LOCALIZED RADIAL SOLUTIONS FOR NONLINEAR P-LAPLACIAN EQUATION IN $\mathbb{R}^{\mathbb{N}}$

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We establish the existence of radial solutions to the p-laplacian equation $\Delta_p u + f(u)=0$ in $\mathbf{R}^{N}$, where f behaves like $|u|^{q-1} u$ when u is large and $f(u) < 0$ for small positive u. We show that for each nonnegative integer n, there is a localized solution u which has exactly n zeros. Also, we look for radial solutions of a superlinear dirichlet problem in a ball. We show that for each nonnegative integer n, there is a solution u which has exactly n zeros. Here we give an alternate proof to that which was given by Castro and Kurepa.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Discussion of the Problems, Methods and Previous Results

In this dissertation we look for radial solutions for a p-Laplacian equation and a nonlinear laplacian equation.

There has been extensive study of the partial differential equation

$$\Delta u + f(u) = 0 \text{ in } \Omega \tag{1}$$

$$u = 0 \text{ on } \partial\Omega \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$ and $f(u)$ is a nonlinear function.

The standard way to proceed is to attempt to find critical points of the nonlinear functional

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)\, dx \tag{3}$$

where $F(u) = \int_0^u f(t)\, dt$ over an appropriate function space. Assuming there is a critical point u_0 with $u_0 = 0$ on $\partial\Omega$ then for any smooth function v with $v = 0$ on $\partial\Omega$ we obtain that

$$0 = \frac{d}{dt}\left[E(u_0 + tv)\right]|_{t=t_0} = \int_\Omega \nabla u_0 \cdot \nabla v - f(u)v\, dx. \tag{4}$$

Applying Green's formula we therefore obtain

$$0 = \int_\Omega [\Delta u_0 + f(u_0)]v\, dx \tag{5}$$

for all v hence it follows that u_0 satisfies (1)-(2). The usual function space used is $W_0^{1,2}(\Omega)$: the closure of $C_0^\infty(\Omega)$ with respect to

$$|u|_{W_0^{1,2}(\Omega)} = |\nabla u|_{L^2(\Omega)}.$$

There are several challenges which arise in attempting to carry this procedure out. First, one needs to have some way of finding a critical point of (3). One standard way of doing this requires applying the Mountain Pass Lemma to the functional $E(u)$ (see [15]). A standard assumption on $f(u)$ which guarantees that the assumptions of the Mountain Pass Lemma apply is that $f(u)$ grows slower than u^p where $p < \frac{N+2}{N-2}$. For this reason $\frac{N+2}{N-2}$ is called the "critical exponent" of the Laplacian operator. See [4] for the Mountain Pass Lemma and the critical exponent.

In fact, for any convex region Ω, the critical and supercritical growth of f is a real obstruction to existence of solutions of (1)-(2). The *Pohozaev Identity* (see [3]) states that any solution of (1)-(2) must satisfy:

$$\int_{\Omega} [NF(u) - \frac{N-2}{2} u f(u)]\, dx = \int_{\partial\Omega} \frac{1}{2} |\nabla u|^2 (x \cdot n)\, dS$$

where n is the outward unit normal to $\partial\Omega$. In particular, if $f(u) = |u|^{p-1}u$ and $p \geq \frac{N+2}{N-2}$ and Ω is a convex region then there are *no nontrivial solutions* of (1)-(2).

However, assuming that a critical point of (3) exists, one then needs to establish the regularity of the critical point. That is, one needs to show that if $u \in W_0^{1,2}(\Omega)$ and (4) holds then in fact $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and is equal to zero on $\partial\Omega$. Once this has been established, one can apply (5) and obtain finally a solution of (1)-(2).

In a groundbreaking paper in 1979, B. Gidas, W. Ni, and L. Nirenberg [7] proved that if Ω is a ball then all positive solutions of (1)-(2) are *spherically symmetric*. That is, $u(x) = u(|x|)$ so that $u(x)$ only depends on the distance of x to the origin. This remarkable result allows one to reduce the study of positive solutions of (1)-(2) when Ω is a ball to the corresponding ordinary differential equation obtained by substituting $u(x) = u(|x|) = u(r)$ where $r = |x|$. This yields:

$$u'' + \frac{N-1}{r} u' + f(u) = 0 \text{ for } 0 < r < R \tag{6}$$

$$u(R) = 0. \tag{7}$$

A variation of this problem is when (7) is replaced by

$$\lim_{r\to\infty} u(r) = 0. \tag{8}$$

Strauss [17] and Berestycki and Lions [1] have proved the existence of infinitely many radially symmetric solutions of (6)-(7) by variational methods when $f(u)$ is odd. Then it was an open question as to whether solutions exist with prescribed number of zeros. Jones and Küpper [10] addressed this question using a dynamical systems approach and an application of the Conley index.

In general, the set of radial solutions has been extensively studied. See Haraux and Weissler [8], Lions [11], McLeod and Serrin [12], and Peletier and Serrin [14].

A standard way to proceed is to use the so-called "shooting" method. That is, one first solves the initial value problem (6) along with

$$u(0) = d \tag{9}$$

and

$$u'(0) = 0. \tag{10}$$

One then varies the parameter d to hopefully obtain a solution satisfying (7) or (8).

One of the well-known papers proving existence of solutions of (6)-(7) was by A. Castro and A. Kurepa [3] in which they proved the existence of infinitely many solutions of (6)-(7) for nonlinearities which are superlinear (i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$) and subcritical. They use a delicate phase plane analysis to establish the existence of solutions of (6)-(7) with many zeros. In this thesis one of the things we present is an alternate proof of this result. The proof presented here seems more natural and more easily applicable to other problems. We feel confident that this method will allow us to prove a result similar to (6)-(7) for the p-Laplacian and the result proved in McLeod, Troy and Weissler [13]. There are numerous technical facts in the Castro and Kurepa paper which can be difficult to follow and the proof that we provide seems much more straightforward. We simply divide the domain into two regions: one where $|u|$ is large and the other when $|u|$ is small. When $|u|$ is small we can approximate u by the solution of a well-understood linear problem and when $|u|$ is large we can easily estimate the size of u.

In mathematics, alternate proofs have always been significant as they can give a simpler proof which indeed can help others to appreciate and also bring more people to the field. For example, in 1799 Gauss gave his first proof to the fundamental theorem of algebra and also stated his objections to other proofs that existed; this was his doctoral thesis. In fact Gauss also gave two more different proofs for the same theorem. In mathematics there is also a history that conjectures are made by someone and proved by someone else. For example, the prime number theorem was conjectured by Legendre in 1796. Around 1850, Chebyshev proved it using the zeta function. However, in 1949 Selberg gave a more elementary proof and around the same time independently Paul Erdos also contributed an elementary proof.

A natural extension of (1)-(2) is

$$\Delta_p u + f(u) = 0 \text{ in } \Omega \tag{11}$$

$$u = 0 \text{ on } \partial\Omega \tag{12}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian of u. Solutions of (11)-(12) arise as critical points of

$$E(u) = \int_\Omega \frac{p-1}{p} |\nabla u|^p - F(u)\, dx. \tag{13}$$

Here the usual function space is $W_0^{1,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ with respect to

$$|u|_{W_0^{1,p}(\Omega)} = |\nabla u|_{L^p(\Omega)}.$$

Equation (11) has been studied in different settings. Gazzola, Serrin and Tang [6] have proved existence of radial solutions to (11) with Dirichlet and Neumann boundary conditions. Calzolari, Filippucci and Pucci [2] have proved existence of radial solutions for the p-Laplacian with weights.

Here we look for radial solutions of (11)-(12) in $\mathbb{R}^N$. That is we attempt to solve

$$|u'|^{p-2}\left[(p-1)u'' + \frac{N-1}{r}u'\right] + f(u) = 0$$

and

$$\lim_{r\to\infty} u(r) = 0$$

where $f(u)$ grows like $|u|^{q-1}u$ with $1 < p < q+1 < \frac{Np}{N-p}$.

This is a generalization of [13] where it is assumed that $p = 2$. Again we use the shooting method and first solve:

$$|u'|^{p-2}\left((p-1)u'' + \frac{N-1}{r}u'\right) + f(u) = 0$$

$$u(0) = d$$

$$u'(0) = 0.$$

Then we vary the parameter d so that $\lim_{r\to\infty} u(r) = 0$.

CHAPTER 2

THE PROBLEMS

2.1. Discussion of the Problems, Methods and Previous Results

In this dissertation we talk about two results. In the first result we look for localized radial solutions of nonlinear p-Laplacian equation in $\mathbb{R}^N$. And in the second result we look for infinitely many radially symmetric solutions for a superlinear dirichlet problem in a ball.

2.1.1. *Result 1*

Here is the introduction and motivation to the first result:

In this paper we look for solutions $u : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ of the nonlinear partial differential equation

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(u) = 0, \tag{14}$$

$$\lim_{|x| \to \infty} u(x) = 0, \tag{15}$$

with $1 < p < N$. We also assume $f(u)$ behaves like $|u|^{q-1}u$ where u is large and $f(u) < 0$ for small positive u.

Motivation: When $p = 2$ then (14) is

$$\Delta u + f(u) = 0.$$

McLeod, Troy and Weissler studied the radial solutions of the above mentioned equation in [13]. In this paper they made a remark that their result could be extended to the p-Laplacian. In this paper we show that their conjecture is true. Also, Castro and Kurepa studied

$$\Delta u + g(u) = 0,$$

subject to Dirichlet boundary conditions on a ball in $\mathbb{R}^{\mathbb{N}}$, where g is superlinear in [3].

We assume that the function f satisfies the following hypotheses:

(H1) f is an odd locally Lipschitz continuous function,

(H2) $f(u) < 0$ for $0 < u < \epsilon_1$ for some $\epsilon_1 > 0$,

(H3) $f(u) = |u|^{q-1}u + g(u)$ with $\dfrac{g(|u|)}{|u|^q} \to 0$ as $|u| \to \infty$ where $1 < p < q+1 < \frac{Np}{N-p}$.

From (H2) and (H3) we see that $f(u)$ has at least one positive zero.

(H4) Let α be the least positive zero of f and β be the greatest positive zero of f,

(H5) Let $F(u) \equiv \int_0^u f(s)ds$ with exactly one positive zero γ, with $\gamma > \beta$,

(H6) If $p > 2$ we also assume for some $\epsilon_2 > 0$

$$\int_0^{\epsilon_2} \frac{1}{\sqrt[p]{|F(u)|}} du = \infty.$$

We assume that $u(x) = u(|x|)$ and let $r = |x|$. In this case (14)-(15) becomes the nonlinear ordinary differential equation

$$\frac{1}{r^{N-1}}(r^{N-1}|u'|^{p-2}u')' + f(u) = |u'|^{p-2}\left((p-1)u'' + \frac{N-1}{r}u'\right) + f(u) = 0 \tag{16}$$

for $0 < r < \infty$, with

$$\lim_{r\to\infty} u(r) = 0, \lim_{r\to 0^+} u'(r) = 0. \tag{17}$$

We would like to find $\mathcal{C}^2$ solutions of (16)-(17) but we will see later that this is not always possible (see the proof of Lemma (3.6)). However multiplying (16) by r^{N-1} and integrating gives

$$r^{N-1}|u'|^{p-2}u' = -\int_0^r t^{N-1}f(u)dt. \tag{18}$$

Instead of looking for solutions of (16)-(17) in $\mathcal{C}^2$ we look for solutions of (17)-(18) in $\mathcal{C}^1$.

Main Theorem

Let the nonlinearity f have the properties (H1)-(H6), and let n be a nonnegative integer. Then there is a solution $u \in \mathcal{C}^1[0,\infty)$ of (17)-(18) such that u has exactly n zeros.

The technique used to solve (17)-(18) is the shooting method. That is, we first solve the initial value problem

$$r^{N-1}|u'|^{p-2}u' = -\int_0^r s^{N-1}f(u(s))ds$$

$$u(0) = d \geq 0.$$

By varying d appropriately, we attempt to find a d such that $u(r,d)$ has exactly n zeros and u solves (18). In chapter 3 section 1, we establish the existence of solutions of this initial value problem by the contraction mapping principle. In section 2.2 we show why we look for $\mathcal{C}^1$ solutions instead of $\mathcal{C}^2$ solutions. In sections 3.2.1 and 3.2.2 we prove some technical lemmas. In section 3.3 we show the uniqueness property of the initial value problem which we use several times. In chapter 4 section 1, we see that after a rescaling of u we get a family of functions $\{u_\lambda\}$, which converges to the solution of

$$-r^{N-1}|v'|^{p-2}v' = \int_0^r s^{N-1}|v|^{q-1}v ds,$$

$$v(0) = 1,$$

where $1 < p < q+1 < \frac{Np}{N-p}$. We will then show that v has infinitely many zeros which will imply that there are solutions, u, of (18) with any given number of zeros. In chapter 5 we prove our Main Theorem.

Note: From (H3) and (H5) we see that

$$F(u) = \frac{1}{q+1}|u|^{q+1} + G(u), \tag{19}$$

where $G(u) = \int_0^u g(s)ds$. Dividing both sides by $|u|^{q+1}$ and taking the limit as $|u| \to \infty$ gives

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^{q+1}} = \lim_{|u|\to\infty} \left(\frac{1}{q+1} + \frac{G(u)}{|u|^{q+1}} \right). \tag{20}$$

Using L'Hopital's rule and (H3) we see that

$$\lim_{u\to\infty} \frac{G(u)}{|u|^{q+1}} = 0. \tag{21}$$

Thus, we have

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^{q+1}} = \frac{1}{q+1}.$$

This implies that $F(u) \geq 0$ for $|u|$ sufficiently large, so $F(u) \geq 0$ for $|u| \geq M$. Also since F is continuous on the compact set $[-M, M]$ we see that F is bounded below and there is a

$-L < 0$ such that

$$F(u) \geq -L \tag{22}$$

for all u.

Note: When $1 < p \leq 2$, then assumption (H6) also holds. This follows from (H1). The details of this are as follows: since f is locally Lipschitz and since $f(0) = 0$ we have

$$|f(u)| = |f(u) - f(0)| \leq c|u - 0| = c|u|$$

for $|u| < \epsilon_2$ for some $\epsilon_2 > 0$, and where $c > 0$ is a Lipschitz constant for f in a neighborhood of $u = 0$. Integrating on $(0, u)$ where $0 \leq u \leq \epsilon_2$ gives:

$$-\int_0^u ctdt \leq \int_0^u f(t)dt \leq \int_0^u ctdt.$$

Thus,

$$\frac{-cu^2}{2} \leq F(u) \leq \frac{cu^2}{2}$$

for $|u| < \epsilon_2$. So, $|F(u)| \leq \dfrac{cu^2}{2}$ for $|u| \leq \epsilon_2$. Thus, $|F(u)|^{\frac{1}{p}} \leq \left(\dfrac{c}{2}\right)^{\frac{1}{p}} u^{\frac{2}{p}}$ for $|u| < \epsilon_2$. Hence,

$$\int_0^{\epsilon_2} \frac{1}{|F(u)|^{\frac{1}{p}}} du \geq \left(\frac{2}{c}\right)^{\frac{1}{p}} \int_0^{\epsilon_2} \frac{1}{u^{\frac{2}{p}}} du = \infty, \text{ if } 1 < p \leq 2.$$

2.1.2. *Result 2*

Here is the introduction and motivation for result 2: We look for radial solutions $u : \mathbb{R}^N \to \mathbb{R}$ of the partial differential equation

$$\begin{cases} \Delta u + f(u) = 0, x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases} \tag{23}$$

for $N \geq 2$ and where Ω is the ball of radius T centered at the origin in $\mathbb{R}^N$, Δ is the Laplacian operator, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivation: A. Castro and A. Kurepa proved existence of solutions of (23) for a wide variety of nonlinearities See [3]. In this paper we give an alternate proof of this result.

We assume that the function f satisfies the following hypotheses:

(H1') $f(0) = 0$, f is a locally lipschitz continuous function and increasing for large u,

(H2')

$$\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty,$$

(H3') Let $F(u) = \int_0^u f(s)ds$,

(H4') There exists a k with $0 < k \leq 1$, such that

$$\lim_{u\to\infty} \left(NF(ku) - \frac{(N-2)}{2} uf(u) \right) \left(\frac{u}{f(u)} \right)^{\frac{N}{2}} = \infty$$

(H4*) There exists a k with $0 < k \leq 1$, such that

$$\lim_{u\to-\infty} \left(NF(ku) - \frac{(N-2)}{2} uf(u) \right) \left(\frac{u}{f(u)} \right)^{\frac{N}{2}} = \infty$$

(H5') There exists an $\mathbf{M} > 0$ such that

$$NF(u) - \frac{N-2}{2} uf(u) > -\mathbf{M}$$

for all u.

Note: It follows from (H2') and by L'Hopitals Rule that

$$\lim_{|u|\to\infty} \frac{F(u)}{u^2} = \infty. \tag{24}$$

We assume that $u(x) = u(|x|)$ and let $r = |x|$. In this case (23) becomes the nonlinear ordinary differential equation

$$u'' + \frac{N-1}{r} u' + f(u) = 0, \text{ for } 0 < r < T, \tag{25}$$

$$u'(0) = 0, u(T) = 0. \tag{26}$$

Main Theorem: If (H1')-(H5') are satisfied then (23) has infinitely many radially symmetric solutions with $u(0) > 0$. If in place of (H4') we have (H4*) then (23) has infinitely many radially symmetric solutions with $u(0) < 0$.

Again we solve the initial value problem

$$u'' + \frac{N-1}{r} u' + f(u) = 0, \text{ for } 0 < r < T$$

$$u(0) = d$$

$$u'(0) = 0.$$

Then we try to find appropriate values of d so that $u(T) = 0$.

In chapter 6 section 1 we the solve the initial value problem and also show that energy increases as d increases. In chapter 7 section 1 we use Bessel's equation. In section 7.2 we show that u has a zero. In section 7.3 we prove an important lemma. In chapter 8 we prove the main result.

CHAPTER 3

EXISTENCE OF SOLUTIONS OF THE INITIAL VALUE PROBLEM

3.1. Existence by Contraction Mapping

In this section we use contraction mapping principle to show the existence of the a solution to the initial value problem

$$r^{N-1}|u'|^{p-2}u' = -\int_0^r s^{N-1} f(u(s))ds, \tag{27}$$

with

$$u(0) = d \geq 0. \tag{28}$$

We define $\Phi_p(x) = |x|^{p-2}x$ for $x \in \mathbb{R}$ and $p > 1$. Note that the inverse of $\Phi_p(x)$ is $\Phi_{p'}(x)$ where $\frac{1}{p} + \frac{1}{p'} = 1$, that is $p' = \frac{p}{p-1}$. Note that both Φ_p and $\Phi_{p'}$ are odd for every p. Now dividing (27) by r^{N-1}, gives

$$|u'|^{p-2}u' = \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u(s))ds. \tag{29}$$

Using the definition of Φ_p, we get

$$\Phi_p(u') = \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u(s))ds.$$

Now applying $\Phi_{p'}$ on both sides, leads to

$$u' = -\Phi_{p'}\left(\frac{1}{r^{N-1}} \int_0^r s^{N-1} f(u(s))ds\right).$$

Integrating this again on $(0, r)$ and using the initial condition $u(0) = d$ gives

$$u = d - \int_0^r \Phi_{p'}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s))ds\right) dt.$$

We will solve (27)-(28) by finding fixed points of

$$Tu = d - \int_0^r \Phi_{p'}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s))ds\right) dt. \tag{30}$$

Note that if $f(d) = 0$, then $u \equiv d$ is a solution of (27)-(28). So, we now assume that

$$f(d) \neq 0. \tag{31}$$

Let $\mathbf{B}_R^\epsilon(d) = \{u \in \mathcal{C}[0,\epsilon], ||u - d|| \leq R\}$ for ϵ and R small enough and where $||\cdot||$ is the supremum norm, and $\mathcal{C}[0,\epsilon]$ is the set of continuous functions on $[0,\epsilon]$.

LEMMA 3.1. *Let $f(d) \neq 0$. Then $T : \mathbf{B}_R^\epsilon(d) \to \mathbf{B}_R^\epsilon(d)$ for ϵ and R small enough.*

PROOF. Let $u \in \mathbf{B}_R^\epsilon(d)$ and $R > 0$, then by (30) we have

$$|Tu - d| \leq \int_0^r \left| \Phi_{p'} \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s)) ds \right) \right| dt.$$

Since f is a continuous function on $[\frac{d}{2}, \frac{3d}{2}]$, there exists an M such that $|f| \leq M$. So, on $[0,\epsilon]$ and for ϵ small enough we have the following

$$|Tu - d| \leq \int_0^r \left[\frac{1}{t^{N-1}} \int_0^t s^{N-1} M ds \right]^{\frac{1}{p-1}} dt$$

$$= \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} \left[\int_0^t s^{N-1} M ds \right]^{\frac{1}{p-1}} dt$$

$$= \left(\frac{M}{N} \right)^{\frac{1}{p-1}} \int_0^r \frac{1}{t^{\frac{N-1}{p-1}}} t^{\frac{N}{p-1}} dt$$

$$= \left(\frac{M}{N} \right)^{\frac{1}{p-1}} \int_0^r t^{\frac{1}{p-1}} dt$$

$$= \frac{p-1}{p} \left(\frac{M}{N} \right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}}$$

$$\leq \frac{p-1}{p} \left(\frac{M}{N} \right)^{\frac{1}{p-1}} \epsilon^{\frac{p}{p-1}}$$

$$< R.$$

Therefore, $T : \mathbf{B}_R^\epsilon(d) \to \mathbf{B}_R^\epsilon(d)$ for ϵ and R small enough. $\square$

LEMMA 3.2. *Let $f(d) \neq 0$. Then $T : \mathbf{B}_R^\epsilon(d) \to \mathbf{B}_R^\epsilon(d)$ is a contraction mapping for R and ϵ chosen small enough.*

PROOF. Since $f(d) \neq 0$, and $d \geq 0$ then $d > 0$. Let u, $v \in \mathbf{B}_R^\epsilon(d)$ and choose R so that $0 < R < \frac{d}{2}$. Also,

$$Tu - Tv = -\left[\int_0^r \Phi_{p'}(X(u))dt - \int_0^r \Phi_{p'}(X(v))dt\right], \tag{32}$$

where

$$X(u) = \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s))ds. \tag{33}$$

Note $p' = \frac{p}{p-1}$, $p' - 1 = \frac{1}{p-1}$ and $p' - 2 = \frac{2-p}{p-1}$. For a fixed t, by the mean value theorem we have

$$\Phi_{p'}(X(u)) - \Phi_{p'}(X(v)) = \Phi'_{p'}(c)(X(u) - X(v)), \tag{34}$$

where

$$c = \frac{1}{t^{N-1}} \int_0^t s^{N-1}[\lambda f(u) + (1-\lambda) f(v)]ds$$

for some $0 < \lambda < 1$, and where

$$\Phi'_{p'}(c) = (p'-1)|c|^{p'-2} = \frac{1}{p-1}|c|^{\frac{2-p}{p-1}}. \tag{35}$$

Thus,

$$\Phi'_{p'}(c) = \frac{1}{p-1}\left|\frac{1}{t^{N-1}} \int_0^r s^{N-1}(\lambda f(u) + (1-\lambda) f(v))ds\right|^{\frac{2-p}{p-1}}$$

$$\Phi'_{p'}(c) = \frac{1}{(p-1)t^{\frac{(N-1)(2-p)}{p-1}}} \left|\int_0^t s^{N-1}|\lambda f(u) + (1-\lambda) f(v)|ds\right|^{\frac{2-p}{p-1}}. \tag{36}$$

We now estimate

$$\int_0^t s^{N-1}|\lambda f(u) + (1-\lambda) f(v)|ds.$$

Case (i): $1 < p \leq 2$. Since $|f| \leq M$ this gives the following

$$\int_0^t s^{N-1}|\lambda f(u) + (1-\lambda) f(v)|ds \leq \int_0^t M s^{N-1} ds = \frac{Mt^N}{N}.$$

For $1 < p \leq 2$, we have

$$\left[\int_0^t s^{N-1}|\lambda f(u) + (1-\lambda)f(v)|ds\right]^{\frac{2-p}{p-1}} \leq \left[\frac{Mt^N}{N}\right]^{\frac{2-p}{p-1}} = C_1 t^{\frac{N(2-p)}{p-1}}, \tag{37}$$

where $C_1 = (\frac{M}{N})^{\frac{2-p}{p-1}}$.

Case (ii): $2 < p$. Since f is continuous and $f(d) \neq 0$ (by assumption), let $\epsilon = \frac{|f(d)|}{2}$. Then there is a $\delta > 0$ such that for every y with $|y - d| < \delta$, $|f(y) - f(d)| < \epsilon = \frac{|f(d)|}{2}$.

By the triangular inequality it follows that

$$|f(d)| \leq |f(y)| + |f(d) - f(y)|,$$

and since $|f(d) - f(y)| \leq \frac{|f(d)|}{2}$ this gives

$$|f(d)| \leq |f(y)| + \frac{|f(d)|}{2}.$$

Thus, we have

$$0 < \frac{|f(d)|}{2} \leq |f(y)| \tag{38}$$

for all y with $|y - d| < \delta$. Note now that if $|u - d| < \delta$ and $|v - d| < \delta$, then both $f(u)$ and $f(v)$ are of the same sign; for if $f(u) > 0$ and $f(v) < 0$ then there is a w between u and v such that $|w - d| < \delta$ and $f(w) = 0$ which contradicts (38). Now using (38) and the fact that $f(u)$ and $f(v)$ are of the same sign we have the following estimate:

$$\begin{aligned}|\lambda f(u) + (1-\lambda)f(v)| &= \lambda|f(u)| + (1-\lambda)|f(v)| \\ &\geq \lambda\frac{|f(d)|}{2} + (1-\lambda)\frac{|f(d)|}{2} \\ &= \frac{|f(d)|}{2}.\end{aligned}$$

For $p > 2$ we have

$$\begin{aligned}\left[\int_0^t s^{N-1}|\lambda f(u) + (1-\lambda)f(v))|ds\right]^{\frac{2-p}{p-1}} &\leq \left[\int_0^t s^{N-1}\frac{|f(d)|}{2}\right]^{\frac{2-p}{p-1}} \\ &= \left(\frac{|f(d)|}{2}\right)^{\frac{2-p}{p-1}}\left(\frac{t^N}{N}\right)^{\frac{2-p}{p-1}}.\end{aligned}$$

So,

$$\left[\int_0^t s^{N-1}|\lambda f(u)+(1-\lambda)f(v))|ds\right]^{\frac{2-p}{p-1}} \le C_2 t^{\frac{N(2-p)}{p-1}}, \tag{39}$$

where $C_2 = (\frac{|f(d)|}{2N})^{\frac{2-p}{p-1}}$.

Now plugging (37) and (39) into (36) we get

$$\Phi'_{p'}(c) \le \frac{1}{p-1}\frac{1}{t^{\frac{(N-1)(2-p)}{p-1}}}Ct^{\frac{N(2-p)}{p-1}} = \frac{Ct^{\frac{2-p}{p-1}}}{p-1}, \tag{40}$$

where $C = C_1$, if $1 < p \le 2$ and $C = C_2$, if $p > 2$. Now using (40) in (34), we have

$$|\Phi_{p'}(X(u)) - \Phi_{p'}(X(v))| \le \frac{C}{p-1}t^{\frac{(2-p)}{p-1}}|X(u)-X(v)|. \tag{41}$$

Then by (33), it follows that

$$|X(u)-X(v)| \le \frac{1}{t^{N-1}}\int_0^t s^{N-1}|f(u)-f(v)|ds.$$

By (H1), f is locally Lipschitz, therefore $|f(u)-f(v)| \le C_3||u-v||$, where C_3 is the Lipschitz constant for f on $[\frac{d}{2}, \frac{3d}{2}]$. Thus,

$$\begin{aligned}|X(u)-X(v)| &\le \frac{1}{t^{N-1}}\int_0^t s^{N-1}C_3||u-v||ds\\ &= \frac{C_3||u-v||t^N}{Nt^{N-1}}\\ &= \frac{C_3||u-v||t}{N}.\end{aligned}$$

Utilizing this in (41), we have

$$\begin{aligned}|\Phi_{p'}(X(u)) - \Phi_{p'}(X(v))| &\le \frac{CC_3t^{\frac{2-p}{p-1}}t||u-v||}{(p-1)N}\\ &= \frac{Kt^{\frac{1}{p-1}}||u-v||}{(p-1)N}\end{aligned}$$

where $K = CC_3$. Then by (32) and the above inequality, we get

$$\begin{aligned}||T(u)-T(v)|| &\le \int_0^r \frac{Kt^{\frac{1}{p-1}}||u-v||}{(p-1)N}dt\\ &= \frac{K||u-v||}{(p-1)N}\int_0^r t^{\frac{1}{p-1}}dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{Kr^{\frac{p}{p-1}}||u-v||(p-1)}{(p-1)Np} \\
&= \frac{Kr^{\frac{p}{p-1}}||u-v||}{pN} \\
&\le \frac{K||u-v||\epsilon^{\frac{p}{p-1}}}{pN} \\
&= \omega||u-v||,
\end{aligned}
$$

where $\omega = \frac{K\epsilon^{\frac{p}{p-1}}}{pN} < 1$ for ϵ small enough. Therefore, T is a contraction mapping on $\mathbf{B}_R^{\epsilon}(d)$. $\square$

Now by the contraction mapping principle, T has a unique fixed point $u \in \mathbf{B}_R^{\epsilon}(d)$ for $R > 0$ and $\epsilon > 0$ small enough. Hence, $u \in \mathcal{C}[0, \epsilon]$ and

$$
u = T(u) = d - \int_0^r \Phi_{p'}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s))ds\right) dt. \tag{42}
$$

Now we show that u is in $\mathcal{C}^1$ on a small interval

LEMMA 3.3.

$$
\lim_{t\to 0} \frac{1}{t^{N-1}} \left(\int_0^t s^{N-1} f(u(s))ds\right) = 0.
$$

PROOF. Since f is a continuous function and so is bounded on say $[0, 1]$, there exists an $M > 0$ such that $|f| \le M$ on $[0, 1]$. Now consider

$$
\begin{aligned}
\left|\frac{1}{t^{N-1}} \left(\int_0^t s^{N-1} f(u(s))ds\right)\right| &\le \frac{1}{t^{N-1}} \left(\int_0^t |s^{N-1} f(u(s))ds|\right) \\
&\le \frac{1}{t^{N-1}} \left(\int_0^t s^{N-1} M ds\right) \\
&= \frac{tM}{N}.
\end{aligned}
$$

Clearly, $\dfrac{tM}{N} \to 0$ as $t \to 0$ and hence the lemma follows. $\square$

Note Hence, $\Phi_{p'}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} f(u(s))ds\right)$ is continuous on $[0, \epsilon]$. So by (27), u' is defined and is continuous on $(0, \epsilon)$.

Lemma 3.4.

$$u'(0) = 0.$$

Proof. By definition,

$$u'(0) = \lim_{h\to 0} \frac{u(h)-u(0)}{h} = \lim_{h\to 0} \frac{\int_0^h -\Phi_{p'}(\frac{1}{t^{N-1}}\int_0^t s^{N-1}f(u(s))ds)dt}{h}.$$

Now we consider

$$\begin{aligned}
\left|\frac{\int_0^h \Phi_{p'}\left(\frac{1}{t^{N-1}}\left(\int_0^t s^{N-1}f(u(s))ds\right)dt\right)}{h}\right| &\leq \frac{\int_0^h (\frac{1}{t^{N-1}}|\int_0^t s^{N-1}f(u(s))ds|)^{\frac{1}{p-1}}dt}{h} \\
&\leq \frac{1}{h}\int_0^h \frac{1}{t^{\frac{N-1}{p-1}}}\left(\frac{M}{N}\right)^{\frac{1}{p-1}} t^{\frac{N}{p-1}}dt \\
&= \frac{1}{h}\int_0^h \left(\frac{M}{N}\right)^{\frac{1}{p-1}} t^{\frac{1}{p-1}}dt \\
&= \frac{p-1}{p}\left(\frac{M}{N}\right)^{\frac{1}{p-1}} \frac{h^{\frac{p}{p-1}}}{h} \\
&= \frac{p-1}{p}\left(\frac{M}{N}\right)^{\frac{1}{p-1}} h^{\frac{1}{p-1}}.
\end{aligned}$$

Clearly, $\frac{(p-1)}{p}\left(\frac{M}{N}\right)^{\frac{1}{p-1}} h^{\frac{1}{p-1}} \to 0$ as $h \to 0$ and hence the result follows. □

Lemma 3.5.

$$\lim_{r\to 0} u'(r) = 0.$$

Proof. Taking absolute values and taking the limit as $r \to 0$ in equation (29) we get

$$\lim_{r\to 0} |u'|^{p-1} = \lim_{r\to 0}\left|\frac{-1}{r^{N-1}}\int_0^r s^{N-1}f(u(s))ds\right|$$

and from Lemma (3.3) we have that the right hand side is 0. Thus, $\lim_{r\to 0} u'(r) = 0$. □

Thus, $u \in \mathcal{C}^1[0,\epsilon]$.

In the following section we explain why we aim at solutions of (17)-(18) instead of solutions of (16)-(17).

3.2. Why $\mathcal{C}^1$ Solutions

Lemma 3.6.

$$u \in \mathcal{C}^2[0,\epsilon)$$

if $1 < p \le 2$ *and*

$$u \in \mathcal{C}^2\{r \in [0,\epsilon) | u'(r) \neq 0\}$$

if $p > 2$.

Proof. Recall that after differentiating (42) we have

$$-u' = \Phi_{p'}\left(\frac{1}{r^{N-1}}\int_0^r t^{N-1}f(u)dt\right).$$

Since $\Phi_{p'}(x) = |x|^{p'-2}x$, so $\Phi'_{p'} = (p'-1)|x|^{p'-2}$. Since $p'-2 = \dfrac{2-p}{p-1}$, we see that $\Phi'_{p'}$ is continuous for all x, if $1 < p \le 2$ and $\Phi'_{p'}$ is continuous at all $x \neq 0$, if $p > 2$.

Let

$$k(r) = \frac{1}{r^{N-1}}\int_0^r t^{N-1}f(u)dt.$$

By Lemma (3.3), k is continuous on $[0,\epsilon)$. Now,

$$k'(r) = \left[\frac{-(N-1)}{r^N}\int_0^r t^{N-1}f(u)dt + f(u)\right]$$

so k' continuous on $(0,\epsilon)$.

Claim: k' is continuous on $[0,\epsilon)$.

Proof of the Claim: We do this in two steps:

Step 1: We show $k'(0) = \frac{f(d)}{N}$.

By definition

$$\begin{aligned} k'(0) &= \lim_{r\to 0}\frac{k(r)-k(0)}{r-0} \\ &= \lim_{r\to 0}\frac{\frac{1}{r^{N-1}}\int_0^r t^{N-1}f(u)dt - 0}{r-0} \\ &= \lim_{r\to 0}\frac{\int_0^r t^{N-1}f(u)dt}{r^N}. \end{aligned}$$

Applying L'Hopital's rule gives $k'(0) = \dfrac{f(d)}{N}$.

Step 2: We show $\lim_{r\to 0} k'(r) = k'(0) = \dfrac{f(d)}{N}$.

Differentiating $k(r)$ and taking the limit as $r \to 0$ gives

$$\begin{aligned}
\lim_{r\to 0} k'(r) &= \lim_{r\to 0} \frac{-(N-1)}{r^N} \int_0^r t^{N-1} f(u) dt + f(u) \\
&= \frac{-(N-1)}{N} f(d) + f(d) \\
\\
&= \frac{f(d)}{N}.
\end{aligned}$$

We get the second equality by using L'Hopital's rule.

Steps 1 and 2 imply that k' is continuous on $[0, \epsilon)$.

Finally, by the chain rule and (27) we see u' is differentiable and that

$$\begin{aligned}
-u'' &= \Phi'_{p'} \left(\frac{1}{r^{N-1}} \int_0^r t^{N-1} f(u) dt \right) k'(r) \\
\\
&= (p'-1) \left| \frac{1}{r^{N-1}} \int_0^r t^{N-1} f(u) dt \right|^{p'-2} \left[\frac{-(N-1)}{r^N} \int_0^r t^{N-1} f(u) dt + f(u) \right] \\
&= \frac{1}{p-1} \left| \frac{1}{r^{N-1}} \int_0^r t^{N-1} f(u) dt \right|^{\frac{2-p}{p-1}} \left[\frac{-(N-1)}{r^N} \int_0^r t^{N-1} f(u) dt + f(u) \right] \\
&= \frac{1}{p-1} |u'|^{2-p} \left[\frac{-(N-1)}{r^N} \int_0^r t^{N-1} f(u) dt + f(u) \right].
\end{aligned}$$

By the previous claim, k' is continuous. Note that $|u'|^{2-p}$ is continuous for $1 < p \le 2$ and $|u'|^{2-p}$ is continuous at all points where $u' \neq 0$ for $p > 2$ and hence the lemma follows. □

Remark: If $p > 2$, $u'(r_0) = 0$, and $f(u(r_0)) \neq 0$, then $u''(r_0)$ is undefined.

To see this, suppose on the contrary that $u''(r_0)$ is defined. Using the fact that $u'(r_0) = 0$, (27) becomes

$$-r^{N-1} |u'|^{p-2} u' = \int_{r_0}^r t^{N-1} f(u) dt.$$

Dividing by $(r - r_0)$ and taking the limit as $r \to r_0$ gives

$$\lim_{r \to r_0} -r^{N-1} |u'|^{p-2} \left(\frac{u'}{r - r_0} \right) = \lim_{r \to r_0} \frac{\int_{r_0}^r t^{N-1} f(u(t)) dt}{(r - r_0)}.$$

Using L' Hopital's rule we obtain

$$0 = -|u'(r_0)|^{p-2}u''(r_0) = f(u(r_0)).$$

Thus, $|f(u(r_0))| = 0$ which is a contradiction to our assumption that $f(u(r_0)) \neq 0$. Thus, $u''(r_0)$ is undefined.

Remark: If $p > 2$, $u'(r_0) = 0$, and $f(u(r_0)) = 0$, then it is not clear whether u is $\mathcal{C}^2$ in a neighborhood of r_0 when $u'(r_0) = 0$. However, for the purposes of this paper a more detailed analysis of this situation is not needed.

To prove the following two lemmas, let $[0, R)$ be the maximal interval of existence for which u is a solution for (27)-(28).

Our goal is to show that u solves (27)-(28) on $[0, \infty)$. So, we aim at proving $R = \infty$, and we will do this in two lemmas. In the first lemma we show that if $R < \infty$ then the limits of u and u' as $r \to R^-$ are defined. Once the limits exist then in the second lemma, we establish that $R = \infty$.

LEMMA 3.7. *Suppose u solves (27)-(28) on $[0, R)$ with $R < \infty$, then there exists $u_0, u_0' \in \mathbb{R}$ such that*

$$\lim_{r \to R^-} u(r) = u_0,$$

$$\lim_{r \to R^-} u'(r) = u_0'.$$

PROOF. The following is the energy equation for (27)-(28)

$$E(r) = \frac{(p-1)|u'|^p}{p} + F(u). \tag{43}$$

Using (27) we see that

$$E'(r) = \frac{-(N-1)|u'|^p}{r} \leq 0. \tag{44}$$

Note that $E'(r) \leq 0$, so E is decreasing, and so $E(r) \leq E(0)$ which is

$$\frac{(p-1)|u'|^p}{p} + F(u) \leq E(0) = F(d).$$

Then by (22)

$$\frac{(p-1)|u'|^p}{p} - L \leq F(d).$$

Further simplification gives

$$|u'|^p \le \frac{p(F(d)+L)}{p-1}.$$

Then $|u'| \le M$ where $M = \left[\frac{p(F(d)+L)}{p-1}\right]^{\frac{1}{p}}$. So, by the mean value theorem we have

$$|u(x) - u(y)| \le M|x-y|$$

for all $x, y \in [0, R)$. This implies that u has a limit as $x \to R^-$. So, there exists a $u_0 \in \mathbb{R}$ such that $\lim_{r\to R^-} u(r) = u_0$. Taking the limit as $r \to R^-$ on both sides of (27), we see that $\lim_{r\to R^-} u'(r)$ exists, and we call it u_0'. □

LEMMA 3.8. *A solution exists for (27)-(28) on* $[0, \infty)$.

PROOF. If $R = \infty$, we are done. Suppose $R < \infty$.

Case(i): If $u'(R) \neq 0$, then by Lemma (3.6), $u \in \mathcal{C}^2$ in a neighborhood of R, so differentiating (18) and then dividing by $|u'|^{p-2}$, we have

$$(p-1)u'' + \frac{N-1}{r}u' + |u'|^{2-p} f(u) = 0.$$

Since $u'(R) \neq 0$, then by the standard existence theorem for ordinary differential equations there exists solution for the differential equation on $[R, R+\epsilon)$ for some $\epsilon > 0$ with $u(R) = u_0$ and $u'(R) = u_0'$. This contradicts the definition of R, hence, $R = \infty$.

Case(ii): If $u'(R) = 0$ and $f(u(R)) \neq 0$, then we can use the contraction mapping principle as in Lemma (3.2) and extend our solution u to $[R, R+\epsilon)$ for some $\epsilon > 0$. This contradicts the definition of R.

Case(iii): If $u'(R) = 0$ and $f(u(R)) = 0$ we can extend $u \equiv u(R)$ for $r > R$. Again this contradicts the definition of R. □

3.2.1. *For d large, $|u| < d$*

LEMMA 3.9. *Let $d > \beta$, then $|u(r)| < d$ for $0 < r < \infty$ and $f(d) \neq 0$.*

PROOF. From (43)-(44) it follows that

$$\frac{(p-1)|u'|^p}{p} + F(u) + \int_0^r \frac{N-1}{t}|u'|^p dt = F(d).$$

If there exists a $r_0 > 0$ such that $|u(r_0)| = d$, then

$$\int_0^{r_0} \frac{N-1}{t} |u'|^p dt = 0.$$

This implies $|u'| = 0$ on $[0, r_0]$. Hence, $u(r) \equiv d$ on $[0, r_0]$. Then by (18), $f(d) = 0$, but this contradicts our assumption that $f(d) \neq 0$. □

3.2.2. *Only one extremum between two consecutive zeros*

LEMMA 3.10. *If $z_1 < z_2$, with $u(z_1) = u(z_2) = 0$, and $|u| > 0$ on (z_1, z_2), then there is exactly one extremum, m, between (z_1, z_2) and also $|u(m)| > \gamma$.*

PROOF. Suppose without loss of generality that $u > 0$ on (z_1, z_2). Then there exists an extremum, m, such that $u'(m) = 0$. And

$$F(u(m)) = E(m) \geq E(z_2) = \frac{p-1}{p} |u'(z_2)|^p \geq 0.$$

Thus $|u(m)| \geq \gamma$ for any extremum. Suppose there exists consecutive extrema $m_1 < m_2 < m_3$ such that at m_1 and m_3 we have local maxima and m_2 is a local minimum with $u' < 0$ on (m_1, m_2) and $u' > 0$ on (m_2, m_3). We have $z_1 < m_1 < m_2 < m_3 < z_2$ and since the energy is decreasing we obtain $E(m_2) \geq E(m_3) \geq E(z_2)$. Since $u'(m_2) = u'(m_3) = 0$ and since $F(u(z_2)) = 0$ this gives

$$F(u(m_2)) \geq F(u(m_3)) \geq \frac{p-1}{p} |u'(z_2)|^p \geq 0. \tag{45}$$

And by (H5) it follows that $u(m_2) \geq \gamma$ and $u(m_3) \geq \gamma$. Also, since m_2 is a local minimum and m_3 is a local maximum we have $\gamma \leq u(m_2) < u(m_3)$. But by (H5), F is increasing for $u > \gamma$ and this implies $F(u(m_2)) < F(u(m_3))$ which is a contradiction to (45). □

3.3. If $u(r_1) = u'(r_1) = 0$, then $u \equiv 0$.

LEMMA 3.11. *If $u(r_0) = u'(r_0) = 0$ then $u \equiv 0$.*

PROOF. Suppose $u(r_0) = 0$ and $u'(r_0) = 0$. First we will do the easy case, and show that $u \equiv 0$ on (r_0, ∞). Since $E' \leq 0$ and $E(r_0) = 0$ then either $E < 0$ for $r > r_0$ or $E \equiv 0$ on

$(r_0, r_0+\epsilon)$ for some $\epsilon > 0$. We will show $E \equiv 0$ on $(r_0, r_0+\epsilon)$. For suppose $E < 0$ for $r > r_0$. Then we see that $|u| > 0$ for $r > r_0$, for if there exists an $r_1 > r_0$ such that $u(r_1) = 0$ then

$$0 \leq \frac{p-1}{p}|u'(r_1)|^p = E(r_1) < 0.$$

This is a contradiction. So suppose without loss of generality that $u > 0$ for $r > r_0$. Then for $r > r_0$ and r close to r_0 and by (H2), $f(u) < 0$ so

$$-r^{N-1}|u'|^{p-2}u' = \int_{r_0}^{r} t^{N-1} f(u)dt < 0.$$

Thus u is increasing on $(r_0, r_0+\epsilon)$ for some $\epsilon > 0$. Now since $E(r) < 0$ on $(r_0, r_0+\epsilon)$ therefore

$$\frac{p-1}{p}|u'|^p + F(u) < 0,$$

and so

$$|u'| < \left(\frac{p}{p-1}\right)^{\frac{1}{p}} |F(u)|^{\frac{1}{p}}.$$

Therefore,

$$\infty = \int_0^{u(r_0+\epsilon)} \frac{ds}{\sqrt[p]{|F(s)|}} = \int_{r_0}^{r_0+\epsilon} \frac{|u'|}{|F(u)|^{\frac{1}{p}}} dt < \int_{r_0}^{r_0+\epsilon} \left(\frac{p}{p-1}\right)^{\frac{1}{p}} dt < \infty.$$

This is a contradiction to (H6) and to the note at the end of the introduction. Then $E \equiv 0$ on $[r_0, r_0+\epsilon)$ and so

$$\frac{-(N-1)}{r}|u'|^p = E' \equiv 0$$

on $[r_0, r_0+\epsilon)$ and thus $u \equiv 0$ on $[r_0, r_0+\epsilon)$. Denote $[r_0, r_1)$ as the maximal half open interval for which $u \equiv 0$. If $r_1 < \infty$, again we can show that $u \equiv 0$ on $[r_1, r_1+\epsilon)$, but this will contradict the definition of r_1. Thus, $E \equiv 0$ on (r_0, ∞). Hence $u \equiv 0$ on $[r_0, \infty)$.

Now we will prove that $u \equiv 0$ on $(0, r_0)$. To prove this we use the idea from [5] and do the required modifications to fit our case. We will use hypothesis (H6). Let

$$r_1 = \inf_{r>0}\{r | u(r) = 0, u'(r) = 0\}.$$

If $r_1 = 0$ then $u \equiv 0$ on $(0, \infty)$ and then by continuity $u \equiv 0$ on $[0, \infty)$ and we are done. So suppose by the way of contradiction that $r_1 > 0$. Let $\frac{r_1}{2} < r < r_1$, so $\frac{2}{r_1} > \frac{1}{r}$. Now consider the derivative of the energy function given in equation (44) and then integrate it between r and r_1 to obtain

$$E(r_1, d) - E(r, d) = -\int_r^{r_1} \frac{(N-1)|u'|^p}{r} dt.$$

Since $u(r_1) = 0$, so $F(u(r_1)) = 0$ and $u'(r_1) = 0$, we get

$$\frac{(p-1)|u'|^p}{p} + F(u) = \int_r^{r_1} \frac{(N-1)|u'|^p}{r} dt. \tag{46}$$

Now let

$$w = \int_r^{r_1} \frac{(N-1)|u'|^p}{t} dt.$$

Differentiating we get

$$w' = -\frac{(N-1)|u'(r)|^p}{r}.$$

Solving this for $|u'|^p$, gives

$$|u'(r)|^p = \frac{-rw'}{N-1}.$$

Substituting this in (46) gives

$$\frac{-(p-1)rw'}{p(N-1)} + F(u) = w \tag{47}$$

and rearranging terms, we get

$$\frac{(p-1)rw'}{p(N-1)} + w = F(u).$$

Letting $\eta = \frac{(N-1)p}{p-1}$ then we have

$$w' + \frac{\eta w}{r} = \frac{\eta F(u)}{r}.$$

Multiplying both sides by r^η, gives

$$(r^\eta w)' = \eta r^{\eta-1} F(u).$$

Integrating between r and r_1 for r sufficiently close to r_1, gives

$$r_1^\eta w(r_1) - r^\eta w = \int_r^{r_1} \eta t^{\eta-1} F(u) dt.$$

Since $w(r_1) = 0$, and by (H2), $F(u(t)) \leq 0$ for t sufficiently close to r_1 we obtain

$$w = \frac{-\eta}{r^\eta} \int_r^{r_1} t^{\eta-1} F(u(t)) dt = \frac{\eta}{r^\eta} \int_r^{r_1} t^{\eta-1} |F(u(t))| dt.$$

Now plugging w and w' in (47) we have

$$\frac{(p-1)|u'|^p}{p} + F(u) = \frac{\eta}{r^\eta}\int_r^{r_1} t^{\eta-1}|F(u(t))|dt.$$

Solving this for $|u'|^p$ gives (for r close to r_1)

$$|u'|^p = \frac{p}{p-1}\left[\frac{\eta}{r^\eta}\int_r^{r_1} t^{\eta-1}|F(u(t))|dt + |F(u(r))|\right]. \tag{48}$$

Observe next that for $r < r_1$ and r sufficiently close to r_1 that $u'(r) \neq 0$; for if there exists $r_2 < r_1$ such that $u'(r_2) = 0$ then from (48), $u \equiv 0$ on (r_2, r_1), this contradicts the definition of r_1. Hence without loss of generality assume that $u'(r) < 0$ for $r < r_1$ and r sufficiently close to r_1. Now for $r < t < r_1$, u is decreasing so $u(r) > u(t) > 0$ which implies $F(u(r)) < F(u(t)) < 0$ and so $|F(u(r))| > |F(u(t))| > 0$, which leads to the following

$$\begin{aligned}
|u'|^p &\leq \frac{p}{p-1}\left[|F(u(r))| + \frac{\eta}{r^\eta}|F(u(r))|\int_r^{r_1} t^{\eta-1}dt\right] \\
&= \frac{p}{p-1}\left[|F(u(r))| + \frac{\eta}{r^\eta}\frac{|F(u(r))|}{\eta}(r_1^\eta - r^\eta)\right] \\
&= \frac{p|F(u(r))|r_1^\eta}{(p-1)r^\eta} \\
&\leq \frac{p2^\eta|F(u(r))|}{p-1}.
\end{aligned}$$

The last inequality follows as $\frac{2}{r_1} > \frac{1}{r}$, so

$$|u'|^p \leq \frac{p2^\eta|F(u(r))|}{p-1}.$$

Solving this for $|u'|$, we get

$$|u'| \leq \sqrt[p]{\frac{p2^\eta}{p-1}}\sqrt[p]{|F(u(r))|}.$$

Dividing by $\sqrt[p]{|F(u(r))|}$, integrating on (r, r_1) and using (H6) and the remark following (H6) we obtain

$$\begin{aligned}
\infty = \int_0^{u(r)} \frac{1}{\sqrt[p]{|F(s)|}}ds = \int_r^{r_1} \frac{|u'|}{\sqrt[p]{|F(u)|}}dt \\
&\leq \sqrt[p]{\frac{p2^\eta}{p-1}}\int_r^{r_1} dt \\
&= \sqrt[p]{\frac{p2^\eta}{p-1}}(r_1 - r)
\end{aligned}$$

$$< \infty.$$

Thus we get a contradiction and so $r_1 = 0$ and hence $u \equiv 0$. □

CHAPTER 4

SOLUTIONS WITH A PRESCRIBED NUMBER OF ZEROS

4.1. Rescaling Argument

In this section we show that there are solutions for (27)-(28) with a large number of zeros. For this we study the behavior of solutions as d grows large. We consider the idea from [13], page 371 and we do the required modifications to fit our case. Given $\lambda > 0$, let $u(r)$ be the solution of (27)-(28) with $d = \lambda^{\frac{p}{q-p+1}}$. Define

$$u_\lambda = \lambda^{\frac{-p}{q-p+1}} u(\frac{r}{\lambda}). \tag{49}$$

Then u_λ satisfies

$$r^{N-1}|u_\lambda'|^{p-2}u_\lambda' = -\int_0^r s^{N-1}\lambda^{\frac{-pq}{q-p+1}} f(\lambda^{\frac{p}{q-p+1}} u_\lambda(s))ds, \tag{50}$$

and

$$u_\lambda(0) = 1. \tag{51}$$

Lemma 4.1. *As $\lambda \to \infty$, $u_\lambda \to v$, uniformly on compact subsets of $[0, \infty)$, where v is a solution of*

$$r^{N-1}|v'|^{p-2}v' = -\int_0^r s^{N-1}|v(s)|^{q-1}v(s)ds, \tag{52}$$

$$v(0) = 1. \tag{53}$$

Proof. Let

$$E(r, \lambda) = \frac{(p-1)|u_\lambda'|^p}{p} + \lambda^{\frac{-pq}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}} u_\lambda)$$

then

$$\frac{\partial}{\partial r}E(r, \lambda) \equiv E'(r, \lambda) = -\frac{(N-1)|u_\lambda'|^p}{r}.$$

This implies $E(r,\lambda)$ is decreasing in r. So for $\lambda > 0$

$$E(r,\lambda) \leq E(0,\lambda)$$
$$= \lambda^{\frac{-pq}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}})$$

Using (19) to simplify the right hand side, gives the following:

$$\lambda^{\frac{-p(q+1)}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}}) = \lambda^{\frac{-p(q+1)}{q-p+1}} \frac{\lambda^{\frac{p(q+1)}{q-p+1}}}{q+1} + \lambda^{\frac{-p(q+1)}{q-p+1}} G(\lambda^{\frac{p}{q-p+1}})$$

(54)
$$\lambda^{\frac{-p(q+1)}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}}) = \frac{1}{q+1} + \lambda^{\frac{-p(q+1)}{q-p+1}} G(\lambda^{\frac{p}{q-p+1}}).$$

Then by (21)

(55)
$$\frac{G(\lambda^{\frac{p}{q-p+1}})}{(\lambda^{\frac{p}{q-p+1}})^{q+1}} \to 0,$$

as $\lambda \to \infty$. Thus, $E(r,\lambda) < \frac{2}{q+1}$ for large λ. Moreover $E(r,\lambda)$ is bounded above independently of r and for large λ.

The usual trick to show the convergence of u_λ is to use Arzela-Ascoli's Theorem. For this it suffices to show u_λ and u'_λ are bounded.

Claim: $u_\lambda(r)$ and $u'_\lambda(r)$ are bounded.

Proof of Claim: By Lemma (3.9), $|u(r)| \leq d = \lambda^{\frac{p}{q-p+1}}$. Thus, by (49), $|u_\lambda(r)| \leq 1$. Also, since $E(r,\lambda) \leq E(0,\lambda)$ we have

$$\frac{(p-1)|u'_\lambda|^p}{p} + \lambda^{\frac{-pq}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}} u_\lambda) \leq \lambda^{\frac{-pq}{q-p+1}} F(\lambda^{\frac{p}{q-p+1}}).$$

Since $F(u) \geq -L$ (proved in introduction) then we get

$$\frac{(p-1)|u'_\lambda|^p}{p} \leq \frac{1}{q+1} + \lambda^{\frac{-p(q+1)}{q-p+1}} G(\lambda^{\frac{p}{q-p+1}}) + L\lambda^{\frac{-pq}{q-p+1}}.$$

By (55) we see that

(56)
$$\frac{(p-1)|u'_\lambda|^p}{p} \leq \frac{2}{q+1}$$

for large λ. Hence, $|u'_\lambda|$ is bounded independent of r and for large λ. By Arzela-Ascoli's theorem and by a standard diagonal argument there is a subsequence of $u_\lambda(r)$, denoted by $u_{\lambda_k}(r)$, such that

$$\lim_{k\to\infty} u_{\lambda_k}(r) = v(r)$$

uniformly on compact subsets of $\mathbb{R}$ and v is continuous. End of proof of Claim.

We have

$$r^{N-1}|u'_\lambda|^{p-2}u'_\lambda = -\int_0^r s^{N-1}\lambda^{\frac{-pq}{q-p+1}} f(\lambda^{\frac{p}{q-p+1}}u_\lambda(s))ds \tag{57}$$

$$r^{N-1}|u'_\lambda|^{p-2}u'_\lambda = -\int_0^r s^{N-1}\left[|u_\lambda|^{q-1}u_\lambda + \lambda^{\frac{-pq}{q-p+1}} g(\lambda^{\frac{p}{q-p+1}}u_\lambda(s))\right] ds$$

also,

$$u'_{\lambda_k} = -\Phi_{p'}\left(\frac{1}{r^{N-1}}\int_0^r s^{N-1}\left[|u_{\lambda_k}|^{q-1}u_{\lambda_k} + \lambda_k^{\frac{-pq}{q-p+1}} g(\lambda_k^{\frac{p}{q-p+1}})u_{\lambda_k}\right] ds\right). \tag{58}$$

Since $u_{\lambda_k}(r) \to v(r)$ uniformly on compact subsets of $\mathbb{R}$ and using (H3), gives

$$\lim_{k\to\infty} u'_{\lambda_k} = -\Phi_{p'}\left(\frac{1}{r^{N-1}}\int_0^r s^{N-1}|v|^{q-1}v ds\right)$$

$$\equiv \phi.$$

Hence, $u'_{\lambda_k} \to \phi$ (pointwise) and since v is continuous it follows that ϕ is continuous. We also have

$$u_{\lambda_k} = 1 + \int_0^r u'_{\lambda_k} ds.$$

Since $u_{\lambda_k} \to v$ uniformly, and $u'_{\lambda_k} \to \phi$ pointwise, and by (56), u'_{λ_k} is uniformly bounded say by, M, applying dominated convergence theorem we get

$$v(r) = 1 + \int_0^r \phi(s)ds.$$

So,

$$v' = \phi.$$

Thus, from (58) we see that

$$v' = -\Phi_{p'}\left(\frac{1}{r^{N-1}}\int_0^r s^{N-1}|v|^{q-1}v ds\right).$$

Hence,

$$-r^{N-1}|v'|^{p-2}v' = \int_0^r s^{N-1}|v|^{q-1}v ds.$$

Note that $v(0) = 1, v'(0) = 0$. Hence, $v \in \mathcal{C}^1[0,\infty)$ and v satisfies (52)-(53) for $1 < p < q+1 < \frac{Np}{N-p}$. □

As u_{λ_k} converges to v uniformly on compact subsets of $\mathbb{R}$, so now we look for zeros of v. This is done in two steps. In step one we show v has a zero and in step two we show v has infinitely many zeros.

4.2. v has a Zero

The following lemma is technical and we use the result in the subsequent lemma.

LEMMA 4.2. *Let v solve (52)-(53). If $1 < p < q+1 < \frac{Np}{N-p}$ and if $v > 0$, then*

$$\int_0^\infty s^{N-1}v^{q+1}ds < \infty.$$

PROOF. By Lemma (4.1), we know that v is continuous and hence bounded on any compact set so to prove this lemma it is sufficient to show $\int_1^\infty s^{N-1}v^{q+1}ds < \infty$. We have

$$-r^{N-1}|v'|^{p-2}v' = \int_0^r s^{N-1}|v|^{q-1}v ds$$

and $v > 0$. So $v' < 0$ and so v is decreasing. Therefore,

$$\begin{aligned} r^{N-1}|v'|^{p-1} &= \int_0^r s^{N-1}v^q ds \\ &\geq v(r)^q \int_0^r s^{N-1} ds \\ &= \frac{v^q r^N}{N}. \end{aligned}$$

Thus,

$$\begin{aligned} |v'|^{p-1} &\geq \frac{v^q r}{N} \\ -v' = |v'| &\geq \frac{r^{\frac{1}{p-1}} v^{\frac{q}{p-1}}}{N^{\frac{1}{p-1}}} \end{aligned}$$

$$\frac{-v'}{v^{\frac{q}{p-1}}} \geq \frac{r^{\frac{1}{p-1}}}{N^{\frac{1}{p-1}}}.$$

Integrating this on $(0, r)$, gives

$$\left[\frac{-v^{\frac{-q}{p-1}+1}}{\frac{-q}{p-1}+1}\right]_0^r \geq \int_0^r \frac{s^{\frac{1}{p-1}}}{N^{\frac{1}{p-1}}} ds$$

further simplification gives

$$\left[\frac{(p-1)v^{\frac{p-q-1}{p-1}}}{q-p+1}\right]_0^r \geq \frac{(p-1)r^{\frac{p}{p-1}}}{pN^{\frac{1}{p-1}}}.$$

Since by assumption $\frac{q-p+1}{p-1} > 0$, multiplying both sides with $\frac{q-p+1}{p-1}$ leads to

$$[v^{\frac{p-1-q}{p-1}}]_0^r \geq \frac{(q-p+1)r^{\frac{p}{p-1}}}{pN^{\frac{1}{p-1}}}.$$

Thus,

$$v(r)^{\frac{p-1-q}{p-1}} - 1 \geq Cr^{\frac{p}{p-1}},$$

where $C = \dfrac{q-p+1}{pN^{\frac{1}{p-1}}}$. So,

$$\frac{1}{v^{\frac{q+1-p}{p-1}}} = v^{\frac{p-1-q}{p-1}} \geq 1 + Cr^{\frac{p}{p-1}} \geq Cr^{\frac{p}{p-1}}.$$

Thus,

$$v^{\frac{q+1-p}{p-1}} \leq C_1 r^{\frac{-p}{p-1}}.$$

So,

$$v \leq C_1 r^{\frac{-p}{q+1-p}}.$$

Thus we see that

$$\int_1^\infty s^{N-1} v^{q+1} ds \leq C_1^{q+1} \int_1^\infty s^{N-1-\frac{p(q+1)}{q+1-p}} ds < \infty.$$

The last inequality is due to our assumption that $1 < p < q+1 < \frac{Np}{N-p}$. □

Lemma 4.3. *Let v be a solution of (52)-(53). Then v has a zero.*

PROOF. To prove this lemma, we use an idea of paper [8]. Suppose $v > 0$ for all r, and consider integrating

$$(r^{N-1}vv'|v'|^{p-2})' = r^{N-1}|v'|^p - r^{N-1}v^{q+1}$$

on $(0, r)$, which leads to

$$r^{N-1}vv'|v'|^{p-2} = \int_0^r s^{N-1}|v'|^p ds - \int_0^r s^{N-1}v^{q+1}ds.$$

After rearranging terms, we have

$$-r^{N-1}vv'|v'|^{p-2} + \int_0^r s^{N-1}|v'|^p ds = \int_0^r s^{N-1}v^{q+1}ds. \tag{59}$$

Since $v > 0, v' < 0$, and since $p < q+1$, it follows from (59) and Lemma (4.2) that

$$\int_0^\infty s^{N-1}|v'|^p \le \int_0^\infty s^{N-1}v^{q+1}ds < \infty. \tag{60}$$

Then using (60) in (59) and taking the limit as $r \to \infty$, gives

$$-\lim_{r\to\infty} r^{N-1}vv'|v'|^{p-2}\text{exists and is finite.} \tag{61}$$

Now integrating the following identity

$$\left(\frac{(p-1)r^N|v'|^p}{p} + \frac{r^N v^{q+1}}{q+1}\right)' = \frac{-(N-p)|v'|^p r^{N-1}}{p} + \frac{Nr^{N-1}v^{q+1}}{q+1}$$

on $(0, r)$, gives

$$\left(\frac{(p-1)r^N|v'|^p}{p} + \frac{r^N v^{q+1}}{q+1}\right) = \int_0^r \frac{-(N-p)|v'|^p s^{N-1}}{p}ds + \int_0^r \frac{Ns^{N-1}v^{q+1}}{q+1}ds. \tag{62}$$

Then by (60), both the integrals on the right hand side of (62) converge, hence

$$\lim_{r\to\infty} \frac{(p-1)r^N|v'|^p}{p} + \frac{r^N v^{q+1}}{q+1}$$

exists. Denote

$$h(r) = \frac{(p-1)r^N|v'|^p}{p} + \frac{r^N v^{q+1}}{q+1}.$$

We have shown that $\lim_{r\to\infty} h(r) = l$ for some $l \ge 0$. Then by (60),

$$\int_0^\infty \frac{h(s)}{s}ds < \infty.$$

Thus, it follows that $l = 0$, so that

$$\lim_{r\to\infty} \frac{(p-1)r^N|v'|^p}{p} + \frac{r^N v^{q+1}}{q+1} = 0.$$

Then taking the limit as $r \to \infty$ in (62) gives

$$0 = \int_0^\infty \frac{-(N-p)|v'|^p s^{N-1}}{p} ds + \int_0^\infty \frac{N s^{N-1} v^{q+1}}{q+1} ds.$$

So,

$$\int_0^\infty s^{N-1}|v'|^p ds = \frac{Np}{(N-p)(q+1)} \int_0^\infty s^{N-1} v^{q+1} ds.$$

But by (60) we have

$$\int_0^\infty s^{N-1}|v'|^p \leq \int_0^\infty s^{N-1} v^{q+1} ds.$$

So it follows that

$$\frac{Np}{(N-p)(q+1)} \leq 1.$$

This contradicts our assumption that $q+1 < \frac{Np}{N-p}$. So, v is not positive for all r. Hence, v has a zero. $\square$

4.3. v has Infinitely Many Zeros

LEMMA 4.4. *Let v be the solution of (52)-(53). Then v has infinitely many zeros.*

PROOF. We have from the above lemma that there exists a z_1 such that $v > 0$ on $[0, z_1)$ and $v(z_1) = 0$. So after z_1 we have two cases, Case(i): v has a first local minimum call it $m_1 > z_1$, or Case (ii): $v' < 0$ for all $r > z_1$. We want to show that the Case(ii) is not possible. Suppose $v' < 0$ for all $r > 0$. Then

$$E \equiv \frac{(p-1)|v'|^p}{p} + \frac{1}{q+1}|v|^{q+1} \geq 0$$

and $E' \leq 0$ so

$$\frac{1}{q+1}|v|^{q+1} \leq E(r,d) \leq E(0,d) = \frac{1}{q+1}.$$

Thus $|v| \leq 1$. So v is bounded and $v' < 0$ and thus $\lim_{r\to\infty} v = J$. Also since E is bounded and since $E' \leq 0$, so $\lim_{r\to\infty} E(r,d)$ exists and thus $\lim_{r\to\infty} v'(r)$ exists.

Claim: $\lim_{r\to\infty} v'(r) = 0$.

Proof of Claim: Suppose not, which means $-v'(r) > m > 0$ for large r. Then integrating from $(0, r)$, gives

$$-v(r) + v(0) > mr.$$

Taking the limit as $r \to \infty$, we see that $-v$ is unbounded, which contradicts our assumption that v is bounded. So, we have the claim. End of proof of Claim.

Consider dividing (52) by r^N and then taking the limit as $r \to \infty$ and using the above claim, gives

$$0 = \lim_{r\to\infty} \frac{-\int_0^r t^{N-1}|v|^{q-1}v dt}{r^N}.$$

Applying L'Hopital's rule on right hand side and using $\lim_{r\to\infty} v(r) = J < 0$ gives

$$0 = \frac{-|J|^{q-1}J}{N}.$$

This contradicts our assumption that $J < 0$. So Case(ii) is not possible.

Hence, v has a first local minimum call it m_1, where $m_1 > z_1$, and let $v_1 = v(m_1) < 0$. Now v satisfies

$$r^{N-1}|v'|^{p-2}v' = -\int_{m_1}^r t^{N-1}|v|^{q-1}v dt$$

and

$$v(m_1) = v_1.$$

We may now use the same argument as in Lemma (4.3) to show that v has a second zero at $z_2 > z_1$. Proceeding inductively, we can show that v has infinitely many zeros. □

As $u_\lambda \to v$ on any fixed compact set when λ is large, this means that the graph of u_λ is uniformly close to the graph of v. Since v has infinitely many zeros, suppose the first ρ zeros of v are on $[0, K]$ for $K > 0$. By uniform convergence on compact subsets u_λ will have at least ρ zeros on $[0, K+1]$ for large λ. By (49), $u_\lambda(r) = \lambda^{\frac{-p}{q-p+1}} u\left(\frac{r}{\lambda}\right)$, so u will have at least ρ zeros on $[0, \infty)$. So now we are ready to shift gears from v to u.

The following lemma is technical and we mimic the idea from [9] and we do necessary changes to fit our case.

4.4. An Important Lemma

Lemma 4.5. *Let $u(r,d)$ be the solution of (27)-(28). Let us suppose that $u(r,d^*)$ has exactly k zeros and $u(r,d^*) \to 0$ as $r \to \infty$. If $|d-d^*|$ is sufficiently small, then $u(r,d)$ has at most $k+1$ zeros on $[0,\infty)$.*

Proof. Our goal is to show that for d close to d^*, $u(.,d)$ has at most $(k+1)$ zeros in $[0,\infty)$. So we suppose there is a sequence of values d_j converging to d^* and such that $u(.,d_j)$ has at least $(k+2)$ zeros on $[0,\infty)$ (if there is no such sequence, we are done). We write $u_j(r) = u(r,d_j)$ and we denote by z_j the $(k+1)$st zero of u_j, counting from the smallest. We will show that if u_j has a $(k+2)$nd zero, then $u(r,d^*)$ is going to have a $(k+1)$st zero, which is a contradiction.

First we show that $u(r,d_j) \to u(r,d^*)$ and $u'(r,d_j) \to u'(r,d^*)$ on compact subsets of $[0,\infty)$ as $d_j \to d^*$ and $j \to \infty$. We prove this in two claims.

Claim 1: If $\lim\limits_{j\to\infty} d_j = d^*$, then $|u(r,d_j)| \le M_1$ and $|u'(r,d_j)| \le M_2$ for some $M_1, M_2 > 0$ for all j.

Proof of Claim 1: We use the fact from (43) and (44) that energy is decreasing and hence E is bounded by $E(0,d_j) = F(d_j)$, we can write the energy at r as the following

$$E(r,d_j) = \frac{(p-1)|u'(r,d_j)|^p}{p} + F(u(r,d_j)) \le F(d_j) \le F(d^*) + 1$$

for large j. Also, by (22), $F(u) \ge -L$ thus

$$\frac{(p-1)|u'(r,d_j)|^p}{p} \le F(d^*) + 1 + L \le C$$

for large j and for some $C > 0$. Thus, for j large, $|u'(r,d_j)| \le M_2$ for some $M_2 > 0$. Also, note that since $\lim\limits_{r\to\infty} E(r,d^*)$ exists and since $\lim\limits_{r\to\infty} u(r,d^*) = 0$ it follows that

$$F(d^*) = E(0,d^*) > \lim_{r\to\infty} E(r,d^*) \ge 0.$$

Thus, $F(d^*) > 0$. Hence by (H4) and (H5), $d^* > \gamma$. By lemma (3.9), $|u(r,d_j)| \le d_j$ and since $\lim\limits_{j\to\infty} d_j = d^*$ we have $|u(r,d_j)| \le d^* + 1 = M_1$ for large j. End of proof of Claim 1.

Claim 2: $u(r,d_j) \to u(r,d^*)$ and $u'(r,d_j) \to u'(r,d^*)$ uniformly on compact subsets of $[0,\infty)$ as $j \to \infty$.

Proof of Claim 2: By Claim 1, $|u(r, d_j)| \leq M_1$ and $|u'(r, d_j)| \leq M_2$. So the $u(r, d_j)$ are bounded and equicontinuous. Then by Arzela-Ascoli's theorem we have a subsequence (still denoted by d_j) such that $u(r, d_j) \to u(r, d^*)$ uniformly on compact subsets of $[0, \infty)$ as $j \to \infty$. Then by (27) and since $u_j \to u$ uniformly on compact subsets of $[0, \infty)$ we have

$$\lim_{j\to\infty} |u_j'|^{p-2}u_j' = \lim_{j\to\infty} \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u_j) ds$$

$$= \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u) ds.$$

Therefore, $|u_j'|^{p-2}u_j'$ converges uniformly on compact subsets of $[0, \infty)$. Thus, $u_j'(r)$ converges uniformly say to $g(r)$. We now show that $g(r) = u'(r, d^*)$. Integrating on $(0, r)$, gives

$$\lim_{j\to\infty} \int_0^r u_j'(t)dt = \int_0^r g(t)dt$$

$$\lim_{j\to\infty} (u_j(r) - u_j(0)) = \int_0^r g(t)dt.$$

Since $u_j(r, d_j) \to u(r, d^*)$, we get

$$u(r, d^*) - u(0, d^*) = \int_0^r g(t)dt.$$

Differentiating this we get $u'(r, d^*) = g(r) = \lim_{j\to\infty} u_j'$. End of proof of Claim 2.

Let t_j be the $(k + 2)$nd zero of u_j. Then there exists an l_j such that $z_j < l_j < t_j$ and l_j is a local extremum. So by Lemma (3.11)

$$F(u(l_j)) = E(l_j, d_j) \geq E(t_j, d_j) = \frac{p-1}{p}|u'(t_j)|^p > 0.$$

Then by (H5), $|u(l_j)| > \gamma$. Now let b_j be the smallest number greater than z_j such that $|u_j(b_j)| = \alpha$. Let a_j be the smallest number greater than z_j such that $|u_j(a_j)| = \frac{\alpha}{2}$. Let m_j be the local extrema between the kth and $(k+1)$st zeros of u_j. So we have $m_j < z_j < a_j < b_j$. Since the energy is decreasing we have $E(z_j, d_j) \leq E(m_j, d_j)$. Since $u_j'(m_j) = 0$, $u_j(z_j) = 0$, $F(u_j(z_j)) = 0$, and by Lemma (3.11), we have

$$0 < \frac{(p-1)|u_j'(z_j)|^p}{p} \leq F(u_j(m_j)).$$

Thus, $|u_j(m_j)| > \gamma$. So there exists a largest number q_j less than z_j such that $|u_j(q_j)| = \gamma$. Note $m_j < q_j < z_j < a_j < b_j < l_j < t_j$.

Claim 3: $b_j - a_j \geq K_1 > 0$, where K_1 is independent of j for sufficiently large j. Also, $\xi_2 - \xi_1 \geq K_2 > 0$ where ξ_1 and ξ_2 are two consecutive zeros of u_j.

Proof of Claim 3: Since the energy is decreasing and since $d_j \to d^*$ for j large we have

$$\frac{p-1}{p}|u_j'|^p + F(u_j) \leq F(d_j) \leq F(d^*) + 1$$

for large j. Rewriting this inequality gives

$$\frac{|u_j'|}{(F(d^*) + 1 - F(u_j))^{\frac{1}{p}}} \leq \left(\frac{p}{p-1}\right)^{\frac{1}{p}}. \tag{63}$$

So integrate (63) on (a_j, b_j) to get

$$\int_{\frac{\alpha}{2}}^{\alpha} \frac{dt}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} = \int_{a_j}^{b_j} \frac{|u_j'|}{(F(d^*) + 1 - F(u_j))^{\frac{1}{p}}} ds \leq \left(\frac{p}{p-1}\right)^{\frac{1}{p}} (b_j - a_j).$$

So letting

$$K_1 \equiv \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\frac{\alpha}{2}}^{\alpha} \frac{dt}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} \tag{64}$$

we see that $K_1 \leq b_j - a_j$ for all j.

Turning to the second part of the claim, using Lemma (3.10), let m be the extremum between ξ_1 and ξ_2. Let us integrate (63) on (ξ_1, m) and using (63) and that $|u(m)| > \gamma$ (by Lemma (3.10)) gives

$$\begin{aligned}\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\gamma} \frac{dt}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} &\leq \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{|u(m)|} \frac{dt}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} \\ &= \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\xi_1}^{m} \frac{|u_j'|}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} ds \\ &\leq m - \xi_1.\end{aligned}$$

Similarly on $[m, \xi_2]$ we have,

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\gamma} \frac{dt}{(F(d^*) + 1 - F(t))^{\frac{1}{p}}} \leq \xi_2 - m.$$

So,

$$K_2 \equiv 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\gamma} \frac{dt}{(F(d^*)+1-F(t))^{\frac{1}{p}}} \leq \xi_2 - \xi_1. \tag{65}$$

End of proof of Claim 3.

In particular $u_j \to u$ uniformly on $\left[0, y^* + \frac{K_2}{2}\right]$ where y^* is the kth zero of $u(r, d^*)$. Along with Lemma (3.11) and the previous claim, it follows that for large j, u_j has exactly k zeros on $\left[0, y^* + \frac{K_2}{2}\right]$. Let y_j be the kth zero of u_j, then by Claim 2, $u_j(r, d_j) \to u(r, d^*)$ as $j \to \infty$ on $\left[0, y^* + \frac{K_2}{2}\right]$, so $y_j \to y^*$ as $j \to \infty$.

Claim 4: $z_j \to \infty$ as $j \to \infty$.

Proof of Claim 4: Suppose not, that is if $|z_j| \leq A$ then there exists a subsequence j_k such that $z_{j_k} \to z$ and $u(r, d_{j_k}) \to u(r, d^*)$ on $[0, A]$ which in turn implies

$$0 = u(z_{j_k}, d_{j_k}) \to u(z, d^*).$$

Since $z_{j_k} > y_{j_k}$ and $y_{j_k} \to y^*$ as $j \to \infty$, then $z \geq y^*$. On the other hand, $u(r, d^*)$ has exactly k zeros, therefore $z = y^*$. Thus $u_j(y_j) = 0 = u_j(z_j)$. By the mean value theorem, $u_j'(w_j) = 0$ for some w_j with $y_j \leq w_j \leq z_j$. Since $u_j \to u$ uniformly on $[0, A]$ and $y_j \to y^* \leftarrow z_j$, so taking the limit gives $u'(y^*) = 0$, but by Lemma (3.11), this implies $u \equiv 0$. This is a contradiction to our assumption that u has exactly k zeros. End of Claim 4.

Now let us show that the q_j are bounded as $j \to \infty$. Since $u_j \to u$ and $u_j' \to u'$ uniformly on compact subsets of $[y^*, m^* + 1]$, where m^* is the local extremum of $u(r, d^*)$ that occurs after y^*, we see that u_j' must be zero on $[y^*, m^* + 1]$ for j large. Thus there exists an m_j with $y_j < m_j < m^* + 1$ such that $u_j'(m_j) = 0$.

Next, we estimate $q_j - m_j$ on $[m_j, q_j]$, since $u \geq \gamma > \beta$ on $[m_j, q_j]$ so we have $f(u) \geq C > 0$. So

$$-r^{N-1}|u_j'|^{p-2}u_j' = -\int_{m_j}^{r} (r^{n-1}|u_j'|^{p-2}u_j')' dt = \int_{m_j}^{r} r^{N-1} f(u_j) dt \geq \frac{C(r^N - m_j^N)}{N} \geq \frac{C}{N}(r - m_j) r^{N-1}.$$

So we have

$$-|u_j'|^{p-2}u_j' \geq \frac{C}{N}(r - m_j).$$

Further simplification and integrating on $[m_j, q_j]$ gives

$$d_j - \gamma \geq u(m_j) - \gamma = \int_{m_j}^{q_j} u_j' dt \geq \left(\frac{C}{N}\right)^{\frac{1}{p-1}} \int_{m_j}^{q_j} (r - m_j)^{\frac{1}{p-1}} dt.$$

Now using Lemma (3.9) and the fact that j is large gives

$$d^* + 1 \geq d_j - \gamma \geq \left(\frac{C}{N}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right) (q_j - m_j)^{\frac{p}{p-1}}$$

for large j. As we saw in a previous paragraph that m_j are bounded by $m^* + 1$, it follows that q_j are bounded.

Claim 5: For sufficiently large j, $|u_j(r)| < \gamma$ for all $r > z_j$.

Proof of Claim 5: Suppose on the contrary that there is a smallest $c_j > z_j$ such that $|u_j(c_j)| = \gamma$. Thus, on (z_j, c_j) we have $0 < |u_j| < \gamma$. Hence, $F(u_j) \leq 0$ on (z_j, c_j). So there exists an a_j and a b_j such that $z_j < a_j < b_j < c_j$ with $|u_j(a_j)| = \dfrac{\alpha}{2}$, $|u_j(b_j)| = \alpha$. Also, F is decreasing on $[\frac{\alpha}{2}, \alpha]$, so that $F(\frac{\alpha}{2}) \geq F(u_j)$ for all $\frac{\alpha}{2} \leq u_j \leq \alpha$.

Now integrating the following identity on (q_j, c_j)

$$(r^{\frac{p(N-1)}{p-1}} E)' = \frac{p(N-1)F(u_j(r))r^{\frac{Np-2p+1}{p-1}}}{p-1}$$

and since $|u_j(q_j)| = |u_j(c_j)| = \gamma$ and $F(u_j(q_j)) = 0 = F(u_j(c_j))$, gives

(66)
$$0 \leq \frac{c_j^{\frac{p(N-1)}{p-1}} |u_j'(c_j)|^p (p-1)}{p} = \frac{q_j^{\frac{p(N-1)}{p-1}} |u_j'(q_j)|^p (p-1)}{p} + \int_{q_j}^{c_j} \frac{p(N-1)F(u_j(r))t^{\frac{Np-2p+1}{p-1}}}{p-1} dt.$$

Since q_j is bounded, for an appropriate subsequence $q_j \to q^*$ where $u(q^*, d^*) = \gamma$ and since $u_j' \to u'$ uniformly on $[0, q^* + 1]$, then $u_j'(q_j) \to u'(q^*, d^*)$. Hence

$$\liminf_{j \to \infty} \int_{q_j}^{c_j} t^{\frac{(N-2)p+1}{p-1}} F(u_j(r)) dt \geq -\frac{(q^*)^{\frac{p(N-1)}{p-1}} |u'(q^*, d^*)|^p (p-1)}{p} > -\infty. \tag{67}$$

Also, since $z_j \to \infty$ and since $z_j < a_j < b_j$, so $a_j \to \infty$.

On other hand, by Claim 3

$$\begin{aligned}\int_{q_j}^{c_j} t^{\frac{(N-2)p+1}{p-1}} F(u_j(t)) dt &\leq \int_{a_j}^{b_j} t^{\frac{(N-2)p+1}{p-1}} F(u_j(t)) dt \\ &\leq F\left(\frac{\alpha}{2}\right) (b_j^{\frac{(N-1)p}{p-1}} - a_j^{\frac{(N-1)p}{p-1}}) \left(\frac{p-1}{(N-1)p}\right)\end{aligned}$$

$$\leq F\left(\frac{\alpha}{2}\right) a_j^{\frac{(N-1)p}{p-1}-1}(b_j - a_j)\left(\frac{p-1}{(N-1)p}\right)$$
$$\leq K_1 F\left(\frac{\alpha}{2}\right) a_j^{\frac{(N-1)p}{p-1}-1}\left(\frac{p-1}{(N-1)p}\right) \to -\infty$$

as $j \to \infty$. (We obtain the last inequality by using $K_1 \leq (b_j - a_j)$ from Claim 3 and also $F\left(\frac{\alpha}{2}\right) < 0$.) Thus,

$$\int_{q_j}^{c_j} t^{\frac{(N-2)p+1}{p-1}} F(u_j(t))dt \to -\infty,$$

but this is a contradiction to (67). Hence, $|u_j(r)| < \gamma$ for large j and for $r > z_j$. End of proof of Claim 5.

Now suppose u_j has another zero, call it $t_j > z_j$. Then there is a local extrema for u_j at a value s_j such that $z_j < s_j < t_j$. Since the energy is decreasing, we have $E(t_j) \leq E(s_j)$. By Lemmas (3.10) and (3.11) we have

$$0 < \frac{(p-1)|u_j'(t_j)|^p}{p} \leq F(u_j(s_j)).$$

Thus $|u_j(s_j)| > \gamma$ (by (H5)). By Claim 5, for sufficiently large j and for all $r > z_j$ we have $|u_j(r)| < \gamma$. In particular $|u_j(s_j)| < \gamma$, a contradiction. Hence, there is no zero of u_j larger than z_j. □

We use the above lemma to prove our Main Theorem in Chapter 4.

CHAPTER 5

PROOF OF THE MAIN THEOREM

5.1. Constructing Sets

To prove the Main Theorem we construct the following sets such that u has any prescribed number of zeros.

Let $\mathcal{S}_k = \{d \geq \gamma | u(r,d)$ has exactly k zeros for $r \geq 0$ $\}$ and let $d_k = \sup \mathcal{S}_k$.

We will then show that $\mathcal{S}_k$ for $k \geq 0$ is nonempty and bounded above.

Let $\mathcal{S}_0 = \{\ d \geq \gamma | u(r,d) > 0$ for all $r \geq 0\ \}$.

5.1.1. *$\mathcal{S}_0$ is nonempty*

Claim: $\gamma \in \mathcal{S}_0$.

Proof of Claim: If $d = \gamma$, then $u(0) = \gamma > 0$. So the energy at $r = 0$ is

$$E(0,\gamma) = \frac{p-1}{p}|u'(0)|^p + F(u(0)) = 0.$$

So $E < 0$ for $r > 0$; for if there is an $r_1 > 0$ such that $E(r_1,\gamma) = 0$ then $E \equiv 0$ on $[0,r_1]$, this implies $u \equiv 0$ on $[0,r_1]$, but $u(0) = \gamma > 0$. Thus $E < 0$ for $r > 0$. If there exsits an r_2 such that $u(r_2) = 0$ then

$$E(r_2,d) = \frac{p-1}{p}|u'(r_2)|^p \geq 0$$

contradicting $E < 0$ for all $r > 0$. Therefore, $u(r,\gamma) > 0$ for all $r \geq 0$. Hence $\gamma \in \mathcal{S}_0$. End of proof of Claim.

Lemma 5.1. *$\mathcal{S}_0 \neq 0$ and $\mathcal{S}_0$ is bounded above.*

Proof. $\mathcal{S}_0$ is nonempty by the above Claim and $\mathcal{S}_0$ is bounded above by the lemmas (4.1) and (4.3). $\square$

Let $d_0 = \sup \mathcal{S}_0$. Since $d > \gamma$ for all $d \in \mathcal{S}_0$ so $d_0 > \gamma > 0$.

Now our goal is to show that $u(r, d_0) > 0$ and that $u(r, d_0)$ satisfies (17). As d_0 is the supremum of $\mathcal{S}_0$ we expect $u(r, d_0) > 0$. We prove this in two lemmas. In the first lemma we show $u(r, d_0) \geq 0$ and in the second lemma we show $u(r, d_0) > 0$.

5.1.2. $u(r, d_0)$ *stays postive*

LEMMA 5.2. $u(r, d_0) \geq 0$ *for all* $r \geq 0$.

PROOF. If $u(r_0, d_0) < 0$ for some r_0, then by continuity with respect to initial conditions on compact sets for d close to d_0 and $d < d_0$, we have $u(r_0, d) < 0$. This contradicts the definition of S_0. □

LEMMA 5.3. $u(r, d_0) > 0$.

PROOF. Suppose there exists an r_1 such that $u(r_1, d_0) = 0$. By Lemma (5.2), we know $u(r, d_0) \geq 0$. So $u(r, d_0)$ has a minimum at r_1 and also since $u \in \mathcal{C}^1[0, \infty)$, this implies $u'(r_1) = 0$. Then by Lemma (3.11), $u \equiv 0$ which is a contradiction to $u(0) = d_0 \neq 0$.

□

Let $d > d_0$. Then $u(r, d)$ has at least one zero, otherwise d would be in $\mathcal{S}_0$ which it is not. Moreover, as d approaches d_0 from above, we expect that the first zero of u, $z_1(d)$, should go to infinity. This is shown in the following lemma.

LEMMA 5.4. $\lim_{d \to d_0^+} z_1(d) = \infty$.

PROOF. Suppose $\lim_{d \to d_0^+} z_1(d) = z_{d_0} < \infty$. Since $u(r, d) \to u(r, d_0)$ uniformly on compact subsets as $d \to d_0$, this implies $u(z_{d_0}) = \lim_{d \to d_0} u(z_1(d), d)$, and which in turn implies $u(z_{d_0}, d_0) = 0$. However, by Lemma (5.3), $u(r, d_0) > 0$, which is a contradiction. □

Next we want to show the energy $E(r, d_0) \geq 0$. This is crucial, as if $E(r, d_0) < 0$ at some point, say n_1, then u will not have any zeros after n_1, and also u will not decay as $r \to \infty$. So we have the following lemma.

5.1.3. $E(r, d_0)$ *is never negative*

LEMMA 5.5. $E(r, d_0) \geq 0$ *for all* $r \geq 0$.

PROOF. If $E(r_0, d_0) < 0$ then by continuity $E(r_0, d) < 0$ for $d > d_0$ and d close to d_0. On other hand, since $d > d_0$ then $u(r, d)$ has a first zero, $z_1(d)$, $(F(u(z_1(d))) = 0)$ so the energy is

$$E(z_1(d), d) = \frac{p-1}{p}|u'|^p \geq 0.$$

But since $E' \leq 0$, we must have that $z_1(d) \leq r_0$. This contradicts Lemma (5.4). Hence the result follows. $\square$

LEMMA 5.6. *$u'(r, d_0) < 0$ on $(0, \infty)$.*

PROOF. Since $u(0) = d$ and $u'(0) = 0$, first we want to show that u is decreasing on $(0, \epsilon)$ for some $\epsilon > 0$. Dividing both sides of (27) by r^N and then taking the limit as $r \to 0$, and applying L'Hopital's rule, gives

$$\lim_{r\to 0} |u'|^{p-2}\left(\frac{u'}{r}\right) = \lim_{r\to 0} \frac{-f(u(r))}{N} = \frac{-f(u(0))}{N} = \frac{-f(d_0)}{N} < 0.$$

The last inequality is true since by the definition of $\mathcal{S}_0$, we have that $d_0 > \gamma$ and then by (H5), $\gamma > \beta$ where β is the largest zero of f. Thus, $f(d_0) > 0$. So, $u' < 0$ on $(0, \epsilon)$ for some $\epsilon > 0$.

Let $[0, R_{d_0}]$ be the maximal interval so that $u' < 0$ on $(0, R_{d_0})$. If $R_{d_0} = \infty$, then $u' < 0$ on $(0, \infty)$ and we are done. Otherwise $R_{d_0} < \infty$ and $u'(R_{d_0}) = 0$.

Claim: $0 < u(R_{d_0}) \leq \beta$.

Proof of Claim: Suppose $f(u(R_{d_0})) > 0$ and let us look at the following identity:

$$-\int_r^{R_{d_0}} (r^{N-1}|u'|^{p-2}u')' = \int_r^{R_{d_0}} t^{N-1}f(u)dt.$$

Using $u'(R_{d_0}) = 0$, this gives

$$r^{N-1}|u'|^{p-2}u' = \int_r^{R_{d_0}} t^{N-1}f(u)dt > 0$$

for $r < R_{d_0}$ and r close to R_{d_0}. We get the last inequality since $f(u(R_{d_0})) > 0$, and by continuity, $f(u) > 0$ for r near R_{d_0}. This implies $u' > 0$ on (r, R_{d_0}) for r close to R_{d_0}. But by assumption $u' < 0$ on $(0, R_{d_0})$. Hence, $f(u(R_{d_0})) \leq 0$ and since we also know $u(R_{d_0}) > 0$, this implies $0 < u(R_{d_0}) \leq \beta$. End of proof of Claim.

The previous claim implies $F(u(R_{d_0})) < 0$. Since $u'(R_{d_0}) = 0$ we obtain

$$E(R_{d_0}, d_0) = F(u(R_{d_0})) < 0,$$

which is a contradiction to Lemma (5.5). Hence, $u'(r, d_0) < 0$ for all $r > 0$. $\square$

5.1.4. $\lim_{r\to\infty} u(r, d_0) = 0$

Since we now know that $u(r, d_0) > 0$ and $u'(r, d_0) < 0$, it follows that $\lim_{r\to\infty} u(r, d_0)$ exists.

LEMMA 5.7. $\lim_{r\to\infty} u(r, d_0) = U \geq 0$ *where* U *is some nonnegative zero of* f.

PROOF. Since E is decreasing and bounded below we see that $\lim_{r\to\infty} E(r, d_0) = E$. Rewriting (43) we obtain:

$$\frac{p-1}{p}|u'|^p = E(r, d_0) - F(u(r, d_0)).$$

The limit of both terms on the right exists and so we have

$$\lim_{r\to\infty} \frac{(p-1)|u'|^p}{p} = E - F(U).$$

Thus, $\lim_{r\to\infty} |u'|$ exists.

Claim 1: $\lim_{r\to\infty} |u'| = 0$.

Proof of Claim 1: Suppose not, then $\lim_{r\to\infty} |u'| = \mathcal{L} > 0$. So $|u'(r)| > \frac{\mathcal{L}}{2} > 0$ if $r \geq R$. Suppose $|u'(r)| = -u'(r)$, now integrating $|u'(r)| > \frac{\mathcal{L}}{2} > 0$, gives

$$\int_R^r -u'(r)dt > \int_R^r \frac{\mathcal{L}}{2}dt$$

for $r \geq R$, this implies

$$d_0 \geq u(R) \geq u(R) - u(r) \geq \frac{\mathcal{L}}{2}(r - R) \to \infty,$$

which is a contradiction. Hence, $\lim_{r\to\infty} |u'| = 0$ and so $\lim_{r\to\infty} u' = 0$. End of proof of Claim 1.

Dividing both sides of (27) by r^N gives

$$|u'|^{p-2}\frac{u'}{r} = \frac{-\int_0^r t^{N-1} f(u)dt}{r^N}.$$

Taking the limit as $r \to \infty$, and then doing simplification on right hand side by L'Hopital's rule, gives

$$0 = \lim_{r\to\infty} |u'|^{p-2}\left(\frac{u'}{r}\right) = \lim_{r\to\infty} \frac{-\int_0^r t^{N-1} f(u(t))dt}{r^N} = -f(U).$$

So, $f(U) = 0$. □

LEMMA 5.8. $U = 0$.

PROOF. Taking the limit as $r \to \infty$ in (27), gives $E = F(U)$. By Lemma (5.4), $E \geq 0$. Hence $F(U) \geq 0$. Also by Lemma (5.7), $f(U) = 0$. Thus by (H5) and (H6), $U \equiv 0$. □

Let $S_1 = \{\ d > d_0 | u(r, d)$ has exactly one zero for all $r \geq 0\ \}$.

LEMMA 5.9. *$S_1 \neq \emptyset$ and S_1 is bounded above.*

PROOF. By Lemma (4.5), if $d > d_0$ and d close to d_0 then $u(r, d)$ has at most one zero. Also, if $d > d_0$ then $d \notin S_0$ so $u(r, d)$ has at least one zero. Therefore, for $d > d_0$ and d close to d_0, $u(r, d)$ has exactly one zero. Hence S_1 is nonempty. Also by Lemmas (4.1)-(4.4), S_1 is bounded above. □

Define $d_1 = \sup S_1$.

As above we can show that $u(r, d_1)$ has exactly one zero and $u(r, d_1) \to 0$ as $r \to \infty$.

Proceeding inductively, we can find solutions that tend to zero at infinity and with any prescribed number of zeros. Hence, we complete the proof of the main theorem. □

CHAPTER 6

PRELIMINARIES FOR RESULT 2

6.1. Initial Value Problem

The technique used to solve (25) - (26) is the shooting method. That is, we first look at the initial value problem

$$u'' + \frac{N-1}{r}u' + f(u) = 0, \text{for } 0 < r < T, \tag{68}$$

$$u(0) = d > 0, u'(0) = 0. \tag{69}$$

By varying d appropriately, we attempt to find a d such that $u(r,d)$ has exactly n zeros on $(0,T)$ and $u(T) = 0$.

Multiplying (68) by r^{N-1} and integrating on $(0,r)$ gives

$$u' = \frac{-1}{r^{N-1}} \int_0^r t^{N-1} f(u) dt \tag{70}$$

Integrating (70) and applying the initial conditions we get

$$u(r) = d - \int_0^r \frac{1}{r^{N-1}} \int_0^r t^{N-1} f(u) dt. \tag{71}$$

Let $\phi(u)$ be equal to the right hand side of (71). It can be shown that $\phi(u)$ is a contraction mapping on $\mathcal{C}[0,\epsilon]$, the set of continuous function on $[0,\epsilon]$ for some $\epsilon > 0$. Then by the contraction mapping principle there exists a $u \in \mathcal{C}[0,\epsilon]$ such that $\phi(u) = u$. Thus, u is continuous solution of (71). Then by (H1'), (69) and (70), u' is continuous on $[0,\epsilon]$. From (H1') and (70) it follows that $\frac{u'}{r}$ is bounded on $(0,\epsilon]$ and then it follows from (68) that u'' is continuous on $[0,\epsilon]$.

We define the energy equation of the (68)-(69) as

$$E = \frac{u'^2}{2} + F(u). \tag{72}$$

Observe that

$$E' = -\frac{N-1}{r}u'^2 \leq 0$$

for $r > 0$ so that

$$\frac{u'^2}{2} + F(u) = E \leq E(0) = F(d). \tag{73}$$

Note that it follows from (H3') that there exists a $J > 0$ such that

$$F(u) \geq -J \tag{74}$$

for all $u \in \mathbb{R}$. So from (72) and (73) it follows that

$$\frac{u'^2}{2} \leq F(d) + J$$

and so $|u'|$ is uniformly bounded where ever it is defined and thus u is uniformly bounded where ever it is defined. It follows from this that u is defined on all of $[0, T]$ and in fact $u \in \mathcal{C}^2[0, T]$.

We will prove two results which in turn will help us to prove an important lemma.

Since $f(u) > 0$ for sufficiently large d, we see from (70) that $u' < 0$ on $(0, r)$ for small $r > 0$. Let k be a number given by (H4). For sufficiently large d it follows that $u' < 0$ on $(0, r_{kd})$ where r_{kd} is the smallest positive value of r such that $u(r_{kd}) = kd$.

Remark 1: First, we want to find a lower bound for r_{kd}. Since f is increasing for large u, we see from (72) that

$$\begin{aligned} -r^{N-1}u' &\leq f(d)\int_0^r t^{N-1}dt \\ &= \frac{f(d)r^N}{N}. \end{aligned}$$

So we have

$$\begin{aligned} \int_0^{r_{kd}} -u'dt &\leq \int_0^{r_{kd}} \frac{rf(d)}{N}dt \\ (1-k)d &\leq \frac{r_{kd}^2 f(d)}{2N} \\ r_{kd}^2 &\geq \frac{2N(1-k)d}{f(d)} \end{aligned}$$

$$(75) \qquad r_{kd} \geq \sqrt{\frac{2N(1-k)d}{f(d)}}.$$

Remark 2: We want to find a lower bound on

$$(76) \qquad \int_0^{r_{kd}} t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right)dt.$$

Then by hypothesis (H3'), $F(u) = \int_0^u f(t)dt$ and $F' = f > 0$ for large u. Therefore, F is increasing for large u. Since u is decreasing, $kd \leq u(t) \leq d$ for $0 \leq t \leq r_{kd}$, this implies $F(kd) \leq F(u) \leq F(d)$. So on $[0, r_{kd}]$ we have

$$(77) \qquad \int_0^{r_{kd}} t^{N-1} NF(u)dt \geq F(kd)r_{kd}^N.$$

By hypothesis (H1'), f is increasing for large u. Using this we have

$$\int_0^{r_{kd}} t^{N-1}\frac{N-2}{2}uf(u)dt \leq \frac{N-2}{2N}df(d)r_{kd}^N$$

so

$$(78) \qquad -\int_0^{r_{kd}} t^{N-1}\frac{N-2}{2}uf(u)dt \geq -\frac{N-2}{2N}df(d)r_{kd}^N.$$

Now using the estimates in (77), (78) and (75) in (76) gives the following:

$$\int_0^{r_{kd}} t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right)dt \geq \left(F(kd) - \frac{N-2}{2N}df(d)\right)r_{kd}^N$$

$$= \left(NF(kd) - \frac{N-2}{2}df(d)\right)\frac{r_{kd}^N}{N}$$

$$\geq \left(NF(kd) - \frac{N-2}{2}df(d)\right)\left(\frac{1}{N}\left(\sqrt{\frac{2N(1-k)d}{f(d)}}\right)^N\right)$$

$$= C(N,k)\left(NF(kd) - \frac{N-2}{2}df(d)\right)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}}$$

where $C(N,k) = \frac{1}{N}(\sqrt{2N(1-k)})^N$. So we have

(79)
$$\int_0^{r_{kd}} t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right)dt \geq C(N,k)\left(NF(kd) - \frac{N-2}{2}df(d)\right)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}}.$$

6.1.1. *As d increases so does the energy*

LEMMA 6.1. *If (H5) and (H4) are satisfied, then*

$$\lim_{d\to\infty} E(r,d) = \infty \tag{80}$$

uniformly for $r \in [0,T]$. If in place of (H4) we have (H4) then $\lim_{d\to-\infty} E(r,d) = \infty$ uniformly for $r \in [0,T]$.*

PROOF. Let us suppose $0 \leq r \leq T$. Consider Pohozaev's identity

$$\left\{r^N\left(\frac{u'^2}{2} + F(u)\right) + \frac{N-2}{2}r^{N-1}uu'\right\}' = r^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right).$$

Integrating Pohozaev's identity on $[0,r]$ and using hypothesis (H5) and (79) gives

$$\begin{aligned} r^N E(r,d) + \frac{N-2}{2}r^{N-1}uu' &= \int_0^r t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right)dt \\ &= \int_0^{r_{kd}} t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u)\right)dt + \int_{r_{kd}}^r t^{N-1}\left(NF(u) - \frac{N-2}{2}uf(u\right. \\ &\geq C(N,k)\left(NF(ku) - \frac{(N-2)}{2}uf(u)\right)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}} - M\left(\frac{r^N - r_{kd}^N}{N}\right). \end{aligned}$$

Ignoring the last term on the right hand side we get

$$r^N E(r,d) + \frac{N-2}{2}r^{N-1}uu' \geq C(N,k)\left(NF(ku) - \frac{(N-2)}{2}uf(u)\right) - \frac{Mr^N}{N}.$$

Now let us estimate uu'.

First, it follows from (24) that $\lim_{u\to\infty}\frac{u^2}{F(u)} = 0$. Further $\lim_{u\to\infty}\frac{u^2}{F(u)+J} = 0$ (Recall $F(u) \geq -J$ for all u).

Thus, there exists a B such that if $|u| \geq B$ then $\frac{u^2}{F(u)+J} \leq 1$ that is if $|u| \geq B$ then $u^2 \leq F(u) + J$. On other hand if $|u| \leq B$ then $u^2 \leq B^2$. So we see that for all u we have

$$u^2 \leq F(u) + J + B^2. \tag{81}$$

By Young's inequality, and then utilizing (81) and (72) gives us the following

$$\begin{aligned} uu' &\leq \frac{1}{2}u^2 + \frac{1}{2}u'^2 \\ &\leq (F(u) + J + B^2) + \frac{1}{2}u'^2 \\ &= \left(\frac{1}{2}u'^2 + F(u)\right) + J + B^2 \\ &= E(r,d) + J + B^2. \end{aligned}$$

Thus,

$$\begin{aligned} r^N E(r,d) + \frac{N-2}{2} r^{N-1} uu' &= r^N \left(E(r,d) + \frac{N-2}{2r} uu' \right) \\ &\leq r^N \left(E(r,d) + \frac{N-2}{2r} (E(r,d) + J + B^2) \right) \\ &= r^{N-1} E(r,d) \left(r + \frac{N-2}{2} \right) + \frac{N-2}{2} r^{N-1} (J + B^2). \end{aligned}$$

Therefore,

$$\begin{aligned} T^{N-1}E(T,d)\left(T + \frac{N-2}{2}\right) + \frac{N-2}{2}T^{N-1}(J+B^2) &\geq r^N E(r,d) + \frac{N-2}{2} r^{N-1} uu' \\ &\geq C(N,k)\left(NF(kd) - \frac{(N-2)}{2} df(d)\right)\left(\frac{d}{f(d)}\right)^{\frac{N}{2}} - \end{aligned}$$

Now J, B, $C(N,k)$, and T do not depend on d, then from (H4) we see that

$$\lim_{d \to \infty} E(T,d) = \infty.$$

$\square$

CHAPTER 7

FINDING ZEROS FOR RESULT 2

7.1. Using Bessel's Equation

We know that $F(u) \to \infty$ as $u \to \pm\infty$. Therefore, since $E(T, d) \to \infty$ as $d \to \infty$, for sufficiently large d there are exactly two solutions of $F(u) = \frac{1}{2}E(T, d)$ which we denote as $h_2(d) < 0 < h_1(d)$. For $d > 0$ sufficiently large we see that $u''(0) = \frac{-f(d)}{N} < 0$ and $u'(0) = 0$ so u is initially decreasing.

Let

$$C(d) = \min_{r \in [0, r_1]} \frac{f(u)}{u} = \min_{u \in [h_1(d), d]} \frac{f(u)}{u}. \tag{82}$$

Then by (H2) we see that $C(d) \to \infty$ as $d \to \infty$.

Lemma 7.1. *$r_1 \to 0$ as $d \to \infty$.*

Proof. To show this we compare

$$u'' + \frac{N-1}{r}u' + \frac{f(u)}{u}u = 0 \tag{83}$$

with initial conditions $u(0) = d > 0$ and $u'(0) = 0$ with

$$v'' + \frac{N-1}{r}v' + C(d)v = 0 \tag{84}$$

with initial conditions $v(0) = d$ and $v'(0) = 0$ where $\frac{f(u)}{u} > C(d)$ on $[0, r_1]$.

Claim: $u < v$ on $[0, r_1]$.

Proof of the Claim: Multiplying (83) by $r^{N-1}v$ and (84) by $r^{N-1}u$ and then taking the difference of the resultant equations gives

$$(r^{N-1}(u'v - uv'))' + r^{N-1}uv\left(\frac{f(u)}{u} - C(d)\right) = 0.$$

Now integrating this from 0 to r where $0 < r \le r_1$ and using $u(0) = v(0) = d$, $u'(0) = v'(0) = 0$ and $\frac{f(u)}{u} - C(d) > 0$ gives

$$r^{N-1}(u'(r)v(r) - v'(r)u(r)) < 0.$$

Then dividing by $v^2(r)$ and integrating between 0 to r where $0 < r \le r_1$ and since $u(0) = v(0) = d$ leads to $\frac{u}{v} < 1$ this implies that $u < v$ on $[0, r_1]$. End of proof of Claim.

Let $z(r) = \left(\frac{r}{\sqrt{C(d)}}\right)^{\frac{N-2}{2}} v\left(\frac{r}{\sqrt{C(d)}}\right)$. Then

$$z'' + \frac{z'}{r} + \left(1 - \frac{\left(\frac{N-2}{2}\right)^2}{r^2}\right) z = 0. \tag{85}$$

The above equation is a Bessel equation of order $\frac{N-2}{2}$. Thus, $z(r) = A_1 J_{\frac{N-2}{2}}(r) + A_2 Y_{\frac{N-2}{2}}(r)$ for constants A_1 and A_2. Since z is bounded at $r = 0$ and $Y_{\frac{N-2}{2}}$ is not, it must be that $z(r) = A_1 J_{\frac{N-2}{2}}(r)$ for some constant A_1.

Denoting $\beta_{\frac{N-2}{2},1}$ as the first positive zero of $J_{\frac{N-2}{2}}(r)$, we see that the first positive zero of v is $\frac{\beta_{\frac{N-2}{2},1}}{\sqrt{C(d)}}$ and since $u < v$ on $[0, r_1]$ we see that

$$r_1 < \frac{\beta_{\frac{N-2}{2},1}}{\sqrt{C(d)}} \to 0$$

as $d \to \infty$. $\square$

7.2. u has a Zero

LEMMA 7.2. *For large d, u has a zero, $z_1(d)$, and $z_1(d) \to 0$ as $d \to \infty$.*

PROOF. First we show that u has a zero. We prove this by contradiction. Suppose $u > 0$ on $[0, T]$ and consider $r > r_1$. Then $0 < u < u(r_1) = h_1(d)$ so $F(u) < F(h_1(d))$. Also since $F(h_1(d)) = \frac{1}{2}E(T, d)$ we obtain

$$\frac{u'^2}{2} + F(h_1(d)) > \frac{u'^2}{2} + F(u) \ge E(T, d) = 2F(h_1(d)).$$

Thus,

$$u'^2 \ge 2F(h_1(d))$$

$$\int_{r_1}^{r} |u'|dt \geq \int_{r_1}^{r} \sqrt{2F(h_1(d))}dt.$$

Since u is decreasing and using $u(r_1) = h_1(d)$ gives

$$h_1(d) - u(r) = u(r_1) - u(r) \geq \sqrt{2F(h_1(d))}(r - r_1). \tag{86}$$

So

$$h_1(d) - \sqrt{2F(h_1(d))}(r - r_1) \geq u(r) > 0.$$

Thus,

$$\frac{h_1(d)}{\sqrt{2F(h_1(d))}} \geq r - r_1. \tag{87}$$

So evaluating at $r = T$ and using (24) and by Lemma (7.1)

$$T \leftarrow T - r_1 \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \to 0$$

as $d \to \infty$. Thus, $T \leq 0$. A contradiction. Thus u has a first zero $z_1(d)$. Then repeating the above argument on $[0, z_1(d)]$ and letting $r = z_1$ in (87) we get

$$0 \leq z_1 - r_1 \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \to 0$$

as $d \to \infty$. Also, since $r_1 \to 0$ as $d \to \infty$ (by Lemma (7.1)) we see that $z_1 \to 0$ as $d \to \infty$. $\square$

Remark: Taking the derivative of (72) with respect to r gives

$$E'(r,d) = -\frac{N-1}{r}u'^2 \leq 0. \tag{88}$$

So E is decreasing and is bounded by $E(0,d) = F(d)$ which is same as

$$E(r,d) = \frac{u'^2}{2} + F(u) \leq F(d). \tag{89}$$

Hence u is bounded.

LEMMA 7.3. *For large d, $|u(r)| < d$ and $f(d) \neq 0$.*

Proof. From (88) and (89) it follows that

$$\frac{u'^2}{2} + F(u) + \int_0^r \frac{N-1}{r} u'^2 = F(d).$$

If there exists a $r_0 > 0$ such that $|u(r_0)| = d$, then

$$\frac{u'^2}{2} + \int_0^{r_0} \frac{N-1}{r} u'^2 dt = 0.$$

This implies $u' = 0$ on $[0, r_0]$, hence $u'' = 0$ on $[0, r_0]$. Then by (68), $f(d) = 0$, but this contradicts our assumption that $f(d) \neq 0$. □

7.2.1. *Using energy estimate*

We next show that for sufficiently large d, u attains the value $h_2(d)$ at some r_2 where $z_1 < r_2 < T$. So we suppose $u' < 0$ on a maximal interval (z_1, r). Here $-u > -h_2(d)$ so $u < h_2(d)$ and this implies $F(u) < F(h_2(d))$. Then as earlier

$$\frac{1}{2}u'^2 + F(h_2(d)) \geq \frac{1}{2}u'^2 + F(u) \geq E(T, d) = 2F(h_2(d)).$$

So

$$u'^2 \geq 2F(h_2(d)),$$

integrating this between (z_1, r) gives

$$\int_{z_1}^r u' dt = \int_{z_1}^r |u'| dt \geq \int_{z_1}^r \sqrt{2F(h_2(d))}$$

since $u(z_1) = 0$ leads to

$$-u(r) \geq \sqrt{2F(h_2(d))}(r - z_1)$$

$$u(r) \leq -\sqrt{2}\sqrt{F(h_2(d))}(r - z_1). \tag{90}$$

Now suppose by the way of contradiction that $u > h_2(d)$ on (z_1, T) and plugging this in (90) gives us

$$h_2(d) \leq u(r) \leq \sqrt{2}\sqrt{F(h_2(d))}(r - z_1)$$

$$-h_2(d) \geq \sqrt{2}\sqrt{F(h_2(d))}(r - z_1).$$

Evaluating this at $r = T$ and then taking limit as $d \to \infty$ and by (24)

$$0 \leftarrow \frac{-h_2(d)}{\sqrt{F(h_2(d))}} \geq \sqrt{2}(T - z_1) \to \sqrt{2}T.$$

Hence $T \leq 0$. A contradiction. Therefore, there exists a small value of r, r_2, such that $z_1 < r_2 < T$ with $u(r_2) = h_2(d)$. Now evaluating (90) at $r = r_2$ and since $u(r_2) = h_2(d)$

$$h_2(d) = u(r_2) \leq -\sqrt{2}\sqrt{F(h_2(d))}(r_2 - z_1)$$

and then taking limit as $d \to \infty$ and by (24) gives

$$0 \leftarrow \frac{-h_2(d)}{\sqrt{F(h_2(d))}} \geq \sqrt{2}(r_2 - z_1).$$

Hence $r_2 - z_1 \to 0$ as $d \to \infty$.

7.2.2. *u has a minimum*

We next want to show that u has a minimum on (r_2, T).

Suppose not. Suppose that u is decreasing on (r_2, T). Now we want to show that there exists an extremum of u after r_2.

Let $G(d) = \min\limits_{[r_2,T]} \frac{f(u)}{u}$. Note that $G(d) \to \infty$ as $d \to \infty$ by (24). Now we compare

$$u'' + \frac{N-1}{r}u' + \frac{f(u)}{u}u = 0 \tag{91}$$

with

$$v'' + \frac{N-1}{r}v' + G(d)v = 0 \tag{92}$$

with initial conditions $v(r_2) = u(r_2)$ and $v'(r_2) = u'(r_2)$. With an argument similar to the Claim in Lemma (7.1) we can show that $u > v$ on $[r_2, T]$. Let $z(r) = \left(\frac{r}{\sqrt{G(d)}}\right)^{\frac{N-2}{2}} v\left(\frac{r}{\sqrt{G(d)}}\right)$. Then

$$z'' + \frac{z'}{r} + \left(1 - \frac{\left(\frac{N-2}{2}\right)^2}{r^2}\right) z = 0. \tag{93}$$

Now it is a well known fact about Bessel functions (see [16]) that there exists a constant such that the distance between two successive zeros of z is less than K. This implies that the distance between two successive zeros of v is less than $\frac{K}{\sqrt{G(d)}} \to 0$ as $d \to \infty$. Thus for large d, v must have a zero on (r_2, T). Since $u > v$ and $0 > h_2(d) = u(r_2)$ we see that u gets positive which contradicts that u is decreasing on (r_2, T). Thus we see that there exists an m_1 with $r_2 < m_1 < T$ such that u decreases on (r_2, m_1) and m_1 is a local minimum of u. Also we see that

$$m_1 - r_2 \leq \frac{K}{\sqrt{G(d)}} \to \infty$$

as $d \to \infty$. Hence $m_1 \to 0$ as $d \to \infty$. Also, $F(u(m_1)) = E(m_1) \geq E(T, d) \to \infty$ as $d \to \infty$. In a similar way we can show that for large d, u has a second zero, z_2, with $m_1 < z_2 < T$ and $z_2 \to 0$ as $d \to \infty$ and u has a second extremum, m_2, with $z_2 < m_2 < T$ and $m_2 \to 0$ as $d \to \infty$. Continuing in this way we can get as many zeros as desired on $(0, T)$ for large d.

LEMMA 7.4. *If $u(T, d) = 0$ then $u'(T, d) \neq 0$. In particular $|u'(T, d)| \geq C > 0$ for $T - \delta \leq r \leq T$.*

PROOF. Suppose on contrary that $u'(T, d) = 0$ and by assumption $u(T, d) = 0$, then by the uniqueness of solutions for initial value problem we have that $u \equiv 0$. This contradicts the initial condition that $u(0) = d > 0$. Thus, $u'(T, d) \neq 0$, so

$$|u'(r, d)| \geq C > 0 \text{ for } T - \delta \leq r \leq T \tag{94}$$

for $\delta > 0$. □

LEMMA 7.5. *If u satisfies equation (68), then u has a finite number of zeros.*

PROOF. Suppose u has an infinite number of zeros, say $\{z_n\}$ on $[0, T]$. Since $[0, T]$ is a compact, there exists a subsequence of $\{z_{n_l}\}$ of $\{z_n\}$ such that $z_{n_l} \to z$ and $u(z) = 0$. Then by mean value theorem $u'(m_{n_l}) = 0$ where $\{m_{n_l}\}$ is a subsequence of the extrema $\{m_n\}$ and where m_{n_l} are between z_{n_l} and $z_{n_{l+1}}$ and so $\lim_{l \to \infty} m_{n_l} = z$. Then taking limit as $l \to \infty$ gives $u'(z) = 0$. So we have $u(z) = 0$ and $u'(z) = 0$, this implies that $u \equiv 0$ but this is a contradiction to $u(0) = d > 0$. □

7.3. An Important Lemma

LEMMA 7.6. *Let $u(r,d)$ be the solution of (25) and (26). Let us suppose that $u(r,d^*)$ has exactly k zeros on $[0,T)$ and $u(r,d^*)=0$. If $|d-d^*|$ is sufficiently small, then $u(r,d)$ has at most $k+1$ zeros on $[0,T)$.*

PROOF. Since $u(r,d^*)$ and $u(r,d)$ have a finite number of zeros by Lemma (7.5), then this result will follow if we can show that $u(r,d) \to u(r,d^*)$ uniformly on $[0,T)$ as $d \to d^*$.

Claim 1: If $\lim_{j\to\infty} d_j = d^*$, then $|u(r,d_j)| \le M_1$ and $|u'(r,d_j)| \le M_2$ for some $M_1, M_2 > 0$.

Proof of Claim 1: We use the fact from (88) and (89) that energy is decreasing and hence E is bounded by $E(0)=F(d_j)$, we can write the energy at r as the following

$$E(r,d_j) = \frac{|u'(r,d_j)|^2}{2} + F(u(r,d_j)) \le F(d_j).$$

Also, by (74) we have $F(u) \ge -J$, thus

$$\frac{|u'(r,d_j)|^2}{2} \le F(d^*) + 1 + J \le C$$

for some $C>0$. Thus, for j large, $|u'(r,d_j)| \le M_2$ for some $M_2 > 0$. By Lemma (7.3) we have $|u(r,d_j)| \le d_j$ and since $\lim_{j\to\infty} d_j = d^*$ we have $|u(r,d_j)| \le d^*+1 = M_1$ for large j. End of proof of Claim 1.

Claim 2: $u(r,d_j) \to u(r,d^*)$ and $u'(r,d_j) \to u'(r,d^*)$ uniformly on $[0,T]$ as $d_j \to d^*$, $j \to \infty$.

Proof of Claim 2: By Claim 1, we have $|u(r,d_j)| \le M_1$ and $|u'(r,d_j)| \le M_2$. So the $u(r,d_j)$ are bounded and equicontinuous. Then by Arzela-Ascoli's theorem we have a subsequence (still denoted by d_j) such that $u(r,d_j) \to u(r,d^*)$ uniformly on compact subsets of $[0,T]$ as $j\to\infty$. Then by (72) and since $u_j \to u$ uniformly on compact subsets of $[0,T]$ we have

$$\lim_{j\to\infty} u_j' = \lim_{j\to\infty} \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u_j)\,ds$$

$$= \frac{-1}{r^{N-1}} \int_0^r s^{N-1} f(u)\,ds.$$

Therefore, u_j' converges compact uniformly on $[0, T]$. Thus, $u_j'(r)$ converges uniformly say to $g(r)$. We now show that $g(r) = u'(r, d^*)$. Integrating on $(0, r)$, gives

$$\lim_{j\to\infty} \int_0^r u_j'(t)dt = \int_0^r g(t)dt$$

$$\lim_{j\to\infty} u_j(r) - u_j(0) = \int_0^r g(t)dt.$$

Since $u_j(r, d_j) \to u(r, d^*)$, we get

$$u(r, d^*) - u(0, d^*) = \int_0^r g(t)dt.$$

Differentiating this we get $u'(r, d^*) = g(r) = \lim_{j\to\infty} u_j'$. End of proof of Claim 2. □

We will use the above lemma to prove the Main Theorem in chapter 8.

CHAPTER 8

MAIN THEOREM FOR RESULT 2

To prove the Main Theorem we construct the following sets such that u has any prescribed number of zeros.

Let $\mathcal{S}_k = \{\ d | u(r,d)$ has exactly k zeros for all $r \in [0,T)\ \}$.

$\mathcal{S}_k \neq 0$ for some k since $u(r,d)$ has a finite number of zeros. Also, $\mathcal{S}_k$ is bounded above by remarks before Lemma (7.4).

Let $d_k = \sup \mathcal{S}_k$.

LEMMA 8.1. *$u(r,d_k)$ has exactly k zeros on* $[0,T)$.

PROOF. Suppose $u(r,d_k)$ has more (less) than k zeros on $[0,T)$. Then for d close to d_k, by continuity with respect to initial conditions $u(r,d)$ also has more (less) than k zeros on $[0,T)$. However, there exists values of $d \in \mathcal{S}_k$ so that $u(r,d)$ has exactly k zeros on $[0,T)$. This is a contradiction to the definition of d_k. □

LEMMA 8.2. $u(T,d_k) = 0$.

PROOF. If $u(T,d_k) \neq 0$ then $u(r,d)$ has same number of zeros as $u(r,d_k)$ for d close to d_k. But if $d > d_k$ then $d \notin \mathcal{S}_k$, so it cannot have same number of zeros as $u(r,d_k)$. This is a contradiction. Thus, $u(T,d_k) = 0$. □

Let $\mathcal{S}_{k+1} = \{\ d > d_k | u(r,d)$ has exactly one zero on $[0,T)\ \}$.

LEMMA 8.3. $\mathcal{S}_{k+1} \neq \emptyset$ *and* $\mathcal{S}_{k+1}$ *is bounded above.*

PROOF. By Lemma (7.6), if $d > d_k$ and d close to d_k then $u(r,d)$ has at most $k+1$ zeros on $[0,T)$. Also, if $d > d_k$ then $d \neq \mathcal{S}_k$ so $u(r,d)$ has at least $k+1$ zeros on $[0,T)$. Therefore, for $d > d_k$ and d close to d_k, $u(r,d)$ has exactly $k+1$ zeros on $[0,T)$. Hence $\mathcal{S}_{k+1}$ is nonempty. Then by remarks before Lemma (7.4), $\mathcal{S}_{k+1}$ is bounded above. □

Define $d_{k+1} = \sup \mathcal{S}_{k+1}$.

As above we can show that $u(r, d_{k+1})$ has exactly $k + 1$ zeros on $[0, T)$. Proceeding inductively, we can find solutions that tend to zero at infinity and with any prescribed number, n, of zeros on $[0, T)$ where $n \geq k$. Hence, we complete the proof of the main theorem. $\square$

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