A RAPID METHOD FOR ESTIMATING THE SEPARATION POINT OF A
COMpressible Laminar Boundary Layer

By Laurence K. Loftin, Jr., and Homer B. Wilson, Jr.

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
February 1953
A RAPID METHOD FOR ESTIMATING THE SEPARATION POINT OF A
COMPRESSIBLE LAMINAR BOUNDARY LAYER

By Laurence K. Loftin, Jr., and Homer B. Wilson, Jr.

SUMMARY

A method has been developed for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method consists of an extension of Von Doenhoff's simplified solution for the incompressible case (NACA TN 671) and makes use of a set of transforms derived by Stewartson (Proc. Roy. Soc., 1949) which permit, under certain assumptions, the expression of compressible laminar layers in terms of equivalent incompressible laminar layers. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. The dependence of the method upon the boundary-layer assumptions, of course, means that it should not be applied to determine whether the pressure rise through a shock wave causes separation.

Calculations of the laminar separation point for a wide range of Mach number and velocity gradient indicate that, for all velocity gradients, the amount of velocity recovery possible before laminar separation occurs decreases as the Mach number increases.

INTRODUCTION

The determination of the position at which the laminar boundary layer separates has been the subject of much theoretical investigation. For the case of incompressible flow, such methods as those of Howarth (ref. 1), Von Kármán and Millikan (ref. 2), and Von Doenhoff's simplified adaptation of the Von Kármán-Millikan theory (ref. 3) may be used for determining the laminar separation point on bodies with velocity distributions of arbitrary shape.

For the case of compressible flows, several investigators have been interested in determining the basic nature of the effect of Mach number on the laminar separation point. Such methods as those of Howarth (ref. 4) and Cope and Hartree (ref. 5) provide means for determining the laminar separation point in a compressible flow. These methods are rather
cumbersome, however, and are not readily adaptable for rapidly estimating the effect of Mach number and velocity gradient on laminar separation. Stewartson (ref. 6) has recently shown that a vast simplification in the theory of compressible laminar boundary layers is possible if certain rather reasonable simplifying assumptions are made. On the basis of these assumptions, he was able to develop transforms which express the compressible laminar boundary layer in terms of an equivalent incompressible laminar boundary layer.

In the present paper the method of Von Doenhoff has been combined with Stewartson's transforms to provide a means for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. With the use of the method developed, the effect of Mach number on the laminar separation point has been calculated for a wide range of Mach number and velocity gradient. The results of these calculations are presented.

SYMBOLS

\begin{itemize}
\item \(c\) reference length
\item \(x\) abscissa
\item \(y\) ordinate
\item \(L\) equivalent flat-plate length
\item \(a\) speed of sound
\item \(U\) velocity just outside boundary layer
\item \(u\) velocity inside boundary layer
\item \(M\) Mach number
\item \(F\) nondimensional velocity gradient, \(\frac{L}{U_0} \frac{dU}{dx}\)
\item \(c_p\) specific heat at constant pressure
\item \(c_v\) specific heat at constant volume
\item \(\gamma\) ratio of specific heats, \(c_p/c_v\)
\item \(\mu\) viscosity
\end{itemize}
\( \rho \) \hspace{1em} \text{mass density} \\
\( \nu \) \hspace{1em} \text{kinematic viscosity} \\
\( k \) \hspace{1em} \text{coefficient of heat conduction} \\

Subscripts:

\( o \) \hspace{1em} \text{reference condition taken at point of maximum velocity} \\
\( l \) \hspace{1em} \text{any point along body just outside boundary layer} \\
\( c \) \hspace{1em} \text{compressible} \\
\( i \) \hspace{1em} \text{incompressible} \\

\section*{ANALYSIS}

Stewartson's Transforms

Stewartson (ref. 6) showed that the equations governing the behavior of two-dimensional laminar boundary layers in a compressible flow are identical to those governing their behavior in an incompressible flow if the variables in the two planes are related by the following transformations:

\[
y_1 = \frac{a_l}{a_0 \nu_0} \int_0^{y_c} \frac{\rho}{\rho_0} \, dy_c \tag{1}
\]

\[
x_1 = \int_0^{x_c} \left( \frac{a_1}{a_0} \right)^{3/\gamma - 1} \, dx_c \tag{2}
\]

\[
U_{1l} = \frac{a_0}{a_1} U_{c1} \tag{3}
\]
where \( x_c, y_c, U_c \) and \( x_1, y_1, U_1 \) are the coordinates and the velocity just outside the boundary layer in the compressible and incompressible planes, respectively. The subscript o refers to some convenient reference condition and the subscript 1 refers to conditions at a variable point just outside the boundary layer. The detailed derivation of the incompressible-boundary-layer equation from the compressible equations with the use of transforms (1) to (3) is treated in reference 6.

The transformation of the compressible-boundary-layer equation to an equivalent incompressible-boundary-layer equation with the use of transforms (1) to (3) depends upon the following assumptions: The boundary is thermally insulated, the viscosity is proportional to the absolute temperature, and the Prandtl number \( \frac{c_p \mu}{k} \) of the fluid is unity. By assuming the boundary thermally insulated, the basic problem of motion of the compressible boundary layer is isolated from extraneous problems associated with heat transfer through the boundary surface. The assumption that the Prandtl number is unity means that the stagnation temperature is reached at the wall. The Prandtl number for air, however, is actually about 0.715 (ref. 4); therefore, the temperature near the wall is lower than that predicted. Since the viscosity increases with the temperature, the assumption that the Prandtl number is unity would indicate that the effects of viscosity are overestimated. With regard to the remaining assumption that the viscosity is proportional to the absolute temperature, experiments have shown that the viscosity for air varies as the eight-ninths power of the absolute temperature between \( 90^\circ \) and \( 300^\circ \) Kelvin (ref. 4). Thus, this assumption also results in an overestimation of the effects of viscosity.

Von Doenhoff's Method

Since the transforms (1) to (3) permit the expression of the compressible laminar boundary layer in terms of an equivalent incompressible laminar boundary layer, it is apparent that all methods for calculating the laminar separation point in incompressible flow are equally applicable in the case of compressible flow. The laminar separation point is defined as the point along the surface at which \( \frac{\partial u}{\partial y} = 0 \) at \( y = 0 \). Of all methods available for calculating the laminar separation point in incompressible flow, that due to Von Doenhoff (ref. 3) is perhaps the simplest over-all method to apply and is used herein as a basis for developing a method for rapidly estimating the laminar separation point in a compressible flow.

With the use of the theory developed by Von Karman and Millikan for computing the separation point of a laminar boundary layer, Von Doenhoff calculated the laminar separation point for a series of velocity distributions which consisted of a region of uniform velocity (flat-plate
flow) followed by a region of linearly decreasing velocity. These calculations were generalized to the case of bodies with velocity distributions of arbitrary shape on the basis of the fundamental assumption that the shape of the boundary-layer velocity distribution at the point along a body at which the outside velocity is a maximum, that is, at the point of application of the adverse velocity gradient, is very nearly the same as that of a Blasius flat-plate distribution. Thus, the condition of the boundary layer at the point of application of the adverse velocity gradient can be represented by the flow over an equivalent length of flat plate with a uniform velocity equal to the maximum velocity. The assumption was also made that, for purposes of estimating the laminar separation point, any velocity gradient likely to be encountered could be approximated by a straight line. Von Doenhoff presented the results of his calculations in the form of the decrement in velocity necessary to cause separation \( \Delta U_{i1}/U_{i0} \) as a function of the nondimensional velocity gradient \( F_i = \frac{L_i}{U_{i0}} \frac{dU_{i1}}{dx_1} \), where \( L_i \) is the equivalent flat-plate length, \( U_{i0} \) is the maximum velocity, and \( dU_{i1}/dx_1 \) is the velocity gradient. The relationship between \( \Delta U_{i1}/U_{i0} \) and \( F_i \) given in reference 3 is presented herein as figure 1. The equivalent length of flat plate corresponding to the flow at the point of application of the adverse velocity gradient can be found with the use of the following equation (ref. 3, eq. (1)):

\[
\frac{L_i}{c_i} = \int_0^{x_{i0}/c_i} \left( \frac{U_{i1}}{U_{i0}} \right)^{8.17} d\left( \frac{x_1}{c_i} \right)
\]  

(4)

where \( U_{i1}/U_{i0} \) is the velocity ratio at any point on the body and the integration is carried from the beginning of the flow to the point of application of the adverse gradient. Equation (4) was obtained from an integration of the Von Kármán momentum relation by making the assumption that the boundary-layer velocity distribution at every point along the body in the region of accelerating flow is the same as a Blasius flat-plate distribution. Calculations of the laminar separation point on the NACA 0012 airfoil at zero lift made by the use of Von Doenhoff's rapid method and calculations made by the use of the more elaborate Von Kármán-Millikan theory are shown in reference 3 to agree very closely.
Application of Stewartson's Transforms to Von Doenhoff's Method

In order to find the position on a body at which laminar separation occurs, the distribution of velocity outside the boundary layer and the Mach number at some typical point must be known. With this information, the laminar separation point can be calculated through the direct application of transforms (2) and (3) and Von Doenhoff's relation given in figure 1. Although relatively easy to apply, this method does involve a rather tedious point-by-point transformation from the compressible to the equivalent incompressible plane.

For this reason, a more rapid procedure has been developed for estimating the separation point of a compressible laminar boundary layer. This procedure consists essentially in the direct application of Von Doenhoff's method to the compressible velocity distribution, the effect of Mach number being accounted for by a simple multiplying factor applied to the measured velocity gradient in the compressible plane.

Consider the velocity gradient \( \frac{dU_c}{dx_c} \) in the compressible plane.

From equation (2) it can be seen that \( dx_1 = \left( \frac{a_1}{a_0} \right)^{\gamma-1} dx_c \) and equation (3) states that \( U_{11} = \frac{a_0}{a_1} U_{c1} \). Consequently, the compressible and incompressible velocity gradients are related as follows:

\[
\frac{dU_{11}}{dx_1} = \frac{d}{dx_c} \left( \frac{a_0}{a_1} U_{c1} \right) dx_c = \frac{d}{dx_c} \left( \frac{a_0}{a_1} U_{c1} \right) \frac{1}{3\gamma-1} \left( \frac{a_1}{a_0} \right)^{\gamma-1} \]

or

\[
\frac{dU_{11}}{dx_1} = \left[ U_{c1} \frac{d\left( \frac{a_0}{a_1} \right)}{dx_c} + \frac{a_0}{a_1} \frac{dU_{c1}}{dx_c} \right] \frac{1}{3\gamma-1} \left( \frac{a_1}{a_0} \right)^{\gamma-1}
\]
If the reference velocity of sound $a_o$ is taken at the point of maximum velocity $U_{co}$, that is, at the point of application of the adverse velocity gradient, then $U_{co} = U_{i_0}$ and

$$\frac{d}{dx_c} \left( \frac{U_{i_1}}{U_{i_0}} \right) = \left[ \frac{U_{c_1}}{U_{co}} \frac{d}{dx_c} \left( \frac{a_o}{a_1} \right) + \frac{a_o}{a_1} \frac{d}{dx_c} \left( \frac{U_{c_1}}{U_{co}} \right) \right] \frac{1}{3\gamma - 1} \left( \frac{a_1}{a_o} \right)^{\gamma - 1} \right] \right.$$ (5)

By the use of the one-dimensional energy equation, the terms $a_1/a_o$ and $d(a_o/a_1)/dx_c$ can be written in terms of the Mach number $M_o$ and the velocity ratio $U_{c_1}/U_{co}$:

$$\frac{a_1}{a_o} = \sqrt{1 + \frac{\gamma - 1}{2} M_o^2 \left[ 1 - \left( \frac{U_{c_1}}{U_{co}} \right)^2 \right]}$$ (6)

and

$$\frac{d}{dx_c} \left( \frac{a_o}{a_1} \right) = \left[ \frac{\gamma - 1}{2} M_o^2 \left( \frac{U_{c_1}}{U_{co}} \right) \frac{d}{dx_c} \left( \frac{U_{c_1}}{U_{co}} \right) \right] \frac{1}{\left( 1 + \frac{\gamma - 1}{2} M_o^2 \left[ 1 - \left( \frac{U_{c_1}}{U_{co}} \right)^2 \right] \right)^{3/2}}$$ (7)

An examination of equations (5) to (7) indicates that a linear velocity distribution in the compressible plane transforms into a nonlinear distribution in the equivalent incompressible plane. The basic assumption
is made, however, that the determination of the approximate laminar separation point in the equivalent incompressible plane can be made from a knowledge of the transformed adverse velocity gradient at the point of maximum velocity. This assumption presupposes that there is a discontinuity in the compressible velocity distribution at the point of application of the adverse gradient or, at least, that the velocity distribution can be approximated in this manner. On the basis of this assumption, $\frac{a_1}{a_0} = 1.0$ and

$$\frac{d\left(\frac{a_1}{a_0}\right)}{dx_c} = \frac{\gamma - 1}{2} M_0^2 \frac{d\left(\frac{U_{c1}}{U_{c0}}\right)}{dx_c} \quad (8)$$

Substitution of equation (8) into equation (5) and the value 1.0 for the terms $U_{c1}/U_{c0}$ and $a_0/a_1$ yields the following relation:

$$\frac{d\left(\frac{U_{11}}{U_{10}}\right)}{dx_1} = \left(1 + \frac{\gamma - 1}{2} M_0^2\right) \frac{d\left(\frac{U_{c1}}{U_{c0}}\right)}{dx_c} \quad (9)$$

Thus, the value of the velocity gradient at the point of maximum velocity in the incompressible plane is obtained by multiplying the measured value of the velocity gradient in the compressible plane by the factor $1 + \frac{\gamma - 1}{2} M_0^2$, where it should be remembered that $M_0$ is the Mach number at the point of application of the adverse gradient in the compressible plane. The factor $1 + \frac{\gamma - 1}{2} M_0^2$ is plotted as a function of Mach number in figure 2.

The velocity gradient in the equivalent incompressible plane (eq. (9)) must be made nondimensional in terms of the equivalent flat-plate length in the incompressible plane if it is to be used with the relation given in figure 1 for finding the velocity recovery before laminar separation occurs in an incompressible flow. The equivalent length of flat plate in the incompressible plane can be determined from the relation
\[ L_1 = \int_0^{x_{co}} \left( \frac{U_{c1}}{U_{co}} \right) \left( \frac{a_0}{a_1} \right) \frac{8.17}{\gamma - 1} \frac{3\gamma - 1}{\gamma - 1} \, dx_c \]  

or, in terms of variables in the compressible plane, from the relation

\[ L_1 = \int_0^{x_{co}} \left( \frac{U_{c1}}{U_{co}} \right) ^{8.17} \left( \frac{a_0}{a_1} \right) ^{0.17} \, dx_c \]  

For air, \( \gamma = 1.4 \); therefore, \( \frac{3\gamma - 1}{\gamma - 1} = 8.0 \) and equation (11) reduces to

\[ L_1 = \int_0^{x_{co}} \left( \frac{U_{c1}}{U_{co}} \right) ^{8.17} \left( \frac{a_0}{a_1} \right) ^{0.17} \, dx_c \]  

Equation (12) can be used directly with the compressible velocity distribution to determine \( L_1 \). In many cases, however, it would appear that the value of \( \frac{a_0}{a_1} \) would be so near unity as to have a negligible effect on \( L_1 \). In this case, the equivalent flat-plate length \( L_1 \) can be determined from the equation

\[ L_1 = \int_0^{x_{co}} \left( \frac{U_{c1}}{U_{co}} \right) ^{8.17} \, dx_c \]  

The complete expression for the nondimensional velocity gradient in the equivalent incompressible plane in terms of variables in the compressible plane is therefore
\[
\frac{L_1}{U_{10}} \frac{dU_{11}}{dx_1} = \left(1 + \frac{\gamma - 1}{2} M_0^2\right) \frac{L_1}{U_{\infty}} \frac{dU_{c1}}{dx_c}
\]  

(14)

In the determination of \( L_1 \) and \( \frac{d}{dx_c} \left( \frac{U_{c1}}{U_{\infty}} \right) \) the value of \( x_c \) can, of course, be expressed in terms of a reference length in the compressible plane.

The determination of the laminar separation point corresponding to a known velocity distribution at a given Mach number involves the following steps:

(a) Graphical measurement of the velocity gradient \( \frac{dU_c}{dx_c} \) in the compressible plane. This process may require fairing of the given velocity distribution in such a way that the adverse velocity gradient begins at a distinct point and is approximated by a straight line.

(b) Calculation of the equivalent flat-plate length by the use of equation (13).

(c) Determination of the factor \( 1 + \frac{\gamma - 1}{2} M_0^2 \) from figure 2.

The nondimensional velocity gradient in the equivalent incompressible plane may then be determined by the use of equation (14). The velocity ratio corresponding to laminar separation in the incompressible plane may be found from the curve of figure 1 which expresses the velocity decrement necessary to cause laminar separation in terms of the nondimensional velocity gradient. The velocity ratio corresponding to laminar separation in the compressible plane is found with the use of the relation \( \frac{U_{11}}{U_{10}} = \frac{a_2}{a_1} \frac{U_{c1}}{U_{\infty}} \). The relationship between \( U_{11}/U_{10} \) and \( U_{c1}/U_{\infty} \) for different Mach numbers is plotted in figure 3.

In some cases, considerable fairing of the given velocity distribution may be required in order to obtain what appears to be a reasonable measure of \( \frac{d}{dx_c} \left( \frac{U_{c1}}{U_{\infty}} \right) \). Under such circumstances, the position of the calculated laminar separation point with respect to the given velocity distribution may indicate that a somewhat different fairing of the
velocity distribution would give a more realistic approximation of that portion of the velocity gradient along which laminar separation occurs. A second calculation of the laminar separation point may then be indicated.

Comparison With Results of Previous Analyses

The effect of Mach number on the position at which laminar separation occurs in a boundary-layer flow progressing against a linearly decreasing velocity from the leading edge has been investigated by Howarth (ref. 4) who employed a Von Kármán-Pohlhausen type of analysis, and by Stewartson (ref. 6) who applied his transforms to the incompressible solution developed by Howarth in 1938 (ref. 1). This type of velocity distribution has a value of $F_c$ equal to zero since the equivalent length of flat plate is zero. In order to provide a comparison between the method of the present investigation and those used in references 4 and 6, the laminar separation point has been calculated by the method presented herein for the case of $F_c$ equal to zero for a range of Mach number. The results of these computations are presented in figure 4 together with the results obtained by Howarth and Stewartson. Examination of figure 4 indicates that considerable difference exists at all Mach numbers between the predictions of the three methods. For zero Mach number, the value of the decrement in velocity ratio of 0.12 given by Howarth in reference 1 is probably the most exact. The magnitude of the differences between the three methods, however, appears to be roughly the same for all Mach numbers. This result can be seen more clearly in figure 5 in which the difference in $\frac{\Delta U_{c_1}}{U_{c_0}}$ at Mach number zero and at some arbitrary Mach number is plotted against Mach number. The trends shown in figure 5 indicate that the predicted effect of Mach number on the laminar separation point as determined by the method of the present investigation is in substantial agreement with the results obtained by Stewartson and Howarth for the case of linearly decreasing velocity from the leading edge.

EFFECT OF MACH NUMBER ON THE LAMINAR SEPARATION POINT

In order to show the effect of Mach number on the position of laminar separation for a series of values of $F_c$, the nondimensional compressible velocity gradient, calculations have been made for Mach numbers varying from 0 to 10 and for nondimensional compressible velocity gradients $F_c$ varying from 0 to -0.12. These calculations are for velocity distributions represented by a region of uniform velocity equal to the maximum velocity followed by a region of uniformly decreasing velocity. The results of these calculations are presented in figure 6.
and show that the velocity recovery obtainable before laminar separation decreases as the Mach number increases for all nondimensional velocity gradients. For a given velocity gradient, therefore, the laminar separation point moves forward as the Mach number increases. Figure 6 also shows that the shape of the curves of velocity recovery against Mach number is not influenced to any large extent by the value of the velocity gradient.

It is important to point out that the individual curves presented in figure 6 show the effect on the laminar separation point of increasing the Mach number for fixed values of the velocity gradient. The problem of most practical interest, however, is that of a body having fixed geometry. In this case, the velocity gradients vary as the Mach number varies. Thus, when the laminar separation point on a given body for different Mach numbers is calculated, the value of the compressible velocity gradient $F_c$ varies with Mach number. In general, adverse velocity gradients become steeper as the Mach number increases so that, for a given body, the forward movement of the laminar separation point with increasing Mach number would be greater than is indicated in figure 6 for a given velocity gradient.

A word of caution should be added with regard to the type of flow field to which the method developed herein should be applied. It will be recalled that the method developed is based on the classical boundary-layer assumptions. These assumptions state that the rate of change of velocity in the $x$-direction must be small with respect to its rate of change in the $y$-direction. The method obviously cannot be applied in the vicinity of a discontinuity in velocity (pressure) such as that associated with the presence of a shock wave.

CONCLUDING REMARKS

A method has been developed for rapidly estimating the separation point of a laminar boundary layer in a compressible flow. The method consists of an extension of Von Doenhoff's simplified solution for the incompressible case (NACA TN 671) and makes use of a set of transforms derived by Stewartson (Proc. Roy. Soc., 1949) which permit, under certain assumptions, the expression of compressible laminar layers in terms of equivalent incompressible laminar layers. The method developed is generally applicable to any two-dimensional flow in which the classical boundary-layer assumptions are satisfied. The dependence of the method upon the boundary-layer assumptions, of course, means that it should not be applied to determine whether the pressure rise through a shock wave causes separation.
Calculations of the laminar separation point for a wide range of Mach number and velocity gradient indicate that, for all velocity gradients, the amount of velocity recovery possible before laminar separation occurs decreases as the Mach number increases.

Langley Aeronautical Laboratory, National Advisory Committee for Aeronautics, Langley Field, Va., November 19, 1952.

REFERENCES


Figure 1.- Incompressible velocity decrement \( \Delta U_{1}/U_{10} \) obtainable before laminar separation occurs as a function of the velocity gradient \( F_{1} \) (ref. 3).
Figure 2. The quantity $1 + \frac{\gamma - 1}{2} M_0^2$ as a function of Mach number.
Figure 3.—Incompressible velocity ratio as a function of compressible velocity ratio for different Mach numbers.
Figure 4.- Comparison of the results of the present method with those of Howarth and Stewartson for the case when $F = 0$. 

Howarth's Kármán-Pohlhausen type of analysis (ref. 4) 

Stewartson's application of Howarth's method (ref. 5) 

Present method
Figure 5.- Comparison of effect of Mach number on laminar separation as predicted by the present method and the methods of Howarth and Stewartson for $F = 0$. 

Howarth's Karman-Pohlhausen type of analysis (ref. 4)

Stewartson's application of Howarth's method (ref. 6)
Figure 6. - Compressible velocity decrement $\Delta U_c / U_c$ obtainable before separation as a function of Mach number for various velocity gradients $F_c$. 