\varphi(U_{l,1})W = WU_{l,1} \Rightarrow \varphi(U_{l,1}) = WU_{l,1}W^*$ for every $1 \leq l \leq n$.

For every $1 \leq i,j \leq n$ we have that $\varphi(U_{i,j}) = \varphi(U_{i,1}U_{1,j}) = \varphi(U_{i,1})\varphi(U_{1,j}) = \varphi(U_{i,1})\varphi(U_{j,1}^*) = \varphi(U_{i,1})(\varphi(U_{j,1}))^* = (WU_{i,1}W^*)(WU_{j,1}W^*)^* = WU_{i,1}W^*WU_{j,1}^*W^* = WU_{i,1}U_{1,j}W^* = WU_{i,j}W^*$.

The system $(U_{i,j})_{1\leq i,j\leq n}$ is linearly independent and the dimension of the linear $\mathrm{span}(U_{i,j})$ is n^2. Since the dimension of $\mathcal{L}(\mathcal{H})$ is n^2, we have that $\mathcal{L}(\mathcal{H}) = \mathrm{span}(U_{i,j})$. Hence, for every $T \in \mathcal{L}(\mathcal{H})$, $T = \sum_{i,j} a_{ij}U_{i,j}$. This implies that $\varphi(T) = \varphi(\sum_{i,j} a_{ij}U_{i,j}) = \sum_{i,j} a_{ij}\varphi(U_{i,j}) = \sum_{i,j} a_{ij}WU_{i,j}W^* = W\sum_{i,j} a_{ij}U_{i,j}W^* = WTW^*$. $\square$

Proposition 5.32. *If $\mathcal{H}$ is a separable Hilbert space, then for every $*-$automorphism φ acting on $\mathcal{L}(\mathcal{H})$ there is an unitary operator W such that $\varphi(T) = WTW^*$ for every $T \in \mathcal{L}(\mathcal{H})$.*

Proof. Let φ be a $*-$automorphism on $\mathcal{L}(\mathcal{H})$. If P is an orthogonal projection, then $(\varphi(P))^2 = \varphi(P)\varphi(P) = \varphi(P^2) = \varphi(P)$ and $(\varphi(P))^* = \varphi(P^*) = \varphi(P)$, and hence $\varphi(P)$ is an orthogonal projection. If P_1 and P_2 are two orthogonal projections such that $P_1 \geq P_2$, then $P_1 - P_2$ is an orthogonal projection, and $\varphi(P_1 - P_2) = \varphi(P_1) - \varphi(P_2)$ is an orthogonal projection, and then $\varphi(P_1) \geq \varphi(P_2)$. Hence φ preserves the order of projections and sends minimal nonzero projections into minimal nonzero projections. If U is a partial isometry, then $(\varphi(U))^*\varphi(U) = \varphi(U^*)\varphi(U) = \varphi(U^*U)$ is an orthogonal projection, since U^*U is, and hence $\varphi(U)$ is a partial isometry.

Let $(U_{i,j})_{i,j\in I}$ be a self-adjoint system of operator units, as in Definition 5.29. Note that since $U_{i,i}^2 = U_{i,i}$, $U_{i,i}^* = U_{i,i}$ and $U_{i,i}U_{j,j} = 0$ for $i \neq j$, then $U_{i,i}$ is a family of nonzero orthogonal projections. Also note that since $U_{i,j}U_{i,j}^* = U_{i,j}U_{j,i} = U_{i,i}$ is an orthogonal projection, then each $U_{i,j}$ is a partial isometry. Since $U_{i,i}$ is a minimal nonzero projection, we have that $U_{i,i}(\mathcal{H})$ is 1-dimensional for every $i \in I$. Since $\varphi(U_{i,i})$ is also a minimal nonzero projection, we have that $\varphi(U_{i,i})(\mathcal{H})$ is 1-dimensional.

Let $e_1 \in U_{1,1}(\mathcal{H})$ and $f_1 \in \varphi(U_{1,1})(\mathcal{H})$ be such that $\|e_1\| = 1$ and $\|f_1\| = 1$. For every $l \geq 1$ let $e_l = U_{l,1}(e_1)$ and $f_l = \varphi(U_{l,1})(f_1)$. If $i \neq j$, then $\langle e_i, e_j\rangle = \langle U_{i,1}(e_1), U_{j,1}(e_1)\rangle = \langle U_{1,j}U_{i,1}(e_1), e_1\rangle = \langle 0(e_1), e_1\rangle = 0$ and $\langle e_i, e_i\rangle = \langle U_{i,1}(e_1), U_{i,1}(e_1)\rangle = \langle e_1, U_{1,i}U_{i,1}(e_1)\rangle = \langle e_1, U_{1,1}(e_1)\rangle = \langle e_1, e_1\rangle = 1$. Hence $\{e_l\}_{l\geq 1}$ is orthonormal. Let $x \in \mathcal{H}$ such that $\langle x, e_l\rangle = 0$ for every $l \geq 1$. Then $\langle U_{l,l}(x), e_l\rangle = \langle x, U_{l,l}(e_l)\rangle = \langle x, e_l\rangle = 0$, and hence $U_{l,l}(x) = 0$ for every $l \geq 1$. Since $\|\sum_{l\geq 1} U_{l,l}(x)\| \leq \sum_{l\geq 1}\|U_{l,l}(x)\| = 0$ and $\|\sum_{l\geq 1} U_{l,l}(x)\| \to \|x\|$, we have that $x = 0$ and therefore that $\{e_l\}_{l\geq 1}$ is an orthonormal basis. A similar argument shows that $\{f_l\}_{l\geq 1}$ is also an orthonormal basis.

Define $W : \mathcal{H} \to \mathcal{H}$ by $W(e_l) = f_l$ for every $l \in I$. It is clear that W is an invertible operator. If $x = \sum a_i e_i$, then $\|W(x)\|^2 = \|W(\sum a_i e_i)\|^2 = \|\sum a_i f_i\|^2 = \sum |a_i|^2 = \|x\|^2$. Hence, W is an isometry and, since it is surjective, W is unitary. Next we will show that $\varphi(U_{i,j}) = WU_{i,j}W^*$.

Note first that $WU_{l,1}(e_1) = W(e_l) = f_l = \varphi(U_{l,1})(f_1) = \varphi(U_{1,l})W(e_1)$. If $l \neq 1$, then $WU_{l,1}(e_l) = WU_{l,1}U_{l,1}(e_1) = W0(e_1) = 0$ and $\varphi(U_{l,1})W(e_l) = \varphi(U_{l,1})(f_l) = \varphi(U_{l,1})\varphi(U_{l,1})(f_1) = \varphi(U_{l,1}U_{l,1})(f_1) = \varphi(0)(f_1) = 0(f_1) = 0$. Since $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$ are orthonormal bases, we have that $\varphi(U_{l,1})W = WU_{l,1} \Rightarrow \varphi(U_{l,1}) = WU_{l,1}W^*$ for every $l \in I$. For every $i, j \in I$ we have that $\varphi(U_{i,j}) = \varphi(U_{i,1}U_{1,j}) = \varphi(U_{i,1})\varphi(U_{1,j}) = \varphi(U_{i,1})\varphi(U_{j,1}^*) = \varphi(U_{i,1})(\varphi(U_{j,1}))^* = (WU_{i,1}W^*)(WU_{j,1}W^*)^* = WU_{i,1}W^*WU_{j,1}^*W^* = WU_{i,1}U_{1,j}W^* = WU_{i,j}W^*$. So the family $U_{i,j}$ satisfy the conclusion of the theorem.

Let $T \in \mathcal{L}(\mathcal{H})$ and let $x = \sum_{l\geq 1} a_l e_l \in \mathcal{H}$. Then $T(x) = \sum_{l\geq 1} b_l e_l \in \mathcal{H}$ and

$$(\sum_{i,j\geq 1} U_{i,i}TU_{j,j})(x) = (\sum_{i\geq 1} U_{i,i}T\sum_{j\geq 1} U_{j,j})(\sum_{l\geq 1} a_l e_l) = (\sum_{i\geq 1} U_{i,i}T)(\sum_{j\geq 1} a_j e_j) = (\sum_{i\geq 1} U_{i,i}T)(x) =$$

$$= (\sum_{i\geq 1} U_{i,i})(\sum_{l\geq 1} b_l e_l) = \sum_{i\geq 1} b_i e_i = T(x)$$

Hence $\sum_{i,j\geq 1} U_{i,i}TU_{j,j} = T$ for every $T \in \mathcal{L}(\mathcal{H})$. If $x = \sum_{l\geq 1} a_l e_l$ and if for every $j \geq 1$ we let $T(e_j) = \sum_{l\geq 1} \alpha_l^j e_l$, then $(U_{i,i}TU_{j,j})(x) = (U_{i,i}TU_{j,j})(\sum_{l\geq 1} a_l e_l) = (U_{i,i}T)(a_j e_j) =$

$a_jU_{i,i}T(e_j) = a_jU_{i,i}(\sum_{l\geq 1}\alpha_l^j e_l) = a_j\alpha_i^j e_i = \alpha_i^j a_j e_i$ for every $i,j \geq 1$. But $U_{i,j}(x) = U_{i,j}(\sum_{l\geq 1} a_l e_l) = \sum_{l\geq 1} a_l U_{i,j} e_l = \sum_{l\geq 1} a_l U_{i,j}U_{l,1}(e_1) = a_jU_{i,j}U_{j,1}(e_1) = a_jU_{i,1}(e_1) = a_je_i$ for every $i,j \geq 1$, and hence $U_{i,i}TU_{j,j} = \alpha_i^j U_{i,j}$ for every $i,j \geq 1$. Therefore for every $T \in \mathcal{L}(\mathcal{H})$ we have that $T = \sum_{i,j\geq 1} \alpha_i^j U_{i,j}$.

For every $T \in \mathcal{L}(\mathcal{H})$ we have that $\varphi(T) = \varphi(\sum_{i,j\geq 1}\alpha_i^j U_{i,j}) = \sum_{i,j\geq 1}\alpha_i^j \varphi(U_{i,j}) = \sum_{i,j\geq 1}\alpha_i^j WU_{i,j}W^* = W(\sum_{i,j\geq 1}\alpha_i^j U_{i,j})W^* = WTW^*$. $\square$

5.4. Main Result

Lemma 5.33. *Let G be a Polish topological group, $H \subset G$ a subgroup such that $H \in \mathcal{BP}$ and G/H is countable. Then H is open in G and therefore closed in G.*

Proof. If H is meager in G, then each coset of G/H is meager in G and then G is meager since G/H is countable. This contradicts the fact that G is Polish. Thus H is nonmeager. By the Theorem of Pettis (Theorem 9.9, page 61, [18]) we have that $H^{-1}H = H$ contains an open neighborhood V of $e \in G$ and since $H = \cup_{x\in H} xV$ we have that H is open.

Let $x \in \mathrm{cl}_G H$. Then xH is an open neighborhood of $x \Rightarrow xH \cap H \neq \emptyset \Rightarrow x \in H \Rightarrow H$ is closed $\Rightarrow H$ is a Polish topological group. $\square$

Lemma 5.34. ★ *$\mathcal{A}ut(\mathcal{L}(\mathcal{H})) = \{\alpha^2 \mid \alpha \in \mathcal{S}\}$, where $\mathcal{S}$ is the group defined in Proposition 5.16.*

Proof. If $\alpha \in \mathcal{S}$, since the square of a $*-$anti-automorphism is a $*-$automorphism, then α^2 is a $*-$automorphism $\Rightarrow \{\alpha^2 \mid \alpha \in \mathcal{S}\} \subset \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$.

Let $\varphi \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$. Then by Proposition 5.32 we have that there exists $U \in \mathcal{U}(\mathcal{H})$ such that $\varphi = \varphi_U$, where $\varphi_U(T) = UTU^*$ for every $T \in \mathcal{L}(\mathcal{H})$. Choose $V \in \mathcal{U}(\mathcal{H})$ such that $V^2 = U$. Such a V exists by the Spectral Theorem. Note that if $\varphi_V(T) = VTV^*$ then $\varphi_V \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$. Since $\varphi(T) = \varphi_U(T) = UTU^* = V(VTV^*)V^* = (\varphi_V)^2(T)$ we have that $\varphi \in \{\alpha^2 \mid \alpha \in \mathcal{S}\} \Rightarrow \mathcal{A}ut(\mathcal{L}(\mathcal{H})) \subset \{\alpha^2 \mid \alpha \in \mathcal{S}\}$. $\square$

Theorem 5.35. ★ *If $\mathcal{H}$ is a separable Hilbert space, then $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is a closed subgroup of $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ and therefore is a Polish topological group.*

Proof. We will prove that $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is closed in $\mathcal{S}$ and hence Polish. Then, since $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ is a Polish topological group by Corollary 5.2 and since $\mathcal{S}$ is closed in $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ by Corollary 5.28, we will have that $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is closed in $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ and hence Polish.

The mapping $\psi \mapsto \psi^2$ from $\mathcal{S}$ to $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is onto by Lemma 5.34 and continuous since multiplication in $\mathcal{H}om(\mathcal{L}(\mathcal{H})_1)$ is continuous. Since $\mathcal{S}$ is Polish, we have that $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is analytic, and hence $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ has the Baire property. $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is a normal subgroup of $\mathcal{S}$ and $|\mathcal{S}/\mathcal{A}ut(\mathcal{L}(\mathcal{H}))| = 2$ by Proposition 5.16. From Lemma 5.33 it follows that $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is open in $\mathcal{S}$ and hence closed in $\mathcal{S}$. $\square$

THEOREM 5.36. ★ *If $\mathcal{H}$ is a complex separable Hilbert space, then $\mathcal{PU}(\mathcal{H})$ and $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ are topologicallly isomorphic.*

Proof. Let $f : \mathcal{U}(\mathcal{H}) \to \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ be defined as $f(U) = \varphi_U$, where $\varphi_U : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is defined as $\varphi_U(T) = UTU^*$. We will first show that if $U \in \mathcal{U}(\mathcal{H})$, then $f(U) \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$.

Let $U \in \mathcal{U}(\mathcal{H})$, and $S, T \in \mathcal{L}(\mathcal{H})$ be such that $\varphi_U(S) = \varphi_U(T)$. Then $USU^* = UTU^* \Rightarrow S = T \Rightarrow \varphi_U$ is one-to-one. If $S \in \mathcal{L}(\mathcal{H})$ let $T = U^*SU \in \mathcal{L}(\mathcal{H})$. Then $\varphi_U(T) = UTU^* = UU^*SUU^* = S \Rightarrow \varphi_U$ is onto and hence φ_U is a bijection. Let $S, T \in \mathcal{L}(\mathcal{H})$ and let $\lambda \in \mathbb{C}$. Then $\varphi_U(ST) = USTU^* = USU^*UTU^* = \varphi_U(S)\varphi_U(T)$; $\varphi_U(S+T) = U(S+T)U^* = USU^* + UTU^* = \varphi_U(S) + \varphi_U(T)$; $\varphi_U(\lambda T) = U(\lambda T)U^* = \lambda UTU^* = \lambda\varphi_U(T)$ and $\varphi_U(T^*) = UT^*U^* = (UTU^*)^* = (\varphi_U(T))^* \Rightarrow f(U) = \varphi_U \in \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ and hence f is well defined.

Let $U, V \in \mathcal{U}(\mathcal{H})$ and let $T \in \mathcal{L}(\mathcal{H})$. Then $f(UV)(T) = \varphi_{UV}(T) = UVT(UV)^* = UVTV^*U = U\varphi_V(T)U^* = \varphi_U\varphi_V(T) = f(U)f(V)(T) \Rightarrow f$ is a homomorphism.

Let $\mathfrak{id} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be the identity on $\mathcal{L}(\mathcal{H})$. Let $U \in \mathcal{U}(\mathcal{H})$ be such that $f(U) = \mathfrak{id}$. Then $\varphi_U(T) = T$ for every $T \in \mathcal{L}(\mathcal{H}) \Rightarrow UTU^* = T$ for every $T \in \mathcal{L}(\mathcal{H}) \Rightarrow UT = TU$ for every $T \in \mathcal{L}(\mathcal{H}) \Rightarrow UW = WU$ for every $W \in \mathcal{U}(\mathcal{H}) \Rightarrow U \in Z(\mathcal{U}(\mathcal{H})) \Rightarrow \ker(f) = Z(\mathcal{U}(\mathcal{H}))$.

Let $\{U_j\}_{j\in J} \subset \mathcal{U}(\mathcal{H})$ be such that $U_j \xrightarrow{wo} U \in \mathcal{U}(\mathcal{H})$. Then $U_j^* \xrightarrow{wo} U^*$ by Lemma 3.4 and hence $U_j \xrightarrow{so} U$ and $U_j^* \xrightarrow{so} U^*$ by Proposition 3.3. Thus, for every $T \in \mathcal{L}(\mathcal{H})_1$ and every

$x, y \in \mathcal{H}$ we have the following

$$|\langle U_j T U_j^* x, y\rangle - \langle U T U^* x, y\rangle| = |\langle U_j^* x, T^* U_j^* y\rangle - \langle T U^* x, U^* y\rangle| \leq$$

$$|\langle U_j^* x, T^* U_j^* y\rangle - \langle U^* x, T^* U_j^* y\rangle| + |\langle T U^* x, U_j^* y\rangle - \langle T U^* x, U^* y\rangle| =$$

$$|\langle (U_j^* - U^*)x, T^* U_j^* y\rangle| + |\langle T U^* x, (U_j^* - U^*)y\rangle| \leq$$

$$\|(U_j^* - U^*)x\| \cdot \|T^*\| \cdot \|U_j^* y\| + \|T\| \cdot \|U^* x\| \cdot \|(U_j^* - U^*)y\| \leq$$

$$\|(U_j^* - U^*)x\| \cdot \|y\| + \|x\| \cdot \|(U_j^* - U^*)y\| \to 0$$

This implies that $|\langle \varphi_{U_j}(T)x, y\rangle - \langle \varphi_U(T)x, y\rangle| \to 0$ uniformly in $T \in \mathcal{L}(\mathcal{H})_1$ for every $x, y \in \mathcal{H} \Rightarrow d(\varphi_{U_j}(T), \varphi_U(T)) \to 0$ uniformly for every $T \in \mathcal{L}(\mathcal{H})_1 \Rightarrow \sup_{T \in \mathcal{L}(\mathcal{H})_1} d(\varphi_{U_j}(T), \varphi_U(T)) \to 0$. Similarly we have that $\sup_{T \in \mathcal{L}(\mathcal{H})_1} d(\varphi_{U_j}^{-1}(T), \varphi_U^{-1}(T)) \to 0$ and hence $\rho(\varphi_{U_j}, \varphi_U) \to 0 \Rightarrow f(U_j) = \varphi_{U_j} \xrightarrow{\rho} \varphi_U = f(U) \Rightarrow f$ is continuous. We also have from Proposition 5.32 that the mapping f is onto. Thus $f : \mathcal{U}(\mathcal{H}) \to \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is a continuous onto homomorphism and $\ker(f) = Z(\mathcal{U}(\mathcal{H}))$.

Let $\pi : \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})/\ker(f) = \mathcal{PU}(\mathcal{H})$ be the natural quotient mapping and let $\psi : \mathcal{PU}(\mathcal{H}) \to \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ be the natural group isomorphism so that $f = \psi \circ \pi$. If $\mathcal{U} \subset \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is open, then $f^{-1}(\mathcal{U}) \subset \mathcal{U}(\mathcal{H})$ is open, since f is continuous. But $f^{-1}(\mathcal{U}) = \pi^{-1}(\psi^{-1}(\mathcal{U})) \Rightarrow \psi^{-1}(\mathcal{U}) = \pi(f^{-1}(\mathcal{U}))$ is open in $\mathcal{PU}(\mathcal{H})$ since π, being the quotient mapping, is open. This implies that ψ is continuous. Thus $\psi : \mathcal{PU}(\mathcal{H}) \to \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ is a continuous isomorphism between two Polish topological groups. From Lusin-Souslin Theorem (page 89, [18]) we have that ψ^{-1} is Borel measurable, and hence it is measurable with respect to the sets with the Baire property. From the result of Banach-Kuratowski-Pettis (Theorem 9.10, page 61, [18]) it follows that ψ^{-1} is continuous and hence ψ is a topological isomorphism. $\square$

Corollary 5.37. ★ *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. From Theorem 5.36 we have that $\mathcal{PU}(\mathcal{H})$ and $\mathcal{A}ut(\mathcal{L}(\mathcal{H}))$ are topologically isomorphic. From Theorem 4.18 we have that if $\phi : G \to \mathcal{PU}(\mathcal{H})$ is an algebraic isomorphism, then ϕ is a topological isomorphism. The conclusion follows. $\square$

CHAPTER 6

THE ORTHOGONAL GROUP

Throughout this section $\mathcal{H}$ is assumed to be a separable real Hilbert space.

DEFINITION 6.1. If $\mathcal{H}$ is a real Hilbert space a unitary operator acting on $\mathcal{H}$ is called an orthogonal operator, the set of orthogonal operators is denoted by $\mathcal{O}(\mathcal{H})$ and is called the orthogonal group of $\mathcal{H}$. If $\mathcal{H}$ is n-dimensional, $\mathcal{O}(\mathcal{H})$ is sometimes denoted by $\mathcal{O}(n)$. If $U \in \mathcal{O}(\mathcal{H})$, the adjoint operation U^* on $\mathcal{O}(\mathcal{H})$ is denoted with U^T and on the finite dimensional case is equivalent to taking transposes of matrices. The center of $\mathcal{O}(\mathcal{H})$ is denoted by $Z(\mathcal{O}(\mathcal{H}))$. If $\mathcal{H}$ is finite dimensional, the special orthogonal group is the set $SO(\mathcal{H}) = \{U \in \mathcal{O}(\mathcal{H}) \mid \det(U) = 1\}$. $SO(\mathcal{H})$ is sometimes denoted $SO(n)$, where n is the dimension of $\mathcal{H}$.

REMARK 6.2. If $\mathcal{M}$ is a closed subspace of the Hilbert space $\mathcal{H}$ and if $\mathcal{O}_{\mathcal{M}} = \{U \in \mathcal{O}(\mathcal{H}) \mid U|_{\mathcal{M}^\perp} = I\}$ then, as in Proposition 3.14, $\mathcal{O}_{\mathcal{M}}$ may be identified with $\mathcal{O}(\mathcal{M})$, and we can consider $\mathcal{O}(\mathcal{M})$ to be a closed subgroup of $\mathcal{O}(\mathcal{H})$.

REMARK 6.3. The proofs of Proposition 3.3 and Proposition 3.6 can be easily adapted to $\mathcal{O}(\mathcal{H})$ if $\mathcal{H}$ is a separable real Hilbert space and we can conclude that weak operator topology, the strong operator topology and the relative topology given by $\mathcal{H}om(\mathcal{H}_1)$ coincide on $\mathcal{O}(\mathcal{H})$.

THEOREM 6.4. ★ *$\mathcal{O}(\mathcal{H})$ is a Polish topological group.*

Proof. If $\mathcal{H}$ is a real separable Hilbert space, in the view of Comment 6.3 we can prove a theorem similar to the Theorem 3.7 to prove that $\mathcal{O}(\mathcal{H})$ is closed in $\mathcal{H}om(\mathcal{H}_1)$. Since $\mathcal{H}om(\mathcal{H}_1)$ is a Polish topological group by Theorem 2.24, the conclusion follows. □

PROPOSITION 6.5. *If $\mathcal{H}$ is a real Hilbert space, then $Z(\mathcal{O}(\mathcal{H})) = \{\pm I\}$.*

Proof. It is clear that $I, -I \in Z(\mathcal{O}(\mathcal{H}))$. Let $U \in Z(\mathcal{O}(\mathcal{H}))$. Let $\{e_l\}_{l\geq 1}$ be an orthonormal basis and let $R : \mathcal{H} \to \mathcal{H}$ be defined as $Re_1 = -e_2$, $Re_2 = e_1$ and $Re_l = e_l$ for every $l \geq 3$. If $x = \sum_{l\geq 1} a_l e_l \in \mathcal{H}$ then $\|Rx\|^2 = \|\sum_{l\geq 1} a_l Re_l\|^2 = |a_1|^2\| - e_2\|^2 + |a_2|^2\|e_1\|^2 + \sum_{l\geq 3} |a_l|^2\|e_l\|^2 = \sum_{l\geq 1} |a_l|^2 = \|x\|^2 \Rightarrow R$ is an isometry. If $y = \sum_{l\geq 1} a_l e_l \in \mathcal{H}$, let $x = -a_2e_1 + a_1e_2 + \sum_{l\geq 3} a_l e_l$. Then $Rx = a_2e_2 + a_1e_1 + \sum_{l\geq 3} a_l e_l = \sum_{l\geq 1} a_l e_l = y \Rightarrow R$ is onto, and hence $R \in \mathcal{O}(\mathcal{H})$. We also have that $R^T e_1 = \sum_{l\geq 1}\langle R^T e_1, e_l\rangle e_l = \sum_{l\geq 1}\langle e_1, Re_l\rangle e_l = e_2$. Thus, since $UR = RU$ we have that $-\langle Ue_2, e_1\rangle = \langle U(-e_2), e_1\rangle = \langle URe_1, e_1\rangle = \langle RUe_1, e_1\rangle = \langle Ue_1, R^T e_1\rangle = \langle Ue_1, e_2\rangle$ and $\langle Ue_1, e_1\rangle = \langle URe_2, e_1\rangle = \langle RUe_2, e_1\rangle = \langle Ue_2, R^T e_1\rangle = \langle Ue_2, e_2\rangle$.

Let V be defined as $Ve_1 = -e_1$ and $Ve_l = e_l$ for every $l \geq 2$. V is obviously an orthogonal operator and $V^T e_l = e_l$ for every $l \geq 2$. Since $UV = VU$ we have that $-\langle Ue_1, e_2\rangle = \langle UVe_1, e_2\rangle = \langle VUe_1, e_2\rangle = \langle Ue_1, V^T e_2\rangle = \langle Ue_1, e_2\rangle \Rightarrow \langle Ue_1, e_2\rangle = 0$ and since $\langle Ue_1, e_2\rangle = -\langle Ue_2, e_1\rangle \Rightarrow \langle Ue_2, e_1\rangle = 0$.

Using similar arguments we can show that $\langle Ue_i, e_j\rangle = 0$ for every $i \neq j$ and that $\langle Ue_i, e_i\rangle = \langle Ue_j, e_j\rangle$ for every $i, j \geq 1$ and hence there exists $\lambda \in \mathbb{R}$ such that $\langle Ue_l, e_l\rangle = \lambda$ for every $l \geq 1 \Rightarrow U = \lambda I$. This implies that $U^T = U \Rightarrow I = UU^T = U^2 = \lambda^2 I \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$ and and hence $U = \pm I$. $\square$

6.1. The Orthogonal Group $\mathcal{O}(n)$

PROPOSITION 6.6. *Let G_1, G_2 be two topological groups and $\phi : G_1 \to G_2$ be a homomorphism. If ϕ is continuous at the origin $e_1 \in G_1$ then ϕ is continuous.*

Proof. Let $x \in H_1$ and $\phi(x) \in U \subset G_2$ be open. Then $e_2 \in [\phi(x)]^{-1}U$ and since ϕ is continuous at the origin there exists $V \subset G_1$ open such that $e_1 \in V$ and $\phi(V) \subset [\phi(x)]^{-1}U$. Then xV is open, $x \in xV$ and if $y \in xV$ then $\phi(y) \in \phi(x)\phi(V) \subset \phi(x)[\phi(x)]^{-1}U = U \Rightarrow \phi(xV) \subset U \Rightarrow \phi$ is continuous at $x \Rightarrow \phi$ is continuous. $\square$

LEMMA 6.7. *Let G_1, G_2 be two Polish topological groups, let $\phi : G_1 \to G_2$ be an algebraic isomorphism, let $H_2 \subset G_2$ be a subgroup with the Baire property and let $H_1 = \phi^{-1}(H_2) \subset G_1$. If G_2/H_2 is countable, H_1 is a set with the Baire property and $\phi|_{H_1} : H_1 \to H_2$ is measurable with respect to the sets with the Baire property, then ϕ is a topological isomorphism.*

Proof. From Lemma 5.33 we have that H_2 is open and closed in G_2 and hence H_2 is a Polish topological group. Since G_1/H_1 is also countable, we have by the same lemma that H_1 is open and closed in G_1 and hence H_1 is a Polish topological group. Since $\phi|_{H_1} : H_1 \to H_2$ is Baire measurable we have by Theorem 9.10, page 61, [18] that $\phi|_{H_1}$ is continuous, and hence $\phi|_{H_1}$ is continuous at $e \Rightarrow \phi$ is continuous by Proposition 6.6. The conclusion follows from Lemma 3.57. □

THEOREM 6.8. ★ *Let G be a Polish topological group, $\mathcal{H}$ a n-dimensional real Hilbert space, with $n \geq 3$, $\mathcal{O}(n)$ the orthogonal group acting on $\mathcal{H}$ and $\phi : G \to \mathcal{O}(n)$ an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. $SO(n) \subset \mathcal{O}(n)$ is a subgroup. Using the result from Chapter I, Section 14, [19], we have that $\mathcal{O}(n) = SO(n) \cup O_0 \cdot SO(n)$, where $O_0 \in \mathcal{O}(n)$ and $\det(O_0) = -1$, and hence the cardinality $|\mathcal{O}(n)/SO(n)| = 2$. Since $\phi^{-1}(SO(n))$ is closed in G by Corollary 6.36 and hence it has the Baire property and since the restriction $\phi|_{\phi^{-1}(SO(n))} : \phi^{-1}(SO(n)) \to SO(n)$ is continuous for $n \geq 3$ by the result from [14], it follows from Lemma 6.7 that ϕ is continuous. □

6.2. The Complexification of $\mathcal{H}$

DEFINITION 6.9. Suppose that $\mathcal{H}$ is a real Hilbert space and let $\mathcal{K}$ be the set of all ordered pairs (x, y) with both $x, y \in \mathcal{H}$. Define the sum of two elements of $\mathcal{K}$ by $(x, y) + (u, v) = (x+u, y+v)$ and the product of an element of $\mathcal{K}$ by a complex number $a+ib$ by $(a+ib)\cdot(x, y) = (ax - by, bx + ay)$.

PROPOSITION 6.10. *The set $\mathcal{K}$ in the previous definition is a complex vector space.*

Proof. $[(x_1, y_1)+(x_2, y_2)]+(x_3, y_3) = (x_1+x_2, y_1+y_2)+(x_3, y_3) = (x_1+x_2+x_3, y_1+y_2+y_3) = (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$.

$(x, y) + (0, 0) = (x, y) = (0, 0) + (x, y)$.

$(x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y)$.

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2, y_2) + (x_1, y_1)$.

$(a+ib)[(x_1,y_1)+(x_2,y_2)] = (a+ib)(x_1+x_2,y_1+y_2) = (ax_1+ax_2-by_1-by_2, bx_1+bx_2+ay_1+ay_2) = (ax_1-by_1, bx_1+ay_1)+(ax_2-by_2, bx_2+ay_2) = (a+ib)(x,y)+(a+ib)(x_2,y_2)$.

$[(a+ib)+(c+id)](x,y) = [(a+c)+i(b+d)](x,y) = (ax+cx-by-dy, bx+dx+ay+cy) = (ax-by, bx+ay)+(cx-dy, dx+cy) = (a+ib)(x,y)+(c+id)(x,y)$.

$[(a+ib)(c+id)](x,y) = [(ac-bd)+i(bc+ad)](x,y) = (acx-bdx-bcy-ady, bcx+adx+acy-bdy) = [a(cx-dy)-b(dx+cy), b(cx-dy)+a(dx+cy)] = (a+ib)(cx-dy, dx+cy) = (a+ib)[(c+id)(x,y)]$.

$1(x,y) = (x-0y, 0x+y) = (x,y)$. $\square$

Definition 6.11. We call the space $\mathcal{K}$ from the previous proposition the complexification of the space $\mathcal{H}$ and denote its elements by $x+iy$.

Proposition 6.12. *If $\mathcal{H}$ is a real inner product space and if $\mathcal{K}$ is its complexification, then the following*

$$\langle x+iy, u+iv\rangle = \langle x,u\rangle + \langle y,v\rangle - i(\langle x,v\rangle - \langle y,u\rangle)$$

defines an inner product on $\mathcal{K}$. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{K}$, together with this inner product is a Hilbert space.

Proof.
$\langle (a+ib)(x+iy)+(c+id)(z+iw), u+iv\rangle = \langle (ax-by+cz-dw)+i(bx+ay+dz+cw), u+iv\rangle = \langle ax-by+cz-dw, u\rangle + \langle bx+ay+dz+cw, v\rangle - i(\langle ax-by+cz-dw, v\rangle - \langle bx+ay+dz+cw, u\rangle) = a\langle x,u\rangle - b\langle y,u\rangle + c\langle z,u\rangle - d\langle w,u\rangle + b\langle x,v\rangle + a\langle y,v\rangle + d\langle z,v\rangle + c\langle w,v\rangle - ia\langle x,v\rangle + ib\langle y,v\rangle - ic\langle z,v\rangle + id\langle w,v\rangle + ib\langle x,u\rangle + ia\langle y,u\rangle + id\langle z,u\rangle + ic\langle w,u\rangle = a(\langle x,u\rangle + \langle y,v\rangle - i\langle x,v\rangle + i\langle y,u\rangle) + ib(i\langle y,u\rangle - i\langle x,v\rangle + \langle y,v\rangle + \langle x,u\rangle) + c(\langle z,u\rangle + \langle w,v\rangle - i\langle z,v\rangle + i\langle w,u\rangle) + id(i\langle w,u\rangle - i\langle z,v\rangle + \langle w,v\rangle + \langle z,u\rangle) = (a+ib)(\langle x,u\rangle + \langle y,v\rangle - i\langle x,v\rangle + i\langle y,u\rangle) + (c+id)(\langle z,u\rangle + \langle w,v\rangle - i\langle z,v\rangle + i\langle w,u\rangle) = (a+ib)\langle x+iy, u+iv\rangle + (c+id)\langle z+iw, u+iv\rangle$.

$\langle x+iy, u+iv\rangle = \langle x,u\rangle + \langle y,v\rangle - i(\langle x,v\rangle - \langle y,u\rangle) = \langle u,x\rangle + \langle v,y\rangle - i(\langle v,x\rangle - \langle u,y\rangle) = \overline{\langle u,x\rangle + \langle v,y\rangle - i(\langle u,y\rangle - \langle v,x\rangle)} = \overline{\langle u+iv, x+iy\rangle}$.

$\langle x+iy, x+iy\rangle = \langle x,x\rangle + \langle y,y\rangle - i(\langle x,y\rangle - \langle y,x\rangle) = \langle x,x\rangle + \langle y,y\rangle \geq 0$.

If $\langle x+iy, x+iy\rangle = 0$ then $\langle x,x\rangle + \langle y,y\rangle = 0 \Rightarrow \langle x,x\rangle = 0$ and $\langle y,y\rangle = 0 \Rightarrow x = 0$ and $y = 0$. $\square$

Proposition 6.13. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ its complexification. If $A \in \mathcal{L}(\mathcal{H})$ define $\tilde{A} : \mathcal{K} \to \mathcal{K}$ to be $\tilde{A}(x+iy) = Ax + iAy$. Then $\tilde{A} \in \mathcal{L}(\mathcal{K})$ and $\|A\| = \|\tilde{A}\|$.*

Proof. $\tilde{A}[(x+iy)+(u+iv)] = \tilde{A}[(x+u)+i(y+v)] = A(x+u)+iA(y+v) = Ax+Au+iAy+iAv = Ax+iAy+Au+iAv = \tilde{A}(x+iy)+\tilde{A}(u+iv)$.

$\tilde{A}[(a+ib)(x+iy)] = \tilde{A}[(ax-by)+i(bx+ay)] = A(ax-by)+iA(bx+ay) = aAx-bAy+i(bAx+aAy) = (a+ib)(Ax+iAy) = (a+ib)\tilde{A}(x+iy)$.

$\|\tilde{A}(x+iy)\|^2 = \|Ax+iAy\|^2 = \langle Ax+iAy, Ax+iAy\rangle = \langle Ax,Ax\rangle + \langle Ay,Ay\rangle - i(\langle Ax,Ay\rangle - \langle Ay,Ax\rangle) = \|Ax\|^2 + \|Ay\|^2 \le \|A\|^2(\|x\|^2+\|y\|^2) = \|A\|^2\|x+iy\|^2 \Rightarrow \|\tilde{A}\| \le \|A\|$.

Note that if $x \in \mathcal{H}$ then $\|x\|^2_{\mathcal{K}} = \langle x+i0, x+i0\rangle = \langle x,x\rangle = \|x\|^2_{\mathcal{H}}$. It follows that $\|Ax\| = \|\tilde{A}x\| \le \|\tilde{A}\| \cdot \|x\|$ and hence $\|A\| \le \|\tilde{A}\|$ $\square$

Proposition 6.14. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ its complexification. If $A \in \mathcal{L}(\mathcal{H})$, then $(\tilde{A})^* = \widetilde{A^T}$*

Proof. $\langle x+iy, (\tilde{A})^*(u+iv)\rangle = \langle \tilde{A}(x+iy), u+iv\rangle = \langle Ax+iAy, u+iv\rangle = \langle Ax,u\rangle + \langle Ay,v\rangle - i(\langle Ax,v\rangle - \langle Ay,u\rangle) = \langle x, A^Tu\rangle + \langle y, A^Tv\rangle - i(\langle x, A^Tv\rangle - \langle y, A^Tu\rangle) = \langle x+iy, A^Tu + iA^Tv\rangle = \langle x+iy, \widetilde{A^T}(u+iv)\rangle$. $\square$

Proposition 6.15. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ its complexification. Define $J : \mathcal{K} \to \mathcal{K}$ as $J(x+iy) = x-iy$. Then $J^2 = I$, J is a real linear isometry, $J(\lambda z) = \overline{\lambda}J(z)$ for every $\lambda \in \mathbb{C}$ and $z \in \mathcal{K}$ and $\langle Jw, Jz\rangle = \langle z,w\rangle$ for every $w,z \in \mathcal{K}$.*

Proof. $J^2(x+iy) = J(x-iy) = x+iy$ for every $x+iy \in \mathcal{K} \Rightarrow J^2 = I$.

$J[(x+iy)+(u+iv)] = J[(x+u)+i(y+v)] = (x+u)-i(y+v) = (x-iy)+(u-iv) = J(x+iy)+J(u+iv)$ and $J[a(x+iy)] = J(ax+iay) = ax-iay = a(x-iy) = aJ(x+iy)$ for every $a \in \mathbb{R}$ and every $x+iy, u+iv \in \mathcal{K} \Rightarrow J$ is real linear. $\|J(x+iy)\|^2 = \langle x-iy, x-iy\rangle = \langle x,x\rangle + \langle -y,-y\rangle - i(\langle x,-y\rangle - \langle -y,x\rangle) = \langle x,x\rangle + \langle y,y\rangle - i(\langle y,x\rangle - \langle x,y\rangle) = \langle x,x\rangle + \langle y,y\rangle - i(\langle x,y\rangle - \langle y,x\rangle) = \langle x+iy, x+iy\rangle = \|x+iy\|^2$ and hence J is an isometry.

$J[(a+ib)(x+iy)] = J[(ax-by)+i(bx+ay)] = ax-by-i(bx+ay) = ax-(-b)(-y)+i[(-b)x+a(-y)] = (a-ib)(x-iy) = (a-ib)J(x+iy)$ for every $a+ib \in \mathbb{C}$ and every $x+iy \in \mathcal{K}$.

$\langle J(x+iy), J(u+iv)\rangle = \langle x-iy, u-iv\rangle = \langle x,u\rangle + \langle -y,-v\rangle - i(\langle x,-v\rangle - \langle -y,u\rangle) = \langle u,x\rangle + \langle v,y\rangle - i(\langle u,y\rangle - \langle v,x\rangle) = \langle u+iv, x+iy\rangle$. □

Proposition 6.16. ★ *If $T \in \mathcal{L}(\mathcal{K})$ and J is the mapping defined in Proposition 6.15, then $JTJ \in \mathcal{L}(\mathcal{K})$, $\|JTJ\| = \|T\|$ and $(JTJ)^* = JT^*J$.*

Proof. Let $z, w \in \mathcal{K}$ and $\lambda \in \mathbb{C}$. Then $JTJ(z+w) = JT(Jz+Jw) = J(TJz+TJw) = JTJz + JTJw$, $JTJ(\lambda z) = JT(\overline{\lambda}Jz) = J(\overline{\lambda}TJz) = \lambda JTJz$ and $\|JTJz\| = \|TJz\| \leq \|T\| \cdot \|Jz\| = \|T\| \cdot \|z\| \Rightarrow \|JTJ\| \leq \|T\|$ and hence $JTJ \in \mathcal{L}(\mathcal{K})$. By replacing T with JTJ in the last inequality we obtain that $\|T\| \leq \|JTJ\|$ and hence $\|T\| = \|JTJ\|$ for every $T \in \mathcal{L}(\mathcal{K})$.

Since $\langle JTJz, w\rangle = \langle JTJz, J^2w\rangle = \langle Jw, TJz\rangle = \langle T^*Jw, Jz\rangle = \langle J^2T^*Jw, Jz\rangle = \langle z, JT^*Jw\rangle$ for every $w, z \in \mathcal{K}$ we have that $(JTJ)^* = JT^*J$. □

Proposition 6.17. ★ *If $E(\cdot)$ is a spectral measure on (X, Σ) with values in $\mathcal{K}$, then $JE(\cdot)J$ is also a spectral measure.*

Proof. $JE(X)J(x+iy) = JE(X)(x-iy) = J(x-iy) = x+iy$ for every $x+iy \in \mathcal{K} \Rightarrow JE(X)J = I$.

$JE(\cup_{l\geq 1}M_l)J(x+iy) = J[\sum_{l\geq 1} E(M_l)J(x+iy)] = \sum_{l\geq 1} JE(M_l)J(x+iy) = \sum_{l\geq 1}[JE(M_l)J](x+iy)$. Thus $JE(\cdot)J$ is countably additive.

$[JE(M)J]^* = J[E(M)]^*J = JE(M)J$ and $[JE(M)J]^2 = JE(M)J^2E(M)J = JE(M)J$ for every $M \in \Sigma$ and hence $JE(\cdot)J$ is an orthogonal projection. □

Proposition 6.18. ★ *If $T \in \mathcal{L}(\mathcal{K})$ is self-adjoint, $E(\cdot)$ is its associated spectral measure, then JTJ is self adjoint and $JE(\cdot)J$ is its associated spectral measure.*

Proof. If $T^* = T$ then $(JTJ)^* = JT^*J = JTJ$ and hence JTJ is self-adjoint. From the Proposition 6.17 we have that $JE(\cdot)J$ is a spectral measure. Since T is self adjoint then for every $x, y \in \mathcal{K}$ there exists $\mu_{x,y}$ a complex-valued measure on (X, Σ) such that $\langle Tx, y\rangle = \int \lambda d\mu_{x,y}$, where $\mu_{x,y}(B) = \langle E(B)x, y\rangle$ for every $B \in \Sigma$ and every $x, y \in \mathcal{K}$.

Since $\langle JTJx, y\rangle = \langle Jy, TJx\rangle = \langle T^* Jy, Jx\rangle = \langle TJy, Jx\rangle = \int \lambda d\mu_{Jy,Jx}$ and since $\mu_{Jy,Jx}(B) = \langle E(B)Jy, Jx\rangle = \langle x, JE(B)Jy\rangle = \langle [JE(B)J]^* x, y\rangle = \langle JE(B)Jx, y\rangle$ for every $B \in \Sigma$ we have that $JE(\cdot)J$ is the spectral measure associated with JTJ. $\square$

Corollary 6.19. ★ *If $T \in \mathcal{L}(\mathcal{K})$ is self-adjoint, $E(\cdot)$ is its associated spectral measure and $T = JTJ$, then $E(B) = JE(B)J$ for every $B \in \Sigma$.*

Proof. From Proposition 6.18 we have that $JE(\cdot)J$ is the spectral measure associated with $JTJ = T$. Since spectral measure associated with T is unique, it follows that $JE(B)J = E(B)$ for every $B \in \Sigma$. $\square$

Lemma 6.20. ★ *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{K}$ its complexification, let J be the mapping defined in Proposition 6.15 and let $z \in \mathcal{K}$. Then $z \in \mathcal{H} \Leftrightarrow Jz = z$.*

Proof. If $z \in \mathcal{H}$ then $Jz = z$ by the definition of J. Let $z = x + iy \in \mathcal{K}$ be such that $Jz = z$. Then $x + iy = z = Jz = x - iy \Rightarrow y = 0 \Rightarrow z = x \in \mathcal{H}$. $\square$

Lemma 6.21. ★ *If P is an orthogonal projection on $\mathcal{K}$ such that $JPJ = P$ then $P(\mathcal{H}) \subset \mathcal{H}$ and $P(\mathcal{K}) = P(\mathcal{H}) + iP(\mathcal{H})$. Therefore, $P(\mathcal{K})$ is the complexification of $P(\mathcal{H})$.*

Proof. If $x \in \mathcal{H}$ then $Px = JPJx = JPx \Rightarrow Px \in \mathcal{H}$ by Lemma 6.20 $\Rightarrow P(\mathcal{H}) \subset \mathcal{H}$. If $z = x + iy \in \mathcal{K}$ then $P(z) = P(x + iy) = Px + iPy \in P(\mathcal{H}) + iP(\mathcal{H}) \subset \mathcal{H} + i\mathcal{H} \Rightarrow P(\mathcal{K}) = (P(\mathcal{K}) \cap \mathcal{H}) + i(P(\mathcal{K}) \cap \mathcal{H}) = P(\mathcal{H}) + iP(\mathcal{H})$. $\square$

Lemma 6.22. *Let $S, T \in \mathcal{L}(\mathcal{K})$ be such that $ST = TS$, $T = T^*$ and let $E(\cdot)$ be the spectral measure on the measurable space (X, Σ) associated with T. Then $SE(B) = E(B)S$ for every $B \in \Sigma$.*

Proof. Let P be any polynomial with complex coefficients. Then for every $x, y \in \mathcal{K}$ we have that $\langle P(T)x, y\rangle = \int P(\lambda) d\mu_{x,y}$, where $\mu_{x,y}(B) = \langle E(B)x, y\rangle$ for every $B \in \Sigma$. Thus $\langle P(T)Sx, y\rangle = \int P(\lambda) d\mu_{Sx,y}$ and $\langle P(T)x, S^*y\rangle = \int P(\lambda) d\mu_{x,S^*y}$. Since S commutes with T, S commutes with $P(T)$ and hence $\langle P(T)Sx, y\rangle = \langle SP(T)x, y\rangle = \langle P(T)x, S^*y\rangle \Rightarrow$

$\int P(\lambda) d\mu_{Sx,y} = \int P(\lambda) d\mu_{x,S^*y}$. This implies that $\mu_{Sx,y}(B) = \mu_{x,S^*y}(B)$ for every $B \in \Sigma \Rightarrow \langle E(B)Sx, y\rangle = \langle E(B)x, S^*y\rangle = \langle SE(B)x, y\rangle \Rightarrow E(B)S = SE(B)$ for every $B \in \Sigma$. □

DEFINITION 6.23. Let $\mathcal{H}$ be a complex or a real Hilbert space. A subspace $\mathcal{M} \subset \mathcal{H}$ is invariant under an operator A if $A(\mathcal{M}) \subset \mathcal{M}$. A subspace $\mathcal{M} \subset \mathcal{H}$ reduces an operator A if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under A.

PROPOSITION 6.24. *Let $\mathcal{H}$ be a complex or a real Hilbert space. If $\mathcal{M} \subset \mathcal{H}$ is a subspace and P is the orthogonal projection on $\mathcal{M}$, then $\mathcal{M}$ reduces an operator A if and only if $AP = PA$.*

Proof. Suppose that $PA = AP$. Then $PAP = AP$ and $PA = PAP \Rightarrow PA^*P = PA^*$ and $A^*P = PA^*P$. If $x \in \mathcal{M}$ and $y \in \mathcal{M}^\perp$ then $Ax = APx = PAPx \in \mathcal{M} \Rightarrow \mathcal{M}$ is invariant under A. Also $A^*x = A^*Px = PA^*Px \in \mathcal{M} \Rightarrow \mathcal{M}$ is invariant under A^*. Since $\langle Ay, x\rangle = \langle y, A^*x\rangle = 0 \Rightarrow Ay \in \mathcal{M}^\perp \Rightarrow \mathcal{M}^\perp$ is invariant under A. Since both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under A, we have that $\mathcal{M}$ reduces A.

Suppose now that $\mathcal{M}$ reduces A. Then $\mathcal{M}$ is invariant under A and $\mathcal{M}^\perp$ is invariant under A. Since $Px \in \mathcal{M}$ for every $x \in \mathcal{H}$ then $APx \in \mathcal{M} \Rightarrow PAPx = APx$ for every $x \in \mathcal{H} \Rightarrow PAP = AP$. Let $y \in \mathcal{M}$ and let $z \in \mathcal{M}^\perp$. Since $\mathcal{M}^\perp$ is invariant under A then $0 = \langle y, Az\rangle = \langle A^*y, z\rangle \Rightarrow A^*y \in \mathcal{M} \Rightarrow \mathcal{M}$ is invariant under $A^* \Rightarrow A^*Px \in \mathcal{M}$ for every $x \in \mathcal{H} \Rightarrow PA^*Px = A^*Px$ for every $x \in \mathcal{H} \Rightarrow PA^*P = A^*P \Rightarrow PAP = PA$ and hence $AP = PA$. □

LEMMA 6.25. *Let $\mathcal{K}$ be a complex Hilbert space and let $E : \Sigma \to \mathcal{L}(\mathcal{K})$ be a spectral measure on the measurable space (X, Σ), where $X \subset \mathbb{R}$ and Σ is the family of Borel subsets of X. If $B \in \Sigma$ is such that $\{0\} \neq E(B)(\mathcal{K})$ is finite dimensional, then there exists at least one $\lambda \in B$ such that $\dim(E(\{\lambda\})(\mathcal{K})) \neq 0$.*

Proof. We will construct a sequence $\{B_n\}_{n\geq 0}$ of Borel subsets of B such that $B_n \supset B_{n+1}$ for every $n \geq 0$ and $\dim(E(B_n)(\mathcal{K})) > 0$. Choose $B_0 = B$ and then cover B_0 with a sequence $\{I_n\}$ of disjoint intervals of length ≤ 1. There is at least one interval I_{n_1} such that

$E(B_0 \cap I_{n_1})(\mathcal{K})$ has positive dimension since otherwise, if $\dim(E(B \cap I_n)(\mathcal{K})) = 0$ for every n, then $E(B) = E(\cup_n (B \cap I_n)) = \sum_n E(B \cap I_n) = 0 \Rightarrow \dim(E(B)(\mathcal{K})) = 0$, a contradiction. Choose $B_1 = B_0 \cap I_{n_1}$. Cover B_1 with disjoint intervals I_n of length $\leq \frac{1}{2}$. By the same reason as before there is at least one interval I_{n_2} such that $E(B_1 \cap I_{n_2})(\mathcal{K})$ has positive dimension. Choose $B_2 = B_1 \cap I_{n_2}$ and continue inductively. Since $B_0 \supset B_1 \supset \ldots \supset B_n \supset \ldots$ we have that $E(B_0) \geq E(B_1) \geq \ldots \geq E(B_n) \geq \ldots > 0$ and hence $\dim(E(B_0)(\mathcal{K})) \geq \dim(E(B_1)(\mathcal{K})) \geq \ldots \geq \dim(E(B_n)(\mathcal{K})) \geq \ldots > 0$. Then there exists $N \geq 0$ such that $\dim(E(B_n)(\mathcal{K})) = C$ for every $n \geq N$, where $C > 0$ is an integer and hence $E(B_n) = E(B_N)$ for all $n \geq N$. Since $|I_n| \leq \frac{1}{n}$ we have that the intersection $\cap_{n\geq 1} B_n$ is at most one point. Since $E(B_n) \xrightarrow{so} E(\cap_{n\geq 1} B_n)$ we have that $E(\cap_{n\geq 1} B_n) = E(B_N) \neq 0$. Hence, there is a $\lambda \in B$ such that $\cap_{n\geq 1} B_n = \{\lambda\}$ and $E(\{\lambda\}) = E(B_N) \neq 0$. $\square$

THEOREM 6.26. ★ *Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space and let $O \in \mathcal{O}(\mathcal{H})$. Then there exists $\mathcal{M} \subset \mathcal{H}$ a reducing subspace for O such that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are infinite dimensional.*

Proof. Let $O \in \mathcal{O}(\mathcal{H})$ and let $A = \frac{O+O^T}{2}$. We will first show that if $\mathcal{K}$ is the complexification of $\mathcal{H}$ and if $\tilde{A}$, $\tilde{O}$ are the extensions to $\mathcal{K}$ of A, respectively O, then $\tilde{A}$ is self-adjoint and that $\tilde{A}$ commutes with $\tilde{O}$. Since $A^T = \left(\frac{O+O^T}{2}\right)^T = \frac{O^T+O}{2} = A$ we have using Proposition 6.14 that $(\tilde{A})^* = \widetilde{A^T} = \tilde{A}$ and hence $\tilde{A}$ is self-adjoint. Since $OA = O\frac{O+O^T}{2} = \frac{O^2+OO^T}{2} = \frac{O^2+O^TO}{2} = \frac{O+O^T}{2}O = AO$ we have that $\tilde{O}\tilde{A}(x+iy) = \tilde{O}(Ax+iAy) = OAx + iOAy = AOx + iAOy = \tilde{A}(Ox+iOy) = \tilde{A}\tilde{O}(x+iy)$ for every $x+iy \in \mathcal{K}$ and hence $\tilde{A}$ and $\tilde{O}$ commute. Also note that $J\tilde{A}J(x+iy) = J\tilde{A}(x-iy) = J(Ax-iAy) = Ax+iAy = \tilde{A}(x+iy)$ for every $x, y \in \mathcal{H}$ and hence $J\tilde{A}J = \tilde{A}$.

Let $E(\cdot)$ be the spectral measure defined on the measurable space (X, Σ) associated with $\tilde{A}$. Since $\tilde{A}$ is self-adjoint, by the spectral theorem we have that $X = [-\|\tilde{A}\|, \|\tilde{A}\|] \subset \mathbb{R}$ and Σ is the collection of Borel subsets of $[-\|\tilde{A}\|, \|\tilde{A}\|]$. Since $J\tilde{A}J = \tilde{A}$, we have by Corollary 6.19 that $JE(B)J = E(B)$ for every $B \in \Sigma$ and hence by Lemma 6.21 that $E(B)(\mathcal{H}) \subset \mathcal{H}$ for every $B \in \Sigma$. Since $\tilde{O}$ commutes with $\tilde{A}$, it follows from Lemma 6.22

that $\tilde{O}E(B) = E(B)\tilde{O}$ for every $B \in \Sigma$. Thus, if $x \in \mathcal{H}$, using the fact that $E(B)(\mathcal{H}) \subset \mathcal{H}$ we have that $E(B)Ox = E(B)\tilde{O}x = \tilde{O}E(B)x = OE(B)x$ for every $B \in \Sigma$. It follows from Proposition 6.24 that $E(B)(\mathcal{H})$ reduces O for every $B \in \Sigma$. If, for some $B \in \Sigma$, both $E(B)(\mathcal{H})$ and $[I - E(B)](\mathcal{H}) = E(B^C)(\mathcal{H})$ are infinite dimensional we are done. We will show that such a B exists.

Let $D = \{\lambda \in X \mid E(\{\lambda\})(\mathcal{H})$ has positive dimension$\}$. Since $\mathcal{H}$ is separable, the set D is countable. If $|D| = \infty$, let $D = F \cup G$, where F, G are disjoint, infinite sets. Let $B = F \subset \Sigma$. Then $G \subset B^C$, and hence both $E(B)(\mathcal{H})$ and $E(B^C)(\mathcal{H})$ have infinite dimension and are invariant under O.

Suppose that $|D| < \infty$ and there exists $\lambda \in D$ so that $\dim(E(\{\lambda\})(\mathcal{H})) = \infty$. Then $\tilde{A}(z) = \lambda z$ for every $z \in E(\{\lambda\})(\mathcal{K})$, where $\lambda \in \mathbb{R}$ since $\tilde{A}$ is self-adjoint and $0 < |\lambda| \leq \|\tilde{A}\| \leq 1$. This implies that $\frac{\tilde{O}+\widetilde{O^T}}{2} = \lambda I \Rightarrow \tilde{O}z + \widetilde{O^T}z = 2\lambda z$ for every $z \in E(\{\lambda\})(\mathcal{K})$. Let $z = x + iy$, with $x, y \in E(\{\lambda\})(\mathcal{H})$. Then $\tilde{O}(x + iy) + \widetilde{O^T}(x + iy) = 2\lambda(x + iy) \Rightarrow Ox + iOy + O^T x + iO^T y = 2\lambda x + i2\lambda y \Rightarrow Ox + O^T x = 2\lambda x \Rightarrow O^2 x + x = 2\lambda Ox$ for every $x \in \mathcal{H}$. Fix $0 \neq x_1 \in \mathcal{H}$ and let $\mathcal{S}_1 \subset \mathcal{H}$ be the subspace spanned by x_1 and Ox_1. If $y \in \mathcal{S}_1$ then there exist $a, b \in \mathbb{R}$ such that $y = ax_1 + bOx_1 \Rightarrow Oy = aOx_1 + bO^2 x_1 = aOx_1 + b(2\lambda Ox_1 - x_1) = -bx_1 + (a + 2b\lambda)Ox_1 \in \mathcal{S}_1 \Rightarrow \mathcal{S}_1$ is invariant under O. Also $O^T y = aO^T x_1 + bx_1 = a(2\lambda x_1 - Ox_1) + bx_1 = (2a\lambda + b)x_1 - Ox_1 \in \mathcal{S}_1 \Rightarrow \mathcal{S}_1$ is invariant under $O^T \Rightarrow \mathcal{S}_1^\perp$ is invariant under $O \Rightarrow \mathcal{S}_1$ reduces O. Fix $0 \neq x_2 \in \mathcal{S}_1^\perp$ and let $\mathcal{S}_2$ be the subspace spanned by x_2 and Ox_2. We show as before that $\mathcal{S}_2$ reduces O. Continue inductively and get an infinite sequence $\{\mathcal{S}_n\}$ of subspaces of $\mathcal{H}$, mutually orthogonal, each of which 1 or 2-dimensional and all reduce O. Split this sequence into two infinite subsequences $\{\mathcal{S}'_n\}$ and $\{\mathcal{S}''_n\}$ and let $\mathcal{M} = \oplus_n \mathcal{S}'_n$. Then $\mathcal{M}$ reduces O and both $\mathcal{M}$ and $\mathcal{M}^\perp = (\oplus_n \mathcal{S}''_n) \oplus E(X \setminus \{\lambda\})(\mathcal{H})$ are infinite dimensional.

Finally, suppose that $|D| < \infty$ and for every $\lambda \in D$, $\dim(E(\{\lambda\})(\mathcal{H})) < \infty$. Then $E(D)(\mathcal{H})$ is finite dimensional. Let $C = \mathbb{R} \setminus D$. Then for every $\lambda \in C$ we have that $E(\{\lambda\}) = 0$ and, since $\mathcal{H} = E(D)(\mathcal{H}) \cup E(C)(\mathcal{H})$ we have that $\dim(E(C)(\mathcal{H})) = \infty$. Cover

X with intervals $\left[\frac{k}{2^1}, \frac{k+1}{2^1}\right)$, where $k \in \mathbb{Z}$ and let $I_1^k = C \cap \left[\frac{k}{2^1}, \frac{k+1}{2^1}\right)$. If there is only one $k_1 \in \mathbb{Z}$ such that $E(I_1^{k_1}) \neq 0$, then $E(I_1^{k_1}) = E(C)$. Cover $I_1^{k_1}$ with intervals $\left[\frac{k}{2^2}, \frac{k+1}{2^2}\right)$, where $k \in \mathbb{Z}$ and let $I_2^k = I_1^{k_1} \cap \left[\frac{k}{2^2}, \frac{k+1}{2^2}\right)$. If there is only one $k_2 \in \mathbb{Z}$ such that $E(I_2^{k_2}) \neq 0$, then $E(I_2^{k_2}) = E(C)$. Cover $I_2^{k_2}$ with intervals $\left[\frac{k}{2^3}, \frac{k+1}{2^3}\right)$, where $k \in \mathbb{Z}$ and let $I_3^k = I_2^{k_2} \cap \left[\frac{k}{2^3}, \frac{k+1}{2^3}\right)$. If it is possible to continue this way, we get a sequence $I_1^{k_1} \supset \ldots \supset I_n^{k_n} \supset I_{n+1}^{k_{n+1}} \supset \ldots$ such that $E(I_n^{k_n}) = E(C)$ and the length $|I_n^{k_n}| \leq \frac{1}{2^n}$ for every $n \geq 1$. This implies that $E(\cap_{n\geq 1} I_n^{k_n}) = E(C) \neq 0 \Rightarrow \cap_{n\geq 1} I_n^{k_n} \neq \emptyset$ consists of at most one point $\Rightarrow$ there exists $\lambda \in C$ such that $\cap_{n\geq 1} I_n^{k_n} = \{\lambda\}$. But then $0 \neq E(C) = E(\{\lambda\}) = 0$, a contradiction. Thus, there exists $n \geq 1$ and $k, l \in \mathbb{Z}$ such that $k \neq l$ and both $\dim(E(I_n^k)(\mathcal{H})) > 0$ and $\dim(E(I_n^l)(\mathcal{H})) > 0$. If $E(I_n^k)(\mathcal{H})$ is finite dimensional then $E(I_n^k)(\mathcal{K})$ is finite dimensional, where $\mathcal{K}$ is the complexification of $\mathcal{H}$ and then, according with Lemma 6.25 we have that there exists $\lambda \in I_n^k$ such that $\dim(E(\{\lambda\})(\mathcal{K})) > 0 \Rightarrow$ by Lemma 6.21 that $\dim(E(\{\lambda\})(\mathcal{H})) > 0$, a contradiction with $\lambda \in C$. Hence $E(I_n^k)(\mathcal{H})$ is infinite dimensional and by similar reasoning we have that $E(I_n^l)(\mathcal{H})$ is infinite dimensional. If we let $B = I_n^k$, then $I_n^l \subset B^C$ and hence both $E(B)(\mathcal{H})$ and $E(B^C)(\mathcal{H})$ are infinite dimensional and invariant under O. $\square$

Corollary 6.27. ★ *Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space and let $O \in \mathcal{O}(\mathcal{H})$. Then $\mathcal{H}$ is the direct sum of an infinite sequence of infinite dimensional subspaces that reduce O.*

Proof. According with Theorem 6.26, there exists $\mathcal{H}_1 \subset \mathcal{H}$ a reducing subspace for O such that both $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ are infinite dimensional. Using the same theorem again for $\mathcal{H}_1^\perp$ we have that there exists $\mathcal{H}_2 \subset \mathcal{H}_1^\perp$ a reducing subspace for O such that both $\mathcal{H}_2$ and $\mathcal{H}_1^\perp \cap \mathcal{H}_2^\perp$ are infinite dimensional. Proceed inductively to obtain an infinite sequence $\{\mathcal{H}_n\}$ of mutually orthogonal infinite dimensional reducing subspaces. If the intersection $\cap_{n\geq 1} \mathcal{H}_n^\perp \neq \{0\}$, adjoin it to $\mathcal{H}_1$. $\square$

Proposition 6.28. ★ *Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space and let $O \in \mathcal{O}(\mathcal{H})$. Then there exists $A, B \in \mathcal{O}(\mathcal{H})$ such that $O = ABA^TB^T$.*

Proof. Let $\mathcal{H} = \oplus_{n\in\mathbb{Z}}\mathcal{H}_n$, where each $\mathcal{H}_n$ is a separable infinite dimensional Hilbert space that reduces O, as in Corollary 6.27. Since all $\mathcal{H}_n$'s are separable and have the same infinite dimension, they all are isomorphic to a fixed separable infinite dimensional Hilbert space $\mathcal{H}'$ and hence for every $n \in \mathbb{Z}$ there exists $W_n : \mathcal{H}_n \to \mathcal{H}'$ a norm preserving isomorphism. Note that each W_n is orthogonal and that $W_n^T = W_n^{-1}$. Let $W = \oplus_{n\in\mathbb{Z}}W_n : \oplus_{n\in\mathbb{Z}}\mathcal{H}_n \to \oplus_{n\in\mathbb{Z}}\mathcal{H}'$. Note that W is a norm preserving isomorphism of $\mathcal{H}$ onto $\oplus_{n\in\mathbb{Z}}\mathcal{H}' \Rightarrow W$ is orthogonal and $W^{-1} = W^T$. If $O \in \mathcal{O}(\mathcal{H})$ then $O' = WOW^T : \oplus_{n\in\mathbb{Z}}\mathcal{H}' \to \oplus_{n\in\mathbb{Z}}\mathcal{H}'$ is a norm preserving surjection and hence $O' \in \mathcal{O}(\oplus_{n\in\mathbb{Z}}\mathcal{H}')$. If $\mathcal{H}'$ is the n-th Hilbert space in $\oplus_{n\in\mathbb{Z}}\mathcal{H}'$ and if $x \in \mathcal{H}'$ then $W_n^T x \in \mathcal{H}_n \Rightarrow OW_n^T x \in \mathcal{H}_n$ since $\mathcal{H}_n$ is invariant under $O \Rightarrow O'x = W_n O W_n^T x \in \mathcal{H}' \Rightarrow \mathcal{H}'$ is invariant under O' and hence each $\mathcal{H}'$ is invariant under O'. We will show that the assertion is true for O', *i.e.* there exist $A', B' \in \mathcal{O}(\oplus_{n\in\mathbb{Z}}\mathcal{H}')$ such that $O' = A'B'A'^T B'^T$. If this is true, then $A = W^T A' W \in \mathcal{O}(\mathcal{H})$, $B = W^T B' W \in \mathcal{O}(\mathcal{H})$ and $O = W^T O' W = W^T A'B'A'^T B'^T W = W^T A' W W^T B' W W^T A'^T W W^T B'^T W = (W^T A' W)(W^T B' W)(W^T A' W)^T (W^T B' W)^T = ABA^T B^T$.

For every $n \in \mathbb{Z}$ let $P_n : \oplus_{n\in\mathbb{Z}}\mathcal{H}' \to \mathcal{H}'$ be the orthogonal projection of $\oplus_{n\in\mathbb{Z}}\mathcal{H}'$ onto the n-th $\mathcal{H}'$. Let $A' : \oplus_{n\in\mathbb{Z}}\mathcal{H}' \to \oplus_{n\in\mathbb{Z}}\mathcal{H}'$ be defined as $A'x = \sum_{n\in\mathbb{Z}} O'^n P_n x$. Note that $P_n A' x = O'^n P_n x$ for every $n \in \mathbb{Z}$. If $a, b \in \mathbb{R}$ and $x, y \in \mathcal{H}$ then $A'(ax + by) = \sum_{n\in\mathbb{Z}} O'^n P_n(ax + by) = a\sum_{n\in\mathbb{Z}} O'^n P_n x + b\sum_{n\in\mathbb{Z}} O'^n P_n y = aA'x + bA'y \Rightarrow A'$ is linear. Since $\|A'x\|^2 = \left\|\sum_{n\in\mathbb{Z}} P_n A'x\right\|^2 = \sum_{n\in\mathbb{Z}} \|P_n A'x\|^2 = \sum_{n\in\mathbb{Z}} \|O'^n P_n x\|^2 = \sum_{n\in\mathbb{Z}} \|P_n x\|^2 = \left\|\sum_{n\in\mathbb{Z}} P_n x\right\|^2 = \|x\|^2 \Rightarrow A'$ is an isometry. Let $y \in \oplus_{n\in\mathbb{Z}}\mathcal{H}'$. For every $n \in \mathbb{Z}$ let $x_n = (O'^T)^n P_n y \in \mathcal{H}'$ and let $x = \sum_{n\in\mathbb{Z}} x_n$. Since $\sum_{n\in\mathbb{Z}} \|x_n\|^2 = \sum_{n\in\mathbb{Z}} \|(O'^T)^n P_n y\|^2 = \sum_{n\in\mathbb{Z}} \|P_n y\|^2 = \left\|\sum_{n\in\mathbb{Z}} P_n y\right\|^2 = \|y\|^2 < \infty$, x is well defined. Note that $P_n x = x_n$ for every $n \in \mathbb{Z}$. Then $A'x = \sum_{n\in\mathbb{Z}} O'^n P_n x = \sum_{n\in\mathbb{Z}} O'^n x_n = \sum_{n\in\mathbb{Z}} O'^n (O'^T)^n P_n y = \sum_{n\in\mathbb{Z}} P_n y = y \Rightarrow A'$ is surjective $\Rightarrow A' \in \mathcal{O}(\oplus_{n\in\mathbb{Z}}\mathcal{H}')$. Since $P_n A' = O'^n P_n$ for every $n \in \mathbb{Z}$ we have that $P_n = O'^n P_n A'^T \Rightarrow (O'^T)^n P_n = P_n A'^T$ for every $n \in \mathbb{Z}$.

For every $x \in \oplus_{n\in\mathbb{Z}}\mathcal{H}'$ let $B'x = y$, where y is such that $P_n y = P_{n-1}x$. Then $B' : \oplus_{n\in\mathbb{Z}}\mathcal{H}' \to \oplus_{n\in\mathbb{Z}}\mathcal{H}'$ is a well defined mapping and $P_n B' x = P_{n-1}x$ for every $x \in \oplus_{n\in\mathbb{Z}}\mathcal{H}'$. If

$a,b\in\mathbb{R}$ and $x,y\in\oplus_{n\in\mathbb{Z}}\mathcal{H}'$ then $B'(ax+by)=\sum_{n\in\mathbb{Z}}P_nB'(ax+by)=\sum_{n\in\mathbb{Z}}P_{n-1}(ax+by)=a\sum_{n\in\mathbb{Z}}P_{n-1}x+b\sum_{n\in\mathbb{Z}}P_{n-1}y=a\sum_{n\in\mathbb{Z}}P_nB'x+b\sum_{n\in\mathbb{Z}}P_nB'y\Rightarrow B'$ is linear. Since $\|B'x\|^2=\left\|\sum_{n\in\mathbb{Z}}P_nB'x\right\|^2=\sum_{n\in\mathbb{Z}}\|P_nB'x\|^2=\sum_{n\in\mathbb{Z}}\|P_{n-1}x\|^2=\left\|\sum_{n\in\mathbb{Z}}P_{n-1}x\right\|^2=\|x\|^2\Rightarrow B'$ is an isometry. Let $y\in\oplus_{n\in\mathbb{Z}}\mathcal{H}'$. For every $n\in\mathbb{Z}$ let $x_n=P_{n+1}y$ and let $x=\sum_{n\in\mathbb{Z}}x_n$. Since $\sum_{n\in\mathbb{Z}}\|x_n\|^2=\sum_{n\in\mathbb{Z}}\|P_{n+1}y\|^2=\left\|\sum_{n\in\mathbb{Z}}P_{n+1}y\right\|^2=\|y\|^2<\infty$, x is well defined. Then $B'x=\sum_{n\in\mathbb{Z}}P_nB'x=\sum_{n\in\mathbb{Z}}P_{n-1}x=\sum_{n\in\mathbb{Z}}x_{n-1}=\sum_{n\in\mathbb{Z}}P_ny=y\Rightarrow B'$ is surjective $\Rightarrow B'\in\mathcal{O}(\oplus_{n\in\mathbb{Z}}\mathcal{H}')$. Since $P_nB'=P_{n-1}$ for every $n\in\mathbb{Z}$ we have that $P_n=P_{n-1}B'^T$ for every $n\in\mathbb{Z}$.

$A'B'A'^TB'^Tx=\sum_{n\in\mathbb{Z}}P_nA'B'A'^TB'^Tx=\sum_{n\in\mathbb{Z}}O'^nP_nB'A'^TB'^Tx=\sum_{n\in\mathbb{Z}}O'^nP_{n-1}A'^TB'^Tx=\sum_{n\in\mathbb{Z}}O'^n(O'^T)^{n-1}P_{n-1}B'^Tx=\sum_{n\in\mathbb{Z}}O'P_nx=\sum_{n\in\mathbb{Z}}P_nO'x=O'x$ for every $x\in\mathcal{H}\Rightarrow O'=A'B'A'^TB'^T$. $\square$

6.3. The Subsets $\mathcal{O}(\mathcal{M})$ and $SO(\mathcal{M})$ of $\mathcal{O}(\mathcal{H})$

Proposition 6.29. ★ *Let G be a Polish topological group, $\mathcal{M}$ a closed subspace of the real Hilbert space $\mathcal{H}$ and $\phi:G\to\mathcal{O}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in G.*

Proof. If $\dim(\mathcal{H})=1$ then $\mathcal{M}=\mathcal{H}\Rightarrow Z(\mathcal{O}(\mathcal{H}))=\mathcal{O}(\mathcal{M})=\{\pm I\}\Rightarrow\phi^{-1}(ZOH)$ is closed. Suppose that $\dim(\mathcal{H})\geq 2$.

We will prove that $Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})=\{W\in\mathcal{O}(\mathcal{H})\mid WV=VW\ \forall V\in\mathcal{O}(\mathcal{M}^\perp)\}$. This will imply that $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))=\phi^{-1}(\{W\in\mathcal{O}(\mathcal{H})\mid WV=VW\ \forall V\in\mathcal{O}(\mathcal{M}^\perp)\})=\{\phi^{-1}(W)\mid\phi^{-1}(W)\phi^{-1}(V)=\phi^{-1}(V)\phi^{-1}(W)\ \forall\phi^{-1}(V)\in\phi^{-1}(\mathcal{O}(\mathcal{M}^\perp))\}$ and then, according with the Proposition 3.26 we will have that $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in G. Note that by Proposition 6.5 we have that $Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})=\{\pm U\mid U\in\mathcal{O}(\mathcal{M})\}$.

Let $U\in\mathcal{O}(\mathcal{M})$, let $V\in\mathcal{O}(\mathcal{M}^\perp)$ and let $x=x_1+x_2\in\mathcal{H}$, with $x_1\in\mathcal{M}$ and $x_2\in\mathcal{M}^\perp$. Then $Ux_2=x_2$, $Vx_1=x_1$ and, by Proposition 3.14, $Ux_1\in\mathcal{M}$ and $Vx_2\in\mathcal{M}^\perp$ and hence $VUx_1=Ux_1$ and $UVx_2=Vx_2$. It follows that $\lambda UVx=\lambda UV(x_1+x_2)=\lambda(UVx_1+UVx_2)=\lambda(Ux_1+Vx_2)=\lambda(VUx_1+VUx_2)=\lambda VUx=V\lambda Ux\Rightarrow\lambda UV=V\lambda U$ for every $V\in\mathcal{O}(\mathcal{M}^\perp)\Rightarrow Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})\subset\{W\in\mathcal{O}(\mathcal{H})\mid WV=VW\ \forall V\in\mathcal{O}(\mathcal{M}^\perp)\}$.

Let $W \in \mathcal{O}(\mathcal{H})$ be such that $WV = VW$ for every $V \in \mathcal{O}(\mathcal{M}^\perp)$. Let $U : \mathcal{M}^\perp \to \mathcal{M}^\perp$ be orthogonal, and let $V : \mathcal{H} \to \mathcal{H}$ be defined as $Vx = x_1 + Ux_2$ for every $x = x_1 + x_2 \in \mathcal{H}$, where $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. V is orthogonal since it is an isometry from $\mathcal{H}$ onto $\mathcal{H}$, and $V|_{\mathcal{M}} = I$. Thus $V \in \mathcal{O}(\mathcal{M}^\perp)$, and hence $VW = WV$. Let $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. Then, by Lemma 3.27 $Wx_1 \in \mathcal{M}$ and $Wx_2 \in \mathcal{M}^\perp$, and hence $Wx_1 + UWx_2 = VWx_1 + VWx_2 = VW(x_1 + x_2) = WV(x_1 + x_2) = W(x_1 + Ux_2) = Wx_1 + WUx_2 \Rightarrow UWx_2 = WUx_2$ for every $x_2 \in \mathcal{M}^\perp \Rightarrow UW|_{\mathcal{M}^\perp} = W|_{\mathcal{M}^\perp}U$. Hence $W|_{\mathcal{M}^\perp}$ is in the center of $\mathcal{O}(\mathcal{M}^\perp)$ and by Proposition 6.5 it follows that $W|_{\mathcal{M}^\perp} = \pm I$.If $W|_{\mathcal{M}^\perp} = I \Rightarrow W \in \mathcal{O}(\mathcal{M}) \Rightarrow W = IW \in Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})$. If $W|_{\mathcal{M}^\perp} = -I \Rightarrow -W \in \mathcal{O}(\mathcal{M}) \Rightarrow W = -(-W) \in Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})$. This implies that $\{W \in \mathcal{O}(\mathcal{H}) \mid WV = VW \ \forall V \in \mathcal{O}(\mathcal{M}^\perp)\} \subset Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})$. □

PROPOSITION 6.30. ★ *Let G be a Polish topological group, $\mathcal{M}$ an infinite dimensional closed subspace of the real Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{O}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is an analytic subset of G.*

Proof. Let $[\cdot,\cdot] : G \times G \to G$ be defined as $[a, b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot,\cdot]$ is continuous. If $a, b \in \phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})) \subset G$ then $\phi(a), \phi(b) \in Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}) \Rightarrow$ there exist $U, V \in \mathcal{O}(\mathcal{M})$ such that $\phi(a) = \pm U$ and $\phi(b) = \pm V$. But then $[a, b] = \phi^{-1}(\pm U)\phi^{-1}(\pm V)\phi^{-1}((\pm U)^{-1})\phi^{-1}((\pm V)^{-1}) = \phi^{-1}(UVU^{-1}V^{-1}) \in \phi^{-1}(\mathcal{O}(\mathcal{M}))$. This proves that $[\cdot,\cdot]|_{\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))\times\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))}$ takes its values in $\phi^{-1}(\mathcal{O}(\mathcal{M}))$. Let $T \in \mathcal{O}(\mathcal{M})$ and denote $T|_{\mathcal{M}} = W$. Since $\mathcal{M}$ is infinite dimensional and since W is orthogonal on $\mathcal{M}$, we have by Proposition 6.28 that there exist orthogonals $U', V' : \mathcal{M} \to \mathcal{M}$ such that $W = U'V'U'^{-1}V'^{-1}$. If $U, V : \mathcal{H} \to \mathcal{H}$ are such that $U|_{\mathcal{M}} = U'$, $U|_{\mathcal{M}^\perp} = I$, $V|_{\mathcal{M}} = V'$ and $V|_{\mathcal{M}^\perp} = I$ then $U, V \in Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})$ and $[\phi^{-1}(U), \phi^{-1}(V)] = \phi^{-1}(UVU^{-1}V^{-1}) = \phi^{-1}(T)$ and hence $[\cdot,\cdot]|_{\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))\times\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))}$ is onto $\phi^{-1}(\mathcal{O}(\mathcal{M}))$. Since G is a Polish topological group, $G \times G$ is a Polish topological group and since $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in G by Theorem 6.29, we have that $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M})) \times \phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in $G \times G$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is the continuous image of a closed subset of a Polish topological group, and therefore an analytic subset of G. $\square$

PROPOSITION 6.31. ★ *Let G be a Polish topological group, $\mathcal{M}$ a closed subspace of the real infinite dimensional Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{O}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in G.*

Proof. If $\mathcal{M} = \mathcal{H}$ then $\mathcal{O}(\mathcal{M}) = \mathcal{O}(\mathcal{H})$ and there is nothing to prove, so we may assume that $\mathcal{M} \neq \mathcal{H}$. Suppose first that $\mathcal{M}$ is infinite dimensional. By Theorem 6.29 we have that $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in G and hence Polish. $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))) = Z(G)$, the center of G is a closed in G and $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is analytic by Proposition 6.30. If $U \in Z(\mathcal{O}(\mathcal{H})) \cap \mathcal{O}(\mathcal{M})$, then $U = \pm I$ and, since $U|_{\mathcal{M}^\perp} = I$, we have that $U = I \Rightarrow Z(\mathcal{O}(\mathcal{H})) \cap \mathcal{O}(\mathcal{M}) = \{I\} \Rightarrow \phi^{-1}(Z(\mathcal{O}(\mathcal{H}))) \cap \phi^{-1}(\mathcal{O}(\mathcal{M})) = \phi^{-1}(Z(\mathcal{O}(\mathcal{H})) \cap \mathcal{O}(\mathcal{M})) = \phi^{-1}(I) = \{e\}$ is closed in G. Using Corollary 3.39 we have that $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ and since $\phi^{-1}(Z(\mathcal{O}(\mathcal{H}))\mathcal{O}(\mathcal{M}))$ is closed in G it follows that $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in G.

Suppose that $\mathcal{M}$ is finite dimensional. Let $\{e_1, e_2, ..., e_n\}$ be a orthonormal basis for $\mathcal{M}$. Extend this to $\{e_1, ..., e_n, ..., e_{n+l}, ...\}$ an orthonormal basis for $\mathcal{H}$. For every $l \geq 1$, let $\mathcal{M}_l = span(\{e_i\}_{i\geq 1} \setminus \{e_{n+l}\})$. Each $\mathcal{M}_l$ is infinite dimensional. Hence, by the previous paragraph we have that $\phi^{-1}(\mathcal{O}(\mathcal{M}_l))$ is closed in G, for every $l \geq 1$.

Since $U \in \mathcal{O}(\mathcal{M}) \Leftrightarrow U|_{\mathcal{M}^\perp} = I \Leftrightarrow Ue_{n+l} = e_{n+l}$ for every $l \geq 1 \Leftrightarrow U \in \mathcal{O}(\mathcal{M}_l)$ for every $l \geq 1 \Leftrightarrow U \in \cap_{l\geq 1}\mathcal{O}(\mathcal{M}_l)$ we have that $\mathcal{O}(\mathcal{M}) = \cap_{l\geq 1}\mathcal{O}(\mathcal{M}_l) \Rightarrow \phi^{-1}(\mathcal{O}(\mathcal{M})) = \phi^{-1}(\cap_{l\geq 1}\mathcal{O}(\mathcal{M}_l)) = \cap_{l\geq 1}\phi^{-1}(\mathcal{O}(\mathcal{M}_l)) \Rightarrow \phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in G. $\square$

DEFINITION 6.32. Let $\mathcal{H}$ be a two dimensional real Hilbert space. An element $R \in \mathcal{L}(\mathcal{H})$ is called a rotation if its associated matrix can be written in the form

$$R = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta \in \mathbb{R}$ is the angle of rotation. If $R \in \mathcal{L}(\mathcal{H})$ is a rotation, since $R^TR = RR^T = I$ we have that $R \in \mathcal{O}(2)$ and since $\det(R) = 1$ it follows that $R \in SO(2)$.

LEMMA 6.33. *Let $\mathcal{M}$ be a finite dimensional real Hilbert space and let $U \in SO(\mathcal{M})$. Then there exist $P, Q \in \mathcal{O}(\mathcal{M})$ such that $U = PQP^{-1}Q^{-1}$.*

Proof. If $U \in SO(\mathcal{M})$, then $U \in \mathcal{O}(\mathcal{M})$ and using a result from [6], §81, page 162, we have that there exists an orthonormal basis for $\mathcal{M}$ such that the matrix representation of U is

$$U = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & -1 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & -1 & & & & & \\ & & & & & & \cos\theta_1 & -\sin\theta_1 & & & \\ & & & & & & \sin\theta_1 & \cos\theta_1 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \cos\theta_n & -\sin\theta_n \\ & & & & & & & & & \sin\theta_n & \cos\theta_n \end{pmatrix}$$

(here, all the other entries are 0). Since $\det(U) = 1$ and since the determinant of every rotation is 1 we must have an even number of -1's on the diagonal of U. Note that every pair of 1's is equivalent to a rotation by 0 and every pair of -1's is equivalent to a rotation by π. Thus, the matrix representation of U consists of rotations on the diagonal if the dimension of $\mathcal{M}$ is even and a 1 and rotations on the diagonal if the dimension of $\mathcal{M}$ is odd. The conclusion will follow if we prove that for every rotation R there exist $P, Q \in \mathcal{O}(2)$ such that $R = PQP^{-1}Q^{-1}$.

Let $R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be a rotation and let $P = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

It is easy to see that $P^2 = I$ and $Q^2 = I \Rightarrow P^{-1} = P$ and $Q^{-1} = Q$ and hence $P, Q \in \mathcal{O}(2)$. By computation we have that

$$PQP^{-1}Q^{-1} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R$$

which completes the proof. □

Proposition 6.34. ★ *Let G be a Polish topological group, $\mathcal{M}$ a finite dimensional closed subspace of the real infinite dimensional Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{O}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(SO(\mathcal{M}))$ is an analytic subset of G.*

Proof. Since $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in G by Proposition 6.31, $\phi^{-1}(\mathcal{O}(\mathcal{M})) \times \phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in $G \times G$. Let $[\cdot,\cdot] : \phi^{-1}(\mathcal{O}(\mathcal{M})) \times \phi^{-1}(\mathcal{O}(\mathcal{M})) \to G$ be defined as $[a,b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot,\cdot]$ is continuous. If $a, b \in \phi^{-1}(\mathcal{O}(\mathcal{M}))$ then $\phi(a), \phi(b) \in \mathcal{O}(\mathcal{M})$, $\phi([a,b]) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)(\phi(a))^{-1}(\phi(b))^{-1} \in \mathcal{O}(\mathcal{M})$ and $\det(\phi([a,b])) = \det(\phi(aba^{-1}b^{-1})) = \det(\phi(a))\det(\phi(b))(\det(\phi(a)))^{-1}(\det(\phi(b)))^{-1} = 1 \Rightarrow \phi([a,b]) \in SO(\mathcal{M}) \Rightarrow [a,b] \in \phi^{-1}(SO(\mathcal{M}))$. This proves that $[\cdot,\cdot]$ takes its values in $\phi^{-1}(SO(\mathcal{M}))$. Let $y \in \phi^{-1}(SO(\mathcal{M}))$. Then $\phi(y) = W \in SO(\mathcal{M})$. By Lemma 6.33 we have that there exist $U, V \in \mathcal{O}(\mathcal{M}))$ such that $W = UVU^{-1}V^{-1}$. Let $a = \phi^{-1}(U) \in \phi^{-1}(\mathcal{O}(\mathcal{M}))$ and $b = \phi^{-1}(V) \in \phi^{-1}(\mathcal{O}(\mathcal{M}))$. a and b exist since ϕ is an isomorphism. Then $y = \phi^{-1}(W) = \phi^{-1}(UVU^{-1}V^{-1}) = \phi^{-1}(U)\phi^{-1}(V)(\phi^{-1}(U))^{-1}(\phi^{-1}(V))^{-1} = aba^{-1}b^{-1} = [a,b] \Rightarrow [\cdot,\cdot]$ is onto $\phi^{-1}(SO(\mathcal{M}))$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(SO(\mathcal{M}))$ is the continuous image of $\phi^{-1}(\mathcal{O}(\mathcal{M}))\times\phi^{-1}(\mathcal{O}(\mathcal{M}))$, a closed set of a Polish space by Proposition 6.31, and therefore $\phi^{-1}(SO(\mathcal{M}))$ is an analytic subset of G. □

Proposition 6.35. *If $\mathcal{M}$ is a finite dimensional real Hilbert space, then*

$$\mathcal{O}(\mathcal{M}) = Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M})$$

Proof. Since $Z(\mathcal{O}(\mathcal{M})), SO(\mathcal{M}) \subset \mathcal{O}(\mathcal{M})$ and $\mathcal{O}(\mathcal{M})$ is a subgroup it follows that $Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M}) \subset \mathcal{O}(\mathcal{M})$.

Let $U \in \mathcal{O}(\mathcal{M})$. Since $1 = \det(I) = \det(UU^T) = \det(U)\det(U^T) = \det(U)^2 \Rightarrow \det(U) = \pm 1$. If $\det(U) = 1$ then $U \in SO(\mathcal{M}) \Rightarrow U = IU \in Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M})$.

If $\det(U) = -1$, consider the matrix representation of U as in Lemma 6.33. Since the determinant of every rotation is 1 and every rotation is a transformation on a two-dimensional Hilbert space, we must have that the dimension of $\mathcal{M}$, n is odd. Let e be a unit vector such that $e \perp \mathcal{M}$ and let $\mathcal{H} = span(\{e\} \cup \mathcal{M})$. Let $V : \mathcal{H} \to \mathcal{H}$ be defined as $V|_{\mathcal{M}} = -I,\ V|_{\{e\}} = I$ and $W : \mathcal{H} \to \mathcal{H}$ be defined as $W|_{\mathcal{M}} = -U,\ W|_{\{e\}} = I$. Then $V \in Z(\mathcal{O}(\mathcal{M}))$ by Proposition 6.5. Since

$$W = \begin{pmatrix} 1 & 0 \\ 0 & -U \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

we have that $\det(W) = \det(-I)\det(U)$, where I is the identity in $\mathcal{U}(\mathcal{M})$. Since $n = \dim(\mathcal{M})$ is odd, we have that $\det(-I) = (-1)^n = -1 \Rightarrow \det(W) = 1$ and hence $W \in SO(\mathcal{M})$. Since $U = (-I)(-U) = V|_{\mathcal{M}} W|_{\mathcal{M}}$ and since $U|_{\{e\}} = I = V|_{\{e\}} W|_{\{e\}}$ we have that $U = VW \in Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M})$ and hence $\mathcal{O}(\mathcal{M}) \subset Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M})$. □

Corollary 6.36. ★ *Let G be a Polish topological group, $\mathcal{M}$ a finite dimensional closed subspace of the real infinite dimensional Hilbert space $\mathcal{H}$ and $\phi : G \to \mathcal{O}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(SO(\mathcal{M}))$ is closed in G.*

Proof. From Corollary 6.31 we have that $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is closed in G and hence Polish. From Proposition 6.35 we have that $Z(\mathcal{O}(\mathcal{M}))\ SO(\mathcal{M}) = \mathcal{O}(\mathcal{M}) \Rightarrow \phi^{-1}(Z(\mathcal{O}(\mathcal{M})))\phi^{-1}(SO(\mathcal{M})) = \phi^{-1}(\mathcal{O}(\mathcal{M}))$. $\phi^{-1}(Z(\mathcal{O}(\mathcal{M}))) = Z(\phi^{-1}(\mathcal{O}(\mathcal{M})))$, the center of $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ is a closed subgroup of $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ and $\phi^{-1}(SO(\mathcal{M}))$ is an analytic subgroup of $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ by Proposition 6.34. Let $C = Z(\mathcal{O}(\mathcal{M})) \cap SO(\mathcal{M})$. Then $C = \{U \in \mathcal{O}(\mathcal{M}) \mid U|_{\mathcal{M}} = \pm I,\ U|_{\mathcal{M}^\perp} = I\} \Rightarrow C$ is finite and since ϕ is an isomorphism, we have that $\phi^{-1}(C)$ is finite and hence $\phi^{-1}(Z(\mathcal{O}(\mathcal{M}))) \cap$

$\phi^{-1}(SO(\mathcal{M})) = \phi^{-1}(Z(\mathcal{O}(\mathcal{M}))\cap SO(\mathcal{M})) = \phi^{-1}(C)$ is closed in $\phi^{-1}(\mathcal{O}(\mathcal{M}))$. It follows from Corollary 3.39 that $\phi^{-1}(SO(\mathcal{M}))$ is closed in $\phi^{-1}(\mathcal{O}(\mathcal{M}))$ and hence closed in G. $\square$

6.4. Main Result

LEMMA 6.37. ★ *Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space, let $\{e_l\}_{l\geq 1} \subset \mathcal{H}$ be an orthonormal basis for $\mathcal{H}$ and let P be the orthogonal projection on span$(\{e_1\})$. Then there exists $\mathcal{M}$ a three dimensional subspace of $\mathcal{H}$ such that for every $U \in \mathcal{O}(\mathcal{H})$ there exists $U_0 \in SO(\mathcal{M})$ such that $PU_0e_1 = PUe_1$.*

Proof. Let $\mathcal{M} = span(\{e_1, e_2, e_3\})$, a three dimensional subspace of $\mathcal{H}$. Note that since P is the orthogonal projection on $span(\{e_1\})$, then $PUe_1 = \lambda e_1$ and since $|\lambda|^2 = |\lambda|^2\|e_1\|^2 = \|\lambda e_1\|^2 = \|PUe_1\|^2 \leq \|PUe_1\|^2 + \|(I-P)Ue_1\|^2 = \|Ue_1\|^2 = \|e_1\|^2 = 1$ we have that $|\lambda| \leq 1$.

Let θ be such that $\cos\theta = \lambda$ and let

$$U_0 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we have that

$$U_0^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence $U_0U_0^T = I$ and $U_0^TU_0 = I$. We also have that $\det(U_0) = 1$ and hence $U_0 \in SO(\mathcal{M})$.

Since $U_0e_1 = \cos\theta e_1 + \sin\theta e_2$ it follows that $PU_0e_1 = \cos\theta e_1 = \lambda e_1 = PUe_1$. $\square$

LEMMA 6.38. ★ *Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space, let $e \in \mathcal{H}$ be such that $\|e\| = 1$ and let $\mathcal{S} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|e - Oe\| < \epsilon\}$. Then there exists $\mathcal{M} \subset \mathcal{H}$ a three dimensional subspace such that $\mathcal{S} = \mathcal{O}(\{e\}^{\perp})\ [SO(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{O}(\{e\}^{\perp})$.*

Proof. Note that if $W \in \mathcal{O}(\{e\}^{\perp})$ and if $O \in \mathcal{S}$ then $\|e - OWe\| = \|e - Oe\| < \epsilon \Rightarrow OW \in \mathcal{S} \Rightarrow \mathcal{S}\ \mathcal{O}(\{e\}^{\perp}) \subset \mathcal{S} \Rightarrow \mathcal{S}\ \mathcal{O}(\{e\}^{\perp}) = \mathcal{S}$ and $\|e - WOe\| = \|We - WOe\| = \|W(e - Oe)\| = \|e - Oe\| < \epsilon \Rightarrow WO \in \mathcal{S} \Rightarrow \mathcal{O}(\{e\}^{\perp})\ \mathcal{S} \subset \mathcal{S} \Rightarrow \mathcal{O}(\{e\}^{\perp})\ \mathcal{S} = \mathcal{S}$ and hence $\mathcal{O}(\{e\}^{\perp})\ \mathcal{S}\ \mathcal{O}(\{e\}^{\perp}) = \mathcal{S}$.

Let $U \in \mathcal{S}$. Let P be the orthogonal projection on $span(\{e\})$ and let $Q = I - P$. By Lemma 6.37 we have that there exists $\mathcal{M}$ a three dimensional subspace and $U_0 \in SO(\mathcal{M})$ such that $PU_0e = PUe$. Since $\|PUe\|^2 + \|QUe\|^2 = \|Ue\|^2 = 1 = \|U_0e\|^2 = \|PU_0e\|^2 + \|QU_0e\|^2$ we have that $\|QUe\|^2 = \|QU_0e\|^2$. Since $QUe \in \{e\}^\perp$ and $QU_0e \in \{e\}^\perp$ there exists $W \in \mathcal{O}(\{e\}^\perp)$ such that $WQU_0e = QUe$. Since by Lemma 3.50 W commutes with P and with Q we have that $WU_0e = PWU_0e + QWU_0e = WPU_0e + WQU_0e = PU_0e + QUe = PUe + QUe = Ue \Rightarrow U_0^TW^TUe = e \Rightarrow U_0^TW^TU = V \in \mathcal{O}(\{e\}^\perp) \Rightarrow U = WU_0V$. We also have that $\|e - U_0e\|^2 = \|e - PU_0e\|^2 + \|QU_0e\|^2 = \|e - PU_0e\|^2 + \|WQU_0e\|^2 = \|e - PUe\|^2 + \|QUe\|^2 = \|P(e - Ue)\|^2 + \|Q(e - Ue)\|^2 = \|e - Ue\|^2 < \epsilon^2 \Rightarrow U_0 \in \mathcal{S}$. Thus $U = WU_0V$, with $W, V \in \mathcal{O}(\{e\}^\perp)$ and $U_0 \in SO(\mathcal{M}) \cap \mathcal{S}$. This implies that $\mathcal{S} \subset \mathcal{O}(\{e\}^\perp)\ [SO(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{O}(\{e\}^\perp) \subset \mathcal{O}(\{e\}^\perp)\ \mathcal{S}\ \mathcal{O}(\{e\}^\perp) = \mathcal{S} \Rightarrow \mathcal{S} = \mathcal{O}(\{e\}^\perp)\ [SO(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{O}(\{e\}^\perp)$. □

LEMMA 6.39. ★ *Let G be a Polish topological group, let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $e \in \mathcal{H}$ be such that $\|e\| = 1$. Let $\mathcal{S} = \{U \in \mathcal{O}(\mathcal{H}) \mid \|e - Ue\| < \epsilon\}$ and let $\phi : G \to \mathcal{O}(\mathcal{H})$ be an algebraic isomorphism. Then $\phi^{-1}(\mathcal{S})$ is analytic in G.*

Proof. Let $\mathcal{M}$ be as in Lemma 6.37 so that $\mathcal{S} = \mathcal{O}(\{e\}^\perp)\ [SO(\mathcal{M}) \cap \mathcal{S}]\ \mathcal{O}(\{e\}^\perp)$. Since $SO(\mathcal{M})$ is a connected compact metric group with a totally disconnected center (Chapter I, Section 14, [19]), using the result from [14] we have that $\phi|_{\phi^{-1}(SO(\mathcal{M}))} : \phi^{-1}(SO(\mathcal{M})) \to SO(\mathcal{M})$ is a homeomorphism. $\mathcal{S} \cap SO(\mathcal{M})$ is a relatively open subset of $SO(\mathcal{M}) \Rightarrow \phi^{-1}(\mathcal{S} \cap SO(\mathcal{M}))$ is relatively open in $\phi^{-1}(SO(\mathcal{M}))$. Since $\phi^{-1}(SO(\mathcal{M}))$ is closed in G by Corollary 6.36, we have that $\phi^{-1}(\mathcal{S} \cap SO(\mathcal{M}))$ is a Borel subset of G. Since $\phi^{-1}(\mathcal{O}(\{e\}^\perp))$ is closed in G by Proposition 6.31, it follows from Lemma 3.53 that $\phi^{-1}(\mathcal{S}) = \phi^{-1}(\mathcal{O}(\{e\}^\perp)\ [\mathcal{S} \cap SO(\mathcal{M})]\ \mathcal{O}(\{e\}^\perp)) = \phi^{-1}(\mathcal{O}(\{e\}^\perp))\phi^{-1}(\mathcal{S} \cap SO(\mathcal{M}))\phi^{-1}(\mathcal{O}(\{e\}^\perp))$ is analytic. □

THEOREM 6.40. ★ *Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{O}(\mathcal{H})$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. Let $\{e_l\}_{l \geq 1}$ be an orthonormal basis for $\mathcal{H}$. Let $\mathcal{U}$ be a basic neighborhood of I in $\mathcal{O}(\mathcal{H})$. According with Proposition 3.11 $\mathcal{U}$ is of the form $\mathcal{U} = \cap_{1 \leq l \leq n}\{U \in \mathcal{O}(\mathcal{H}) \mid \|Ue_l - e_l\| < \epsilon\}$

for some $\epsilon > 0$. $\phi^{-1}(\mathcal{O})$ is analytic by Lemma 6.39 and, since analytic sets have the Baire property, $\phi^{-1}(\mathcal{U})$ is a set with the Baire property. The conclusion follows from Lemma 3.57.

□

CHAPTER 7

THE PROJECTIVE ORTHOGONAL GROUP

Throughout this section $\mathcal{H}$ is assumed to be a real Hilbert space.

Definition 7.1. If H is a real Hilbert space, the projective orthogonal group is the group $\mathcal{PO}(\mathcal{H}) = \mathcal{O}(\mathcal{H})/Z(\mathcal{O}(\mathcal{H}))$. If $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ is the natural quotient mapping and if $\mathcal{S} \subset \mathcal{O}(\mathcal{H})$ then $\pi(\mathcal{S}) = \{\pm O \mid O \in \mathcal{S}\}$. Throughout this section $\mathcal{H}$ is assumed to be a real Hilbert space.

Proposition 7.2. *$\mathcal{PO}(\mathcal{H})$ is a topological group.*

Proof. $Z(\mathcal{O}(\mathcal{H}))$ is a normal subgroup of $\mathcal{O}(\mathcal{H})$ and use Proposition 4.2. $\square$

Corollary 7.3. ★ *If $\mathcal{H}$ is separable, $\mathcal{PO}(\mathcal{H})$ is a Polish topological group.*

Proof. $\mathcal{PO}(\mathcal{H})$ is metrizable by Theorem 4.4. If $\mathcal{H}$ is separable, then $\mathcal{H}om(\mathcal{H}_1)$, the homeomorphism group of the unit ball, is completely metrizable by Corollary 2.25 and since $\mathcal{O}(\mathcal{H})$ is a closed subgroup of $\mathcal{H}om(\mathcal{H}_1)$ by Theorem 3.7, we have that $\mathcal{O}(\mathcal{H})$ is completely metrizable. Since the mapping π is continuous and onto, using a theorem of Hausdorff [8] we have that $\mathcal{PO}(\mathcal{H})$ is completely metrizable. $\mathcal{PO}(\mathcal{H})$ is separable by Proposition 4.5. $\square$

Theorem 7.4. ★ *Let $\mathcal{M}$ be a closed subspace of the infinite dimensional Hilbert space $\mathcal{H}$ and let $W \in \mathcal{O}(\mathcal{H})$ be such that $WOW^TO^T \in Z(\mathcal{O}(\mathcal{H}))$ for every $O \in \mathcal{O}(\mathcal{M})$. Then $WO = OW$ for every $O \in \mathcal{O}(\mathcal{M})$.*

Proof. Let $W \in \mathcal{O}(\mathcal{H})$ be such that $WOW^TO^T \in Z(\mathcal{O}(\mathcal{H}))$ for every $O \in \mathcal{O}(\mathcal{M})$. Then $WO = \pm OW$. For every $O \in \mathcal{O}(\mathcal{H})$ let $\lambda(O) = \pm 1$ be such that $WO = \lambda(O)OW$. If $O_1, O_2 \in \mathcal{O}(\mathcal{H})$ then $\lambda(O_1O_2)O_1O_2 = WO_1O_2 = \lambda(O_1)O_1WO_2 = \lambda(O_1)\lambda(O_2)O_1O_2W \Rightarrow \lambda(O_1O_2) = \lambda(O_1)\lambda(O_2) \Rightarrow \lambda : \mathcal{O}(\mathcal{H}) \to \{\pm 1\}$ is a homomorphism of groups. If $O \in \mathcal{O}(\mathcal{H})$ then $O^T \in \mathcal{O}(\mathcal{H})$ and $1 = \lambda(I) = \lambda(O^TO) = \lambda(O^T)\lambda(O) \Rightarrow \lambda(O^T) = \lambda(O)$.

If $\mathcal{M}$ is infinite dimensional and if $O \in \mathcal{O}(\mathcal{M})$, according to Proposition 6.28, there exist $P, Q \in \mathcal{O}(\mathcal{M})$ such that $O = PQP^TQ^T$ and then $\lambda(O) = \lambda(P)\lambda(Q)\lambda(P)\lambda(Q) = 1$ for every $O \in \mathcal{O}(\mathcal{M}) \Rightarrow WO = OW$ for every $O \in \mathcal{O}(\mathcal{M})$.

Suppose first that $\mathcal{M}$ is one-dimensional, that $\mathcal{M} = span(\{e_1\})$ and that $\{e_l\}_{l\geq 1}$ is an orthonormal basis for $\mathcal{H}$. Let $O \in \mathcal{O}(\mathcal{M})$. Then $Oe_l = e_l$ for every $l \geq 2$ and either $Oe_1 = e_1$ or $Oe_1 = -e_1$. If $Oe_1 = e_1$ then $O = I \Rightarrow WO = OW$ and we are done. So suppose that O is such that $Oe_1 = -e_1$ and $Oe_l = e_l$ for every $l \geq 2$ and that $WO = -OW$. Note that in this case $O^T = O$. Since $\langle We_1, e_1\rangle = -\langle We_1, Oe_1\rangle = -\langle OWe_1, e_1\rangle = \langle WOe_1, e_1\rangle = -\langle We_1, e_1\rangle$ we have that $\langle We_1, e_1\rangle = 0$. Since for every $i, j \geq 2$ we have that $\langle We_i, e_j\rangle = \langle We_i, Oe_j\rangle = \langle OWe_i, e_j\rangle = -\langle WOe_i, e_j\rangle = -\langle We_i, e_j\rangle \Rightarrow \langle We_i, e_j\rangle = 0$ for every $i, j \geq 2$. Thus $We_2 = \sum_{l\geq 1}\langle We_2, e_l\rangle e_l = \langle We_2, e_1\rangle e_1 \Rightarrow W^TWe_2 = \sum_{l\geq 1}\langle W^TWe_2, e_l\rangle e_l = \sum_{l\geq 1}\langle We_2, We_l\rangle e_l = \sum_{l\geq 1}\langle\langle We_2, e_1\rangle e_1, We_l\rangle e_l = \langle We_2, e_1\rangle \left(\sum_{l\geq 1}\langle e_1, We_l\rangle e_l\right) = \langle We_2, e_1\rangle \left(\sum_{l\geq 1}\langle We_l, e_1\rangle e_l\right) \Rightarrow \langle W^TWe_2, e_2\rangle = \langle We_2, e_1\rangle^2$ and $\langle W^TWe_2, e_3\rangle = \langle We_2, e_1\rangle\langle We_3, e_1\rangle$. Similar computation shows that $\langle W^TWe_3, e_3\rangle = \langle We_3, e_1\rangle^2$. But then, since $W^TW = I$ we must have that $\langle We_2, e_1\rangle^2 = 1$, $\langle We_3, e_1\rangle^2 = 1$ and $\langle We_2, e_1\rangle\langle We_3, e_1\rangle = 0$, which is a contradiction.

Suppose now that $\mathcal{M}$ is n-dimensional and that $O \in \mathcal{O}(\mathcal{M})$. Using a result from [6], §81, page 162, we have that there exists $\{e_l\}_{1\leq l\leq n}$ an orthonormal basis for $\mathcal{M}$ such that the matrix representation of O is

$$O = \begin{pmatrix} 1 & & & & & & & & & & & \\ & \ddots & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & -1 & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & -1 & & & & & & \\ & & & & & & \cos\theta_1 & -\sin\theta_1 & & & & \\ & & & & & & \sin\theta_1 & \cos\theta_1 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & \cos\theta_n & -\sin\theta_n \\ & & & & & & & & & \sin\theta_n & \cos\theta_n \end{pmatrix}$$

(here, all the other entries are 0). Since the determinant of every rotation is 1 we must have that $\det(O) = \pm 1$. If $\det(O) = 1$ then $O \in SO(\mathcal{M}) \Rightarrow$ by Lemma 6.33 that there exists $P, Q \in \mathcal{O}(\mathcal{M})$ such that $O = PQP^TQ^T \Rightarrow \lambda(O) = \lambda(P)\lambda(Q)\lambda(P^T)\lambda(Q^T) = \lambda(P)^2\lambda(Q)^2 = 1$. If $\det(O) = -1$ then we must have an odd number of -1's on the diagonal of O. Without loss of generality we may assume that $Oe_1 = -e_1$. If we let $V \in \mathcal{O}(\mathcal{M})$ to be such that $Ve_1 = -e_1$ and $Ve_l = e_l$ for $2 \leq l \leq n$, then $O = VW$, where $W \in SO(\mathcal{M})$. But then $\lambda(W) = 1$ and, by the previous paragraph, $\lambda(V) = 1$ and hence $\lambda(O) = \lambda(V)\lambda(W) = 1$. □

THEOREM 7.5. ★ *Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$, G a Polish topological group and $\phi : G \to \mathcal{PO}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ is closed in G, where $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ is the natural quotient mapping.*

Proof. We will prove that $\pi(\mathcal{O}(\mathcal{M})) = \{\hat{W} \in \mathcal{PO}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W}$ for all $\hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))\}$. This will imply that $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M}))) = \{\phi^{-1}(\hat{W}) \mid \phi^{-1}(\hat{W})\phi^{-1}(\hat{V}) = \phi^{-1}(\hat{V})\phi^{-1}(\hat{W})\ \forall\ \phi^{-1}(\hat{V}) \in \phi^{-1}(\pi(\mathcal{O}(\mathcal{M}^{\perp})))\}$ and then, according with the Proposition 3.26 we will have that $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ is closed in G. Note that if $\mathcal{S} \subset \mathcal{O}(\mathcal{H})$ and $\hat{O} \in \pi(\mathcal{S})$ then there exists $O \in \mathcal{S}$ such that $\pi(O) = \hat{O}$.

Let $\hat{U} \in \pi(\mathcal{O}(\mathcal{M}))$ and $\hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))$. Let $U \in \mathcal{O}(\mathcal{M})$ be such that $\pi(O) = \hat{O}$ and $V \in \mathcal{O}(\mathcal{M}^{\perp})$ be such that $\pi(V) = \hat{V}$. According with Theorem 6.29 we have that $UV = VU \Rightarrow \pi(U)\pi(V) = \pi(V)\pi(U) \Rightarrow \hat{U}\hat{V} = \hat{V}\hat{U} \Rightarrow \pi(\mathcal{O}(\mathcal{M}))\pi(\mathcal{O}(\mathcal{M}^{\perp})) = \pi(\mathcal{O}(\mathcal{M}^{\perp}))\pi(\mathcal{O}(\mathcal{M})) \Rightarrow \pi(\mathcal{O}(\mathcal{M})) \subset \{\hat{W} \in \mathcal{PO}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W} \text{ for all } \hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))\}$.

Let $\hat{W} \in \mathcal{PO}(\mathcal{H})$ be such that $\hat{W}\hat{V} = \hat{V}\hat{W}$ for all $\hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))$. Let $W \in \mathcal{O}(\mathcal{H})$ be such that $\pi(W) = \hat{W}$ and, for every $\hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))$, let $V \in \mathcal{O}(\mathcal{M}^{\perp})$ be such that $\pi(V) = \hat{V}$. Then $\pi(W)\pi(V) = \pi(V)\pi(W) \Rightarrow \pi(WV) = \pi(VW) \Rightarrow WVW^TV^T \in Z(\mathcal{O}(\mathcal{H})) \Rightarrow WV = VW$ by Theorem 7.4. Using Theorem 6.29 we have that $W \in Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}(\mathcal{M}) \Rightarrow$ there exists $U \in \mathcal{O}(\mathcal{M})$ such that $W = \pm U \Rightarrow \pi(W) = \pi(U) \Rightarrow \hat{W} \in \pi(\mathcal{O}(\mathcal{M})) \Rightarrow \{\hat{W} \in \mathcal{PO}(\mathcal{H}) \mid \hat{W}\hat{V} = \hat{V}\hat{W} \text{ for all } \hat{V} \in \pi(\mathcal{O}(\mathcal{M}^{\perp}))\} \subset \pi(\mathcal{O}(\mathcal{M}))$. □

PROPOSITION 7.6. *If $\mathcal{M} \subset \mathcal{H}$ is a finite dimensional subspace, then*

$$\pi(\mathcal{O}(\mathcal{M})) = \pi(Z(\mathcal{O}(\mathcal{M})))\pi(SO(\mathcal{M}))$$

Proof. Since $Z(\mathcal{O}(\mathcal{M})), SO(\mathcal{M}) \subset \mathcal{O}(\mathcal{M})$ and $\mathcal{O}(\mathcal{M})$ is a subgroup we have that $Z(\mathcal{O}(\mathcal{M}))SO(\mathcal{M}) \subset \mathcal{O}(\mathcal{M}) \Rightarrow \pi(Z(\mathcal{O}(\mathcal{M})))\pi(SO(\mathcal{M})) \subset \pi(\mathcal{O}(\mathcal{M}))$.

Let $\hat{U} \in \pi(\mathcal{O}(\mathcal{M}))$. Then there exists $U \in \mathcal{O}(\mathcal{M})$ such that $\pi(U) = \hat{U}$ and by Proposition 6.35 we have that there exist $V \in Z(\mathcal{O}(\mathcal{M}))$ and $W \in SO(\mathcal{M})$ such that $U = VW \Rightarrow \pi(U) = \pi(VW) = \pi(V)\pi(W) \subset \pi(Z(\mathcal{O}(\mathcal{M})))\pi(SO(\mathcal{M})) \Rightarrow \pi(\mathcal{M}) \subset \pi(Z(\mathcal{O}(\mathcal{M})))\pi(SO(\mathcal{M}))$. □

PROPOSITION 7.7. ★ *Let G be a Polish topological space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{PO}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(SO(\mathcal{M})))$ is an analytic subset of G.*

Proof. Since $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ is closed in G by Theorem 7.5, $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M}))) \times \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ is closed in $G \times G$. Let $[\cdot, \cdot] : \phi^{-1}(\pi(\mathcal{O}(\mathcal{M}))) \times \phi^{-1}(\pi(\mathcal{O}(\mathcal{M}))) \to G$ be defined as $[a, b] = aba^{-1}b^{-1}$. Since the group operations are continuous, $[\cdot, \cdot]$ is continuous. If $a, b \in \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ then $\phi(a), \phi(b) \in \pi(\mathcal{O}(\mathcal{M})) \Rightarrow$ there exist $U, V \in \mathcal{O}(\mathcal{M})$ such that $\phi(a) = \pi(U)$, $\phi(b) = \pi(V)$ and $(\phi(a))^{-1} = (\pi(U))^{-1} = \pi(U^T)$ and similarly $(\phi(b))^{-1} = \pi(V^T)$. Since $\phi([a, b]) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)(\phi(a))^{-1}(\phi(b))^{-1} = \pi(U)\pi(V)\pi(U^T)\pi(V^T) = \pi(UVU^TV^T) \in \pi(\mathcal{O}(\mathcal{M}))$

and since $\det(UVU^TV^T) = \det(U)^2\det(V)^2 = 1$, we have that $\phi([a,b]) \in \pi(SO(\mathcal{M})) \Rightarrow [a,b] \in \phi^{-1}(\pi(SO(\mathcal{M})))$. This proves that $[\cdot,\cdot]$ takes its values in $\phi^{-1}(\pi(SO(\mathcal{M})))$.

Let $y \in \phi^{-1}(\pi(SO(\mathcal{M})))$. Then $\phi(y) \in \pi(SO(\mathcal{M})) \Rightarrow$ there exists $W \in SO(\mathcal{M})$ such that $\phi(y) = \pi(W)$. By Lemma 6.33 we have that there exist $U, V \in \mathcal{O}(\mathcal{M}))$ such that $W = UVU^TV^T$. Let $a = \phi^{-1}(\pi(U)) \in \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ and $b = \phi^{-1}(\pi(V)) \in \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$. Then $y = \phi^{-1}(\pi(W)) = \phi^{-1}(\pi(UVU^TV^T)) = \phi^{-1}(\pi(U))$
$\phi^{-1}(\pi(V))(\phi^{-1}(\pi(U)))^{-1}(\phi^{-1}(\pi(V)))^{-1} = aba^{-1}b^{-1} = [a,b] \Rightarrow [\cdot,\cdot]$ is onto $\phi^{-1}(\pi(SO(\mathcal{M})))$. Since $[\cdot,\cdot]$ is continuous, it follows that $\phi^{-1}(\pi(SO(\mathcal{M})))$ is the continuous image of $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))\times \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$, a closed subset of a Polish space, and therefore $\phi^{-1}(\pi(SO(\mathcal{M})))$ is an analytic subset of G. $\square$

Lemma 7.8. ★ *If $\mathcal{M} \subset \mathcal{H}$ is a finite dimensional subspace, then $\pi(Z(\mathcal{O}(\mathcal{M}))) = Z(\pi(\mathcal{O}(\mathcal{M})))$.*

Proof. Let $\hat{U} \in \pi(Z(\mathcal{O}(\mathcal{M})))$. Then there exists $U \in Z(\mathcal{O}(\mathcal{M}))$ such that $\pi(U) = \hat{U}$. Let $\hat{V} \in \pi(\mathcal{O}(\mathcal{M}))$ and $V \in \mathcal{O}(\mathcal{M})$ be such that $\pi(V) = \hat{V}$. Then, since U and V commute, we have that $\hat{U}\hat{V} = \pi(U)\pi(V) = \pi(UV) = \pi(VU) = \pi(V)\pi(U) = \hat{V}\hat{U} \Rightarrow \hat{U} \in Z(\pi(\mathcal{O}(\mathcal{M}))) \Rightarrow \pi(Z(\mathcal{O}(\mathcal{M}))) \subset Z(\pi(\mathcal{O}(\mathcal{M})))$.

Let $\hat{U} \in Z(\pi(\mathcal{O}(\mathcal{M})))$ and let $U \in \mathcal{O}(\mathcal{H})$ be such that $\pi(U) = \hat{U}$. We will show that $U \in Z(\mathcal{O}(\mathcal{M}))$. This will imply that $\hat{U} \in \pi(Z(\mathcal{O}(\mathcal{M})))$ and therefore that $Z(\pi(\mathcal{O}(\mathcal{M}))) \subset \pi(Z(\mathcal{O}(\mathcal{M})))$. Let $V \in \mathcal{O}(\mathcal{M})$. Then $\pi(V) \in \pi(\mathcal{O}(\mathcal{M}))$ and hence $\hat{U}\pi(V) = \pi(V)\hat{U} \Rightarrow \pi(U)\pi(V) = \pi(V)\pi(U) \Rightarrow \pi(UVU^TV^T) = Id \in \mathcal{PO}(\mathcal{H}) \Rightarrow UVU^TV^T \in Z(\mathcal{O}(\mathcal{H})) \Rightarrow$ from Theorem 7.4 that $UV = VU \Rightarrow U \in Z(\mathcal{O}(\mathcal{M}))$. $\square$

Corollary 7.9. ★ *Let G be a Polish topological space, $\mathcal{M} \subset \mathcal{H}$ a finite dimensional closed subspace and $\phi : G \to \mathcal{PO}(\mathcal{H})$ an algebraic isomorphism. Then $\phi^{-1}(\pi(SO(\mathcal{M})))$ is closed in G.*

Proof. From Corollary 7.5 we have that $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ is closed in G and hence Polish. From Proposition 7.6 we have that $\phi^{-1}(\pi(Z(\mathcal{O}(\mathcal{M}))))\phi^{-1}(\pi(SO(\mathcal{M}))) = \phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$. By Lemma 7.8 we have that $\pi(Z(\mathcal{O}(\mathcal{M}))) = Z(\pi(\mathcal{O}(\mathcal{M})))$ and, since ϕ is an isomorphism, it follows that $\phi^{-1}(\pi(Z(\mathcal{O}(\mathcal{M}))))$ is the center of $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ and therefore $\phi^{-1}(\pi(Z(\mathcal{O}(\mathcal{M}))))$

is closed in $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$. $\phi^{-1}(\pi(SO(\mathcal{M})))$ is an analytic subgroup of G by Proposition 7.7, and hence analytic subgroup of $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$. Let $C = \pi(Z(\mathcal{O}(\mathcal{M}))) \cap \pi(SO(\mathcal{M}))$ and let $\hat{U} \in C$. Then there exist $U \in Z(\mathcal{O}(\mathcal{M}))$ and $V \in SO(\mathcal{M})$ such that $\pi(U) = \hat{U} = \pi(V) \Rightarrow \pi(UV^T) = Id \in \mathcal{PO}(\mathcal{H}) \Rightarrow UV^T \in Z(\mathcal{O}(\mathcal{H})) \Rightarrow UV^T = \pm I \Rightarrow U = \pm V$. Since $U|_{\mathcal{M}^\perp} = I$ and $V|_{\mathcal{M}^\perp} = I$ we have that $U = V \Rightarrow C = \{\pi(U) \mid U \in Z(\mathcal{O}(\mathcal{M})) \cap SO(\mathcal{M})\} = \{\pi(U) \mid U|_{\mathcal{M}} = \pm I,\ U|_{\mathcal{M}^\perp} = I\} \Rightarrow C$ is finite. Since ϕ is an isomorphism we have that $\phi^{-1}(C)$ is finite and hence closed in $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$. It follows from Corollary 3.39 that $\phi^{-1}(\pi(SO(\mathcal{M})))$ is closed in $\phi^{-1}(\pi(\mathcal{O}(\mathcal{M})))$ and hence closed in G. □

Proposition 7.10. ★ *Let G be a Polish topological group, let $\mathcal{H}$ be a separable real Hilbert space and let $e \in \mathcal{H}$ be such that $\|e\| = 1$. Let $\mathcal{S} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|e - Oe\| < \epsilon\} \subset \mathcal{O}(\mathcal{H})$ and let $\phi : G \to \mathcal{PO}(\mathcal{H})$ be an algebraic isomorphism. Then $\phi^{-1}(\pi(\mathcal{S}))$ is analytic in G.*

Proof. Note first that the quotient mapping $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ is open and continuous. Let $\mathcal{M} \subset \mathcal{H}$ be a three dimensional subspace as in Lemma 6.38 so that $\mathcal{S} = \mathcal{O}(\{e\}^\perp) \cdot [SO(\mathcal{M}) \cap \mathcal{S}] \cdot \mathcal{O}(\{e\}^\perp)$. Then $\pi(\mathcal{S}) = \pi(\mathcal{O}(\{e\}^\perp))\pi[SO(\mathcal{M}) \cap \mathcal{S}]\pi(\mathcal{O}(\{e\}^\perp))$. Since $SO(\mathcal{M})$ is a connected compact metric group with a totally disconnected center (Chapter I, Section 14, [19]), then $\pi(SO(\mathcal{M}))$ is a connected compact metric group. A proof similar to the proof of Proposition 7.8 shows that $Z(\pi(SO(\mathcal{M}))) = \pi(Z(SO(\mathcal{M})))$ and hence the center of $\pi(SO(\mathcal{M}))$ is finite. Using the result from [14] we have that $\phi|_{\phi^{-1}(\pi(SO(\mathcal{M})))} : \phi^{-1}(\pi(SO(\mathcal{M}))) \to \pi(SO(\mathcal{M}))$ is a homeomorphism. $SO(\mathcal{M}) \cap \mathcal{S}$ is a relatively open subset of $SO(\mathcal{M})$ and hence Borel $\Rightarrow \pi[SO(\mathcal{M}) \cap \mathcal{S}]$ is analytic in $\pi(SO(\mathcal{M})) \Rightarrow \phi^{-1}(\pi[SO(\mathcal{M}) \cap \mathcal{S}])$ is analytic in $\phi^{-1}(\pi(SO(\mathcal{M})))$. Since $\phi^{-1}(\pi(\mathcal{O}(\{e\}^\perp)))$ is closed in G by Theorem 7.5 and therefore analytic, it follows from Lemma 3.53 that $\phi^{-1}(\pi(\mathcal{S})) = \phi^{-1}(\pi(\mathcal{O}(\{e\}^\perp))\pi[SO(\mathcal{M}) \cap \mathcal{S}]\pi(\mathcal{O}(\{e\}^\perp))) = \phi^{-1}(\pi(\mathcal{O}(\{e\}^\perp)))\phi^{-1}(\pi[SO(\mathcal{M}) \cap \mathcal{S}])\phi^{-1}(\pi(\mathcal{O}(\{e\}^\perp)))$ is analytic. □

Proposition 7.11. ★ *Let $\{e_m\}_{m\geq 1}$ be an orthonormal basis for the separable infinite dimensional Hilbert space $\mathcal{H}$. For every $m, n \geq 1$ let $\mathcal{O}_{m,n} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|e_m - Oe_m\| < \frac{1}{n}\}$.*

Let $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ be the natural quotient mapping. Then

$$\bigcap_{m,n\geq 1} \pi^{-1}(\pi(\mathcal{O}_{m,n})) = \{W \in \mathcal{O}(\mathcal{H}) \mid We_m = \pm e_m \textit{ for every } m \geq 1\}$$

Proof. Note first that $\pi^{-1}(\pi(\mathcal{O}_{m,n})) = Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}_{m,n}$ for every $m, n \geq 1$. Let $W \in \mathcal{O}(\mathcal{H})$ be such that $We_m = \pm e_m$ for every $m \geq 1$. Then $\|e_1 - We_1\| = 0 < \frac{1}{n}$ for every $n \geq 1$ or $\|e_1 + We_1\| = 0 < \frac{1}{n}$ for every $n \geq 1 \Rightarrow W \in \mathcal{O}_{1,n}$ for every $n \geq 1$ or $-W \in \mathcal{O}_{1,n}$ for every $n \geq 1 \Rightarrow W \in Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}_{1,n}$ for every $n \geq 1$. Similarly we have that $W \in Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}_{m,n}$ for every $m, n \geq 1 \Rightarrow W \in \cap_{m,n\geq 1} Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}_{m,n} = \cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{O}_{m,n}))$.

Let $W \in \cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{O}_{m,n})) = \cap_{m,n\geq 1} Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{O}_{m,n}$. Then for every $m, n \geq 1$ there exists $W_{m,n} \in \mathcal{O}_{m,n}$ such that $W = \pm W_{m,n} \Rightarrow W_{m,n} = \pm W$. If we fix m, since $\|e_m - W_{m,n}e_m\| < \frac{1}{n}$ for every $m, n \geq 1$, we have that $\|e_m + We_m\| < \frac{1}{n}$ or $\|e_m - We_m\| < \frac{1}{n}$. If both $\|e_m + We_m\| < \frac{1}{n}$ and $\|e_m - We_m\| < \frac{1}{n}$, then $2 = 2\|e_m\| = \|2e_m\| \leq \|e_m - We_m\| + \|e_m + We_m\| < \frac{2}{n} \to 0$ as $n \to \infty$, a contradiction. Thus, either $\|e_m + We_m\| < \frac{1}{n}$ or $\|e_m - We_m\| < \frac{1}{n} \Rightarrow We_m = \pm e_m$. $\square$

Corollary 7.12. ★ *Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space and $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ be the natural quotient mapping. Then there exists $\{\mathcal{S}_l\}_{l\geq 1} \subset \mathcal{O}(\mathcal{H})$ a sequence of subbasic open neighborhoods of I such that $\cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) = Z(\mathcal{O}(\mathcal{H}))$.*

Proof. Let $\{e_m\}_{m\geq 1}$ be an orthonormal basis for $\mathcal{H}$. Let $f_1 = \frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}$. Then $\|f_1\|^2 = \frac{6}{\pi^2}\sum_{m\geq 1}\frac{1}{m^2} = 1$ and expand $\{f_1\}$ to an orthonormal basis $\{f_m\}_{m\geq 1}$. Let $\mathcal{U}_{m,n} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|e_m - Oe_m\| < \frac{1}{n}\}$ and let $\mathcal{V}_{m,n} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|f_m - Of_m\| < \frac{1}{n}\}$. Let $\{\mathcal{S}_l\}_{l\geq 1} = \{\mathcal{U}_{m,n}, \mathcal{V}_{m,n} \mid m, n \geq 1\}$. According with the Proposition 3.11 $\{\mathcal{S}_l\}_{l\geq 1}$ is a sequence of subbasic open neighborhoods of I in $\mathcal{O}(\mathcal{H})$.

Let $W \in \cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) = [\cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{U}_{m,n}))] \cap [\cap_{m,n\geq 1}\pi^{-1}(\pi(\mathcal{V}_{m,n}))]$. Then, according with the Proposition 7.11 we have that $We_m = \pm e_m$ and $Wf_m = \pm f_m$ for every $m \geq 1$. Since $Wf_1 = W\left(\frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}\right) = \frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{We_m}{m}$ and also $Wf_1 = \pm f_1 = \pm\left(\frac{\sqrt{6}}{\pi}\sum_{m\geq 1}\frac{e_m}{m}\right) \Rightarrow$ either $We_m = e_m$ for every $m \geq 1$ or $We_m = -e_m$ for every $m \geq 1 \Rightarrow W = \pm I \Rightarrow W \in Z(\mathcal{O}(\mathcal{H})) \Rightarrow \cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) \subset Z(\mathcal{O}(\mathcal{H}))$.

If $W \in Z(\mathcal{O}(\mathcal{H}))$ then $W = \pm I$ and since $I \in \mathcal{U}_{m,n}$ and $I \in \mathcal{V}_{m,n}$ for every $m, n \geq 1 \Rightarrow W \in Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{U}_{m,n} = \pi^{-1}(\pi(\mathcal{U}_{m,n}))$ and $W \in Z(\mathcal{O}(\mathcal{H})) \cdot \mathcal{V}_{m,n} = \pi^{-1}(\pi(\mathcal{V}_{m,n}))$ for every $m, n \geq 1 \Rightarrow W \in \cap_{l \geq 1} \pi^{-1}(\pi(\mathcal{S}_l))$. $\square$

Corollary 7.13. ★ *Let $\mathcal{H}$ be a separable infinite dimensional real Hilbert space, let G be a Polish topological group and $\phi : G \to \mathcal{PO}(\mathcal{H})$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. Let $\pi : \mathcal{O}(\mathcal{H}) \to \mathcal{PO}(\mathcal{H})$ be the natural quotient mapping. Let $\{\mathcal{S}_l\}_{l \geq 1}$ be the sequence defined in Proposition 7.12, $\{\mathcal{S}_l\}_{l \geq 1} = \{\mathcal{U}_{m,n}, \mathcal{V}_{m,n} \mid m, n \geq 1\}$, where $\mathcal{U}_{m,n} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|e_m - Oe_m\| < \frac{1}{n}\}$, $\mathcal{V}_{m,n} = \{O \in \mathcal{O}(\mathcal{H}) \mid \|f_m - Of_m\| < \frac{1}{n}\}$ and $\{e_m\}_{m \geq 1}$, $\{f_m\}_{m \geq 1}$ are two orthonormal bases for $\mathcal{H}$. We will prove that the sequence $\{\pi(\mathcal{S}_l)\}_{l \geq 1}$ of subsets of $\mathcal{PO}(\mathcal{H})$ satisfy the hypothesis of Theorem 4.16 and the conclusion will follow from the same theorem. Since the projection mapping is open we have that $\pi(\mathcal{S}_l)$ is open for every $l \geq 1$. Also note that each $\phi^{-1}(\pi(\mathcal{S}_l))$ is analytic in G by Proposition 7.10 and hence each $\phi^{-1}(\pi(\mathcal{S}_l))$ is a set with the Baire property.

Since $\|e_m - O^T e_m\| = \|O^T(Oe_m - e_m)\| = \|Oe_m - e_m\|$ we have that $O^T \in \mathcal{U}_{m,n}$ whenever $O \in \mathcal{U}_{m,n}$. Let $\hat{O} \in \pi(\mathcal{U}_{m,n})$ and $O \in \mathcal{U}_{m,n}$ be such that $\pi(O) = \hat{O}$. Then $O^T \in \mathcal{U}_{m,n} \Rightarrow \hat{O}^{-1} = (\pi(O))^{-1} = \pi(O^T) \in \pi(\mathcal{U}_{m,n}) \Rightarrow (\pi(\mathcal{U}_{m,n}))^{-1} \subset \pi(\mathcal{U}_{m,n})$. By replacing $\mathcal{U}_{m,n}$ with $\mathcal{U}_{m,n}^{-1}$ we have that $(\pi(\mathcal{U}_{m,n}^{-1}))^{-1} \subset \pi(\mathcal{U}_{m,n}^{-1}) \Rightarrow \pi(\mathcal{U}_{m,n}) \subset (\pi(\mathcal{U}_{m,n}))^{-1} \Rightarrow (\pi(\mathcal{U}_{m,n}))^{-1} = \pi(\mathcal{U}_{m,n})$ for every $m, n \geq 1$. Similarly $(\pi(\mathcal{V}_{m,n}))^{-1} = \pi(\mathcal{V}_{m,n})$ for every $m, n \geq 1 \Rightarrow (\pi(\mathcal{S}_l))^{-1} = \pi(\mathcal{S}_l)$ for every $l \geq 1$.

Let $U, V \in \mathcal{U}_{m,2n}$. Then $\|e_m - Ue_m\| < \frac{1}{2n}$ and $\|e_m - Ve_m\| < \frac{1}{2n}$ and hence $\|e_m - UVe_m\| \leq \|e_m - Ue_m\| + \|Ue_m - UVe_m\| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow UV \in \mathcal{U}_{m,n} \Rightarrow \mathcal{U}_{m,2n}^2 \subset \mathcal{U}_{m,n} \Rightarrow (\pi(\mathcal{U}_{m,2n}))^2 = \pi(\mathcal{U}_{m,2n}^2) \subset \pi(\mathcal{U}_{m,n})$ and hence for every $m_0, n_0 \geq 1$ there exists $m_1 = m_0$ and $n_1 = 2n_0$ such that $(\pi(\mathcal{U}_{m_1,n_1}))^2 \subset \pi(\mathcal{U}_{m_0,n_0})$. Similarly for every $m_0, n_0 \geq 1$ there exists $m_1 = m_0$ and $n_1 = 2n_0$ such that $(\pi(\mathcal{V}_{m_1,n_1}))^2 \subset \pi(\mathcal{V}_{m_0,n_0})$ and therefore for every $l_0 \geq 1$ there exists l_1 such that $(\pi(\mathcal{S}_{l_1})^2 \subset \pi(\mathcal{S}_{l_0})$.

From Corollary 7.12 we have that $\cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l)) = Z(\mathcal{O}(\mathcal{H}))$. From Lemma 4.17 we have that $\pi(\cap_{l\geq 1}\pi^{-1}(\pi(\mathcal{S}_l))) = \cap_{l\geq 1}\pi(\pi^{-1}(\pi(\mathcal{S}_l))) = \cap_{l\geq 1}\pi(\mathcal{S}_l) \Rightarrow \cap_{l\geq 1}\pi(\mathcal{S}_l) = \pi(Z(\mathcal{O}(\mathcal{H}))) = Z(\mathcal{O}(\mathcal{H}))$ and hence $\cap_{l\geq 1}\pi(\mathcal{S}_l)$ is the identity in $\mathcal{PO}(\mathcal{H})$. $\square$

CHAPTER 8

THE ISOMETRY GROUP

Definition 8.1. Let $\mathcal{H}$ be a complex Hilbert space. For every $(U,a) \in \mathcal{U}(\mathcal{H}) \times \mathcal{H}$ and every $x \in \mathcal{H}$ we define $(U,a)(x) = Ux + a$. If $\mathcal{H}$ is a real Hilbert space and if $(O,a) \in \mathcal{O}(\mathcal{H}) \times \mathcal{H}$ we define $(O,a)(x) = Ox + a$ for every $x \in \mathcal{H}$.

Proposition 8.2. *If $\mathcal{H}$ is a complex Hilbert space, the semidirect product $\mathcal{U}(\mathcal{H}) \times_\alpha \mathcal{H}$ together with the operation $(U,a)(V,b) = (UV, U(b)+a)$ is a group. We call this group the complex isometry group and denote it by $\mathbb{I}_\mathbb{C}$. If $\mathcal{H}$ is real, the real isometry group $\mathcal{O}(\mathcal{H}) \times_\alpha \mathcal{H}$ is defined in a similar way and is denoted $\mathbb{I}_\mathbb{R}$.*

Proof. Let $(U,a), (V,b), (W,c) \in \mathbb{I}_\mathbb{C}$. Then
$(U,a)(V,b) = (UV, U(b)+a) \in \mathbb{I}_\mathbb{C}$;
$[(U,a)(V,b)](W,c) = (UV, U(b)+a)(W,c) = (UVW, UV(c)+U(b)+a) = (UVW, U[V(c)+b]+a) = (U,a)(VW, V(c)+b) = (U,a)[(V,b)(W,c)]$;
$(U,a)(I,0) = (U,a) = (I,0)(U,a)$ and
$(U,a)(U^*, U^*(-a)) = (UU^*, UU^*(-a)+a) = (I,0) = (U^*U, U^*(a)+U^*(-a)) = (U^*, U^*(-a))(U,a)$.

The proof for the real isometry group is similar. □

Lemma 8.3. *Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{U}(\mathcal{H})$ is given the weak operator topology and if $\mathcal{H}$ is given the norm topology, then the mapping $\mathcal{U}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$, $(U,a) \mapsto U(a)$ is continuous. Same result holds if $\mathcal{H}$ is a real Hilbert space and $\mathcal{U}(\mathcal{H})$ is replaced with $\mathcal{O}(\mathcal{H})$.*

Proof. Let $(U_j)_{j\in J} \subset \mathcal{U}(\mathcal{H})$ be such that $U_j \xrightarrow{wo} U$ and $(a_k)_{k\in K} \subset \mathcal{H}$ be such that $a_k \xrightarrow{\|\cdot\|} a$. Since the weak operator topology on $\mathcal{U}(\mathcal{H})$ and the strong operator topology are equivalent we have that $\|(U_j - U)(x)\| \to 0$ for every $x \in \mathcal{H}$. Then $\|U_j(a_k) - U(a)\| \le \|U_j(a_k) - U_j(a)\| + \|U_j(a) - U(a)\| = \|U_j(a_k - a)\| + \|(U_j - U)(a)\| = \|a_k - a\| + \|(U_j - U)(a)\| \to 0 \Rightarrow U_j(a_k) \xrightarrow{\|\cdot\|} U(a) \Rightarrow$ the mapping $(U,a) \mapsto U(a)$ is continuous.

If $\mathcal{H}$ is a real Hilbert space, the continuity of the mapping $\mathcal{O}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ is proved similarly. □

Proposition 8.4. ★ *Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{U}(\mathcal{H})$ is given the weak operator topology and if $\mathcal{H}$ is given the norm topology, then $\mathbb{I}_{\mathbb{C}}$ with the product topology is a Polish topological group. $\mathcal{U}(\mathcal{H}) \times \{0\}$ is the centralizer of $\{(-I,0)\}$ in $\mathbb{I}_{\mathbb{C}}$ and $\{I\} \times \mathcal{H}$ is maximal abelian and therefore both $\mathcal{U}(\mathcal{H}) \times \{0\}$ and $\{(-I,0)\}$ are closed subgroups of $\mathbb{I}_{\mathbb{C}}$. If $\mathcal{H}$ is a real Hilbert space then $\mathbb{I}_{\mathbb{R}}$ is a Polish topological group. $\mathcal{O}(\mathcal{H}) \times \{0\}$ is the centralizer of $\{(-I,0)\}$ in $\mathbb{I}_{\mathbb{R}}$ and $\{I\} \times \mathcal{H}$ is maximal abelian and therefore both $\mathcal{O}(\mathcal{H}) \times \{0\}$ and $\{(-I,0)\}$ are closed subgroups of $\mathbb{I}_{\mathbb{R}}$.*

Proof. Since both $\mathcal{U}(\mathcal{H})$ and $\mathcal{H}$ are Polish spaces, $\mathbb{I}_{\mathbb{C}}$ is a Polish space. To show that $\mathbb{I}_{\mathbb{C}}$ is a topological group, let $(U,a),(V,b) \in \mathbb{I}_{\mathbb{C}}$. Since the mappings $U \mapsto U^*$, $(U,V) \mapsto U^*V$ and $a \mapsto -a$ are continuous, and since the mapping $(U,a) \mapsto U(a)$ is continuous by Lemma 8.3 we have that $((U,a),(V,b)) \mapsto (U^*V, U^*(b) + U^*(-a)) = (U^*, U^*(-a))(V,b) = (U,a)^{-1}(V,b)$ is continuous.

To show directly that $\mathcal{U}(\mathcal{H}) \times \{0\}$ and $\{I\} \times \mathcal{H}$ are closed subgroups of $\mathbb{I}_{\mathbb{C}}$, let $(U_j)_{j\in J} \subset \mathcal{U}(\mathcal{H})$ be such that $U_j \to U$. Then $(U_j,0) \to (U,0) \Rightarrow \mathcal{U}(\mathcal{H}) \times \{0\}$ is closed in $\mathbb{I}_{\mathbb{C}}$. If $(a_j)_{j\in J} \subset \mathcal{H}$ is such that $a_j \to a$ then $(I,a_j) \to (I,a) \Rightarrow \{I\} \times \mathcal{H}$ is closed in $\mathbb{I}_{\mathbb{C}}$.

If $U \in \mathcal{U}(\mathcal{H})$ then $(U,0)(-I,0) = (-U,0) = (-I,0)(U,0)$. Conversely, if $(U,a)(-I,0) = (-I,0)(U,a)$ then, since $(U,a)(-I,0) = (-U,a)$ and $(-I,0)(U,a) = (-U,-a)$, we have that $a = -a \Rightarrow a = 0 \Rightarrow (U,a) \in \mathcal{U}(\mathcal{H}) \times \{0\} \Rightarrow \mathcal{U}(\mathcal{H}) \times \{0\}$ is the centralizer of $\{(-I,0)\}$,

To show that $\{I\} \times \mathcal{H}$ is maximal abelian, let $(U,a) \in \mathbb{I}_{\mathbb{C}}$ be such that $(U,a)(I,b) = (I,b)(U,a)$ for every $b \in \mathcal{H}$. Then $(U,a)(I,b) = (U, U(b)+a)$ and $(I,b)(U,a) = (U, a+b) \Rightarrow U(b) = b$ for every $b \in \mathcal{H} \Rightarrow U = I \Rightarrow (U,a) \in \{I\} \times \mathcal{H} \Rightarrow \{I\} \times \mathcal{H}$ is maximal abelian.

The proof for $\mathbb{I}_{\mathbb{R}}$ is similar. □

Remark 8.5. Since the mapping $\mathcal{U}(\mathcal{H}) \to \mathbb{I}_{\mathbb{C}}$, $U \mapsto (U,0)$ is an isomorphism of topological groups, we may identify $\mathcal{U}(\mathcal{H})$ with $\mathcal{U}(\mathcal{H}) \times \{0\} \subset \mathbb{I}_{\mathbb{C}}$ and we can consider $\mathcal{U}(\mathcal{H})$ as being a closed subgroup of $\mathbb{I}_{\mathbb{C}}$. Similarly, if $\mathcal{H}$ is a real Hilbert space then $\mathcal{O}(\mathcal{H})$ is a closed subgroup of $\mathbb{I}_{\mathbb{R}}$. Since the mapping $\mathcal{H} \to \mathbb{I}_{\mathbb{C}}$, $x \mapsto (I, x)$ is an isomorphism of topological groups, we may identify $\mathcal{H}$ with $\{I\} \times \mathcal{H} \subset \mathbb{I}_{\mathbb{C}}$ and we can consider $\mathcal{H}$ as being a closed subgroup of $\mathbb{I}_{\mathbb{C}}$. Similarly, if $\mathcal{H}$ is a real Hilbert space then $\mathcal{H}$ is a closed subgroup of $\mathbb{I}_{\mathbb{R}}$.

LEMMA 8.6. ★ *Let G be a Polish topological group and let $\phi : G \to \mathbb{I}_{\mathbb{C}}$ be an algebraic isomorphism. Then $\phi^{-1}(\mathcal{U}(\mathcal{H}))$ and $\phi^{-1}(\mathcal{H})$ are closed in G. If $\mathcal{H}$ is a real Hilbert space and if $\phi : G \to \mathbb{I}_{\mathbb{R}}$ is an algebraic isomorphism, then $\phi^{-1}(\mathcal{O}(\mathcal{H}))$ and $\phi^{-1}(\mathcal{H})$ are closed in G.*

Proof. Since by Proposition 8.4, $\mathcal{U}(\mathcal{H}) = \{(U, a) \in \mathbb{I}_{\mathbb{C}} \mid (U, a)(-I, 0) = (-I, 0)(U, a)\}$, we have that $\phi^{-1}(\mathcal{U}(\mathcal{H})) = \{\phi^{-1}(U) \mid \phi^{-1}(U)\phi^{-1}((-I, 0)) = \phi^{-1}((-I, 0))\phi^{-1}(U)\}$ and the conclusion will follow from Proposition 3.26.

Since $\{I\} \times \mathcal{H}$ is maximal abelian by Proposition 8.4 we have that $\phi^{-1}(\mathcal{H})$ is maximal abelian and therefore closed in G.

The proof in real case is similar. □

LEMMA 8.7. ★ *Let G be a Polish topological group, let $\phi : G \to \mathbb{I}_{\mathbb{C}}$ be an algebraic isomorphism and let $0 \neq a \in \mathcal{H}$. Then $\phi^{-1}(\{(I, b) \in \mathbb{I}_{\mathbb{C}} \mid \|b\| = \|a\|\}$ is an analytic subset of G. Same result holds if $\mathcal{H}$ is a real Hilbert space and if $\mathbb{I}_{\mathbb{C}}$ is replaced with $\mathbb{I}_{\mathbb{R}}$.*

Proof. Let $T_a = \{(I, b) \in \mathbb{I}_{\mathbb{C}} \mid \|b\| = \|a\|\}$. We will prove that $T_a = \{(U, 0)(I, a)(U, 0)^{-1} \mid U \in \mathcal{U}(\mathcal{H})\}$. This will imply that $\phi^{-1}(T_a) = \{\phi^{-1}((U, 0))\phi^{-1}((I, a))\phi^{-1}((U, 0)^{-1}) \mid U \in \mathcal{U}(\mathcal{H})\} = \{R\phi^{-1}((I, a))R^{-1}) \mid R \in \phi^{-1}(\mathcal{U}(\mathcal{H}))\}$ and then, since the multiplication in G is continuous and since $\phi^{-1}(\mathcal{U}(\mathcal{H}))$ is closed by Lemma 8.6, the conclusion follows from Lemma 3.53.

Let $U \in \mathcal{U}(\mathcal{H})$. Then $(U, 0)(I, a)(U, 0)^{-1} = (U, 0)(I, a)(U^*, 0) = (U, 0)(U^*, a) = (UU^*, U(a)) = (I, U(a)) \in T_a$ since $\|U(a)\| = \|a\| \Rightarrow \{(U, 0)(I, a)(U, 0)^{-1} \mid U \in \mathcal{U}(\mathcal{H})\} \subset T_a$. If $(I, b) \in T_a$ then there exists $U \in \mathcal{U}(\mathcal{H})$ such that $U(a) = b \Rightarrow (I, b) = (I, U(a)) = (U, 0)(I, a)(U, 0)^{-1} \Rightarrow T_a \subset \{(U, 0)(I, a)(U, 0)^{-1} \mid U \in \mathcal{U}(\mathcal{H})\}$.

The proof in real case is similar. □

LEMMA 8.8. ★ *Let $\mathcal{H}$ be a complex Hilbert space and let $a \in \mathcal{H}$. If $b, c \in \mathcal{H}$ then $\{(I, b - c) \mid \|b\| = \|c\| = \|a\|\} = \{(I, d) \mid \|d\| \leq 2\|a\|\}$.*

Proof. Let $b, c \in \mathcal{H}$ be such that $\|b\| = \|c\| = \|a\|$ and let $d = b - c$. Then $\|d\| = \|b - c\| \leq \|b\| + \|c\| = 2\|a\| \Rightarrow \{(I, b - c) \mid \|b\| = \|c\| = \|a\|\} \subset \{(I, d) \mid \|d\| \leq 2\|a\|\}$.

Let $d \in \mathcal{H}$ be such that $\|d\| \leq 2\|a\|$. Let $\mu : \mathbb{R} \to \mathcal{H}$ be defined as $\mu(\theta) = e^{i\theta}a$. Then μ is continuous, $\mu(0) = a$ and $\mu(\pi) = -a$. The mapping $\theta \mapsto \|a - \mu(\theta)\|$ is also continuous, $\|a - \mu(0)\| = 0$ and $\|a - \mu(\pi)\| = 2\|a\|$. By the intermediate value theorem we have that there exists θ_0 such that $\|a - \mu(\theta_0)\| = \|d\| \Rightarrow$ there exists $U \in \mathcal{U}(\mathcal{H})$ such that $U(a - \mu(\theta_0)) = d$. Let $b = U(a)$ and $c = U(e^{i\theta_0}a)$. Then $\|b\| = \|c\| = \|a\|$ and $d = b - c \Rightarrow \{(I, d) \mid \|d\| \leq 2\|a\|\} \subset \{(I, b - c) \mid \|b\| = \|c\| = \|a\|\}$. $\square$

LEMMA 8.9. ★ *Let G be a Polish topological group, let $\phi : G \to \mathbb{I}_{\mathbb{C}}$ be an algebraic isomorphism and let $0 \neq a \in \mathcal{H}$. Then $\phi^{-1}(\{(I, d) \in \mathbb{I}_{\mathbb{C}} \mid \|d\| \leq 2\|a\|\}$ is an analytic subset of G.*

Proof. Let $T_a = \{(I, b) \in \mathbb{I}_{\mathbb{C}} \mid \|b\| = \|a\|\}$ be the set defined in Lemma 8.7. Then $T_a \cdot T_a^{-1} = \{(I, b)(I, c)^{-1} \mid \|b\| = \|c\| = \|a\|\} = \{(I, b)(I, -c) \mid \|b\| = \|c\| = \|a\|\} = \{(I, b - c) \mid \|b\| = \|c\| = \|a\|\} = \{(I, d) \mid \|d\| \leq 2\|a\|\}$ by Lemma 8.8 $\Rightarrow \phi^{-1}(\{(I, d) \mid d \in \mathcal{U}\}) = \phi^{-1}(T_a)\phi^{-1}(T_a)^{-1}$. Since $\phi^{-1}(T_a)$ is an analytic subset of G by Lemma 8.7 we have that $\phi^{-1}(\{(I, d) \mid d \in \mathcal{U}\})$ is analytic. $\square$

THEOREM 8.10. ★ *Let G be a Polish topological group and let $\phi : G \to \mathbb{I}_{\mathbb{C}}$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. The case when $\dim(\mathcal{H}) = 1$ was done by Kallman in [15].

Since $\phi^{-1}(\mathcal{H})$ is closed in G by Lemma 8.6, it is Polish and hence $\phi|_{\phi^{-1}(\mathcal{H})} : \phi^{-1}(\mathcal{H}) \to \mathcal{H}$ is an isomorphism between two Polish topological groups. Let $\delta > 0$ and let $\mathcal{U} = \{x \in \mathcal{H} \mid \|x\| < \delta\}$ be an open neighborhood of 0 in $\mathcal{H}$. Then $\mathcal{U} = \cup_{n\geq 1}\{x \in \mathcal{H} \mid \|x\| \leq \frac{\delta(n-1)}{n}\} \Rightarrow \phi^{-1}(\mathcal{U}) = \cup_{n\geq 1}\phi^{-1}(\{x \in \mathcal{H} \mid \|x\| \leq \frac{\delta(n-1)}{n}\})$ and each of the sets $\phi^{-1}(\{x \in \mathcal{H} \mid \|x\| \leq \frac{\delta(n-1)}{n}\})$ is analytic by Lemma 8.9 $\Rightarrow \phi^{-1}(\mathcal{U})$ is analytic and hence it has the Baire property. It follows from Lemma 3.57 that $\phi|_{\phi^{-1}(\mathcal{H})}$ is a topological isomorphism.

Since by Lemma 8.6 $\phi^{-1}(\mathcal{U}(\mathcal{H}))$ is closed in G and therefore Polish, $\phi|_{\phi^{-1}(\mathcal{U}(\mathcal{H}))} : \phi^{-1}(\mathcal{U}(\mathcal{H})) \to \mathcal{U}(\mathcal{H})$ is an algebraic isomorphism between two Polish topological groups. Let $\{h_n\}_{n\geq 1}$ be a dense subset of $\mathcal{H}$. Let $\Psi : \phi^{-1}(\mathcal{U}(\mathcal{H})) \to \prod_{n\geq 1} \phi^{-1}(\mathcal{H})$ be defined as $\Psi(\phi^{-1}((U,0))) = \prod_{n\geq 1} \phi^{-1}((U,0))\phi^{-1}((I,h_n))\phi^{-1}((U,0))^{-1} = \prod_{n\geq 1} \phi^{-1}((I,U(h_n)))$. If $U_1, U_2 \in \mathcal{U}(\mathcal{H})$ are such that $\prod_{n\geq 1} \phi^{-1}((I,U_1(h_n))) = \prod_{n\geq 1} \phi^{-1}((I,U_2(h_n)))$ then $U_1(h_n) = U_2(h_n)$ for every $n \geq 1 \Rightarrow U_1 = U_2$ since $\{h_n\}_{n\geq 1}$ is dense $\Rightarrow \Psi$ is one-to-one. Since the group operations are continuous in G, Ψ is continuous onto its range. If $\Phi : \prod_{n\geq 1} \phi^{-1}(\mathcal{H}) \to \prod_{n\geq 1} \mathcal{H}$ is the mapping $\Phi(\prod_{n\geq 1} \phi^{-1}((I,x_n))) = \prod_{n\geq 1}(I,x_n)$, then Φ is continuous since each $\phi|_{\phi^{-1}(\mathcal{H})}$ is a topological isomorphism. For each $n \geq 1$ let $F_n : \mathcal{U}(\mathcal{H}) \to \mathcal{H}$ be defined as $F_n((U,0)) = (U,0)(I,h_n)(U,0)^{-1} = (I,U(h_n))$. Since the group operations are continuous, each F_n is continuous. Let $F : \mathcal{U}(\mathcal{H}) \to \prod_{n\geq 1} \mathcal{H}$ be defined as $F((U,0)) = \prod_{n\geq 1} F_n((U,0)) = \prod_{n\geq 1}(I,U(h_n))$. Note that the range of F is the same as the range of $\Phi \circ \Psi$. If $U_1, U_2 \in \mathcal{U}(\mathcal{H})$ are such that $F((U_1,0)) = F((U_2,0))$ then $\prod_{n\geq 1}(I,U_1(h_n)) = \prod_{n\geq 1}(I,U_2(h_n)) \Rightarrow U_1(h_n) = U_2(h_n) \Rightarrow U_1 = U_2 \Rightarrow F$ is one-to-one. F is continuous onto its range since the group multiplication is continuous. By Lusin-Souslin Theorem (page 89, [18]) we have that $F^{-1} : F(\mathcal{U}(\mathcal{H})) \to \mathcal{U}(\mathcal{H})$ defined on the range of $\Phi \circ \Psi$ is Borel measurable. Thus the mapping $F^{-1} \circ \Phi \circ \Psi : \phi^{-1}(\mathcal{U}(\mathcal{H})) \to \mathcal{U}(\mathcal{H})$ is Borel measurable. Since $(F^{-1} \circ \Phi \circ \Psi)(\phi^{-1}((U,0))) = (U,0) = \phi(\phi^{-1}((U,0)))$ we have that $\phi|_{\phi^{-1}(\mathcal{U}(\mathcal{H}))} = F^{-1} \circ \Phi \circ \Psi \Rightarrow \phi|_{\phi^{-1}(\mathcal{U}(\mathcal{H})}$ is Borel measurable. It follows from Lemma 3.57 that $\phi|_{\phi^{-1}(\mathcal{U}(\mathcal{H}))}$ is a topological isomorphism. Note that if $\mathcal{H}$ is infinite dimensional this is true by Theorem 3.58. However, the proof from this paragraph works independent of the dimension of $\mathcal{H}$.

Let $f : \phi^{-1}(\mathcal{H}) \times \phi^{-1}(\mathcal{U}(\mathcal{H})) \to G$ be defined as $f(\phi^{-1}((I,a)), \phi^{-1}((U,0))) = \phi^{-1}((I,a)(U,0)) = \phi^{-1}((U,a))$. f is obviously one-to-one. Since the group operations are continuous, f is continuous onto its range. It follows from Lusin-Souslin Theorem (page 89, [18]) that $f^{-1} : G \to \phi^{-1}(\mathcal{H}) \times \phi^{-1}(\mathcal{U}(\mathcal{H}))$ is Borel measurable. The mapping $g : \phi^{-1}(\mathcal{H}) \times \phi^{-1}(\mathcal{U}(\mathcal{H})) \to \mathcal{H} \times \mathcal{U}(\mathcal{H})$ defined as $g(\phi^{-1}(I,a), \phi^{-1}(U,0)) = \phi(\phi^{-1}((I,a)))\phi(\phi^{-1}((U,0))) = (U,a)$ is a topological isomorphism since the restrictions of ϕ to $\phi^{-1}(\mathcal{H})$ and $\phi^{-1}(\mathcal{U}(\mathcal{H}))$ are topological isomorphisms. The mapping $h : \mathcal{H} \times \mathcal{U}(\mathcal{H}) \to \mathbb{I}_{\mathbb{C}}$ defined as $h((a,U)) = (U,a)$ is obviously a topological isomorphism. Thus $h \circ g \circ f^{-1}$ is Borel measurable. Since $(h \circ g \circ f^{-1})(\phi^{-1}((U,a))) = h(g(\phi^{-1}((I,a)), \phi^{-1}((U,0)))) = h((a,U)) = (U,a) = \phi(\phi^{-1}((U,a)))$ we have that $\phi = h \circ g \circ f^{-1} \Rightarrow \phi$ is a Borel isomorphism and therefore a topological isomorphism by Lemma 3.57. $\square$

LEMMA 8.11. ★ *Let $\mathcal{H}$ be a real Hilbert space with* $\dim(\mathcal{H}) \geq 2$ *and let $a \in \mathcal{H}$. If $b, c \in \mathcal{H}$ then $\{(I, b-c) \mid \|b\| = \|c\| = \|a\|\} = \{(I,d) \mid \|d\| \leq 2\|a\|\}$.*

Proof. Let $b, c \in \mathcal{H}$ be such that $\|b\| = \|c\| = \|a\|$ and let $d = b - c$. Then $\|d\| = \|b - c\| \leq \|b\| + \|c\| = 2\|a\| \Rightarrow \{(I, b-c) \mid \|b\| = \|c\| = \|a\|\} \subset \{(I,d) \mid \|d\| \leq 2\|a\|\}$.

Let $d \in \mathcal{H}$ be such that $\|d\| \leq 2\|a\|$. Since $\dim(\mathcal{H}) \geq 2$ there exists at least one $e \in \mathcal{H}$ such that $\|e\| = \|a\|$ and $\langle a, e\rangle = 0$. Let $\psi : \mathbb{R} \to \mathcal{H}$ be defined as $\psi(\theta) = (\cos\theta)a + (\sin\theta)e$. Then ψ is continuous, $\|\psi(\theta)\| = \|a\|$ for every $\theta \in \mathbb{R}$, $\psi(0) = a$ and $\psi(\pi) = -a$. The mapping $\theta \mapsto \|a - \psi(\theta)\|$ is also continuous, $\|a - \psi(0)\| = 0$ and $\|a - \psi(\pi)\| = 2\|a\|$. By the intermediate value theorem we have that there exists θ_0 such that $\|a - \psi(\theta_0)\| = \|d\| \Rightarrow$ there exists $O \in \mathcal{O}(\mathcal{H})$ such that $O(a - \psi(\theta_0)) = d$. Let $b = O(a)$ and $c = O(\psi(\theta_0))$. Then $\|b\| = \|c\| = \|a\|$ and $d = b - c \Rightarrow \{(I,d) \mid \|d\| \leq 2\|a\|\} \subset \{(I, b-c) \mid \|b\| = \|c\| = \|a\|\}$. $\square$

LEMMA 8.12. ★ *Let $\mathcal{H}$ be a real Hilbert space with* $\dim(\mathcal{H}) \geq 2$*, let G be a Polish topological group, let $\phi : G \to \mathbb{I}_{\mathbb{R}}$ be an algebraic isomorphism and let $0 \neq a \in \mathcal{H}$. Then $\phi^{-1}(\{(I,d) \in \mathbb{I}_{\mathbb{C}} \mid \|d\| \leq 2\|a\|\}$ is an analytic subset of G.*

Proof. The proof is identical with the proof of Lemma 8.9, with the exception that instead of Lemma 8.8 we use Lemma 8.11. $\square$

THEOREM 8.13. ★ *Let $\mathcal{H}$ be a real Hilbert space with* $\dim(\mathcal{H}) \geq 2$*, let G be a Polish topological group and let $\phi : G \to \mathbb{I}_{\mathbb{R}}$ be an algebraic isomorphism. Then ϕ is a topological isomorphism.*

Proof. The proof is identical with the proof of Theorem 8.10 with a few exceptions. In the second paragraph instead of Lemma 8.9 we use Lemma 8.12. $\square$

Remark 8.14. It follows from [23] that on a real Hilbert space the surjective isometries coincide with $\mathbb{I}_{\mathbb{R}}$.

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