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## DYNAMIC LOADS ON AIRPLANE STRUCTURES DURING LANDING

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------------ REPORT

## DYNAMIC LOADS ON AIRPLANE STRUCTURES DURING LANDING

### By M. A. Biot and R. L. Bisplinghoff

## 1. SUMMARY

The application of transient theory to the rational determination of dynamic loads on airplane structures during initial landing impact is discussed. Simplified procedures are described by which the distribution of the maximum attainable value of the dynamic stresses in the structure may be obtained. Illustrations of the procedure are given by numerical examples for the case of airplane wings. This indicates approximate orders of magnitude to be expected in a typical problem. The validity of the underlying simplifications and assumptions is discussed. A brief outline of the general mathematical theory of transients in undamped elastic system is presented.

## 2. INTRODUCTION

During landing, the airplane structure undergoes transient oscillations which are excited by the initial landing impact. Recent experience has shown that in the case of large aircraft these oscillations may produce critical design conditions for the structure; whereas heretofore design loads for the landing condition have been based upon calculations which assume the structure to be rigid. Since the advent of larger aircraft has resulted in heavier and more flexible structures, these calculations are considerably in error. This together with the fact that flight load factors are reduced in the case of large aircraft makes it necessary that methods be developed for predicting dynamic landing loads. The present theoretical investigation of this problem is of a preliminary nature. It has the twofold purpose first of presenting methods by which the designer may predict the dynamic loads and second of serving as a guide in the experimental

investigation by defining the significant factors involved
and determining the approximate magnitude of the quantities
to be measured.

It appears that the problem may be approached in two
different ways:

In the case of the landplane to consider the airplane
structure and its landing gear as a whole, and to introduce
the actual force displacement characteristics of the landing
into the theory.  In this procedure the dynamic stresses re-
sult from the sudden application of moving constraints im-
posed on the airplane during landing.  Similarly for the
seaplane the elastic structure and the water surrounding the
hull may be considered as interacting bodies.  While this
method is not precluded in the investigation of specific
cases or for research purposes, it involves inherent complex-
ities, such as those resulting from the nonlinear properties
of the landing gear and the variable-mass effects of the
water surrounding the hull of a seaplane, which tends to make
this type of approach less adequate for design purpose.

In the other procedure, in which stresses in the struc-
ture are considered to be caused by a landing impact force
applied directly to the structure, it is assumed that the
time history of the impact force may be investigated inde-
pendently of the elastic properties of the structure.  In
this way the investigation involves two separate phases - a
study of the landing forces and a study of the dynamic be-
havior  of the structure under such forces.  This procedure
involves the assumption that a landing impact force may be
defined in such a way that its time history is for all prac-
tical purposes independent of the elastic properties of the
structure.  Since it is believed that the relative simplicity
of the latter approach overshadows the approximations in-
volved, it has been adopted as the basis for the present work.
This procedure was described previously in reference 1 in
connection with the problem of dynamic stresses in buildings
during an earthquake and the present work is essentially an
adaptation to aircraft structures of the methods described
in this reference.

It is assumed that in first approximation the damping
and the aerodynamic forces may be neglected.  The landing
impact force is applied directly to the elastic structure as
an external force of given time history.  The response of
the structure is represented as a superposition of natural
modes excited by the landing impact.  A first step in the

analysis is therefore to obtain the natural modes of the air-
plane either by calculation or by a shake test.  Calculation
methods have recently been developed by which natural modes
of airplanes may be derived in a relatively simple way.

An important feature of the designer's approach to the
landing loads problem is the fact that he is not so much in-
terested in the actual time history of the structure as he
is in the magnitude of the highest attainable stresses during
the operation of the airplane.  This viewpoint was introduced
in the procedure by using a statistical approach.  The stress
amplitudes of each mode are superposed with their positive or
negative values irrespective of phase and the worst possible
combination is used as a basis for design.  Furthermore, the
stress history in each mode is not actually computed but the
stress amplitude is obtained directly from a graph represent-
ing what is designated as a "dynamic response factor."  This
factor itself results from a statistical analysis of the ef-
fect of forces of various time histories on a single degree
of freedom oscillator, using a sufficient number of such time
histories to represent all possible types of landing condi-
tions.  Values of the dynamic response factors are obtained
by applying typical time history excitations to a torsional
pendulum (described in reference 1) and measuring the maxi-
mum amplitude of its response.  In this way, when the natural
modes of the airplane are known, the landing loads are read-
ily obtained without the necessity of integrating the dif-
ferential equations which govern the behavior of the elastic
structure in the transient condition.

The method has its limitations and is not necessarily
applicable to all types of transient problems.  Some of
these limitations are pointed out in the discussion, but the
extent to which the methods are valid for some particular
class of problems will have to be determined experimentally.

## 3. NOTATION

$M, m$     mass

$k$        spring constant

$Q(t)$     generalized force

$\omega$        circular frequency

T        period

t,τ      time variable

γ        dynamic response factor

q        generalized coordinate

z        deformation displacement of any point on the wing

φ        normal function describing the wing mode shape

α        normal function describing the mode shape of twisting
         about the elastic axis

h        normal function describing the mode shape of bending
         of the elastic axis

I        moment of inertia

S        static mass moment

j        subscript denoting jth mode

k        subscript denoting kth spanwise wing station

ρ        mass per unit volume of prismatic beam

A        cross-sectional area of prismatic beam

E        modulus of elasticity

n        vertical landing load factor

F(t)     landing reaction

N(t)     shock strut axial force - time relation

D(t)     drag force - time relation caused by wheel spin-up

a(t)     observed acceleration time history in a drop or
         landing test

W        gross weight of airplane

$W_L$    gross weight of airplane less landing gear weight

$r_e$    effective rolling radius of wheel

$V_l$    landing speed

$\mu$    coefficient of sliding friction of tire on runway

s    stress

$I_W$    moment of inertia of wheel and tire about axis of
    rotation

$T_{IN}$    period of shock strut axial impulse or seaplane
    vertical impulse

$T_{ID}$    period of drag impulse

$\theta$    angular displacement of wheel

## 4. THE EVALUATION OF TRANSIENT MOTION OF ELASTIC BODIES

### BY THE USE OF GENERALIZED COORDINATES

As pointed out in the introduction, the theory proceeds
on the assumption that the landing impact force is known.
In this way the theoretical problem is reduced to the evalu-
ation of the response of an elastic structure to a force of
given time history. Methods for the determination of this
landing impact force from test data will be discussed later
in section 5.

In order to introduce the fundamental concepts involved
in the present theory, the simplest possible elastic system
will first be considered. This system is illustrated in fig-
ure 1 as a single mass oscillator.
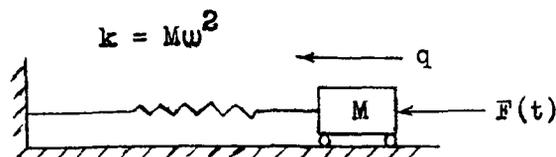


Figure 1

Denoting by  M  the mass and  ω  the natural frequency of
oscillation (ω  is the circular frequency expressed in
radians/sec), the spring constant is  $k = M\omega^2$.  The expres-
sion giving the displacement  q  of this mass under a force
F(t)  of arbitrary time history is well known (references 2,
3, and 4).  It may be written as:

$$q = \frac{1}{M\omega} \int_0^t F(\tau) \sin \omega(t - \tau)d\tau \qquad (1)$$

where  τ  is a variable of integration.  It is usually desig-
nated as Duhamel's integral.  According to this formula the
computation of the displacement  q  at the instant  t  re-
quires the evaluation of a definite integral between the
limits of integration  0  and  t,  and the time history of  q
is obtained by repeating this process for every value of  t.
It is noted that even in the simple case of a single-mass
system the process of computing the transient response is
quite elaborate.  Fortunately, this difficulty may be avoided
in adapting the theory to practical problems of design by
considering only the maximum value of  q  instead of its com-
plete time history.  How this is achieved will be explained
later (sec. 7).

Assume for the present that the complete time history
of the deformation in the airplane structure is desired.
Such a structure differs from the simple system of figure 1
by two features:

(a) It is a free body.

(b) It is an elastic body with many degrees of freedom.

In order to show how the previous discussion may be ex-
tended to include these features, consider a free system of
two masses  M  and  m  connected by a spring of constant  k
as illustrated by figure 2.
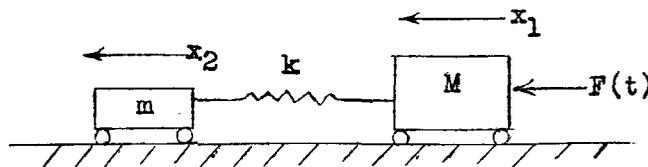


Figure 2

The mass  M  is under the action of a force  F(t)  of arbi-
trary time history.  The straightforward way of describing
the motion of this system is in terms of the displacements
$x_1$  and  $x_2$  of each mass.  However, there is a more general
approach.  By an elementary analysis it is shown in appendix
II that this motion may be described as the superposition of
two configurations.  One in which the two masses move to-
gether as a rigid body (fig. 3a), the other in which the cen-
ter of gravity remains fixed with the masses moving in oppo-
site phase and with amplitudes inversely proportional to the
respective masses (fig. 3b).



(a) Rigid configuration          (b) Motion about c.g.
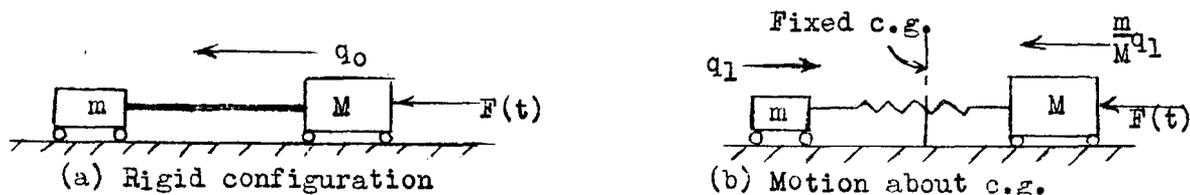
Figure 3

Each of these configurations has only one degree of freedom.
The displacement in the first configuration is measured by
the quantity  $q_0$  and in the other by the quantity  $q_1$.  The
actual displacements of the masses  M  and  m  in terms of
the motion of each configuration are, respectively,

$$x_1 = q_0 + \frac{m}{M} q_1$$

$$\tag{2}$$

$$x_2 = q_0 - q_1$$

These two configurations may be interpreted physically as
representing the natural modes of oscillations of the system.
From this viewpoint the rigid translation  $q_0$  may be thought
of as a mode of zero frequency or "zero mode."  The other
configuration where the center of gravity remains fixed and
the masses move in opposite phase represents a mode of fre-
quency  $\omega_1$.  Since the coordinate  $q_0$  of the zero mode rep-
resents the motion of the center of gravity, it is determined
by the motion of a single mass  $M_0 = M + m$  under the force
F(t)  (fig. 4a).  As shown in appendix II, the motion in the
mode defined by  $q_1$  may be determined from that of an equiv-
alent system which is constituted of a single mass  $M_1$

elastically restrained and under the action of a force $Q_1(t)$ proportional to $F(t)$ (fig. 4b).
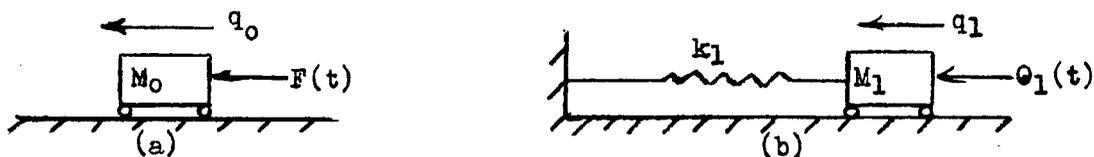


Figure 4

The natural frequency of this equivalent system is the same $\omega_1$ as the natural frequency of the mode which it represents in the actual physical system. Such coordinates as $q_0$ and $q_1$ are called <u>generalized coordinates</u>. The mass $M_1$ of the equivalent system is referred to as the <u>generalized mass</u> of the corresponding mode and $Q_1(t)$ is referred to as the <u>generalized force</u> for this same mode. It is also shown (see appendix II) that the value of this generalized mass is derived quite simply by expressing that the kinetic energy $T$ in the equivalent system is the same as in the corresponding mode

$$T = \frac{1}{2} M_1 \dot{q}_1{}^2 = \frac{1}{2} m \dot{q}_1{}^2 + \frac{1}{2} M \phi^2 \dot{q}_1{}^2 \qquad (3)$$

where $\phi = \dfrac{m}{M}$ is the ratio of the amplitudes of the masses $m$ and $M$ in the $q_1$ mode.

Hence $\qquad\qquad M_1 = m + M \phi^2 \qquad\qquad\qquad (4)$

Similarly the generalized force $Q_1(t)$ is derived by expressing that the work done by $F(t)$ on the actual system is equal to the work done by $Q_1(t)$ on the equivalent system — that is,

$$\phi\, q_1\, F(t) = Q_1(t)\, q_1$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5)$

$$Q_1(t) = \phi\, F(t)$$

The problem of finding the transient motion of two masses connected elastically has thus been reduced to that of two independent single masses for which the motion may be expressed by Duhamel's integral (equation (1)).

Consider now a simplified airplane in which the wing is
represented by two masses elastically connected to a rigid
fuselage as illustrated in figure 5a.
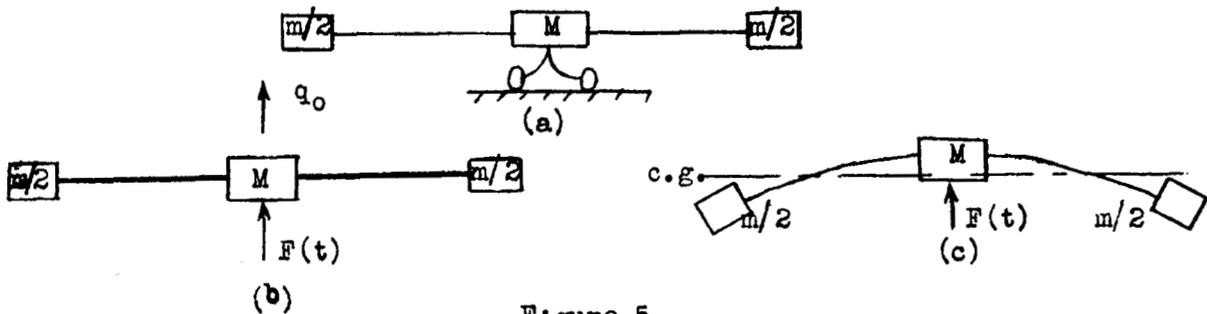


Figure 5

Obviously, for symmetric deformations this simplified
structure is equivalent to the two-mass free system of fig-
ure 2. The motion under the landing impact force may be
described as the superposition of a rigid translation (fig.
5b) which represents the motion of the center of gravity and
a natural mode (fig. 5c) which represents the motion about
the center of gravity. The equivalent single mass systems
are the same as in figure 4.

This procedure may be immediately generalized to a com-
plex airplane structure. Instead of a single deformation
mode as in the case of the simplified airplane discussed
above, there are actually an infinite number. It can be
shown that the deformation of the structure may be repre-
sented by a superposition of these modes. If damping is
neglected as is the case here, these modes are uncoupled.
The behavior of each one under the landing impact force is
independent of the other. The motion of each mode is repre-
sented by that of an equivalent single mass oscillator of
the same natural frequency and excited by a generalized
force. As an example, take the case of a wing in pure bend-
ing. There are an infinite number of bending modes, a few
of which are represented in figure 6. The motion of the
center of gravity represented by the rigid airplane with the
generalized coordinate $q_0$ is referred to in the present
text as the "zero mode."

Zero mode

First mode
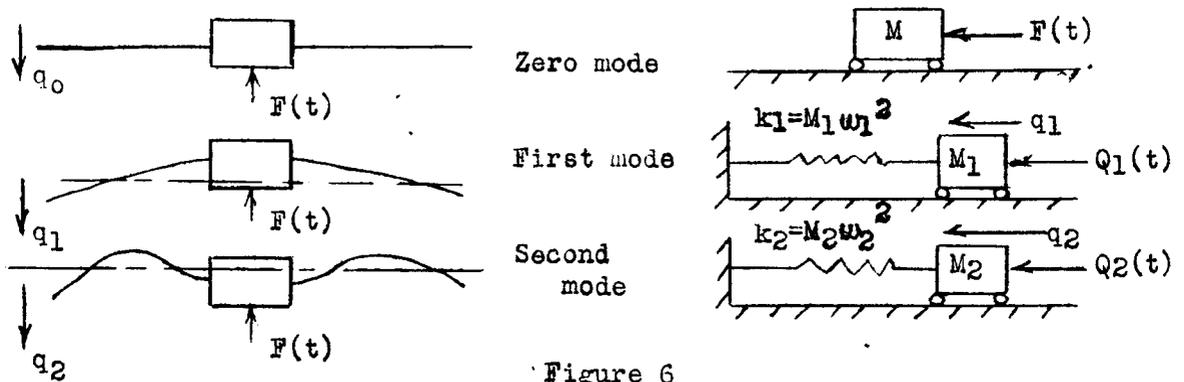
Second
mode

·Figure 6

The amplitudes of the wing tip are usually selected as generalized coordinate and denoted by $q_1$ for the first mode, $q_2$ for the second mode, and so forth. The corresponding generalized masses are determined by the condition that the kinetic energy be the same for the mode and its equivalent system. This yields the expression:

$$M_1 = \sum^k \left[ \phi_k^{(1)} \right]^2 m_k$$

$$M_2 = \sum^k \left[ \phi_k^{(2)} \right]^2 m_k, \quad \text{and so forth} \tag{6}$$

In these expressions, $\phi_k^{(j)}$ represents the amplitude at station $k$ of the mass $m_k$ of that station in the jth mode for a unit deflection of the wing tip. Similarly, the generalized forces determined by the conditions that the work done by $F(t)$ be the same in the particular mode as the work done by $Q(t)$ in the equivalent system are

$$Q_1(t) = F(t) \, \phi_F^{(1)}$$

$$Q_2(t) = F(t) \, \phi_F^{(2)}, \quad \text{and so forth} \tag{7}$$

The subscript $F$ refers to the station at which the landing impulse force is applied.

## 5. THE LANDING IMPACT FORCE  F(t)

In the preceding discussion, the time history  F(t)  of the externally applied landing impact force is assumed to be known.  In the case of a landplane, the forces are applied to the wing through an oleo strut.  In the case of a flying boat, they are transmitted to the wing through a hull or pontoon.

For the landplane, a convenient source of information of axial strut characteristics is the oleo drop test.  In the drop test, a mass–oleo strut system is dropped in a jig, and the acceleration time history of the mass is measured. The mass corresponds to the zero mode mass.

$$F(t) = M \, a(t) \tag{8}$$

An additional source of landing force data is from actual landings with accelerometers installed in the airplane which are capable of recording time history.  This method provides the only present source of seaplane data.  A sketch of such a record is shown in figure 7.

————————— oscillograph record

– – – – – – – assumed a(t) curve
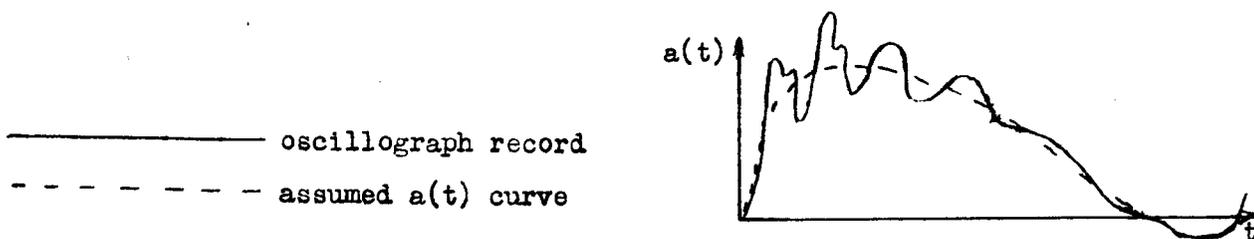


Figure 7

The high-frequency oscillations must be disregarded since they represent oscillations of the structure relative to its center of gravity.  The average dotted line which is shown would be considered as  a(t)  in the computation of the applied landing force  F(t)  in equation (8).

## 6. EVALUATION OF THE STRESSES

Having derived the time history of the deformation of the structure, the time history of the stresses is obtained by a straightforward procedure.  In this discussion, stress

is used as a general term which refers to shear or moment.
It is convenient to consider the total stress as resulting
from the superposition of the stresses due to the deformation
in each mode. In this approach, the zero mode which is a
rigid motion does not contribute any stress. Only the first
mode, second mode, and so forth, have to be considered.
From a practical viewpoint it is also important to note that
the stresses in each mode are conveniently calculated by us-
ing the inertia forces of the natural oscillation rather
than by trying to calculate the strain from the space curva-
ture of the mode shapes. Consider, for example, a pure
bending mode as represented in figure (8).
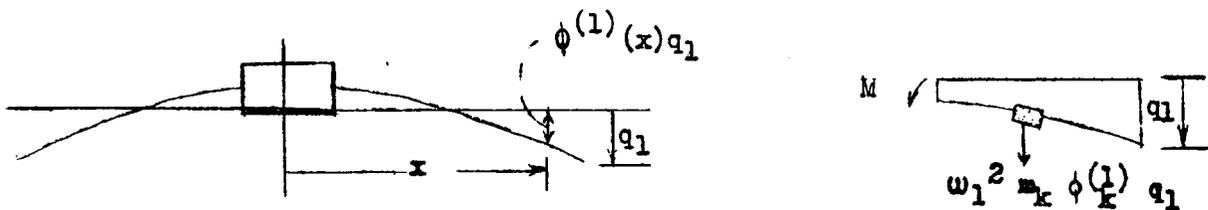


Figure 8

The shape of this mode is represented by a function
$\phi^{(1)}(x)$ such that if $q_1$ is the tip deflection $\phi^{(1)}x q_1$
is the deflection at the location $x$. The bending moment
in this mode is

$$M(x) = E I \frac{d^2 \phi^{(1)}(x)}{d x^2} q_1 \qquad (9)$$

$M$ is proportional to $q_1$ and varies along the span as

$E I \dfrac{d^2 \phi^{(1)}(x)}{d x^2}$. Obviously the process of computing the

second derivative of $\phi^{(1)}(x)$ is elaborate and inaccurate.
It is therefore preferable to derive the bending moment in
each mode directly from d'Alembert's principle by expressing
the equilibrium condition which exists in the natural mode
between the bending moment at station $k$ and the inertia
forces $\omega_1^2 m_k \phi^{(1)}_k q_1$ due to each mass $m_k$ located out-
board from $x$. Similarly the shear in each mode may be ob-
tained from the summation of all inertia forces.

It is seen that the stress $s\binom{j}{k}$ in the jth mode at the kth wing station may be expressed as

$$s\binom{j}{k} = A\binom{j}{k} q_j \qquad (10)$$

each of these stresses being proportional to the coordinate $q_j$ of the corresponding mode. The total stress is obtained by superposition and is

$$s_k = \sum_{1}^{j} A\binom{j}{k} q_j \qquad (11)$$

## 7. THE DYNAMIC RESPONSE FACTOR Y

It has been shown that the response of each mode is defined by the motion of an equivalent simple oscillator acted upon by a generalized force $Q(t)$. The motion of a simple oscillator under the action of an arbitrary force $F(t)$ is given by an evaluation of Duhamel's integral as shown by equation (1). Consider, for example, the motion of a simple oscillator under the influence of the isosceles triangle force-time impulse shown in figure 9.
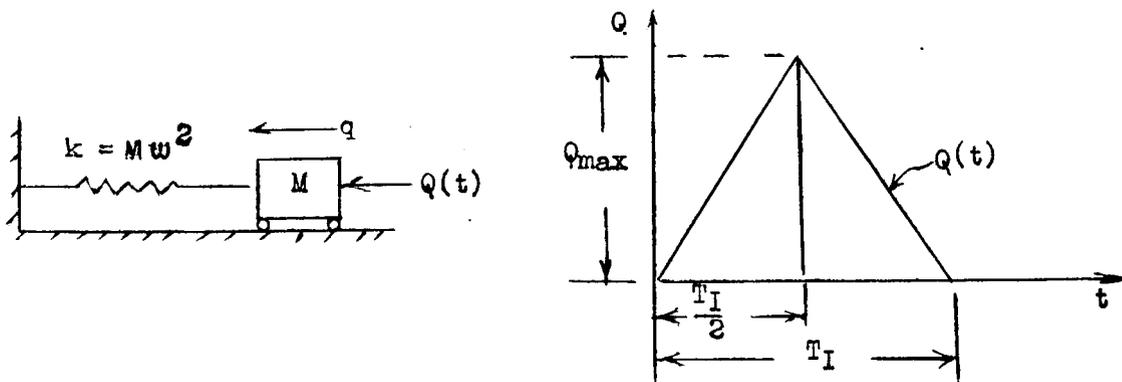


Figure 9

Let $q_s$ = static displacement caused by $Q_{max}$

$\quad$ T $= 2\pi/\omega$ = natural period of the oscillator

$T_I$ = period of the triangular impulse

The complete time histories of the motion of the oscillator have been evaluated for the isosceles triangular impulse for two ratios of $T_I/T$, and are plotted in figure 10. The response of the oscillator is expressed as a ratio of its actual displacement to its static displacement under $Q_{max}$.



Figure 10
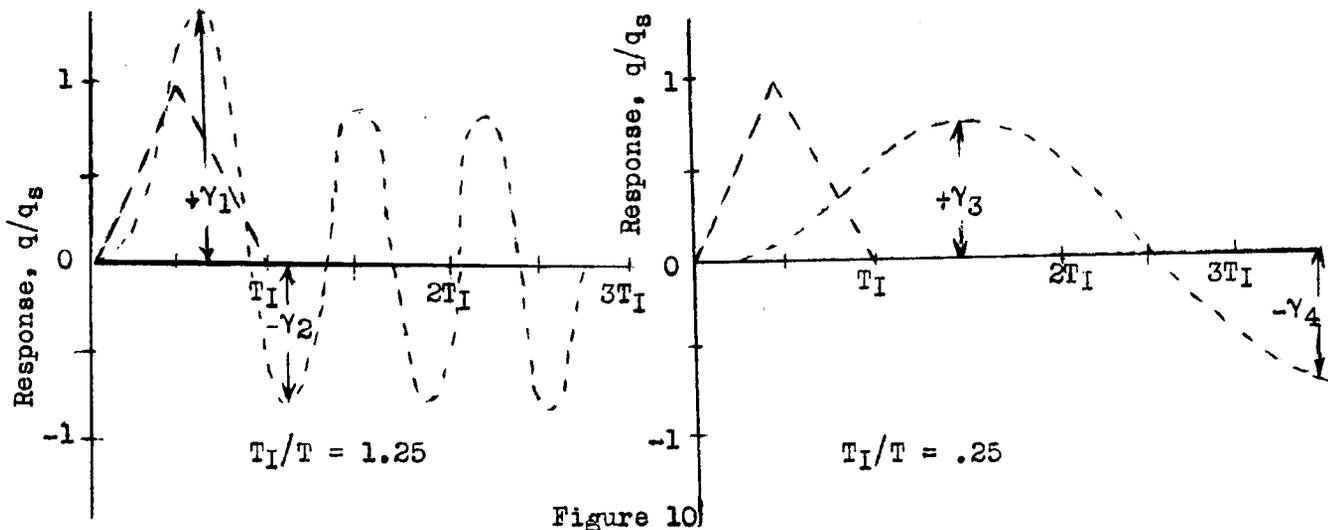
For each period ratio $T_I/T$ there is a maximum value of $q/q_s$ in the positive direction, and a maximum value in the negative direction. These maximum values are designated here as dynamic response factors and denoted by $\gamma$ with the proper sign affixed. There are two values of $\gamma$ associated with each period ratio as shown by figure 10. Consequently a curve may be drawn showing the variation of dynamic response factor with period ratio. Such a curve for the isosceles triangular impulse is shown by figure 11.

Figure 11

$$\frac{T_I}{T_j} \sim \frac{\text{Period of impulse}}{\text{Period of oscillator}}$$
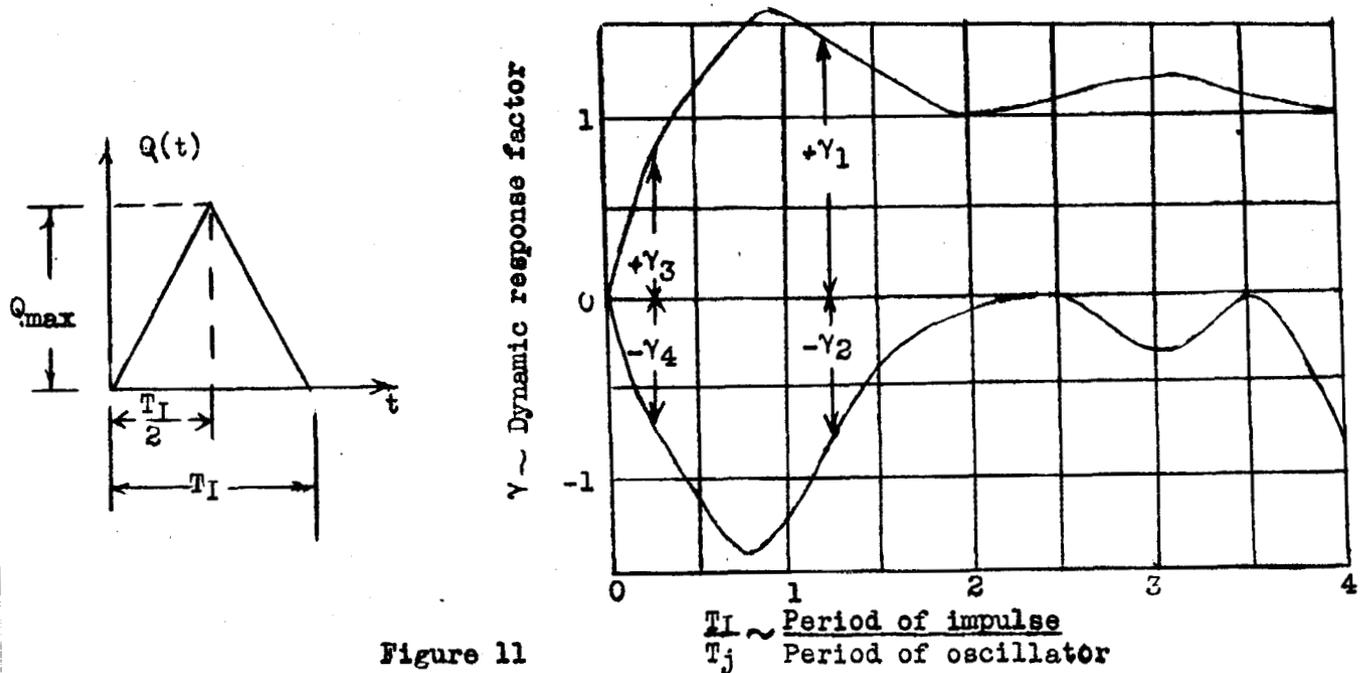
The determination of a dynamic response factor-period ratio curve may be accomplished for any arbitrary $Q(t)$ variation by a numerical or analytical evaluation of Duhamel's integral. However, both processes are quite lengthy, and require plotting of the time history as shown by figure 10. A mechanical analyzer consisting of a torsional pendulum has been developed (reference 1) which may be used to measure the dynamic response factor without recourse to an evaluation of the complete time history of the motion. By such means, a dynamic response factor diagram for any shape of $Q(t)$ curve may be evaluated in a relatively short time.

## 8. STATISTICAL APPROACH TO THE LANDING PROBLEM

The methods outlined above are applicable when the time history of the externally applied landing impact forces are accurately known. Actually, the shape of the landing impact force-time curve is apt to vary considerably from one landing to the next, and with different operating conditions of the airplane. Furthermore, the responses of the various modes are sensitive to small variations in the shape of the $F(t)$ curve. For these reasons, it is not practical to design for

a single mathematical landing, but instead it is desirable
to employ a statistical approach to the problem. In the
previous section it was shown that the extreme positions of
oscillation of a simple oscillator for any one type of ap-
plied force-time curve may be conveniently represented by
means of a dynamic response factor-period ratio curve. It
is possible to consider a large number of shapes of landing
force-time curves varying from soft to harsh landings, and
evaluate a dynamic response factor-period ratio curve for
each of them. These curves may be plotted on the same graph,
and an envelope curve may be drawn which bounds all of them.
This envelope would represent conditions which exceed in
severity every type of landing which was considered. Such a
statistical basis may be used to establish design landing
dynamic response factor envelopes for the landplane and the
seaplane. (For example, see fig. 13.) By means of the dy-
namic response factor envelope, the maximum deflection of
the structure in each mode during the landing may be evalu-
ated quite simply. Considering any single mode, it is as-
sumed that the structure is restrained to deflect in that
mode only, while the maximum value of the generalized force
is applied statically. The deflection may be computed under
these conditions and then multiplied by the dynamic response
factor in order to obtain the maximum dynamic deflection
during the landing.

$$q_{j(max)} = \gamma_j\, q_{j(static)} = \gamma_j\, \frac{Q_{j(max)}}{M_j\, \omega_j^2} = \gamma_j\, \frac{F(t)_{max}\, \phi_F^{(j)}}{M_j\, \omega_j^2} \qquad (12)$$

It is apparent that the phase relations between the modes
are lost when an approach of this type is employed. However,
this is not serious since for design purposes it must be
assumed that sometime during the life of the airplane the
phasing between the modes will be such as to produce the
worst combination of stresses. For this reason, the maximum
deflections are superposed without regard for phase in order
to yield the most critical combination.

## 9. THE LANDPLANE WING

In the case of the landplane, the forces and moments
are applied to the airplane structure through the shock

strut-tire system. These consist of a force applied in an
axial direction along the shock strut caused by the landing
reaction component parallel to the strut, and a moment about
the landing gear attachment point caused by the landing re-
action component perpendicular to the strut. Two dynamic
response factors must be determined. One for the axial strut
force, and one for the moment about the landing gear attach-
ment point.

The shock strut axial force.- A study of force-time
curves for axial strut reactions obtained from drop and
flight test data shows that they differ considerably with
pilot technique and landing attitude. A group of six types
of force-time curves are considered which would represent
various types of strut characteristics. These are shown in
figure 12.
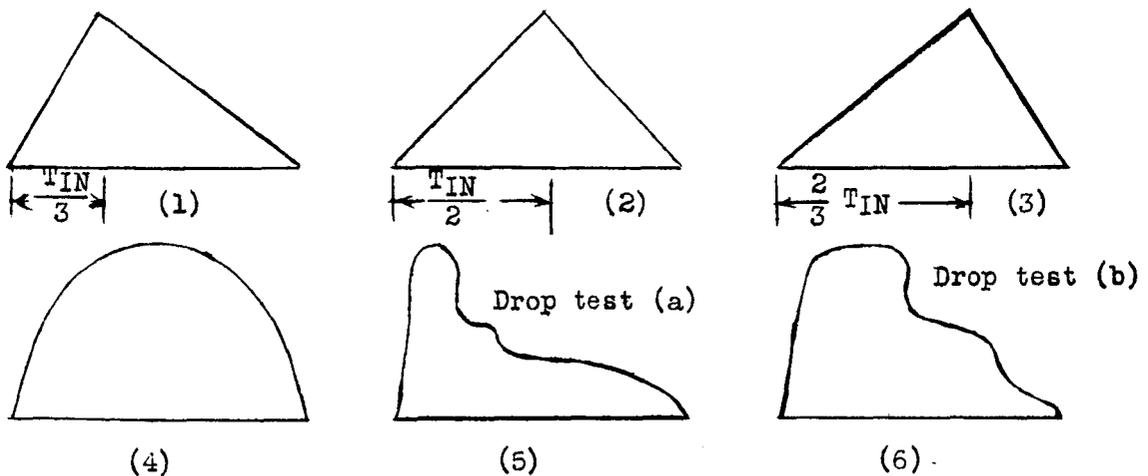


Figure 12

Dynamic response factor curves similar to figure 11 have
been determined for each of these six curves by means of the
mechanical analyzer (reference 1). The points which define
these curves have been plotted on one graph in figure 13, and
a smooth envelope curve has been drawn which bounds all of
them. The condition, which the envelope shown in figure 13
represents, exceeds in severity the condition of the six
types of landings considered. It is possible that after con-
sideration of a large number of shapes of axial shock strut
force-time curves taken from drop and flight test records, an
envelope of this type may well represent conditions which ex-
ceed in severity every probable landing which would be exper-
ienced during the normal operation of a land-type airplane.

In order to apply the landplane dynamic response factor envelope, the impulse period must be known. A plot of limited data available at the Bureau of gross weight against landing gear vertical impulse period during the first impact of landing as determined by flight test is shown on log-log paper by figure 14. This graph has been determined from accelerometer records of various types of landings made by 11 airplanes of various weights. Each point represents an average of several landings. The equation which fits the curve drawn in figure 14 is

$$T_{IN} = 0.25 \left\{ \frac{W}{1000} \right\}^{0.1475} \tag{13}$$

where

$T_{IN}$    axial shock strut impulse period in seconds

$W$    gross weight of airplane in pounds

   The wheel drag force.- The characteristics of the drag force on the wheel are not as well known as those of the axial strut force. The drag force is produced by the spinning up of the wheel when ground contact is made. If it is assumed that the tire is slipping or on the verge of slipping on the runway at all times during the wheel spin-up time, and that the coefficient of sliding friction $\mu$ is constant, the following differential equation may be written.

$$D(t) = \mu N(t) = \frac{I_w \ddot{\theta}}{r_e} \tag{14}$$

Integrating once gives

$$\frac{V_l I_w}{r_e^2 \mu} = \int_0^{T_{ID}} N(t) \, dt \tag{15}$$

where

$V_l$    landing speed, feet per second

$I_w$    moment of inertia of wheel and tire, slug-feet square

$r_e$    effective rolling radius of wheel under impact loading, feet
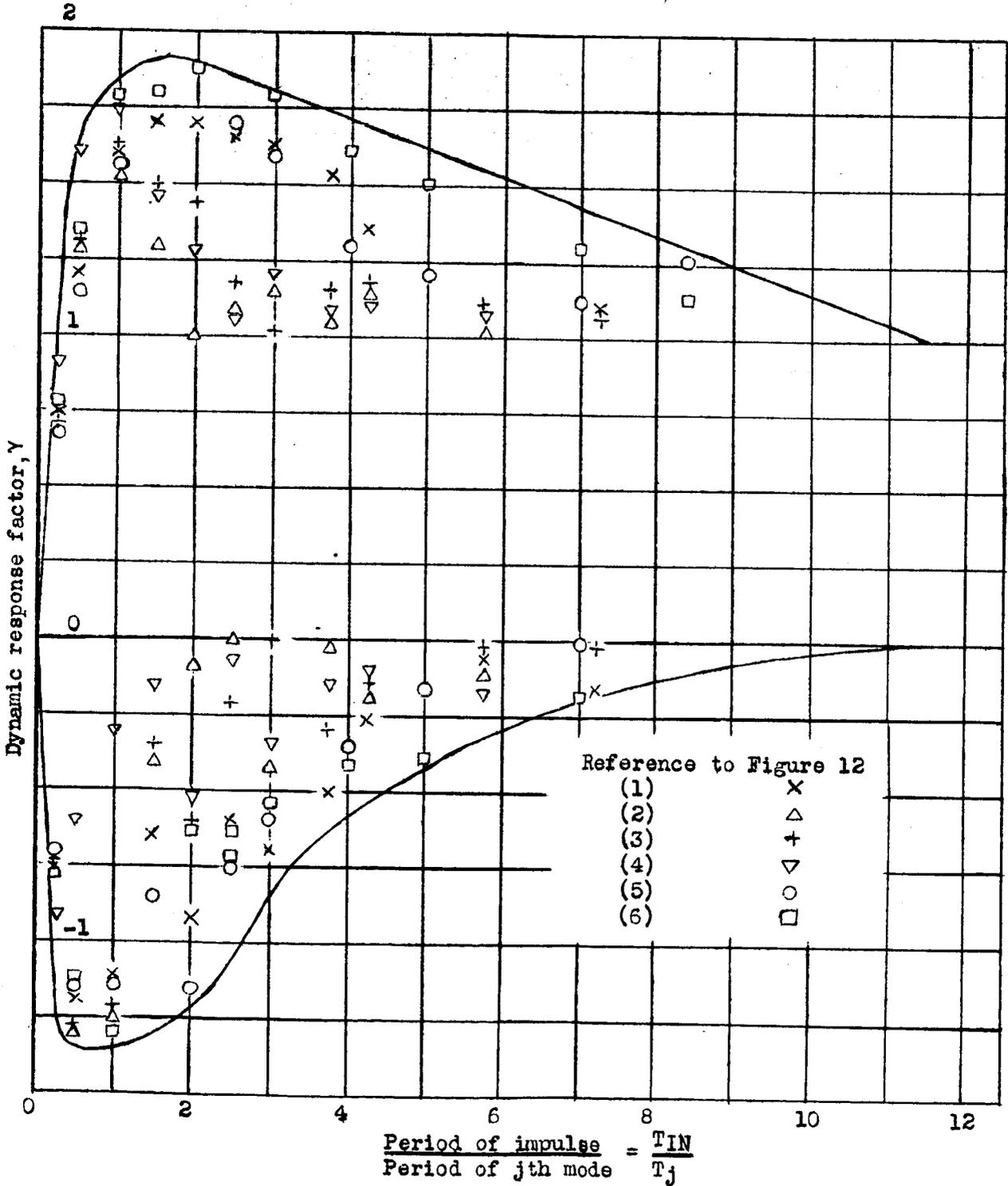
Figure 13.- Dynamic response factor envelope for landplane axial shock
          strut loads.
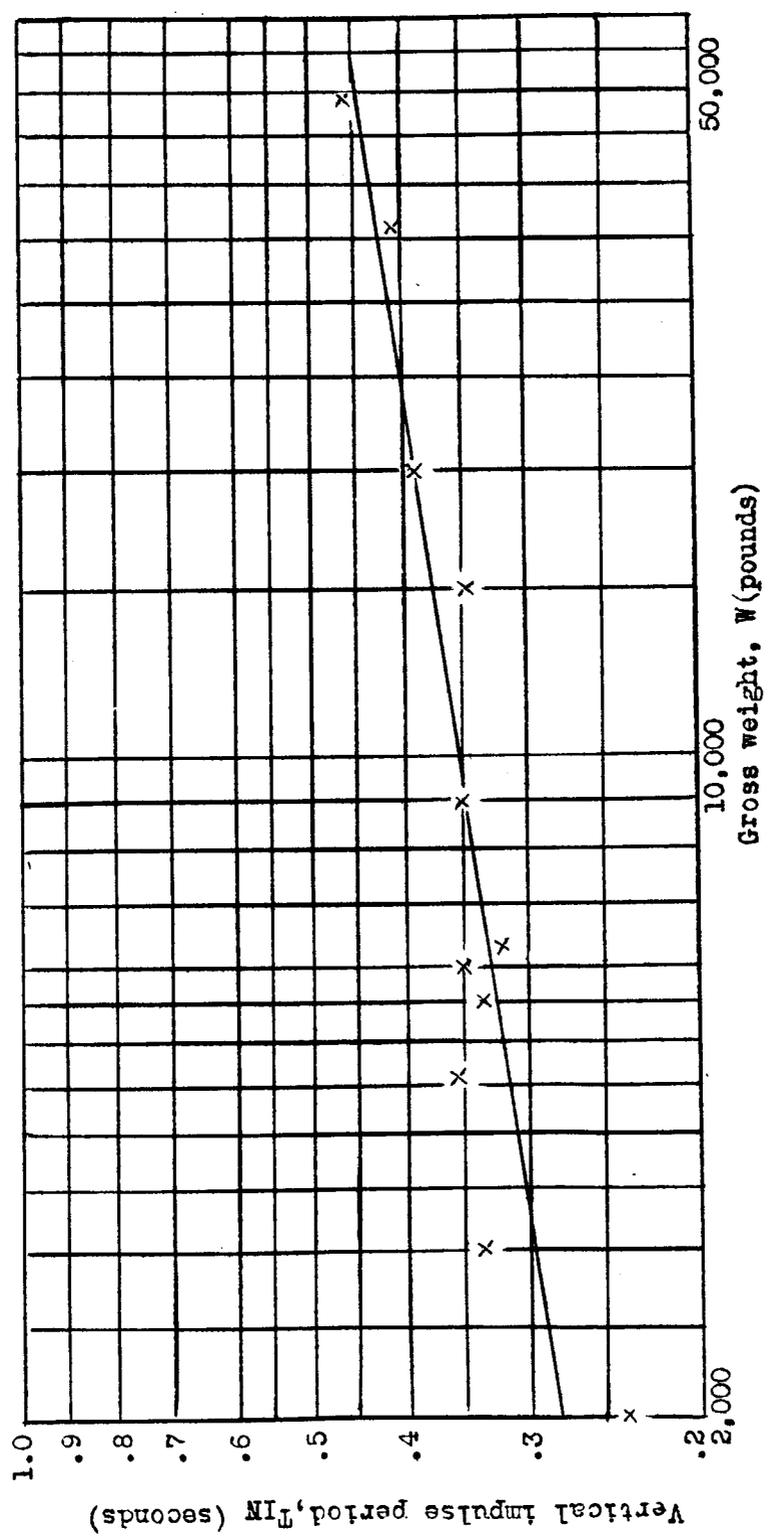
Figure 14.- Vertical impulse period during first impact of landing vs. gross weight.

$T_{ID}$   period of drag force impulse, seconds

If the shock strut axial force is assumed to grow linearly
with time during the wheel spin-up period, then equation
(15) may be solved for the maximum value of the drag force
in terms of the period of the drag force impulse.

$$D_{max} = \frac{2V_l \, I_w}{r_e^2 T_{ID}} \qquad (16)$$

With the assumptions used to write equation (16), the drag
force-time variation is of the type shown in figure 15.
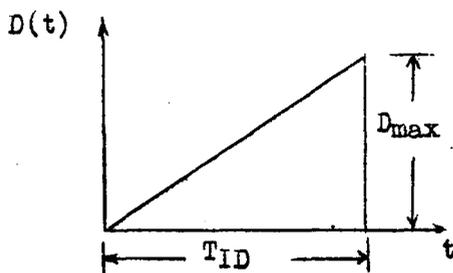


Figure 15

The dynamic response factor curve for the force-time
relation shown by figure 15 is given in figure 16.  The
wheel spin-up time is of the order of one-fourth to one-
fifth of the axial strut impulse period, and the wheel is
often  up to speed before the axial strut force reaches its
maximum value.

In the case of modern large aircraft with retractable
landing gears, the assumption that the landing gear leg is
rigid in fore and aft bending may be considerably in error.
Because of this lack of rigidity, there is an additional vi-
bration mode to be considered which involves large wheel and
strut amplitude and very little wing torsional motion.  This
lack of strut rigidity may be particularly troublesome if
the fundamental fore and aft bending frequency of the land-
ing gear is coincident with some other mode of the structure,
and resonance is established.

In a general consideration of the landplane landing
problem, drift and one wheel landings which excite antisym-
metric wing oscillations should be considered as well as sym-
metric landings.  The principles discussed heretofore are
quite general and apply equally well in either case.

Dynamic response factor curve for
landplane drag force impulse of
the type shown by figure 15.
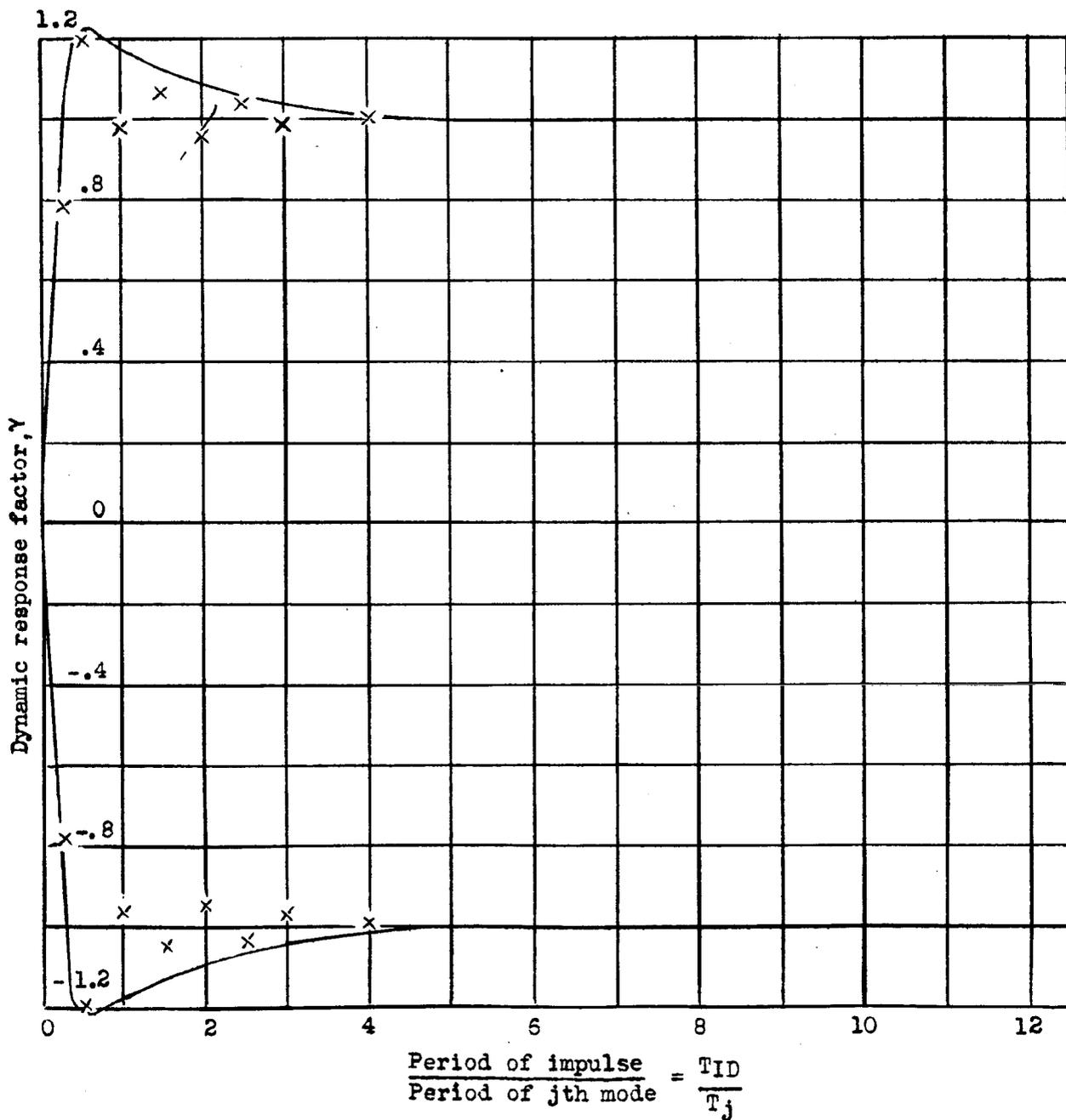


Figure 16.

## 10. THE FLYING BOAT WING

In the flying boat problem, the forcing impulses are
transmitted to the wing through the hull structure, and the
problem may be assumed identical to the landplane problem
except that the force and moment is applied at the center
line of the airplane rather than outboard at a landing gear
station. The problem of the impact of a flying boat hull in
water is considerably more complex than the landplane impact
problem. A limited amount of test data is available showing
time histories of center of gravity vertical and drag ac-
celerations, and pitching acceleration for flying boats.
These records indicate that the time history of the landing
reaction varies widely according to landing attitude, pilot
technique, condition of seaway, and detailed characteristics
of hull. Theory on the seaplane impact problem is extensive;
however, its applicability to the complex hull shapes of
modern flying boats has not yet been demonstrated, and hence
little attempt is made to use it for design purposes. Not
only the force-time relations of the vertical and drag forces
are necessary, but also their lines of action on the hull
bottom must be known. Of these necessary items, the charac-
teristics of the vertical force are more completely known
than any of the others. Very little general information is
available concerning the drag force and how the exact line
of action of the vertical and drag forces vary throughout
the impact period. Because of this lack of information, it
is difficult to determine exactly how the impulses are ap-
plied to the flying boat wing. Furthermore, the influence
of the moment on the wing vibration is undetermined unless
the effect of fuselage and tail oscillations are considered.
Such a consideration is beyond the scope of this discussion.
A first approximation may be obtained by considering only
the effect of the vertical force applied at the elastic axis
of the free wing. In the absence of more complete test data
it may be assumed that the vertical force on the seaplane
hull, during the initial landing impact, varies as a half
cycle of a sine wave. A dynamic response factor envelope
has been determined for a half cycle of a sine wave impulse
and is given in figure 17. The variation in the impulse
period is as wide as the variation in the shape of the im-
pulse curve, and hence it is not possible to derive an em-
pirical relation for the impulse period from test data, as
was the case with the landplane. In order to apply the dy-
namic response factor curve, an impulse period must be as-
sumed, or a value taken from flight test data on an airplane
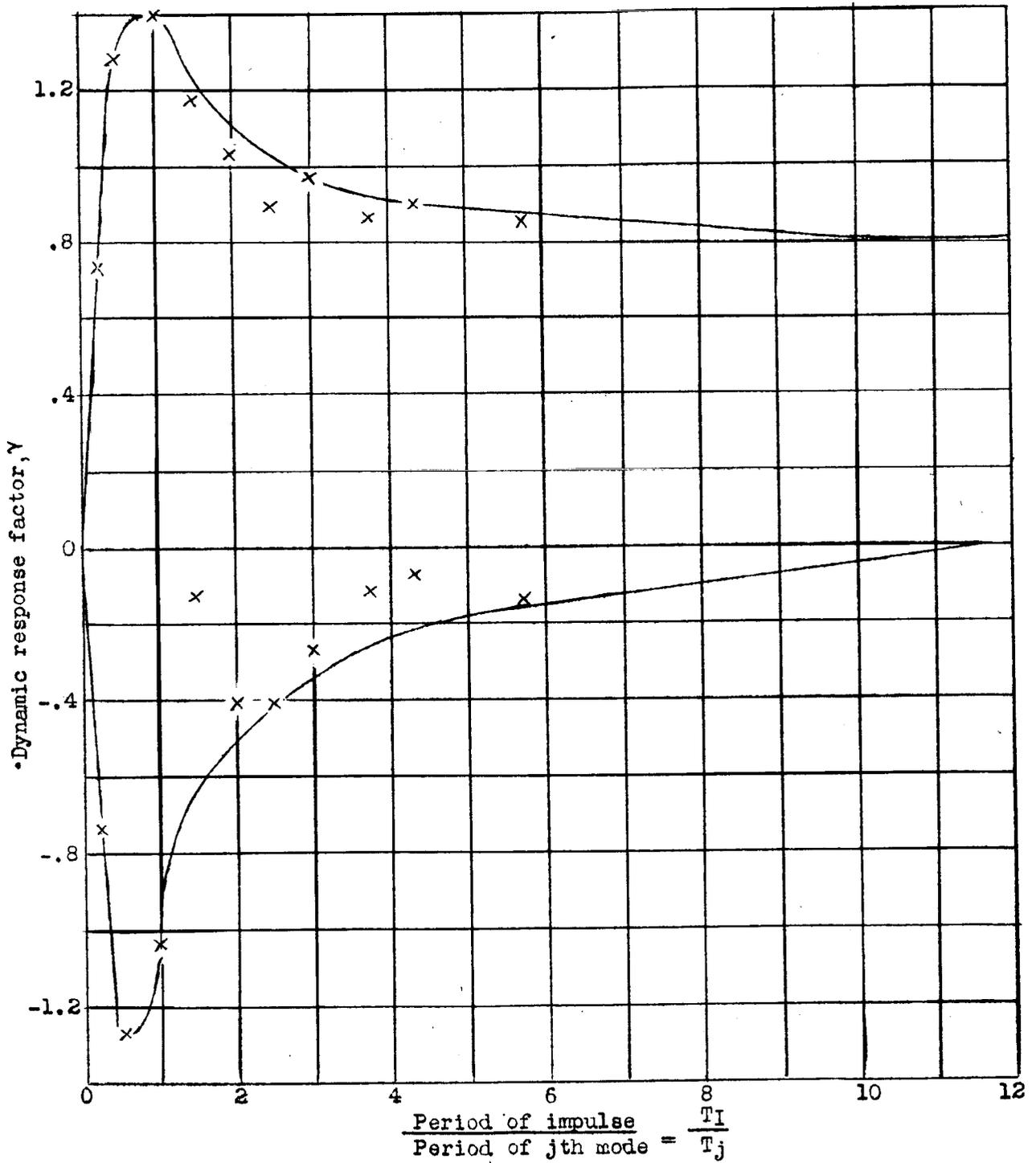
Figure 17.- Dynamic response factor curve for seaplane vertical landing force-time curve consisting of a half cycle of a sine wave.

of similar size to the one being investigated.  A conservative procedure, in any case, would be to choose a period ratio $T_I/T_j$ so as to yield the most critical combination of stresses.

## 11. ILLUSTRATIVE NUMERICAL EXAMPLE FOR LANDPLANE WING

The principles discussed in section 9 are illustrated by a numerical example in which the stresses due to landing in a four-engine land type bomber are computed.  Mode shapes and frequencies of the free wing of this airplane were computed and transmitted in a report prepared for the Army Air Forces at C.I.T. by M. A. Biot in 1941.  The bending and torsional moments are computed at seven wing stations.  In this example, the wing chord is assumed parallel to the ground during the landing, and the landing gear strut is assumed to be perpendicular to the wing chord line.  The landing gear strut is assumed to be rigid and rigidly attached to the wing.  Figure 18 shows the assumed conditions during the landing.
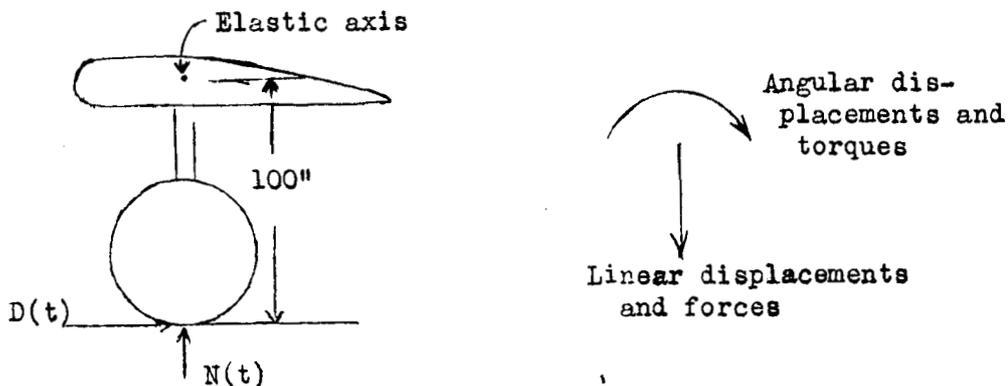


Figure 18

In this particular example, the elastic axis passes directly over the landing gear leg as is shown by figure 18. The computations may be carried out by table 1 which is self-explanatory when reference is made to appendix I.  A table similar to table 1 is required for each wing mode.  The mode shapes and frequencies of the free wing are taken from the Army Air Forces report prepared at C.I.T. and tabulated in tables 2, 3, and 4, where the first three wing modes are considered.  The following additional data are required to complete the tables.

Gross weight, $W$ .................................... 47,200 lb
Gross weight less weight of landing gear, $W_L$ ..... 44,426 lb
Moment of inertia of wheel and tire, $I_W$ ........ 28 slug-ft$^2$
Effective rolling radius, $r_e$ ..................... 2.08 ft
Landing speed, $V_s$ ............................... 124.8 fps

From equation (13), the period of the vertical impact force is,

$$T_{IN} = 0.25 \left\{\frac{W}{1000}\right\}^{0.1475} = 0.25 \left\{\frac{47200}{1000}\right\}^{0.1475} = 0.441 \text{ second}$$

The maximum value of the vertical impact force is

$$N_{max} = \frac{1}{2} W_L n = \frac{1}{2} \times 44,426n = 22,213n$$

where $n$ is the ultimate vertical load factor. In this example, it is assumed that $n = 4$.

$$N_{max} = (22,213)(4) = 88,852 \text{ pounds}$$

It is assumed that the period of the drag force is one-fifth that of the vertical force.

$$T_{ID} = \frac{1}{5} T_{IN} = \frac{0.441}{5} = 0.0882 \text{ second}$$

From equation (16), the maximum value of the drag force is

$$D_{max} = \frac{2V_1 I_W}{r_e^2 T_{ID}} = \frac{(2)(124.8)(28)}{(2.08)^2(0.0882)} = 18,290 \text{ pounds}$$

From the data computed in columns 5, 6, 13, and 14 in tables 2, 3, and 4, wing bending and torsional moments are plotted in figures 19 and 20, respectively, for each mode in the first and second extreme positions of oscillation. Critical values of bending moment and torsional moment at each station are obtained by adding corresponding ordinates of the three bending moment curves on the same side of the zero axis. For example, at the airplane center line, the critical negative bending moment is obtained by adding:

TABLE 1

Mode number _____  Frequency $f$ = _____ cps

| Station | $m_k$ | $S_k$ | $I_k$ | $b_k$ | $a_k$ | $m_k b_k^2$ | $I_k a_k^2$ | $2a_k b_k S_k$ |
|---|---|---|---|---|---|---|---|---|
| | Mass at station | Static mass moment about elastic axis | Moment of inertia about elastic axis | Bending deflection of elastic axis | Angular deflection about elastic axis | | | |
| | Distance outboard from center line | | | | | | | |
| 0 | | | | | | | | |
| 1 | | | | | | | | |
| 2 | | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | | | | |
| 6 | | | | | | | | |

See equation I-15.

$a = \sum$  
$b = \sum$  
$c = \sum$  
$M_j = a + b + c$

| | Item or $M_{max}$ | Axial force | Drag moment |
|---|---|---|---|
| ① | $F_{max}$ | | |
| ② | $M_j$ | ⟶ | ⟶ |
| ③ | $\phi_F(j)$ | Equation (I-15) | Equation (I-16) |
| ④ | $T_I$ | Equation (13) | |
| ⑤ | $T_j$ | $1/f$ | $1/f$ |
| ⑥ | $T_I/T_j$ | ④/⑤ | ④/⑤ |
| ⑦ | $\gamma_j^{(+)}$ | Figure 13 | Figure 16 |
| ⑧ | $\gamma_j^{(-)}$ | Figure 13 | Figure 16 |
| ⑨ | $\eta = \dfrac{F_{max}\,\phi_F}{M_j}$ | ①×③/② | ⟶ |
| ⑩ | $\eta\gamma^{(+)}$ | ⑨×⑦ | ⟶ |
| ⑪ | $\eta\gamma^{(-)}$ | ⑨×⑧ | ⟶ |
| ⑫ | $a^{(+)} =$ | ⑩(axial)+⑩(drag) | |
| ⑬ | $a^{(-)} =$ | ⑪(axial)+⑪(drag) | |

| Station | ① $a^{(+)} \times b_k$ | ② $a^{(-)} \times b_k$ | ③ $F_k^{(+)} = -m_k \times$① | ④ $F_k^{(-)} = -m_k \times$② | ⑤ Bending moment in positive maximum position | ⑥ Bending moment in negative maximum position | ⑦ $a^{(+)} \times a_k$ | ⑧ $a^{(-)} \times a_k$ | ⑨ $T_k^{(+)} = -I_k \times$⑦ | ⑩ $T_k^{(-)} = -I_k \times$⑧ | ⑪ $\Delta T_k^{(+)} = -S_k \times$① | ⑫ $\Delta T_k^{(-)} = -S_k \times$② | ⑬ Total torque in positive maximum position ⑨+⑪ | ⑭ Total torque in negative maximum position ⑩+⑫ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k = 0$ | | | | | | | | | | | | | | |
| $k = 1$ | | | | | | | | | | | | | | |
| $k = 2$ | | | | | | | | | | | | | | |
| $k = 3$ | | | | | | | | | | | | | | |
| $k = 4$ | | | | | | | | | | | | | | |
| $k = 5$ | | | | | | | | | | | | | | |
| $k = 6$ | | | | | | | | | | | | | | |

## TABLE 2

### Mode number 1    Frequency f = 3.365 cps

### Landplane

| Station | $m_k$ | $B_k$ | $I_k$ | $h_k$ | $\alpha_k$ | $m_k h_k^2$ | $I_k \alpha_k^2$ | $2\alpha_k h_k B_k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 28.5 | 0 | ∞ | -0.078 | 0 | 0.1731 | ----- | |
| 1 | 16.3 | -640 | 85,234 | -0.031 | -0.00084 | 0.0157 | 0.0601 | -0.0333 |
| 2 | 5.27 | 0 | 1,288 | +0.047 | -0.0016 | 0.01169 | 0.00330 | |
| 3 | 9.15 | -569 | 61,717 | +0.164 | -0.00183 | 0.246 | 0.2065 | +0.342 |
| 4 | 0.974 | 0 | 536 | +0.374 | -0.00185 | 0.1361 | 0.00184 | |
| 5 | 0.686 | 0 | 287 | +0.670 | -0.00187 | 0.308 | 0.0010 | |
| 6 | 0.153 | 0 | 34.1 | +0.936 | -0.00188 | 0.134 | 0 | |

$$a = 1.025 \qquad b = 0.873 \qquad c = 0.309$$

$$a + b + c = M_1 = 1.607$$

| | Item | $F_{max}$ or $M_{max}$ | Axial force | Drag moment |
|---|---|---|---|---|
| ① | $F_{max}$ or $M_{max}$ | 88,852 | 1,829,000 | |
| ② | $M_1$ | 1.607 | 1.607 | |
| ③ | $\Phi_F^{(1)}$ | -0.031 | -0.00084 | |
| ④ | $T_I$ | 0.441 | 0.088 | |
| ⑤ | $T_1$ | 0.297 | 0.297 | |
| ⑥ | $T_I/T_1$ | 1.485 | 0.298 | |
| ⑦ | $\gamma_1^{(+)}$ | +1.90 | +0.96 | |
| ⑧ | $\gamma_1^{(-)}$ | -1.30 | -0.96 | |
| ⑨ | $\eta = \dfrac{F_{max}\,\Phi_F^1}{M_1}$ | -1711 | -958 | |
| ⑩ | $\eta\,\gamma_1^{(+)}$ | -3251 | -918 | |
| ⑪ | $\eta\,\gamma_1^{(-)}$ | +2224 | +918 | |
| ⑫ | $a^{(+)} = $ ⑩ (axial) + ⑩ (drag) | -4,169 | | |
| ⑬ | $a^{(-)} = $ ⑪ (axial) + ⑪ (drag) | +3,142 | | |

| Station | ① $a^{(+)} \times h_k$ | ② $a^{(-)} \times h_k$ | ③ $F_k^{(+)} = -m_k \times$ ① | ④ $F_k^{(-)} = -m_k \times$ ② | ⑤ Bending moment in positive maximum position | ⑥ Bending moment in negative maximum position | ⑦ $a^{(+)} \times \alpha_k$ | ⑧ $a^{(-)} \times \alpha_k$ | ⑨ $T_k^{(+)} = -I_k \times$ ⑦ | ⑩ $T_k^{(-)} = -I_k \times$ ⑧ | ⑪ $\Delta T_k^{(+)} = -B_k \times$ ① | ⑫ $\Delta T_k^{(-)} = -B_k \times$ ② | ⑬ Total torque in positive maximum position | ⑭ Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | +326 | -246 | -9300 | +7010 | -3,940,000 | +3,985,000 | 0 | 0 | ------- | ------- | ------- | 0 | -1,092,150 | +815,602 |
| 1 | +129 | -97 | -2100 | +1581 | -2,719,000 | +3,060,000 | +3.502 | -2.64 | -299,000 | +216,200 | +82,600 | -62,100 | -875,750 | +661,502 |
| 2 | -196 | +148 | +1033 | -781 | -1,769,000 | +1,340,000 | +6.68 | -5.04 | -8,600 | +6,490 | | | -867,150 | +655,012 |
| 3 | -685 | +516 | +6260 | -4720 | -844,000 | +638,500 | +7.64 | -5.76 | -471,000 | +356,000 | -389,500 | +294,000 | -6,650 | +5,012 |
| 4 | -1560 | +1179 | +1520 | -1148 | -356,000 | +269,000 | +7.72 | -5.82 | -4,140 | +3,120 | | | -2,510 | +1,892 |
| 5 | -2800 | +2110 | +1921 | -1450 | -53,850 | +40,550 | +7.81 | -5.89 | -2,242 | +1,690 | | | -268 | +202 |
| 6 | -3910 | +2950 | +598 | -451 | 0 | 0 | +7.85 | -5.92 | -268 | +202 | | | 0 | 0 |

## TABLE 3

### Mode number 2          Frequency f = 4.61 cps

### Landplane

| Station | $m_k$ | $S_k$ | $I_k$ | $h_k$ | $\alpha_k$ | $m_k h_k^2$ | $I_k \alpha_k^2$ | $a\,\alpha_k\,b_k\,S_k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 28.5 | 0 | ∞ | -0.1237 | 0 | 0.436 | — | |
| 1 | 16.3 | -640 | 85,234 | -0.0693 | +0.0057 | 0.0783 | 2.77 | +0.505 |
| 2 | 5.27 | 0 | 1,288 | +0.0330 | +0.0073 | 0.0058 | .0685 | |
| 3 | 9.15 | -569 | 61,717 | +0.229 | +0.0095 | 0.480 | 5.57 | -2.48 |
| 4 | 0.974 | 0 | 536 | +0.756 | -0.00956 | 0.556 | 0.0491 | |
| 5 | 0.686 | 0 | 387 | +1.741 | -0.00964 | 2.085 | 0.0266 | |
| 6 | 0.153 | 0 | 34.1 | +2.862 | -0.00968 | 1.273 | 0.0032 | |

$$a = 4.914 \qquad b = 8.488 \qquad c = -1.975$$
$$a + b + c = M_2 = +11.427$$

| | Item or $F_{max}$ | Axial force | Drag moment |
|---|---|---|---|
| (1) | $F_{max}$ or $M_{max}$ | 88,852 | 1,829,000 |
| (2) | $M_2$ | 11.427 | 11.487 |
| (3) | $\phi_r^{(2)}$ | -0.0693 | +0.0057 |
| (4) | $T_I$ | 0.441 | 0.088 |
| (5) | $T_2$ | 0.2165 | 0.2165 |
| (6) | $T_I/T_2$ | 2.04 | 0.406 |
| (7) | $\gamma_2^{(+)}$ | +1.90 | +1.90 |
| (8) | $\gamma_2^{(-)}$ | -1.20 | -1.20 |
| (9) | $\eta = \dfrac{F_{max}\,\phi_r^{(2)}}{M_2}$ | -537 | +911 |
| (10) | $\eta\,\gamma_2^{(+)}$ | -1020 | +1093 |
| (11) | $\eta\,\gamma_2^{(-)}$ | +644 | -1093 |
| (12) | $a^{(+)} = $ (10) (axial) + (10) (drag) | +73 | |
| (13) | $a^{(-)} = $ (11) (axial) + (11) (drag) | -449 | |

| Station | (1) $a^{(+)} \times b_k$ | (2) $a^{(-)} \times b_k$ | (3) $F_k^{(+)} = -m_k \times$ (1) | (4) $F_k^{(-)} = -m_k \times$ (2) | (5) Bending moment in positive maximum position | (6) Bending moment in negative maximum position | (7) $a^{(+)} \times \alpha_k$ | (8) $a^{(-)} \times \alpha_k$ | (9) $T_k^{(+)} = -I_k \times$ (7) | (10) $T_k^{(-)} = -I_k \times$ (8) | (11) $\Delta T_k^{(+)} = -S_k \times$ (1) | (12) $\Delta T_k^{(-)} = -S_k \times$ (2) | (13) Total torque in positive maximum position | (14) Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -9.04 | +55.5 | +258 | -1581 | +92,400 | -790,000 | 0 | 0 | -------- | -------- | -------- | -------- | -73,316 | +450,909 |
| 1 | -5.06 | +31.1 | +82.5 | -506.5 | +68,300 | -583,000 | +0.416 | -2.56 | -35,500 | +218,100 | -3240 | +19,900 | -34,576 | +212,909 |
| 2 | +2.41 | -14.8 | -12.71 | +78 | +48,100 | -411,000 | +0.533 | -3.28 | -686 | +4,220 | | | -33,890 | +208,689 |
| 3 | +16.71 | -102.9 | -153 | +941 | +27,190 | -232,500 | +0.694 | -4.26 | -48,800 | +363,500 | +9510 | -58,500 | -600 | +3,689 |
| 4 | +55.2 | -340 | -53.8 | +331 | +12,290 | -105,000 | +0.698 | -4.29 | -374 | +2,300 | | | -226 | +1,389 |
| 5 | +127.1 | -782 | -87.4 | +537 | +2,905 | -17,810 | +0.704 | -4.32 | -202 | +1,241 | | | -34.1 | +148 |
| 6 | +211 | -1295 | -32.3 | +198 | 0 | 0 | +0.706 | -4.34 | -24.1 | +148 | | | 0 | 0 |

## TABLE 4

Mode number 3  Frequency  $f$ = 8.46 cps

Landplane

| Station | | $m_k$ | $S_k$ | $I_k$ | $\alpha_k$ | $h_k$ | $m_k h_k^2$ | $I_k \alpha_k^2$ | $2\alpha_k h_k S_k$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 28.5 | 0 | ∞ | 0 | +0.0426 | 0.0519 | ------- | |
| 1 | 133 | 16.3 | -640 | 85,234 | -0.00213 | -0.0064 | 0.000668 | 0.386 | -0.0175 |
| 2 | 217 | 5.27 | 0 | 1,288 | -0.00132 | -0.0710 | 0.0266 | 0.00224 | |
| 3 | 307 | 9.15 | -569 | 61,717 | -0.00018 | -0.1250 | 0.1430 | 0.002 | -0.0256 |
| 4 | 428 | 0.974 | 0 | 536 | -0.00014 | -0.0370 | 0.0013 | 0.00001052 | |
| 5 | 548 | 0.686 | 0 | 287 | -0.00010 | +0.3890 | 0.104 | 0.000003 | |
| 6 | 638 | 0.153 | 0 | 34.1 | -0.00008 | +1.045 | 0.167 | 0 | |
| | | | | | | | $a$ = 0.494 | $b$ = 0.390 | $c$ = -0.0431 |
| | | | | | | | | $a + b + c$ = $M_3$ = +0.841 | |

| | Item | $F_{max}$ or $M_{max}$ | Axial force | Drag moment |
|---|---|---|---|---|
| ① | $F_{max}$ or $M_{max}$ | | 88,852 | 1,829,000 |
| ② | $M_3$ | | 0.841 | 0.841 |
| ③ | $\dfrac{\phi(3)}{F}$ | | -0.0064 | -0.00213 |
| ④ | $T_I$ | | 0.441 | 0.088 |
| ⑤ | $T_3$ | | 0.118 | 0.118 |
| ⑥ | $T_I/T_3$ | | 3.74 | 0.745 |
| ⑦ | $\gamma_3^{(+)}$ | | +1.73 | +1.21 |
| ⑧ | $\gamma_3^{(-)}$ | | -0.63 | -1.21 |
| ⑨ | $\eta = \dfrac{F_{max}\,\phi_3}{M_3}$ | | -675 | -463 |
| ⑩ | $\eta\,\gamma_3^{(+)}$ | | -1168 | -560 |
| ⑪ | $\eta\,\gamma_3^{(-)}$ | | +425 | +560 |
| ⑫ | $a^{(+)}$ = ⑩(axial) + ⑩(drag) | | | -1728 |
| ⑬ | $a^{(-)}$ = ⑪(axial) + ⑪(drag) | | | +985 |

| Station | ① $a^{(+)} \times h_k$ | ② $a^{(-)} \times h_k$ | ③ $F_k^{(+)}$ = $-m_k \times$ ① | ④ $F_k^{(-)}$ = $-m_k \times$ ② | ⑤ Bending moment in positive maximum position | ⑥ Bending moment in negative maximum position | ⑦ $a^{(+)} \times \alpha_k$ | ⑧ $a^{(-)} \times \alpha_k$ | ⑨ $T_k^{(+)}$ = $-I_k \times$ ⑦ | ⑩ $T_k^{(-)}$ = $-I_k \times$ ⑧ | ⑪ $\Delta T_k^{(+)}$ = $-S_k \times$ ① | ⑫ $\Delta T_k^{(-)}$ = $-S_k \times$ ② | ⑬ Total torque in positive maximum position | ⑭ Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -73.6 | +42 | +2100 | -1198 | +372,000 | 212,200 | 0 | 0 | --------- | --------- | | | -142,525 | +117,587 |
| 1 | +110.7 | -6.3 | -1802 | +102.9 | +87,900 | -49,850 | +3.68 | -2.1 | -314,000 | +179,000 | +70,900 | -4,040 | +100,575 | -57,373 |
| 2 | +122.9 | -70 | -648 | +369 | -77,000 | +43,700 | +2.28 | -1.30 | -2,940 | +1,672 | | | +103,515 | -59,045 |
| 3 | +216 | -123.1 | -1977 | +1129 | -195,000 | +110,900 | +0.311 | -0.1771 | -19,200 | +10,950 | +122,900 | -70,100 | -184.32 | +104.99 |
| 4 | +63.9 | -36.4 | -62.2 | +35.4 | -113,200 | +64,400 | +0.242 | -0.138 | -130 | +74.1 | | | -54.32 | +30.89 |
| 5 | -671 | +383 | +461 | -263 | -24,950 | +14,190 | +0.1728 | -0.0985 | -49.6 | +28.2 | | | -4.72 | +2.69 |
| 6 | -1809 | +1030 | +276.5 | -157.5 | 0 | 0 | -0.1381 | -0.0789 | -4.72 | +2.69 | | | 0 | 0 |

                                                      Inch-pounds

First mode - positive maximum . . . . . . . . . . . . . -3,940,000
Second mode - negative maximum  . . . . . . . . . . . .   -790,000
Third mode - negative maximum . . . . . . . . . . . . .   -212,000
Maximum negative bending moment at the center line =    -4,942,000

        Similarly, for the maximum negative torsional moment at
the center line, the following are added:

                                                      Inch-pounds

First mode - positive maximum . . . . . . . . . . . . . -1,092,150
Second mode - positive maximum  . . . . . . . . . . . .    -73,316
Third mode - positive maximum . . . . . . . . . . . . .   -142,525
Maximum negative torsional moment at the center line = -1,307,991

        The frequency of the fourth mode is approximately 1350
cpm, and hence its contribution to the stress would be small.
This is true because of the tendency for $\phi_F^{(j)}$ to be reduced
to small values in the higher modes, and because of the in-
hibitive effect which the aerodynamic and structural damping
has upon the higher modes.  It is important to remember that
the stresses, shown here must be superposed upon the steady
stresses produced by the aerodynamic loads on the wing dur-
ing landing.


        12.  ILLUSTRATIVE NUMERICAL EXAMPLE FOR SEAPLANE WING


        The principles discussed for the seaplane are illus-
trated by a numerical example.  In this example, a four-
engine land type patrol bomber is considered as being a
flying boat in order that the same mode shapes and fre-
quencies may be used.  The bending and torsional moments
are computed at seven wing stations for a vertical load
factor of 1, and the effect of the drag force is neglected.
The first three deformation modes of the free wing are
considered, and the computations are carried out by means
of table 1.  The vertical force on the hull is assumed
perpendicular to the wing chord during impact.

Figure 19.- Landplane-wing bending moment vs. wing station. First three modes for 4G landing.
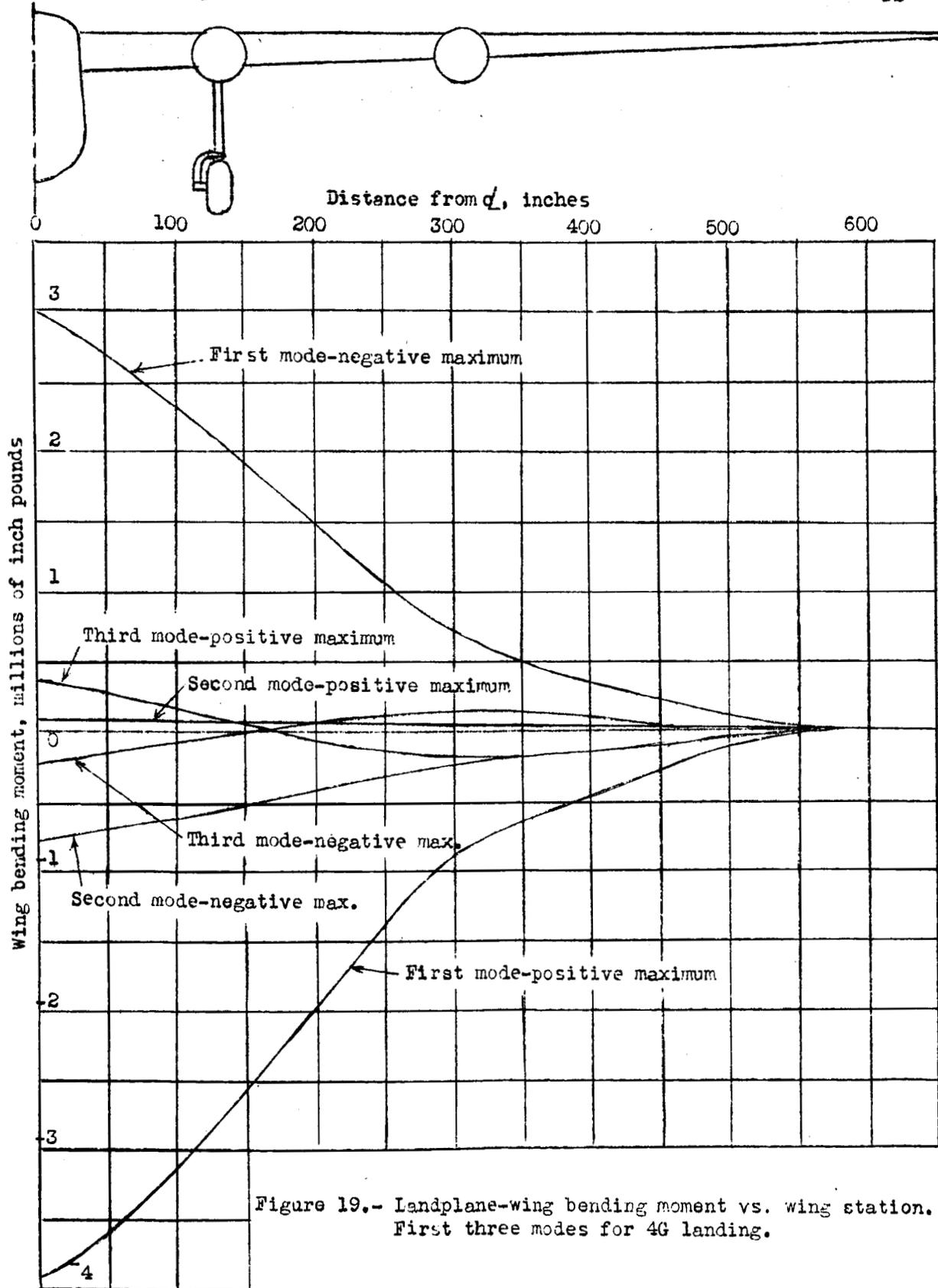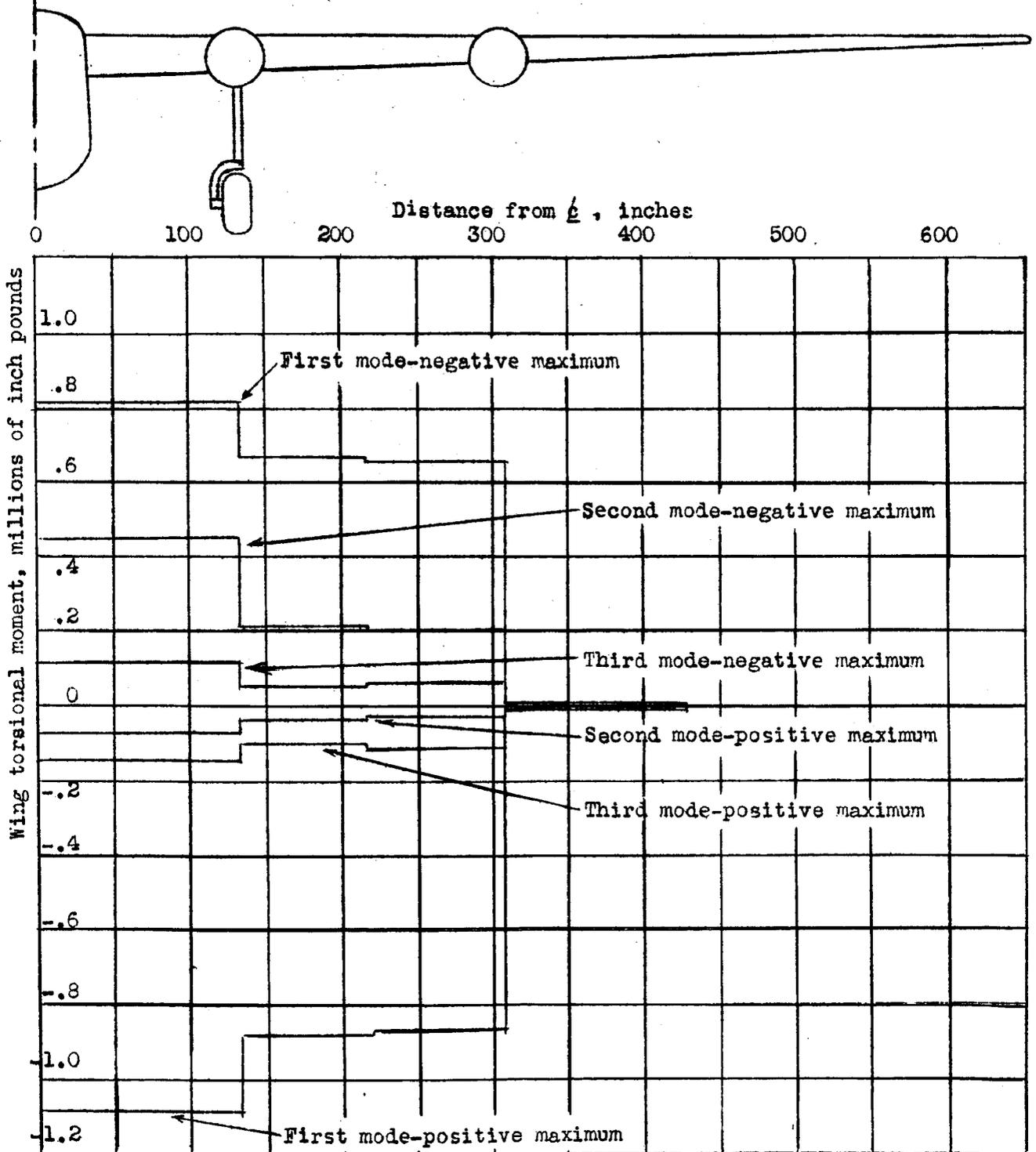
Figure 20.- Landplane-wing torsional moment vs. wing station. First three
modes for 4G landing.

The following sketch shows the assumed conditions during the landing.

Elastic axis

N(t)    Figure 21
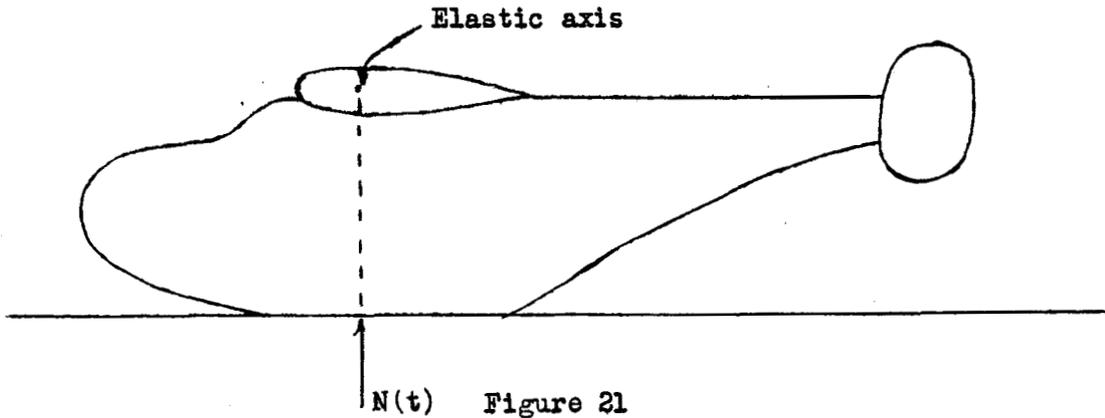
For purpose of this example, an impulse period of 0.2 second is arbitrarily assumed. From the data computed in columns 5, 6, 13, and 14 in tables 5, 6, and 7, wing bending and torsional moments are plotted in figures 22 and 23, respectively, for each mode in the first and second extreme positions of oscillation for a vertical load factor of 4. The values plotted in figures 22 and 23 are four times the values shown in columns 5, 6, 13, and 14 in tables 5, 6, and 7.

## 13. DISCUSSION

In the usual landing with large aircraft there is a period of initial impact followed by a short interval during which the airplane is airborne. It is to conditions existing during the initial impact and shortly thereafter that the present theory of transients is applicable. In the ensuing motion during run-out, the wheels are substantially in contact with the ground or the hull with the water. During this latter phase of the motion, the structure has imposed upon it a series of alternating loads depending upon the ground contour or the condition of the seaway. It may be possible that a resonance condition during the latter phase of the landing may produce stresses more critical than those produced during and shortly after the initial impact.

A preliminary analytical investigation into the effect of aerodynamic damping has shown that it may be neglected with small error; however, this is subject to experimental

## TABLE 5

Mode number 1          Frequency f = 3.365 cps

Flying boat

| Station | $m_k$ | $s_k$ | $I_k$ | $h_k$ | $\alpha_k$ | $m_k h_k^2$ | $I_k \alpha_k^2$ | $2\alpha_k h_k s_k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 28.5 | 0 | ∞ | -0.078 | 0 | 0.1731 | ---- | ---- |
| 1 | 16.3 | -640 | 85,234 | -0.031 | -0.00084 | 0.0157 | 0.0601 | -0.0333 |
| 2 | 5.27 | 0 | 1,288 | +0.047 | -0.0016 | 0.01169 | 0.0033 | |
| 3 | 9.15 | -569 | 61,717 | +0.164 | -0.00183 | 0.246 | 0.2065 | +0.342 |
| 4 | 0.974 | 0 | 536 | +0.374 | -0.00185 | 0.1361 | 0.00184 | |
| 5 | 0.686 | 0 | 287 | +0.670 | -0.00187 | 0.308 | 0.0010 | |
| 6 | 0.153 | 0 | 34.1 | +0.936 | -0.00188 | 0.134 | 0 | |

(Station column values: 0, 133, 217, 307, 428, 548, 638)

$a = 1.025$   $b = 0.273$   $c = 0.309$

$a + b + c = M_1 = 1.607$

| | Item | Vertical force |
|---|---|---|
| (1) | $F_{max}$ | 25,600 |
| (2) | $M_1$ | 1.607 |
| (3) | $\Phi_F^{(1)}$ | -0.078 |
| (4) | $T_I$ | 0.200 |
| (5) | $T_1$ | 0.297 |
| (6) | $T_I/T_1$ | 0.674 |
| (7) | $\gamma_1^{(+)}$ | +1.72 |
| (8) | $\gamma_1^{(-)}$ | -1.57 |
| (9) | $\eta = \dfrac{F_{max}\,\Phi}{M_1}$ | -1,147 |
| (10) | $\eta\times\gamma_1^{(+)}$ | -1,973 |
| (11) | $\eta\times\gamma_1^{(-)}$ | +1,801 |

| Station | (1) $\eta\times\gamma_1^{(+)}\times b_k$ | (2) $\eta\times\gamma_1^{(-)}\times b_k$ | (3) $F_k^{(+)}=-m_k\times$①| (4) $F_k^{(-)}=-m_k\times$② | (5) Bending moment in positive maximum position | (6) Bending moment in negative maximum position | (7) $\eta\times\gamma_1^{(+)}\times\alpha_k$ | (8) $\eta\times\gamma_1^{(-)}\times\alpha_k$ | (9) $T_k^{(+)}=-I_k\times$⑦ | (10) $T_k^{(-)}=-I_k\times$⑧ | (11) $\Delta T_k^{(+)}=-s_k\times$① | (12) $\Delta T_k^{(-)}=-s_k\times$② | (13) Total torque in positive maximum position | (14) Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | +154 | -140.9 | -4390 | +4010 | -1,870,000 | +1,709,000 | 0 | 0 | -------- | -------- | | | -516,305 | +471,779 |
| 1 | +61.2 | -55.9 | -999 | +911 | -1,288,000 | +1,172,000 | +1.66 | -1.51 | -141,700 | +128,900 | +39,200 | -35,800 | -413,805 | +378,679 |
| 2 | -92.9 | +84.6 | +489 | -446 | -836,400 | +264,000 | +3.16 | -2.88 | -4,060 | +3,705 | | | -409,745 | +374,974 |
| 3 | -324 | +296 | +2962 | -2705 | -398,800 | +364,400 | +3.61 | -3.30 | -222,500 | +204,000 | -184,100 | +168,100 | -3,145 | +2,874 |
| 4 | -739 | +674 | +720 | -656 | -168,200 | +153,700 | +3.65 | -3.34 | -1,959 | +1,791 | | | -1,186 | +1,083 |
| 5 | -1321 | +1209 | +908 | -830 | -25,400 | +23,200 | +3.69 | -3.37 | -1,060 | +967 | | | -126 | +116 |
| 6 | -1850 | +1690 | +283 | -258 | 0 | 0 | +3.71 | -3.39 | -126.5 | +115.7 | | | 0 | 0 |

## TABLE 8

### Deformation mode number 2    Frequency f = 4.61 cps

#### Seaplane

| Item | Vertical force |
|---|---|
| ① $F_{max}$ | 23,600 |
| ② $M_2$ | 11.487 |
| ③ $\phi_F^{(2)}$ | -0.1237 |
| ④ $T_I$ | 0.300 |
| ⑤ $T_2$ | 0.2165 |
| ⑥ $T_I/T_2$ | 0.923 |
| ⑦ $\gamma_2^{(+)}$ | +1.75 |
| ⑧ $\gamma_2^{(-)}$ | -1.45 |
| ⑨ $\eta = \dfrac{F_{max}\,\phi_F^{(2)}}{M_2}$ | -255.5 |
| ⑩ $\eta\,\gamma_2^{(+)}$ | -447 |
| ⑪ $\eta\,\gamma_2^{(-)}$ | +370 |

| Station | $m_k$ | $s_k$ | $I_k$ | $\alpha_k$ | $b_k$ | $m_k h_k^2$ | $I_k \alpha_k^2$ | $2\alpha_k b_k s_k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 28.5 | 0 | ∞ | 0 | -0.1237 | 0.436 | ---- | 0 |
| 1 | 16.3 | -840 | 85,234 | +0.0057 | -0.0693 | 0.0783 | 2.77 | +0.505 |
| 2 | 5.27 | 0 | 1,288 | +0.0073 | +0.0330 | 0.0058 | 0.0685 | 0 |
| 3 | 9.15 | -569 | 81,717 | +0.0095 | +0.229 | 0.480 | 5.57 | -2.48 |
| 4 | 0.974 | 0 | 536 | +0.00956 | +0.756 | 0.556 | 0.0491 | 0 |
| 5 | 0.686 | 0 | 287 | +0.00984 | +1.741 | 2.085 | 0.0266 | 0 |
| 6 | 0.153 | 0 | 34.1 | +0.00968 | +3.882 | 1.273 | 0.0032 | 0 |
|  |  |  |  |  |  | $b = 4.914$ | $b = 8.488$ | $c = -1.975$ |

$$a + b + c = M_2 = 11.487$$

| Station | ① $\eta\times\gamma_2^{(+)}\times b_k$ | ② $\eta\times\gamma_2^{(-)}\times b_k$ | ③ $F_k^{(+)} = -s_k\times①$ | ④ $F_k^{(-)} = -s_k\times②$ | ⑤ Bending moment in positive maximum position | ⑥ Bending moment in negative maximum position | ⑦ $\eta\times\gamma_2^{(+)}\times\alpha_k$ | ⑧ $\eta\times\gamma_2^{(-)}\times\alpha_k$ | ⑨ $T_k^{(+)} = -I_k\times⑦$ | ⑩ $T_k^{(-)} = -I_k\times⑧$ | ⑪ $\Delta T_k^{(+)} = -s_k\times①$ | ⑫ $\Delta T_k^{(-)} = -s_k\times②$ | ⑬ Total torque in positive maximum position | ⑭ Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | +55.4 | -45.8 | -1580 | +1305 | -796,000 | +656,000 | 0 | 0 | ------ | ------ |  |  | +448,893 | -372,208 |
| 1 | +31.0 | -25.8 | -505 | +417 | -589,000 | +485,000 | -2.55 | +2.11 | +217,500 | -180,000 | +19,820 | -16,390 | +211,573 | -175,818 |
| 2 | -14.8 | +12.2 | +78.1 | -64.4 | -414,000 | +343,000 | -3.26 | +2.70 | +4,190 | -3,475 |  |  | +207,383 | -172,343 |
| 3 | -102.5 | +84.8 | +934 | -776 | -234,000 | +193,500 | -4.25 | +3.52 | +362,100 | -217,500 | -58,400 | +48,200 | +3,683 | -3,043 |
| 4 | -338 | +280 | +329 | -272 | -105,900 | +87,200 | -4.28 | +3.54 | +2,297 | -1,900 |  |  | +1,386 | -1,143 |
| 5 | -780 | +645 | +535 | -443 | -17,910 | +14,810 | -4.31 | +3.56 | +1,239 | -1,021 |  |  | +147 | -122 |
| 6 | -1291 | +1089 | +197 | -163 | 0 | 0 | -4.33 | +3.58 | +147 | -122 |  |  | 0 | 0 |

## TABLE 7

### Deformation mode number 3     Frequency $f = 8.46$ cps

### Seaplane

| Station | $m_k$ | $S_k$ | $I_k$ | $b_k$ | $\alpha_k$ | $m_k b_k^2$ | $I_k \alpha_k^2$ | $2\,\alpha_k b_k S_k$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 28.5 | 0 | ∞ | +0.0426 | 0 | 0.0519 | ----- | 0 |
| 1 | 16.3 | -640 | 85,234 | -0.0064 | -0.00313 | 0.000668 | 0.386 | -0.0175 |
| 2 | 5.27 | 0 | 1,288 | -0.0710 | -0.00132 | 0.0266 | 0.00224 | 0 |
| 3 | 9.15 | -569 | 61,717 | -0.1250 | -0.00018 | 0.1430 | 0.002 | -0.0256 |
| 4 | 0.974 | 0 | 536 | -0.0370 | -0.00014 | 0.0013 | 0.0000105 | 0 |
| 5 | 0.688 | 0 | 287 | +0.3890 | -0.00010 | 0.104 | 0.000003 | 0 |
| 6 | 0.153 | 0 | 34.1 | +1.045 | -0.00008 | 0.167 | 0 | $c = -0.0431$ |
| | | | | | | $a = 0.494$ | $b = 0.390$ | |
| | | | | | | | $a + b + c = M_3 = 0.841$ | |

| | Item | Vertical force |
|---|---|---|
| ① | $F_{max}$ | 23,600 |
| ② | $M_3$ | 0.841 |
| ③ | $\Phi_F^{(3)}$ | +0.0426 |
| ④ | $\tau_I$ | 0.200 |
| ⑤ | $\tau_3$ | 0.118 |
| ⑥ | $\tau_I/\tau_3$ | 1.695 |
| ⑦ | $\gamma_3^{(+)}$ | +1.475 |
| ⑧ | $\gamma_3^{(-)}$ | -0.725 |
| ⑨ | $\eta = \dfrac{F_{max}\,\Phi_F^{3}}{M_3}$ | +1197 |
| ⑩ | $\eta\,\gamma_3^{(+)}$ | +1766 |
| ⑪ | $\eta\,\gamma_3^{(-)}$ | -868 |

| Station | ① $\eta\times\gamma_3^{(+)}\times b_k$ | ② $\eta\times\Phi_3^{(-)}\times b_k$ | ③ $F_k^{(+)}=-m_k\times$① | ④ $F_k^{(-)}=-m_k\times$② | ⑤ Bending moment in positive maximum position | ⑥ Bending moment in negative maximum position | ⑦ $\eta\times\gamma_3^{(+)}\times\alpha_k$ | ⑧ $\eta\times\gamma_3^{(-)}\times\alpha_k$ | ⑨ $T_k^{(+)}=-I_k\times$⑦ | ⑩ $T_k^{(-)}=-I_k\times$⑧ | ⑪ $\Delta T_k^{(+)}=-S_k\times$① | ⑫ $\Delta T_k^{(-)}=-S_k\times$② | ⑬ Total torque in positive maximum position | ⑭ Total torque in negative maximum position |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | +75.4 | -37 | -2145 | +1055 | -381,000 | +187,000 | 0 | 0 | ----- | | | | +210,569 | -103,851 |
| 1 | -11.3 | +5.55 | +184 | -90.5 | -89,300 | +44,100 | -3.76 | +1.85 | +320,500 | -157,900 | -7,240 | +3,555 | -108,691 | +50,494 |
| 2 | -125.3 | +61.8 | +661 | -325 | +78,400 | -38,700 | -2.33 | +1.145 | +3,000 | -1,474 | | | -105,691 | +51,968 |
| 3 | -220.5 | -108.3 | +2020 | -993 | +198,800 | -98,000 | -0.318 | +0.156 | +19,620 | -9,640 | -125,500 | +61,700 | +189 | -92 |
| 4 | -65.4 | -32.1 | +63.8 | -31.8 | +115,300 | -57,000 | -0.247 | +0.121 | +133 | -65 | | | +58 | -27 |
| 5 | +686 | -338 | -471 | +232 | +25,400 | -12,520 | -0.177 | +0.087 | +51 | -25 | | | +5 | -8 |
| 6 | +1849 | -907 | -883 | +139 | 0 | 0 | -0.141 | +0.069 | +4.81 | -2.36 | | | 0 | 0 |
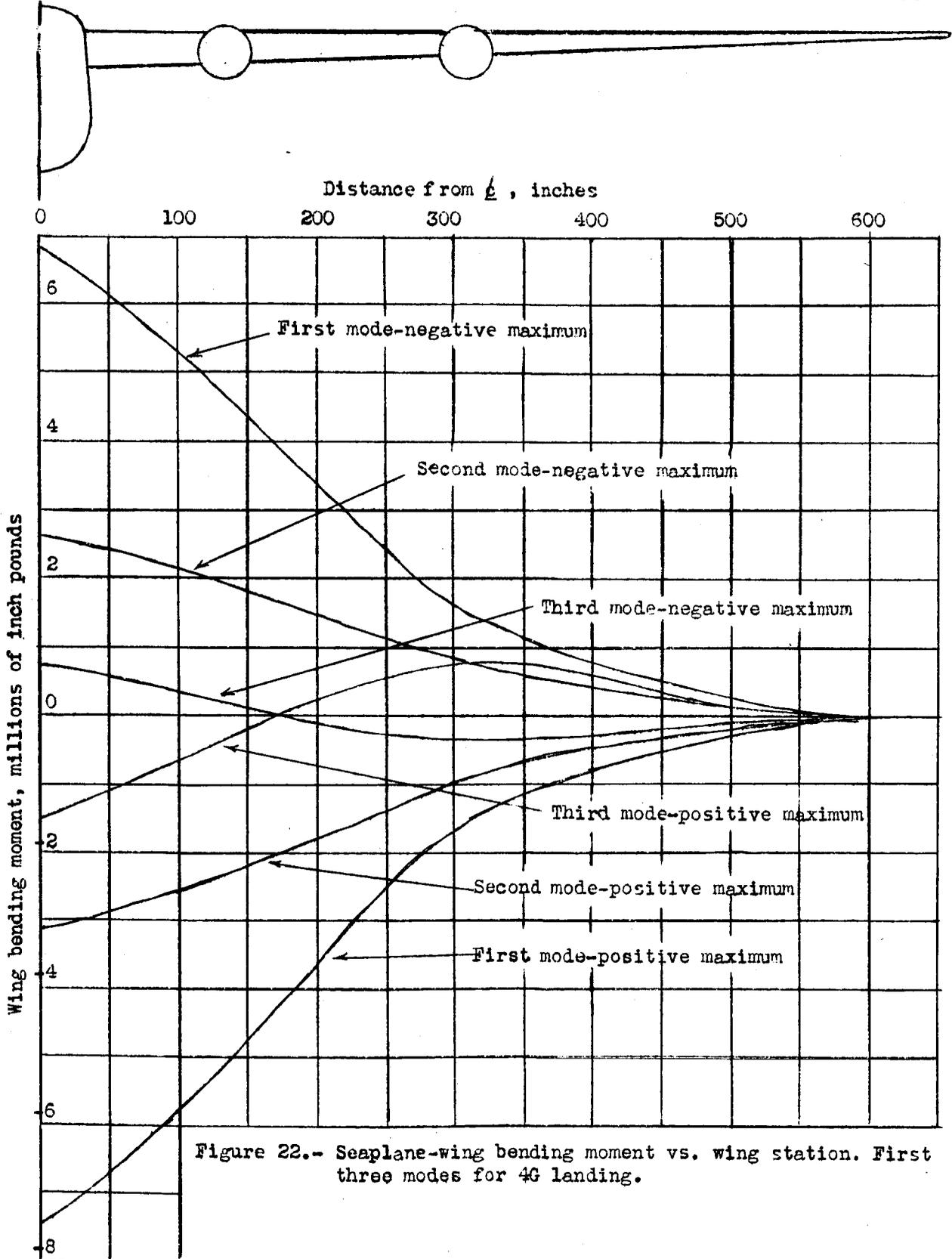
Figure 22.— Seaplane-wing bending moment vs. wing station. First
three modes for 4G landing.

Distance from ₵ , inches

First mode-negative maximum

Second mode-positive maximum

Third mode-positive maximum

Third mode-negative maximum

Second mode-negative maximum

First mode-positive maximum
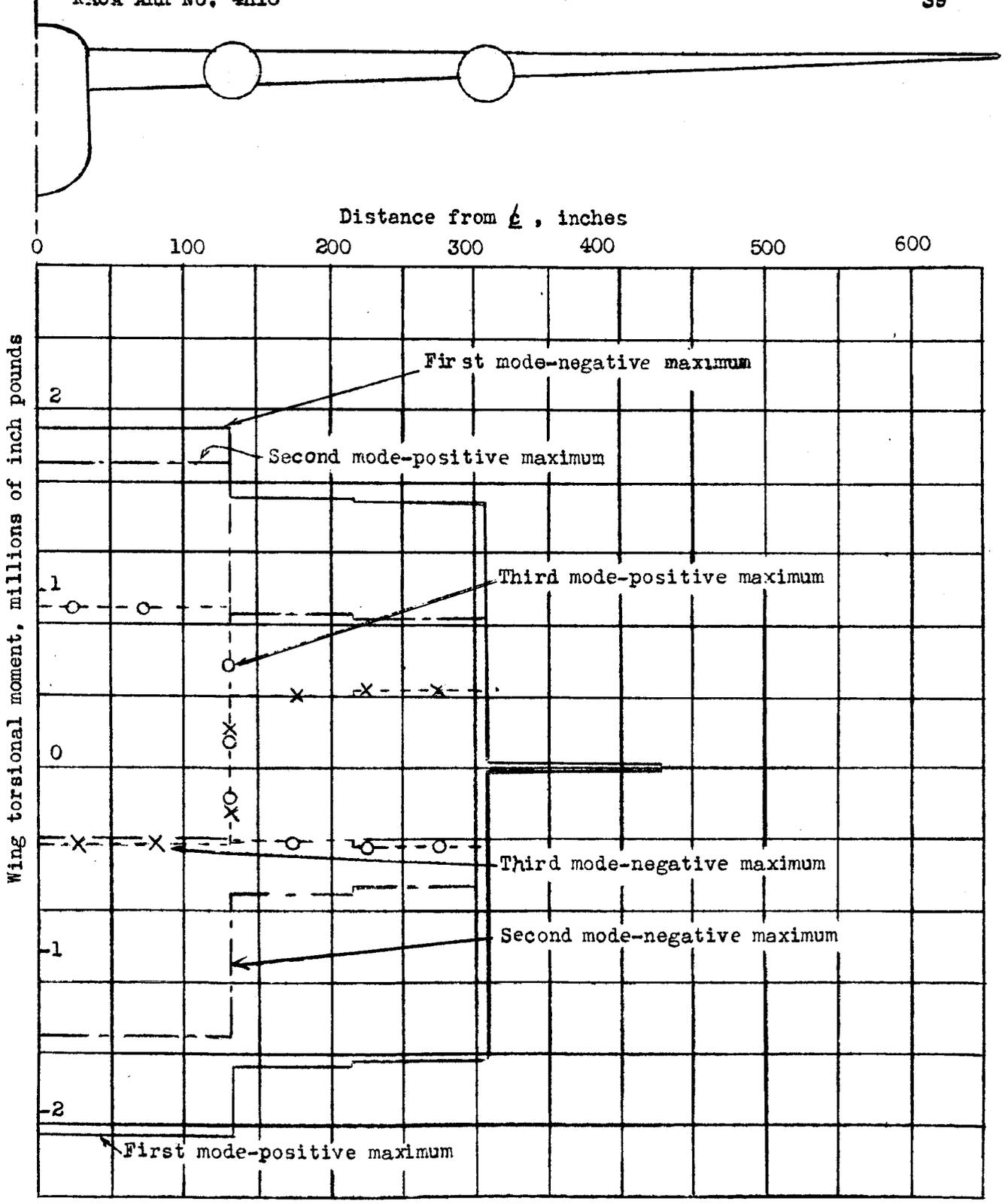
Figure 23.- Seaplane-wing torsional moment vs. wing station. First three modes
for 4G landing.

NACA ARR No. 4H10
check.  Damping has the effect of reducing the amplitudes
and coupling the modes.

Coupling between the motion of the structure and the
external force is neglected in this discussion; however, this
assumption is also subject to experimental check.

The methods discussed here are not universally appli-
cable to all types of transient problems, and should be used
with discretion.  For example, appendix III discusses the
case of a simple beam subjected to a unit impulse applied at
the center.  In this case, the method is not applicable,
since reference to equations (III-11) and (III-13) shows
that the series describing the moment and shear does not
converge.  However, in practical problems, the applied forces
do not behave like the impulse type, but rather behave like
the step type.  In this case, reference to equations (III-10)
and (III-12) shows that the series describing the moment con-
verges, whereas the series describing the shear does not con-
verge when maximum values are added regardless of sign.  In
the case of the procedure discussed here, convergence of the
series describing the moment and shear is dependent upon di-
minishing values of $\phi_F^{(j)}$.  This is assured because of the
characteristic taper of the airplane wing from center line
to wing tip, and because $\phi_F^{(j)}$ is measured at the inboard
side of the wedge in its region of greatest mass per unit
length.  These limitations regarding the convergence of the
series apply even more so if one attempts to predict the
local accelerations in the structures.  This aspect of the
problem becomes significant when evaluating the dynamic
stresses on the attachments of small localized masses.  It
is not possible to formulate at this stage a general rule as
to how many modes should be taken, and each application of
the procedure will present a different problem depending
upon the mode shapes and frequencies of the wing.

In the landplane case, the position of the landing gear
leg has an important effect upon the wing dynamic loads dur-
ing impact.  For example, if the leg intersects a nodal line
for one of the modes, that particular mode is not excited.
The effect of various landing gear positions on the stresses
may be readily studied by these methods.

In studying the present day large aircraft with con-
ventional wings, it appears as though wing dynamic loads will

produce torsional moments inboard of the nacelles which,
aggravated by the overhanging engines, may be critical in
severe impact.  The design of the nacelle carry through
structure in bending may be controlled by dynamic loads,
and the possibility of critical wing bending stresses is
not precluded in very large aircraft.

A comparison between the landplane and flying boat ex-
amples given in sections 12 and 13 shows that the wing dy-
namic loads in the flying boat are more severe than in the
case of the landplane.  This is attributed to two causes:

(a) In the case of the second mode in the landplane
    example, the drag force actually has a reliev-
    ing effect on the stresses produced by the
    vertical force.

(b) In the case of all modes $\phi_F^{(j)}$ at the center line
    is greater than $\phi_F^{(j)}$ at station 1 - that is,
    the forces introduced at the landing gear are
    applied nearer to the nodal lines than forces
    introduced through the hull at the airplane
    center line.

The present work is of a preliminary nature and many
questions are left for further investigation.  It is evident
that the methods here presented are not restricted to the
evaluation of landing loads, since it is possible at least
theoretically to handle in the same way dynamic loads due to gun
recoil and "flak."  It must be remembered, however, that in
flight the aerodynamic forces become of primary importance
and cannot be generally neglected.  This is especially true
in the determination of dynamic loads due to gusts in which
case the flutter characteristics of the airplane must have a
preponderant effect.  It must also be kept in mind that the
possibility that the representation of the transient motion
as a superposition of natural modes is not necessarily the
best procedure in all cases.  Considering the dynamic stresses
from the standpoint of wave propagation in the elastic system
might turn out to be a more direct and significant viewpoint
in the case of high frequency transients.  This viewpoint
also eliminates the convergence difficulties mentioned above
in connection with the determination of local accelerations.
Another case where natural modes lose their significance
is that of resonance between loosely coupled parts of the
structure, in which case the vibrational energy at one loca-
tion is gradually transferred to another.

Examples shown in the present work are limited to the
wing structure under the assumptions of a symmetric landing.
Such a landing condition is exceptional. For a landplane
the degree to which the unsymmetric modes are excited by an
unsymmetric landing depends a great deal on the time inter-
val between the instant at which the left and right wheel
enter in contact with the ground. Statistical data on this
time interval can only be obtained by flight testing. No
example has been presented of an application of the proced-
ure to the evaluation of landing loads in the fuselage and
tail. However, the same methods are directly applicable to
this case, provided the natural modes of the fuselage and
tail have been determined. Data obtained during landing
tests of flying boats have shown that modes of the fuselage
and tail are excited and result in a "tail whip" effect
causing considerable dynamic overstress in the tail and aft
portion of fuselage.


Bureau of Aeronautics,
    U. S. Navy Department,
        Washington, D. C., August 10, 1944.

APPENDIX I

GENERAL MATHEMATICAL THEORY OF TRANSIENTS IN AN

UNDAMPED ELASTIC STRUCTURE

The general transient theory of linear systems with lumped or distributed parameters is well known and has been extensively developed in the case of electrical network theory (references 2 and 3). The problem of transients in airplane structures is identical in its mathematical form. In an elastic system with distributed parameters there is a space as well as a time variation in the variables. The problem may be considered with two viewpoints. The motion may be considered to be made up of a series of traveling waves, or it may be considered to be made up of a superposition of natural oscillations, in which case to be rigorous, an infinite number is required. The airplane structures problem is treated here from the standpoint of a superposition of natural oscillations.

In a transient problem of this type where maximum values occur very soon after the motion starts, the effect of damping may be justifiably neglected. The motion of an undamped elastic system may be shown to be composed of a superposition of normal modes which are orthogonal. The airplane structure vibrates in a series of normal modes when excited by a random impulse as is the case of any elastic system. These normal modes are each characterized by a certain mode shape and a certain frequency. For the airplane they are composed of coupled oscillations of the wing, fuselage, and empennage system. These calculated mode shapes and frequencies may be obtained from a ground vibration survey of the airplane.

If each normal mode shape is considered to be represented by the space function $\phi^{(i)}$ the displacement of any point of the structure at any time may be written as

$$z = \sum_{i=o}^{n} \phi^{(i)} q_i \qquad (I-1)$$

where  n  modes are considered, and where the  $q_i$  terms are regarded as generalized coordinates.

The kinetic energy of the structure is

$$T = \frac{1}{2} \int \left\{ \sum_{i=0}^{n} \phi(i) \; \dot{q}_i \right\}^2 dm \qquad (I-2)$$

Considering the  jth  generalized coordinate, and the orthoganality condition, the following equation may be written

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) = \ddot{q}_j \int \left[ \phi(j) \right]^2 dm = M_j \; \ddot{q}_j \qquad (I-3)$$

where

$$M_j = \int \left[ \phi(j) \right]^2 dm$$

The potential energy of the structure is

$$U = \frac{1}{2} \sum_{i=0}^{n} M_i \; \omega_i^2 \; q_i^2 \qquad (I-4)$$

Considering the  jth  generalized coordinate, write

$$\frac{\partial U}{\partial q_j} = M_j \; \omega_j^2 \; q_j \qquad (I-5)$$

By Lagrange's equation

$$M_j \; \ddot{q}_j + M_j \; \omega_j^2 \; q_j = Q_j \qquad (I-6)$$

where  $Q_j$  is the generalized force corresponding to the  jth mode, and is evaluated by virtual work principles.

If $\delta W_j$ is the virtual work produced when all the external forces are allowed to move through displacements corresponding to a virtual displacement $\delta q_j$, then the generalized force $Q_j$ is defined by

$$Q_j = \frac{\delta W_j}{\delta q_j} \tag{I-7}$$

The complete motion of the structure is then defined by a series of differential equations of the form

$$\ddot{q}_j + \omega_j{}^2 q_j = \frac{Q_j}{M_j}$$

where the form of the right-hand side is dependent upon the character of the applied forces.

In general, the aerodynamic applied forces on an airplane structure vary with deflection, velocity, and acceleration of the structure, and the landing reactions vary with time in a manner which is determined by experiment. In the case where the external forces are landing reactions assumed to be given functions of time, the equations governing the response are

$$\ddot{q}_j + \omega_j{}^2 q_j = \frac{Q_j(t)}{M_j} \tag{I-8}$$

This is the differential equation for the undamped motion of a simple oscillator of mass $M_j$ and natural frequency $\omega_j$, which is under the influence of an arbitrary forcing impulse $Q_j(t)$.

If $Q_j(t)$ is a unit step function $1(t)$, the response $q_j$ which is called the indicial admittance (references 3 and 4) is

$$A(t) = \frac{1}{M_j \omega_j{}^2} (1 - \cos \omega_j t) \, 1(t) \tag{I-9}$$

The response $q_j$ to any arbitrary forcing impulse $Q_j(\tau)$ may be written by the superposition theorem as

$$q_j = \frac{1}{M_j \, \omega_j} \int_0^t Q_j(\tau) \, \sin \, \omega_j (t - \tau) \, d\tau \qquad (I\text{-}10)$$

$\tau$ is a variable of integration.

When this integration with respect to $\tau$ is carried out between the limits 0 and t, a function of time results which is the time history of the deformation of the jth mode. The stress at any point in the structure in the jth mode is proportional to the deformation of the jth mode.

$$s^{(j)} = A^{(j)} \, q_j = \frac{A^{(j)}}{M_j \, \omega_j} \int_0^t Q_j(\tau) \, \sin \, \omega_j (t - \tau) d\tau \quad (I\text{-}11)$$

When the constant $A^{(j)}$ is properly chosen, equation (I-11) yields the stress time history of some particular point in the structure caused by the deformation of the jth mode. The stress $s_k^{(j)}$ in the jth mode at the kth wing station may be written as

$$s_k^{(j)} = A_k^{(j)} \, q_j \qquad (I\text{-}12)$$

The total stress at the kth wing station for n modes is obtained by superposition as

$$s_k = \sum_{i=1}^{n} A_k^{(i)} \, q_i \qquad (I\text{-}13)$$

Note on the Computation of $M_j$ for the Wing

From equation (I-3)

$$M_j = \int \left[ \phi^{(j)} \right]^2 dm$$

For the case of the wing, the mode shape is conveniently described by considering the wing deformation to be made up of a bending of the elastic axis, and a twisting about the elastic axis.  Considering the wing to be divided into k spanwise stations, the normal function describing any point on the chord of the kth station is  (see fig. I-1)

$$\phi_k^{(j)} = h_k^{(j)} + x\, \alpha_k^{(j)} \tag{I-14}$$
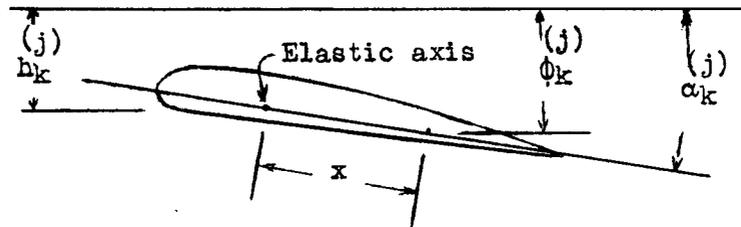


Figure I-1

In figure I-1 positive bending deflections are downward and positive pitching deflections are stalling.

Using equations (I-3) and (I-12), the following equations may be written

$$M_j = \int \left[\phi^{(j)}\right]^2 dm$$

$$M_j = \sum^k \left\{ \left[h_k^{(j)}\right]^2 m_k + \left[\alpha_k^{(j)}\right]^2 I_k + 2h_k^{(j)} \alpha_k^{(j)} S_k\right\} \tag{I-15}$$

Note on the Computation of the Generalized Force  $Q_j$

From equation (I-7)

$$Q_j = \frac{\delta W}{\delta q_j}$$

If the external force $F(t)$ is a landing reaction considered to be applied at one point to the wing structure, the virtual work may be written as

$$\delta W = F(t) \; \delta z_F$$

where $z_F$ is the value of $z$ in the direction of the applied force evaluated at the point $F$, the point of application of the landing force.

From equation (I-1)

$$z = \phi^{(1)} q_1 + \phi^{(2)} q_2 + \ldots \phi^{(n)} q_n$$

hence

$$\delta W = F(t) \left\{ \phi_F^{(1)} \; \delta q_1 + \phi_F^{(2)} \; \delta q_2 + \ldots \phi_F^{(n)} \; \delta q_n \right\}$$

where $\phi_F^{(j)}$ is the normal function evaluated at the point of application of the force $F$.

Then

$$Q_j = \frac{\delta W}{\delta q_j} = \phi_F^{(j)} \; F(t) \qquad\qquad (I-16)$$

The factor $\phi_F^{(j)}$ is a measure of the contribution of the external force to the generalized force in the $j$th mode.

## APPENDIX II

## ANALYSIS OF THE MOTION OF A SYSTEM CONSISTING OF TWO SPRING-CONNECTED MASSES EXCITED BY AN ARBITRARY FORCE
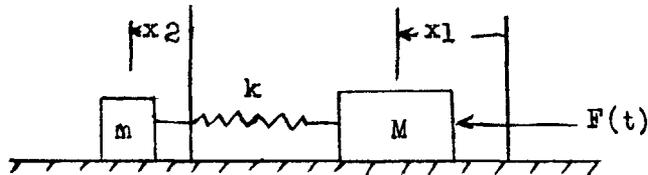


Figure II-1

It is shown below that the motions of the mass  m  are given by a superposition of the motion of the center of gravity of the system, and the motion of a simple oscillator which will be defined.  Denote by  $x_1$, $x_2$  the coordinates of the masses  M  and  m.

The differential equations defining the motion are:

$$M\ddot{x}_1 + k(x_1 - x_2) = F(t) \qquad (II-1)$$

$$m\ddot{x}_2 - k(x_1 - x_2) = 0 \qquad (II-2)$$

The motion of the center of gravity of the system is determined first.  Adding equations (II-1) and (II-2) gives

$$M\ddot{x}_1 + m\ddot{x}_2 = F(t)$$

Let  $q_0$  be the displacement of the center of gravity of the system

$$mx_2 + Mx_1 = (M + m) q_0$$

and

$$m\ddot{x}_2 + M\ddot{x}_1 = (M + m) \ddot{q}$$

Hence

$$(M + m) \ddot{q}_0 = F(t)$$

is the differential equation defining the motion of the center of gravity of the system.

The motion of the mass $m$ relative to the center of gravity of the system is determined next.

Multiplying equations (II-1) and (II-2) by $m$ and $M$, respectively, and subtracting, the following equation is obtained

$$Mm(\ddot{x}_1 - \ddot{x}_2) + k(M + m)(x_1 - x_2) = mF(t)$$

and substituting

$$q_1 = \frac{M}{M + m} (x_1 - x_2)$$

which represents the motion of $m$ relative to the center of gravity; the differential equation may be rewritten as

$$\frac{m}{M} (m + M) \ddot{q}_1 + k \left( \frac{m + M}{M} \right)^2 q_1 = \frac{m}{M} F(t) \qquad (II-4)$$

Let

$$M_1 = \frac{m}{M} (m + M)$$

$$M_1 \omega_1{}^2 = k \left( \frac{m + M}{M} \right)^2$$

$$Q_1(t) = \frac{m}{M} F(t)$$

and write

$$M_1 \ddot{q}_1 + M_1 \omega_1{}^2 q_1 = Q_1(t) \qquad (II-5)$$

This is the differential equation defining the motion of $m$ relative to the center of gravity of the system. It is also

the differential equation for a simple oscillator.  (See fig.
II-2.)
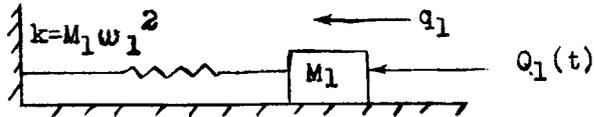


Figure II-2

The actual displacements $x_1$ and $x_2$ may be derived by
solving for $x_1$, $x_2$.

From the relations

$$mx_2 + Mx_1 = (M + m)\, q_0$$

$$(x_1 - x_2)\, \frac{M}{M + m} = q_1$$

may be found

$$x_1 = q_0 + \frac{m}{M}\, q_1$$

$$x_2 = q_0 - q_1$$

The motion may be considered as the superposition of two
configurations, one defined by $q_0$,  a rigid motion of the
system, the other defined by $q_1$ represents a configuration
in which the center of gravity remains fixed while the masses
M and m move in opposite phase with amplitudes inversely
proportional to their masses.  The equivalent system of fig-
ure II-2 represents the motion in the latter configuration.
It will be noted that the generalized mass $M_1$ may be derived
quite simply by considering the kinetic energy T in the
corresponding configuration

$$T = \frac{1}{2}\, m\dot{q}_1{}^2 + \frac{1}{2}\, M\left(\frac{m}{M}\right)^2 \dot{q}_1{}^2$$

or

$$T = \frac{1}{2} \frac{m}{M}(m + M)\ \dot{q}_1^{\ 2}$$

Equating this to the kinetic energy of the equivalent oscillation

$$T = \frac{1}{2} M_1\ \dot{q}_1^{\ 2}$$

there is found

$$M_1 = \frac{m}{M}\ (m + M)$$

Similarly $Q_1(t)$ may be derived by equating the work done by $F(t)$ in the corresponding configuration and the work done by $Q_1$ in the equivalent system.

$$Q_1(t)\ q_1 = F(t)\ \frac{m}{M}\ q_1$$

Hence

$$Q_1(t) = \frac{m}{M}\ F(t)$$

## APPENDIX III

### TRANSIENTS IN A PRISMATIC BEAM SIMPLY SUPPORTED AT THE ENDS
### WITH A FORCE APPLIED AT THE CENTER
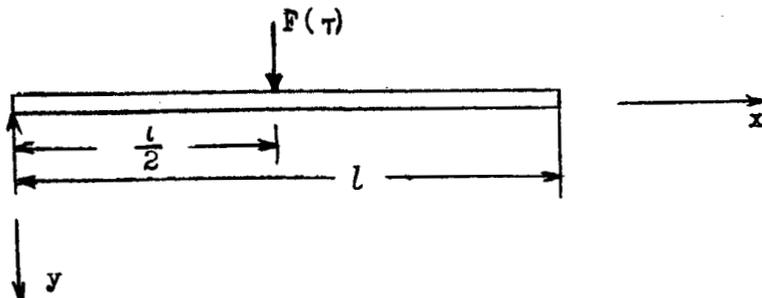


Figure III-1

The natural mode shapes and frequencies of a simply supported beam with constant cross section may be shown to be:

$$\phi^{(i)} = \sin \frac{i\pi x}{l} \tag{III-1}$$

$$\omega_i = i^2 \pi^2 \sqrt{\frac{EI}{\rho A l^4}} \tag{III-2}$$

where

E     modulus of elasticity

I     moment of inertia

$\rho$     mass per unit volume

A     cross-sectional area

$l$     length

From equation (I-1) of appendix I, the displacement of any point on the beam may be written as

$$y = \sum_{i=0}^{\infty} \phi^{(i)} q_i = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l} q_i \tag{III-3}$$

From equation (I-10) of appendix I, the generalized coordinate $q_i$ is expressed by

$$q_i = \frac{1}{M_i \omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau) \, d\tau \tag{III-4}$$

where

$$M_i = \int \left[ \phi^{(i)} \right]^2 dm = \rho A \int_0^l \sin^2 \frac{i\pi x}{l} \, dx = \frac{l\rho A}{2} \tag{III-5}$$

and

$$Q_1(\tau) = \phi \frac{(1)}{F} F(\tau) = \sin \frac{1\pi}{2} F(\tau) \qquad \text{(III-6)}$$

Substituting equations (III-4), (III-5), and (III-6) into equation (III-3) yields,

$$y = \frac{2}{l\rho A} \sum_{i=1}^{\infty} \frac{1}{\omega_1} \sin \frac{i\pi}{2} \sin \frac{i\pi x}{l} \int_0^t F(\tau) \sin \omega_1 (t - \tau) d\tau \quad \text{(III-7)}$$

When $F(\tau)$ is a unit step, the response is:

$$y = \frac{2}{l\rho A} \sum_{i=1}^{\infty} \frac{1}{\omega_1{}^2} \sin \frac{i\pi}{2} \sin \frac{i\pi x}{l} (1 - \cos \omega_1 t) \qquad \text{(III-8)}$$

When $F(\tau)$ is a unit impulse [1], the response is obtained by a time differentiation of equation (III-8). The response to a unit impulse is:

$$y = \frac{2}{l\rho A} \sum_{i=1}^{\infty} \frac{1}{\omega_1} \sin \frac{i\pi}{2} \sin \frac{i\pi x}{l} \sin \omega_1 t \qquad \text{(III-9)}$$

Since $M = -EI \frac{\partial^2 y}{\partial x^2}$, the moment in the beam at any time after being subjected to a unit step force is,

$$M = \frac{2l}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \sin \frac{i\pi}{2} \sin \frac{i\pi x}{l} (1 - \cos \omega_1 t) \qquad \text{(III-10)}$$

The moment in the beam at any time after being subjected to a unit impulse is,

---

[1] A unit impulse is an infinite force applied during a vanishingly small interval of time in such a way that the time integral $\int F(\tau) d\tau = 1$ (reference 4).

$$M = \frac{2}{l}\sqrt{\frac{EI}{\rho A}}\sum_{i=1}^{\infty}\sin\frac{i\pi}{2}\sin\frac{i\pi x}{l}\sin\omega_1 t \qquad (III\text{-}11)$$

The shear in the beam at any time after being subjected to a unit step force is,

$$\frac{\partial M}{\partial x} = \frac{2}{\pi}\sum_{i=1}^{\infty}\frac{1}{i}\sin\frac{i\pi}{2}\cos\frac{i\pi x}{l}(1 - \cos\omega_1 t) \qquad (III\text{-}12)$$

The shear in the beam at any time after being subjected to a unit impulse is,

$$\frac{\partial M}{\partial x} = \frac{2\pi}{l^2}\sqrt{\frac{EI}{\rho A}}\sum_{i=1}^{\infty} i\sin\frac{i\pi}{2}\cos\frac{i\pi x}{l}\sin\omega_1 t \qquad (III\text{-}13)$$

A study of equations (III-10), (III-11), (III-12), and (III-13) indicates the shortcomings of the procedure when adding contributions of each of the modes to moment and shear for a simple prismatic beam with a force at the center. This is illustrated in table III-1. Limitations of a similar nature are encountered in all problems where the motion is described as a superposition of modes.

TABLE III-1

| | UNIT STEP | | UNIT IMPULSE |
|---|---|---|---|
| | Percent of 1st mode | | Percent of 1st mode |
| Moment | | Moment | |
| 1st mode | 100 | 1st mode | 100 |
| 2nd mode | 11.1 | 2nd mode | 100 |
| 3rd mode | 4 | 3rd mode | 100 |
| 4th mode | 2.04 | 4th mode | 100 |
| Shear | | Shear | |
| 1st mode | 100 | 1st mode | 100 |
| 2nd mode | 33.3 | 2nd mode | 300 |
| 3rd mode | 20 | 3rd mode | 500 |
| 4th mode | 14.3 | 4th mode | 700 |

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