NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3331

ANALYSIS OF LAMINAR FORCED-CONVECTION HEAT TRANSFER IN ENTRANCE REGION OF FLAT RECTANGULAR DUCTS

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Washington
January 1955

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SUMMARY

The simultaneous development of temperature and velocity profiles in the entrance region of a flat rectangular duct is studied. The flow is assumed to be laminar with negligible dissipation. All fluid properties are taken to be constant, and the wall temperatures are uniform. Thermal and velocity boundary layers are calculated using the Kármán-Pohlhausen method. Nusselt numbers are reported for Prandtl numbers in the range 0.01 to 50.

When the Nusselt numbers are plotted against Graetz number $\left(\frac{Re_d Pr}{x/D_e}\right)$, there is a separate curve for each Prandtl number. In contrast, the Nusselt numbers for the case of a parabolic velocity profile throughout can be plotted against Graetz number as a single curve which is applicable for all Prandtl numbers. Beyond the position where the boundary layer analysis can no longer be used, curves have been faired to connect the Nusselt number results of the present analysis with those of Norris and Streid, who assumed a parabolic velocity profile throughout the length of the duct.

Calculations were also made for a duct having one wall at uniform temperature and the other wall insulated. Nusselt numbers are reported for the range of Prandtl numbers from 0.01 to 50.

The results of the present analysis are compared with those for a single flat plate. For purposes of comparison with the results of Norris and Streid, the boundary layer analysis was also used to calculate Nusselt numbers for the case of an unchanging parabolic velocity profile.

INTRODUCTION

Until recently, analytical studies of laminar forced-convection flow in tubes and ducts have been concerned primarily with three groups of problems. In the first group, a fluid with a velocity profile that is already fully developed enters a section of pipe having a wall temperature different from the temperature of the entering fluid. The initial analysis for the development of the temperature profile was made by Graetz in 1883 for a round tube. In 1910, Graetz's solution was obtained independently by Nusselt. Since then, a number of investigators have recalculated and extended Graetz's results; for example, Drew, Jakob, Lee, and Groeber. A different approach to Graetz's problem was made by Lévêque in an effort to obtain better numerical values in the entrance region of the tube section, where Graetz's results are inaccurate. A presentation of the aforementioned investigations is given in reference 1 and reference 2. The problem analogous to that of Graetz for flow between two parallel planes (flat rectangular duct) has been treated by Norris and Streid (ref. 3) and by Prins, Mulden, and Schenk (ref. 4). 3349

The second class of problems is concerned with the development of the velocity profile in the entrance region of a pipe with no heat transfer. The fluid is assumed to enter the pipe with a uniform velocity. In the course of its flow through the pipe, the fluid is retarded by friction at the pipe wall, so that the velocity profile is distorted and finally becomes parabolic far from the entrance. For the round pipe, the problem has been studied by Schiller, Atkinson and Goldstein, Boussinesq (all reported in ref. 5, pp. 301-308), and Langharr (ref. 6). The first three used the boundary layer concept in at least part of the analysis. The analogous problem for the flat duct was treated by Schlichting (ref. 7) and by Schiller (ref. 5, p. 309), both of whom assumed the existence of boundary layers.

In the third group are so-called fully developed problems in which the heat-transfer and friction parameters do not change along the length of the duct. The results are applied to long ducts. Examples of fully developed solutions may be found in the work of Clark and Kays (ref. 8), where the flow and heat transfer in noncircular ducts is studied for the cases of uniform wall temperature and uniform heat flux.

Only recently has some attention been given to the case of simultaneous development of velocity and temperature profiles in the entrance region of a pipe. Solutions for air flowing in round tubes have been obtained by Kays (ref. 9) and by Toong (ref. 10). With the assumption that air is an incompressible, constant-property fluid, a direct numerical integration of a simplified energy equation by the use of finite differences is obtained in reference 9. The velocity values required in this integration were taken from the results of reference 6. The analysis of reference 10, which is for high-speed flow, assumes that the air is compressible while the other properties are constant. The boundary layer differential equations are transformed and then solved on a differential analyzer for specific values of inlet Mach number and wall-to-inlet temperature ratio.

In the present analysis, simultaneous development of the velocity and temperature profiles in the entrance region of a flat rectangular duct is studied. Incompressible, constant-property fluids having Prandtl numbers in the range 0.01 to 50 are considered. The presence of velocity and thermal boundary layers is assumed, and these boundary layers are taken to have definite thicknesses, which are calculated by the Kármán-Pohlhausen method. Nusselt numbers are computed for the length of duct over which the boundary layer analysis is made. Beyond the position where the boundary layer treatment can no longer be used, curves have been faired to connect the Nusselt number results of the present analysis with those of reference 3, in which a parabolic velocity profile is assumed throughout the entire duct length.

In addition to the symmetrical situation in which both duct walls have the same uniform temperature, a second case is treated, in which one wall is insulated and the opposite wall is held at a uniform temperature.

The Nusselt numbers obtained in the present analysis should also be applicable to annuli in which the curvature effects are small (i.e., where the ratio of the diameters of the concentric cylinders forming the walls of the annulus is close to 1).

So that a comparison could be made with the results of reference 3, the boundary layer analysis was used to study the development of the temperature profile associated with a parabolic velocity profile throughout. The development of the temperature profile was also calculated for the case of a uniform velocity profile throughout (slug flow).

ASSUMPTIONS

The physical model selected for a study of the development of the velocity and temperature profiles in the entrance region of a flat rectangular duct is the following:

(1) The flow is laminar.

(2) All fluid properties are constant.

(3) The velocity and temperature profiles are uniform across the entrance section.

(4) There exists a velocity boundary layer of definite thickness δ. As on a flat plate, the action of viscocity in the entrance region of the duct is confined primarily to the fluid layers near the wall, although in actuality small effects exist everywhere in the flow. It will be assumed in this analysis that viscosity plays a role only in a definite region adjacent to the wall called the velocity boundary

layer δ. The fluid outside the velocity boundary layer is called the core. Inasmuch as the fluid adjacent to the wall is retarded by friction, it is necessary that the flow in the core be accelerated in order that the same mass pass through every cross section of the duct. The velocity profile is taken to be flat throughout the core. The velocity boundary layer thickness is assumed zero at the entrance section. The boundary layer grows in thickness along the length of the duct until it reaches the center line, where it meets the boundary layer from the other wall of the duct. Figure 1 shows diagrammatically the development of the velocity boundary layer.

(5) Two temperature conditions at the duct walls are to be considered:

(a) Both duct walls have the same uniform temperature, which differs from the temperature of the fluid entering the duct; this will be called the case of heat transfer at both walls.

(b) One of the duct walls is maintained at a uniform temperature which differs from the entering fluid temperature while the other wall is insulated; this will be called the case of heat transfer at one wall.

(6) The viscous dissipation and work of compression are negligible compared with the heat conduction.

(7) There exists a thermal boundary layer of definite thickness Δ. Since the temperature of the fluid entering the duct differs from the temperature of the duct walls, there will be a heat transfer to (or from) the fluid. The principal effects of this heat transfer will be felt by the fluid layers close to the wall, although small effects will exist everywhere. The assumption will be made that the effects of the heat transfer play a role only in a definite region adjacent to the wall called the thermal boundary layer Δ. The fluid outside the thermal boundary layer will be uninfluenced by the heat transfer and will therefore have a uniform temperature identical to the value at the entrance of the duct. The thermal boundary layer thickness will be assumed zero at the entrance section. For the symmetrical case where both duct walls have the same uniform temperature, thermal boundary layers will grow in thickness along the length of the duct in an identical manner for both walls. These boundary layers will meet at the center line, where $\Delta = a$. For the case where one duct wall is maintained at a uniform temperature and the opposite wall is insulated, a thermal boundary layer will develop only at the wall at which the heat transfer occurs. The boundary layer will grow until it reaches the opposite wall, where $\Delta = 2a$. In general, the velocity and thermal boundary layers may be of different thicknesses.

(8) The flow is two-dimensional.

(9) The flow is steady.

BASIC EQUATIONS

The laws of conservation of momentum, energy, and mass for steady
3349 laminar boundary layer flow with constant properties and negligible dissipation and work of compression are as follows:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1a}$$

$$\frac{\partial p}{\partial y} = 0 \tag{1b}$$

$$u \frac{\partial t}{\partial x} + v \frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2} \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

(The symbols used herein are defined in appendix A.) For the core flow, which has been assumed nonviscous and uniform across the section, the momentum equation reduces to

$$U_1 \frac{dU_1}{dx} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{4}$$

The pressure term is eliminated from equation (1a) by means of equations (4) and (1b). Then, using the assumption that definite values Δ and δ are associated with the thermal and velocity boundary layers, respectively, equations (1a), (2), and (3) are integrated across the section to yield

$$\frac{d}{dx}\left[\int_0^\delta (U_1 - u)u \, dy \right] + \frac{dU_1}{dx} \int_0^\delta (U_1 - u)dy = \nu \left. \frac{\partial u}{\partial y} \right|_{y=0} \tag{5}$$

$$\frac{d}{dx}\left[\int_0^\Delta (t_1 - t)u \, dy \right] = \alpha \left. \frac{\partial t}{\partial y} \right|_{y=0} \tag{6}$$

This integration may be found in reference 5, page 136 and pages 614-615.

$$\int_0^{\delta} (u - U_1)dy + a(U_1 - \overline{\overline{U}}) = 0 \tag{7}$$

These equations require only that the conservation laws be satisfied for the cross section as a whole. This is a much less stringent requirement than that imposed by equations (1), (2), and (3), which ask that the conservation laws be satisfied at every point in the cross section.

According to the Kármán-Pohlhausen method, the velocity and temperature are approximated by polynomials in y having coefficients that are functions of x. The coefficients are determined by satisfying boundary conditions at the duct wall and at the edge of the boundary layer and by using equations (5), (6), and (7). 3349

A simple and convenient expression for the velocity profile in the boundary layer which gives reasonably good agreement with Schlichting's more exact results is the following one first used by Schiller:

$$u = U_1\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] \qquad 0 \leqslant y \leqslant \delta \tag{8}$$

This equation satisfies the conditions

$$\left.\begin{aligned} &u = 0 \text{ at } y = 0 \\ &\left.\begin{aligned} u &= U_1 \\ \frac{\partial u}{\partial y} &= 0 \end{aligned}\right\} \text{at } y = \delta \end{aligned}\right. \tag{9}$$

but does not satisfy the condition that at the wall $(y = 0)$, $\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$. This requirement arises from the evaluation of equation (1a) at the wall, where $u = v = 0$.

For the temperature profile in the boundary layer, the following relation is chosen:

$$\frac{t - t_w}{t_1 - t_w} = \frac{3}{2}\left(\frac{y}{\Delta}\right) - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \qquad 0 \leqslant y \leqslant \Delta \tag{10}$$

The boundary conditions satisfied by this expression are

$$\left.\begin{aligned} t &= t_w \\ \frac{\partial^2 t}{\partial y^2} &= 0 \end{aligned}\right\} \text{at } y = 0 \tag{11}$$

$$\left.\begin{aligned} t &= t_1 \\ \frac{\partial t}{\partial y} &= 0 \end{aligned}\right\} \text{at} \quad y = \Delta \tag{11}$$

Cont.

The condition $\frac{\partial^2 t}{\partial y^2} = 0$ at $y = 0$ arises from the evaluation of equation (2) at the wall.

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It is seen from equations (8) and (10) that δ, U_1, and Δ remain to be determined as functions of x. The assumption of constant fluid properties means that the solution for the velocity (i.e., δ and U_1) is independent of the heat transfer. (The velocity solution therefore will apply for both the case of heat transfer at both walls and the case of heat transfer at one wall.) However, the solution of the temperature problem (i.e., Δ) requires a prior knowledge of the velocity.

VELOCITY PROBLEM

In a manner similar to that followed by Schiller, the velocity expression (8) is introduced into the integrated momentum and mass equations (5) and (7), respectively, to give the following relations between δ, U_1, and x:

$$dx = \frac{3}{10} \frac{a^2}{\nu} \frac{(U_1 - \overline{U})(9U_1 - 7\overline{U})}{U_1^2} dU_1 \tag{12}$$

$$\frac{\delta}{a} = 3\left(1 - \frac{\overline{U}}{U_1}\right) \tag{13}$$

It is convenient to introduce the dimensionless variables

$$\left.\begin{aligned} x^* &= \frac{x}{a} \frac{\nu}{a\overline{U}} = \frac{16 \frac{x}{D_e}}{Re_d} \\ U_1^* &= \frac{U_1}{\overline{U}} \\ \delta^* &= \frac{\delta}{a} \end{aligned}\right\} \tag{14}$$

where the equivalent diameter D_e is equal to $4a$ for the duct with spacing $2a$. Equations (12) and (13) become

$$dx^* = \frac{3}{10}\frac{(U_1^* - 1)}{U^{*2}}(9U_1^* - 7)\ dU^* \tag{12a}$$

$$\delta^* = 3\left(1 - \frac{1}{U_1^*}\right) \tag{13a}$$

At the inlet $U_1^* = 1$, since $U_1 = \overline{U}$; and for fully developed flow $U_1^* = 1.5$, since $U_1 = 1.5\ \overline{U}$. The integration of equation (12a) is carried through numerically to $U_1^* = 1.5$ $(\delta^* = 1)$, at which point the velocity given by equation (8) coincides with the fully developed parabolic profile. It should be noted that the velocity solution given here does not join smoothly to the fully developed parabolic profile, since $\frac{dU_1^*}{dx^*}$ does not approach zero as U_1^* approaches 1.5. In the neighborhood of the entrance section, that is, near $x^* = 0$, the solution for the velocity is

$$U_1^* - 1 = \sqrt{\frac{10\ x^*}{3}} \tag{15}$$

TEMPERATURE PROBLEM FOR HEAT TRANSFER AT BOTH WALLS

The cases of $\Delta > \delta$ and $\Delta < \delta$ will be treated separately. Another section will be devoted to solutions for the temperature for cases where the velocity profile is unchanging throughout the duct length. The heat-transfer parameters will be defined in the final section.

Case of $\Delta > \delta$. - For this case, the velocity profile in the thermal boundary layer is flat for values of $\delta \leqslant y \leqslant \Delta$. In the region $0 \leqslant y \leqslant \Delta$, which represents the range of integration in the energy equation (6), the velocity profile is given by two expressions.

$$u = U_1\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] \qquad 0 \leqslant y \leqslant \delta \tag{8}$$

$$u = U_1 \qquad 0 \leqslant y \leqslant \Delta \tag{16}$$

As a consequence, the energy equation can be written as

$$\frac{d}{dx}\left[\int_0^\delta (t_1 - t)u\ dy + U_1\int_\delta^\Delta (t_1 - t)dy\right] = \alpha\left.\frac{\partial t}{\partial y}\right|_{y=0} \tag{6a}$$

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Introduction of the temperature expression (10) and the velocity expression (8) yields the following dimensionless first-order differential equation:

$$d\left[U_1^*\left(\frac{3}{8}\,\Delta^* - \frac{1}{3}\,\delta^* + \frac{1}{8}\,\frac{\delta^{*2}}{\Delta^*} - \frac{1}{120}\,\frac{\delta^{*4}}{\Delta^{*3}}\right)\right] = \frac{3}{2Pr}\,\frac{dx^*}{\Delta^*} \tag{17}$$

The quantities U_1^*, δ^*, and x^* have already been defined in equation (14), and

$$\Delta^* = \frac{\Delta}{a} \tag{14a}$$

The integration of equation (17) to determine the relation between Δ^* and x^* is discussed in appendix B. Calculations were made over the range from $\Delta^* = 0$ at $x^* = 0$ to $\Delta^* = 1$ $(\Delta = a)$. In the neighborhood of the entrance, that is, near $x^* = 0$, it was found that $\Delta^* = C\sqrt{x^*}$, where C is a function of Prandtl number.

Case of $\Delta < \delta$. - Since for this case the velocity boundary layer extends beyond the thermal boundary layer, only a single expression, equation (8), is required to give the velocity profile for the region $0 \leq y \leq \Delta$. Using the temperature expression (10) and the velocity expression (8) in the energy equation (6) gives the following dimensionless differential equation:

$$d\left[U_1^*\left(\frac{1}{5}\,\frac{\Delta^{*2}}{\delta^*} - \frac{1}{24}\,\frac{\Delta^{*3}}{\delta^{*2}}\right)\right] = \frac{3}{2Pr}\,\frac{dx^*}{\Delta^*} \tag{18}$$

The integration of this equation also is discussed in appendix B. The remarks already made for equation (17) concerning the range of integration and the behavior near $x^* = 0$ also apply to equation (18).

Cases of unchanging velocity profile. - The two unchanging velocity profiles to be considered are the parabolic and the uniform.

Parabolic velocity profile: The equation for the parabolic profile with y measured from the duct wall is

$$u = 1.5\,\overline{U}\left[2\left(\frac{y}{a}\right) - \left(\frac{y}{a}\right)^2\right] \tag{19}$$

Using this relation and the temperature expression (10) in evaluating the energy equation (6) leads to the following dimensionless differential equation:

$$d\left(\frac{3}{10}\Delta^{*2} - \frac{1}{16}\Delta^{*3}\right) = \frac{3}{2\mathrm{Pr}}\frac{dx^*}{\Delta^*} \tag{20}$$

The solution for the initial condition $\Delta^* = 0$ at $x^* = 0$ is

$$\frac{\Delta^{*3}}{5} - \frac{3}{64}\Delta^{*4} = \frac{3}{2\mathrm{Pr}}x^* \tag{21}$$

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Uniform velocity profile: The equation for the uniform velocity profile is simply

$$u = \overline{U}$$

and the solution of the energy equation (6) for initial conditions $\Delta^* = 0$ at $x^* = 0$ is

$$\Delta^* = \sqrt{\frac{8x^*}{\mathrm{Pr}}} \tag{22}$$

Heat-transfer parameters. - The heat-transfer results will be presented in dimensionless form utilizing the average Nusselt number, which is defined in the usual way as

$$\overline{\mathrm{Nu}} = \frac{\overline{h}D_e}{k} \tag{23}$$

For a duct with spacing 2a, the equivalent diameter D_e is equal to 4a. The average heat-transfer coefficient for a length of duct x (and unit width) in which there is heat transfer at both walls is given by

$$\overline{h} = \frac{Q}{A(\Delta t)} = \frac{Q}{2x(1)(\Delta t)} \tag{24}$$

The temperature difference to be used remains to be specified. Heat-transfer coefficients for three definitions of Δt will be calculated, and accordingly there will be a set of Nusselt numbers corresponding to each of the three definitions of Δt. These definitions are

$$\left.\begin{aligned}
(\Delta t)_1 &= \frac{(t_w - t_1) - (t_w - t_{b,x})}{\ln\dfrac{t_w - t_1}{t_w - t_{b,x}}} \\
(\Delta t)_2 &= (t_w - t_1) \\
(\Delta t)_3 &= \frac{(t_w - t_1) + (t_w - t_{b,x})}{2}
\end{aligned}\right\} \tag{25}$$

In the first and third definitions, $t_{b,x}$ represents the bulk temperature corresponding to any position x. The bulk temperature is most conveniently obtained from the steady-flow energy equation

$$(t_{b,x} - t_1) = \frac{Q}{mc_p} = \frac{Q}{\rho 2a(1)\overline{U}c_p} \tag{26}$$

in which the kinetic energy and gravity differences have been taken to be negligible.

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The total heat transferred at both duct walls in a length x and width equal to unity is

$$Q = -2 \int_0^x k\left(\frac{\partial t}{\partial y}\right)_0 dx \tag{27}$$

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The derivative in equation (27) is evaluated from the temperature expression (10) to give

$$Q = + 3k(t_w - t_1) \int_0^x \frac{dx}{\Delta} \tag{28}$$

All the quantities necessary to evaluate the Nusselt number are given in equations (24) to (26) and (28); the results are as follows:

$$\overline{Nu}_1 = 0.25 \text{ Gz} \ln\left(\frac{1}{1-B}\right) \tag{29a}$$

$$\overline{Nu}_2 = 0.25 \text{ Gz } B \tag{29b}$$

$$\overline{Nu}_3 = 0.25 \text{ Gz} \frac{B}{(1-0.5B)} \tag{29c}$$

The symbol Gz represents the Graetz number, which is defined by

$$\text{Gz} \equiv \frac{Re_d Pr}{x/D_e} = \frac{16a^2\rho c_p\overline{U}}{kx} = \frac{16Pr}{x^*} \tag{30}$$

The other dimensionless group appearing in equation (29) is given by

$$B = \frac{3}{2Pr} \int_0^{x^*} \frac{dx^*}{\Delta^*} \tag{31}$$

In general, B depends separately upon Prandtl number and x^*. By virtue of the relation between Graetz number and x^* given in equation (30), B may also be considered a function of Prandtl number and of Graetz number.

Inspection of equations (29) shows that in the most general case, the Nusselt number will depend separately upon two other dimensionless groups. For example,

$$\overline{Nu} = f(Gz,Pr)$$

or, alternatively,

$$\left.\begin{aligned}\overline{Nu} &= f(x^*,Pr) = f\left(\frac{x/D_e}{Re_d},Pr\right)\end{aligned}\right\} \tag{32}$$

For the case of simultaneous development of the velocity and temperature profiles, the Nusselt number depends separately on two other groups, as shown in equation (32). However, for the case of the parabolic velocity throughout (and for the case of the uniform velocity profile as well), reference to equations (20) and (21) shows that B is a function of Graetz number alone. Hence, for these unchanging profiles, the Nusselt number depends on only the Graetz number.

Although the boundary layer nature of the present analysis does not allow calculations to be made for very long ducts, it is interesting to notice the behavior of the various Nusselt numbers at great distances from the entrance. To this end, expression (24) for h is introduced into the defining equation (23) for the Nusselt number to give

$$\overline{Nu} = \frac{Q\ D_e}{kA(\Delta t)} \tag{33}$$

For very long ducts, the duct surface area A becomes very large. Also, the bulk temperature of the stream will approach the wall temperature. Inspection of the definitions (25) shows that as A approaches infinity,

$$\left.\begin{aligned}\Delta t_1 &\rightarrow 0\\ \Delta t_2 &\rightarrow (t_w-t_1)\\ \Delta t_3 &\rightarrow \frac{(t_w-t_1)}{2}\end{aligned}\right\} \tag{34}$$

Equation (33), then, indicates that as A approaches infinity, $\overline{Nu_2}$ and $\overline{Nu_3}$ approach zero. $\overline{Nu_1}$ appears to be an indeterminate form, the limiting value of which has been determined in reference 3 as 7.60.

TEMPERATURE PROBLEM FOR HEAT TRANSFER AT ONE WALL

The temperature problem, as before, must be formulated separately for $\Delta > \delta$ and $\Delta < \delta$. It should be noted that this separate formulation

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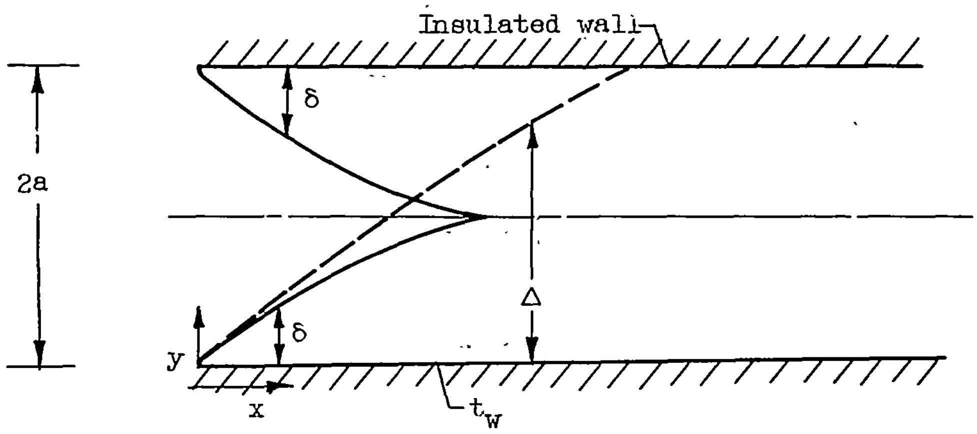
is necessary only for $\delta < a$, where different formulas are required to describe the velocity profiles in the boundary layer and in the core. When $\delta = a$, the velocity profile is fully developed and the parabolic formula gives the velocity over the entire cross section. In all cases, the calculations are limited to $\Delta < 2a$.

Case of $\Delta > \delta$. - A sketch of the development of the velocity and thermal boundary layers for this case is given here.



It is seen that the thermal boundary layer, growing into the duct from the wall at temperature t_w, eventually will meet the velocity boundary layer developing along the opposite wall. The meeting will occur when $\Delta = 2a - \delta$. In the region preceding this meeting, that is, for values of $\delta < \Delta < (2a - \delta)$, the analysis already given for the temperature problem with heat transfer at both walls for $\Delta > \delta$ is valid.

For $(2a - \delta) < \Delta < 2a$, it is necessary to use three formulas to express the velocity profile.

$$\left.\begin{aligned} u &= U_1\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] && 0 \leqslant y \leqslant \delta \\ u &= U_1 && \delta \leqslant y \leqslant (2a-\delta) \\ u &= U_1\left[2\left(\frac{2a-y}{\delta}\right) - \left(\frac{2a-y}{\delta}\right)^2\right] && (2a-\delta) \leqslant y \leqslant \Delta \end{aligned}\right\} \tag{35}$$

Introducing these relations together with the temperature expression (10) into the energy equation (6) leads to the following result:

$$d\left[\left(U_1^*\left\{-\frac{1}{3}\delta^* + \frac{1}{8}\frac{\delta^{*2}}{\Delta^*} - \frac{1}{120}\frac{\delta^{*4}}{\Delta^{*3}} + \frac{3}{8}e\Delta^* + \frac{1}{10}f\Delta^{*2} + \frac{1}{24}g\Delta^{*3} + \right.\right.\right.$$

$$(2-\delta^*)(1-e) - (2-\delta^*)^2\left[(1-e)\frac{3}{4\Delta^*} + \frac{f}{2}\right] - (2-\delta^*)^3\left[\frac{g}{3} - \frac{1}{2}\frac{f}{\Delta^*}\right] +$$

$$\left.\left.\left.(2-\delta^*)^4\left[(1-e)\frac{1}{8\Delta^{*3}} + \frac{3}{8}\frac{g}{\Delta^*}\right] - (2-\delta^*)^5\frac{f}{10\Delta^{*3}} - (2-\delta^*)^6\frac{g}{12\Delta^{*3}}\right\}\right)\right]$$

$$= \frac{3}{2Pr}\frac{dx^*}{\Delta^*} \tag{36}$$

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where

$$e = \frac{4}{\delta^*} - \frac{4}{\delta^{*2}}$$

$$f = \frac{4}{\delta^{*2}} - \frac{2}{\delta^*}$$

$$g = \frac{1}{\delta^{*2}}$$

The solution of equation (36) is discussed in appendix B.

Case of $\Delta < \delta$. - The formulation of this case is identical to the analysis already made for the temperature problem with heat transfer at both walls for $\Delta < \delta$.

Heat-transfer parameters. - The Nusselt number $\overline{Nu}_2$, based on the temperature difference $(\Delta t)_2 = (t_w - t_1)$, is chosen for presentation of the heat-transfer results, since it is the one most conveniently used in calculations. If the equivalent diameter D_e is taken as 4a as before, the expression for $\overline{Nu}_2$ for the case of heat transfer at one wall becomes identical to equation (29b).

PRESENTATION OF RESULTS

Heat-Transfer Results

Nusselt numbers for heat transfer at both walls. - The results for $\overline{Nu}_1$ are plotted in figure 2 as a function of Graetz number for Prandtl numbers in the range 0.01 to 50. The curves obtained from the boundary layer analysis (solid lines) terminate at the Graetz numbers corresponding to $\Delta^* = 1$ ($\Delta = a$). The curve of reference 3 for a parabolic velocity profile throughout (dot-dash line) is given for Graetz numbers below 100. Curves have been faired (dotted lines) to connect the results of the present analysis with those of reference 3. Figure 3 shows the results of $\overline{Nu}_1$ replotted with $\frac{x/D_e}{Re_d}$ as abscissa and Prandtl number as parameter.

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The $\overline{Nu}_2$ results are plotted against Graetz number on figure 4 and are replotted against $\frac{x/D_e}{Re_d}$ on figure 5. Prandtl number appears as a parameter on the curves. For practical applications, $\overline{Nu}_2$ will be the most convenient Nusselt number to use since it is based on the simplest temperature difference. The solid lines on figure 4, which represent the results of the boundary layer analysis, can be fitted to within 3 percent by the following equations:

$$\frac{\bar{h}D_e}{k} = \frac{0.644\ Gz^{\frac{1}{2}}}{Pr^{1/2}\varphi}\left[1+7.3\left(\frac{Pr}{Gz}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \qquad 0.01 \leqslant Pr \leqslant 2 \tag{37a}$$

$$\frac{\bar{h}D_e}{k} = \frac{0.664\ Gz^{\frac{1}{2}}}{Pr^{1/6}}\left[1+6.27\left(\frac{Pr}{Gz}\right)^{\frac{4}{9}}\right]^{\frac{1}{2}} \qquad Pr = 10,\ 50 \tag{37b}$$

The function φ appearing in equation (37a) is equal to $(Pr)^{-1/3}$ for values of $Pr \geqslant 1$ and is obtained from figure 11 for $Pr < 1$. The faired curves for $\overline{Nu}_2$ shown on figure 4 are consistent with the faired curves drawn on figure 2 for $\overline{Nu}_1$ in accordance with the relation

$$\overline{Nu}_2 = \frac{Gz}{4}\left[1 - e^{-\frac{4\overline{Nu}_1}{Gz}}\right] \tag{38}$$

The results for $\overline{Nu}_3$ are plotted in figure 6 on separate grids as a function of Graetz number and $\frac{x/D_e}{Re_d}$. The heat-transfer results have also been presented in the form of $\overline{Nu}_3$ because they are often given this way in the literature.

Nusselt numbers for heat transfer at one wall. - The $\overline{Nu}_2$ values for heat transfer at one wall are given on figure 7 as a function of Graetz number for Prandtl numbers in the range 0.01 to 50.

Comparison of Nusselt numbers with those of other analyses. - Two comparisons are made.

(a) In reference 3, it is suggested that entrance region Nusselt numbers for simultaneously developing velocity and temperature profiles might be obtained using the results for heat transfer from a flat plate to a fluid having velocity $\overline{U}$. The flat-plate results can be put into the following form to facilitate a comparison with the Nusselt numbers of the present analysis: 3349

$$\overline{Nu}_2 = \frac{0.664\ Gz^{\frac{1}{2}}}{Pr^{1/2}\varphi} \tag{39}$$

For $Pr \geqslant 1$, many investigators have found that $\varphi = Pr^{-1/3}$. For $Pr < 1$, φ does not appear in the literature; it is calculated in appendix C of this report and is given in figure 11. The flat-plate results of equation (39) are plotted in figure 8 and are compared with the $\overline{Nu}_2$ results of the present analysis for three representative values of Pr - 0.1, 2, and 50.

(b) The boundary layer analysis has been used herein to calculate the development of the temperature profile for an unchanging parabolic velocity profile throughout the duct length. The Nusselt numbers (for heat transfer at both walls) obtained from this analysis are compared in figure 9 with those of reference 3, in which a different method of analysis is used. The values of $\overline{Nu}_1$ and $\overline{Nu}_3$ are almost coincident in the range of Graetz numbers considered and appear as a single line on figure 9. Comparison of the $\overline{Nu}_2$ values from the two investigations is also made in figure 9.

Velocity Results

Velocity values calculated from equation (12a) are plotted in figure 10 as a function of $\frac{x\nu}{a^2\overline{U}} = x^*$ with the dimensionless distance from the wall y/a as a parameter. Included for comparison on the same figure are curves representing the results of Schlichting (ref. 7).

DISCUSSION

Nusselt numbers for heat transfer at both walls. - Figure 2 shows that when Nusselt numbers for the case of simultaneously developing velocity and temperature profiles are plotted against Graetz number, there is a separate curve corresponding to each Prandtl number. In contrast, when the Nusselt numbers for the case of a parabolic velocity profile throughout are plotted against Graetz number, a single curve results which is applicable for all Prandtl numbers. For any given Prandtl number, the Nusselt number for a flow with a developing velocity profile is greater than (or, at least, equal to) that for a flow where the velocity profile is parabolic throughout (at a fixed Re_d and x/D_e) because the velocities near the wall are higher for the developing velocity profile than for the parabolic profile.

The curve of reference 3, based on a parabolic velocity profile throughout, is shown in figure 2 for the smaller Graetz numbers, where the boundary layer analysis can no longer be used. The previously mentioned limiting value of $\overline{Nu}_1$ of 7.6 is attained in the solution of reference 3 at $Gz = \frac{Re_d Pr}{x/D_e} = 4$. If fully developed heat transfer is defined by the criterion that the limiting value of $\overline{Nu}_1$ is achieved, then it is seen that for a given Reynolds number, fluids of low Prandtl number attain fully developed heat transfer in shorter lengths of run (i.e., lower x/D_e) than do fluids of higher Prandtl number.

It is expected that at a sufficiently large distance from the beginning of the duct, the velocity conditions in the entrance region will cease to play an important role in determining the average Nusselt number. It is realized that only at infinity will the Nusselt numbers based on a developing velocity profile coincide precisely with those for a parabolic profile throughout the duct length. For practical purposes, however, the differences should become negligible at some finite location; so, curves have been faired in figure 2 connecting the Nusselt numbers from the boundary layer analysis with those of reference 3. Although inherently inexact, this extrapolation at least provides an estimate of the location at which the effect of velocity development is negligible. Figure 2 indicates that the influence of velocity development ceases at lower Graetz numbers for the fluids of lower Prandtl number.

The use of the Graetz number as an independent variable allows the convenient comparison on one graph of results for the case of simultaneous development of the thermal and velocity profiles for different Prandtl numbers with a single curve representing the results for a parabolic velocity throughout. Further, the extrapolation of the curves for simultaneous development of the boundary layers to join with the curve for the parabolic velocity can be made conveniently for all Prandtl numbers on a single graph with Graetz number as abscissa.

Figure 3 is a replot of figure 2 in which $\frac{x/D_e}{Re_d}$ is used as abscissa and Prandtl number appears as parameter. The curves shown represent the results for the case of simultaneous development of velocity and temperature profiles. The results for the case of a parabolic velocity profile throughout, which can be plotted as a single curve as a function of Graetz number, would appear as a series of curves with Prandtl number as parameter if they were plotted against $\frac{x/D_e}{Re_d}$. These curves do not appear on figure 3.

The influence of Prandtl number on Nusselt number is clearly shown in figure 3. At a given value of $\frac{x/D_e}{Re_d}$, the fluids of higher Prandtl number have higher Nusselt numbers. This result is in qualitative agreement with the usual heat-transfer correlations, which show Nusselt number increasing with Prandtl number for a fixed Reynolds number. The dependence of Nusselt number upon x/D_e for a fixed Reynolds number (and a fixed Prandtl number) is also clearly shown in figure 3. As has already been noted, the value of x/D_e corresponding to fully developed heat transfer decreases with decreasing Prandtl number. 3349

Figures 4 and 5 show trends similar to those which have been pointed out for figures 2 and 3. The faired curves shown on figure 4 are consistent with those drawn on figure 2.

Nusselt numbers for heat transfer at one wall. - Nusselt numbers for a channel in which one wall is held at a uniform temperature while the other wall is insulated are shown in figure 7. The results are given in the form of $\overline{Nu}_2$, based on the temperature difference $(\Delta t)_2 = t_w - t_1$, because it is the most convenient one to use in calculations. There are at present no solutions of this case for very long ducts. It is suggested that engineering estimates of the heat transfer for ducts with Graetz numbers of 3 or less be obtained from the relation

$$\overline{Nu}_2 = 0.5\ Gz$$

Comparison with other calculations. - Nusselt numbers of the present analysis for simultaneous development of velocity and temperature profiles are compared in figure 8 with those for a single flat plate immersed in a fluid moving with a velocity $\overline{U}$ parallel to the plate. It is suggested in reference 3 that the flat-plate results might be used in the entrance region of a duct. For given Reynolds numbers, the agreement between the flat-plate results and those for the duct becomes poorer as x/D_e increases. This is to be expected, since the velocity

U_1 of the fluid in the core departs more and more from the free-stream velocity U_∞ $(= \overline{U})$ of the plate as x/D_e increases. The agreement appears to be poorer for the fluids of higher Prandtl number because at a given Graetz number, higher values of x/D_e are assocated with the higher Prandtl numbers at a given Reynolds number.

A check on the boundary layer analysis was sought by calculating Nusselt numbers for the case of a parabolic velocity profile throughout the duct length. This case had already been treated in reference 3 using Lévêque's method for Graetz numbers above 400 and a direct numerical integration of the energy equation for Graetz numbers below this value. The agreement of the curves in figure 9 is considered very good in view of the approximations in the boundary layer analysis and those of Lévêque's method.

Longitudinal heat flow in fluid. - The boundary layer energy equation (eq. (2)) is derived from the general energy equation for a constant-property, nondissipative flow by neglecting the longitudinal conduction of heat $k \dfrac{\partial^2 t}{\partial x^2}$ compared with the transverse conduction $k \dfrac{\partial^2 t}{\partial y^2}$. The condition $\Delta/x \ll 1$ is usually assumed sufficient justification for deleting this term. However, it is still possible that the longitudinal conduction can be neglected even if Δ/x is not small compared with 1. Thus, the condition $\Delta/x \ll 1$ is sufficient, but not necessary, for deleting the longitudinal conduction term. When this condition is not satisfied, the rigorous check as to whether the longitudinal conduction term is negligible is to obtain a solution of a problem in which it has not been deleted.

For the lowest Prandtl numbers considered in this report, the condition $\Delta/x \ll 1$ is not satisfied in the length of duct for which the boundary layer analysis is made. For example, inspection of figure 5 shows that the solid curve for $Pr = 0.01$ terminates at $\dfrac{x/D_e}{Re_d} = 0.7\times10^{-4}$, at which point $\Delta = a$. Since $D_e = 4a$, this would correspond to $x/\Delta = 0.6$ for a Reynolds number of 2000.

Since no solution for the energy equation containing the term $k \dfrac{\partial^2 t}{\partial x^2}$ has been obtained, there is no rigorous basis for deleting it in those cases where the condition $\dfrac{\Delta}{x} \ll 1$ has not been satisfied. Neglecting the term $k \dfrac{\partial^2 t}{\partial x^2}$ for a fluid of very low Prandtl number can be justified only on the basis of the resulting simplification of the energy equation. For this reason, the results for the fluids of very low Prandtl number may be less accurate than the other results of this report.

Velocity profile. - The expression (8) chosen to describe the velocity profile in the boundary layer had the virtue of simplicity and gave reasonably good results, as may be seen by comparison in figure 10 with the more exact values of Schlichting. It has already been noted that although the velocity solution obtained in this report coincides with the fully developed parabolic profile when $\delta = a$, its approach to the fully developed parabola is not asymptotic. However, it is not expected that any other polynomial expression for the velocity could be used throughout the entire length of the channel and asymptotically approach a parabola. (Basically, it would not be expected that any boundary layer analysis for the velocity should lead to a solution which approached the fully developed parabola asymptotically.)

A solution for the velocity profile which approaches the parabola asymptotically at infinity can be obtained by using a boundary layer analysis at the beginning of the duct and another method of analysis at great distances from the entrance. This is the method used by Schlichting. Such refinements in the calculation of the velocity profile would have destroyed the simplicity of the analysis for the temperature problem which has been used in this report.

The final check of the influence of the approximate velocity profile on the heat-transfer results can be made only when a more exact solution to the energy equation is available. For the simpler problem of heat transfer from a flat plate, the author has verified that good Nusselt number results can be obtained with the Kármán-Pohlhausen method even though the velocity profiles are approximate.

Validity of thermal boundary layer calculations as Δ approaches a. - Whether or not good heat-transfer results can be obtained from a boundary layer calculation for Δ values near a depends upon the extent of the interaction which is actually taking place in the duct as a result of heat transfer from the opposite walls. It is felt that the boundary layer model will give good results only if this interaction is small. The values of Δ are in a sense arbitrary, since they depend to some extent upon the degree of the polynomial which is taken to approximate the temperature profile. For one choice of polynomial, the condition $\Delta = a$ may be achieved at a position in the duct where the actual interaction of the heat transfer from the opposite walls is negligible. On the other hand, for another polynomial, $\Delta = a$ may occur at a location at which there is considerable interaction actually taking place in the duct.

For a parabolic velocity profile throughout the duct length, the Nusselt numbers calculated from the boundary layer analysis up to $\Delta = a$ agree quite well with the results of reference 3. In fact, when the boundary layer calculation was extended to $\Delta = 1.3a$, the Nusselt numbers still agreed to within 3 percent. This good agreement seems to imply that the boundary layer model is still valid for Δ approaching a, at least when the velocity profile throughout is parabolic.

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For simultaneously developing temperature and velocity profiles, no other solution presently exists with which the results of the present analysis can be compared. So, no completely conslusive statement as to whether it is valid to carry the calculation of the temperature profile to $\Delta = a$ can be made.

Practical departures from conditions of analysis. - Various conditions of the present analysis which may not be completely satisfied in practical application will be discussed.

(a) Variable fluid properties: In some fluids, the variability of properties has a marked effect on the heat-transfer results. The property variations tend to alter the shape of the velocity profile as well as to create free-convection currents. Corrections for the Nusselt number to take into account property changes are discussed in references 3 and 11.

(b) Entrance conditions: In any real apparatus, the entrance velocity and temperature profiles will be determined by the shape of the transition section between the header and the duct and by the heat transfer to the fluid in the header. There may be nonzero values for the velocity and thermal boundary layer thicknesses at the entrance section associated with nonuniform profiles.

(c) Uniformity of wall temperature: In almost any real apparatus, it is not expected that a uniform wall temperature can be maintained in the neighborhood of the entrance section; since, in addition to the possibility of heat conduction from the duct walls to the header, the extremely high heat-transfer coefficients near the entrance section tend to make the temperature nonuniform.

(d) Heat transfer at one wall: In practice, for sufficiently long ducts, there may be deviations from the condition of zero heat transfer from one wall to the fluid even though that wall may be insulated from the external surroundings. This may occur when the thermal boundary layer growing from the heated wall contacts the opposite wall, thereby causing a temperature rise of the opposite wall in the region where the contact takes place. There will then be a conduction of heat from the warmer parts of the wall to the cooler parts which are located nearer the entrance, and some of this conducted heat will be transferred to the fluid. Except for cases of thick walls made of high conductivity material, this secondary heat transfer to the fluid is expected to have very little influence on the results of the analysis. Of course, for those ducts which end before $\Delta = 2a$, the condition of no heat transfer to the fluid at one wall is achieved by insulating the wall from its external surroundings.

CONCLUDING REMARKS

The simultaneous development of temperature and velocity profiles in the entrance region of a flat rectangular duct has been studied using a boundary layer analysis. The Kármán-Pohlhausen method was used to calculate the thickness of the velocity and thermal boundary layers. Incompressible, constant-property fluids having Prandtl numbers in the range 0.01 to 50 were considered. Both duct walls were assumed to have the same uniform temperature. 3349

Nusselt numbers were calculated for the length of duct for which the boundary layer analysis was made. It was found that when Nusselt numbers were plotted against Graetz number, a separate curve for each Prandtl number resulted. In contrast, when the Nusselt numbers for the case of a parabolic velocity profile throughout were plotted against Graetz number, a single curve resulted which is applicable for all Prandtl numbers. For any given Prandtl number, the Nusselt number for a flow with a developing velocity profile is greater than (or, at least, equal to) that for a flow where the velocity profile is parabolic throughout the duct length. Beyond the position where the boundary layer treatment can no longer be used, curves have been faired to connect the Nusselt number results of the present analysis with those of reference 3, in which a parabolic velocity profile was assumed throughout the length of the duct.

Calculations were also made for a duct having one wall at uniform temperature while the other wall was insulated. Nusselt numbers are reported over the range of Prandtl numbers from 0.01 to 50.

Nusselt numbers from the present analysis were compared with those for a single flat plate immersed in a fluid moving with velocity U_∞ which is equal to $\overline{U}$, the average velocity in the duct. The Nusselt numbers for the duct exceeded those of the plate. This difference increased with distance from the inlet. So that this comparison could be made, the Nusselt numbers for a flat plate for Prandtl numbers substantially less than 1 were calculated.

The boundary layer analysis was used to calculate Nusselt numbers for a parabolic velocity throughout the duct length. These results were in good agreement with those of reference 3.

It is expected that the Nusselt numbers obtained for a flat rectangular duct will apply for an annulus for which the diameter ratio is near 1.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, September 30, 1954

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APPENDIX A

SYMBOLS

The following symbols are used in this report:

A	surface area of channel walls through which heat is being transferred, sq ft
a	half-spacing between parallel walls in a duct or annulus, ft
B	dimensionless group defined by eq. (31)
c_p	specific heat at constant pressure, Btu/(lb)(°F)
D_e	equivalent diameter for a duct or annulus, 4a, ft
e	dimensionless group defined by eq. (36)
f	dimensionless group defined by eq. (36)
Gz	Graetz number, $\frac{Re_d\ Pr}{x/D_e}$
g	dimensionless group defined by eq. (36)
$\bar{h}$	average heat-transfer coefficient, $\frac{Q}{A(\Delta t)}$, Btu/(sec)(sq ft)(°F)
k	thermal conductivity of the fluid, Btu/(sec)(ft)(°F)
m	mass rate of flow through cross section, lb/sec
$\overline{Nu}$	average Nusselt number, $\frac{\bar{h}D_e}{k}$
Pr	Prandtl number, $\frac{c_p\mu}{k} = \frac{\nu}{\alpha}$
p	static pressure, lb/sq ft
Q	heat-transfer rate, Btu/sec
Re_d	diameter Reynolds number, $\frac{\bar{U}D_e}{\nu} = \frac{\bar{U}4a}{\nu}$
t	static temperature, °F
$t_{b,x}$	bulk temperature of fluid at position x, °F

t_w	wall temperature of channel or annulus, $^{\circ}F$
t_1	temperature of fluid at entrance section of channel or annulus, $^{\circ}F$
t_∞	temperature of fluid far from surface of a flat plate, $^{\circ}F$
$\overline{\overline{U}}$	average velocity of fluid, ft/sec
U_1	velocity in core, ft/sec
U_1^*	dimensionless velocity in core, $\frac{U_1}{\overline{\overline{U}}}$
U_∞	free-stream velocity for flat plate, ft/sec
u	velocity component in x-direction, ft/sec
v	velocity component in y-direction, ft/sec
x	coordinate measuring distance from entrance of channel, ft
x^*	dimensionless coordinate, $\left(\frac{x}{a}\right)\left(\frac{\nu}{a\overline{\overline{U}}}\right)$
y	transverse coordinate measuring distance from channel wall, ft
α	thermal diffusivity of fluid, $k/c_p\rho$, ft^2/sec
Δ	thermal boundary layer thickness, ft
Δ^*	dimensionless thermal boundary layer thickness, Δ/a
δ	velocity boundary layer thickness, ft
δ^*	dimensionless velocity boundary layer thickness, δ/a
ν	kinematic viscosity of fluid, ft^2/sec
ρ	density, lb/cu ft
φ	dimensionless group defined by equation (C4)

APPENDIX B

CALCULATION PROCEDURE

It was desired to use a rapid and accurate calculation procedure for equations (17) and (18) which also provided a continuous check. Equation (18) will be considered in detail. An identical procedure was used for equation (17). 3349

If the differentiation indicated in equation (18) and the substitution

$$dx^* = \frac{dx^*}{dU_1^*}\,dU_1^* = \frac{dU_1^*}{U_1^{*\prime}} \tag{B1}$$

are made, the following first-order differential equation is obtained:

$$\frac{dU_1^*}{d\Delta^*} = \frac{U_1\left(\frac{2}{5}\,\frac{\Delta^*}{\delta^*} - \frac{1}{8}\,\frac{\Delta^{*2}}{\delta^{*2}}\right)}{\frac{3}{2\Delta^*\,Pr\,U_1^{*\prime}} - \left(\frac{1}{5}\,\frac{\Delta^{*2}}{\delta^*} - \frac{1}{24}\,\frac{\Delta^{*2}}{\delta^{*2}} - \frac{3}{5}\,\frac{\Delta^{*2}}{U_1^*\delta^{*2}} + \frac{1}{4}\,\frac{\Delta^{*3}}{U^*\delta^{*3}}\right)} \tag{B2}$$

Since δ^* and $U_1^{*\prime}$ are simple functions of U_1^* from equations (12a) and (13a), the numerical integration of equation (B2) to find U_1^* as a function of Δ^* can be carried out without recourse to x^*. This integration was performed using the Runge-Kutta method. In order to provide a continuous check at each step of the calculation, the value of U_1^* obtained for a Δ^* was immediately used to compute an x^* from equation (13a) by the Runge-Kutta method. Alternatively, a check value of x^* was calculated from the following integrated form of equation (18):

$$x^* = \frac{2Pr}{3}\left[U_1^*\Delta^*\left(\frac{1}{5}\,\frac{\Delta^{*2}}{\delta^*} - \frac{1}{24}\,\frac{\Delta^{*3}}{\delta^{*2}}\right) - \int_0^{\Delta^*} U_1^*\left(\frac{1}{5}\,\frac{\Delta^{*2}}{\delta^*} - \frac{1}{24}\,\frac{\Delta^{*3}}{\delta^{*2}}\right)d\Delta^*\right] \tag{B3}$$

using Simpson's rule and the values of U_1^* and δ^* obtained from the integration of equation (B2).

The starting value of $dU_1^*/d\Delta^*$ was obtained by using the two results $U_1^* - 1 = \sqrt{\frac{10x^*}{3}}$ and $\Delta^* = C\,\sqrt{x^*}$, which satisfy equations (13a) and (18), respectively, as $x^* \to 0$. It should be noted that equation (B2) can be used only until $\delta^* \approx 1$ ($U_1^* = 1.5$), at which point the velocity is assumed to be fully developed and $U_1^{*\prime}$ becomes equal to zero. After the velocity profile has become fully developed and U_1^* and δ_1^* become constants, equation (18) can be integrated directly.

Other integration procedures were attempted for equations (17) and (18) involving iteration and interpolation of U_1^* against x^*, but these were found to be less accurate and no faster than the method described, and they did not provide a simple continuous check.

It was necessary to calculate only a few points with equation (36). Performing the indicated differentiation would have produced a large number of terms requiring extensive setup time. It was decided to iterate equation (36), since fairly good guesses could be made for the U_1^* values, and interpolation of U_1^* against x^* was fairly accurate in the region considered.

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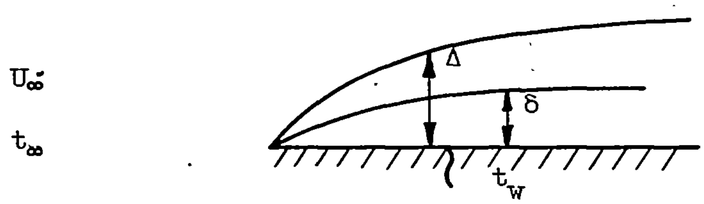
APPENDIX C

LAMINAR FORCED CONVECTION ON A FLAT PLATE FOR Pr < 1

A flat plate in an infinite domain is assumed to be alined parallel to a flowing stream having uniform velocity and temperature upstream of the plate. The plate temperature t_w is held at a value different from that of the approaching stream. Distinct thermal and velocity boundary layers, Δ and δ, respectively, are assumed to grow along the plate. The case of $\Delta > \delta$ is considered here. A diagrammatic sketch of the physical model is given.



The integrated momentum and energy equations for a constant-property, nondissipative laminar flow in a boundary layer are

$$\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad (5a)$$

$$\frac{d}{dx}\int_0^\Delta (t_\infty - t)u\,dy = \alpha \left.\frac{\partial t}{\partial y}\right|_{y=0} \qquad (6b)$$

The velocity in the boundary layer is expressed as a polynomial in y with coefficients that are functions of x.

$$u = U_\infty\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] \qquad 0 \leq y \leq \delta \qquad (C1)$$

This expression satisfies the following boundary conditions:

$$\left.\begin{aligned} u &= 0 \\ \frac{\partial^2 u}{\partial y^2} &= 0 \end{aligned}\right\} y = 0$$

$$\left.\begin{aligned} u &= U \\ \frac{\partial u}{\partial y} &= 0 \end{aligned}\right\} y = \delta$$

The solution of equation (5a) can be obtained immediately without recourse to the temperature by using the velocity expression (C1) and is as follows (it may also be found in ref. 12, pp. 69-70):

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\frac{U_\infty x}{\nu}}} = \frac{4.64}{\sqrt{Re_x}} \tag{C2}$$

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The temperature distribution in the boundary layer is also expressible as a polynomial in y having coefficients that are functions of x:

$$\frac{t - t_w}{t_\infty - t_w} = \frac{3}{2}\frac{y}{\Delta} - \frac{1}{2}\left(\frac{y}{\Delta}\right)^3 \qquad 0 \leqslant y \leqslant \Delta \tag{C3}$$

This expression satisfies the following conditions:

$$\left.\begin{aligned} t &= t_w \\ \frac{\partial^2 t}{\partial y^2} &= 0 \end{aligned}\right\} y = 0$$

$$\left.\begin{aligned} t &= t_\infty \\ \frac{\partial t}{\partial y} &= 0 \end{aligned}\right\} y = \Delta$$

In obtaining the solution to equation (6b), it is necessary to notice that in the range $0 \leqslant y \leqslant \Delta$, the velocity must be expressed by two separate functions. In the range $0 \leqslant y \leqslant \delta$, expression (C1) applies; while $\delta \leqslant y \leqslant \Delta$, the velocity is given by $u = U_\infty$. Equation (6b) may be rewritten as

$$\frac{d}{dx}\int_0^\delta (t_\infty - t)u\,dy + U_\infty \frac{d}{dx}\int_\delta^\Delta (t_\infty - t)dy = \alpha \left.\frac{\partial t}{\partial y}\right|_{y=0} \tag{C6}$$

Inserting expressions (C1) and (C3) in equation (6c) yields a first-order differential equation for Δ with a solution conveniently given as

$$\Delta = \varphi\,\delta = \frac{4.64\,\varphi}{\sqrt{\frac{U_\infty x}{\nu}}} \tag{C4}$$

where φ is the following function of Prandtl number:

$$\varphi^2 - \varphi = \frac{3}{8Pr} - 0.4 \tag{C5}$$

Values of φ are plotted against Prandtl number in figure 11. Also plotted in figure 11 is the line $\left(0.5 + \frac{0.61}{Pr^{1/2}}\right)$, which is a good approximation of φ for $Pr \leq 0.1$.

The average Nusselt number is calculated from the definition 3349

$$\overline{Nu} = \frac{\overline{h}x}{k} \tag{23a}$$

where

$$\overline{h} = \frac{Q}{A(t_w - t_\infty)} = -\frac{k \int_0^x \left(\frac{\partial t}{\partial y}\right)_0 dx}{x(t_w - t_\infty)} = \frac{3}{2}\frac{k}{x}\int_0^x \frac{dx}{\Delta} \tag{24a}$$

Performing the indicated integration and inserting the value of $\overline{h}$ in equation (23a) give the final result for the average Nusselt number.

$$\overline{Nu} = \frac{\overline{h}x}{k} = 0.664\,\frac{\sqrt{Re_x}}{\varphi} \tag{C6}$$

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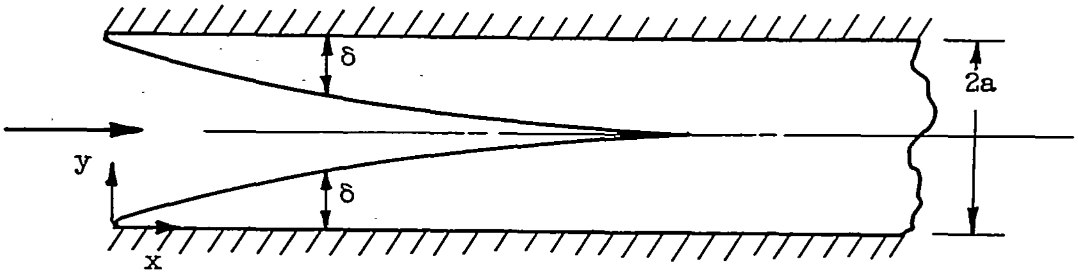


Figure 1. - Development of velocity boundary layer in entrance region of a flat rectangular duct.

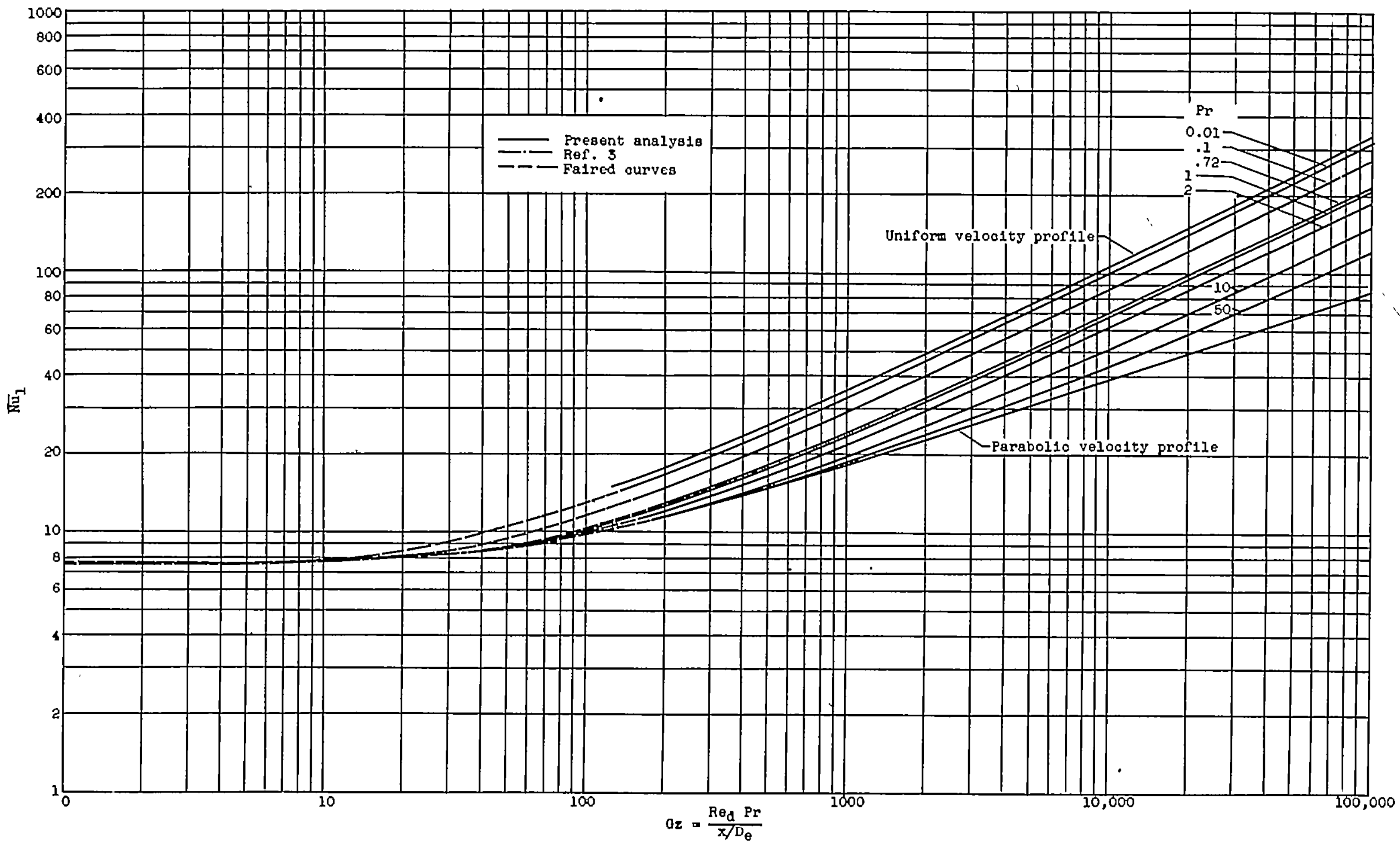


Figure 2. - Variation of average Nusselt number $\overline{Nu}_1$ with Graetz number for flat rectangular duct. Prandtl number appears as a parameter on curves. Fluid properties are constant. Both duct walls have same uniform temperature. $\overline{Nu}_1$ is based on logarithmic mean temperature difference defined by equation (25).

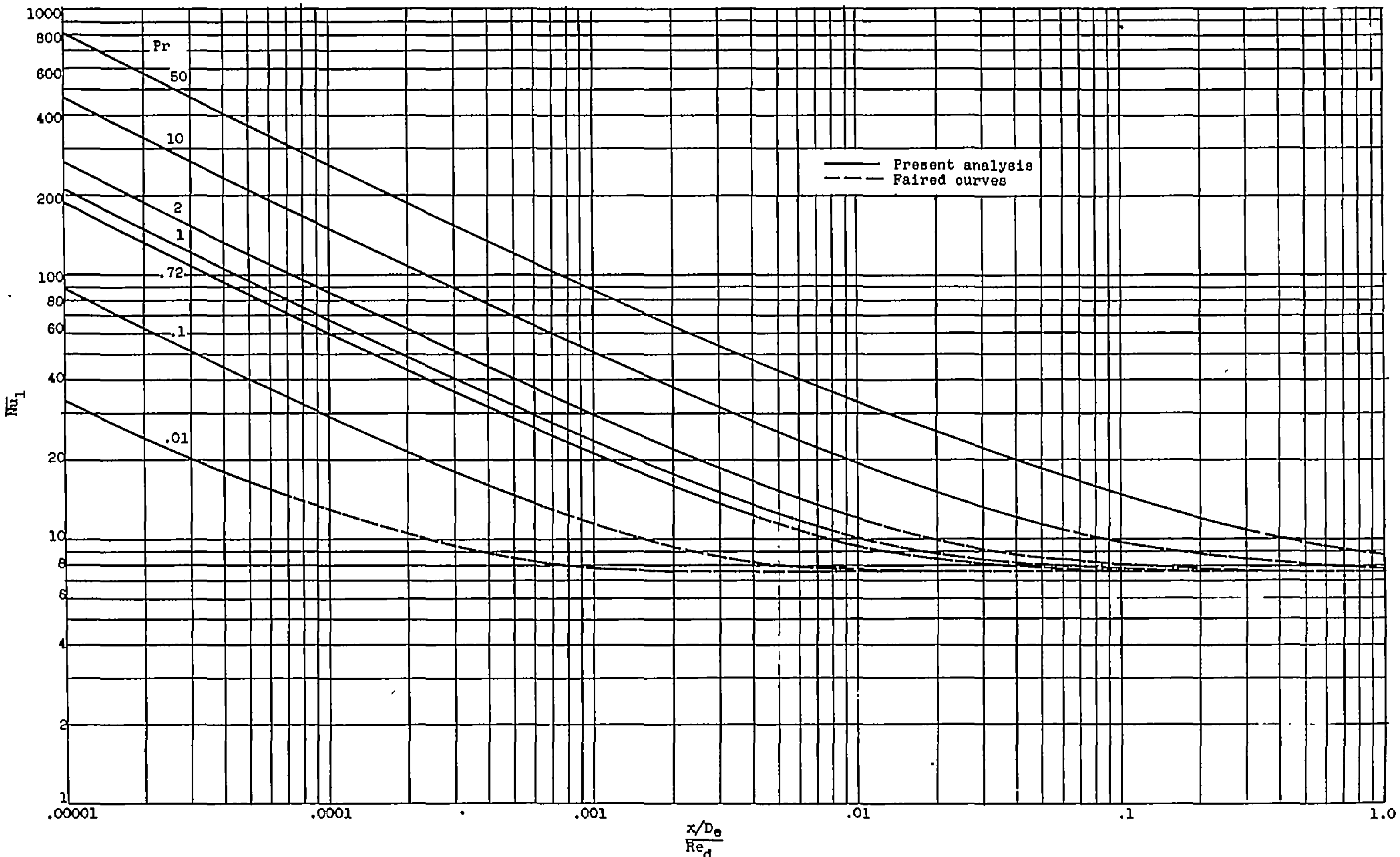


Figure 3. - Replot of average Nusselt number $\overline{Nu}_1$ results from figure 2 using $\frac{x/D_e}{Re_d}$ as independent variable.

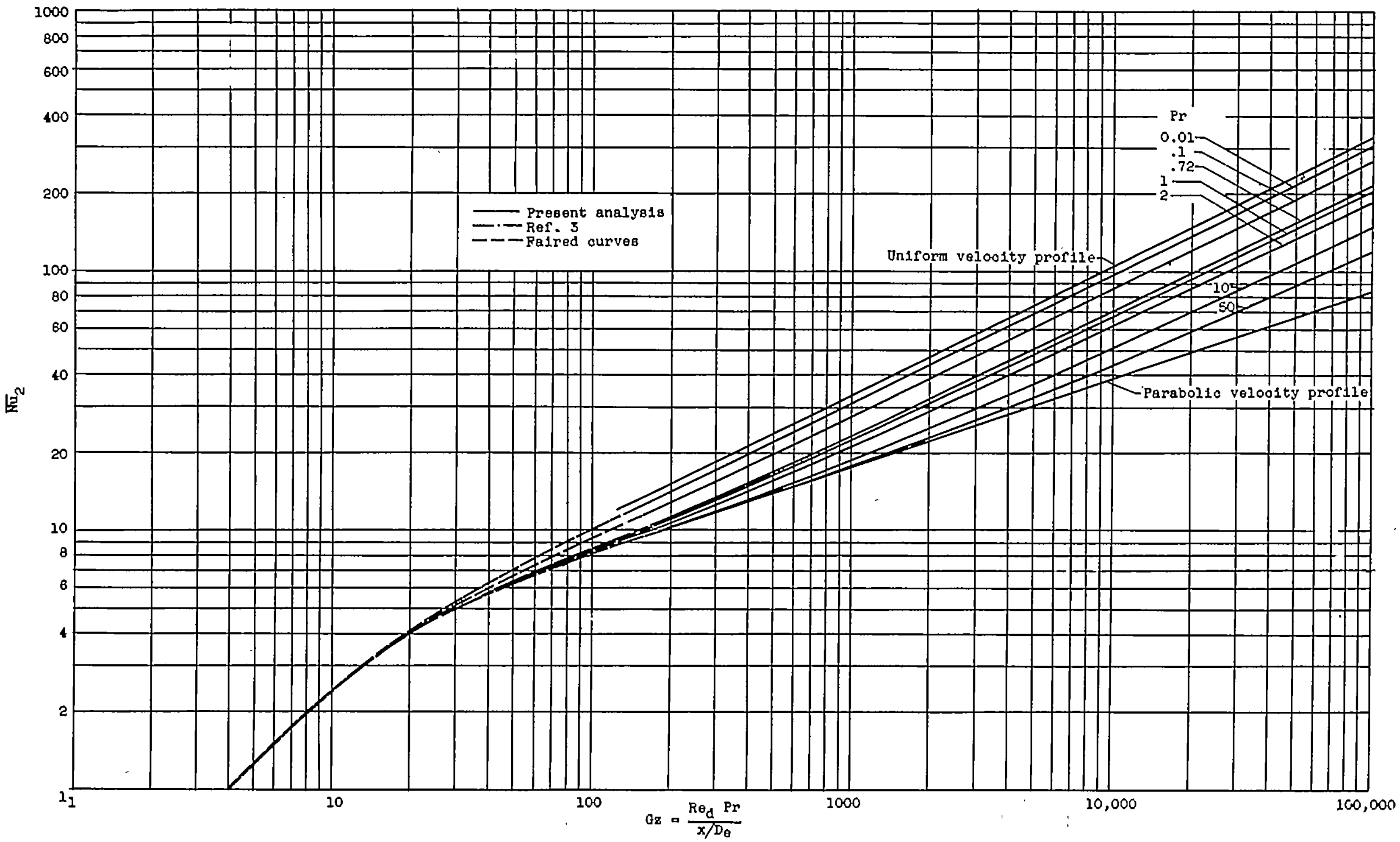


Figure 4. - Variation of average Nusselt number $\overline{Nu}_2$ with Graetz number for flat rectangular duct. Prandtl number appears as parameter on curves. Fluid properties are constant. Both duct walls have same uniform temperature t_w. $\overline{Nu}_2$ is based on temperature difference (t_w-t_1), where t_1 is entering fluid temperature.

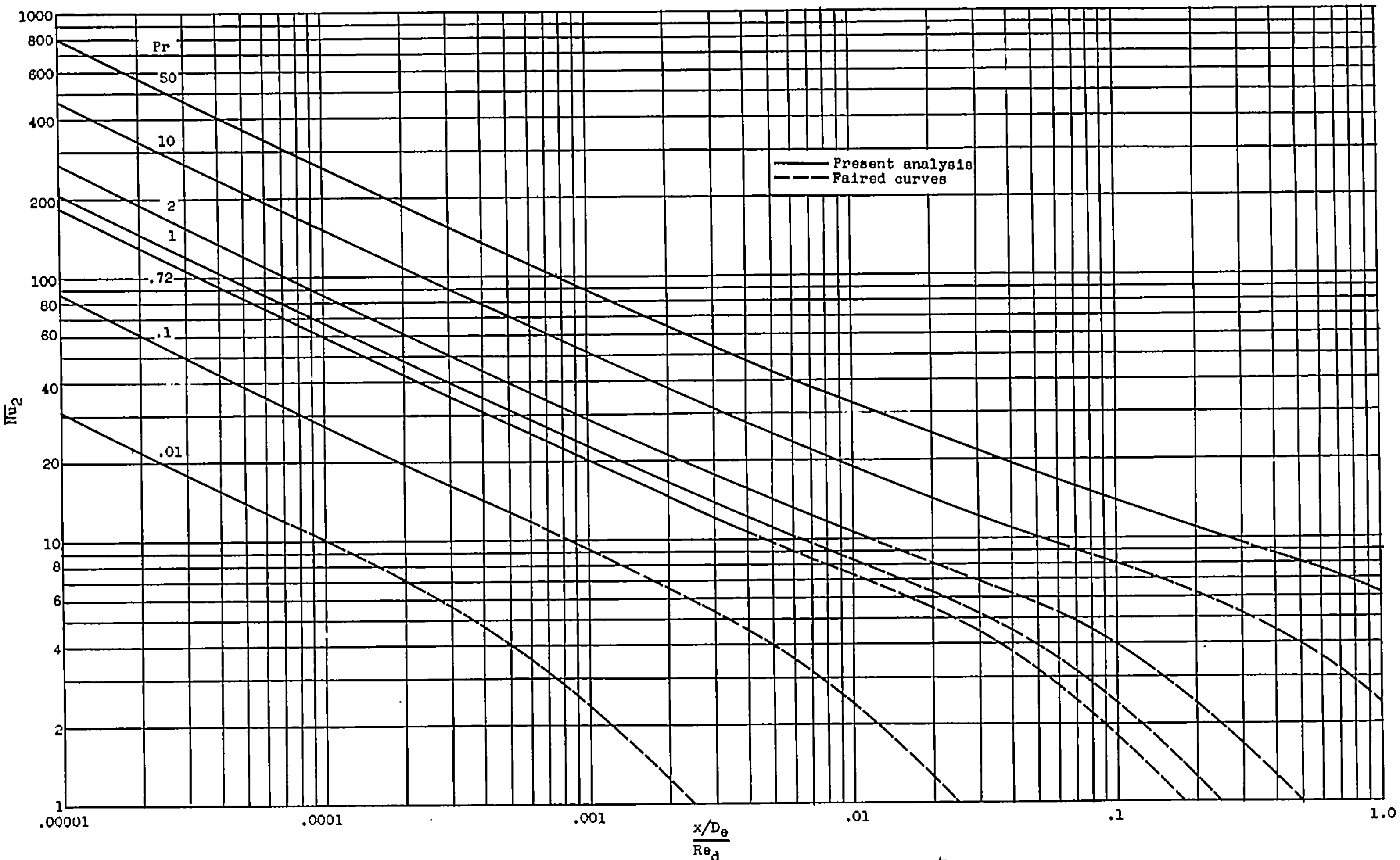
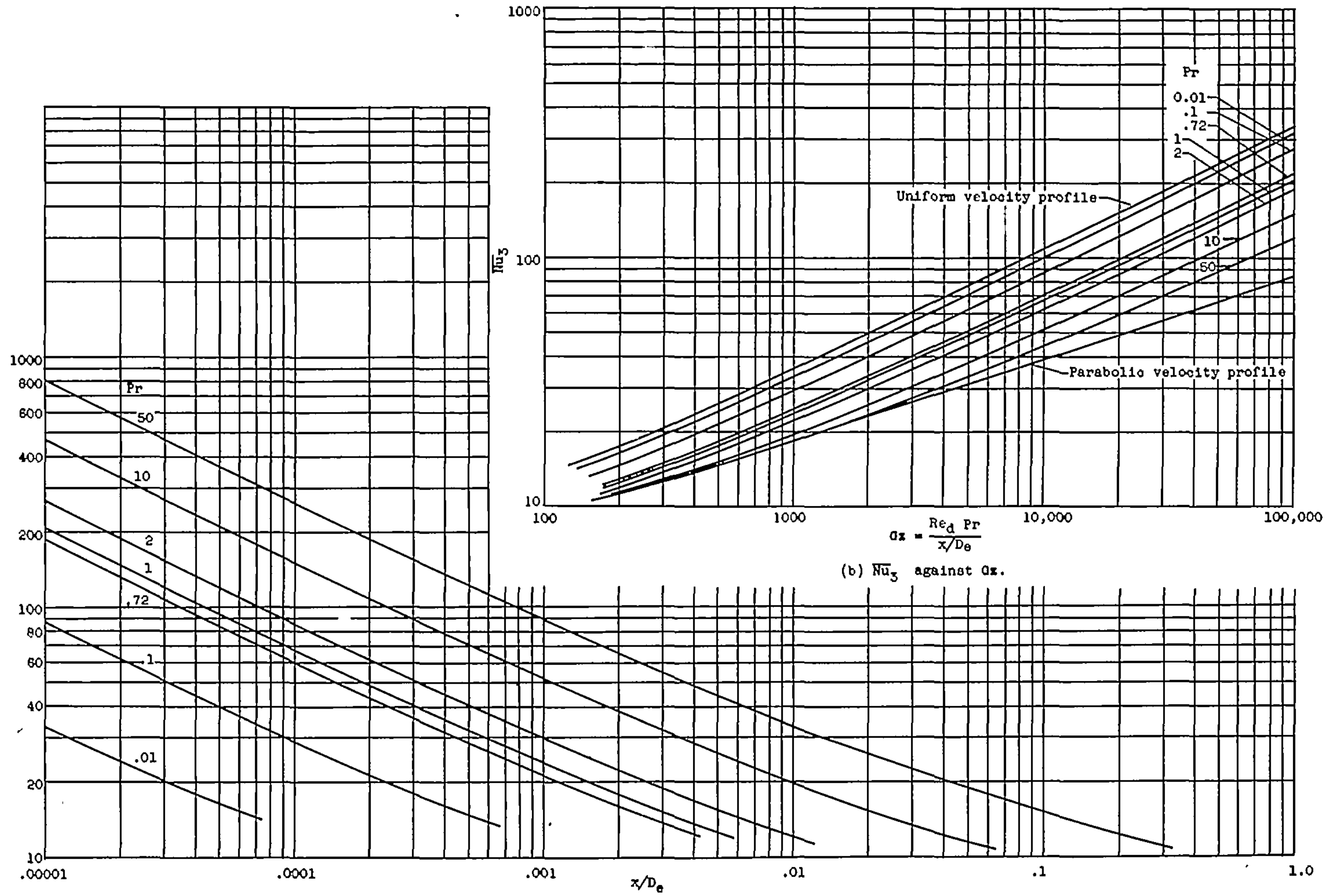


Figure 5. - Replot of average Nusselt number $\overline{Nu}_2$ results from figure 4 using $\frac{x/D_e}{Re_d}$ as independent variable.



(b) $\overline{Nu}_3$ against Gz.

(a) $\overline{Nu}_3$ against $\frac{x/D_e}{Re_d}$

Figure 6. - Variation of average Nusselt number $\overline{Nu}_3$ with $\frac{x/D_e}{Re_d}$ and with Graetz number for flat rectangular duct.

Prandtl number appears as parameter on curves. Fluid properties are constant. Both duct walls have same uniform temperature. $\overline{Nu}_3$ is based on arithmetic mean temperature difference defined by equation (25).

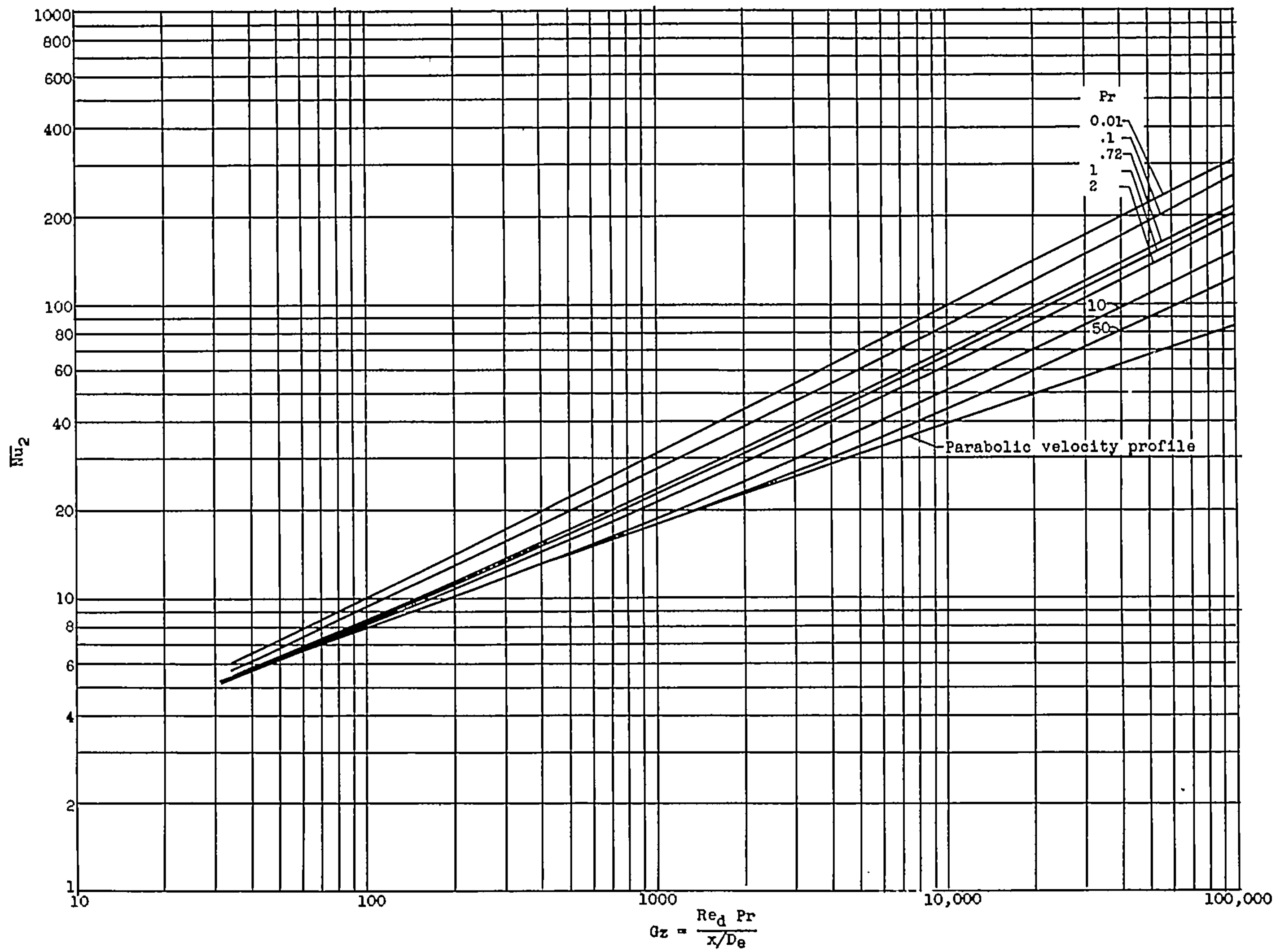


Figure 7. - Plot of average Nusselt number $\overline{Nu}_2$ against Graetz number for flat rectangular duct with heat transfer at only one wall. One duct wall is insulated, the other is maintained at temperature t_w. Fluid properties are constant. $\overline{Nu}_2$ is based on temperature difference (t_w-t_1) where t_1 is entering fluid temperature.

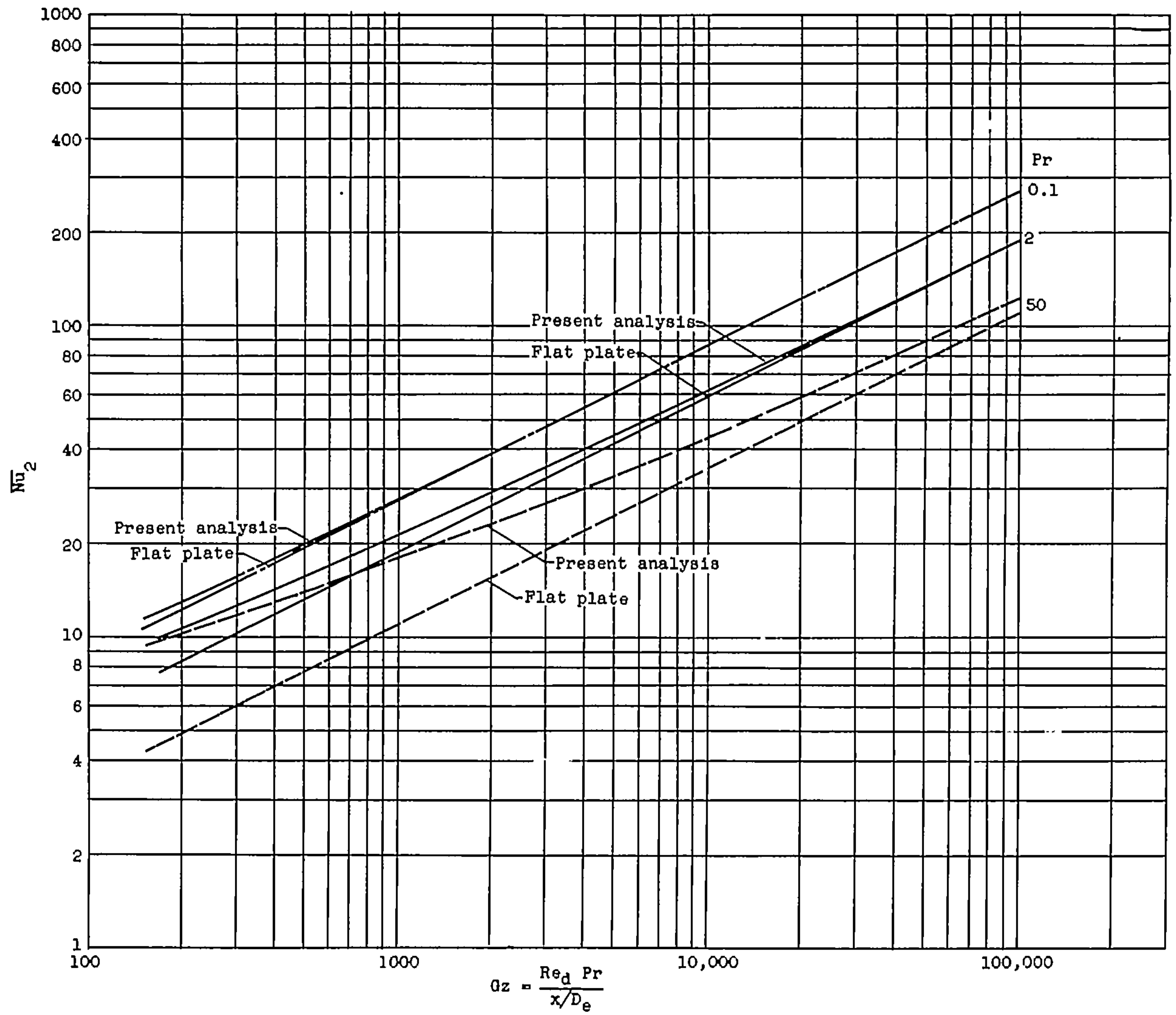


Figure 8. - Comparison of results for forced convection in entrance region of flat rectangular duct with those for single flat plate in stream with velocity U. Fluid properties are constant and wall temperatures are uniform.

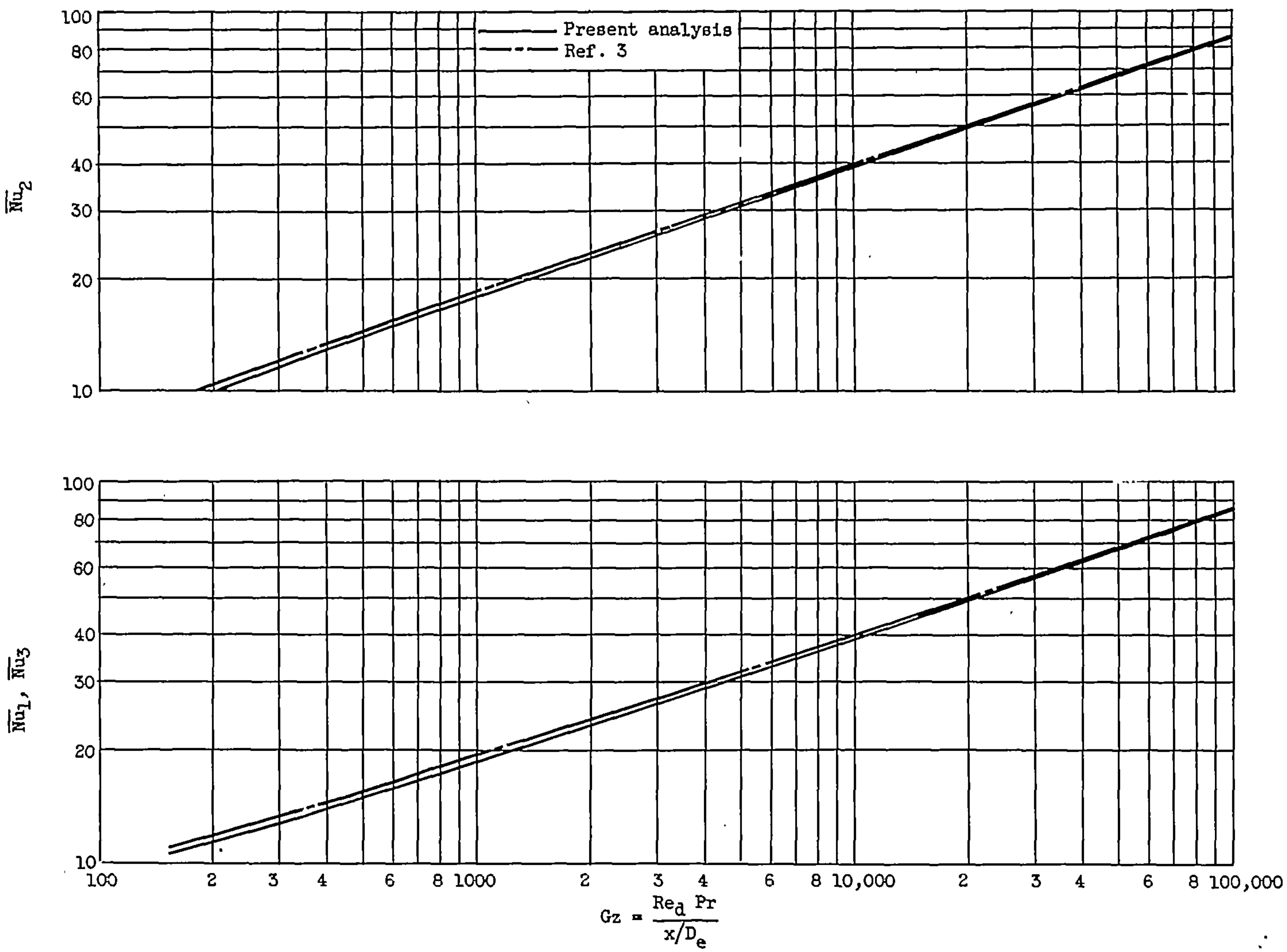


Figure 9. - Comparison of results of present analysis for parabolic velocity profile throughout duct length with those of reference 3.

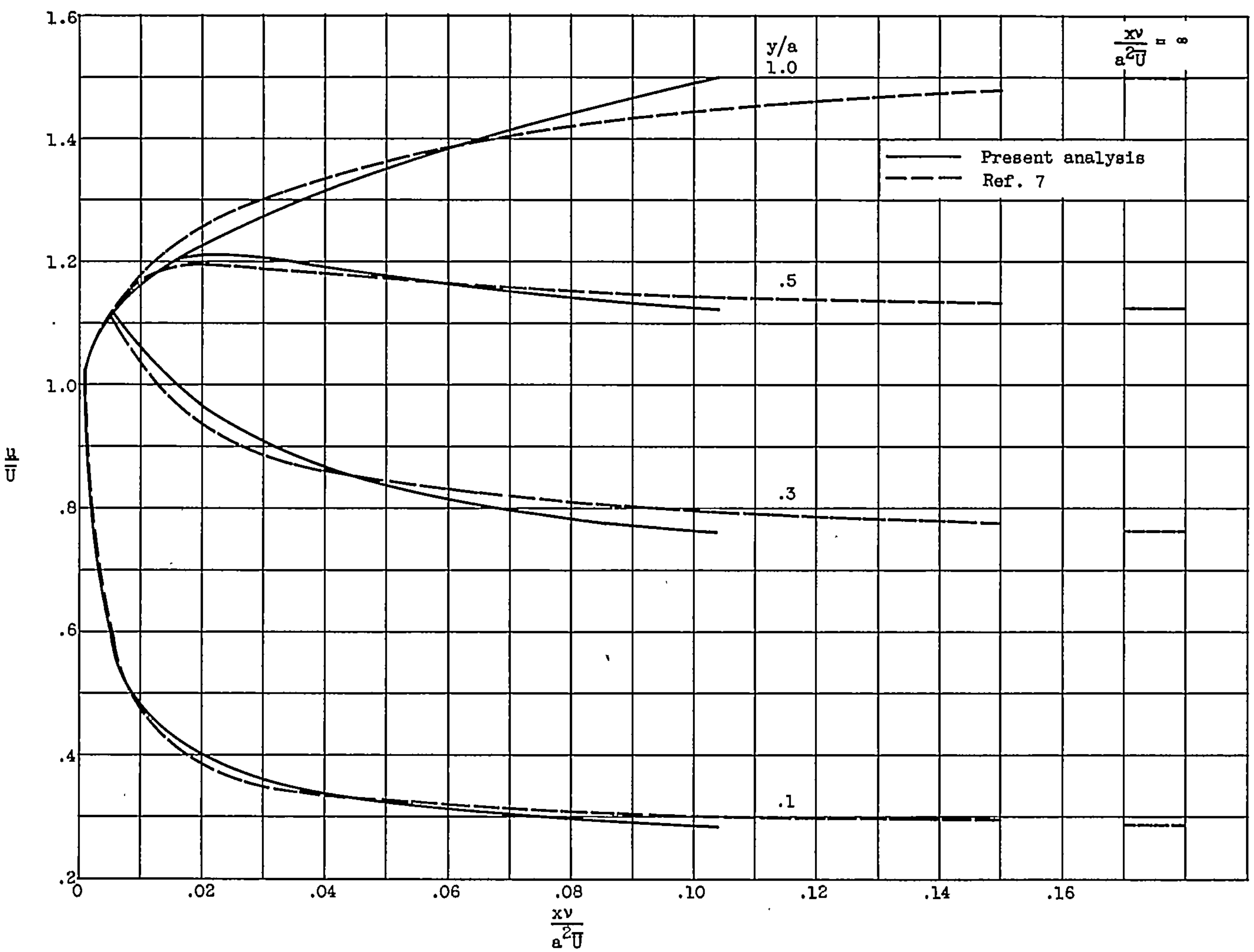


Figure 10. - Comparison of velocity results of present analysis with more exact analysis of reference 7.

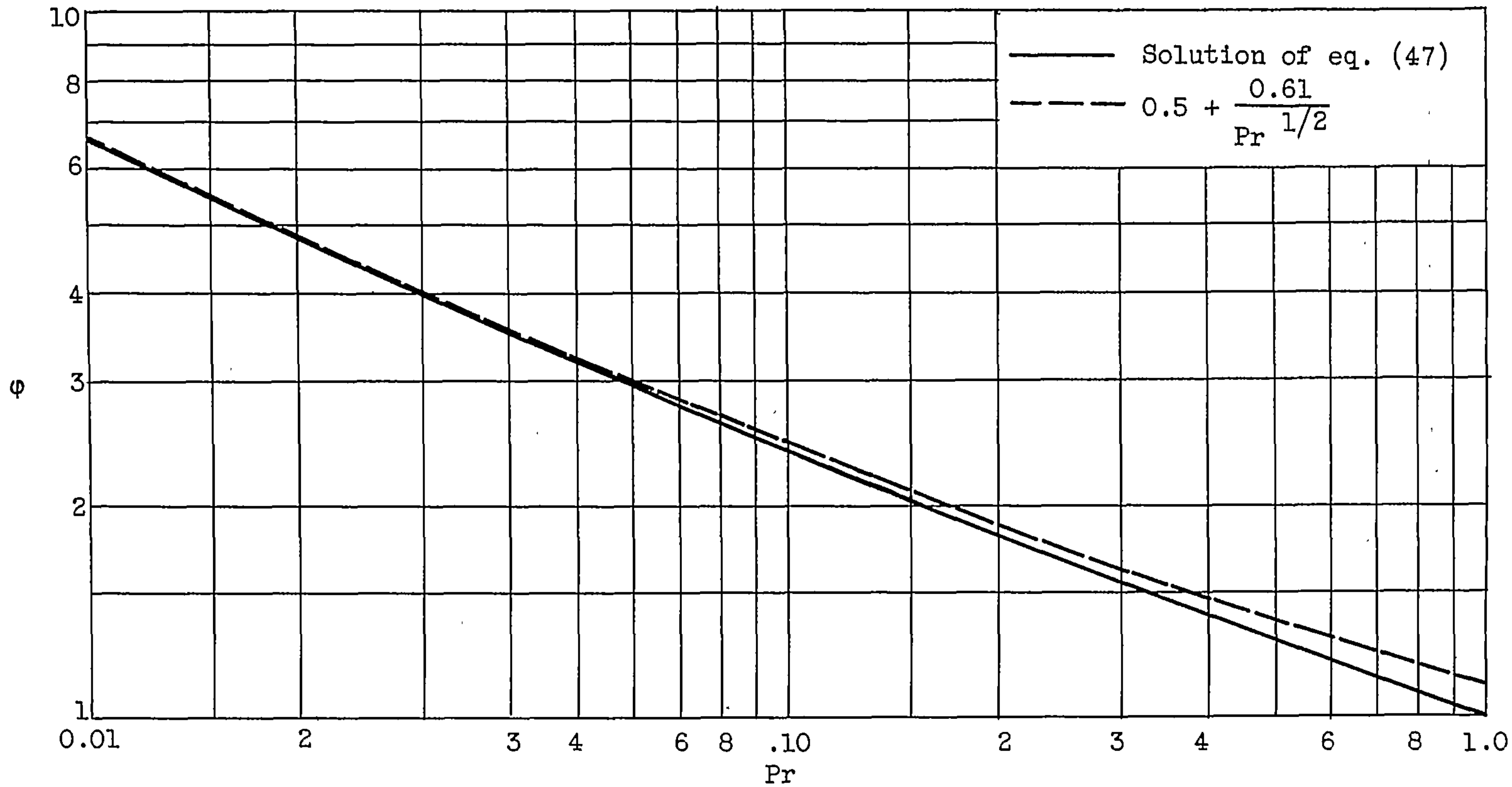


Figure 11. - Numerical values of function φ defined by relation $\Delta = \varphi\delta$, where Δ and δ are, respectively, thermal and velocity boundary layer thickness on single flat plate.

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