NACA

RESEARCH MEMORANDUM

EFFECT OF CENTRIFUGAL FORCE ON CRITICAL FLUTTER SPEED

ON A UNIFORM CANTILEVER BEAM

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SUMMARY

Theoretical calculations were made by means of semirigid-flutter
theory to determine the effect of centrifugal force on the critical
flutter speed of a uniform cantilever beam. Centrifugal force can
under certain conditions be detrimental, decreasing the critical
flutter speed. The higher the ratio of natural torsional frequency
to natural bending frequency the greater the decrease in critical
flutter speed due to centrifugal effect. The conclusion is drawn
that compressor and turbine blades that are flutter-free at zero rpm
might become unstable at high rotative speeds. The centrifugal force
effect can therefore not be ignored in any flutter calculations on
compressor and turbine blades.

INTRODUCTION

The vibration problems of compressor and turbine blades have
become more important as the use of axial-flow compressors and turbines
has increased. These problems involve resonant vibrations due to
mechanical excitation from other parts of the machine or from pulsating
air flows from previous rows of blades in addition to nonresonant
aerodynamic excitation. The aerodynamic excitation involves a type of
vibration that is self-sustained by the continual absorption of energy
from the air stream. This vibration is called flutter.

Classical flutter is more specifically defined as a self-sustained
oscillation due to the coupling of inertia forces, elastic forces,
damping forces, and dynamic aerodynamic forces. This type of flutter
usually occurs on airplane wings at low angles of attack when the
velocity reaches a certain value called the critical flutter speed.
It is to be distinguished from a radically different type of flutter
called stalling flutter, which occurs on profiles at high angles of
attack, such as propeller blades near the stall point (reference 1).
Stalling flutter can be caused by an aerodynamic hysteresis effect,
the negative slope of the lift curve, or excitation by a system of
Karman vortices (references 2 to 4).

   Classical-flutter theory has been developed by many investigators
(references 5 to 7) and the results have been found to agree with
experiment. A direct application of this theory to analysis of com-
pressor and turbine blades would seem to indicate little likelihood
of such blades fluttering at zero angle of attack. This improbability
of flutter is due to the stiffness of compressor and turbine blades
compared to airplane wings. Two important factors, however, must be
considered: the effect of centrifugal force and the effect of
cascading. These effects may be either beneficial, that is, increase
the critical flutter speed, or detrimental, that is, decrease the
critical flutter speed.

   An approximate method of solution developed at the NACA Cleveland
laboratory of the effect of centrifugal force on the critical flutter
speed of a uniform cantilever beam and the results for beams with
various fundamental frequencies are presented herein. The effect of
cascading is yet to be investigated.

                              SYMBOLS

   The following symbols are used in the analysis:

a   coordinate of elastic axis measured from midchord in
                      units of half chord, positive towards trailing edge

a_1, b_1, c_1, functions of reduced frequency k and fundamental bending
b_2, c_2 frequency ω_b

a_y, a_y,   aerodynamic coefficients
a_θ, a_θ

b   half chord, used as reference unit length

b_y, b_y,   aerodynamic coefficients
b_θ, b_θ

c $\mu r_g{}^2$

e μr

F   function of k derived by Theodorsen (reference 5)

f_1, f_2 functions of ω_t/ω_b'

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G	function of k derived by Theodorsen (reference 5)
I	moment of inertia about elastic axis per unit span length
i	$\sqrt{-1}$
K	constant of proportionality
k	reduced frequency $\dfrac{\omega b}{v}$
L	oscillatory aerodynamic lift force per unit span
M	oscillatory aerodynamic moment per unit span
m	mass of profile per unit span
m_a	mass of cylinder of air (diameter of cylinder equal to profile chord) per unit span
N	rotative speed
r	location of center of gravity of profile measured from a in units of half chord
r_g	radius of gyration referred to a in units of half chord
S	static mass unbalance of profile, mrb
t	time
v	critical flutter speed
y	vertical displacement
y_0	maximum bending amplitude
β	phase angle by which torsion vibrational mode lags bending mode during flutter
θ	angle of torsional displacement
θ_0	maximum torsional amplitude
μ	ratio of mass of airfoil per unit span to mass of surrounding air cylinder per unit span, $\dfrac{m}{m_a} = \dfrac{m}{\pi \rho b^2}$

ρ density of surrounding air

ω frequency of flutter vibration

ω_b fundamental bending frequency of airfoil

$\omega_b{}'$ fundamental bending frequency of airfoil corrected for
                   centrifugal-force effect

ω_t fundamental torsional frequency of airfoil

     Dots over the symbols represent derivatives with respect to
time.

ANALYSIS

     Flutter equations. - The analysis of the effect of centrifugal
force is based on the usual semirigid theory. Reference 8 shows that
this theory, which assumes two degrees of freedom, that is, the
uncoupled fundamental bending and torsional modes, gives results in
close agreement with the exact solution, which assumes an infinite
number of degrees of freedom. The assumption is made herein that
these results also apply to the case in which centrifugal forces are
added. The procedure followed is then to correct the fundamental
bending mode for the centrifugal effect and to proceed in the usual
manner to obtain the critical speeds and frequencies. The effect of
centrifugal force on the fundamental torsional mode is assumed to be
small enough to be neglected. The assumption is also made that the
oscillatory air forces and moments acting on the blade are the same
as those acting on an isolated two-dimensional airfoil in an uniform
air stream. Incompressible flow is assumed and structural frictions
are ignored.

     The equations for dynamic equilibrium (reference 5) are

$$\left.\begin{array}{c} m\ddot{y} + S\ddot{\theta} + m(\omega_b{}')^2\, y - L = 0 \\[2mm] I\ddot{\theta} + S\ddot{y} + I\,\omega_t{}^2\,\theta - M = 0 \end{array}\right\} \qquad (1)$$

The oscillatory aerodynamic lift L and moment M are given by

$$\left.\begin{array}{l} L = -\, m_a\omega^2 \left[(a_y + i\bar{a}_y)\, y + (a_\theta + i\bar{a}_\theta)\, b\theta \right] \\[4mm] M = -\, m_a\omega^2 b \left[(b_y + i\bar{b}_y)\, y + (b_\theta + i\bar{b}_\theta)\, b\theta \right] \end{array}\right\} \qquad (2)$$

where

$$a_y = -\left(1 + \frac{2}{k}G\right)$$

$$\ddot{a}_y = \frac{2}{k}F$$

$$a_\theta = a + \frac{2}{k^2}F - \frac{2}{k}\left(\frac{1}{2} - a\right)G$$

$$\ddot{a}_\theta = \frac{1}{k} + \frac{2}{k^2}G + \frac{2}{k}\left(\frac{1}{2} - a\right)F$$

$$b_y = a + \frac{2}{k}\left(a + \frac{1}{2}\right)G$$

$$\bar{b}_y = -\frac{2}{k}\left(a + \frac{1}{2}\right)F$$

$$b_\theta = -\left[\frac{1}{8} + a^2 + \frac{2}{k^2}\left(a + \frac{1}{2}\right)F - \frac{2}{k}\left(\frac{1}{4} - a^2\right)G\right]$$

$$\bar{b}_\theta = -\left[\frac{2}{k^2}\left(a + \frac{1}{2}\right)G + \frac{2}{k}\left(\frac{1}{4} - a^2\right)F - \frac{1}{k}\left(\frac{1}{2} - a\right)\right]$$

(3)

where F and G are functions of reduced frequency k given in reference 5.

Let

$$y = y_0\, e^{i\omega t}$$

$$\theta = \theta_0\, e^{i(\omega t - \beta)}$$

(4)

then

$$\ddot{y} = -\omega^2 y$$

$$\ddot{\theta} = -\omega^2 \theta$$

(5)

By combining equations (1), (2), and (5) and dividing through by $m_a\omega^2$

$$\left\{a_y + i\bar{a}_y + \mu\left[\frac{(\omega_b{}')^2}{\omega^2} - 1\right]\right\}\, y + \left\{a_\theta + i\bar{a}_\theta - c\right\} b\theta = 0$$

$$\left(b_y + i\bar{b}_y - c\right) by + \left[b_\theta + i\bar{b}_\theta + e\left(\frac{\omega_t{}^2}{\omega^2} - 1\right)\right] b^2\theta = 0$$

(6)

Equations (6) have solutions other than $y = \theta = 0$ if and only if the determinant of the coefficients of y and θ vanishes. By setting this determinant equal to zero, separating real and imaginary parts, and rearranging

$$\left(\frac{\omega}{\omega_t}\right)^2 = \frac{b_2}{c_2}$$

and

$$a_1\left(\frac{\omega}{\omega_t}\right)^4 + b_1\left(\frac{\omega}{\omega_t}\right)^2 + c_1 = 0$$

(7)

where

$$b_2 = c\bar{a}_y + \mu\left(\frac{\omega_b{}'}{\omega_t}\right)^2 \bar{b}_\theta$$

$$c_2 = c\bar{a}_y + \mu\,\bar{b}_\theta + \bar{a}_\theta(b_y - e) + \bar{b}_y(a_\theta - e) - b_\theta\,\bar{a}_y - a_y\,\bar{b}_\theta$$

$$a_1 = (c - b_\theta)(\mu - a_y) + \bar{b}_y\,\bar{a}_\theta - \bar{b}_\theta\,\bar{a}_y - (a_\theta - e)(b_y - e)$$

$$b_1 = ca_y + \mu\left(\frac{\omega_b{}'}{\omega_t}\right)^2 b_\theta - c\mu\left[\left(\frac{\omega_b{}'}{\omega_t}\right)^2 + 1\right]$$

$$c_1 = c\mu\left(\frac{\omega_b{}'}{\omega_t}\right)^2$$

For a given profile, a, b, and c are functions only of the reduced frequency k and the fundamental bending frequency corrected for centrifugal force effects $\omega_b{}'$, which is, in turn, a function of the rotative speed N. For a given rotative speed N, a, b, and c are functions only of the reduced frequency k. Equations (7) can then be solved graphically for the reduced frequency k and the ratio of flutter frequency to fundamental torsional frequency ω/ω_t.

Bending frequency. - The effect of centrifugal force on fundamental bending frequency ω_b was obtained from reference 9.

$$(\omega_b{}')^2 = \omega_b{}^2 + KN^2$$

(8)

where ω_b is the fundamental bending frequency without rotation and K is a function of the ratio of blade length to root radius and of the blade taper. By combining equations (7) and (8), the reduced frequency k and flutter frequency ω can be obtained as a function of the rotative speed N. The critical flutter speed v can then be obtained from the relation

$$v = \frac{\omega b}{k} \tag{9}$$

RESULTS AND DISCUSSION

Equations (7) were solved for the reduced frequency k and the frequency ratio ω/ω_t as functions of the ratio of torsional frequency to bending frequency corrected for centrifugal-force effect ω_t/ω_b'; the results are shown in figure 1. A typical profile with the following constants was chosen for the calculations:

$$a = -0.40$$

$$r = 0.10$$

$$r_g = 0.50$$

$$\mu = 100$$

From figure 1 and equation (8) it is possible to obtain the reduced frequency k and the frequency ratio ω/ω_t as a function of the rotative speed N, once a value for the fundamental bending frequency ω_b is chosen. For this purpose, equation (8) can be conveniently rewritten in the form

$$\left(\frac{\omega_t}{\omega_b'}\right)^2 = \frac{\left(\frac{\omega_t}{\omega_b}\right)^2}{1 + \frac{KN^2}{\omega_b^2}} \tag{10}$$

For a given profile, the frequency ratio ω_t/ω_b' will therefore decrease as the rotative speed N increases. From figure 1, for values of frequency ratio ω_t/ω_b' less than approximately 0.90, the value of reduced frequency k will decrease and the value of ω/ω_t will decrease as the rotative speed N increases (or as ω_t/ω_b' decreases). At values of frequency ratio ω_t/ω_b' greater than approximately 0.90, both reduced frequency k and frequency ratio ω/ω_t increase as the rotative speed N increases (or as ω_t/ω_b' decreases). But for values of ω_t/ω_b' close to 0.90, the

reduced frequency k increases at a faster rate than ω/ω_t. The
critical flutter speed v can be expressed as

$$v = \frac{b\left(\dfrac{\omega}{\omega_t}\right)\omega_t}{k} \tag{11}$$

The critical flutter speed v will therefore increase with increasing
rotational speed N for values of frequency ratio ω_t/ω_b' less
than 0.90 and will decrease with increasing rotative speed N for
values of ω_t/ω_b' greater than 0.90. The existence of a minimum
value for the critical flutter speed v is therefore indicated.

The foregoing relations can perhaps be more clearly seen if
equation (10) is used with figure 1 to plot directly the reduced
frequency k and the frequency ratio ω/ω_t as a function of the
rotative speed N. This variation is shown in figures 2 and 3 for a
fundamental bending frequency ω_b of 100 radians per second and
frequency-ratio ω_t/ω_b' values ranging from 1 to 5. The constant K
was assumed to be $\left(\dfrac{2\pi}{60}\right)^2$. This value corresponds to a non-
tapered blade with a ratio of root radius to blade length of zero.
As shown in figures 2 and 3, the frequency ratio ω/ω_t increases
as the rotative speed N increases but the reduced frequency k
at first increases and then decreases. If the rate of increase of
reduced frequency k in any part of the region where it is
increasing is greater than the rate of increase of the frequency
ratio ω/ω_t, then from equation (11) a minimum critical flutter
speed v can be expected.

In order to verify this conclusion, the critical flutter
speed v was calculated assuming a blade chord of 2 inches. The
results are shown in figure 4 for values of fundamental bending
frequency ω_b of 100, 500, 1000, and 2000 radians per second,
respectively. The critical flutter speed v decreases to a minimum
value as the rotative speed N is increased. This minimum value
depends on the bending frequency ω_b, on the ratio of torsional to
bending frequencies ω_t/ω_b, and on the rotative speed N. The
rotative speed N at which the critical flutter speed v is a
minimum increases as the ratio of torsional to bending frequency
ω_t/ω_b increases and as the natural bending frequency ω_b increases;
also, the higher this ratio, the greater the decrease in critical
flutter speed v. This decrease in critical flutter speed v as
the rotative speed N is increased is due to the centrifugal force
causing only the bending frequency ω_b to increase. Although
increasing both the bending frequency ω_b and torsional
frequency ω_t will increase the critical flutter speed v, increasing

the bending frequency ω_b alone may decrease the critical flutter speed (reference 10). This relation can be seen if the curves of figure 1 are expressed as

$$\frac{\omega}{\omega_t} = f_1\left(\frac{\omega_t}{\omega_b{}'}\right)$$

$$k = f_2\left(\frac{\omega_t}{\omega_b{}'}\right)$$

Then by equation (11)

$$v = \frac{bf_1\left(\frac{\omega_t}{\omega_b{}'}\right)\omega_t}{f_2\left(\frac{\omega_t}{\omega_b{}'}\right)}$$

If the torsional frequency ω_t and bending frequency $\omega_b{}'$ are both increased so that the ratio of torsional to bending frequency $\omega_t/\omega_b{}'$ remains constant, $\dfrac{f_1\left(\frac{\omega_t}{\omega_b{}'}\right)}{f_2\left(\frac{\omega_t}{\omega_b{}'}\right)}$ will remain constant. The value of critical flutter speed v, however, increases because of the increase in torsional frequency ω_t. A change in the value of bending frequency $\omega_b{}'$ alone, however, may either raise or lower the value

of $\dfrac{f_1\left(\frac{\omega_t}{\omega_b{}'}\right)}{f_2\left(\frac{\omega_t}{\omega_b{}'}\right)}$ and produce a corresponding change in critical flutter speed v.

At high frequency ratios ω_t/ω_b and at high values of the natural bending frequency ω_b, the critical flutter speeds v obtained are above the lower compressibility limit for which the calculations are valid and the results are therefore meaningless. This limit has been arbitrarily chosen as 800 feet per second and all curves above that speed are shown as dashed lines.

For natural bending frequencies ω_b higher than 1000 radians per second, classical flutter cannot occur in the subsonic range except at low frequency ratios ω_t/ω_b and possibly at high rotative speeds N as shown by figure 4(c) and 4(d). Because current compressor and

turbine blades usually have bending frequencies greater than
1000 radians per second, little danger of such blades fluttering
classically would exist if the torsional frequency ω_t were several
times the bending frequency ω_b. However, at high rotative speeds N,
the critical flutter speed v can be greatly reduced and blades that
would be safe at zero rpm can become unstable. As an example, fig-
ure 4(c) shows that at a frequency ratio ω_t/ω_b of 2 the flutter speed v
is reduced from a value that is out of the subsonic range to about
615 feet per second as the rotative speed is increased from 0 to
10,000 rpm.

CONCLUSIONS

From the calculations made to determine the effect of centrifugal
force on the flutter of a uniform cantilever beam, the conclusion is
drawn that under certain conditions centrifugal force can be detri-
mental, decreasing the critical flutter speed. A compressor or tur-
bine blade that would be stable at zero rpm might become unstable at
high rotative speeds. The flutter characteristics of blades must
therefore be investigated at the operating engine speeds. The
validity of the semirigid flutter theory with the centrifugal force
effect included must still be verified. The effect of cascading must
also be investigated.

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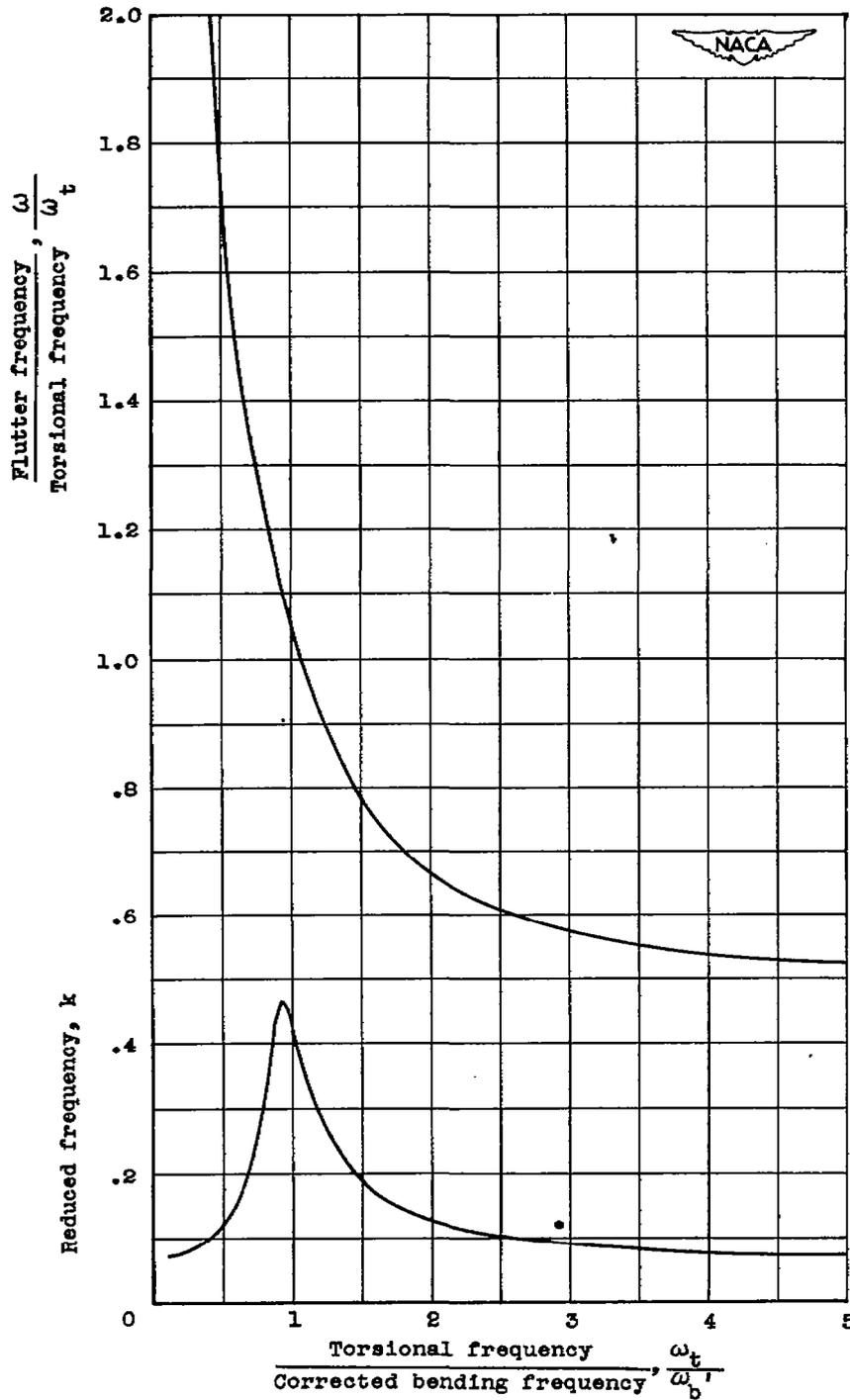
Figure 1. - Variation of ratio of flutter frequency to
torsional frequency and of reduced frequency with
ratio of torsional frequency to corrected bending
frequency.

616
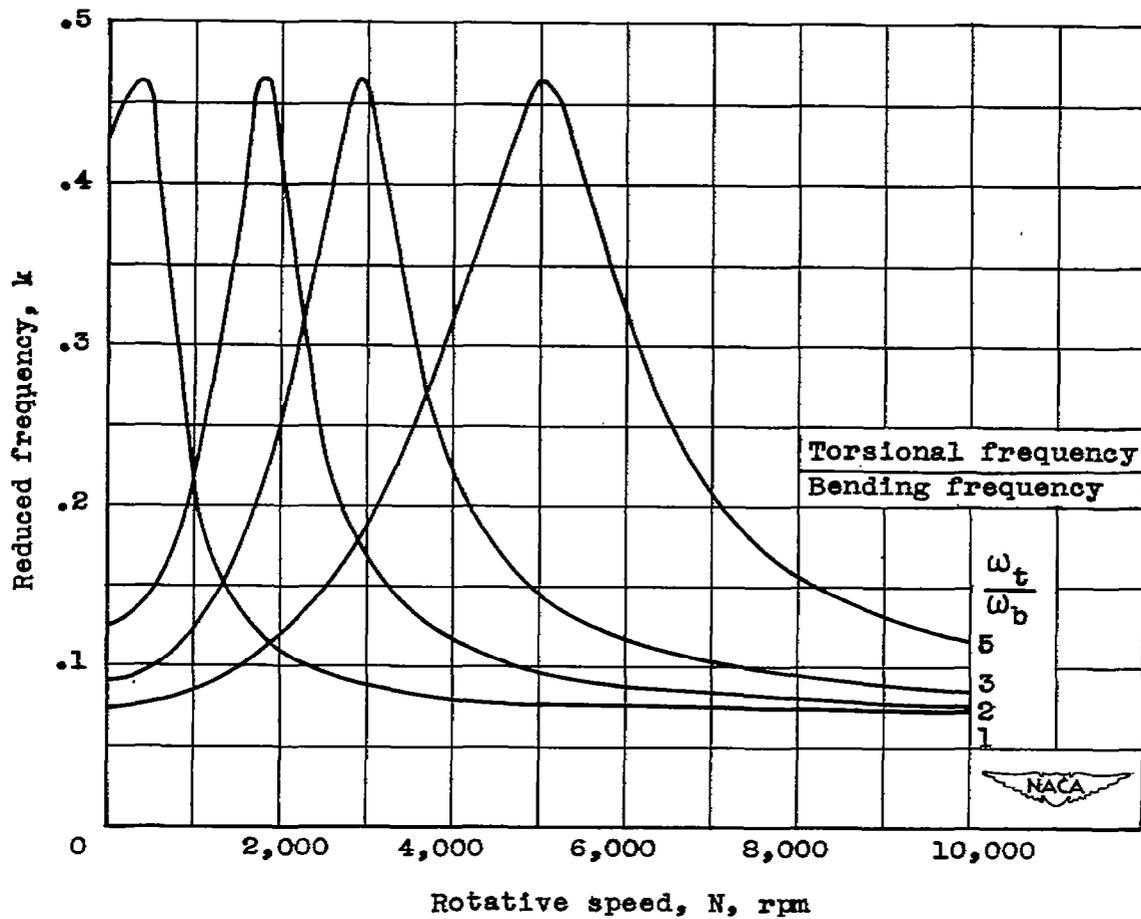
Figure 2. - Variation of reduced frequency with rotative speed
for various ratios of torsional to bending frequencies. Funda-
mental bending frequency, 100 radians per second.
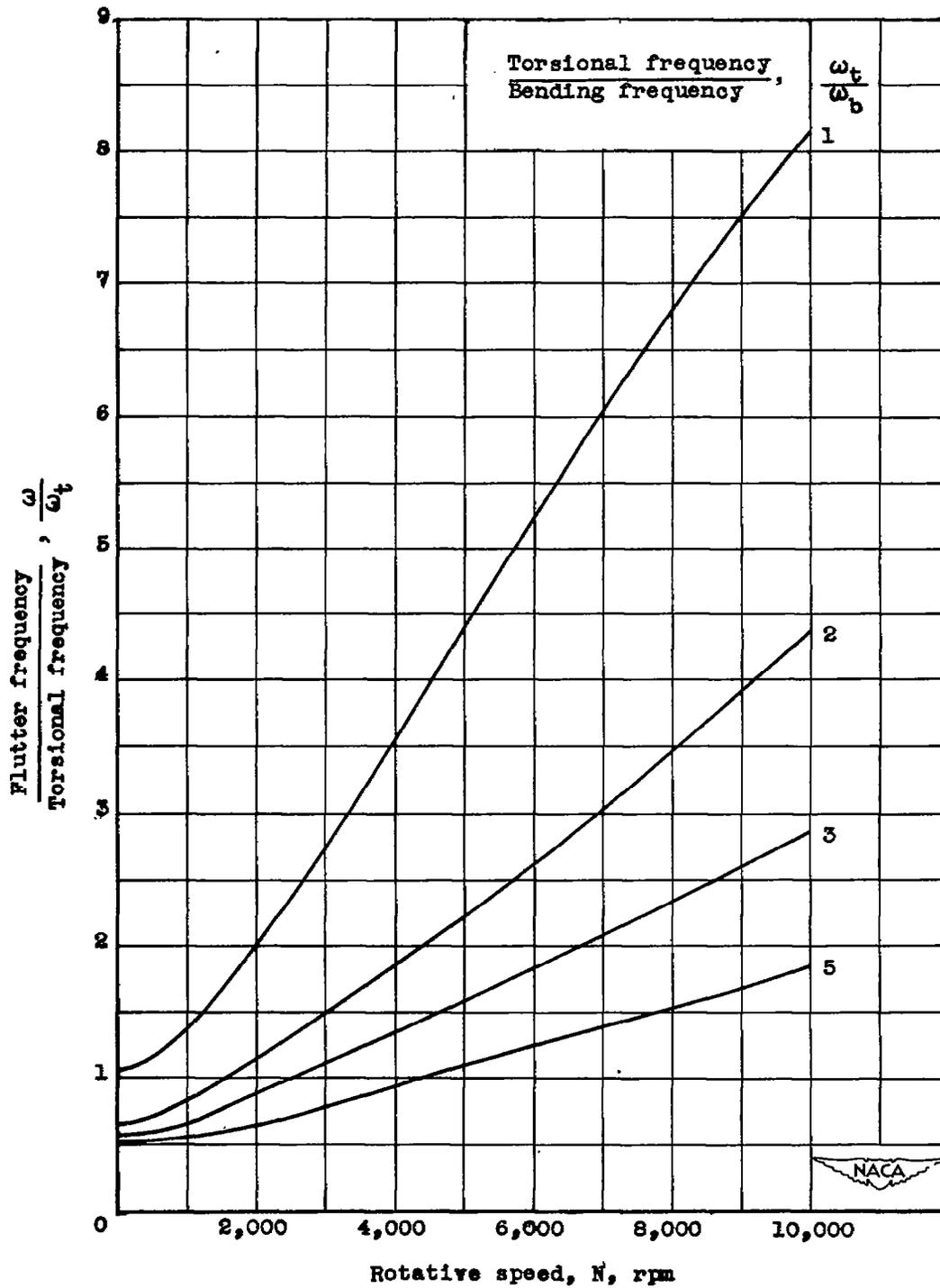
Figure 5. - Variation of ratio of flutter frequency to torsional
frequency with rotative speed for various ratios of torsional
to bending frequencies. Fundamental bending frequency, 100
radians per second.
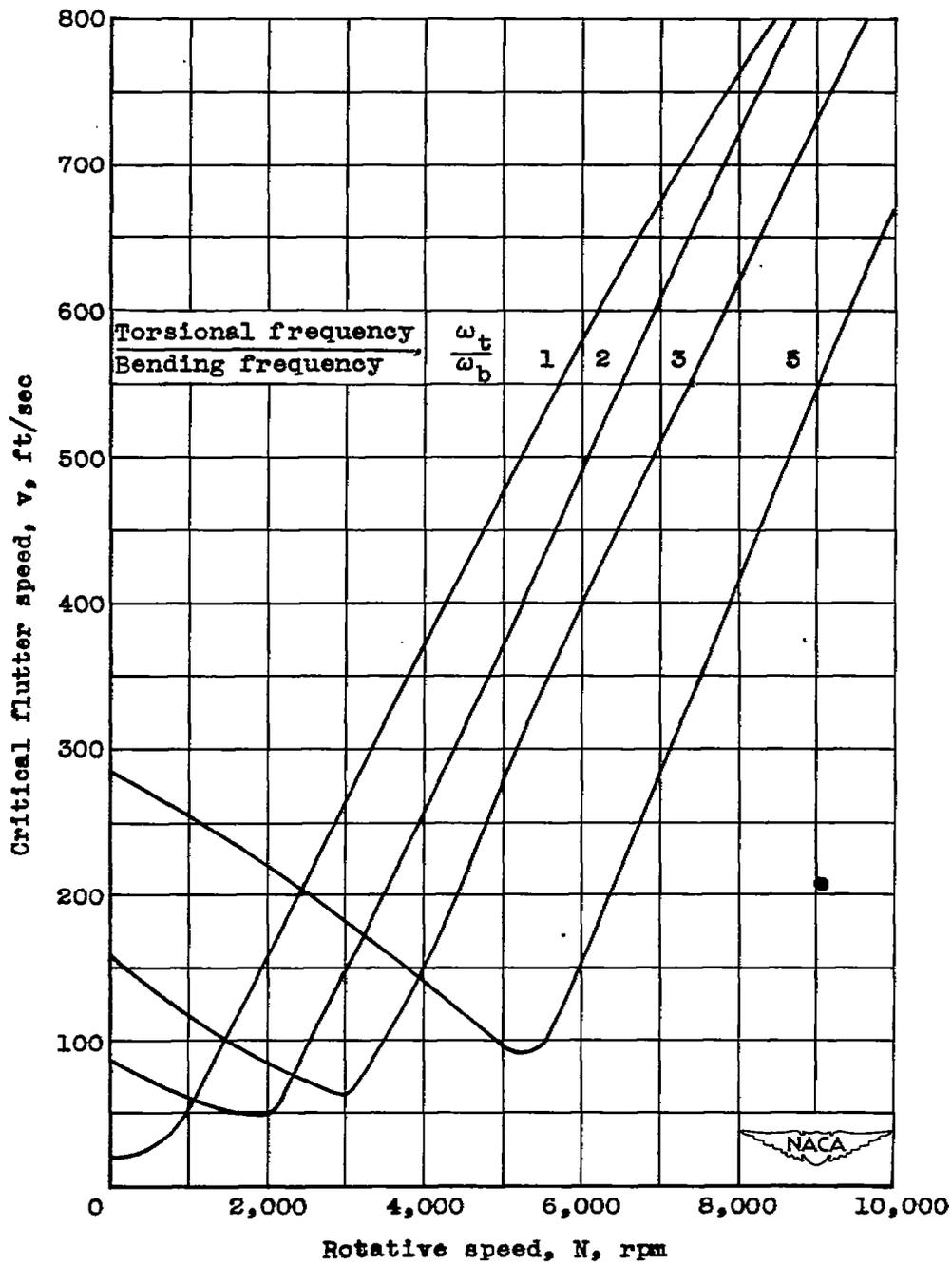
(a) Fundamental bending frequency, 100
radians per second.

Figure 4. - Variation of critical flutter speed with
rotative speed for various ratios of torsional to
bending frequencies.
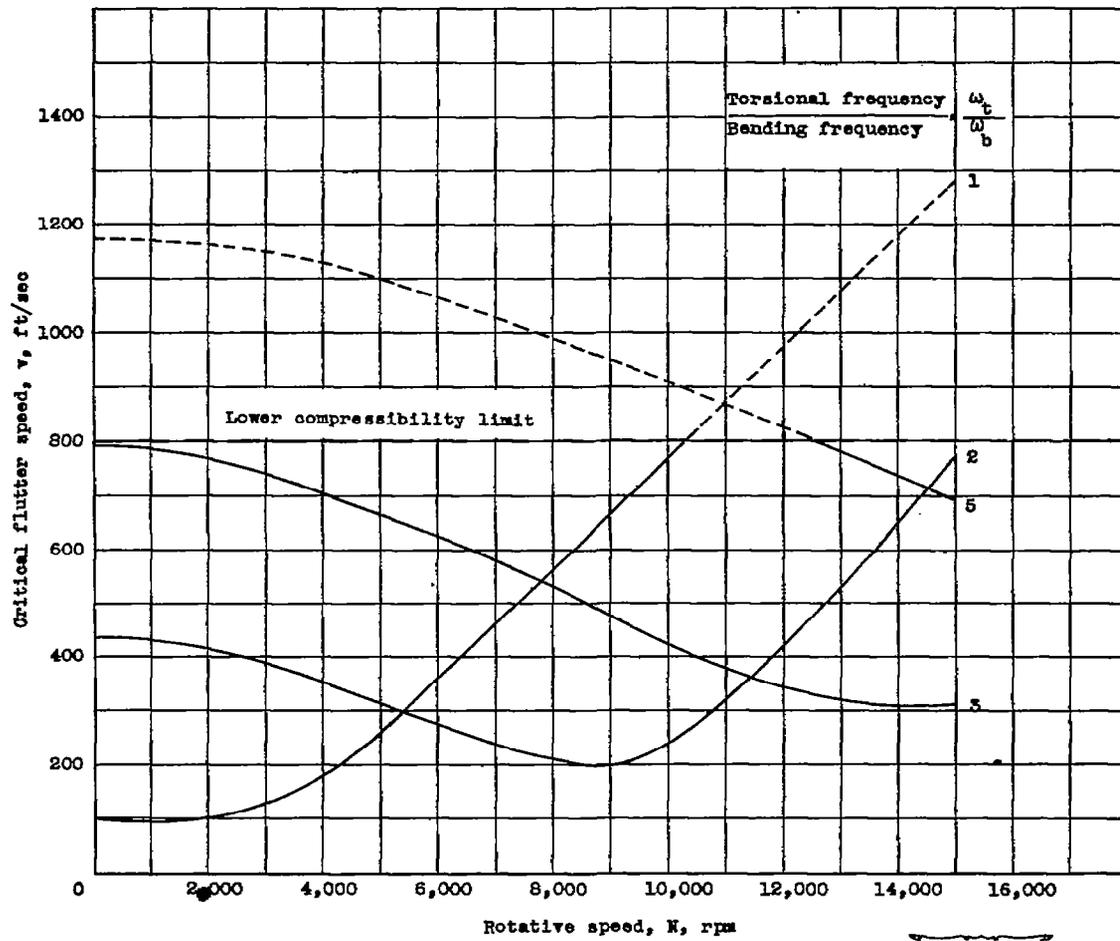
(b) Fundamental bending frequency, 500 radians per second.

Figure 4. - Continued.   Variation of critical flutter speed with rotative speed for various
ratios of torsional to bending frequencies.
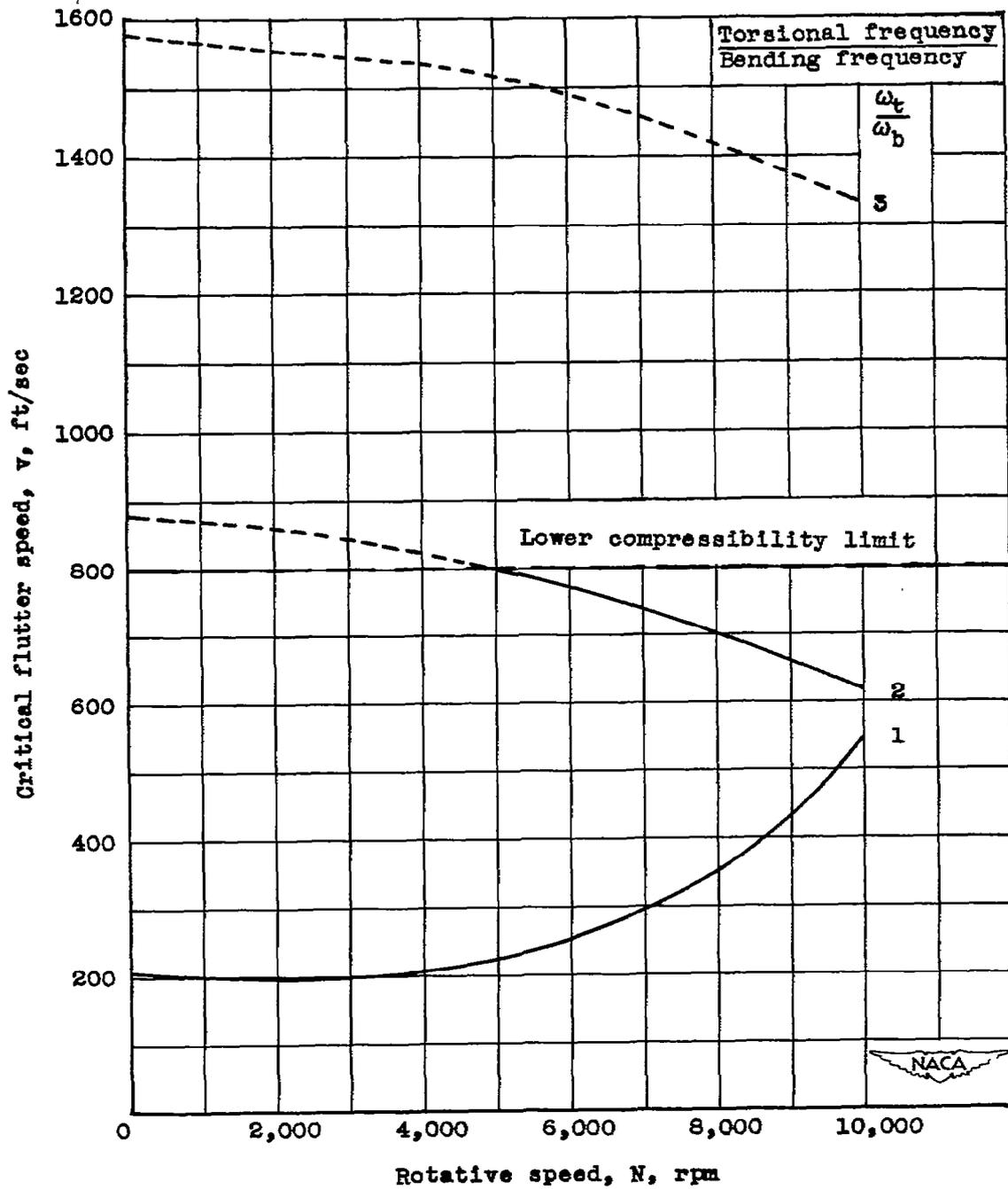
(c) Fundamental bending frequency, 1000 radians
per second.

Figure 4. - Continued. Variation of critical flutter speed with
rotative speed for various ratios of torsional to bending
frequencies.
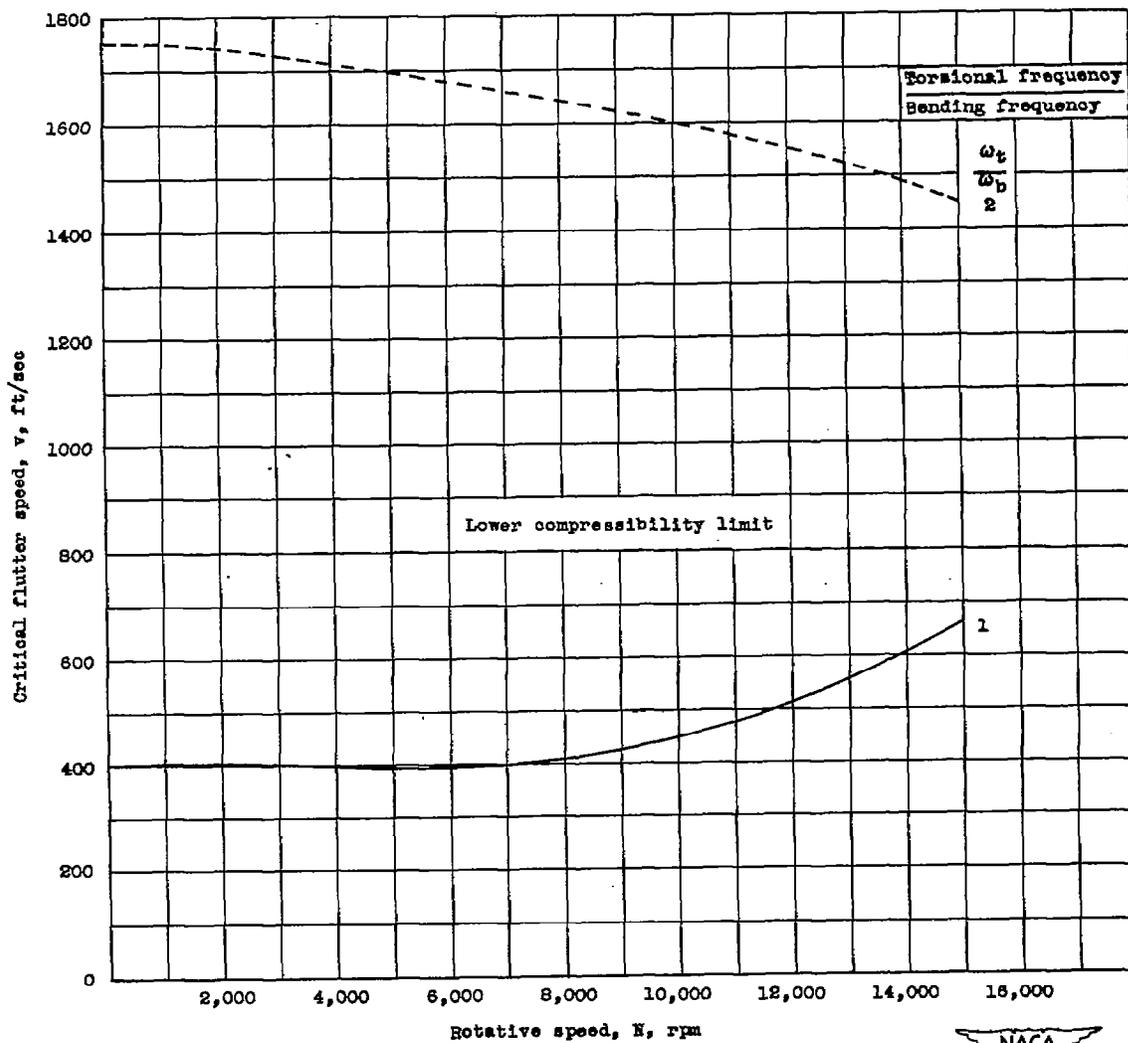
(d) Fundamental bending frequency, 2000 radians per second.

Figure 4. – Concluded. Variation of critical flutter speed with rotative speed for various
ratios of torsional to bending frequencies.