NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3105

AERODYNAMICS OF SLENDER WINGS AND WING-BODY COMBINATIONS HAVING SWEPT TRAILING EDGES

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SUMMARY

A general method, based on two-dimensional crossflow concepts, is presented for obtaining the lift and moments on highly swept wings. Emphasis is placed on obtaining solutions for wings having swept trailing edges. The method is applicable for all problems where the velocity boundary conditions can be made homogeneous by differentiation in the streamwise or spanwise directions.

Lift, roll, and pitch solutions, for highly swept wings, are presented. Both direct problems (where the plan form is given) and inverse problems (where the shed vortex sheet is given) are considered. The solutions of the direct problems are expressed in terms of functions which are evaluated from integral equations. Some limiting solutions of the integral equations are indicated. Numerical results are given for wings having parallel leading and trailing edges.

The transformation of a wing-body problem to an equivalent isolated wing problem is discussed and the application for finding the lift of a wing-body combination is indicated.

Application of the method for solving unsteady two-dimensional incompressible flow problems is also indicated. In particular, the Wagner problem is formulated in terms of the techniques developed herein.

INTRODUCTION

In 1924, Munk (ref. 1) published a remarkable paper concerning the calculation of the aerodynamic forces on airships. His theory was based on the idea that the velocity field induced by a slender body is essentially two-dimensional in planes transverse to the body axis. This reduced the complicated three-dimensional problem to an equivalent two-dimensional unsteady-flow problem and permitted the use of very elegant methods of solution.

With the advent of transonic and supersonic flight, interest in slender bodies was renewed. In 1946, Jones (ref. 2) revived Munk's ideas and used them to compute the forces on low-aspect-ratio pointed wings. He indicated that for such wings, compressibility has no effect and his results applied equally at both subsonic and supersonic speeds. Following Jones' example, many papers were written on the aerodynamics of slender wings, bodies, and wing-body combinations. See, for example, references 3 to 11. One of the most notable of these is Ward's paper (ref. 5) which provided a rigorous justification for the Munk-Jones approach. Ward showed that their solution may be considered as the first term of an expansion in terms of a "slenderness" parameter. Ward considered supersonic flight speeds. More recently, Adams and Sears (ref. 10) gave a similar result for the subsonic case.

Relatively few investigations have been made for cases where the two-dimensional crossflow contains a shed vortex sheet. This occurs, for example, when a slender wing has a swept trailing edge. In this case the crossflow generally contains a shed vortex sheet of unknown strength. The crossflow is not independent of upstream conditions and the problem is considerably more complicated than those considered by the early investigators. References 6 to 8 considered wings having swept trailing edges. In these references the distribution of vorticity in the shed vortex sheet is assumed and the wing plan form which would give rise to such a distribution is then found. This is the so-called inverse problem of aerodynamics and is generally simpler to solve than the direct problem. The direct problem is one in which the wing is completely specified and the flow field is to be determined. Robinson (ref. 11) appears to be the only one to have presented a solution of the direct problem for wings with swept trailing edges[1]. He has treated the lift problem. In reference 12 it is indicated that Robinson's solution is applicable only when the trailing edge is slightly swept.

In the present paper, a general method is developed for solving low-aspect-ratio problems involving shed vortex sheets. Both direct and inverse problems are considered. The method is applicable for all planar problems where the velocity boundary conditions can be made homogeneous by differentiation in either the streamwise or the

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[1]A private communication from K. W. Mangler, in connection with reference 12, served to call the author's attention to reference 13. In reference 13, Mangler has independently obtained the lift, roll, and pitch solutions for a highly swept wing (direct problem) by a method similar to that used herein. A comparison of reference 13 with the present report is given in appendix E.

spanwise directions. The lift, roll, and pitch solutions, for a highly swept wing, are presented and some numerical examples are worked out. Applications to wing-body interference problems and to unsteady two-dimensional incompressible flows are also indicated.

The research reported herein was conducted at the Graduate School of Aeronautical Engineering, Cornell University. This paper is based on material which was originally presented to the faculty of the Graduate School of Cornell University, in June 1953, as a thesis for the degree of Doctor of Philosophy. The author wishes to express his sincere gratitude to Professor N. Rott for his advice and criticism during the course of the study. The author also wishes to thank the other members of the staff of the Graduate School of Aeronautical Engineering and Drs. H. K. Cheng and M. C. Adams (both formerly at Cornell University) for stimulating discussions.

SECTION I - BASIC CONCEPTS

In the following section, the basic results of previous investigators, primarily reference 5, are summarized.

The assumption of slenderness is introduced into the equations of motion. The general features of the crossflow and formulas for lift, drag, and moments are discussed. Finally, symmetry of the velocity components, in planar problems, is mentioned.

<u>1.1 Equation of motion</u>. - Consider a slender body in a free stream of velocity U_0, Mach number M_0, pressure p_0, and density ρ_0 (fig. 1). The coordinate system is stationary with respect to the body and is defined such that the x-axis is parallel to the free stream (i.e., wind axes).

If the body is assumed to perturb the main stream only slightly, the equation of motion can be linearized and reduced to the Prandtl-Glauert equation:

$$\left(1 - M_0^2\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \qquad (1.1.1)$$

where φ is the perturbation velocity potential. Thus, if u, v, and w represent the perturbation velocities in the x, y, and z directions, respectively, then

$$u = \frac{\partial \varphi}{\partial x} \qquad v = \frac{\partial \varphi}{\partial y} \qquad w = \frac{\partial \varphi}{\partial z}$$

Equation (1.1.1) applies for both subsonic and supersonic flight. To as good an approximation, the pressure at any point in the flow field is given by

$$p - p_0 = -\rho_0 \left\{ U_0 \frac{\partial \varphi}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \right\} \tag{1.1.2}$$

where p_0 and ρ_0 are the pressure and density, respectively, in the undisturbed flow. 3120

A body is considered slender if

$$\frac{\left| \sqrt{M_0^2 - 1} \right| b_0}{c_0} \ll 1 \tag{1.1.3}$$

where c_0 characterizes the length of the body and b_0 characterizes its width. Under this condition, equation (1.1.1) becomes

$$\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{1.1.4}$$

which is the governing equation in slender body theory. Equation (1.1.4) indicates that φ can be found, to within a function of x, by considering the flow in each yz-plane to be a two-dimensional incompressible flow.

The boundary conditions for equation (1.1.4) are usually expressed in terms of the perturbation velocities. Let ν and μ be orthogonal coordinates which are normal and tangential, respectively, to the curve defining the body cross-sectional area in a particular yz-plane (fig. 2). Let $v_n \equiv \partial\varphi/\partial\nu$ be the perturbation velocity in the ν-direction. Then, the condition that the resultant flow be tangent to the body surface requires that, at the body surface (ref. 5),

$$v_n = U_0 \frac{\partial \nu}{\partial x} \tag{1.1.5}$$

<u>1.2 Asymptotic form of crossflow.</u> - Two-dimensional incompressible flows are well understood and are most easily handled in terms of the complex variable $\zeta \equiv y + iz$. Let $W = \varphi + i\psi$ where W is the complex potential function, φ is the velocity potential, and ψ is the stream function for the crossflow. (The functions W, φ, and ψ contain x as a parameter.) The Laurent expansion for W, valid everywhere outside the smallest circle, center at $\zeta = 0$, enclosing all the singularities of the flow, is

$$W = f + f_0 \ln \zeta + \sum_{m=1}^{\infty} f_m \zeta^{-m} \qquad (1.2.1)$$

where f, f_0, f_1, . . . are functions of x. The function f introduces a constant pressure at each crossflow. Formulas for its evaluation are given in reference 5 (for $M_0 > 1$) and reference 10 (for
3120 $M_0 < 1$). Since f does not contribute to lift, and since only the lift problem is of interest here, those formulas will not be repeated. The function f_0 is generally complex. The real part is proportional to the source strength required to simulate the expansion or contraction of the body and equals $\frac{U_0}{2\pi}\frac{dA_{cs}}{dx}$ (ref. 5) where A_{cs} is the cross-sectional area of the body. The imaginary part equals $-\Gamma/2\pi$ where Γ is the net circulation at each section. Equation (1.2.1) can then be written

$$W = f + \frac{1}{2\pi}\left(U_0 \frac{dA_{cs}}{dx} - i\Gamma\right) \ln \zeta + \sum_{m=1}^{\infty} f_m \zeta^{-m} \qquad (1.2.2)$$

The complex velocity $V \equiv v - iw$ is found by differentiating equation (1.2.2) with respect to ζ and equals

$$V = \frac{1}{2\pi}\left(U_0 \frac{dA_{cs}}{dx} - i\Gamma\right) \frac{1}{\zeta} - \sum_{m=1}^{\infty} m f_m \zeta^{-(m+1)} \qquad (1.2.3)$$

Differentiation of equation (1.2.3) with respect to ζ gives

$$\frac{\partial V}{\partial \zeta} = -\frac{1}{2\pi}\left(U_0 \frac{dA_{cs}}{dx} - i\Gamma\right) \frac{1}{\zeta^2} + \frac{2f_1}{\zeta^3} + \frac{6f_2}{\zeta^4} + \cdots \qquad (1.2.4)$$

Similarly, if equation (1.2.3) is differentiated with respect to x

$$\frac{\partial V}{\partial x} = \frac{1}{2\pi}\left(U_0 \frac{d^2A_{cs}}{dx^2}\right) \frac{1}{\zeta} - \frac{df_1/dx}{\zeta^2} - 2\,\frac{df_2/dx}{\zeta^3} - \cdots \qquad (1.2.5)$$

where it has been assumed that $d\Gamma/dx = 0$ so as to satisfy the law of conservation of circulation. Equations (1.2.2) to (1.2.5) define the asymptotic behavior of W and its derivatives. These expressions will be useful in later developments.

<u>1.3 Lift, drag, and moments</u>. - Expressions for computing the forces acting on a body are summarized herein. They are based primarily on the results of reference 5. It will be assumed that $\Gamma = 0$, which is the case for all problems treated in the present report (except for section 2.3).

The net force which has acted on a body upstream of any section x = constant is obtained by a contour integration about the body (fig. 2) and equals

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$$F_y + iF_z = -i\rho_0 U_0 \oint_{C_1} \varphi \, d\zeta \qquad (1.3.1)$$

Equation (1.3.1) can be evaluated by Cauchy's theorem for the path $C_1 + C_2$ indicated in figure 2. The cut is introduced to make W single-valued. The path C_2 is sufficiently far from the body so that the Laurent expansion for W can be used along this path. Since there are no singularities inside the path $C_1 + C_2$, Cauchy's theorem gives

$$\oint_{C_1} W \, d\zeta = \oint_{C_2} W \, d\zeta$$

Then equation (1.3.1) can be written

$$F_y + iF_z = -i\rho_0 U_0 \left(\oint_{C_2} W \, d\zeta - i \oint_{C_1} \psi \, d\zeta \right) \qquad (1.3.2)$$

But

$$\oint_{C_2} W \, d\zeta = i \left(2\pi f_1 + \zeta_0 U_0 \frac{dA_{cs}}{dx} \right)$$

$$\oint_{C_1} \psi \, d\zeta = \zeta_0 U_0 \frac{dA_{cs}}{dx} - \oint_{C_1} \zeta \, d\psi \qquad \text{(by parts)}$$

where ζ_0 is the point common to paths C_1 and C_2, as indicated in figure 2. Let $\zeta_g \equiv y_g + iz_g$ be the center of area of the cross-sectional area A_{cs}. Then, by definition of the center of area

$$\zeta_g A_{cs} \equiv \iint_{A_{cs}} \zeta \, dy \, dz$$

Let ν and μ be coordinates normal and tangential to the path C_1 and let v_n be the local velocity component normal to the path. Then

$$\frac{d}{dx}\left(\zeta_g A_{cs}\right) = \frac{d}{dx}\iint_{A_{cs}} \zeta \, dy \, dz = \oint_{C_1} \zeta \frac{d\nu}{dx} d\mu$$

$$= \frac{1}{U_0}\oint_{C_1} \zeta v_n \, d\mu = -\frac{1}{U_0}\oint_{C_1} \zeta \, d\psi$$

Thus, equation (1.3.2) can be expressed in the form

$$F_y + iF_z = 2\pi\rho_0 U_0\left[f_1 + \frac{U_0}{2\pi}\frac{d}{dx}\left(A_{cs}\zeta_g\right)\right] \tag{1.3.3}$$

The force per unit x is

$$\frac{dF_y}{dx} + i\frac{dF_z}{dx} = 2\pi\rho_0 U_0\left[\frac{df_1}{dx} + \frac{U_0}{2\pi}\frac{d^2}{dx^2}\left(A_{cs}\zeta_g\right)\right] \tag{1.3.4}$$

The net moment about the z- and y-axes is then given by

$$M_z + iM_y = \int_{-\infty}^{\infty} x\left(\frac{dF_y}{dx} + i\frac{dF_z}{dx}\right) dx \tag{1.3.5}$$

which can be evaluated in terms of equation (1.3.4). Equations (1.3.3) to (1.3.5) were first derived in reference 5.

A similar expression can be obtained for the rolling moment about the x-axis. From consideration of impulse, the net rolling moment which has acted on a body up to a certain section x is

$$M_x = - \rho_0 U_0 \oint_{C_1} \varphi(y\ dy + z\ dz) \qquad (1.3.6)$$

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The author was not able to evaluate equation (1.3.6) by a procedure similar to that used for equation (1.3.1) except for the case of zero thickness. Thus, when $dz = 0$ in the integrand of equation (1.3.6),

$$M_x = \text{R.P.}\left[-\rho_0 U_0 \oint_{C_1} \zeta\ W\ d\zeta\right] = \text{R.P.}\left[-\rho_0 U_0 \oint_{C_2} \zeta\ W\ d\zeta\right]$$

$$= \text{I.P.}\ (2\pi\rho_0 U_0 f_2) \qquad (1.3.7)$$

where R.P. and I.P. indicate the real and imaginary parts, respectively. The rolling moment, per unit x, for this case, is then

$$\frac{dM_x}{dx} = \text{I.P.}\left(2\pi\rho_0 U_0 \frac{df_2}{dx}\right) \qquad (1.3.8)$$

From equations (1.3.3) to (1.3.8) it is seen that the forces and moments acting on the body can be obtained from a knowledge of the asymptotic form of W and the cross-section geometry. The results are equivalent to integrating the pressure distribution over the body surface, providing the quadratic terms in equation (1.1.2) are retained. The use of equations (1.3.3) to (1.3.8) is considerably simpler than the surface integration of pressures.

For lifting surfaces, there is an induced drag D_i associated with the lift, which can be determined from a consideration of the suction force along the airfoil leading edge. Thus

$$D_i = \alpha L + \oint_{L.E.} dF_x \qquad (1.3.9)$$

where F_x is the suction force acting in the positive x-direction and the contour integral is taken along the airfoil leading edge. (The symbols L and F_z will be used interchangeably to indicate net lift.) The magnitude of the suction force is obtained by considering the flow at a subsonic leading edge to act locally like

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that around a two-dimensional airfoil (fig. 3). In figure 3, the subscripts n and t represent components normal and tangent to the leading edge. The suction force per unit span on the leading edge of a two-dimensional airfoil in a free stream of Mach number $M_{0,n}$ is

$$\frac{dF_n}{dx_t} = - \rho_0 \pi \sqrt{1 - M_{0,n}^2} \lim_{x_n \to 0} u_n^2 x_n$$

Assuming this to apply locally, for a swept wing in a free stream of Mach number M_0,

$$\frac{dF_n}{dx_t} = - \rho_0 \pi \sqrt{1 - M_0^2 \sin^2\theta} \lim_{x_n \to 0} u_n^2 x_n$$

But, from figure 3,

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$$\frac{dF_n}{dx_t} = \frac{dF_x}{dy_2}$$

$$u_n = - \frac{v}{\cos \theta}$$

$$x_n = (y_2 - y)\cos \theta$$

Then

$$\frac{dF_x}{dy_2} = - \rho_0 \pi \frac{\sqrt{1 - M_0^2 \sin^2\theta}}{\cos \theta} \lim_{y \to y_2} v^2(y_2 - y) \qquad (1.3.10)$$

Up to this point, the slender body assumption has not been made in equation (1.3.10). Since v in equation (1.3.10) will be obtained from slender body theory, it is consistent to expand $\sqrt{1 - M_0^2 \sin^2\theta}/\cos \theta$ in terms of the slenderness parameter. Thus,

$$\frac{\sqrt{1 - M_0^2 \sin^2\theta}}{\cos \theta} = 1 + O\left(\frac{\left|\sqrt{M_0^2 - 1}\right| b_0}{c_0}\right)^2$$

where $O(\ )$ indicates order of magnitude. For a slender configuration, equation (1.3.10) becomes

$$\frac{dF_x}{dy_2} = - \rho_0\pi \lim_{y\to y_2} v^2(y_2 - y) = - \frac{\rho_0\pi}{(dy_2/dx)^2} \lim_{y\to y_2} u^2(y_2 - y) \tag{1.3.11}$$

The suction force term in equation (1.3.9) is obtained by integrating equation (1.3.11) along the airfoil leading edge. An alternate method for computing D_i is to find the kinetic energy in the wake behind the wing. Both methods give the same result.

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1.4 Symmetry in planar problems. - The solution of aerodynamic problems is greatly simplified by imposing, at an early stage, whatever symmetry the velocity field must have. The symmetry which exists in problems involving zero thickness wings will now be noted.

If a wing has moderate camber and twist and is at a small angle of attack to the main stream, the solution can be obtained by specifying the boundary conditions in the $z = 0$ plane rather than on the surface of the wing. The w velocity is symmetric while the u and v velocities are antisymmetric about the $z = 0$ plane. That is,

$$\left.\begin{aligned} u(x,y,z) &= - u(x,y,-z) \\ v(x,y,z) &= - v(x,y,-z) \\ w(x,y,z) &= w(x,y,-z) \end{aligned}\right\} \tag{1.4.1}$$

Thus the w velocity is continuous while the u and v velocities are discontinuous (equal and opposite) or zero for corresponding points on the upper and lower surface of the $z = 0$ plane.

When the w boundary condition on the wing is symmetric with respect to the $y = 0$ plane (as in the case of a lifting or pitching wing) and the corresponding plan form edges have the same type singularities, then the following symmetry also applies:

$$\left.\begin{aligned} u(x,y,z) &= u(x,-y,z) \\ v(x,y,z) &= - v(x,-y,z) \\ w(x,y,z) &= w(x,-y,z) \end{aligned}\right\} \tag{1.4.2}$$

When the w boundary condition on the wing is antisymmetric with respect to the $y = 0$ plane (as in the case of a rolling wing), then equations (1.4.2) are replaced by

$$\left.\begin{aligned} u(x,y,z) &= -\,u(x,-y,z) \\ v(x,y,z) &= v(x,-y,z) \\ w(x,y,z) &= -\,w(x,-y,z) \end{aligned}\right\} \qquad (1.4.3)$$

again assuming that corresponding edges have the same singularities.

The symmetry of the derivatives of the perturbation velocities is found by differentiating equations (1.4.1) to (1.4.3).

SECTION 2 - GENERATING FUNCTIONS

The functions $\partial V/\partial \zeta$ and $\partial V/\partial x$ are termed generating functions herein since they are not of particular interest in themselves, but their integration leads to the solution of flow problems[2]. Generating functions can be used to solve all planar problems for which the boundary conditions in the $z = 0$ plane are made homogeneous by differentiation in the x- or y-direction. Expressions for the generating functions are derived in the following section.

2.1 Evaluation of branch points. - The functions which arise in wing problems usually have branch points on the y-axis. It is essential to develop a systematic procedure for introducing cuts so as to make these functions single-valued. In all cases, the cuts will be introduced along the y-axis to the left of the branch points.

As an example, consider the function $(\zeta - y_n)^{N+1/2}$ where N is any integer $(0, \pm 1, \pm 2, \ldots)$ and y_n is an arbitrary point on the y-axis. This function has a branch point at y_n. To make the function single-valued, a cut is introduced along the y-axis from $-\infty$ to y_n (fig. 4). Define the function to be real and positive for

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[2]The application of generating functions to aerodynamic problems was first brought to the author's attention by Dr. H. K. Cheng who used a similar approach in his thesis "Thin Wings in Conical Flow," Cornell University, 1952. The use of the generating function $\partial V/\partial \zeta$ is a classical approach to solving Laplace's equation, for a certain class of boundary conditions, by means of the complex variable. To the author's knowledge, the application of the generating function $\partial V/\partial x$ does not appear explicitly in the literature except for Mangler's use (ref. 13) of an equivalent function $\partial U/\partial \zeta$, (see appendix E).

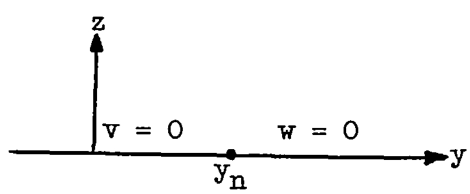
$\zeta = y > y_n$. To evaluate it along the cut, let $\zeta = y_n + \epsilon e^{i\theta}$ and let θ go from 0 to $\pm\pi$. The result is

$$\begin{aligned}(\zeta - y_n)^{N+1/2} &= (y - y_n)^{N+1/2} \quad \text{for} \quad \zeta = y > y_n \\ &= \pm i(y - y_n)^N \sqrt{y_n - y} \quad \text{for} \quad \zeta = y < y_n \qquad (2.1.1)\end{aligned}$$

Thus the function goes from a purely real function for $\zeta = y > y_n$ to a purely imaginary function (discontinuous across the cut) for $\zeta = y < y_n$. 3120

2.2 Behavior of flow near boundary edges. - A boundary edge is defined herein as a point in the $z = 0$ plane where the boundary condition changes from a specification of v to a specification of w. Such a point generally corresponds to the edge of a wing panel.

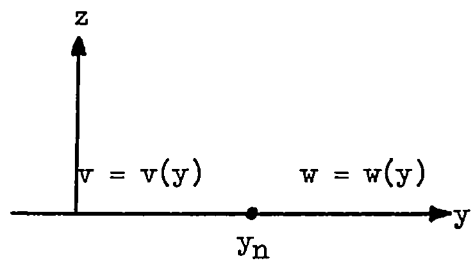
Consider a boundary edge y_n with the boundary conditions $w = 0$ for $y > y_n$ and $v = 0$ for $y < y_n$ (sketch 1).



Sketch 1

These boundary conditions are satisfied by $V = (\zeta - y_n)^{N+1/2}$ where N is an integer. (Because of the uniform procedure prescribed for introducing cuts, this solution for V should be considered as valid for the upper half plane $z \geqslant 0$. The solution for the lower half plane is obtained from symmetry considerations.) For physical reasons the smallest permissible value of N is -1. A smaller value would make W infinite at y_n and this is unrealistic. If the v and w velocities are to be continuous at y_n (Kutta condition), the minimum permissible value of N is 0.

The boundary conditions can now be generalized so that $w = w(y)$ for $y > y_n$ and $v = v(y)$ for $y < y_n$ (sketch 2). In the immediate



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Sketch 2

vicinity of the edge, V must have the form

$$V \sim \left[v(y_n) - iw(y_n)\right] + B(\zeta - y_n)^{N+1/2} \tag{2.2.1}$$

where B is a real constant. The first term of equation (2.2.1) satisfies the inhomogeneous boundary condition at y_n while the second term satisfies the homogeneous boundary conditions indicated in sketch 1. The exponent N has the same limitations as for the flow in sketch 1.

Thus, the homogeneous part of V has at most a half-order singularity at an edge and behaves like

$$\left.\begin{aligned} V &\sim \frac{B}{(\zeta - y_n)^{1/2}} \\ \frac{\partial V}{\partial \zeta} &\sim -\frac{1}{2}\frac{B}{(\zeta - y_n)^{3/2}} \end{aligned}\right\} \tag{2.2.2}$$

where B is real or imaginary, depending on whether the w boundary condition is specified for $y > y_n$ or $y < y_n$. If the Kutta condition applies, the homogeneous part of V behaves like

$$\left.\begin{aligned} V &\sim B(\zeta - y_n)^{1/2} \\ \frac{\partial V}{\partial \zeta} &\sim \frac{1}{2}\frac{B}{(\zeta - y_n)^{1/2}} \end{aligned}\right\} \tag{2.2.3}$$

Differentiation of V with respect to ζ increases the order of the singularity at y_n. If y_n and B are considered functions of x, differentiation of equations (2.2.2) and (2.2.3) yields, respectively,

$$\frac{\partial V}{\partial x} \sim + \frac{1}{2} \frac{B\left(\frac{dy_n}{dx}\right)}{(\zeta - y_n)^{3/2}}$$

$$\frac{\partial V}{\partial x} \sim - \frac{1}{2} \frac{B\left(\frac{dy_n}{dx}\right)}{(\zeta - y_n)^{1/2}} \qquad \text{(Kutta)} \qquad (2.2.4)$$

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where only the leading term is retained in each expression. Thus, differentiation with respect to x also increases the order of the singularity. If y_n does not vary with x, the derivatives in equation (2.2.4) become, respectively,

$$\frac{\partial V}{\partial x} \sim \frac{\frac{dB}{dx}}{(\zeta - y_n)^{1/2}}$$

$$\frac{\partial V}{\partial x} \sim \frac{dB}{dx} (\zeta - y_n)^{1/2} \qquad \text{(Kutta)}$$

so that the order of the singularity is not increased by the differentiation.

2.3 Determination of generating functions. -

(a) Special class of flows: In order to lead smoothly to the discussion of generating functions, it is convenient to solve the two-dimensional incompressible flow associated with one or more flat plates in uniform translation.

Consider a flat plate to be, at a given instant, on the y-axis between y_1 and y_2 and to be moving vertically downward with the velocity $w = - \alpha U_0$ (fig. 5). The boundary conditions in the $z = 0$ plane are indicated in the figure. The complex velocity must equal

$$V = - i\alpha U_0 \left(\frac{\zeta + A}{\sqrt{\zeta - y_1}\sqrt{\zeta - y_2}} \right) + i\alpha U_0 \qquad (2.3.1)$$

where A is an arbitrary real constant which defines the net circulation. (From the asymptotic form of equation (2.3.1) it can be shown that $\Gamma = 2\pi\alpha U_0\left[A + (y_1 + y_2)/2\right]$.)

Equation (2.3.1) is a well-known result. It can be constructed, by inspection, in the following way. The second term on the right-hand side satisfies the nonhomogeneous boundary condition $w = -\alpha U_0$ for $y_1 < y < y_2$. The bracketed term on the right-hand side must
3120 therefore satisfy the homogeneous boundary conditions $w = 0$ for $y_1 < y < y_2$ and $v = 0$ for $y < y_1$ and $y > y_2$. From equation (2.2.2) it is known that V can have half-order singularities at y_1 and y_2. These are introduced as the product $i/\sqrt{\zeta - y_1}\sqrt{\zeta - y_2}$ so that, in the $z = 0$ plane, the term is purely real for $y_1 < y < y_2$ and purely imaginary for $y < y_1$ and $y > y_2$, thereby satisfying the required homogeneous boundary conditions. The factor $\alpha U_0(\zeta + A)$ is then introduced into the numerator to satisfy the condition that V behave like $1/\zeta$ for $\zeta \to \infty$.

The permissibility of introducing A into equation (2.3.1) corresponds to the fact that, mathematically, the circulation about a given airfoil is arbitrary. Taking A equal to $-y_1$ or $-y_2$ is equivalent to applying the Kutta condition at y_1 or y_2, respectively. Either choice gives the same value of Γ, but with different signs. The net circulation is zero when $A = -(y_1 + y_2)/2$.

Equation (2.3.1) can be generalized. This is the problem of m wing panels, each moving downward with velocity $w = -\alpha U_0$ (fig. 6(a)). There are $2m$ boundary edges and the solution is

$$V = -i\alpha U_0\left(\frac{\zeta^m + \sum_{n=0}^{m-1} A_n\zeta^n}{\prod_{n=1}^{2m}\sqrt{\zeta - y_n}}\right) + i\alpha U_0 \qquad (2.3.2)$$

where the A_n are real. The construction of equation (2.3.2) is similar to that of equation (2.3.1). The second term on the right-hand side satisfies the nonhomogeneous boundary condition $w = -\alpha U_0$ for points on the wing panels. The bracketed term satisfies homogeneous boundary conditions in the $z = 0$ plane, ($w = 0$ on the wing panels and $v = 0$ off the wing panels) since it is purely real for points on the wing panels and purely imaginary for points off the wing panels. The numerator is a polynomial of order m since this is the highest order polynomial which will still satisfy the boundary condition $V \sim 1/\zeta$ for $\zeta \rightarrow \infty$.

Equation (2.3.2) has m arbitrary constants, which corresponds to the fact that the circulation about each of the m wing panels is arbitrary. The constants are determined by (1) specifying the circulation about each wing panel, (2) specifying the Kutta condition at m edges, or (3) combinations of (1) and (2). If the Kutta condition is specified at m' edges, designated by y_n', and the remaining m'' edges are designated by y_n'', then equation (2.3.2) can be written 3120

$$V = -i\alpha U_0 \left\{ \frac{\left[\prod_{n=1}^{m'} \sqrt{\zeta - y_n'}\right]\left[\zeta^{(m-m')} + \sum_{n=0}^{(m-m'-1)} A_n \zeta^n\right]}{\prod_{n=1}^{m''} \sqrt{\zeta - y_n''}} \right\} + i\alpha U_0 \tag{2.3.3}$$

Note that $m' \leq m$ and $m' + m'' = 2m$. There are now $(m - m')$ constants to be determined by satisfying the circulation boundary conditions. (If the problem has symmetry in respect to $y = 0$, the number of unknown constants can be reduced by inspection.)

(b) The generating function $\partial V/\partial\zeta$: The boundary conditions in the $z = 0$ plane can be further generalized to the case where the v and w velocities are constants on segments of the y-axis (fig. 6(b)). The problem is one with homogeneous boundary conditions, both in the $z = 0$ plane and for $\zeta \rightarrow \infty$, if $\partial V/\partial\zeta$ rather than V is considered. The solution for $\partial V/\partial\zeta$ can be found by inspection and equals

$$\frac{\partial V}{\partial \zeta} = i\left[\frac{\sum_{n=0}^{3m-2} A_n \zeta^n}{\prod_{n=1}^{2m} (\zeta - y_n)^{3/2}}\right] \tag{2.3.4}$$

where the A_n are real. The denominator of equation (2.3.4) has 3/2-order singularities since differentiation of V with respect to ζ increases the order of the singularity at an edge (eq. (2.2 2)). The boundary conditions $\partial v/\partial y = 0$ and $\partial w/\partial y = 0$, alternately, in the $z = 0$ plane, are satisfied since equation (2.3.4) becomes imaginary and real, alternately. The boundary condition at infinity is satisfied since the leading term of the asymptotic expansion of equation (2.3.4) is of the form $1/\zeta^2$, which is appropriate for problems having a net circulation. There are $3m - 1$ constants in equation (2.3.4) of which only m are arbitrary. The $3m - 1$ constants correspond to the m specified w boundary conditions on the wing panels, the $m - 1$ specified v boundary conditions in the segments between the wing panels, and the m arbitrary circulations which can be imposed. The circulation boundary conditions can be replaced by the Kutta condition (now a half-order singularity) at m, or less, edges. If the Kutta condition is imposed at m' edges (designated by y_n') and the remaining m'' edges are designated by y_n'', equation (2.3.4) becomes

$$\frac{\partial V}{\partial \zeta} = i\left\{\frac{\sum_{n=0}^{(3m-m'-2)} A_n \zeta^n}{\left[\prod_{n=1}^{m'} (\zeta - y_n')^{1/2}\right]\left[\prod_{n=1}^{m''} (\zeta - y_n'')^{3/2}\right]}\right\} \tag{2.3.5}$$

As a further extension, the wing panels may have constant downwash specified but with a finite number of jumps in w occurring across each wing panel (see fig. 6(c)). The solution can be found, from the previous equations, if each jump is artificially separated by a vortex sheet (with the Kutta condition applied at each edge of the sheet) and the limit is then taken as the intervening vortex sheet width goes to zero. Similarly, if there is a discontinuity in v at a point in a vortex sheet, an intervening wing panel is introduced and then made to go to zero. Thus, if such discontinuities occur at m''' points (designated by y_n'''), equation (2.3.5) becomes

$$\frac{\partial V}{\partial \zeta} = i\left\{\frac{\sum_{n=0}^{1/2(m'+3m''+2m'''-4)} A_n \zeta^n}{\left[\prod_{n=1}^{m'} (\zeta - y_n')^{1/2}\right]\left[\prod_{n=1}^{m''} (\zeta - y_n'')^{3/2}\right]\left[\prod_{n=1}^{m'''} (\zeta - y_n''')\right]}\right\} \tag{2.3.6}$$

It is noted that the discontinuities introduce singularities of order one in the expression for $\partial V/\partial \zeta$. This is due to the fact that they correspond to $\log(\zeta - y_n''')$ type terms in the expression for V.

A final generalization will be introduced. Equation (2.3.6) satisfies homogeneous boundary conditions, for $\partial V/\partial \zeta$, both in the $z = 0$ plane and at ∞. A nonhomogeneous constant boundary condition in the $z = 0$ plane can be readily incorporated. Thus, if the w boundary condition on the n^{th} wing panel is $w = C_n + \left(\frac{\partial w}{\partial y}\right)_0 y$, where $\left(\frac{\partial w}{\partial y}\right)_0$ is a constant and the same for all wing panels, equation (2.3.6) becomes

$$\frac{\partial V}{\partial \zeta} = i\left\{\frac{\left(\frac{\partial w}{\partial y}\right)_0 \zeta^{1/2(m'+3m''+2m''')} + \sum_{n=0}^{1/2(m'+3m''+2m'''-2)} A_n\zeta^n}{\left[\prod_{n=1}^{m'} (\zeta - y_n')^{1/2}\right]\left[\prod_{n=1}^{m''} (\zeta - y_n'')^{3/2}\right]\left[\prod_{n=1}^{m'''} (\zeta - y_n''')\right]}\right\} - i\left(\frac{\partial w}{\partial y}\right)_0 \tag{2.3.7}$$

where the additional constant introduced into the numerator of the bracketed term is eliminated by setting equal to zero the coefficient of the $1/\zeta$ term in the asymptotic expansion.

The inverse problem of slender wing theory is to find the trailing edge corresponding to a given shed vortex distribution. The generating function $\partial V/\partial \zeta$ will be used, in later sections, to solve such problems.

(c) The generating function $\partial V/\partial x$: In the previous paragraphs the v or w velocity was specified for all points in the $z = 0$ plane. However, in the direct problem of slender wing theory, the w is specified on the wing but the v distribution off the wing is generally unknown. Consider the case of a swept wing (fig. 7) at angle of attack α in a free stream of velocity U_0. Behind the trailing edge there is a shed vortex sheet whose strength varies only with y. The crossflow then contains two wing panels having the boundary condition $w = -\alpha U_0$ with an intermediate vortex sheet of unknown strength. Previously V was differentiated with respect to ζ in order to obtain a homogeneous boundary-value problem. It is apparent that, for the direct problem, differentiation with respect to x will yield the same result. Thus $\partial V/\partial x$ has homogeneous boundary conditions and the solution for $\partial V/\partial x$ can be constructed in terms of the singularities at the boundary edges.

Differentiation of V with respect to x increases the order of the singularity at an edge, provided the edge varies with x (section 2.2). Thus, for a multiwing panel problem having the boundary condition $w = f(y) + g(x)$ specified for the wing panels, and $v = v(y)$ off the wing panels, the generating function is

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$$\frac{\partial V}{\partial x} = i\left\{\frac{\left[\frac{dg(x)}{dx}\zeta^{1/2(m'+3m''+2m''')} + \sum_{n=0}^{1/2(m'+3m''+2m'''-2)} A_n\zeta^n\right]}{\left[\prod_{n=1}^{m'}(\zeta - y_n')^{1/2}\right]\left[\prod_{n=1}^{m''}(\zeta - y_n'')^{3/2}\right]\left[\prod_{n=1}^{m'''}(\zeta - y_n''')\right]}\right\} - i\frac{dg(x)}{dx} \tag{2.3.8}$$

where the notation is the same as that used in equation (2.3.7). As indicated by equation (1.2.5), the coefficient of $1/\zeta$, in the asymptotic expansion of equation (2.3.8), should be zero. This permits elimination of one of the A_n in equation (2.3.8). Recall that y_n' and y_n'' are boundary edges, that is, points at which the boundary conditions change from a specification of $\partial v/\partial x$ to a specification of $\partial w/\partial x$. If a boundary edge does not vary with x, the corresponding singularity in the denominator and the polynomial in the numerator are each of one order lower than indicated in equation (2.3.8). The symbol y_n''' again represents points at which there is a discontinuity in the v or w boundary condition. If y_n''' does not vary with x, however, the corresponding term does not appear in equation (2.3.8). That this is the case may be verified by replacing the discontinuity by a continuous change over a narrow interval and then letting the interval go to zero. This process does not alter the form of the generating function $\partial V/\partial x$ and hence this generating function is insensitive to discontinuities in the v or w boundary conditions provided these occur along lines of constant y. The discontinuities are reintroduced into the flow field by the integration with respect to x.

For the wing plan form shown in figure 7(a), with the general wing boundary condition $w = f(y) + g(x)$, equation (2.3.8) becomes (for $x > c$)

$$\frac{\partial V}{\partial x} = i\left[\frac{A_0 + A_1\zeta + A_2\zeta^2 + \frac{dg(x)}{dx}\zeta^4}{\sqrt{\zeta^2 - y_2^2}\,(\zeta^2 - y_2^2)^{3/2}}\right] - i\frac{dg(x)}{dx} \tag{2.3.9}$$

where $\Gamma = 0$ is assumed (which makes $A_3 = 0$) and the Kutta condition is imposed at the trailing edge. Equation (2.3.9) is used in sections 4 to 6 to obtain the aerodynamics of swept wings.

For a semi-infinite swept wing (fig. 8) with the general wing boundary condition $w = f(y) + g(x)$, the generating function is

$$\frac{\partial V}{\partial x} = i\left[\frac{A_0 + A_1\zeta + \frac{dg(x)}{dx}\zeta^2}{\sqrt{\zeta - y_1}\,(\zeta - y_2)^{3/2}}\right] - i\,\frac{dg(x)}{dx} \qquad (2.3.10)$$

assuming the Kutta condition at y_1. Moreover, $A_1 = -\frac{1}{2}\frac{dg(x)}{dx}(y_1 + 3y_2)$ so as to satisfy the condition of no net circulation. There is a discontinuity in v across the x-axis (edge of shed vortex sheet) which does not appear in equation (2.3.10). This is due to the fact that the boundary conditions on both sides of the x-axis are in terms of v (i.e., $\partial v/\partial x = 0$) so that the x-axis corresponds to a y_n''' type point which is independent of x. Equation (2.3.10) is used in section 11 to solve unsteady two-dimensional airfoil problems. 3120

SECTION 3 - INTEGRAL EXPRESSIONS FOR FLOW FIELD

As mentioned previously, the generating functions $\partial V/\partial x$ and $\partial V/\partial \zeta$ are used to solve the direct and inverse problems, respectively, of slender wing theory. The final solution of a given problem involves the integration of the generating function and the elimination of the arbitrary A_n so as to satisfy the boundary conditions. The integral expressions for the direct and inverse problem are indicated in sections 3.1 and 3.2, respectively. The solutions should be considered valid for only the upper half-plane ($z \geqslant 0$) because of the arbitrary procedure used to evaluate the branch points (section 2.1). The solution for the lower half-plane is found from symmetry considerations.

The boundary conditions to be satisfied are that there be no lift acting across the shed vortex sheet and that the specified w distribution exist over the wing plan form. The condition of zero loading across the shed vortex sheet can usually be imposed without difficulty. However, in the case of a direct problem, satisfying the specified w boundary condition on the wing requires the solution of an integral equation. Two forms of the integral equation are indicated in section 3.3.

<u>3.1 Integral expressions (direct problem)</u>. - Replacing x by ξ as an integration variable yields

$$W = \int_{\infty}^{\zeta} d\zeta \int_{-\infty}^{x} \left(\frac{\partial V}{\partial \xi}\right) d\xi \qquad (3.1.1a)$$

$$V = \int_{-\infty}^{x} \left(\frac{\partial V}{\partial \xi}\right) d\xi \tag{3.1.1b}$$

Note that W and V must be continuous functions of x for all three-dimensional flow problems where slender body theory is applicable.

3120 The complex velocity V is obtained by differentiating W with respect to ζ. A corresponding complex function U can be defined as the derivative of W with respect to x. That is,

$$U \equiv \frac{\partial W}{\partial x} = u + i \left(\frac{\partial \psi}{\partial x}\right) \tag{3.1.2}$$

so that the real part of U is proportional to the loading in the $z = 0$ plane. Moreover, $\partial^2 W/\partial x\, \partial\zeta = \frac{\partial V}{\partial x} = \frac{\partial U}{\partial \zeta}$, so that

$$U = \int_{\infty}^{\zeta} \left(\frac{\partial V}{\partial x}\right) d\zeta \tag{3.1.3}$$

Equation (3.1.3) can be integrated directly and is used to satisfy the condition of zero loading in the shed vortex sheet.

3.2 Integral expressions (inverse problem). - The corresponding expressions for the inverse problem are

$$\left.\begin{aligned} W &= \int_{\infty}^{\zeta} d\zeta \int_{\infty}^{\zeta} \left(\frac{\partial V}{\partial \zeta}\right) d\zeta \\ V &= \int_{\infty}^{\zeta} \left(\frac{\partial V}{\partial \zeta}\right) d\zeta \\ U &= \frac{\partial}{\partial x} \int_{\infty}^{\zeta} d\zeta \cdot \int_{\infty}^{\zeta} \left(\frac{\partial V}{\partial \zeta}\right) d\zeta \end{aligned}\right\} \tag{3.2.1}$$

3.3 Integral equations (direct problem). - Assume that the cross-flow at some section $x = c$ is known. This is true, for example, for the swept wing indicated in figure 7. Then, from equation (3.1.1b)

$$w(x,y,0) - w(c,y,0) = \text{I.P.}\left[-\int_c^x \left(\frac{\partial V}{\partial \xi}\right) d\xi\right]_{z=0} \tag{3.3.1}$$

where the notation $w(x,y,z)$ means that w is evaluated at (x,y,z). If the value of y is such that $(x,y,0)$ corresponds to a point on the wing surface (as in fig. 9(a)), then the left-hand side of equation (3.3.1) is known and equation (3.3.1) is an integral equation for $\partial V/\partial \xi$.

An alternate form of the integral equation can be found by integrating both sides of equation (3.3.1) in respect to y. Let η be the integration variable in the y-direction. Integrating both sides between the limits y and $y_2(x)$ yields

$$\int_y^{y_2(x)} w(x,\eta,0)d\eta + \left[\psi(c,y_2(x),0) - \psi(c,y,0)\right]$$

$$= \text{I.P.}\left[-\int_c^x d\xi \int_y^{y_2(x)} \left(\frac{\partial V}{\partial \xi}\right) d\eta\right]_{z=0} \tag{3.3.2}$$

where the order of integration on the right-hand side has been reversed. The area of integration is indicated in figure 9(b). Equation (3.3.2) can always be reduced to a function of x plus a function of y equal to a function of x plus a function of y. The functions of x or of y can then be equated, providing an alternate form of the integral equation for $\partial V/\partial \xi$.

Numerical methods are usually required to solve equation (3.3.1) or (3.3.2). For the case of a swept wing, the numerical solution of equation (3.3.1) requires an integration by parts in order to reduce the order of the singularity of the integrand and this introduces derivatives of the unknown functions A_n into the integrand. However, the numerical solution of equation (3.3.2) does not require the integration by parts since the order of the singularity in the integrand is reduced by the η-integration. Thus the numerical solution of equation (3.3.2) is usually more straightforward than that of equation (3.3.1). The alternate forms of the integral equation, for the swept-wing lift problem, are derived in section 4.2.

SECTION 4 - LIFT OF SWEPT WINGS (DIRECT PROBLEM)

The solution for a swept wing at angle of attack is derived in terms of S, a function of x, which must be evaluated from an integral equation. Some limiting solutions of the integral equation are obtained.

4.1 Load distribution. - For the lift case, the w boundary condition on the wing of figure 7 is $w = - \alpha U_0$. The solution for $x < c$ is well known and equals

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$$W = - i\alpha U_0 \left(\sqrt{\zeta^2 - y_2^2} - \zeta\right) \tag{4.1.1a}$$

$$V = - i\alpha U_0 \left(\frac{\zeta}{\sqrt{\zeta^2 - y_2^2}} - 1\right) \tag{4.1.1b}$$

$$\frac{\partial V}{\partial x} = - i\alpha U_0 \left[\frac{y_2\left(\frac{dy_2}{dx}\right)\zeta}{(\zeta^2 - y_2^2)^{3/2}}\right] \tag{4.1.1c}$$

$$\frac{dL}{dx} = 4\pi q\alpha y_2 \left(\frac{dy_2}{dx}\right) \tag{4.1.1d}$$

Equations (4.1.1a) and (4.1.1b) are valid at $x = c$ since W and V are continuous functions of x. However, $\partial V/\partial x$ and dL/dx may be discontinuous at $x = c$.

For $x > c$, the generating function is (from eq. (2.3.9))

$$\frac{\partial V}{\partial x} = - i\alpha U_0 \left[\frac{A_0 + A_2\zeta^2}{\sqrt{\zeta^2 - y_1^2}\,(\zeta^2 - y_2^2)^{3/2}}\right] \tag{4.1.2}$$

where the symmetry with respect to y has been imposed (i.e., $A_1 = 0$). The functions A_0 and A_2 are eliminated by satisfying the w boundary condition on the wing and the boundary condition that there be no loading in the wake.

The load distribution in the $z = 0$ plane is proportional to u (eq. (1.1.2))[3]. The value of u on the upper surface of the right-hand wing panel is, from equation (3.1.3),

$$u = \text{R.P.}\left[-\,i\alpha U_0 \int_{\infty}^{y} \frac{(A_0 + A_2\zeta^2)\,d\zeta}{\sqrt{\zeta^2 - y_1^{\,2}}\,(\zeta^2 - y_2^{\,2})^{3/2}}\right] \tag{4.1.3}$$

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The integration is conducted along the upper surface of the $z = 0$ plane with a suitable indentation at y_2. If η is used as the integration variable in the y-direction, equation (4.1.3) can be written as

$$u = \alpha U_0 \mathop{\int\!\!\!\!\!\!-}_{y_2}^{y} \frac{(A_0 + A_2\eta^2)\,d\eta}{\sqrt{\eta^2 - y_1^{\,2}}\,(y_2^{\,2} - \eta^2)^{3/2}} \tag{4.1.4}$$

where $\int\!\!\!\!\!\!-$ indicates the infinite part of the improper integral (appendix B). Integration of equation (4.1.4) yields (see appendix C)

$$\frac{u}{-\alpha U_0} = \left[\frac{A_0 + A_2 y_1^{\,2}}{y_2(y_2^{\,2} - y_1^{\,2})}\right] F(\beta',k') - \left[\frac{A_0 + A_2 y_2^{\,2}}{y_2(y_2^{\,2} - y_1^{\,2})}\right]\left(E(\beta',k') + \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^{\,2}}{y_2^{\,2} - y^2}}\right) \tag{4.1.5}$$

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[3]Strictly speaking, equation (1.1.2) is applicable when the boundary conditions are satisfied on the wing surface. When the boundary conditions are satisfied in the $z = 0$ plane, as is done herein, the appropriate expression for pressure is

$$p - p_0 = -\,\rho_0\left(uU_0 + \alpha wU_0 + \frac{v^2 + w^2}{2}\right)$$

for a configuration at angle of attack α. At any rate, the loading is proportional to u, for a wing of zero thickness, since all the other terms are symmetric with respect to z.

where $F(\beta',k')$ and $E(\beta',k')$ are incomplete elliptic integrals of the first and second kind, respectively, with amplitude β' and modulus k'. These have the values

$$\beta' = \sqrt{\frac{y_2^2 - y^2}{y_2^2 - y_1^2}} \qquad k' = \sqrt{1 - \left(\frac{y_1}{y_2}\right)^2} \tag{4.1.6}$$

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To satisfy the condition of zero loading in the wake, equation (4.1.5) is set equal to zero for $y = y_1$. This gives a relation between A_0 and A_2. If S, a function of x, is introduced according to

$$\left(\frac{dy_2}{dx}\right) S = \frac{A_0 + A_2 y_2^2}{y_2(y_2^2 - y_1^2)}$$

then equation (4.1.5) becomes

$$u = \alpha U_0 \left(\frac{dy_2}{dx}\right) S \left[E(\beta',k') - F(\beta',k')\frac{E'}{K'} + \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right] \tag{4.1.7}$$

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where K' and E' are complete elliptic integrals of first and second kind with modulus k'. Equation (4.1.7) describes the spanwise variation of loading on the wing. The unknown function S appears only as a scale factor. The generating function can now be written as

$$\frac{\partial V}{\partial x} = - i\alpha U_0 y_2 \left(\frac{dy_2}{dx}\right) S \left(\frac{\zeta^2 - y_1^2}{\zeta^2 - y_2^2} - \frac{E'}{K'}\right) \frac{1}{\sqrt{(\zeta^2 - y_2^2)(\zeta^2 - y_1^2)}} \tag{4.1.8}$$

From equations (1.3.4) and (1.2.5) and the asymptotic form of equation (4.1.8), the lift per unit x is, in terms of S,

$$\frac{dL}{dx} = 4\pi\alpha q y_2 \left(\frac{dy_2}{dx}\right) S \left(1 - \frac{E'}{K'}\right) \tag{4.1.9}$$

From equations (4.1.7) and (1.3.11) the suction force at the leading edge is

$$\frac{dF_x}{dy_2} = - \pi\alpha^2 q S^2 \left(\frac{y_2^2 - y_1^2}{y_2}\right) \tag{4.1.10}$$

The function S remains to be determined.

4.2 Integral equations for S. - The function S is determined from equations (3.3.1) or (3.3.2). Each of these equations is considered separately.

(a) Equation (3.3.1): Substitution into equation (3.3.1) yields

$$\frac{-\alpha U_0 y}{\sqrt{y^2 - b^2}} = \text{I.P.}\left\{i\alpha U_0 \int_c^x \frac{y_2 \left(\frac{dy_2}{d\xi}\right) S\left(\frac{\zeta^2 - y_1^2}{\zeta^2 - y_2^2} - \frac{E'}{K'}\right) d\xi}{\sqrt{(\zeta^2 - y_2^2)(\zeta^2 - y_1^2)}}\right\}_{z=0} \quad (4.2.1a)$$

$$= \alpha U_0 \int_c^{x_2(y)} \frac{y_2 \left(\frac{dy_2}{d\xi}\right) S\left(\frac{y^2 - y_1^2}{y^2 - y_2^2} - \frac{E'}{K'}\right) d\xi}{\sqrt{(y^2 - y_2^2)(y^2 - y_1^2)}} \quad (4.2.1b)$$

where b is the value of y_2 at $x = c$ and $y > b$ is assumed. The path of integration for equations (4.2.1) is indicated in figure 9(a). Transforming from ξ to y_2 as the variable of integration gives

$$\frac{-y}{\sqrt{y^2 - b^2}} = \int_b^y \frac{y_2 S\left(\frac{y^2 - y_1^2}{y^2 - y_2^2} - \frac{E'}{K'}\right) dy_2}{\sqrt{(y^2 - y_2^2)(y^2 - y_1^2)}} \quad (4.2.2)$$

From equation (4.2.2) it is found that $S \rightarrow 1$ for $y_2 \rightarrow b$. Thus $\partial V/\partial x$ and dL/dx are continuous functions of x at $x = c$. It is rather remarkable that dL/dx is continuous at $x = c$ since it can be shown that there is a pressure discontinuity at this section. Equation (4.2.2) is a Volterra type integral equation for S, which must, in general, be solved numerically. To permit numerical solution, the finite-part operation must be eliminated. This can be done by integrating the 3/2-order singularity by parts; the result is

$$0 = \int_b^y \frac{Sy_2 dy_2}{\sqrt{(y^2 - y_1{}^2)(y^2 - y_2{}^2)}} \left(\frac{E'}{K'} + \frac{y^2 - y_1{}^2}{Sy_2} \frac{dS}{dy_2} - \frac{y_1}{y_2} \frac{dy_1}{dy_2}\right) \tag{4.2.3}$$

It is noted that the integrand of equation (4.2.3) contains both the unknown function S and its derivative.

(b) Equation (3.3.2): Substituting into equation (3.3.2) and noting that $y > b$ yield

$$- \sqrt{y_2{}^2(x) - b^2} + \sqrt{y^2 - b^2}$$

$$= \text{I.P.} \left[i \int_c^x y_2 \left(\frac{dy_2}{d\xi}\right) S \, d\xi \int_y^{y_2(x)} \frac{\left(\frac{\zeta^2 - y_1{}^2}{\zeta^2 - y_2{}^2} - \frac{E'}{K'}\right) d\eta}{\sqrt{(\zeta^2 - y_1{}^2)(\zeta^2 - y_2{}^2)}} \right]_{z=0} \tag{4.2.4}$$

It is desirable to express the right-hand side of equation (4.2.4) as a function of x plus a function of y so that the functions of x or y can be equated. Taking the imaginary part of the right-hand side and utilizing the finite-part technique result in an area of integration as indicated in figure 10(a). This integration can be decomposed into two separate integrations as indicated in figures 10(b) and 10(c). The right-hand side of equation (4.2.4) then becomes

$$\text{R.H.S.} = \int_c^x y_2 \left(\frac{dy_2}{d\xi}\right) S \, d\xi \; -\!\!\!\!\!\!\int_{y_2(\xi)}^{y_2(x)} \frac{\left(\frac{\eta^2 - y_1{}^2}{\eta^2 - y_2{}^2} - \frac{E'}{K'}\right) d\eta}{\sqrt{(\eta^2 - y_2{}^2)(\eta^2 - y_1{}^2)}} \; -$$

$$\int_c^{x_2(y)} y_2 \left(\frac{dy_2}{d\xi}\right) S \, d\xi \; -\!\!\!\!\!\!\int_{y_2(\xi)}^{y} \frac{\left(\frac{\eta^2 - y_1{}^2}{\eta^2 - y_2{}^2} - \frac{E'}{K'}\right) d\eta}{\sqrt{(\eta^2 - y_2{}^2)(\eta^2 - y_1{}^2)}} \tag{4.2.5}$$

The first term in equation (4.2.5) is a function of x only while the second term is a function of y only. Equating the functions of y gives

$$\sqrt{y^2 - b^2} = -\int_c^{x_2(y)} y_2\left(\frac{dy_2}{d\xi}\right) S\, d\xi \int_{y_2(\xi)}^{y} \frac{\left(\frac{\eta^2 - y_1{}^2}{\eta^2 - y_2{}^2} - \frac{E'}{K'}\right) d\eta}{\sqrt{(\eta^2 - y_2{}^2)(\eta^2 - y_1{}^2)}} \tag{4.2.6}$$

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Integrating the inner integral and converting from ξ to y_2 as the integration variable give

$$\sqrt{y^2 - b^2} = \int_b^y S\, dy_2 \left(\Phi + \frac{y_2}{y}\sqrt{\frac{y^2 - y_1{}^2}{y^2 - y_2{}^2}}\right) \tag{4.2.7}$$

where

$$\Phi = \left(\frac{E'}{K'} - 1\right)\left[K - F(\beta,k)\right] + E - E(\beta,k)$$

$$= \frac{\pi}{2K'} + \left(1 - \frac{E'}{K'}\right) F(\beta,k) - E(\beta,k) \quad \text{(by Legendre's relation)}$$

$$\beta = y_2/y \qquad k = y_1/y_2$$

and K and E are complete elliptic integrals with modulus k. Equation (4.2.7) provides an alternate Volterra type integral equation for S which does not contain the derivative of S in the integrand.

Equation (4.2.7) differs from equation (15) in reference 11. The discrepancy arises from Robinson's treatment of his equation (12). He properly states that a function of x can be added to his equation (12), but takes this function to be zero. This function of x actually does not equal zero except for $x = c$. Hence Robinson's integral equation is correct only for the limiting case of a wing whose trailing edge is only slightly swept. This limitation of Robinson's work was previously pointed out by this author in reference 12.

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4.3 Limiting solutions of integral equations. -

(a) Special solution: It has already been established that $S = 1$ at $x = c$. A special solution of equation (4.2.3) is to assume that $S = 1$ for all x. Then

$$\frac{E'}{K'} = k \frac{dy_1}{dy_2} \tag{4.3.1}$$

Integrating

$$\frac{y_1}{b} = \frac{k}{E' - k^2 K'} \tag{4.3.2}$$

which gives the equation of the trailing edge. This result was obtained in reference 6, by other methods, and corresponds to the case where there is no shed vortex sheet. A more detailed discussion will be given in section 7.2.

(b) $k \ll 1$: For $k \ll 1$, the solution for S is (eq. (D4b) of appendix D)

$$S = 1 + 4 \int_0^{k/4} \frac{dt}{\gamma \ln t} + \frac{1}{2} k^2 + \left[O(k^4) + O\left(\frac{k^2}{\gamma^2 \ln k}\right) + O\left(\frac{k^3}{\gamma}\right) \right] \tag{4.3.3}$$

where $\gamma \equiv dy_1/dy_2$ is the ratio of the trailing-edge slope to the leading-edge slope. For γ equal to a constant, the integral in equation (4.3.3) becomes the logarithmic integral which is tabulated in reference 14.

(c) $\gamma \to \infty$: When the ratio of the trailing-edge to leading-edge slope is large, the solution for S is (eq. (D5))

$$S = \frac{1}{\sqrt{1 - k^2}} \tag{4.3.4}$$

which is valid for all values of k. Equation (4.3.4) is equivalent to that obtained by Robinson since his integral equation is correct when $\gamma \to \infty$.

(d) $\gamma = 1, k \to 1$: This case corresponds to a two-dimensional swept wing. The solution for S is $S = 1$, 4.4 Numerical solutions. - The y-coordinate will be nondimensionalized with respect to b by means of the notation $Y \equiv y/b$, $Y_2 \equiv y_2/b$, and so forth. Equation (4.2.7) becomes

$$\sqrt{Y^2 - 1} = \int_1^Y S\, dY_2 \left(\Phi + \frac{Y_2}{Y}\sqrt{\frac{Y^2 - Y_1{}^2}{Y^2 - Y_2{}^2}}\right) \qquad (4.4.1)$$

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Divide the integration interval into m parts (fig. 11). Let $\eta_{2,n}$ represent a mean value between $Y_{2,n-1}$ and $Y_{2,n}$. Let $\eta_{1,n}$ represent a mean value between $Y_{1,n-1}$ and $Y_{1,n}$. When the mean value theorem is used, equation (4.4.1) becomes

$$\sqrt{Y^2 - 1} = \sum_{n=1}^{m} S_n\Phi_n(Y_{2,n} - Y_{2,n-1}) +$$

$$S_n \frac{\sqrt{Y^2 - \eta_{1,n}^2}}{Y}\left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right) \qquad (4.4.2)$$

where S_n and Φ_n are evaluated at $\eta_{2,n}$. Thus

$$\beta_n = \eta_{2,n}/Y$$

$$k_n = \eta_{1,n}/\eta_{2,n}$$

and so forth. The values of S_1, S_2, S_3, . . . are found by successively letting $m = 1, 2, 3, \ldots$ in equation (4.4.2). The explicit expression for S_m is

$$S_m = \frac{\sqrt{Y^2 - 1} - \sum_{n=1}^{m-1} S_n\Phi_n(Y_{2,n} - Y_{2,n-1}) + \frac{S_n\sqrt{Y^2 - \eta_{1,n}^2}}{Y}\left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}}\right)}{\Phi_m(Y - Y_{2,m-1}) + \frac{\sqrt{Y^2 - \eta_{1,m}^2}}{Y}\sqrt{Y^2 - Y_{2,n-1}^2}} \qquad (4.4.3)$$

An alternate expression for S_m can be obtained from equation (4.2.3).

Equation (4.4.3) was evaluated for the $\gamma = 1$ case with intervals $(Y_{2,n} - Y_{2,n-1}) = 0.20$. The intermediate points $\eta_{1,n}$ and $\eta_{2,n}$ were

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taken to be in the middle of each interval. The results are compared with equation (4.3.3) in figure 12. The two agree within 2 percent for $y_2/b < 2.5$. According to figure 12, S decreases from 1 at $y_2/b = 1$ to a minimum value of 0.94 at $y_2/b = 2$ and then increases slowly to its asymptotic value of 1.

The aerodynamic forces can be found from a numerical integration with the previously found values of S. Thus, the total lift is given by

$$\frac{L}{2\pi q\alpha b^2} = 1 + 2\int_1^{b_0/b} S(1 - E'/K')Y_2 dY_2$$

$$= 1 + \sum_{n=1}^{m} S_n(1 - E_n'/K_n')(Y_{2,n}^2 - Y_{2,n-1}^2) \qquad (4.4.5)$$

where $m = \dfrac{b_0/b - 1}{Y_{2,n} - Y_{2,n-1}}$ when equal intervals are used. Similar expressions can be deduced for pitching moment, center of pressure, and induced drag. Some of these coefficients are presented in figure 13 for the case $\gamma = 1$ and dy_2/dx = constant.

SECTION 5 - ROLLING SWEPT WING (DIRECT PROBLEM)

The solution for a rolling swept wing is presented using the procedure of section 4.

5.1 Load distribution and rolling moment. - The boundary condition on the wing is $w = -\omega_x y$ where ω_x is the angular velocity. The solution for $x < c$ is

$$W = -i\frac{\omega_x}{2}\left(\zeta\sqrt{\zeta^2 - {y_2}^2} - \zeta^2\right) \qquad (5.1.1a)$$

$$V = -i\frac{\omega_x}{2}\left(\frac{2\zeta^2 - {y_2}^2}{\sqrt{\zeta^2 - {y_2}^2}} - 2\zeta\right) \qquad (5.1.1b)$$

$$\frac{\partial V}{\partial x} = - i \frac{\omega_x}{2}\left[\frac{{y_2}^3\left(\frac{dy_2}{dx}\right)}{(\zeta^2 - {y_2}^2)^{3/2}}\right] \tag{5.1.1c}$$

$$\frac{dM_x}{dx} = \pi q \frac{\omega_x}{U_0} {y_2}^3 \left(\frac{dy_2}{dx}\right) \tag{5.1.1d}$$

where dM_x/dx is the rolling moment per unit x. Again, equations (5.1.1a) and (5.1.1b) are valid at $x = c$. 3120

From equation (2.3.9), the generating function for $x > c$ is

$$\frac{\partial V}{\partial x} = - i \frac{\omega_x}{2} \frac{A_1 \zeta}{\sqrt{\zeta^2 - {y_1}^2}\,(\zeta^2 - {y_2}^2)^{3/2}} \tag{5.1.2}$$

For a point on the right-hand wing panel

$$u = \frac{\omega_x A_1}{2} \int_{y_2}^{y} \frac{\eta \, d\eta}{\sqrt{\eta^2 - {y_1}^2}\,({y_2}^2 - \eta^2)^{3/2}}$$

$$= \frac{\omega_x A_1}{2({y_2}^2 - {y_1}^2)} \sqrt{\frac{y^2 - {y_1}^2}{{y_2}^2 - y^2}} \tag{5.1.3}$$

Define R a function of x, such that

$$y_2 \left(\frac{dy_2}{dx}\right) R = \frac{A_1}{{y_2}^2 - {y_1}^2}$$

Then equation (5.1.3) becomes

$$u = \frac{\omega_x}{2} y_2 \left(\frac{dy_2}{dx}\right) R \sqrt{\frac{y^2 - {y_1}^2}{{y_2}^2 - y^2}} \tag{5.1.4}$$

The generating function, in terms of R, is

$$\frac{\partial V}{\partial x} = - i \frac{\omega_x}{2} y_2({y_2}^2 - {y_1}^2) R \left[\frac{\zeta}{\sqrt{\zeta^2 - {y_1}^2}\,(\zeta^2 - {y_2}^2)^{3/2}}\right]\left(\frac{dy_2}{dx}\right) \tag{5.1.5}$$

The rolling moment is (from eq. (1.3.8)).

$$\frac{dM_x}{dx} = \pi q \frac{\omega_x y_2}{U_0} (y_2^2 - y_1^2)\left(\frac{dy_2}{dx}\right) R$$

5.2 Integral equation for R. - Substituting into equation (3.3.2) and equating functions of y yield (for $y > b$)

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$$y\sqrt{y^2 - b^2} = -\int_c^{x_2(y)} y_2(y_2^2 - y_1^2)\, R \left(\frac{dy_2}{d\xi}\right) d\xi \int_{y_2(\xi)}^{y} \frac{\eta \, d\eta}{\sqrt{\eta^2 - y_1^2}\,(\eta^2 - y_2^2)^{3/2}} \tag{5.2.1}$$

Integrating the inner integral in respect to η, and transforming from ξ to y_2 as the integration variable in the outer integral, gives

$$y\sqrt{y^2 - b^2} = \int_b^{y} R \sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}}\, y_2 dy_2 \tag{5.2.2}$$

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From equation (5.2.2) it is seen that $R = 1$ at $y_2 = b$ so that dM_x/dx and $\partial V/\partial x$ are continuous at $x = c$.

5.3 Limiting solutions of integral equation. -

(a) $k << 1$: For $k << 1$, the solution for R is (eq. (D7b) of appendix D)

$$R = 1 + \frac{1}{2} k^2 + \left[O(k^4) + O(k^3/\gamma)\right] \tag{5.3.1}$$

(b) $\gamma \to \infty$: In this case, from equation (D8),

$$R = \frac{1}{\sqrt{1 - k^2}} \tag{5.3.2}$$

valid for all k.

(c) $\gamma = 1,\ k \to 1$: The solution is $R = 2$.

5.4 Numerical solution. - When equation (5.2.2) is nondimensionalized with respect to b, it becomes

$$Y\sqrt{Y^2 - 1} = \int_1^Y R\sqrt{\frac{Y^2 - Y_1{}^2}{Y^2 - Y_2{}^2}}\, Y_2 dY_2 \qquad (5.4.1)$$

Dividing the integration interval into m parts and using the mean value theorem as in section 4.4 give

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$$Y\sqrt{Y^2 - 1} = \sum_{n=1}^{m} R_n \sqrt{Y^2 - \eta_{1,n}^2}\left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right)$$

$$R_m = \frac{Y\sqrt{Y^2 - 1} - \sum_{n=1}^{m-1} R_n\sqrt{Y^2 - \eta_{1,n}^2}\left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right)}{\sqrt{Y^2 - \eta_{1,m}^2}\,\sqrt{Y^2 - Y_{2,m-1}^2}} \qquad (5.4.2)$$

Values of R were computed from equation (5.4.2) for the $\gamma = 1$ case, with intervals $(Y_{2,n} - Y_{2,n-1}) = 0.20$. The results are compared with equation (5.3.1) in figure 14. The two agree within 2 percent for $y_2/b < 3.0$. From figure 14 it is seen that, for $\gamma = 1$, R increases monotonically from a value of 1 at $y_2/b = 1$ to its asymptotic value of 2 at $y_2/b \to \infty$.

The total rolling moment is

$$M_x = \pi q \frac{b\omega_x}{U_0}\frac{b^3}{4}\left[1 + 4\int_1^{b_0/b} R(1 - Y_1{}^2/Y_2{}^2)Y_2{}^3\, dY_2\right]$$

$$= \pi q \frac{b\omega_x}{U_0}\frac{b^3}{4}\left[1 + \sum_{n=1}^{m} R_n\left(1 - \frac{\eta_{1,n}^2}{\eta_{2,n}^2}\right)(Y_{2,n}^4 - Y_{2,n-1}^4)\right] \qquad (5.4.3)$$

The rolling moment for the case $\gamma = 1$ and dy_2/dx = constant was computed and the results are in figure 15.

SECTION 6 - PITCHING SWEPT WING (DIRECT PROBLEM)

The solution for a swept wing, pitching about the y-axis, is presented following the procedure of section 4.

6.1 Load distribution. - The boundary condition on the wing is $w = -\omega_y x$. The solution for $x < c$ is

$$W = -i\omega_y x\left(\sqrt{\zeta^2 - y_2^2} - \zeta\right) \tag{6.1.1a}$$

$$V = -i\omega_y x\left(\frac{\zeta}{\sqrt{\zeta^2 - y_2^2}} - 1\right) \tag{6.1.1b}$$

$$\frac{\partial V}{\partial x} = -i\omega_y\left\{\frac{\zeta\left[xy_2\left(\frac{dy_2}{dx}\right) - y_2^2\right] + \zeta^3}{(\zeta^2 - y_2^2)^{3/2}} - 1\right\} \tag{6.1.1c}$$

$$\frac{dL}{dx} = 2\pi q\,\frac{\omega_y}{U_0}\,\frac{d(y_2^2 x)}{dx} \tag{6.1.1d}$$

where equations (6.1.1a) and (6.1.1b) are valid at $x = c$.

The generating function, for $x > c$, is (from eq. (2.3.9))

$$\frac{\partial V}{\partial x} = -i\omega_y\left\{\frac{A_0 + A_2\zeta^2 + \zeta^4}{\sqrt{\zeta^2 - y_1^2}\,(\zeta^2 - y_2^2)^{3/2}} - 1\right\} \tag{6.1.2}$$

From equation (3.1.3), the expression for u on the right-hand wing panel is

$$u = -\omega_y\left\{\frac{A_0 + A_2y_1^2 + y_1^2y_2^2}{y_2(y_2^2 - y_1^2)}F(\beta',k') - \frac{A_0 + A_2y_2^2 + y_2^4(1 + k'^2)}{y_2(y_2^2 - y_1^2)}E(\beta',k') - \frac{A_0 + A_2y_2^2 + y_2^4}{y_2(y_2^2 - y_1^2)}\,\frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right\} \tag{6.1.3}$$

To satisfy the condition of zero loading in the wake,

$$\frac{A_0 + A_2y_1^2 + y_1^2y_2^2}{y_2(y_2^2 - y_1^2)}K' = \frac{A_0 + A_2y_2^2 + y_2^4(1 + k'^2)}{y_2(y_2^2 - y_1^2)}E'$$

Define a new variable Q such that

$$2y_2Q \equiv \frac{A_0 + A_2y_2^2 + y_2^4(1 + k'^2)}{y_2(y_2^2 - y_1^2)}$$

Then

$$A_0 = 2y_2^2Q(y_2^2E'/K' - y_1^2) + y_1^2y_2^2$$

$$A_2 = 2y_2^2\left[Q(1 - E'/K') - 1\right]$$

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and equation (6.1.3) becomes

$$u = 2y_2\omega_yQ\left[E(\beta',k') - \frac{E'}{K'}F(\beta',k') + \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right] - \omega_y y\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}} \tag{6.1.4}$$

The generating function, in terms of Q, is

$$\frac{\partial V}{\partial x} = -i\omega_y\left[\frac{2y_2^2Q\left(\frac{\zeta^2 - y_1^2}{\zeta^2 - y_2^2} - \frac{E'}{K'}\right) + (\zeta^2 - y_2^2) - y_2^2\left(\frac{y_2^2 - y_1^2}{\zeta^2 - y_2^2}\right)}{\sqrt{(\zeta^2 - y_2^2)(\zeta^2 - y_1^2)}} - 1\right] \tag{6.1.5}$$

and the lift per unit x, from equations (1.2.5), (1.3.4), and (6.1.5) is

$$\frac{dL}{dx} = 2\pi q\frac{\omega_y}{U_0}\left[4y_2^2Q(1 - E'/K') - (y_2^2 - y_1^2)\right] \tag{6.1.6}$$

6.2 Integral equation for Q. - Substituting equation (6.1.5) into equation (3.3.2), utilizing equation (6.1.1a), and equating functions of y yield, for $y > b$,

$$c\sqrt{y^2 - b^2} = -\int_c^{x_2(y)} d\xi \int_{y_2(\xi)}^{y} \left[\frac{2y_2^2 Q\left(\frac{\eta^2 - y_1^2}{\eta^2 - y_2^2} - \frac{E'}{K'}\right) + (\eta^2 - y_2^2) - y_2^2\left(\frac{y_2^2 - y_1^2}{\eta^2 - y_2^2}\right)}{\sqrt{(\eta^2 - y_1^2)(\eta^2 - y_2^2)}}\right] d\eta \qquad (6.2.1)$$

Integrating the inner integral and transforming from ξ to y_2 as the integration variable of the outer integral give

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$$c\sqrt{y^2 - b^2} + y\int_b^y \frac{dy_2}{(dy_2/d\xi)}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}} = 2\int_b^y \frac{y_2 Q}{(dy_2/d\xi)}\left\{\Phi + \frac{y_2}{y}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}}\right\}dy_2 \qquad (6.2.2)$$

where Φ is defined by equation (4.2.7). To investigate equation (6.2.2), it is convenient to consider Q as the sum of two functions $Q^{(1)}$ and $Q^{(2)}$ defined according to the relation

$$Q \equiv \frac{1}{2}\frac{c}{y_2}\left(\frac{dy_2}{d\xi}\right)\left(Q^{(1)} + Q^{(2)}\right) \qquad (6.2.2a)$$

such that

$$\sqrt{y^2 - b^2} = \int_b^y Q^{(1)}\left(\Phi + \frac{y_2}{y}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}}\right)dy_2 \qquad (6.2.3)$$

$$\frac{y}{c}\int_b^y \frac{dy_2}{(dy_2/d\xi)}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}} = \int_b^y Q^{(2)}\left(\Phi + \frac{y_2}{y}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}}\right)dy_2 \qquad (6.2.4)$$

Comparing equation (6.2.3) with equation (4.2.7) shows that $Q^{(1)} \equiv S$. Thus only equation (6.2.4) requires further study. It can be shown that

$$Q = \frac{1}{2}\frac{c}{b}\left(\frac{dy_2}{dx}\right)\left[1 + \frac{b}{c(dy_2/dx)}\right]$$

at $x = c$ so that dL/dx and $\partial V/\partial x$ are continuous at that section.

6.3 Limiting solutions of integral equation. - It will be assumed that $dy_2/d\xi$ is constant so that $dy_2/d\xi = b/c$.

(a) $k \ll 1$: From equation (D9b) of appendix D,

$$Q = 1 + \left[O(k^2) + O(k/\gamma)\right] \qquad (6.3.1)$$

(b) $\gamma \to \infty$: The solution for Q is (eq. (D10))

$$Q = \frac{1}{2}\left[1 + \frac{1}{\sqrt{1 - k^2}}\right] \qquad (6.3.2)$$

which is valid for all k.

(c) $\gamma = 1$, $k \to 1$: In this case, the solution for Q is $Q = 1$.

6.4 Numerical solution. - In nondimensional coordinates, equation (6.2.4) becomes

$$\frac{b}{c} Y \int_1^Y \frac{dY_2}{(dy_2/d\xi)} \sqrt{\frac{Y^2 - Y_1{}^2}{Y^2 - Y_2{}^2}} = \int_1^Y Q^{(2)} \left(\Phi + \frac{Y_2}{Y}\sqrt{\frac{Y^2 - Y_1{}^2}{Y^2 - Y_2{}^2}}\right) dY_2 \qquad (6.4.1)$$

In some cases, the integral on the left-hand side of equation (6.4.1) can be evaluated analytically but it is convenient, and also consistent, to evaluate it numerically. From the mean value theorem

$$\frac{b}{c} Y \sum_{n=1}^{m} \frac{\sqrt{Y^2 - \eta_{1,n}^2}}{\left(\frac{dy_2}{d\xi}\right)_n \eta_{2,n}} \left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right)$$

$$= \sum_{n=1}^{m} Q_n{}^{(2)} \left[\Phi_n (Y_{2,n} - Y_{2,n-1}) + \frac{\sqrt{Y^2 - \eta_{1,n}^2}}{Y} \left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right)\right] \qquad (6.4.2)$$

The solution for $Q_m{}^{(2)}$ is then

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$$Q_m^{(2)} = \frac{\frac{b}{c}\frac{\sqrt{Y^2 - \eta_{1,m}^2}\sqrt{Y^2 - Y_{2,m-1}^2}}{(dy_2/d\xi)_m}\frac{Y}{\eta_{2,m}} - \sum_{n=1}^{m-1} Q_n^{(2)}\Phi_n(Y_{2,n} - Y_{2,n-1}) - \left[\frac{bY^2}{c\eta_{2,n}(dy_2/d\xi)_n} - Q_n^{(2)}\right]\frac{\sqrt{Y^2 - \eta_{1,n}^2}}{Y}\left(\sqrt{Y^2 - Y_{2,n-1}^2} - \sqrt{Y^2 - Y_{2,n}^2}\right)}{\Phi_m(Y_{2,m} - Y_{2,m-1}) + \frac{\sqrt{Y^2 - \eta_{1,m}^2}\sqrt{Y^2 - Y_{2,m-1}^2}}{Y}} \qquad (6.4.3)$$

Similarly, the net lift is

$$\frac{L}{2q\pi \frac{\omega_y c}{U_0} b^2} = 1 - \int_1^{b_0/b} ({Y_2}^2 - {Y_1}^2)dY + \frac{4}{3}\sum_{n=1}^{m} Q_n \left(1 - \frac{E'_n}{K'_n}\right)(Y_{2,n}^3 - Y_{2,n-1}^3) \qquad (6.4.4)$$

The pitching moment about the y-axis is, for $dy_2/d\xi$ constant,

$$\frac{M_y}{\frac{3}{2}\pi q \frac{\omega_y c}{U_0} b^2 c} = 1 - \frac{4}{3}\int_1^{b_0/b} ({Y_2}^2 - {Y_1}^2)Y_2\, dY_2 + \frac{4}{3}\sum_{n=1}^{m} Q_n \left(1 - \frac{E'_n}{K'_n}\right)(Y_{2,n}^4 - Y_{2,n-1}^4) \qquad (6.4.5)$$

Numerical values of Q were computed, for the $\gamma = 1$ case, with intervals $(Y_{2,n} - Y_{2,n-1}) = 0.20$. The results are indicated in figure 16. According to these results, Q starts from a value of 1 at $y_2/b = 1$, increases very slightly and then decreases to a value of about 0.90 for $3 < y_2/b < 4$. The curve appears to have reached its minimum and will then presumably increase to its asymptotic value of 1.

Numerical calculations of the net lift, pitching moment, and center of pressure are presented in figure 17 for $\gamma = 1$ and dy_2/dx constant.

SECTION 7 - LIFT OF SWEPT WINGS (INVERSE PROBLEM)

In the inverse problem, the shed vortex sheet is specified and the corresponding trailing edge is determined. Since the shed vorticity is antisymmetric about the $y = 0$ plane (for a lifting wing) the general form for v on the top surface of the shed vortex sheet is

$$v = \sum_{n=1}^{\infty} v_n \frac{|y|^n}{y}$$

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where $v_1, v_2, \ldots$ are constants. The solution corresponding to the first term of the expansion is presented in the following sections.

7.1 Determination of crossflow. - The boundary conditions on the upper surface of the $z = 0$ plane are:

For $y_1^2 < y^2 < y_2^2$

$$w = -\alpha U_0$$

$$\frac{dw}{dy} = 0$$

$0 < y^2 < y_1^2$

$$v = v_1 |y| / y$$

$$\frac{dv}{dy} = 0$$

The boundary conditions are homogeneous for the generating function $\partial V/\partial \zeta$. There is a discontinuity in v at $y = 0$ and the Kutta condition is applied at $y = \pm y_1$. The generating function is then, from equation (2.3.6),

$$\frac{\partial V}{\partial \zeta} = -i\alpha U_0 \frac{A_0 + A_2\zeta^2}{\zeta\sqrt{\zeta^2 - y_1^2}\,(\zeta^2 - y_2^2)^{3/2}} \tag{7.1.1}$$

and the complex velocity equals

$$V = -i\alpha U_0 \int_{\infty}^{\zeta} \frac{(A_0 + A_2\zeta^2)d\zeta}{\zeta\sqrt{\zeta^2 - y_1^2}\,(\zeta^2 - y_2^2)^{3/2}} \tag{7.1.2}$$

where A_0 and A_2 are functions of x. Equation (7.1.2) can be integrated directly to give

$$V = \frac{i\alpha U_0}{2}\left[\frac{2(A_0 + A_2 y_2^2)}{y_2^2(y_2^2 - y_1^2)}\left(\sqrt{\frac{\zeta^2 - y_1^2}{\zeta^2 - y_2^2}} - 1\right) - \frac{A_0}{y_1 y_2^3}\ln\frac{\zeta^4 - \left(y_1 y_2 + \sqrt{(\zeta^2 - y_1^2)(\zeta^2 - y_2^2)}\right)^2}{\zeta^2(y_2 - y_1)^2}\right] \tag{7.1.3}$$

When the appropriate branches are taken, equation (7.1.3) gives V at all points in the flow field. The functions A_0 and A_2 are eliminated so as to satisfy $w = -\alpha U_0$ on the wing and $v = v_1|y|/y$ in the vortex sheet. The result is

$$A_0 = -\frac{2}{\pi}\frac{v_1}{\alpha U_0} y_1 y_2^3$$

$$A_2 = -(y_2^2 - y_1^2)\left(1 - \frac{1}{\pi}\frac{v_1}{\alpha U_0}\ln\frac{y_2 + y_1}{y_2 - y_1} - \frac{2}{\pi}\frac{v_1}{\alpha U_0}\frac{y_1 y_2}{y_2^2 - y_1^2}\right).$$

Substituting into equation (7.1.3) gives

$$V = -i\alpha U_0\left[\left(1 - \frac{1}{\pi}\frac{v_1}{\alpha U_0}\ln\frac{y_2 + y_1}{y_2 - y_1}\right)\left(\sqrt{\frac{\zeta^2 - y_1^2}{\zeta^2 - y_2^2}} - 1\right) - \frac{1}{\pi}\frac{v_1}{\alpha U_0}\ln\frac{\zeta^4 - \left(y_1 y_2 + \sqrt{(\zeta^2 - y_1^2)(\zeta^2 - y_2^2)}\right)^2}{\zeta^2(y_2 - y_1)^2}\right] \tag{7.1.4}$$

For a point on the wing panel:

$$V = \alpha U_0\left[\left(\frac{1}{\pi}\frac{v_1}{\alpha U_0}\ln\frac{y_2 + y_1}{y_2 - y_1} - 1\right)\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}} + \frac{1}{\pi}\frac{v_1}{\alpha U_0}\cos^{-1}\frac{y^2(y_2^2 + y_1^2) - 2y_1^2 y_2^2}{y^2(y_2^2 - y_1^2)} + i\right] \tag{7.1.5}$$

For a point on the vortex sheet:

$$V = v_1 + i\alpha U_0 \left\{1 - \sqrt{\frac{y_1^2 - y^2}{y_2^2 - y^2}} + \frac{1}{\pi} \frac{v_1}{\alpha U_0}\left[\sqrt{\frac{y_1^2 - y^2}{y_2^2 - y^2}} \ln \frac{y_2 + y_1}{y_2 - y_1} - \ln \frac{\left(y_1 y_2 - \sqrt{y_1^2 - y^2}\sqrt{y_2^2 - y^2}\right)^2 - y^4}{y^2(y_2 - y_1)^2}\right]\right\} \tag{7.1.6}$$

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The equation of the trailing edge must now be determined.

7.2 Equation of trailing edge. - The equation of the trailing edge can be found by calculating the potential at the trailing edge by two different methods and then matching the solutions. The upper surface of the $z = 0$ plane is considered.

The potential along the x-axis, for $x \geq c$, is constant and equals $\alpha U_0 b$. Integrating in the positive y-direction gives the following formula for the potential at the right-hand trailing edge

$$\varphi_1 = \alpha U_0 b + v_1 y_1 \tag{7.2.1}$$

The expression for φ_1, obtained by integrating v from the leading to the trailing edge, is

$$\varphi_1 = \int_{y_2}^{y_1} v\, dy = -\alpha U_0 \left[y_2\left(\frac{1}{\pi} \frac{v_1}{\alpha U_0} \ln \frac{y_2 + y_1}{y_2 - y_1} - 1\right)(E' - k^2 K') + \frac{y_1}{\pi} \frac{v_1}{\alpha U_0} (2K' - \pi)\right] \tag{7.2.2}$$

Equating equations (7.2.1) and (7.2.2) gives

$$\frac{y_1}{b} = \frac{k}{\left(1 - \frac{1}{\pi} \frac{v_1}{\alpha U_0} \ln \frac{1 + k}{1 - k}\right)(E' - k^2 K') - \left(\frac{2}{\pi} \frac{v_1}{\alpha U_0} kK'\right)} \tag{7.2.3}$$

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where $k = y_1/y_2$ as before. The right-hand side of equation (7.2.3) is a function of k with $v_1/\alpha U_0$ appearing as a parameter. If $v_1/\alpha U_0$ is specified, equation (7.2.3) defines the trailing edge. For $\frac{v_1}{\alpha U_0} = 0$, equation (7.2.3) becomes

$$\frac{y_1}{b} = \frac{k}{E' - k^2 K'} \tag{7.2.4}$$

which was the special result indicated by equation (4.3.2). For this case the leading and trailing edges tend to become parallel as $x \to \infty$. The spacing is given by $\lim_{x \to \infty} (y_2 - y_1)/b = 2/\pi$.

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The wing plan forms corresponding to several values of $v_1/\alpha U_0$ are indicated in figure 18. For all nonzero values of $v_1/\alpha U_0$ the trailing edge has a cusp at $x = c$ (too small to appear on the figure). This result indicates that when the trailing edge is specified so as not to have a cusp (direct problem), the sidewash v will be zero along the x-axis (for all x) but, for $x > c$, will probably increase quite rapidly with y. Note that for $v_1 \neq 0$, the apex of the trailing edge is a triple point for v. As $x \to \infty$ (or $y_2 \to \infty$), equation (7.2.3) becomes

$$\frac{v_1}{\alpha U_0} = \frac{\pi}{\ln \frac{1 + k}{1 - k} + \frac{2kK'}{E' - k^2 K'}} \tag{7.2.5}$$

which is an equation for the limiting value of k in terms of $v_1/\alpha U_0$. Thus, as $x \to \infty$, the trailing edge becomes straight (assuming the leading edge is straight) and the ratio of the trailing-edge slope to leading-edge slope equals the limiting value of k obtained from equation (7.2.5). In effect, the flow field tends to become conical as x increases. From figure 18 it appears that taking the shed vortex sheet equal to $v_1 |y| / y$ is adequate to simulate a swept wing having straight leading and trailing edges such that $\gamma < 1$.

7.3 Lift. - The asymptotic form of equation (7.1.4) is

$$V = - i \frac{\alpha U_0}{\zeta^2} \left[\left(1 - \frac{1}{\pi} \frac{v_1}{\alpha U_0} \ln \frac{y_2 + y_1}{y_2 - y_1} \right) \left(\frac{y_2{}^2 - y_1{}^2}{2} \right) - \frac{1}{\pi} \frac{v_1}{\alpha U_0} y_1 y_2 \right] + \ldots$$

The net lift acting on the airfoil is then

$$\frac{L}{2\pi q\alpha} = (y_2^2 - y_1^2)\left[1 - \frac{1}{\pi}\frac{v_1}{\alpha U_0}\left(\frac{2y_1y_2}{y_2^2 - y_1^2} + \ln\frac{y_2 + y_1}{y_2 - y_1}\right)\right] \tag{7.3.1}$$

where y_1 and y_2 are evaluated at $x = c_0$.

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SECTION 8 - ROLLING SWEPT WING (INVERSE PROBLEM)

The shed vortex sheet, for a rolling wing, is symmetric about the $y = 0$ plane and v may be expressed as

$$v = \sum_{n=1}^{\infty} v_n |y|^{n-1}$$

where $v_1, v_2, \ldots$ are constants. In the following it will be assumed that $v = v_1$ and the corresponding solution will be obtained.

8.1 Determination of crossflow. - The boundary conditions are that $w = -\omega_x y$ on the wing panels and $v = v_1$ in the shed vortex sheet. The generating function $\partial V/\partial \zeta$ has the nonhomogeneous boundary condition $dw/dy = -\omega_x$ on the wing panels and therefore must be expressed as (eq. (2.3.7))

$$\frac{\partial V}{\partial \zeta} = -i\omega_x\left[\frac{A_0 + A_2\zeta^2 + \zeta^4}{\sqrt{\zeta^2 - y_1^2}\,(\zeta^2 - y_2^2)^{3/2}} - 1\right] \tag{8.1.1}$$

Since there is no net lift, $\partial V/\partial \zeta$ behaves like $1/\zeta^4$ as $\zeta \to \infty$ so that

$$A_2 = -\frac{1}{2}(y_1^2 + 3y_2^2)$$

The function A_0 is eliminated by integrating equation (8.1.1) and satisfying the w boundary condition on the wing. The result is

$$A_0 = \frac{y_2^2}{2}(3y_1^2 - y_2^2) + \frac{(y_2^2 - y_1^2)^2}{2}\left(\frac{K}{E}\right)$$

The expression for v on the wing can then be shown to equal

$$v = \frac{-\omega_x}{2} y_2 k'^2 \left\{ \left[F(\beta',k') - E(\beta',k')\right] \frac{K}{E} - F(\beta',k') - \frac{|y|}{y_2} \left[\frac{K}{E} - \frac{2}{k'^2}\right] \sqrt{\frac{y^2 - y_1{}^2}{y_2{}^2 - y^2}} \right\} \tag{8.1.2}$$

3120 The value of v at the trailing edge is

$$v_1 = \frac{\pi}{4} \frac{\omega_x y_2 k'^2}{E} \tag{8.1.3}$$

8.2 Determination of trailing edge. - If v_1 is specified, equation (8.1.3) provides an expression for the trailing edge. Considering the limiting case of $x = c$, however, shows that v_1 must equal $\omega_x b/2$, which is the value of v for the basic wing at $x = c$ and $y = 0$. Thus v_1 is no longer a free parameter. The reason for this is as follows. The lift is zero along the x-axis of the basic wing. Hence, no modification of the trailing edge, at $x = c$, can change the spanwise lift distribution on the wing sufficiently to change the value of the shed vorticity at $x = c$. Thus v is continuous along the x-axis, at $x = c$, and must equal $\omega_x b/2$. The equation for the trailing edge is then

$$\frac{y_1}{b} = \frac{2E}{\pi} \frac{k}{(k')^2} \tag{8.2.1}$$

which is plotted in figure 19. As $x \to \infty$, equation (8.2.1) becomes

$$\lim_{x\to\infty} \frac{y_2 - y_1}{b} = \frac{1}{\pi}$$

Equations (8.2.1) and (8.2.2) were previously obtained in reference 7 by other methods.

8.3 Rolling moment. - The asymptotic form of $\partial V/\partial \zeta$ is

$$\frac{\partial V}{\partial \zeta} = i \frac{\omega_x (y_2{}^2 - y_1{}^2)}{8\zeta^4} \left[7y_2{}^2 + y_1{}^2 - 4(y_2{}^2 - y_1{}^2) \frac{K}{E}\right]$$

The net rolling moment is then, utilizing equation (1.3.7),

$$\frac{M_x}{\pi q \frac{\omega_x}{U_0}} = \frac{(y_2^2 - y_1^2)}{12}\left[7y_2^2 + y_1^2 - 4(y_2^2 - y_1^2)\frac{K}{E}\right] \quad (8.3.1)$$

where y_1 and y_2 are evaluated at $x = c_0$.

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SECTION 9 - PITCHING SWEPT WING (INVERSE PROBLEM)

The shed vortex sheet is assumed to be $v = v_1|y|/y$, as in the lift problem, and the resulting solution is obtained.

9.1 Determination of crossflow. - The crossflow is obtained by replacing αU_0 by $\omega_y x$ in equations (7.1.1) to (7.1.6).

9.2 Equation of trailing edge. - The potential along the x-axis, for $x > c$, is $\varphi = \omega_y cb$. The potential at the right-hand trailing edge can then be expressed as

$$\varphi_1 = \omega_y cb + v_1 y_1 \quad (9.2.1)$$

Integrating v from the leading edge gives

$$\varphi_1 = -\omega_y x\left\{y_2\left(\frac{1}{\pi}\frac{v_1}{\omega_y x}\ln\frac{1+k}{1-k} - 1\right)(E' - k^2K') + \frac{y_1}{\pi}\frac{v_1}{\omega_y x}(2K' - \pi)\right\} \quad (9.2.2)$$

Equating equations (9.2.1) and (9.2.2) and assuming that the leading edge is a straight line $y_2 = bx/c$ give

$$\frac{y_1}{b} = k\left[\frac{1}{2\pi}\frac{v_1}{c\omega_y}\left(\ln\frac{1+k}{1-k} + \frac{2kK'}{E' - k^2K'}\right) + \sqrt{\left(\frac{1}{2\pi}\frac{v_1}{c\omega_y}\right)^2\left(\ln\frac{1+k}{1-k} + \frac{2kK'}{E' - k^2K'}\right)^2 + \frac{1}{E' - k^2K'}}\right] \quad (9.2.3)$$

which is an expression for the trailing edge in terms of k and the parameter $v_1/c\omega_y$. The plan forms corresponding to different values of $v_1/c\omega_y$ are plotted in figure 20. As in the lift case, the trailing edge has a slight cusp at $x = c$ (except for $v_1 = 0$). However, for $x \to \infty$, the trailing edge approaches the leading edge. When $v_1 = 0$, equation (9.2.3) becomes

$$\frac{y_1}{b} = \frac{k}{\sqrt{E' - k^2 K'}}$$

which is one of the curves included in figure 20.

9.3 Pitching moment. - The net lift acting on the wing upstream of a given section x is obtained by replacing αU_0 in equation (7.3.1) by $\omega_y x$ and equals

$$\frac{L}{2\pi q \frac{\omega_y}{U_0}} = x(y_2^2 - y_1^2)\left[1 - \frac{1}{\pi}\frac{v_1}{x\omega_y}\left(\frac{2y_1y_2}{y_2^2 - y_1^2} + \ln\frac{y_2 + y_1}{y_2 - y_1}\right)\right] \tag{9.3.1}$$

The pitching moment is obtained by integration. Thus, the pitching moment about the leading edge is

$$M_y = \int_0^{c_0} x\left(\frac{dL}{dx}\right)dx = Lx\Big]_{x=0}^{x=c_0} - \int_0^{c_0} L\,dx \tag{9.3.2}$$

which can be evaluated by means of equation (9.3.1).

SECTION 10 - WING-BODY COMBINATIONS

To solve nonplanar problems, such as the flow about wing-body combinations, it is necessary to transform the given problem into one with planar boundary conditions in order for the previously developed generating functions to be applicable. In the following section the Joukowski transformation is discussed. The solution for the flow about a highly swept wing mounted on a circular cylinder is then indicated.

The x,y,z-coordinate system is considered, herein, as a body axis system rather than a wind axis system in order that the developments parallel those of the isolated swept wing problem. The perturbation velocities u, v, and w are considered as parallel to the x, y, and z coordinates, respectively. This creates no essential change except for the fact that the pressure formula (eq. (1.1.2)) becomes

$$p - p_0 = - \rho_0 \left(u\, U_0 + \alpha w U_0 + \frac{v^2 + w^2}{2} \right)$$

when the body axis is at angle of attack α in respect to the free stream. Also, the lift and moment formulas (eqs. (1.3.3) to (1.3.5) and (1.3.7)) must be evaluated after transforming from the body axis system to a wind axis system.

10.1 Joukowski transformation. - Consider the problem of a highly swept wing on a circular cylinder of radius a (figs. 21(a) and 21(b)). The Joukowski transformation

$$\overline{\zeta} = \zeta - a^2/\zeta \tag{10.1.1}$$

transforms the configuration such that the body becomes a vertical cut (of width 4a) on the $\overline{z}$-axis of the $\overline{\zeta}$-plane (fig. 21(c)). The wing panels remain as cuts in the $\overline{z} = 0$ plane. The velocities at corresponding points in the two planes are related by

$$V = \overline{V} \left(\frac{d\overline{\zeta}}{d\zeta} \right) \tag{10.1.2}$$

where $\overline{V} = \overline{v} - i\overline{w}$ is the complex velocity in the $\overline{\zeta}$-plane. But

$$\frac{d\overline{\zeta}}{d\zeta} = 1 + \frac{a^2}{\zeta^2} = \left[1 + \left(\frac{a}{r} \right)^2 \cos 2\theta \right] - i \left(\frac{a}{r} \right)^2 \sin 2\theta$$

where r and θ are defined by the relation $\zeta = re^{i\theta}$. Thus

$$\begin{aligned} v &= \overline{v} \left[1 + \left(\frac{a}{r} \right)^2 \cos 2\theta \right] - \overline{w} \left(\frac{a}{r} \right)^2 \sin 2\theta \\ w &= \overline{w} \left[1 + \left(\frac{a}{r} \right)^2 \cos 2\theta \right] + \overline{v} \left(\frac{a}{r} \right)^2 \sin 2\theta \\ v_r &= \left[\overline{v} \cos \theta + \overline{w} \sin \theta \right] \left[1 + \left(\frac{a}{r} \right)^2 \right] - 2\, \overline{w} \left(\frac{a}{r} \right)^2 \sin \theta \end{aligned} \tag{10.1.3}$$

where v_r is the radial velocity in the ζ-plane. Equations (10.1.3) are particularly useful for relating the velocity boundary conditions in the ζ- and $\overline{\zeta}$-planes.

Since body axes are being used, the boundary condition on the body surface, in the ζ-plane, is that $v_r = 0$. The corresponding boundary condition in the $\overline{\zeta}$-plane is that $\overline{v} = 0$ along the vertical cut $(- i2a < \overline{\zeta} = i\overline{z} < i2a)$. This boundary condition is automatically

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satisfied, by symmetry, for all problems when the $\bar{w}$ boundary conditions on the wing panels are symmetric about the $\bar{y} = 0$ plane (i.e., lift and pitch problems). For these problems, only the boundary conditions in the $\bar{z} = 0$ plane and at infinity require attention. Thus, the wing-body problem is transformed, by equation (10.1.1) into an equivalent isolated swept-wing problem (for the symmetric case)[4].

The asymptotic form of the complex velocity in the $\bar{\zeta}$-plane is

$$\bar{V} = -\frac{\bar{f}_1}{(\bar{\zeta})^2} + O\left[\frac{1}{(\bar{\zeta})^3}\right]$$

The lift in the $\bar{\zeta}$-plane is then (from eqs. (1.3.4) and (1.3.3))

$$\frac{d\bar{L}}{dx} = \text{I.P.}\left(2\pi\rho_0 U_0 d\bar{f}_1/dx\right) \tag{10.1.4a}$$

$$\bar{L} = \text{I.P.}\left(2\pi\rho_0 U_0 \bar{f}_1\right) \tag{10.1.4b}$$

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Equation (10.1.4b) corresponds to a configuration which is pointed at its upstream end. The complex velocity in the physical plane is

$$V = \bar{V}\left(1 + \frac{a^2}{\zeta^2}\right) = -\frac{\bar{f}_1}{\zeta^2} + O\left(\frac{1}{\zeta^3}\right)$$

Note that the coefficient of the leading term in the asymptotic expansion for V is unaffected by a transformation from body axes to wind axes. The lift per unit x in the physical plane is then (eq. (1.3.4))

$$\frac{dL}{dx} = \text{I.P.}\left(2\pi\rho_0 U_0 \frac{d\bar{f}_1}{dx} + \rho_0 U_0^2 \frac{d^2 A_{cs}\zeta_g}{dx^2}\right) \tag{10.1.5}$$

But $\zeta_g = -i\alpha x + \text{constant}$ and A_{cs} is constant since a cylindrical body is assumed. Equation (10.1.5) then becomes

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[4]When $\bar{w}$ is not symmetric about the $\bar{y} = 0$ plane (as in a roll problem) the boundary conditions along the vertical cut are not automatically satisfied. In these cases it may be advisable to use the transformation $\bar{\bar{\zeta}} = \zeta + a^2/\zeta$ which transforms the circle into a cut $(-2a < \bar{\bar{\zeta}} = \bar{\bar{y}} < 2a)$ along the $\bar{\bar{y}}$-axis. Thus the problem is transformed into one wherein the boundary conditions are specified along the $\bar{\bar{y}}$-axis and at infinity. The problem is now completely planar but involves three "panels." In some cases it can be handled by the methods of section 2.

$$\frac{dL}{dx} = \text{I.P.}\left(2\pi\rho_0 U_0 d\bar{f}_1/dx\right) \tag{10.1.6a}$$

$$L = \text{I.P.}\left(2\pi\rho_0 U_0 \bar{f}_1\right) \tag{10.1.6b}$$

Equation (10.1.6b) represents the net lift of the configuration in the physical plane provided the body remains cylindrical upstream of the wing. For pointed forebodies, the additional term $\pi\rho_0 {U_0}^2 a^2 \alpha$, representing forebody lift, must be added to the right-hand side of equation (10.1.6b). Comparing equations (10.1.4b) and (10.1.6b) shows that $L = \bar{L}$. That is, the lifts of the configurations in the physical and transformed planes are equal (providing the body remains cylindrical upstream of the wing). This result was derived in reference 15 by another method.

10.2 Lift of swept wing on cylindrical body. - A swept wing is mounted on a circular cylinder and the configuration is at angle of attack α. The x,y,z-coordinate system is based on the body axis, and the following transformations are made:

(1) The flow is transformed from the ζ-plane to the $\bar{\zeta}$-plane by equation (10.1.1).

(2) A uniform flow $\bar{w} = -\alpha U_0$ is added so that the configuration is translating, in the $\bar{\zeta}$-plane, with velocity $\bar{w} = -\alpha U_0$ in a fluid otherwise at rest.

The solution to flow (2) can be found by the methods of sections 4 and 7. The solution to the original problem is then found by reversing this procedure.

(a) Direct problem: The generating function for flow (2) is, from equation (4.1.8),

$$\frac{\partial \bar{V}}{\partial x} = -i\alpha U_0 \bar{y}_2 \frac{d\bar{y}_2}{dx} S\left[\frac{(\bar{\zeta})^2 - (\bar{y}_1)^2}{(\bar{\zeta})^2 - (\bar{y}_2)^2} - \frac{\bar{E}'}{\bar{K}'}\right] \frac{1}{\sqrt{\left[(\bar{\zeta})^2 - (\bar{y}_2)^2\right]\left[(\bar{\zeta})^2 - (\bar{y}_1)^2\right]}} \tag{10.2.1}$$

where $\bar{K}'$ and $\bar{E}'$ are complete elliptic integrals of first and second kinds with modulus $\bar{k}' = \sqrt{1 - (\bar{y}_1/\bar{y}_2)^2}$. The lift of the configuration is (eq. (4.1.9))

$$\frac{d\bar{L}}{dx} = 4\pi\alpha q \bar{y}_2 \left(\frac{d\bar{y}_2}{dx}\right) S \left(1 - \frac{\bar{E}'}{\bar{K}'}\right) \tag{10.2.2}$$

The unknown function S is found from the integral equation (eq. (4.2.7))

$$\sqrt{(\bar{y})^2 - (\bar{b})^2} = \int_{\bar{b}}^{\bar{y}} S \, d\bar{y}_2 \left[\bar{\Phi} + \frac{\bar{y}_2}{\bar{y}} \sqrt{\frac{(\bar{y})^2 - (\bar{y}_1)^2}{(\bar{y})^2 - (\bar{y}_2)^2}} \right] \tag{10.2.3}$$

where

$$\bar{\Phi} = \left(\frac{\bar{E}'}{\bar{K}'} - 1\right)\left[\bar{K} - F(\bar{\beta},\bar{k})\right] + \bar{E} - E(\bar{\beta},\bar{k})$$

$$\bar{\beta} = \bar{y}_2/\bar{y}$$

$$\bar{k} = \bar{y}_1/\bar{y}_2$$

and $\bar{K}$ and $\bar{E}$ are complete elliptic integrals with modulus $\bar{k}$. Limiting solutions for S are given in section 4.3. The solution of the original problem is found by reversing steps (2) and (1). Note that

$$\bar{\zeta} = \zeta - a^2/\zeta$$

$$\bar{b} = b - a^2/b$$

$$\bar{y}_1 = y_1 - a^2/y_1$$

$$\bar{y}_2 = y_2 - a^2/y_2$$

$$\bar{u} = u$$

$$\frac{\partial \bar{v}}{\partial x} = \frac{\partial v}{\partial x} \Bigg/ \left(\frac{d\bar{\zeta}}{d\zeta}\right)$$

$$\bar{L} = L$$

The details need not be given.

(b) Inverse problem: From equation (10.1.2), the relation between the shed vortex sheets in the physical and transformed planes is

$$v = \bar{v}\left[1 + \left(\frac{a}{y}\right)^2\right]$$

Therefore the sidewash on the top surface of the shed vortex sheet in the physical plane will be assumed to be

$$v = v_1 \left[1 + \left(\frac{a}{y}\right)^2\right] \frac{|y|}{y}$$

where v_1 is a constant. The shed vortex sheet in the transformed plane is then

$$\bar{v} = v_1 \frac{|\bar{y}|}{\bar{y}}$$

and the solution follows directly from the results of section 7.

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SECTION 11 - UNSTEADY TWO-DIMENSIONAL INCOMPRESSIBLE FLOWS

The flow field due to the motion of a two-dimensional body in an incompressible fluid, otherwise at rest, is discussed. The application of generating functions for the solution of unsteady airfoil problems is indicated.

11.1 General considerations. - The equations which arise in studies of unsteady two-dimensional incompressible flows are closely analogous to those employed in slender body theory. The analogy between these two classes of flow will be established by comparing the classical equations associated with two-dimensional unsteady incompressible flows to the equations, derived by Ward, for the flow around a slender body.

Assume a two-dimensional coordinate system, fixed in space, such that the fluid, far from the body, is at rest. The velocity potential satisfies

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \qquad (11.1.1)$$

where φ contains the time t as a parameter. The velocity potential, in this case, is not a perturbation potential and the velocities v and w represent the net velocities in the flow field. The boundary condition on the body is that

$$v_n = \frac{d\nu}{dt} \qquad (11.1.2)$$

where v_n is the velocity normal to the body surface and ν is the normal coordinate of an orthogonal coordinate system chosen to be normal and tangential to the body surface at a given instant. The pressure at any point, from the Bernoulli equation for unsteady flow, is

$$p - p(t) = -\rho_0\left\{\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\right\} \tag{11.1.3}$$

where $p(t)$ is an arbitrary function of time. The Laurent expansions for W and V are

$$W = f + \frac{1}{2\pi}\left(\frac{dA_{cs}}{dt} - i\Gamma\right) \ln \zeta + \sum_{m=1}^{\infty} f_m\zeta^m \tag{11.1.4}$$

$$V = \frac{1}{2\pi}\left(\frac{dA_{cs}}{dt} - i\Gamma\right)\frac{1}{\zeta} - \sum_{m=1}^{\infty} mf_m\zeta^{-(m+1)} \tag{11.1.5}$$

where $f, f_1, f_2, \ldots$ are functions of t. The impulse required to generate the motion, at any instant, is found from a contour integral about the body and equals (assuming $\Gamma = 0$)

$$I_y + iI_z = -i\rho_0 \oint_{c_1} \varphi\, d\zeta \tag{11.1.6}$$

Equation (11.1.6) is a classical result derived by Kelvin. Neumark (ref. 16) appears to have been the first to evaluate equation (11.1.6) by replacing φ by W and using Cauchy's theorem. However, Neumark solved the special case of a nondeforming body. For the general case, following the procedure used to obtain equation (1.3.3) from equation (1.3.1), equation (11.1.6) becomes

$$I_y + iI_z = 2\pi\rho_0\left[f_1 + \frac{1}{2\pi}\frac{d(A_{cs}\zeta_g)}{dt}\right] \tag{11.1.7}$$

The force per unit span $(l_y + il_z)$, at any instant, is then

$$l_y + il_z = -i\rho_0 \frac{d}{dt}\oint_{c_1} \varphi\, d\zeta = 2\pi\rho_0\left[\frac{df_1}{dt} + \frac{1}{2\pi}\frac{d^2(A_{cs}\zeta_g)}{dt^2}\right] \tag{11.1.8}$$

Equations (11.1.1) to (11.1.8) are similar to equations (1.1.4), (1.1.5), (1.1.2), (1.2.2), (1.2.3), (1.3.1), (1.3.3), and (1.3.4). The two sets of equations become identical if the transformation $x = U_0 t + \text{constant}$ is introduced provided the arbitrary functions $p(t)$ and f in equations (11.1.3) and (11.1.4) are taken equal to p_0 and f in equations (1.1.2) and (1.2.2), respectively. The equivalence of equations (11.1.6) and (1.3.1) is established by

$$I_y + iI_z = \int_{-\infty}^{t} (l_y + il_z)dt = \frac{1}{U_0}\int_{-\infty}^{t} (l_y + il_z)dx = \frac{F_y + iF_z}{U_0}$$

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Thus, the steady three-dimensional flow about a slender body can be transformed into an equivalent two-dimensional unsteady flow about a cylinder whose cross section varies with $U_0 t$ in the same way that the cross section of the original slender body varies with x, and vice versa, provided $p(t)$ and f (eqs. (11.1.3) and (11.1.4)), which are essentially boundary conditions, are taken equal to p_0 and f (eqs. (1.1.2) and (1.2.2))[5]. Note that $p(t)$, p_0, and f do not contribute to the lift. Munk used this equivalence in his studies of airships and many researchers have since referred to it.

The generating function approach can thus be used for solving unsteady two-dimensional airfoil problems. The generating function is now $\partial V/\partial t$. When τ is used as the integration variable for t, equations (3.1.1) become

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[5]Another approach to this analog is to consider the differential equations and boundary conditions for the problem of two-dimensional unsteady motion in a slightly compressible fluid. In this case $p(t)$ and f are no longer arbitrary boundary conditions and the analog with the three-dimensional steady flow past a slender body is even more striking. For example, the wave drag of the slender body corresponds to the energy radiated as sound in the equivalent unsteady two-dimensional problem.

$$W = \int_{\infty}^{\zeta} d\zeta \int_{-\infty}^{t} \frac{\partial V}{\partial \tau}\, d\tau \qquad (11.1.9)$$

$$V = \int_{-\infty}^{t} \frac{\partial V}{\partial \tau}\, d\tau$$

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Moreover, U is now defined as $U \equiv \partial W/\partial t$ so that equation (3.1.3) becomes

$$U = \int_{\infty}^{\zeta} \frac{\partial V}{\partial t}\, d\zeta \qquad (11.1.10)$$

One complication arises; namely, a finite impulse can be imparted to the system so that W and V may be discontinuous functions of t. Such a discontinuity occurs at $t = 0$ for the problem discussed in section 11.3. The integrals in equations (11.1.9) must then be considered as Stieltges integrals.

Assuming that the flow at time $t = t_a$ is known, the integral equations, corresponding to equations (3.3.1) and (3.3.2), are

$$w(t,y,0) - w(t_a,y,0) = \text{I.P.}\left(-\int_{t_a}^{t} \frac{\partial V}{\partial \tau}\, d\tau\right)_{z=0} \qquad (11.1.11)$$

$$\int_{y}^{y_2(x)} w(t,\eta,0)d\eta + \left[\psi(t_a,y\ (t),0) - \psi(t_a,y,0)\right]$$

$$= \text{I.P.}\left[-\int_{t_a}^{t} d\tau \int_{y}^{y_2(t)} \left(\frac{\partial V}{\partial \tau}\right) d\eta\right]_{z=0} \qquad (11.1.12)$$

<u>11.2 Two-dimensional airfoils</u>. - The flow about a zero thickness airfoil, moving with velocity V_0 along the y-axis in the positive y-direction, will be discussed. For convenience, let the chord of the airfoil be b and let its trailing edge be the origin of the coordinate system at $t = 0$. If, at $t = 0$, the airfoil starts to

move with velocity V_0, it occupies a strip of width b in the ty-plane (fig. 22). The w boundary condition on the strip is defined by the prescribed motion of the airfoil. There is a vortex sheet of unknown strength behind the trailing edge. Therefore, the generating function $\partial V/\partial t$ is applicable. For many problems the w boundary condition on the wing is of the form $w = f(y) + g(t)$. The generating function for these cases can be written, from equation (2.3.10),

$$\frac{\partial V}{\partial t} = i\left[\frac{A_0 - \frac{\zeta}{2}\frac{dg(t)}{dt}(y_1 + 3y_2) + \frac{dg(t)}{dt}\zeta^2}{\sqrt{\zeta - y_1}\,(\zeta - y_2)^{3/2}} - \frac{dg(t)}{dt}\right] \qquad (11.2.1)$$

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where the Kutta condition has been applied at y_1 and $\Gamma = 0$ is assumed. If $w = f(y)$, the generating function is

$$\frac{\partial V}{\partial t} = i\,\frac{A_0}{\sqrt{\zeta - y_1}\,(\zeta - y_2)^{3/2}} \qquad (11.2.2)$$

The problem of an airfoil starting impulsively from rest, and maintaining a constant angle of attack, can be solved by equations (11.2.1) and (11.2.2). (If the airfoil is accelerating, $dg(t)/dt = -\alpha dV_0/dt$ in eq. (11.2.1), and if the airfoil moves with constant velocity, eq. (11.2.2) is used.)

11.3 Wagner problem. - The problem of an airfoil, starting impulsively from rest, and moving with constant velocity and angle of attack, was first solved by Wagner (ref. 17). Recently, it was discussed, from the point of view of slender wing theory, in reference 18. The present approach differs from reference 18 in that the generating function is used to formulate the problem.

If the motion is considered to start impulsively, at $t = 0$, as indicated in figure 23, the crossflow is discontinuous across the y-axis. Approaching the y-axis from $t < 0$ gives a zero velocity field. Approaching the y-axis from $t > 0$ gives a velocity field equivalent to that about a flat plate translating with velocity $w = -\alpha V_0$. The solution for this flow field is

$$W = -i\alpha V_0\left[\sqrt{\zeta(\zeta - b)} - (\zeta - b/2)\right] \qquad (11.3.1a)$$

$$V = -i\alpha V_0\left[\frac{\zeta - b/2}{\sqrt{\zeta(\zeta - b)}} - 1\right] \qquad (11.3.1b)$$

For $t > 0$, the generating function is

$$\frac{\partial v}{\partial t} = -i\alpha V_0 \left[\frac{A_0}{\sqrt{\zeta - y_1}\,(\zeta - y_2)^{3/2}}\right] \tag{11.3.2}$$

The loading on the wing is proportional to $\partial\varphi/\partial t$. For a point on the upper wing surface, from equation (11.1.10),

$$\frac{\partial\varphi}{\partial t} = \alpha V_0 A_0 \int_{y_2}^{y} \frac{d\eta}{\sqrt{\eta - y_1}\,(y_2 - \eta)^{3/2}} = \frac{2\alpha V_0 A_0}{b}\sqrt{\frac{y - y_1}{y_2 - y}} \tag{11.3.3}$$

Substituting equation (11.3.2) into equation (10.1.12), with $t_a = 0$, utilizing equation (11.3.1a), and equating functions of y yield

$$\sqrt{y(y - b)} = -\int_0^{t_2(y)} d\tau \int_{y_2(\tau)}^{y} \frac{A_0\,d\eta}{\sqrt{\eta - y_1}\,(\eta - y_2)^{3/2}}$$

Integrating the inner integral and transforming from τ to y_2 as the integration variable yield

$$\sqrt{y(y - b)} = \int_b^y \left(\frac{2A_0}{bV_0}\right)\sqrt{\frac{y - y_1}{y - y_2}}\,dy_2 \tag{11.3.4}$$

Equation (11.3.4) is an integral equation for A_0. The lift on the airfoil, from equations (11.1.5) and (11.1.8) and the asymptotic form of equation (11.3.2), is

$$l_z = \pi\rho_0 V_0^2\, b\alpha \left(\frac{2A_0}{bV_0}\right) \tag{11.3.5}$$

It can be shown that $\dfrac{2A_0}{bV_0} = \dfrac{1}{2}$, for $t = 0$, and increases monotonically to the value 1 for $t \to \infty$. Equation (11.3.4) has been inverted, by the Laplace transform, and the results are identical to those of reference 19.

It is noted that the problem is formulated, from the beginning, in terms of lift, which is the variable of primary interest. In reference 18, the shed vortex sheet is first determined and an additional integration is required to obtain the lift.

SECTION 12 - CONCLUDING REMARKS

The use of the generating functions $\partial V/\partial \zeta$ and $\partial V/\partial x$ has been described for a wide variety of applications.

Problems for which $\partial V/\partial \zeta$ is applicable may possibly be solved, with equal facility, by other methods. But, the use of the generating function $\partial V/\partial x$ seems to have several advantages over other possible approaches to the direct problem of slender wings having swept trailing edges. First, the dependence of the flow on upstream conditions is initially removed and certain general features of the flow field can be determined immediately. Thus, for the lifting and rolling wing, the pressure distribution at each chordwise station can be expressed, to within a scale factor, without consideration of upstream conditions. Second, the problem is formulated directly in terms of quantities which define the lift and moments. Finally, the method delays until the last stages the problem of solving an integral equation.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, December 14, 1953

APPENDIX A

SYMBOLS

The following symbols are used in this report:

$A, A_0, A_1, A_2, \ldots$	functions of x (or t)
A_{cs}	cross-sectional area
A_{pf}	plan form area
a	radius of circular cylinder
b	semispan of swept wing at $x = c$ (fig. 7)
b_0	maximum semispan of swept wing (fig. 7)
$C, C_1, C_2, \ldots$	constants
C_L	lift coefficient $(= L/q\ A_{pf})$
C_l	rolling moment coefficient $(= M_x/q\ b_0 A_{pf})$
C_m	pitching moment (about $x = 0$) coefficient $(= M_y/q\ c_0 A_{pf})$
c	root chord of swept wing (fig. 7)
c_0	over-all length of swept wing (fig. 7)
$D, D_1, D_2, \ldots$	constants
D_i	induced drag
E	complete elliptic integral of second kind with modulus k
E'	complete elliptic integral of second kind with modulus k'
$E(\beta,k)$	incomplete elliptic integral of second kind with amplitude β and modulus k
$F(\beta,k)$	incomplete elliptic integral of first kind with amplitude β and modulus k

F_x,F_y,F_z	net forces in x,y,z-directions, respectively
$f,f_0,f_1,f_2, \ldots$	functions of x (or t)
I.P.	imaginary part of complex function
K	complete elliptic integral of first kind with modulus k
K'	complete elliptic integral of first kind of modulus k'
k	y_1/y_2 $(k' = \sqrt{1 - k^2})$
L	net lift force acting on configuration
l_y,l_z	sectional force in y,z-directions, respectively
M_x,M_y,M_z	moment about x,y, and z-axes, respectively
M_0	free stream Mach number
p	pressure
p_0	free stream pressure
$Q,Q^{(1)},Q^{(2)}$	functions of x arising in solution for pitching swept wings (section 6)
q	dynamic pressure $(\rho_0 U_0^2/2)$
R	function of x arising in solution for rolling swept wing (section 5)
R.P.	real part of complex function
S	function of x arising in solution for lift of swept wing (section 4)
t	time
U	derivative of W in respect to x (or t)
U_0	free stream velocity
u	perturbation velocity in x-direction
V	complex velocity $(v - iw)$

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V_O	flight velocity in y-direction
v	perturbation velocity in y-direction
W	potential function $(= \varphi + i\psi)$
w	perturbation velocity in z-direction
3120 x	coordinate axis parallel to free stream
$x_{c.p.}$	center of pressure
y	coordinate axis in span direction
z	coordinate axis
α	angle of attack
β	y_2/y
β'	$\sqrt{\dfrac{{y_2}^2 - y^2}{{y_2}^2 - {y_1}^2}}$
Γ	net circulation in yz-plane
γ	ratio of trailing edge to leading edge slope $\left(= \dfrac{dy_1/dx}{dy_2/dx}\right)$
ζ	$y + iz$
ζ_g	centroid of cross-sectional area $(= y_g + iz_g)$
η	integration variable in y-direction
λ	$\ln (4/k)$
ξ	integration variable in x-direction
ρ_O	free stream density
τ	integration variable for t
Φ	see equation (4.2.7)
φ	perturbation velocity potential

ψ	crossflow stream function
ω_x, ω_y	angular velocities about x- and y-axes, respectively

Special notation:

$⨍$	finite part of improper integral (appendix B)
$(\bar{\ \ })$	quantity in $\bar{\zeta}$ plane
$\left.\begin{matrix} y = y_1(x) \\ x = x_1(y) \end{matrix}\right\}$	equation of trailing edge
$\left.\begin{matrix} y = y_2(x) \\ x = x_2(y) \end{matrix}\right\}$	equation of leading edge

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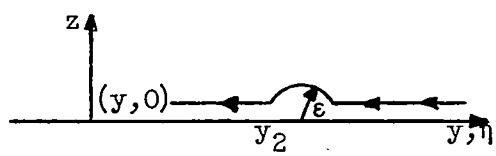
APPENDIX B

FINITE PART OF IMPROPER INTEGRALS

Many of the integrals occurring in the body of the report are of the form

$$I_1 = \text{R.P.}\left[\int_{\infty}^{y} \frac{f(\zeta)}{(\zeta - y_2)^{3/2}}\, d\zeta\right] \tag{B1}$$

to be evaluated for $y < y_2$. The path indicated in the following figure is chosen



(assuming $f(\zeta)$ analytic along the path), and equation (B1) becomes

$$I_1 = \lim_{\varepsilon\to 0}\left[\text{R.P.}\int_{y_2+\varepsilon}^{y_2-\varepsilon} \frac{f(\zeta)}{(\zeta - y_2)^{3/2}}\, d\zeta + \int_{\infty}^{y_2+\varepsilon} \frac{f(\eta)}{(\eta - y_2)^{3/2}}\, d\eta\right] \tag{B2}$$

Letting $\zeta = y_2 + \varepsilon e^{i\theta}$ in the first of the integrals, integrating, and taking the real part give

$$I_1 = \lim_{\varepsilon\to 0}\left[\frac{2f(y_2)}{\sqrt{\varepsilon}} + \int_{\infty}^{y_2+\varepsilon} \frac{f(\eta)}{(\eta - y_2)^{3/2}}\, d\eta\right] \tag{B3}$$

Equations of this type (eq. (B1)) occur quite frequently and the explicit representation of equations (B2) and (B3) becomes tedious. Therefore, to economize, the "finite part" concept is introduced. The finite part of an improper integral, having a 3/2-order singularity at a limit of integration, is defined by

$$\int_{x}^{a} \frac{f(\xi)}{(a - \xi)^{3/2}} d\xi \equiv \lim_{\varepsilon \to 0} \left[\frac{-2f(a)}{\sqrt{\varepsilon}} + \int_{x}^{a-\varepsilon} \frac{f(\xi)}{(a - \xi)^{3/2}} d\xi \right]$$

$$\int_{a}^{x} \frac{f(\xi)}{(\xi - a)^{3/2}} d\xi \equiv \lim_{\varepsilon \to 0} \left[\frac{-2f(a)}{\sqrt{\varepsilon}} + \int_{a+\varepsilon}^{x} \frac{f(\xi)}{(\xi - a)^{3/2}} d\xi \right] \tag{B4}$$

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Equation (B1) can then be written

$$I_1 = \int_{\infty}^{y_2} \frac{f(\eta)}{(\eta - y_2)^{3/2}} d\eta \tag{B5}$$

Similarly, for $y < y_2$,

$$\text{I.P.} \int_{\infty}^{y} \frac{f(\zeta)d\zeta}{(\zeta - y_2)^{3/2}} = \int_{y_2}^{y} \frac{f(\eta)}{(y_2 - \eta)^{3/2}} d\eta \tag{B6}$$

Thus, the finite-part technique is essentially mathematical shorthand, in the present report, since it avoids an explicit representation of the limiting processes required to obtain the real or imaginary part of complex integrals. The finite-part technique can be generalized and has many applications in aerodynamics (see, for example, refs. 20 and 21).

APPENDIX C

ELLIPTIC INTEGRALS

(a) Interval $y_2 \leq y \leq \infty$: Let

$$\beta = y_2/y \qquad k = y_1/y_2 \qquad k' = \sqrt{1 - k^2}$$

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Then:

$$\int_y^\infty \frac{d\eta}{\sqrt{(\eta^2 - y_2^2)(\eta^2 - y_1^2)}} = \frac{1}{y_2} F(\beta,k)$$

$$\int_y^\infty \frac{d\eta}{\sqrt{\eta^2 - y_1^2}\,(\eta^2 - y_2^2)^{3/2}} = \frac{1}{y_2^3 k'^2}\left[\frac{y_2}{y}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}} - E(\beta,k)\right]$$

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$$\int_{y_2}^\infty \frac{d\eta}{\sqrt{\eta^2 - y_1^2}\,(\eta^2 - y_2^2)^{3/2}} = \frac{-E}{y_2^3 k'^2}$$

$$\int_{y_2}^y \frac{\eta^2 d\eta}{\sqrt{(\eta^2 - y_2^2)(\eta^2 - y_1^2)}} = y_2\left\{\left[K - F(\beta,k)\right] - \left[E - E(\beta,k)\right] + \frac{\sqrt{(y^2 - y_2^2)(y^2 - y_1^2)}}{yy_2}\right\}$$

$$\int_{y_2}^y \frac{\sqrt{\eta^2 - y_2^2}}{\sqrt{\eta^2 - y_1^2}}\,d\eta = y_2\left\{\left[E(\beta,k) - E\right] + \frac{\sqrt{(y^2 - y_2^2)(y^2 - y_1^2)}}{yy_2}\right\}$$

<u>(b) Interval $y_1 \leqslant y \leqslant y_2$:</u> Let

$$\beta' = \sqrt{\frac{y_2^2 - y^2}{y_2^2 - y_1^2}}$$

Then

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$$\int_y^{y_2} \frac{d\eta}{\sqrt{(y_2^2 - \eta^2)(\eta^2 - y_1^2)}} = \frac{1}{y_2} F(\beta',k')$$

$$\int_y^{y_2} \sqrt{\frac{\eta^2 - y_1^2}{y_2^2 - \eta^2}}\, d\eta = y_2 \left[E(\beta',k') - k^2 F(\beta',k')\right]$$

$$\int\!\!\!\!\!\!-_{y}^{y_2} \frac{d\eta}{\sqrt{\eta^2 - y_1^2}\,(y_2^2 - \eta^2)^{3/2}} = \frac{1}{y_2^3 k'^2}\left[F(\beta',k') - E(\beta',k') - \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right]$$

$$\int\!\!\!\!\!\!-_{y}^{y_2} \frac{\eta^2 d\eta}{\sqrt{\eta^2 - y_1^2}\,(y_2^2 - \eta^2)^{3/2}} = \frac{1}{y_2 k'^2}\left[k^2 F(\beta',k') - E(\beta',k') - \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right]$$

$$\int_{y}^{y_2} \frac{\eta^4 d\eta}{\sqrt{\eta^2 - y_1^2}\,(y_2^2 - \eta^2)^{3/2}} = \frac{y_2}{k'^2}\left[k^2 F(\beta',k') - (1 + k'^2) E(\beta',k') - \frac{y}{y_2}\sqrt{\frac{y^2 - y_1^2}{y_2^2 - y^2}}\right]$$

(c) Expansions: For $k^2 << 1$: Let

$$\lambda = \ln (4/k) \qquad (k = y_1/y_2)$$

$$\varphi = \sin^{-1} \beta \qquad (\beta = y_2/y)$$

Then:

$$K = \frac{\pi}{2}\left[1 + 2\left(\frac{k^2}{8}\right) + 9\left(\frac{k^2}{8}\right)^2 + \ldots\right]$$

$$E = \frac{\pi}{2}\left[1 - 2\left(\frac{k^2}{8}\right) - 3\left(\frac{k^2}{8}\right)^2 + \ldots\right]$$

$$K' = \lambda + \frac{\lambda - 1}{4} k^2 + \frac{9}{64}\left(\lambda - \frac{7}{6}\right) k^4 + \ldots$$

$$E' = 1 + \frac{1}{2}\left(\lambda - \frac{1}{2}\right) k^2 + \frac{3}{16}\left(\lambda - \frac{13}{12}\right) k^4 + \ldots$$

$$\frac{E'}{K'} = \frac{1}{\lambda}\left[1 + \left(\frac{\lambda}{2} - \frac{1}{2} + \frac{1}{4\lambda}\right) k^2 + O(k^4)\right]$$

$$K - F(\beta,k) = K\left(1 - \frac{2}{\pi}\varphi\right) + \sin\varphi \cos\varphi\left[\frac{1}{4} k^2 + O(k^4)\right]$$

$$E - E(\beta,k) = E\left(1 - \frac{2}{\pi}\varphi\right) - \sin\varphi \cos\varphi\left[\frac{1}{4} k^2 + O(k^4)\right]$$

$$\Phi = (\cos\varphi)/\lambda + O(k^2 \cos\varphi)$$

APPENDIX D

LIMITING SOLUTIONS OF INTEGRAL EQUATIONS

Some limiting solutions of the integral equations occurring in sections 4 to 6 are obtained. The coordinates are nondimensionalized with respect to b so that $Y = y/b$, $Y_2 = y_2/b$ and $Y_1 = y_1/b$.

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(a) Lift case: Equation (4.2.7) may be written

$$\sqrt{Y^2 - 1} = \int_1^Y S \; dY_2 \left(\Phi + \frac{Y_2}{Y} \sqrt{\frac{Y^2 - Y_1^2}{Y^2 - Y_2^2}}\right) \tag{D1}$$

For $y_1/y_2 \equiv k << 1$, $\Phi = \sqrt{Y^2 - Y_2^2}/Y\lambda + O\left(k^2 \sqrt{Y^2 - Y_2^2}/Y\right)$, from appendix C. Equation (D1) can then be written as

$$0 = \int_1^Y dY_2 \left\{\frac{S\sqrt{Y^2 - Y_2^2}}{Y\lambda} + \frac{SY_2}{Y}\sqrt{\frac{Y^2 - Y_1^2}{Y^2 - Y_2^2}} - \frac{Y_2}{\sqrt{Y^2 - Y_2^2}} + O\left(\frac{Sk^2\sqrt{Y^2 - Y_2^2}}{Y}\right)\right\}$$

Integrating the term containing λ by parts yields

$$0 = \int_1^Y \frac{y_2 dY_2}{\sqrt{Y^2 - Y_2^2}}\left[\int_1^{Y_2} \frac{S}{\lambda Y} dY_2 + \frac{S}{Y}\sqrt{Y^2 - Y_1^2} - 1 + O\left(\frac{Sk^2(Y^2 - Y_2^2)}{Y}\right)\right] \tag{D2}$$

Define $\gamma \equiv dy_1/dy_2$, where γ is the ratio of the trailing-edge slope to the leading-edge slope. Then, for $k << 1$,

$$Y_2 = 1 + \int_0^k \frac{Y_2 dk}{\gamma - k} = 1 + O\left(\frac{k}{\gamma}\right)$$

$$dY_2 = \frac{Y_2 dk}{\gamma - k} = \frac{dk}{\gamma}\left[1 + O(k/\gamma)\right]$$

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Assume $S = 1 + O\left(\frac{k}{\gamma \ln k}\right) + O(k^2)$, which will be verified later (eq. (D4)). Then

$$\int_1^{Y_2} \frac{S}{\lambda Y}\, dY_2 = -4\int_0^{k/4} \frac{dt}{\gamma \ln t} + O\left(\frac{k^2}{\gamma^2 \ln k}\right) + O\left(\frac{k^3}{\gamma \ln k}\right)$$

$$O\left[\frac{Sk^2(Y^2 - Y_2{}^2)}{Y}\right] = O\left(\frac{k^3}{\gamma}\right)$$

$$\sqrt{1 - (y_1/y)^2} = \sqrt{1 - k^2} + O(k^3/\gamma)$$

Equation (D2) can then be written

$$0 = \int_1^{Y} \frac{Y_2 dY_2}{\sqrt{Y^2 - Y_2{}^2}}\left\{-4\int_0^{k/4} \frac{dt}{\gamma \ln t} + S\sqrt{1 - k^2} - 1 + O\left(\frac{k^3}{\gamma}\right) + O\left(\frac{k^2}{\gamma^2 \ln k}\right)\right\} \quad \text{(D3)}$$

The solution of equation (D3) is obtained by setting the integrand equal to zero. The resulting expression for S is

$$S = \frac{1}{\sqrt{1 - k^2}}\left[1 + 4\int_0^{k/4} \frac{dt}{\gamma \ln t} + O\left(\frac{k^3}{\gamma}\right) + O\left(\frac{k^2}{\gamma^2 \ln k}\right)\right] \quad \text{(D4a)}$$

$$= 1 + 4\int_0^{k/4} \frac{dt}{\gamma \ln t} + \frac{1}{2}k^2 + O(k^4) + O\left(\frac{k^3}{\gamma}\right) + O\left(\frac{k^2}{\gamma^2 \ln k}\right) \quad \text{(D4b)}$$

which is the solution for S valid for small k.

For the case where the trailing edge is only slightly swept $(\gamma \to \infty)$, equation (D4a) gives

$$S = \frac{1}{\sqrt{1 - k^2}} \tag{D5}$$

which can be shown to be valid for all k.

(b) Roll case: Equation (5.2.2) can be written

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$$\sqrt{Y^2 - 1} = \int_1^Y R\sqrt{\frac{Y_2^2 - Y_1^2}{Y^2 - Y_2^2}}\left[1 + O\left(\frac{k^3}{\gamma}\right)\right]dY_2 \tag{D6}$$

The solution of equation (D6) is

$$R = \frac{1}{\sqrt{1 - k^2}}\left[1 + O\left(\frac{k^3}{\gamma}\right)\right] \tag{D7a}$$

$$= 1 + \frac{1}{2}k^2 + O(k^4) + O\left(\frac{k^3}{\gamma}\right) \tag{D7b}$$

For $\gamma \to \infty$, equation (D7a) becomes

$$R = \frac{1}{\sqrt{1 - k^2}} \tag{D8}$$

which can be shown to be valid for all k.

(c) Pitch case: It will be assumed that $dy_2/d\dot{x}$ is constant. Then equation (6.2.2a) becomes

$$Q = \frac{1}{2}\frac{1}{Y_2}\left[Q^{(1)} + Q^{(2)}\right]$$

where $Q^{(1)} \equiv S$. The integral equation for $Q^{(2)}$ may be written, from equation (6.2.4) and with the expanded form for Φ,

$$0 = \int_1^Y \frac{Y_2 dY_2}{\sqrt{Y^2 - Y_2^2}} \left\{ \int_1^{Y_2} \frac{Q^{(2)}}{\lambda Y} dY_2 + \left[Q^{(2)} - \frac{Y^2}{Y_2} \right] \frac{\sqrt{Y^2 - Y_1^2}}{Y} + O\left(\frac{k^3}{\gamma}\right) \right\}$$

$$= \int_1^Y \frac{Y_2 dY_2}{\sqrt{Y^2 - Y_2^2}} \left\{ \left(Q^{(2)} - 1\right) \sqrt{1 - k^2} + O\left(\frac{k}{\gamma}\right) \right\}$$

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the solution for $Q^{(2)}$ is then $Q^{(2)} = 1 + O(k/\gamma)$. But $Q^{(1)} = \frac{1}{\sqrt{1 - k^2}} + O\left(\frac{k}{\gamma \ln k}\right)$; then, for small k,

$$Q = \frac{1}{2}\left[1 + \frac{1}{\sqrt{1 - k^2}} + O\left(\frac{k}{\gamma}\right)\right] \tag{D9a}$$

$$= 1 + O(k^2) + O\left(\frac{k}{\gamma}\right) \tag{D9b}$$

For $\gamma \rightarrow \infty$, equation (D9a) yields

$$Q = \frac{1}{2}\left[1 + \frac{1}{\sqrt{1 - k^2}}\right] \tag{D10}$$

valid for all values of k.

APPENDIX E

COMPARISON WITH REFERENCE 13

In reference 13, Mangler finds the lift, roll, and pitch solutions for highly swept wings (direct problem) by a method similar to that of the present report. The two papers are compared herein.

From equation (3.1.3)

$$\frac{\partial V}{\partial x} = \frac{\partial U}{\partial \zeta} = \frac{1}{i}\frac{\partial U}{\partial z} \tag{E1}$$

so that the generating function $\partial V/\partial x$ can also be considered as $\frac{1}{i}\frac{\partial U}{\partial z}$. Mangler writes, by inspection, the expression for $\frac{1}{i}\frac{\partial U}{\partial z}$ for the lift, roll, and pitch problems. Each is in terms of a single unknown function of x which is determined from an integral equation. The integral equation is, in effect, the one that is obtained by using equation (3.3.1).

The basic lift solution in both papers is identical. Mangler's H is the same as the S introduced in section 4. The pitch solutions are also in agreement. Mangler's H_q is related to the Q of section 6 by

$$Q = \frac{1}{2}\left(\frac{xH_q}{y_2}\frac{dy_2}{dx} + 1\right) \tag{E2}$$

His integral equation for the pitch case can be obtained from equation (3.3.1) but does not appear in the present paper. For the roll problem, Mangler's generating function is the same as that used herein and his H_p is identical to the R of section 5. However, Mangler appears to have made an error in determining his final integral equation for the roll case. His equation (72) should be (in the present notation)

$$\frac{\sqrt{y^2 - b^2}}{y} = \int_b^y \frac{dR}{dy_2}\sqrt{\frac{y^2 - y_1^2}{y^2 - y_2^2}}\,dy_2 - \int_b^y \frac{R\left[y_1\left(\frac{dy_1}{dy_2}\right) - y_2\right]dy_2}{\sqrt{y^2 - y_1^2}\,\sqrt{y^2 - y_2^2}} \tag{E3}$$

Equation (E3) differs from Mangler's equation (72) in that the left-hand side of the latter is zero. Equation (E3) can be obtained from equation (3.3.1) or by the direct integration of Mangler's equation (71), using $R = 1$ for $y_2 \leq b$. (The numberless equation which precedes equation (72) in Mangler's report is valid only for $y \leq b$ and is thus used incorrectly by Mangler to obtain his equation (72).) Equation (E3) is considerably more complicated than equation (5.2.2) in section 5.

The treatment of the swept wing problem in the present paper differs from Mangler's in several other respects. For example, explicit expressions for $\partial V/\partial x$ are given herein which permit the solution of all problems which can be handled by the generating function approach. (Mangler does not discuss how he goes about getting his expression for the generating functions. However, he does show that they satisfy the boundary conditions and are unique.) Also, the forces and moments are evaluated, herein, by considering the leading terms in the asymptotic expansion of the generating function. This is considerably simpler than integrating pressures over the wing surface, as is done by Mangler. On the other hand, Mangler has presented numerical results for a wider range of plan forms.

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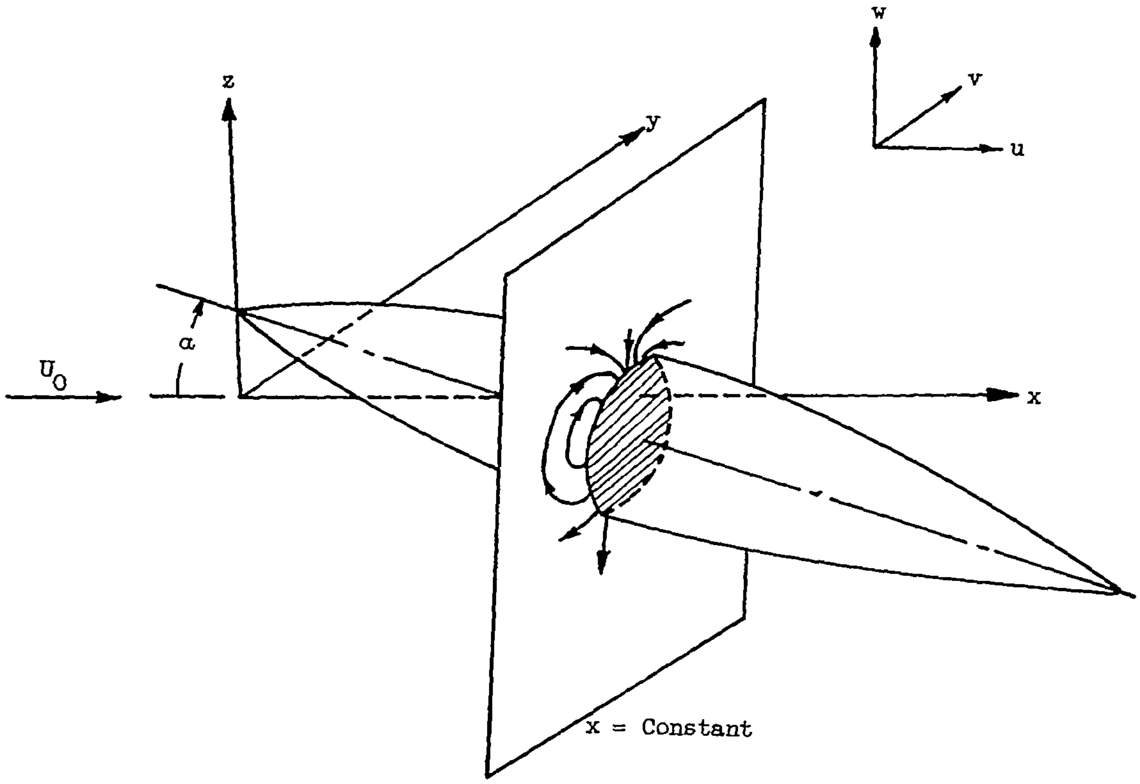
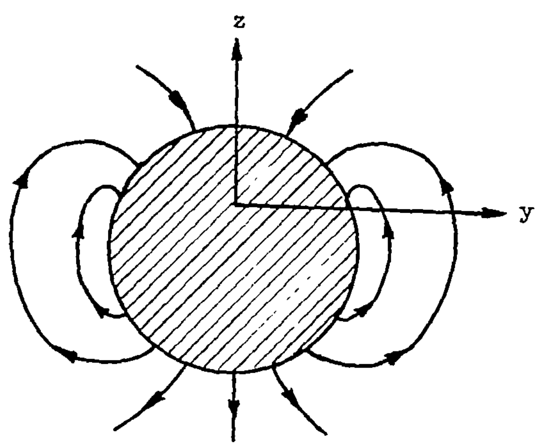


(a) Coordinate system.



(b) Crossflow.

Figure 1. - Slender body in free stream U_0.

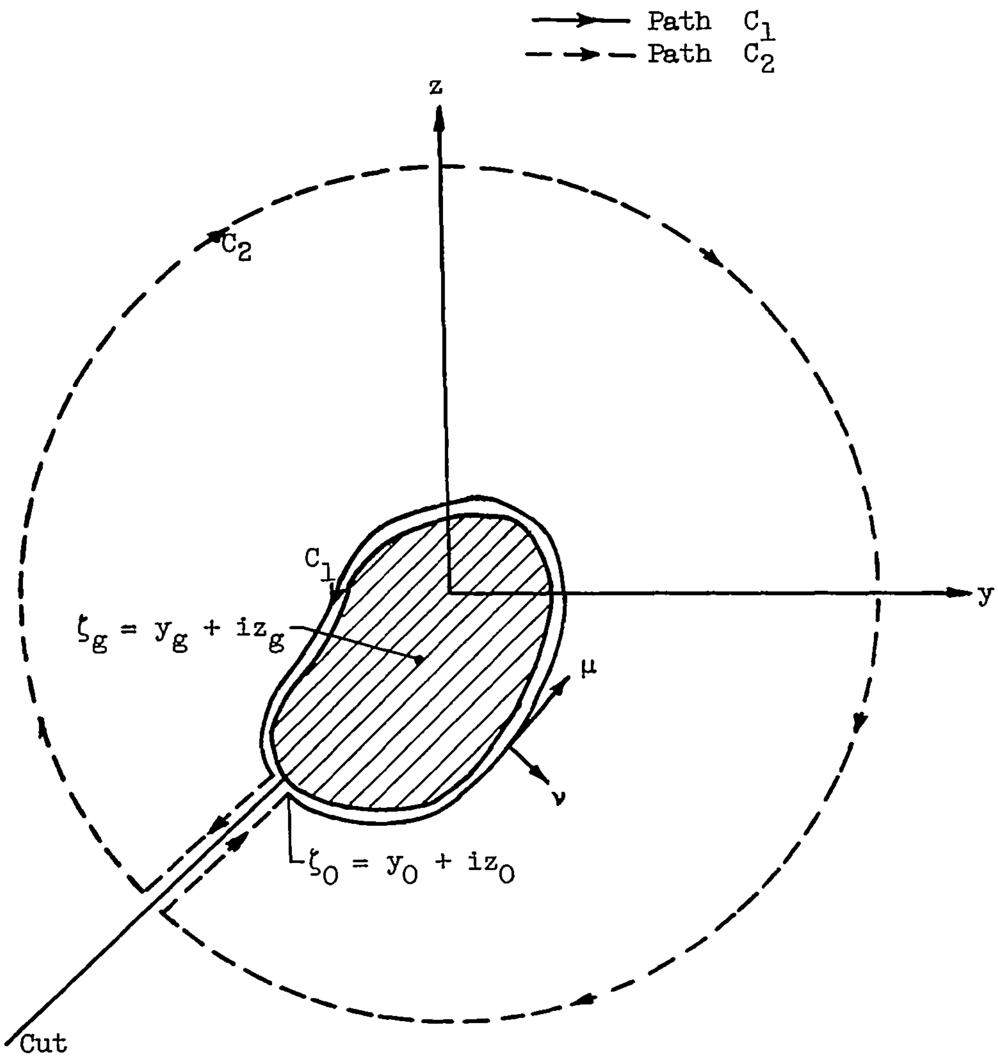
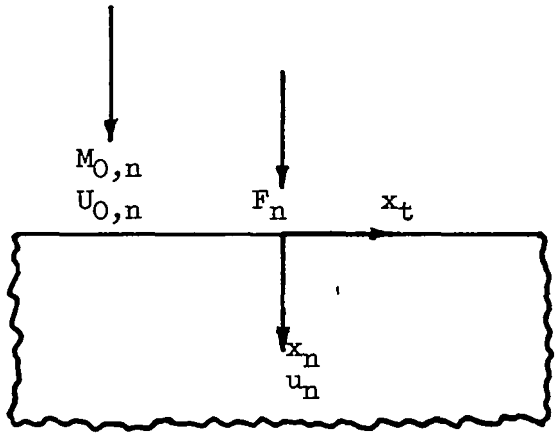
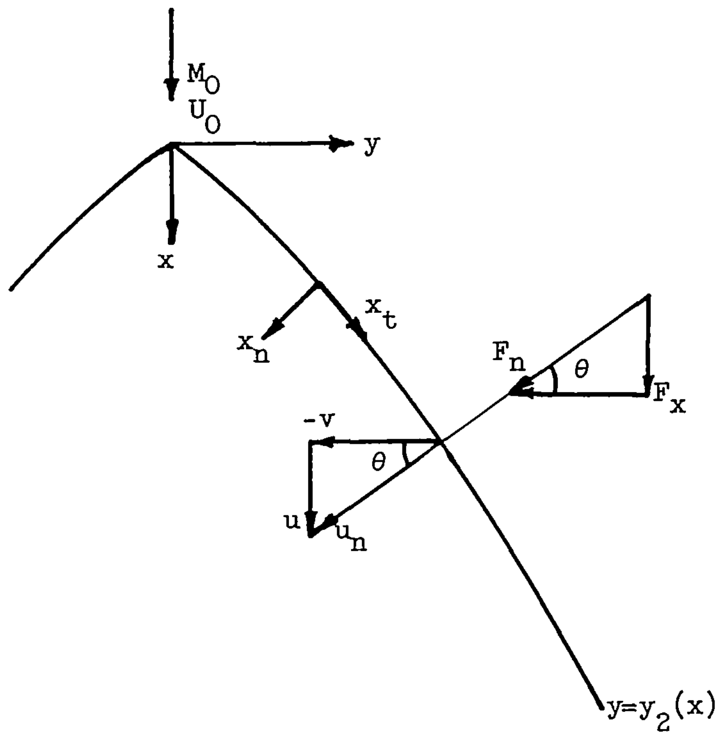


Figure 2. - Contours for obtaining forces on body.



(a) Two-dimensional wing.



(b) Swept leading edge.

Figure 3. - Notation for obtaining suction force.

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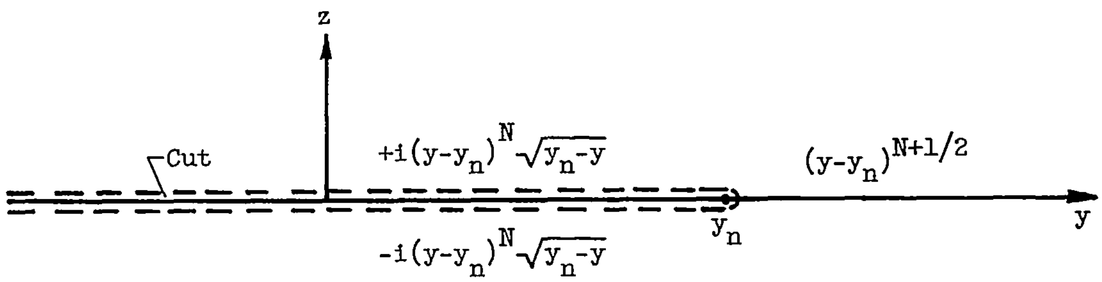

Figure 4. - Evaluation of function $(\zeta-y_n)^{N+1/2}$

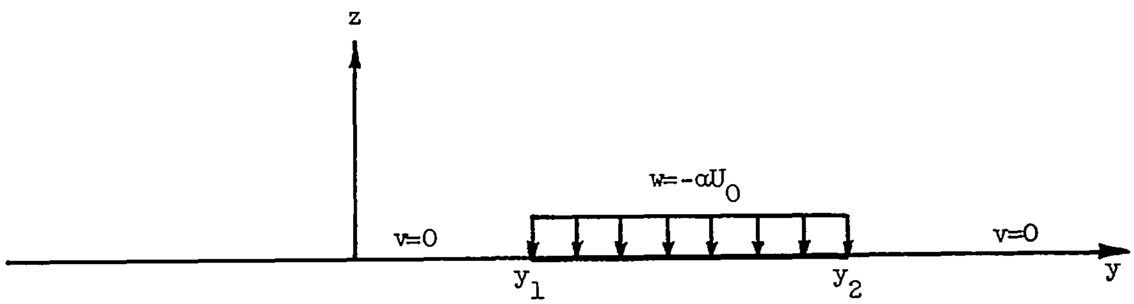

Figure 5. - Plate translating in fluid otherwise at rest.

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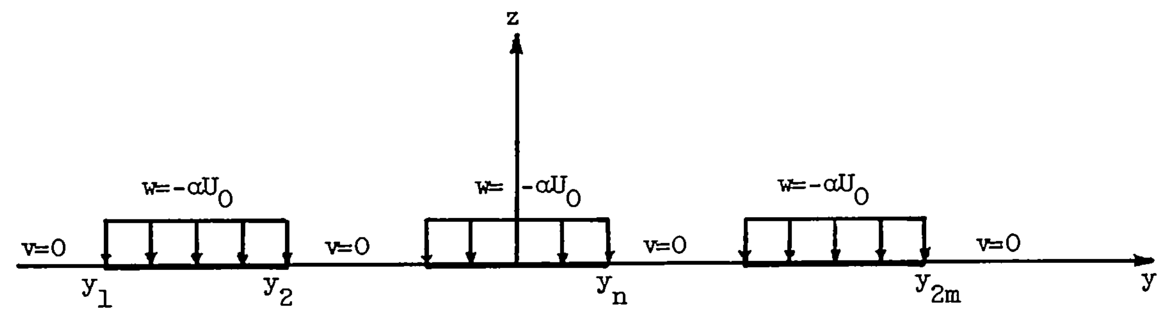
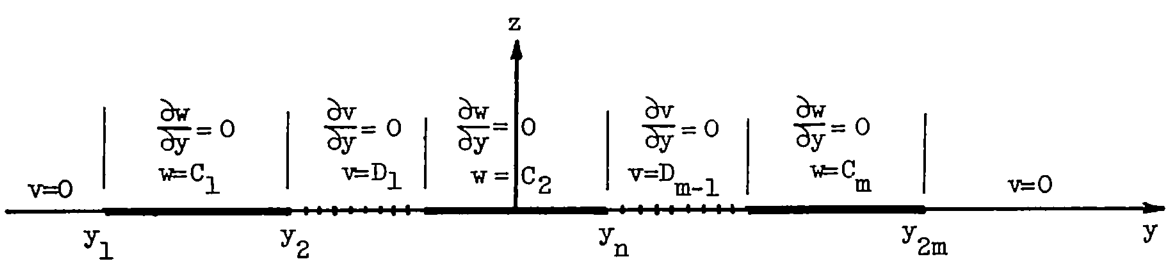
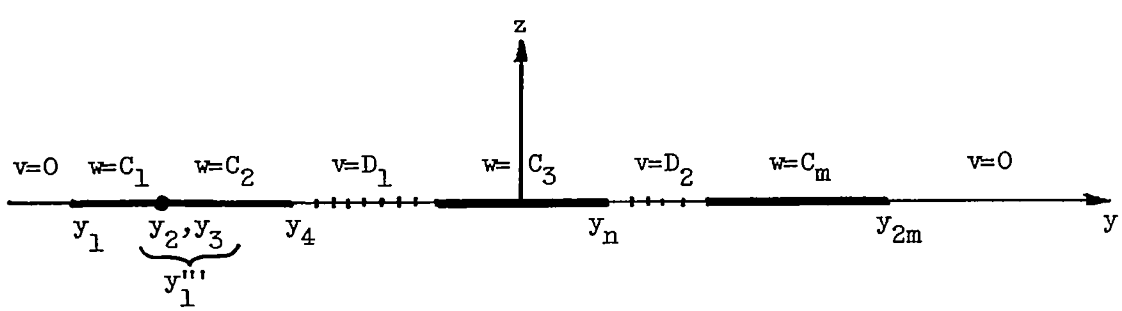


(a) Uniform translation of m wing panels without intermediate vortex sheets.



(b) m Wing panels with constant values of v and w specified in z=0 plane.



(c) Discontinuity in w boundary condition on wing panel.

Figure 6. - Notation and boundary conditions for solution of multiwing panel problems.

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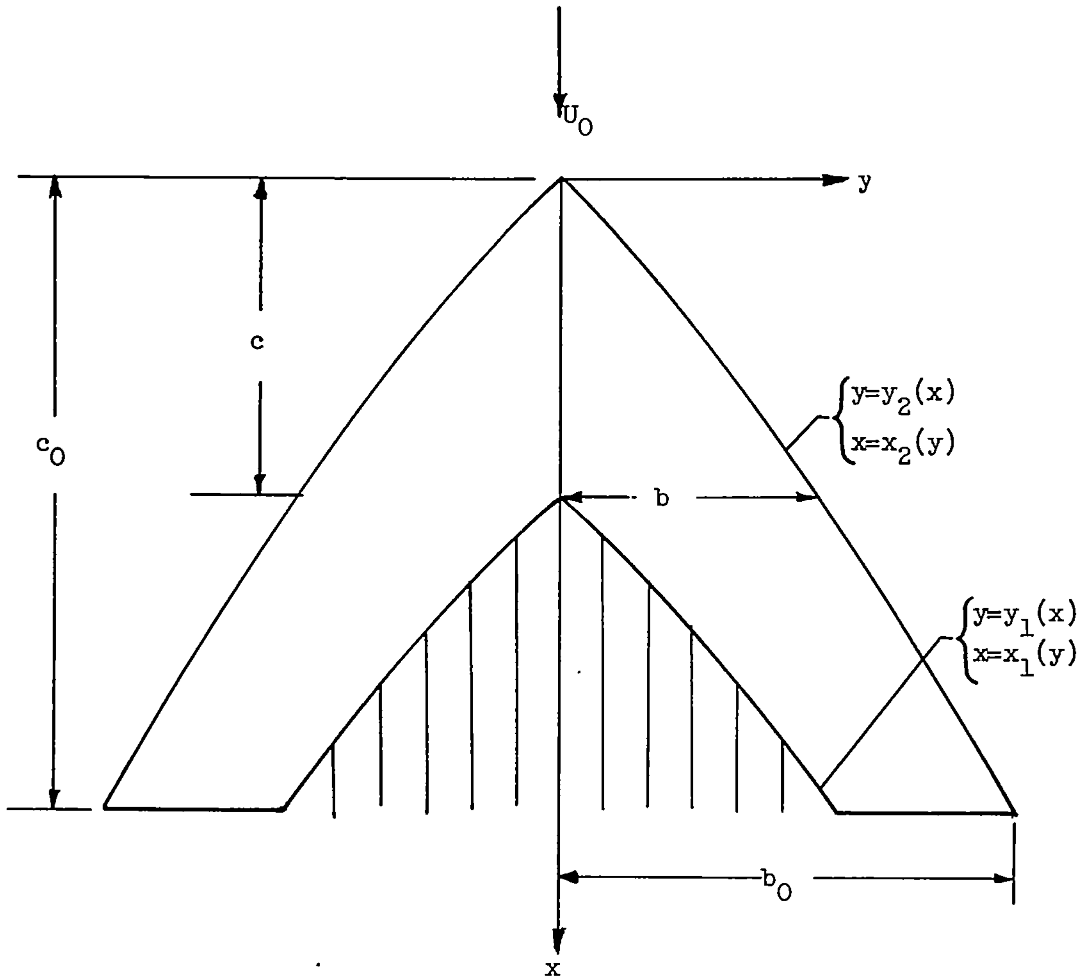
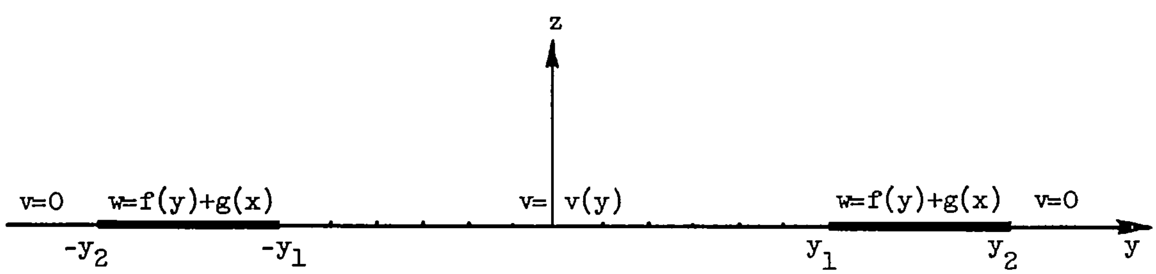


(a) Notation.



(b) Boundary conditions.

Figure 7. - Swept wing.

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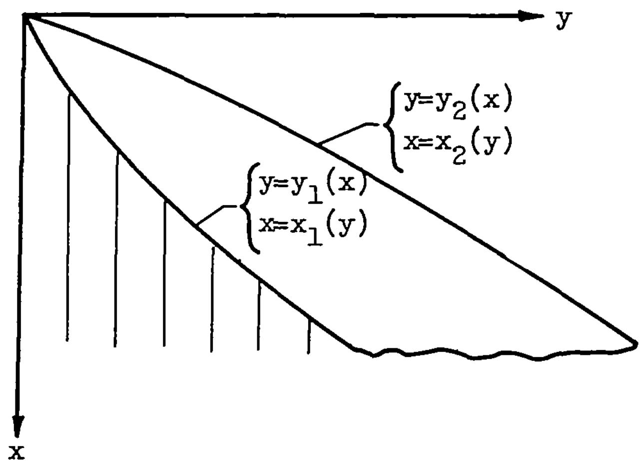
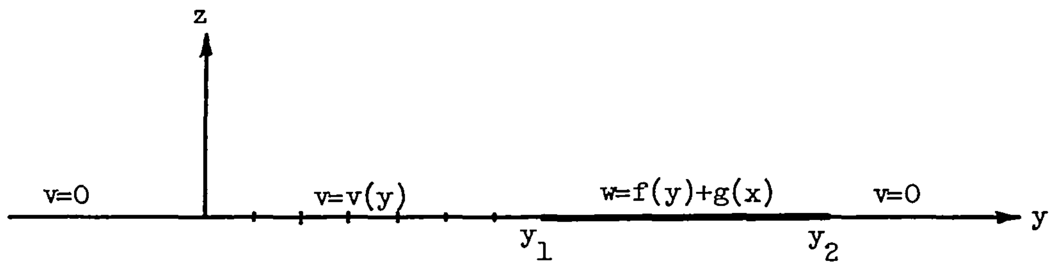
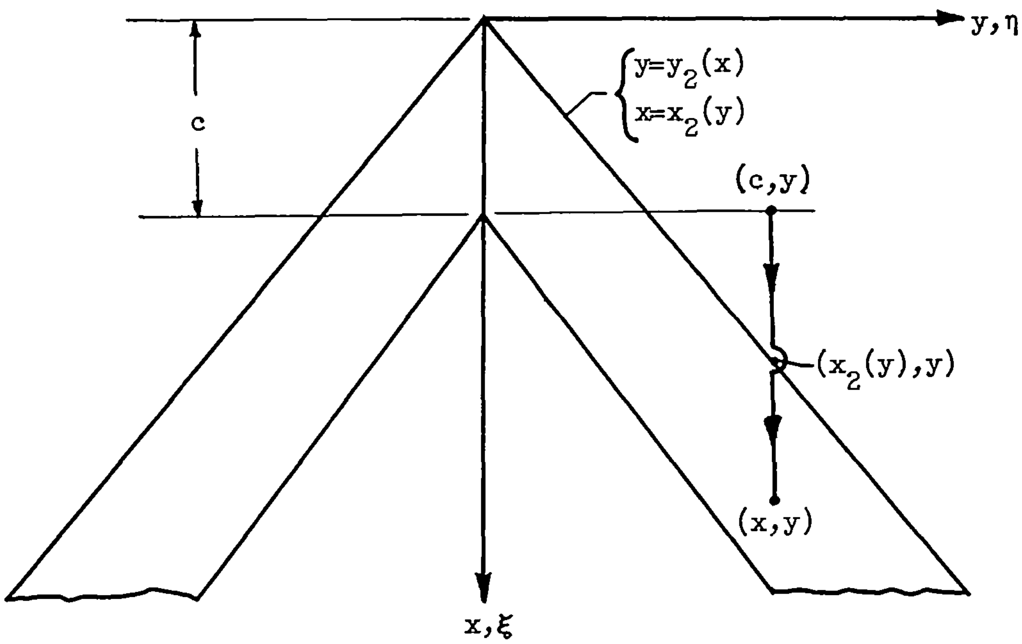


(a) Notation.

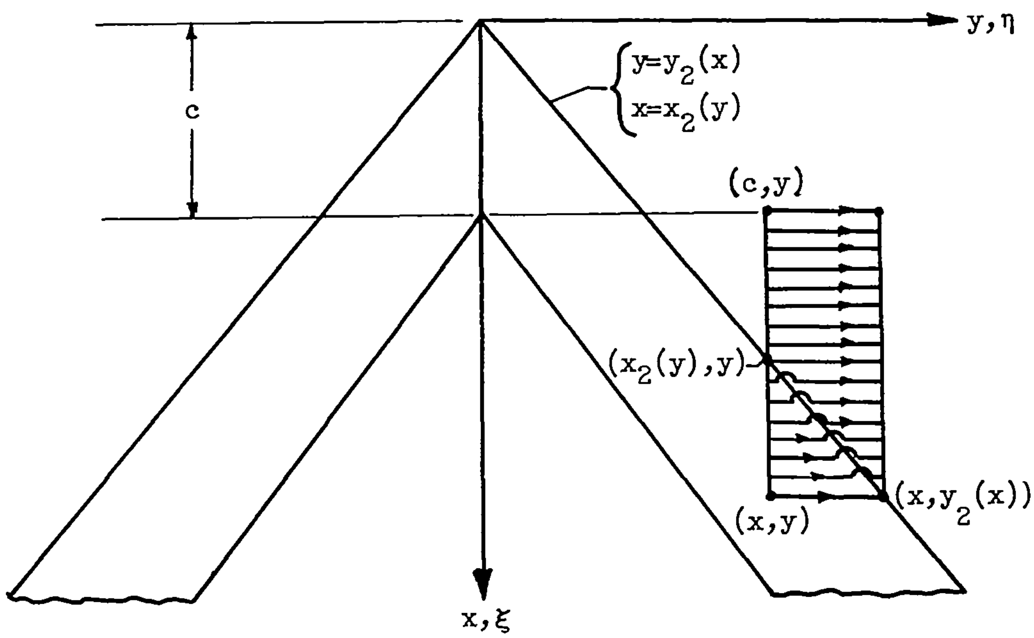


(b) Boundary conditions.

Figure 8. - Semi-infinite swept wing.



(a) Integration path for equation (3.3.1).



(b) Area of integration for equation (3.3.2).

Figure 9. - Integrations in equations (3.3.1) and (3.3.2).

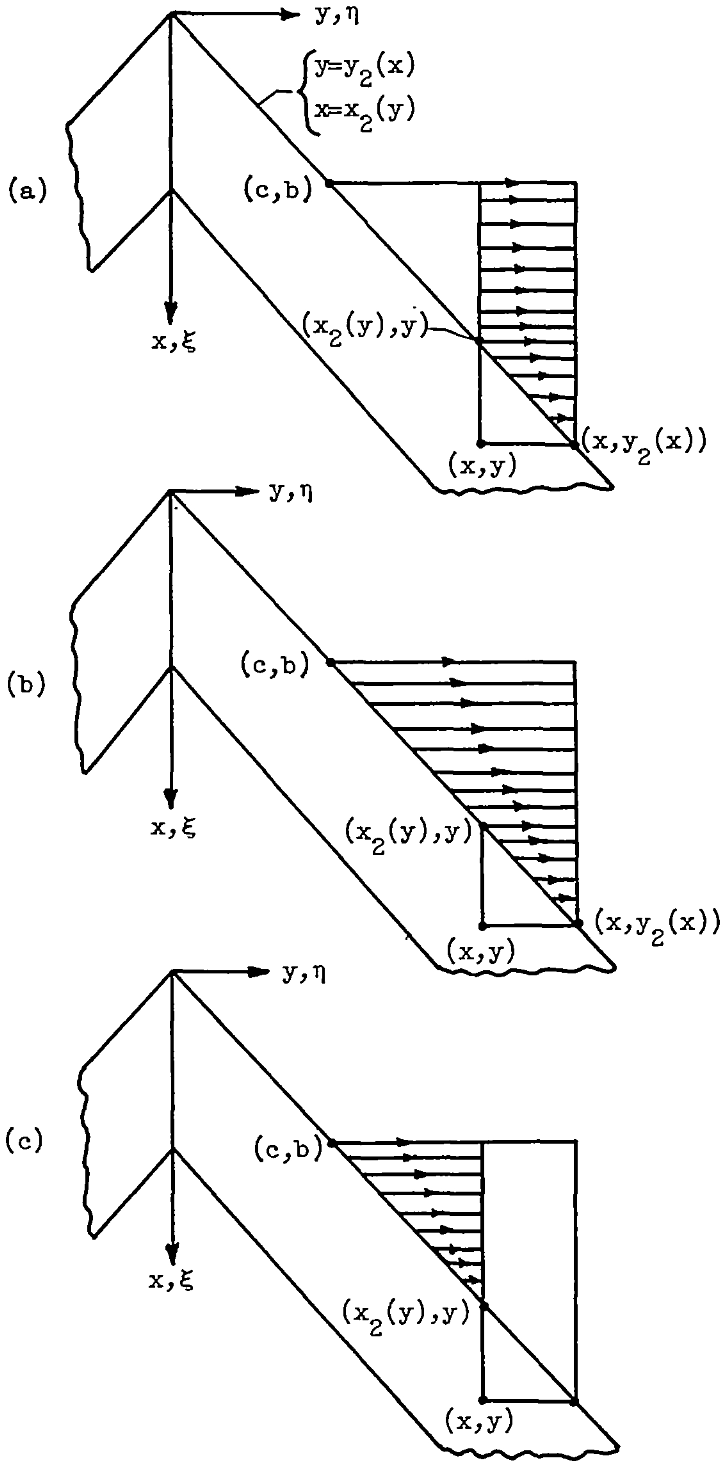


Figure 10. - Integration of equation (4.2.4). ((a) = (b) - (c)).

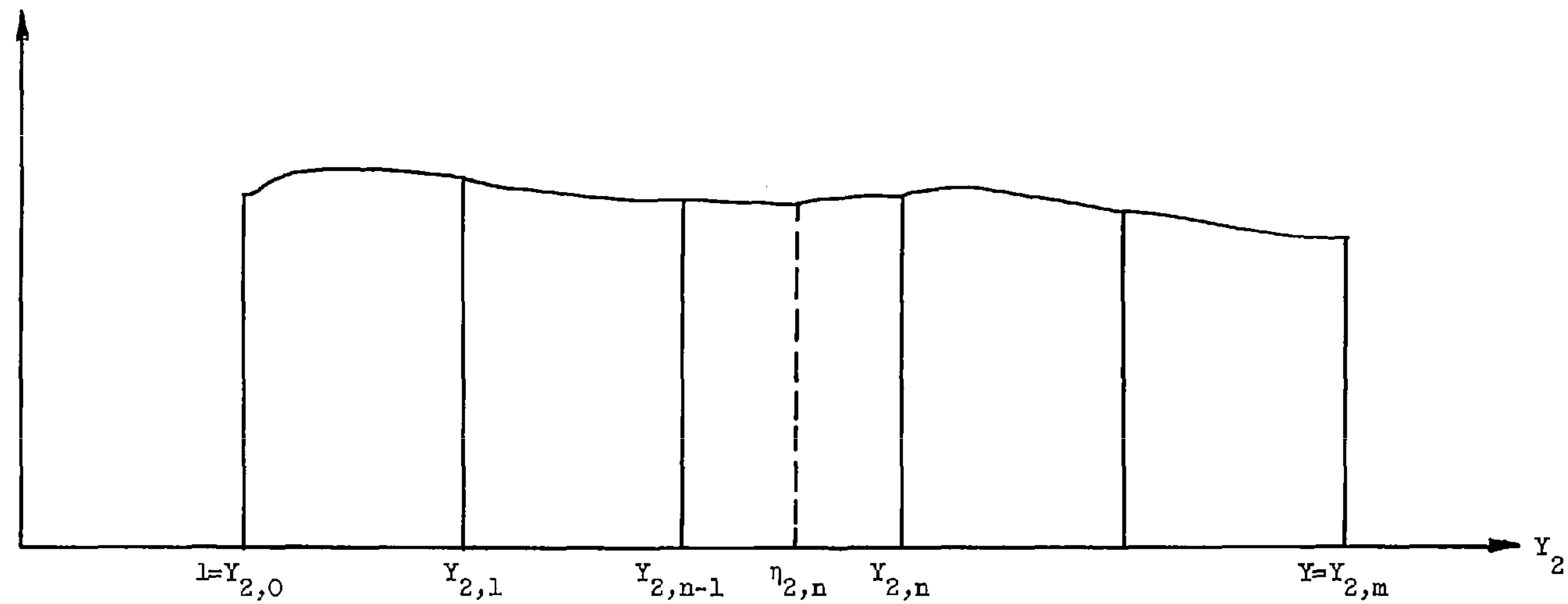


Figure 11. - Notation for solution of integral equations.

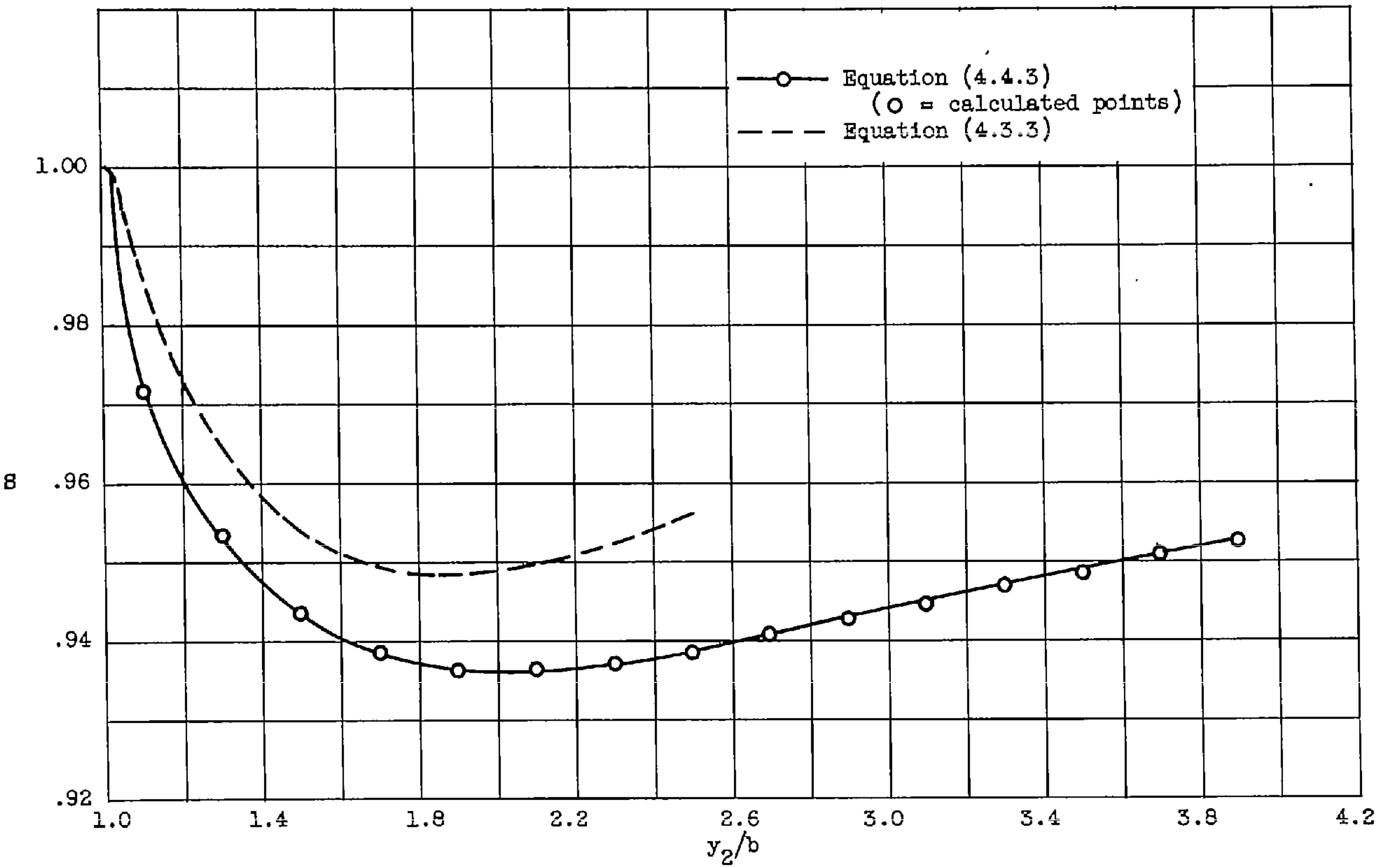


Figure 12. - Numerical calculation for S and comparison with equation (4.3.3), (γ=1).

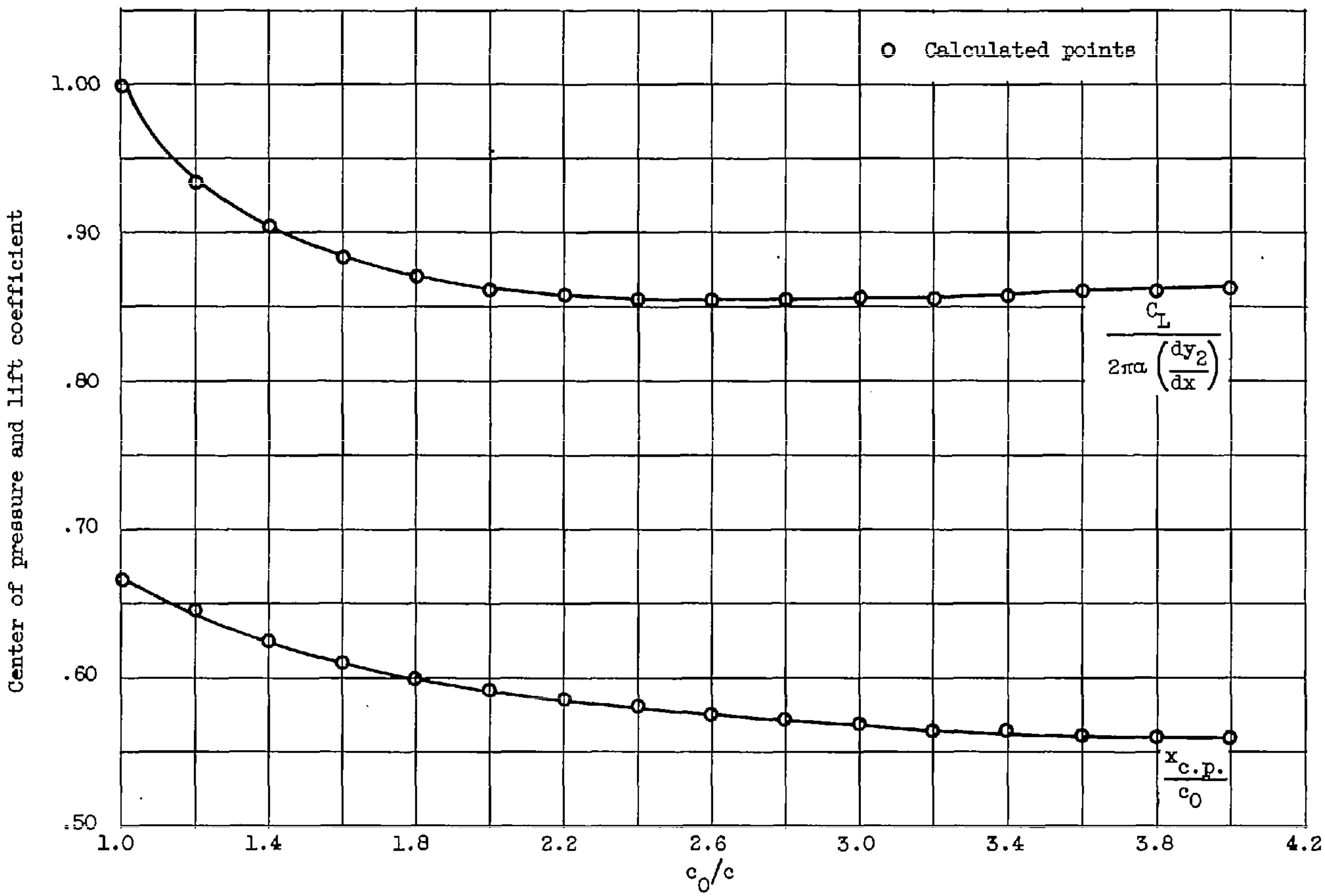


Figure 13. - Center of pressure and lift for swept wing at angle of attack ($\gamma=1$, dy_2/dx = constant).

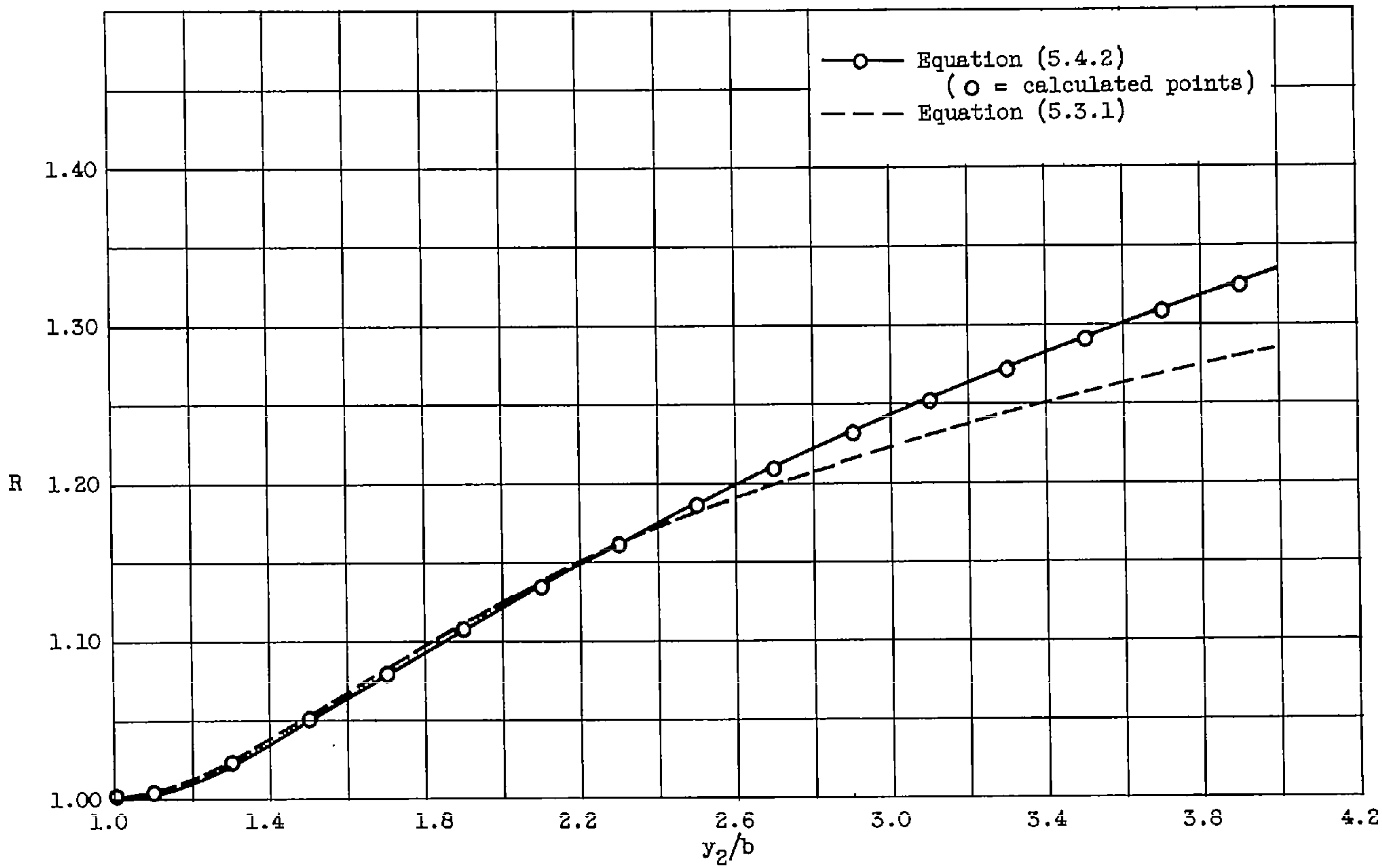


Figure 14. - Numerical calculation for R and comparison with equation (5.3.1). (γ=1).

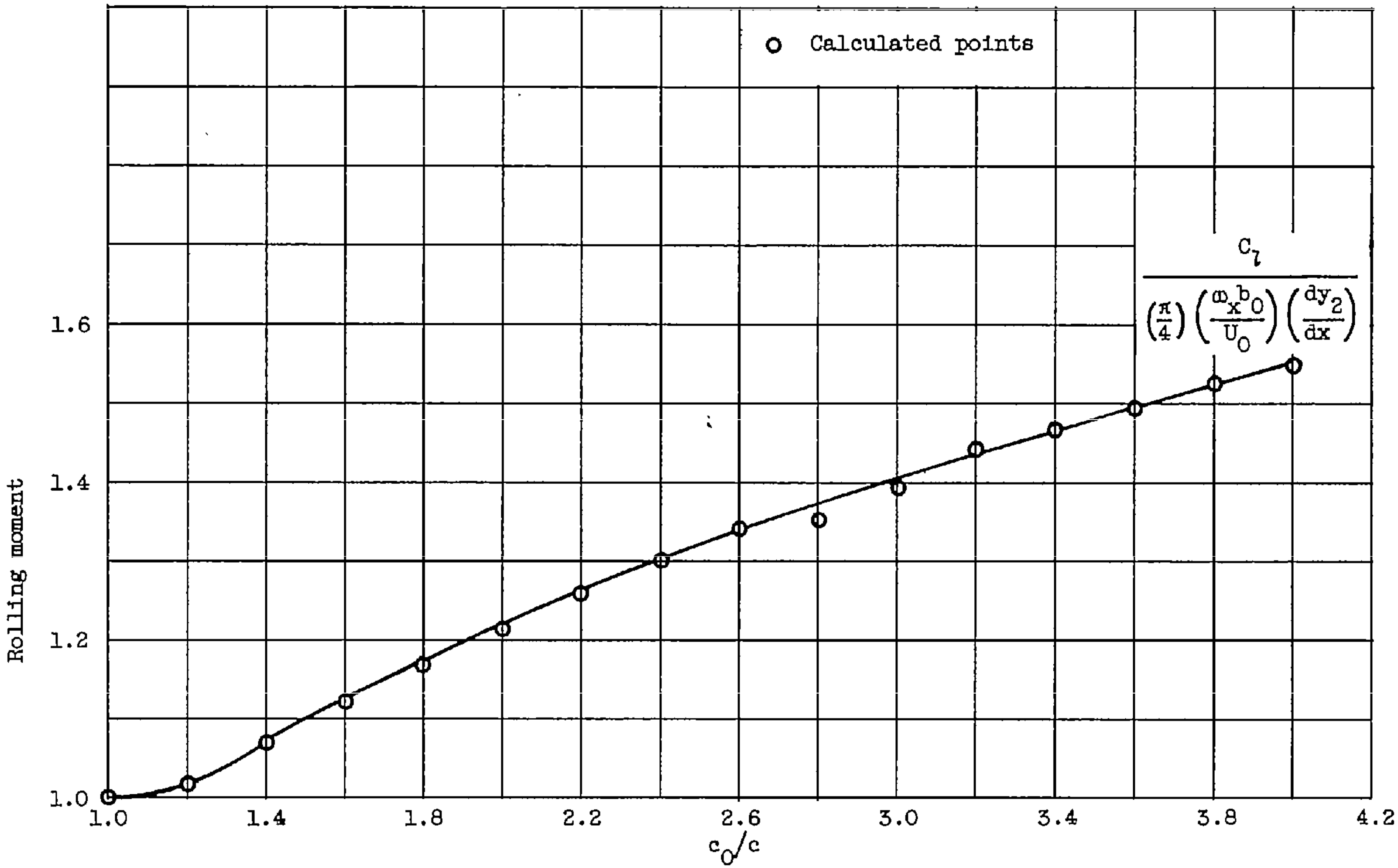


Figure 15. - Rolling moment for swept wing ($\gamma=1$, dy_2/dx = constant).

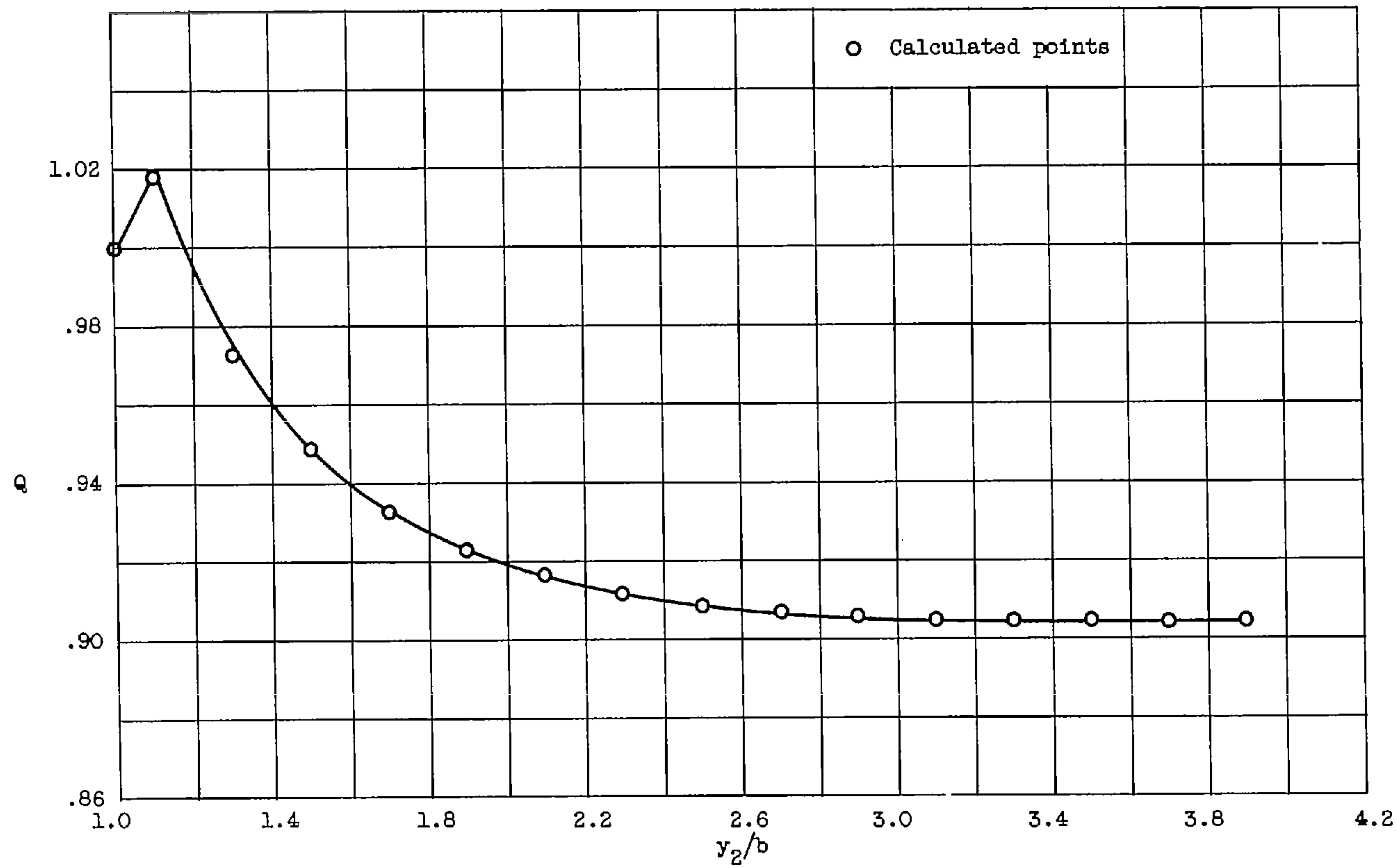


Figure 16. - Numerical calculation for Q based on section 6.4 (γ=1).

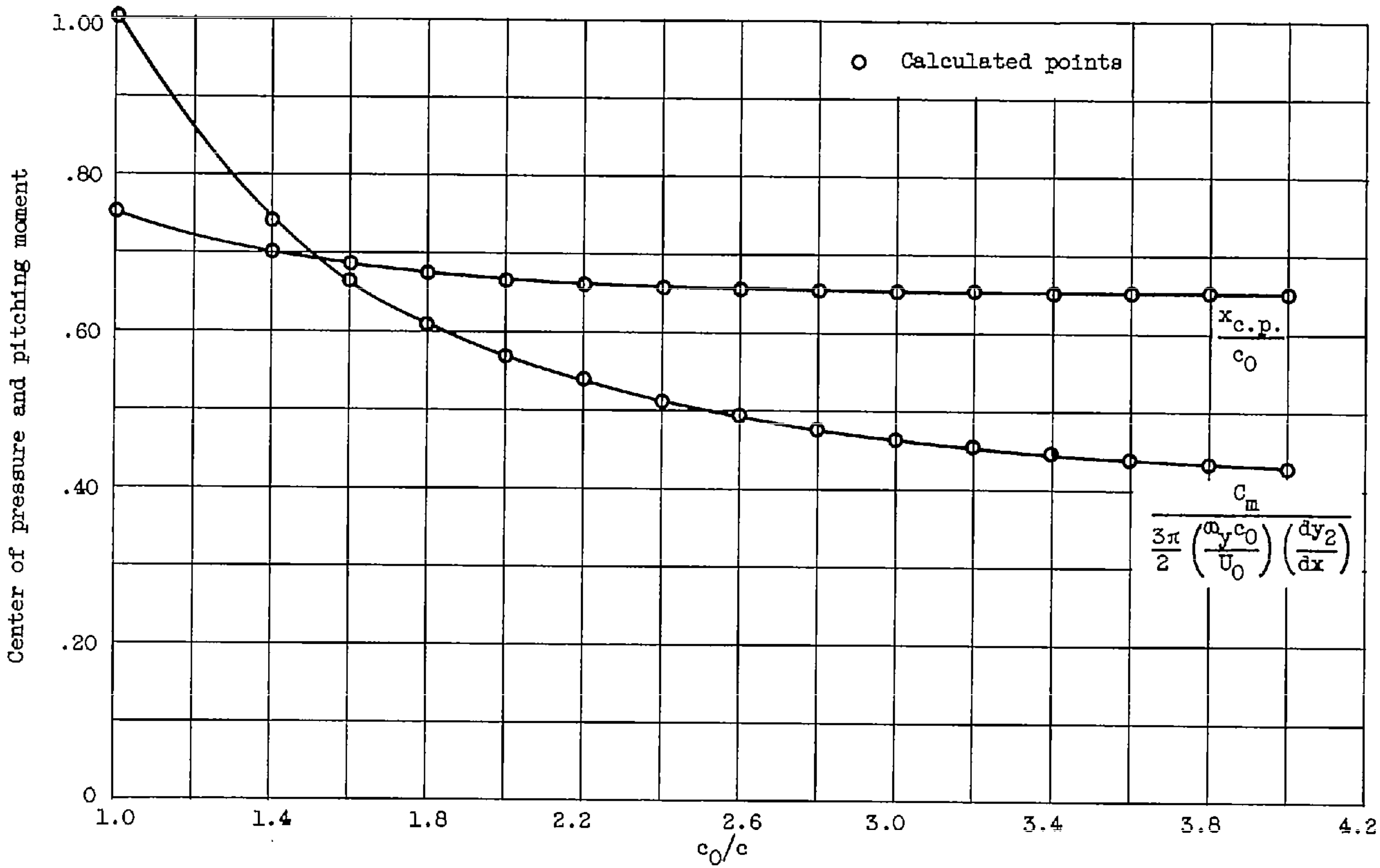


Figure 17. - Swept wing in steady pitch ($\gamma=1$, dy_2/dx = constant).

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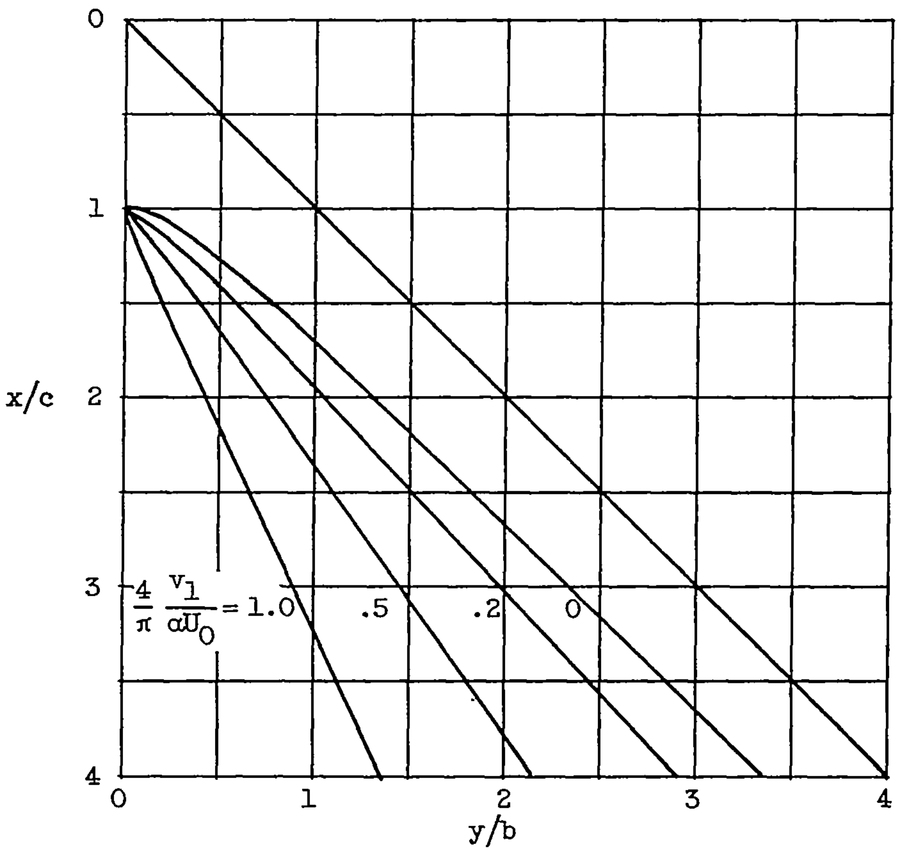

Figure 18. - Trailing-edge shapes (inverse lift problem).

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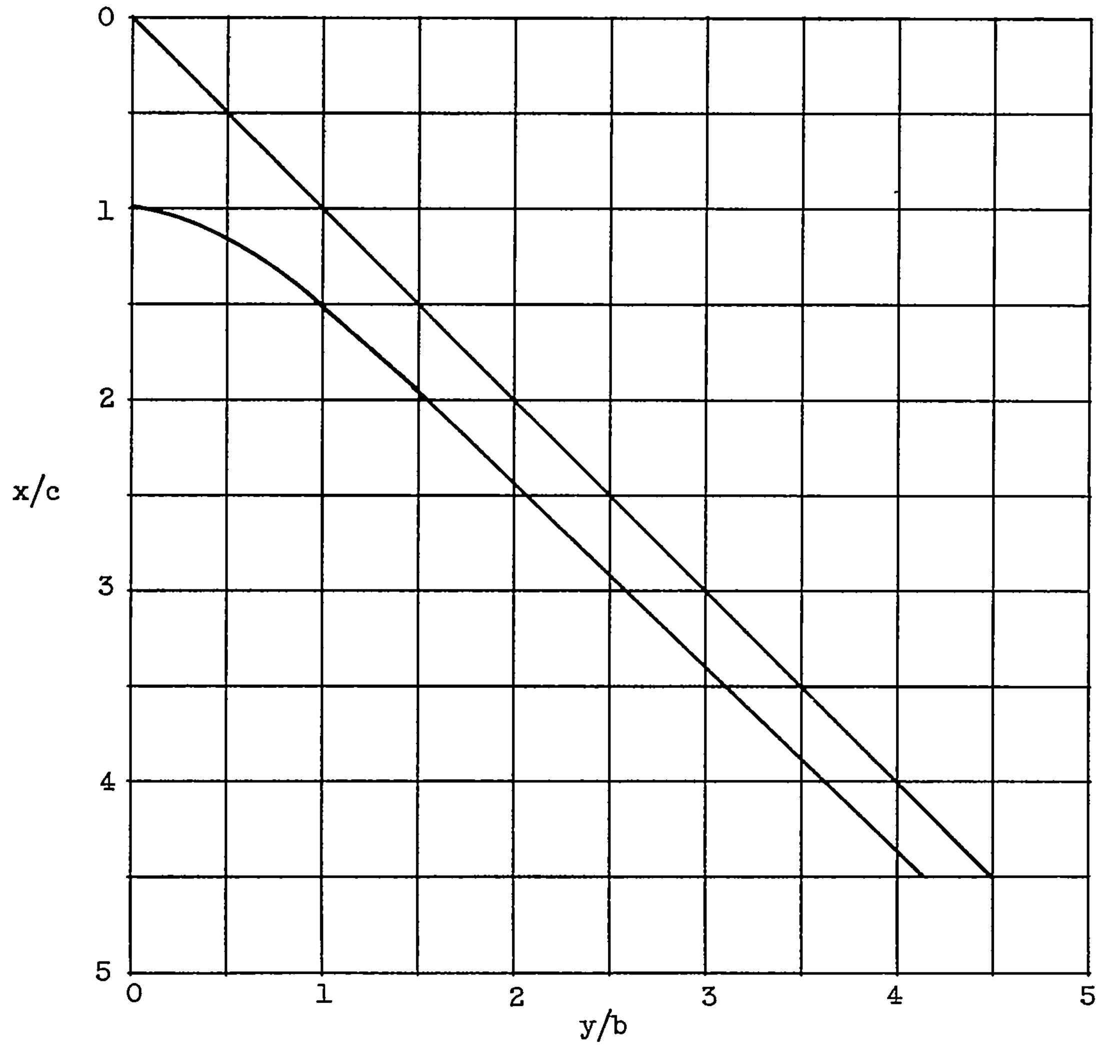

Figure 19. - Trailing-edge shape (inverse roll problem).

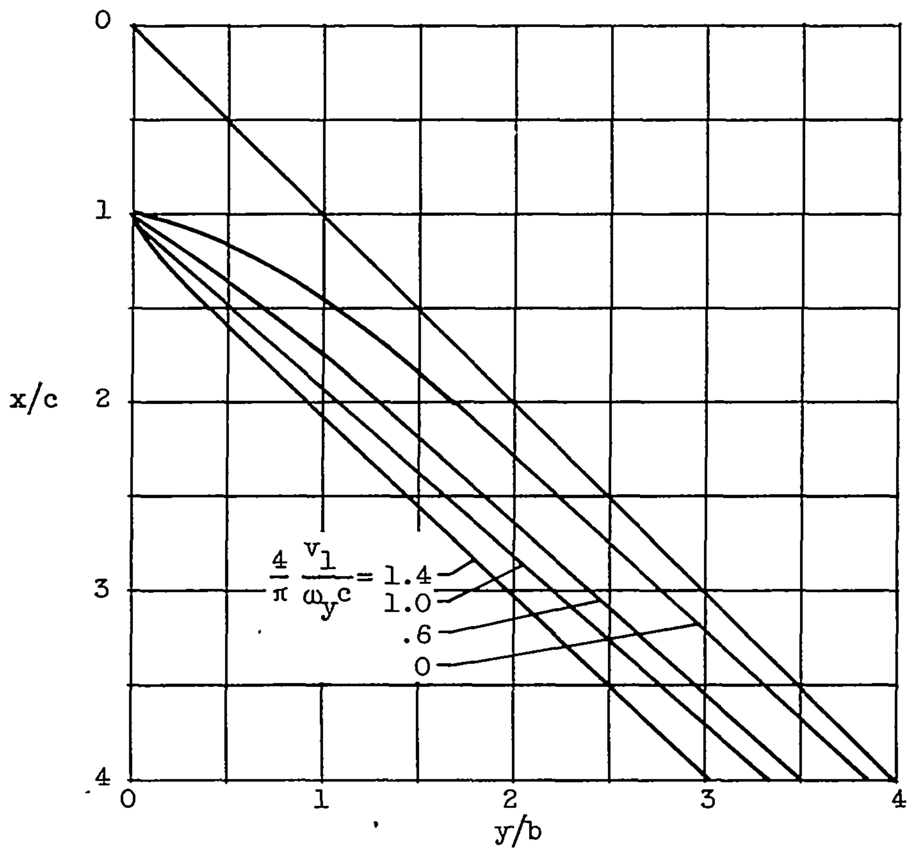

Figure 20. - Trailing-edge shapes (inverse pitch problem).

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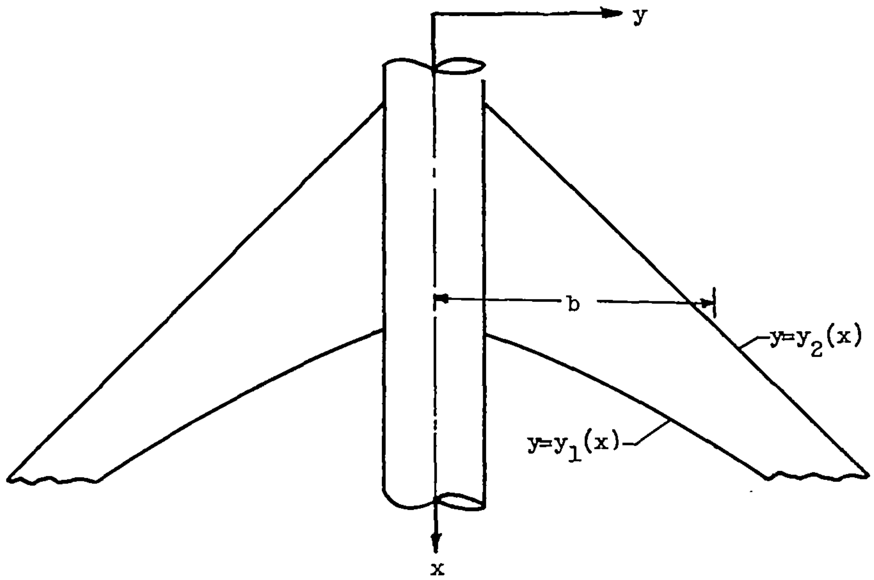
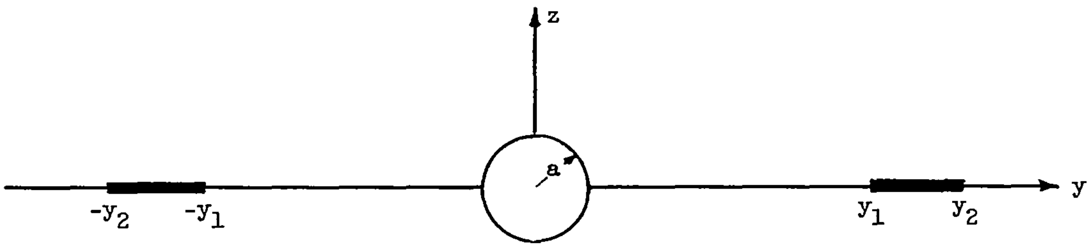
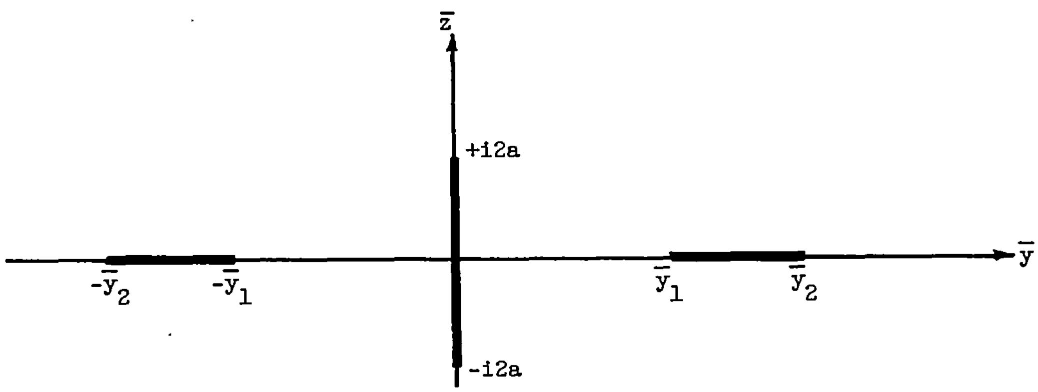


(a) Top view (physical plane).



(b) End view (physical plane).



(c) End view (transformed plane).

Figure 21. - Transformation for solution of swept wing on circular cylinder ($\bar{\zeta}=\zeta-a^2/\zeta$).

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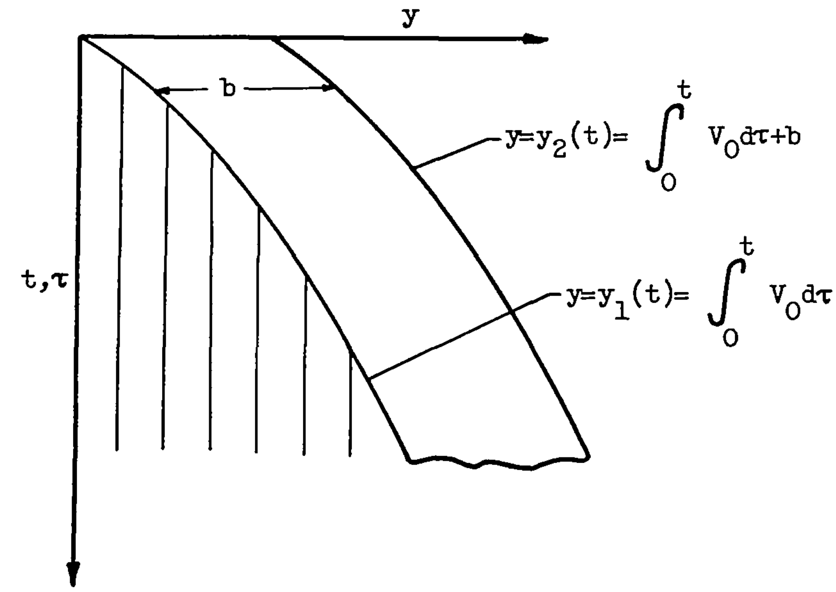

Figure 22. - Motion of two-dimensional wing.

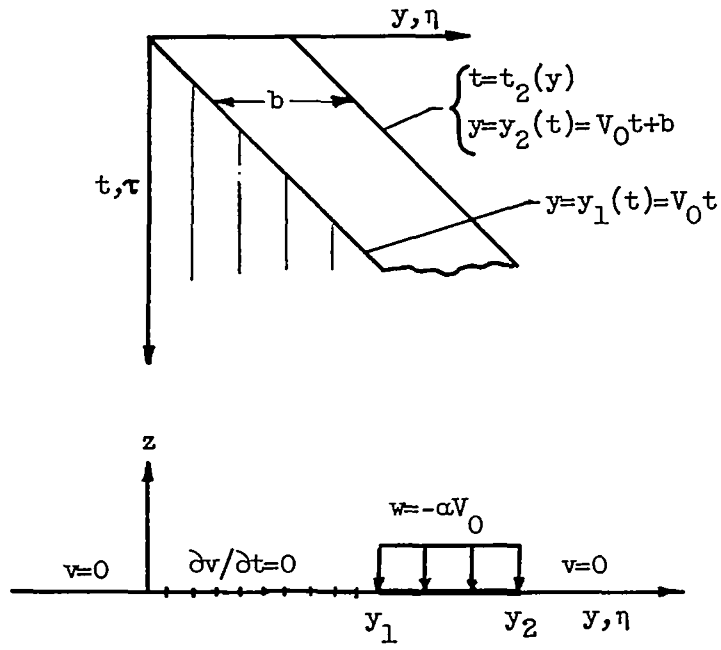

Figure 23. - Formulation of Wagner problem.

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