FLUTTER AND OSCILLATING AIR-FORCE CALCULATIONS FOR AN AIRFOIL IN A TWO-DIMENSIONAL SUPersonic FLOW

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SUMMARY

A connected account is given of the Possio theory of nonstationary flow for small disturbances in a two-dimensional supersonic flow and of its application to the determination of the aerodynamic forces on an oscillating airfoil. Further application is made to the problem of wing flutter in the degrees of freedom - torsion, bending, and aileron torsion. Numerical tables for flutter calculations are provided for various values of the Mach number greater than unity. Results for bending-torsion wing flutter are shown in figures and discussed. The static instabilities of divergence and aileron reversal are examined as is a one-degree-of-freedom case of torsional oscillatory instability.

INTRODUCTION

The problem of flutter or aerodynamic instability for high-speed aircraft is of considerable importance and hence interest is directed to the aerodynamic problem of the oscillating airfoil moving forward at high speed. Although for conventional aircraft the subsonic and the near-sonic or transonic speed ranges are still of main interest, the purely supersonic speed range is becoming increasingly significant.

A theoretical treatment of the oscillating airfoil, of infinite aspect ratio, moving at supersonic speed has been given by Possio (reference 1). This treatment is based on the theory of small perturbations to the main stream, thus is essentially an acoustic theory, and leads to linearization of the equation satisfied by the velocity potential. The airfoil is therefore assumed to
be very thin, at small angle of attack, and the flow is assumed nonviscous, unseparated, and free from strong shocks.

The small-disturbance linearized theory, being much less complicated than a more rigorous nonlinear theory, is to be regarded as an expedient which allows an initial theoretical solution. The theory permits the occurrence of weak (infinitesimally small) shocks and thus the basic trends and effects of the parameters of the simplified problem can be indicated. The theory reduces to that of Ackeret in the stationary (static) case and, like it, is not expected to be valid too near \( M = 1 \). In view of the restrictions and assumptions in the analysis important modifications may be required in certain cases for thick finite airfoils, but even here the simple theory for thin wing sections may serve as a basis.

In addition to Possio's brief work an equivalent extended treatment has been given by Borbely (reference 2) which utilizes contour integrations to carry out the solution of the partial differential equation for the velocity potential according to the Heaviside operator method or Laplace transform method. Recently, another equivalent treatment has been given in England by Temple and Jahn employing the method of characteristics. In reference 1 a few curves are given for the aerodynamic coefficients but no numerical values are tabulated. Reference 2 contains no numerical results. Temple and Jahn recognize the lack of numerical results and supply some initial calculations for the functions necessary for flutter calculations.

A paper has recently appeared by Schwarz (reference 3) devoted to computing and tabulating the key mathematical functions that arise in the theory. The present paper makes use of reference 3 to supply more extensive numerical tables for application of the theory. The formulas of the theory are recast in more familiar form for application to the flutter problem and a series of calculations on bonding-torsion flutter are carried out and discussed. The performance of similar calculations for wing-aileron flutter is indicated. Brief discussions also are given of the static instabilities, divergence and aileron reversal, and of a one-degree-of-freedom torsional oscillatory instability.
For completeness, a connected account of the Possio theory is presented since the original presentation in Italian is quite terse and also since it is believed that this treatment is the simplest and most suitable for general extensions. The extension of its application to include the aileron is given.

AIR FORCES AND MOMENTS ON AN OSCILLATING AIRFOIL MOVING AT SUPERSONIC SPEED IN TWO-DIMENSIONAL FLOW

Differential Equation for the Velocity Potential

The differential equation satisfied by the velocity potential in fixed coordinates in the case of infinitesimal disturbances is the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi \quad (1)$$

where $c$ is the velocity of sound in the undisturbed medium. (For the adiabatic equation of state

$$c^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho}.$$ )

Referred to a system of rectangular coordinates moving forward at a constant supersonic speed $v$ in the negative x-direction the wave equation satisfied by the velocity potential in two-dimensional flow becomes

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{2v}{c^2} \frac{\partial^2 \phi}{\partial x \partial t} + \left[ \frac{v^2}{c^2} - 1 \right] \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (2)$$

It is proposed to treat the effect of a slightly cambered thin airfoil moving forward at a supersonic speed $v$ at small (zero) angle of attack as that of a distribution of small disturbances placed along the x-axis and hence to utilize equation (2). The velocity components in the x- and y-directions relative to the moving airfoil are, respectively,
\[ v_x = \frac{\partial \phi}{\partial x} \]

and

\[ v_y = \frac{\partial \phi}{\partial y} \]

which may be considered the additional components to the main stream due to the disturbance created by the presence of the airfoil. Relative to coordinates fixed in space the velocity components are \( v + v_x \) and \( v_y \).

Effect of a Source

Equation (2) is linear and solutions are therefore additive. An important particular solution of equation (2) having the property of a source pulse is

\[ \phi_0 = \frac{A(\xi, \eta, T)}{\sqrt{c^2(t - T)^2 - [x - \xi - v(t - T)]^2 - (y - \eta)^2}} \]  

(3)

This solution may be considered to give the effect at a point \( (x, y) \) at time \( t \) of a disturbance of magnitude \( A \) originating at a point \( (\xi, \eta) \) at an earlier time \( T \). The potential \( \phi_0 \) is thus a retarded potential and the elapsed time at \( (x, y) \) since the creation of the disturbance is \( \tau = t - T \).

Unlike the situation for a subsonic flow, for a supersonic flow the effect of the disturbance is propagated only downstream, that is, the point being influenced \( (x, y) \) is always considered to be aft of the point of disturbance \( (\xi, \eta) \). Equation (3) is thus valid in the angular region with vertex at \( (\xi, \eta) \) and bounded by two straight lines making the Mach angles

\[ \pm \sin^{-1} \frac{12}{11} \]

with respect to the x-axis.

(See fig. 1.) Upstream from this angular region the value of \( \phi_0 \) is zero. It follows also that disturbances in the wake need not be considered and the solution to
the boundary problem may be attempted by a distribution of potentials of the type $\tilde{\phi}$ taken along the projection of the airfoil on the $x$-axis.

A disturbance at $(\xi, \eta)$ created at time $T$ is first felt at a point $(x, y)$ after a certain time $T_1$ has elapsed. The point $(x, y)$ penetrates the wave front of the disturbed region and because it is moving at a speed greater than that of the wave front it emerges from the disturbed region at a later time $T_2$.

Thus, the duration of this initial disturbance at $(x, y)$ is $T_2 - T_1$. (See fig. 2.) The transition at $(x, y)$ from a region of quiescence to a region of disturbance and vice versa is associated with the vanishing of the denominator in equation (3). The values of $T_1$ and $T_2$ for a disturbance created on the axis $\eta = 0$ are thus given by

$$T_{1,2} = \frac{\tilde{w}(x - \xi) \pm \sqrt{(x - \xi)^2 - y^2(M^2 - 1)}}{c(M^2 - 1)} \quad (4)$$

where the minus sign is associated with $T_1$ and the plus sign with $T_2$, and where $M = \frac{v}{\tilde{c}}$. It may also be observed that a negative quantity under the radical sign in equation (3) is to be interpreted as associated with an undisturbed region. (that is, with $\tilde{\phi} = 0$).

Potential for a Distribution of Sources

The total effect at any point $(x, y)$ is the sum of the effects of disturbances originating between the leading edge $\xi = 0$ and the intersection of the Mach line through $(x, y)$ with the $\xi$-axis

$$\xi = \xi_1 = x - y\sqrt{\tilde{w}^2 - 1}$$

(since only disturbances created forward of the Mach angle region can affect $(x, y)$; see fig. 3).
The total potential at \((x, y)\) at any time \(t\) is thus given by

\[
\phi(x, y, t) = \int_0^{\tau_2} \int_{\tau_1}^{\tau_2} \frac{A(\xi, 0, t - \tau)}{\sqrt{c^2 \tau_2 - (x - \xi - \tau v)^2 - y^2}} \, d\tau \, d\xi
\]

\[
= \frac{1}{\sqrt{v^2 - c^2}} \int_0^{\tau_2} \int_{\tau_1}^{\tau_2} \frac{A(\xi, 0, t - \tau)}{\sqrt{(\tau - \tau_1)(\tau_2 - \tau)}} \, d\tau \, d\xi \quad (5)
\]

Boundary Condition and Strength of Distribution

The function \(A(\xi, 0, t - \tau)\) giving the magnitude of the source distribution is now to be determined by the usual boundary condition of tangential flow along the airfoil. If the ordinate of any point of the mean line defining the airfoil is given as \(y = y_m(x, t)\) the boundary condition may be written

\[
\left( \frac{\partial \phi}{\partial y} \right)_{y=0} = w(x, t) = \frac{\partial y_m}{\partial t}
\]

\[
= v \frac{\partial y_m}{\partial x} + \frac{\partial y_m}{\partial t} \quad (6)
\]

where \(w(x, t)\) thus represents the vertical velocity induced by the source distribution in order to realize tangential flow at the airfoil boundary. (In the stationary case - Ackeret treatment - the two surfaces of the airfoil may be considered as acting independently, which can also be done for the nonstationary case. However, for the purpose of obtaining the oscillating forces in the linear treatment it is sufficient to consider separately the upper and lower sides of only the mean line.)
The evaluation of \( \frac{\partial \phi}{\partial y} \) as \( y \) approaches zero may be readily obtained by use of the variable \( \theta \) instead of \( \tau \) where \( 2\tau = (\tau_2 - \tau_1) \cos \theta + \tau_2 + \tau_1 \). This substitution in equation (5) yields
\[
\phi = \frac{1}{\sqrt{v^2 - c^2}} \int_0^\infty \int_0^\pi A(\xi, 0, t - \frac{\tau_2 + \tau_1}{2} - \frac{\tau_2 - \tau_1}{2} \cos \theta) d\theta \, d\xi
\]

By differentiation with regard to \( y \) and with the aid of an integration by parts,
\[
\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{v^2 - c^2}} \frac{\partial \xi'}{\partial y} \pi A(\xi, 0, t - \frac{My}{c\sqrt{M^2 - 1}})
\]
\[
+ \frac{1}{\sqrt{v^2 - c^2}} \frac{y}{c\sqrt{M^2 - 1}} \int_0^\infty \int_0^\pi \frac{\partial^2 A}{\partial t^2} \sin^2 \theta \, d\theta \, d\xi
\]

Since \( \xi' = x - y\sqrt{M^2 - 1} \), there results in the limit as \( y \) approaches zero on the positive side, the important relation
\[
\left( \frac{\partial \phi}{\partial y} \right)_{y=0} = -\frac{\pi}{c} A(x, 0, t)
\]
or, briefly,
\[
A(x, t) = -\frac{c}{\pi} w(x, t) \quad (7)
\]

For \( y \) approaching 0 on the negative side an equal and opposite result is obtained and hence the distribution of singularities to be utilized to replace the airfoil is of the source-sink type. Thus \( \phi \) is to be understood in the subsequent analysis to be prefixed
by a ± sign, + for the upper side and - for the lower side.

The total potential for \( y = 0 \) may now be expressed by means of equations (5) and (7) as

\[
\phi(x, t) = -\frac{1}{\pi} \frac{1}{\sqrt{m^2 - 1}} \int_0^{\gamma_2} \int_{\tau_1}^{\gamma_2} \frac{w(\xi, t - \tau)}{\sqrt{\tau - \tau_1}(\gamma_2 - \tau)} \, d\tau \, d\xi \tag{8}
\]

where, from equation (4) with \( y = 0 \),

\[
\tau_1 = \frac{x - \xi}{c} \frac{1}{M + 1}
\]

and

\[
\tau_2 = \frac{x - \xi}{c} \frac{1}{M - 1}
\]

**Application to Oscillating Airfoil**

The general result given by equation (8) may now be applied for definiteness to the case of an airfoil performing small sinusoidal oscillations in several degrees of freedom. Let the wing undergo the following motions: a motion due to displacement \( h \) (velocity \( \dot{h} \)) in a vertical direction; a torsional motion consisting of a turning about \( x = x_0 \) with instantaneous angle of attack \( \alpha \); a rotation of an aileron about its hinge at \( x = x_1 \) with instantaneous aileron angle \( \beta \) measured with respect to \( \alpha \). (See fig. 1.)

In accordance with equation (6) the vertical velocity at any point \( x \) of the airfoil situated at \( 0 \leq x \leq 2b \) (of chord 2b and leading edge at \( x = 0 \)) is easily recognized to be

\[
w(x, t) = -[\ddot{h} + v\dot{\alpha} + (x - x_0)\ddot{\alpha} + v\ddot{\beta} + (x - x_1)\dddot{\beta}] \tag{9}
\]

where the \( \beta \)-terms are to be interpreted as zero for \( x < x_1 \) (and where the minus sign is introduced because
the vertical velocity \( w \) is positive upwards whereas the terms within the brackets are positive downwards).

It is convenient in treating sinusoidal motion to utilize the complex notation

\[
\begin{align*}
    h &= h_0 e^{i\omega t} \\
    a &= a_0 e^{i\omega t} \\
    \beta &= \beta_0 e^{i\omega t}
\end{align*}
\]

(10)

where \( h_0, a_0, \) and \( \beta_0 \) are complex amplitudes and hence include phase angles.

Since the further analysis is concerned only with exponential time variations of the type given in equation (10), the function \( w(\xi, t - \tau) \) occurring in equation (8) is of the form \( w(\xi) e^{i\omega(t-\tau)} \), which may also be written for convenience as \( w(\xi, t) e^{-i\omega \tau} \). The potential \( \phi \) given by equation (8) may now be written as

\[
\phi(x, t) = -\frac{1}{\sqrt{M^2 - 1}} \int_0^x w(\xi, t) I(\xi, x) d\xi \tag{11}
\]

where

\[
I(\xi, x) = \frac{1}{\pi} \int_{\tau_1}^{\pi/2} \frac{e^{-i\omega \tau}}{\sqrt{(\tau - \tau_1)(\tau_2 - \tau)}} d\tau
\]

The integration with regard to \( \tau \) may be readily performed by substitution of the variable \( \theta \) where

\[
2\tau = (\tau_2 - \tau_1) \cos \theta + \tau_2 - \tau_1.
\]

Then

\[
I(\xi, x) = \frac{1}{\pi} e^{-i\omega(\tau_2 + \tau_1)/2} \int_0^{\pi/2} e^{-i\omega \cos \theta(\tau_2 - \tau_1)/2} d\theta
\]
With \( \tau_1 \) and \( \tau_2 \) replaced by their values as given for equation (8) and with the aid of the Bessel function relation

\[
\frac{1}{\pi} \int_0^\pi e^{-i\lambda \cos \theta} \, d\theta = J_0(\lambda)
\]

it is recognized that

\[
I(\xi, x) = e^{-\frac{i\omega(x-\xi)}{c}} \frac{M}{M^2-1} J_0 \left( \frac{x-\xi}{c} \frac{\omega}{M^2-1} \right)
\]

(12)

Throughout the subsequent analysis it is convenient to employ the variables \( x \) and \( \xi \) in a new sense to mean nondimensional quantities obtained by dividing the old variables by the chord 2b. The retaining of the symbols \( x \) and \( \xi \) for the nondimensional variables should lead to no confusion.

The potential \( \phi \) of equation (11) is then

\[
\phi(x, t) = \frac{2b}{\sqrt{k_2 - 1}} \int_0^x \left[ \nu \alpha + \nu + 2b(\xi - x_0) \dot{\alpha} + \nu \dot{x} + 2b(\xi - x_1) \ddot{\xi} \right] I(\xi, x) \, d\xi
\]

(13)

where with the introduction of the important frequency parameters

\[
k = \frac{\omega b}{v}
\]

\[
\omega = \frac{2 \pi \nu^2}{M^2 - 1}
\]

the function \( I(\xi, x) \) becomes

\[
I(\xi, x) = e^{-\frac{i\omega(x-\xi)}{c}} J_0 \left( \frac{\omega}{M} (x - \xi) \right)
\]

(12')
Thus, $I(\xi, x)$ is a function of the variable $x - \xi$ and of two parameters $M$ and $\omega$, or, alternatively, $M$ and $k$.

It is desirable to express the potential $\phi$ as the sum of the separate effects due to position and motion of the airfoil associated with the individual terms in equation (13). Thus

$$\phi(x, t) = \phi_a + \phi_h + \phi_\dot{a} + \phi_\dot{\beta} + \phi_\ddot{\beta}$$

where

$$\phi_a = \frac{2b}{\sqrt{m^2 - 1}} \sqrt{a} \int_0^x I(\xi, x) \, d\xi$$

$$\phi_h = \frac{2b}{\sqrt{m^2 - 1}} \sqrt{h} \int_0^x I(\xi, x) \, d\xi$$

$$\phi_\dot{a} = \frac{1b^2}{\sqrt{m^2 - 1}} \dot{a} \int_0^x (\xi - x_0) I(\xi, x) \, d\xi$$

$$\phi_\dot{\beta} = \frac{2b}{\sqrt{m^2 - 1}} \sqrt{v_\beta} \int_{x_1}^x I(\xi, x) \, d\xi$$

$$\phi_\ddot{a} = \frac{1b^2}{\sqrt{m^2 - 1}} \ddot{a} \int_{x_1}^{x_1} (\xi - x_1) I(\xi, x) \, d\xi$$

$$\phi_\ddot{\beta} = \frac{2b}{\sqrt{m^2 - 1}} \sqrt{v_\beta} \int_{x_1}^{x_1} I(\xi, x) \, d\xi$$
Forces and Moments

The basic pressure formula in the theory of small disturbances is

\[ p = -\rho \frac{\partial \phi}{\partial t} \]

which in the present case of the moving airfoil may be expressed as

\[ p = -\rho \left( \frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x} \right) \]

where \( \rho \) is the density in the undisturbed medium. The local pressure difference on the airfoil surface between the upper and lower surfaces at any point \( x \) (nondimensional) is

\[ p' = -2\rho \left( \frac{\partial \phi}{\partial t} + \frac{v}{2b} \frac{\partial \phi}{\partial x} \right) \quad (15) \]

The total force (positive downward) on the airfoil is

\[ p = 2b \int_{x=0}^{1} p' \, dx \]

\[ = -2\rho v \int_{x=0}^{1} \frac{\partial \phi}{\partial x} \, dx - 4\rho b \int_{x=0}^{1} \phi \, dx \quad (16) \]

The moment (positive clockwise; fig. 4) on the entire airfoil about any point \( x_0 \) is

\[ M_x = 4b^2 \int_{x=0}^{1} (x - x_0) p' \, dx \]

\[ = -4\rho bv \int_{x=0}^{1} \frac{\partial \phi}{\partial x} (x - x_0) \, dx - 8\rho b^2 \int_{x=0}^{1} \phi (x - x_0) \, dx \quad (17) \]
Similarly, the moment (positive clockwise; fig. 1) on the aileron about the hinge point \( x_1 \) is

\[
M_p = \int_{x_1}^{1} (x - x_1) p' \, dx
\]

\[
= -4\rho bv \int_{x_1}^{1} \frac{\delta \phi}{\delta x} (x - x_1) \, dx - 8\rho b^2 \int_{x_1}^{1} \phi (x - x_1) \, dx \quad (18)
\]

In the further reduction of equations (16) to (18), with the potential \( \phi \) replaced by its separated form given in equation (14), the following sets of integral evaluations are required:

\[
\int_{0}^{1} \frac{\delta \phi}{\delta x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \varphi_1(M, k)
\]

\[
\int_{0}^{1} \frac{\delta \phi}{\delta x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \hat{a} \left[ r_2(M, k) - x_0 r_1(M, k) \right]
\]

\[
\int_{x_1}^{1} \frac{\delta \phi}{\delta x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \psi_1(M, k, x_1)
\]

\[
\int_{x_1}^{1} \frac{\delta \phi}{\delta x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \psi_2(M, k, x_1)
\]

\[
\int_{0}^{1} \phi \, dx = \frac{2b}{\sqrt{M^2 - 1}} \varphi_2(M, k)
\]

\[
\int_{0}^{1} \phi \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \hat{a} \left[ \frac{1}{2} r_3(M, k) - x_0 r_2(M, k) \right]
\]
\[ \int_{x_1}^{l} \phi_{\beta} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v_{\beta} \, t_2(M, k, x_1) \]

\[ \int_{x_1}^{l} \phi_{\beta} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \hat{v} \, \frac{1}{2} \, t_3(M, k, x_1) \]

\[ \int_{0}^{l} \frac{\partial \phi_{\alpha}}{\partial x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v a \, q_1(M, k) \]

\[ \int_{0}^{l} \frac{\partial \phi_{\beta}}{\partial x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \hat{v} \left[ \frac{1}{2} \, q_2(M, k) - x_0 \, q_1(M, k) \right] \]

\[ \int_{x_1}^{l} \frac{\partial \phi_{\beta}}{\partial x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v_{\beta} \left[ q_2(M, k, x_1) + x_1 \, t_1(M, k, x_1) \right] \]

\[ \int_{x_1}^{l} \frac{\partial \phi_{\beta}}{\partial x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \hat{v} \left[ \frac{1}{2} \, q_2(M, k, x_1) + x_1 \, t_2(M, k, x_1) \right] \]

\[ \int_{0}^{l} \phi_{\alpha} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v a \, \frac{1}{2} \, q_2(M, k) \]

\[ \int_{0}^{l} \phi_{\alpha} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \hat{v} \left[ \frac{1}{6} \, q_3(M, k) - x_0 \, q_2(M, k) \right] \]
\[
\int_{x_1}^{1} \varphi_x \ dx = \frac{2b}{\sqrt{M^2 - 1}} \nu \beta \left[ \frac{1}{2} s_2(M, k, x_1) + x_1 t_2(M, k, x_1) \right]
\]

\[
\int_{x_1}^{1} \varphi \ dx = \frac{4b^2}{\sqrt{\mu^2 - 1}} \beta \left[ \frac{1}{2} s_3(M, k, x_1) + \frac{1}{2} x_1 t_3(M, k, x_1) \right]
\]

\[
\int_{x_1}^{1} \frac{\partial \varphi_a}{\partial x} (x - x_1) \ dx = \frac{2b}{\sqrt{M^2 - 1}} \nu \alpha p_1(\hat{M}, k, x_1)
\]

\[
\int_{x_1}^{1} \frac{\partial \varphi_a}{\partial x} (x - x_1) \ dx = \frac{4b^2}{\sqrt{\mu^2 - 1}} \alpha \left[ \frac{1}{2} p_2(M, k, x_1) - x_0 p_1(\nu, k, x_1) \right]
\]

\[
\int_{x_1}^{1} \frac{\partial \varphi_a}{\partial x} (x - x_1) \ dx = \frac{2b}{\sqrt{\mu^2 - 1}} \nu \beta s_1(M, k, x_1)
\]

\[
\int_{x_1}^{1} \frac{\partial \varphi_a}{\partial x} (x - x_1) \ dx = \frac{4b^2}{\sqrt{\mu^2 - 1}} \beta \ \frac{1}{2} s_2(\hat{\nu}, k, x_1)
\]
\[
\int_{x_1}^{1} \phi_a(x - x_1) \, dx = \frac{2b}{\sqrt{M^2 - 1}} \nu \frac{1}{2} q_a(M, k, x_1)
\]

\[
\int_{x_1}^{1} \phi_x(x - x_1) \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \alpha \left[ \frac{1}{6} q_3(M, k, x_1) - \frac{1}{2} p_2(M, k, x_1) \right]
\]

\[
\int_{x_1}^{1} \phi_\beta(x - x_1) \, dx = \frac{2b}{\sqrt{M^2 - 1}} \nu \beta \frac{1}{2} q_2(M, k, x_1)
\]

\[
\int_{x_1}^{1} \phi_\gamma(x - x_1) \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \beta \frac{1}{6} q_4(M, k, x_1)
\]

The functions defined by the foregoing integral evaluations are further discussed in the following section; first, however, the force and moments (equations (16) to (18)) are given in their final forms as

\[
P = -\frac{4b^2}{\sqrt{M^2 - 1}} \left[ v(va + \hat{h} - 2bx_0 \hat{a}) r_1 + 2b(2v \hat{a} + \hat{h} - 2bx_0 \hat{a}) r_2 
\right.
\]

\[
+ 4b^2 \hat{a} \frac{r_3}{2} + v^2 \beta t_1 + 4bv \hat{a} t_2 + 4b^2 \beta \frac{t_3}{2} \right] \quad (16')
\]

\[
M_\alpha = -\frac{4b^2}{\sqrt{M^2 - 1}} \left[ v(va + \hat{h} - 2bx_0 \hat{a}) q_1 + 2b(2v \hat{a} + \hat{h} - 2bx_0 \hat{a}) \frac{q_3}{2} 
\right.
\]

\[
+ 4b^2 \hat{a} \frac{q_2}{2} + v^2 \beta (s_1 + z_1 t_1) + 4bv \hat{a} \left( \frac{s_2}{2} + x_1 t_2 \right) 
\]

\[
+ 4b^2 \beta \left( \frac{3}{6} + x_1 \frac{t_2}{2} \right) \right] - 2bx_0 P \quad (17')
\]
\[ M_\rho = -\frac{6 pb^2}{\sqrt{M^2 - 1}} \left[ n (va + h - 2bx_0 \dot{\alpha}) p_1 + 2b (2v \dot{a} + h - 2bx_0 \dot{\alpha}) \frac{p_2}{2} \right. \\
\left. + b^2 \dot{a} \frac{P_3}{6} + v^2 p_3 + 4b v \dot{\phi} \frac{\beta_2}{2} + 4b^2 \ddot{\phi} \frac{\beta_3}{6} \right] \]  \\
(18')

Reduction and Evaluation of Foregoing Integrals

It is convenient to introduce the substitution \( u = x - \frac{\zeta}{\xi} \) and to express the function \( I(\xi, x) \) (equation (12')) as

\[ I(\xi, x) = I(u) = e^{-i\omega u} \int_0^{\infty} \frac{e^{-i\omega u}}{\omega} \, \omega \, u \, I(u) \, du \]  \\
(19)

The various functions defined by the foregoing sets of integrals may now be expressed as follows:

\[ r_1(M, k) = \int_0^1 I(u) \, du \]

\[ r_2(M, k) = \int_0^1 \int_0^x I(u) \, du \, dx \]

\[ r_3(M, k) = 2 \int_0^1 \int_0^x (x - u) I(u) \, du \, dx \]

\[ q_1(M, k) = \int_0^1 u \, I(u) \, du \]

\[ q_2(M, k) = 2 \int_0^1 \int_0^x x \, I(u) \, du \, dx \]

\[ q_3(M, k) = 6 \int_0^1 \int_0^x x(x - u) I(u) \, du \, dx \]
\[ p_1(M, k, x_1) = \int_{x_1}^{1} (u - x_1) I(u) \, du \]

\[ p_2(M, k, x_1) = 2 \int_{x_1}^{1} \int_{0}^{x} (x - x_1) I(u) \, du \, dx \]

\[ p_3(M, k, x_1) = 6 \int_{x_1}^{1} \int_{0}^{x} (x - x_1)(x - u) I(u) \, du \, dx \]

\[ t_1(M, k, x_1) = \int_{0}^{l-x_1} I(u) \, du \]

\[ t_2(M, k, x_1) = \int_{0}^{l-x_1} \int_{0}^{x} I(u) \, du \, dx \]

\[ t_3(M, k, x_1) = 2 \int_{0}^{l-x_1} \int_{0}^{x} (x - u) I(u) \, du \, dx \]

\[ s_1(M, k, x_1) = \int_{0}^{l-x_1} u \, I(u) \, du \]

\[ s_2(M, k, x_1) = 2 \int_{0}^{l-x_1} \int_{0}^{x} x \, I(u) \, du \, dx \]

\[ s_3(M, k, x_1) = 6 \int_{0}^{l-x_1} \int_{0}^{x} x(x - u) \, I(u) \, du \, dx \]
Borobely (reference 2) has shown by means of reduction formulas that the six r- and q-functions may be obtained from a single integral. In a similar manner it may be indicated how the foregoing 15 functions may be obtained from the evaluation of the same integral. The reduction is accomplished in two stages. First, consider integrals of the following type:

\[ f_\lambda = f_\lambda(M, \bar{w}) = \int_0^1 I(u) u^\lambda \, du \]
\[ g_\lambda = g_\lambda(M, \bar{w}x_1) = \frac{1}{\lambda + 1} \int_0^{x_1} I(u) u^\lambda \, du \]  \( \quad (20) \)
\[ h_\lambda = h_\lambda(M, \bar{w}(1 - x_1)) = \frac{1}{(1 - x_1)^{\lambda + 1}} \int_0^{1-x_1} I(u) u^\lambda \, du \]

By integration by parts it can be readily verified that the following relations hold:

\[ r_1 = f_0 \]
\[ r_2 = f_0 - f_1 \]
\[ r_3 = f_0 - 2f_1 + f_2 \]
\[ q_1 = f_1 \]
\[ q_2 = f_0 - f_2 \]
\[ q_3 = 2f_0 - 3f_1 + f_3 \]
\begin{align*}
p_1 &= q_1 - x_1 r_1 + x_1^2 (g_0 - g_1) \\
p_2 &= q_2 - 2x_1 r_2 + x_1^3 (g_0 - 2g_1 + g_2) \\
p_3 &= q_3 - 3x_1 r_3 + x_1^4 (g_0 - 3g_1 + 3g_2 - g_3)
\end{align*}

\begin{align*}
t_1 &= (1 - x_1) h_0 \\
t_2 &= (1 - x_1)^2 (h_0 - h_1) \\
t_3 &= (1 - x_1)^3 (h_0 - 2h_1 + h_2) \\
s_1 &= (1 - x_1)^2 h_1 \\
s_2 &= (1 - x_1)^3 (h_0 - h_2) \\
s_3 &= (1 - x_1)^4 (2h_0 - 3h_1 + h_2)
\end{align*}

The final stage in the reduction of these functions is to utilize the following recursion formula (reference 2) obtained by integration by parts:

\[
\frac{M^2 - 1}{M^2} \frac{1}{\bar{\omega}} f_{\lambda}(M, \bar{\omega}) = \left[ 1 + (1 - \lambda) \frac{1}{\bar{\omega}} \right] e^{-i\bar{\omega}} J_0 \left( \frac{\bar{\omega}}{M} \right) - \frac{1}{M} e^{-i\bar{\omega}} J_1 \left( \frac{\bar{\omega}}{M} \right) \\
+ i(1 - 2\lambda) f'_{\lambda-1}(M, \bar{\omega}) \\
+ (1 - \lambda \frac{1}{\bar{\omega}}) f'_{\lambda-2}(M, \bar{\omega})
\]  

(21)

where \( \lambda \geq 1 \) and \( f \) with a negative subscript is to be interpreted as zero.
The function $f_\lambda(M, \bar{\omega})$ may clearly refer also to the foregoing $g$- and $h$-functions, if $\bar{\omega}$ is replaced by the appropriate parameter, namely, $\bar{\omega}x_0$ for $g_\lambda$ and $\bar{\omega}(1 - x_0)$ for $h_\lambda$. (See equations (20).) The recursion relation (equation (21)) thus reduces the various functions to the single function

$$f_0(M, \bar{\omega}) = \frac{1}{\bar{\omega}} \int_0^\infty e^{-iu} J_0\left(\frac{u}{M}\right) du \quad (22)$$

which is therefore the only integral needed in the evaluation of the forces and moments.

The important integral in equation (22) has been recently made the subject of a mathematical investigation by Schwarz (reference 1). Schwarz gives tables of the values of its real and imaginary parts to eight decimal places for $0 \leq \bar{\omega} \leq 5$ and for $1 \leq M \leq 10$ for conveniently small intervals. For values of $\bar{\omega} > 5$ not given in Schwarz' tables, the function $f_0$ may be evaluated by means of the following series development (reference 2):

$$f_0(M, \bar{\omega}) = e^{-i\bar{\omega}} \sum_{n=0}^{\infty} \left(\frac{\bar{\omega}^2 - 1}{\bar{\omega}^2}\right)^n \frac{1}{2^n n!(2n+1)} \left[J_n(\bar{\omega}) + iJ_{n+1}(\bar{\omega})\right] \quad (23)$$

Table I gives values of the functions $f_C(N, \bar{\omega})$ based on the tables of Schwarz and on equation (23) for selected values of the Mach number $M = \frac{10}{9}, \frac{5}{4}, \frac{10}{7}, \frac{5}{3}, 2, \frac{5}{3}, \frac{10}{3}$, and 5 and for various appropriate values of $\bar{\omega}$ (or $\frac{1}{k}$). Later use is made of the values given in table I for obtaining tables for flutter calculations.
EQUATIONS OF MOTION AND DETERMINANTAL EQUATION FOR FLUTTER CONDITION

The equations of motion and the border-line condition of unstable equilibrium yielding the flutter speed and frequency may be obtained exactly as in the incompressible case treated, for example, in reference 4. The two-dimensional treatment (infinite aspect ratio) is retained herein. Modifications due to assumed vibration modes of the finite wing may of course be introduced as in current practice (for example, reference 5). The modification of the forces and moments due to the three-dimensional nature of the flow is a more difficult problem which remains to be studied.

The equilibrium of the vertical forces, of the moments about the torsional axis \( x = x_0 \), and of the moments on the aileron about its hinge \( x = x_1 \) yields the three equations,

\[
\begin{align*}
\ddot{h} M + \dot{\beta} S_\alpha + \dot{\alpha} S_\beta + \dot{c}_h = & p \\
\ddot{\beta} I_\alpha + \ddot{\alpha} \left[ I_\beta + 2b(x_1 - x_0) S_\beta \right] + \dot{h} S_\alpha + \alpha c_\alpha = & M_\alpha \\
\ddot{\beta} I_\beta + \ddot{\beta} \left[ I_\beta + 2b(x_1 - x_0) S_\beta \right] + \dot{h} S_\beta + \beta c_\beta = & M_\beta
\end{align*}
\]

(24)

where the various parameters are defined in the list of notation. (See appendix.)

In order to define the border-line condition of unstable equilibrium separating damped and undamped oscillations, the variables \( h_0 \), \( \alpha_0 \), and \( \beta_0 \) are used in the sinusoidal exponential form given in equation (10). For the desired condition, it is necessary that the equations (24) have a (nontrivial) solution for the complex amplitudes \( h_0 \), \( \alpha_0 \), and \( \beta_0 \), or that the following determinantal equation hold:
\[
\begin{vmatrix}
\overline{A}_{ch} & A_{c\alpha} & A_{c\beta} \\
A_{ah} & \overline{A}_{a\alpha} & A_{a\beta} \\
A_{bh} & A_{b\alpha} & \overline{A}_{b\beta}
\end{vmatrix} = 0
\] (25)

where the complex elements of the determinant in separated form are

\[
\begin{align*}
\overline{A}_{ch} &= \Omega_n x - \mu + L_1 + iL_2 \\
A_{c\alpha} &= -i\mu x_\alpha + L_3 + iL_4 \\
A_{c\beta} &= -i\mu x_\beta + L_5 + iL_6 \\
A_{ah} &= -i\mu x_\alpha + M_1 + iM_2 \\
\overline{A}_{a\alpha} &= \Omega_{a\alpha} x - \mu r_\alpha^2 + M_3 + iM_4 \\
A_{a\beta} &= -i\mu \left[ r_\beta^2 + 2(x_1 - x_0)x_\beta \right] + M_5 + iM_6 \\
A_{bh} &= -i\mu x_\beta + N_1 + iN_2 \\
A_{b\alpha} &= -i\mu \left[ r_\beta^2 + 2(x_1 - x_0)x_\beta \right] + N_3 + iN_4 \\
\overline{A}_{b\beta} &= \Omega_{b\beta} x - \mu r_\beta^2 + N_5 + iN_6
\end{align*}
\]

and where the \( L \) 's, \( M \) 's, and \( N \) 's are defined by the force and moment equations (16'), (17'), and (18') expressed in the following forms:
Hence,

\[ \begin{align*}
L_1 + iL_2 &= \frac{1}{\sqrt{\eta^2 - 1}} \left( -2r_2 + \frac{i}{k} r_1 \right), \\
L_3 + iL_4 &= \frac{1}{\sqrt{\eta^2 - 1}} \left[ -2r_3 + \frac{21}{k} r_2 - \frac{i}{k} \left(-2r_2 + \frac{i}{k} r_1 \right) \\
&\quad - 2x_0 \left(-2r_2 + \frac{i}{k} r_1 \right) \\
L_5 + iL_6 &= \frac{1}{\sqrt{\eta^2 - 1}} \left[-2t_3 + \frac{21}{k} t_2 - \frac{i}{k} \left(-2t_2 + \frac{i}{k} t_1 \right) \right] \\
M_1 + iM_2 &= \frac{1}{\sqrt{\eta^2 - 1}} \left[ -2q_2 + \frac{21}{k} q_1 - 2x_0 \left(-2r_2 + \frac{i}{k} r_1 \right) \right] \\
M_3 + iM_4 &= \frac{1}{\sqrt{\eta^2 - 1}} \left[ \frac{1}{3} q_3 + \frac{21}{k} q_2 - \frac{i}{k} \left(-2q_2 + \frac{21}{k} q_1 \right) \\
&\quad - 2x_0 \left(-2r_3 + \frac{21}{k} r_2 - \frac{i}{k} \left(-2r_2 + \frac{i}{k} r_1 \right) \right) - 2q_2 + \frac{21}{k} q_1 \\
&\quad - 2x_0 \left(-2r_2 + \frac{i}{k} r_1 \right) \right] \\
M_5 + iM_6 &= \frac{1}{\sqrt{\eta^2 - 1}} \left[ \frac{1}{3} s_3 + \frac{21}{k} s_2 - \frac{i}{k} \left(-2s_2 + \frac{21}{k} s_1 \right) \\
&\quad + 2(x_1 - x_0) \left[-2t_3 + \frac{21}{k} t_2 - \frac{i}{k} \left(-2t_2 + \frac{i}{k} t_1 \right) \right] \right].
\end{align*} \]
The determinantal equation (25) with the foregoing complex elements is equivalent to two real simultaneous equations and hence may be solved for two unknowns. In a given case the usual unknowns are the flutter speed \( v \) and the flutter frequency \( \omega \) or, more conveniently, the related nondimensional parameters \( X \) and \( 1/k \). The parameter \( X \) appears linearly and only in the major diagonal elements (with bars), while the parameter \( 1/k \) appears transcendentally in every element of the determinant. Hence an obvious procedure though not the simplest for obtaining the simultaneous solutions of the two equations is to fix values of \( 1/k \), to solve for the roots of the two polynomials in \( X \), to plot graphically these roots against \( 1/k \), and to note the points of intersection.

In a systematic numerical study of flutter any two parameters may be utilized as unknowns instead of \( X \) and \( 1/k \), which is often more convenient. A discussion of such procedure and the use of a method of elimination for simplifying the calculations is given in the appendix of reference 6.

The application to the two-degree-of-freedom subcase of bending-torsion flutter is treated more fully in the following section.
APPLICATION TO BENDING-TORSION FLUTTER

The determinantal equation in the two degrees of freedom $h$ and $\alpha$ is

$$\begin{vmatrix} \bar{A}_{ch} & \bar{A}_{c\alpha} \\ A_{ah} & \bar{A}_{a\alpha} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \Omega_n \Omega_a X^2 - \mu + L_1 + iL_2 & -\mu x_\alpha + L_3 + iL_4 \\ -\mu x_\alpha + M_1 + iM_2 & \Omega_a X - \mu r_\alpha^2 + M_3 + iM_4 \end{vmatrix} = 0 \quad (27)$$

The two equations in $X$ obtained by equating the real and imaginary parts separately to zero are

$$\Omega_n \Omega_a X^2 + \left[ \Omega_a (L_1 - \mu) + \Omega_n (M_3 - \mu r_\alpha^2) \right] X + C_R = 0$$

and

$$\left( \Omega_a L_2 + \Omega_n M_4 \right) X + C_I = 0 \quad (27')$$

where

$$C_R = \mu \left[ x_\alpha (M_1 + L_3) - (M_3 - \mu r_\alpha^2) - L_1 r_\alpha^2 - \mu x_\alpha^2 \right] + D_R$$

and

$$C_I = \mu \left[ x_\alpha (L_2 - M_4) - M_4 - L_2 r_\alpha^2 \right] + D_I$$

where

$$D_R = L_1 M_3 - L_3 M_1 - L_2 M_4 + L_4 M_2$$

and

$$D_I = L_1 M_4 - L_4 M_1 + L_2 M_3 - L_3 M_2$$
For convenience in numerical tabulation, it is desirable to introduce primed quantities, independent of the parameter $x_0$, defined by the following relations:

\[
\begin{align*}
L_3 &= L_3' - 2x_0L_1 \\
L_4 &= L_4' - 2x_0L_2 \\
M_1 &= M_1' - 2x_0L_1 \\
M_2 &= M_2' - 2x_0L_2 \\
M_3 &= M_3' - 2x_0\left[(M_1' + L_3') - 2x_0L_1\right] \\
M_4 &= M_4' - 2x_0\left[(M_2' + L_4') - 2x_0L_2\right]
\end{align*}
\]

In table II convenient expressions for the quantities $L_1$, $L_2$, $L_3'$, $L_4'$, $M_1'$, $M_2'$, $M_3'$, and $M_4'$ are given and tabulated together with the combinations $M_1' + L_3'$ and $M_2' + L_4'$. Clearly these quantities depend on the function $f_0$ given in table I and hence the tabulation is made for the same values of $M$ and $l/k$ (or $\bar{W}$). In addition, table II contains values for the quantities $D_R$ and $D_I$ which, in fact, are independent of $x_0$ and may be expressed as

\[
D_R = L_1M_3' - L_3'M_1' - L_2M_4' + L_4'M_2'
\]

and

\[
D_I = L_1M_4' - L_4'M_1' + L_2M_3' - L_3'M_2'
\]

The numerical application in the case of bending-torsion flutter has been performed for various selected examples. In most of the calculations the numerical procedure was to fix values of $l/k$, eliminate $X$, and
solve for the parameter $x_\alpha$. Interpolation was also used to obtain additional points in order to improve the fairing of some of the curves. Values of $1/k$ less than 1 did not yield any flutter points in this procedure. Results are shown plotted in a number of figures (figs. 5 to 20); however, before these figures are discussed, it is desirable to explain the significance of the parameters and the numerical values assigned to them.

The parameter $\mu$ may be considered to signify the wing density and three selected values 3.927, 7.851, and 15.708 in the order of increasing wing density have been mainly used in the calculations. (These values correspond to values of $1/k = 5, 10$, and 20 in the notation of reference 4.) Alternatively, an increase in $\mu$ may be interpreted as an increase in altitude for a fixed wing density. The parameter $\mu$ may be expected to range up to high values for actual supersonic wings at high altitude. Only a few calculations, however, have been made for high values of $\mu$ ($\mu = 58.54$, $1/k = 100$; see fig. 18).

The parameter $\omega_n/\omega_\alpha$ is the ratio of the wing bending frequency to the wing torsional frequency and may be expected normally to be less than unity. The three values 0, 0.707, and 1 have been largely used in the calculations although other values up to 2 have also been studied.

The parameter $x_0$ represents the position of the elastic axis measured from the leading edge and the three values 0.4, 0.5, and 0.6 represent, respectively, positions at 40, 50, and 60 percent chord. (These values correspond to values of $a = -0.2, 0$, and 0.2 in the notation of reference 4.)

The parameter $x_\alpha$ represents the distance of the center of gravity from the elastic axis. For example, $x_\alpha = 0.2$ represents a position of the center of gravity 10 percent of the chord behind the elastic axis. In many of the calculations $x_\alpha$ has been regarded as variable.
The parameter $r_c^2$ represents the radius of gyration of the wing about the elastic axis and has been kept fixed at the value $r_c^2 = 0.25$.

The ordinate in figures 5 to 20 is the nondimensional flutter coefficient $v/b\omega_c$ where $b\omega_c$ is a convenient reference speed. This coefficient is also a function of the Mach number $M = \frac{v}{c}$ and several values of $M$ have been employed in the calculations.

In a plot of the flutter coefficient $v/b\omega_c$ against $M$, straight lines drawn from the origin at angle $\delta$ and intersecting the curves may be given an interesting interpretation (fig. 17). The slope of the line is given by $\frac{v/b\omega_c}{v/c} = \frac{b\omega_c}{c}$ or $\cot \delta = \frac{b\omega_c}{c}$. Thus, $\cot \delta$ is directly proportional to the product of the chord and the torsional frequency. The question of whether at a given value of $M$ the value of $b\omega_c$ which will just prevent flutter is also sufficient to prevent flutter at neighboring higher values of $M$ is answered by the simple criterion of whether $\cot \delta$ increases or decreases. In figure 17 two typical flutter curves are shown. In curve B the value of $b\omega_c$ just necessary to prevent flutter at a speed corresponding to the value of $M$ at $P_2$ is insufficient to prevent flutter at any higher value of $M$ for which the flutter curve is below the straight line $OP_2$. For the type of curve A a maximum value of $\delta$ occurs at the "design critical points" $P_1$. The values of $b\omega_c$ just necessary to prevent flutter at a speed corresponding to the value of $M$ at $P_1$ is also sufficient to prevent flutter at all higher speeds.

The following salient facts may be extracted by inspection of the figures. Flutter exists or is possible for various ranges of the parameters but, in general, compared with subsonic cases the ranges of the parameters yielding flutter are more restricted.

The chordwise position of the aerodynamic center, the center of the oscillating pressure, is an important
factor in the consideration of flutter. In the simple theory the midchord is the aerodynamic center for $M \gg 1$. For subsonic speeds, $M \ll 1$, the linearized theory indicates the quarter-chord position as the aerodynamic center. It should be expected that in the transonic region near $M = 1$ the aerodynamic center may shift considerably. From this point of view alone conclusions drawn from the simple theory for the range near $M = 1$ may require large modifications. The nature of the modifications may be roughly inferred by further experimental and theoretical study of the behavior of center-of-pressure locations.

For low values of the ratio of bending frequency to torsional frequency $\frac{\omega_b}{\omega_\alpha} \approx 0$ the position of the center of gravity relative to the aerodynamic center is important. For center-of-gravity positions forward of the midchord no flutter exists, whereas for positions behind the midchord there is a sharp decrease in the flutter coefficient from infinity; the position of the elastic axis influences the value of the flutter coefficient in this region, forward positions being more favorable (figs. 5(a) to 16(a)).

For values of $\frac{\omega_b}{\omega_\alpha} \approx 1$ the position of the center of gravity relative to the elastic axis becomes of more importance. For center-of-gravity positions forward of the elastic axis no flutter exists, whereas for positions behind the elastic axis flutter does occur, and a relative minimum coefficient appears for center-of-gravity positions only slightly (a few percent of the chord) behind the elastic axis.

The intermediate case, for which $\frac{\omega_b}{\omega_\alpha} = 0.707$, shows a blending of the effects in which the center-of-gravity position relative both to the aerodynamic center and to the elastic axis is significant.

In figures 12 and 14 there are shown, for reference, some numerical values of $\frac{\omega_b}{\omega_\alpha}$, the ratio of the flutter frequency to the torsional frequency.
The effect of the wing-density parameter \( \mu \) is rather complicated but, in general, an increase in \( \mu \) yields a corresponding increase in the flutter coefficient. For low values of \( \frac{\omega_n}{\omega_\alpha} \) and for high wing densities this increase is expected to be proportional to \( \sqrt{\mu} \). In the resonance-like region near \( \frac{\omega_n}{\omega_\alpha} = 1 \) and for small values of \( x_\alpha \), the flutter coefficient is relatively unaffected by the value of \( \mu \), and in this region the structural damping may be expected to be particularly effective in increasing the flutter coefficient.

For values of the Mach number near unity (for example \( M = \frac{10}{9} \), a value for which the validity of the theory is in question), the flutter calculations become difficult to plot because of the appearance of other branches. In some cases (for instance, \( x_0 = 0.6 \)) the flutter instability appears limited to a definite range of speed. Calculations to include damping were performed to verify the existence of the range. (The appearance of these other branches seems to involve values of \( l/k \) for which the quantity \( M_l \) is negative. The condition of negative \( M_l \) is significant for the one-degree-of-freedom instability discussed in the next section.

A plot of the flutter coefficient against Mach number for two values of \( x_\alpha \) is shown in figure 17. The significance of the straight lines drawn from the origin has already been discussed. The type of curve A is representative of the effect of forward location of the center of gravity and the type of curve B is representative of rearward locations of the center of gravity. Figure 18 gives a plot of the flutter coefficient against \( M \) for various values of the wing-density parameter \( \mu \) and for a rearward location of the center of gravity. The subsonic values for \( M = 0 \) and \( M = 0.7 \) shown on these curves and on some of the other figures have been either taken from reference 7 or calculated in the manner outlined therein. The subsonic and supersonic parts of the curves (figs. 17 and 18) have been arbitrarily joined by a dashed smooth curve in
the transonic range. In figure 19 there is given a cross plot of flutter coefficient against frequency ratio $\omega_n/\omega_\alpha$, for various values of $M$, and in figure 20 is given a similar cross plot for three values of the elastic-axis parameter $x_0$.

An indication of the effect of structural damping in increasing the flutter speed in a few examples may be obtained from the following table, where $g_\alpha$ and $g_h$ are the torsional and flexural damping coefficients, respectively, and where $M = 10/7$, $\mu = 7.854$, $a = 0$, and $x_\alpha = 0.2$:

<table>
<thead>
<tr>
<th>$\omega_n/\omega_\alpha$</th>
<th>$g_\alpha$</th>
<th>$g_h$</th>
<th>$\omega/\omega_\alpha$</th>
<th>$v/b\omega_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.673</td>
<td>2.438</td>
</tr>
<tr>
<td>0</td>
<td>0.05</td>
<td>0</td>
<td>0.648</td>
<td>2.551</td>
</tr>
<tr>
<td>0</td>
<td>0.10</td>
<td>0</td>
<td>0.628</td>
<td>2.669</td>
</tr>
<tr>
<td>0.707</td>
<td>0</td>
<td>0</td>
<td>0.771</td>
<td>1.535</td>
</tr>
<tr>
<td>0.707</td>
<td>0.05</td>
<td>0</td>
<td>0.771</td>
<td>1.535</td>
</tr>
<tr>
<td>0.707</td>
<td>0.10</td>
<td>0</td>
<td>0.766</td>
<td>1.569</td>
</tr>
<tr>
<td>0.707</td>
<td>0</td>
<td>0.05</td>
<td>0.788</td>
<td>1.592</td>
</tr>
<tr>
<td>0.707</td>
<td>0</td>
<td>0.10</td>
<td>0.797</td>
<td>1.642</td>
</tr>
<tr>
<td>0.707</td>
<td>0.05</td>
<td>0.05</td>
<td>0.782</td>
<td>1.623</td>
</tr>
<tr>
<td>0.707</td>
<td>0.10</td>
<td>0.10</td>
<td>0.784</td>
<td>1.725</td>
</tr>
</tbody>
</table>

**STATIC CASES - WING DIVERGENCE AND ALERON REVERSAL**

It is of some interest to examine the expressions for the forces and moments in the limit case in which the frequency approaches zero. There follow then for the mean-line wing section the well-known static-case results which may of course be obtained directly without the use of a limiting process, as originally treated by Ackeret. Thus, with the use of the following relation easily verified from equations (20),

$$\lim_{k \to 0} f_\alpha(m, k) = \frac{1}{\lambda + 1}$$
there are obtained from equations (16' ) to (18') for the lift and moments in the static case,

\[ L = -\frac{4\rho bv^2}{\sqrt{M^2 - 1}} \left[ \alpha + (1 - x_1)\beta \right] \]

\[ M_\alpha = -\frac{4\rho b^2v^2}{\sqrt{M^2 - 1}} \left[ (1 - 2x_0)\alpha + (1 - x_1)(1 + x_1 - 2x_0)\beta \right] \]

\[ M_\beta = -\frac{4\rho b^2v^2}{\sqrt{M^2 - 1}} (1 - x_1)^2(\alpha + \beta) \]

These relations for the mean-line wing section are now used to obtain the critical speeds for the static instabilities - wing divergence and wing-surface reversal (for wing of infinite span). At the wing divergence speed the effective torsional stiffness of the wing vanishes, hence the total moment about the elastic axis is zero. The sum of the structural restoring moment and the aerodynamic twisting moment is

\[ \alpha C_\alpha + \frac{4\rho b^2v^2}{\sqrt{M^2 - 1}} \alpha (1 - 2x_0) \]

which when equated to zero yields the divergence speed

\[ v_D = b\omega(\sqrt{M^2 - 1})^{1/4} \sqrt{\frac{r_d}{\sqrt{2x_0 - 1}}} \]

Thus, the divergence speed is real only for positions of the elastic axis behind the aerodynamic center (mid-chord, in the simple theory). This formula obviously should not be used for values of \( M \) too near unity.

For comparison it is of interest to note the corresponding result for the divergence speed in the subsonic case, where the aerodynamic center is (approximately) at the quarter-chord point. Thus,
\[ v_D = b \omega_a (1 - M^2)^{1/4} \sqrt{\frac{r_0}{k}} \left( \frac{1}{\sqrt{x_0 - 1}} \right) \]

where \( M \leq \text{about 0.7}. \)

The aileron reversal speed is determined by the condition that the change in angle of attack due to wing torsion nullifies the effect of movement of the aileron so as to yield no change in lift (in rolling moment, in the case of finite wing span). There are two equations to be satisfied for this condition; namely,

\[ \alpha + (1 - x_1) \beta = 0 \]

(that is, \( L = 0 \)) and

\[ \alpha \alpha + \frac{4 \rho b^2 v^2}{\sqrt{\beta^2 - 1}} \left[ (1 - 2x_0) \alpha + (1 - x_1)(1 + x_1 - 2x_0) \beta \right] = 0 \]

The aileron reversal speed, obtained by elimination of \( \alpha \) and \( \beta \), is

\[ v_R = b \omega_a (M^2 - 1)^{1/4} \sqrt{r_0} \left( \frac{1}{\sqrt{x_1}} \right) \]

For hinge positions aft of the midchord, the factor \( 1/\sqrt{x_1} \) in this expression varies from 1.4 to 1.0. The aileron reversal speed is thus relatively unaffected by the position of the hinge. In general \( v_R \) may be expected to be lower than \( v_D \).

**ONE-DEGREE-OF-FREEDOM OSCILLATORY INSTABILITY**

As was pointed out by Rossin, the theory indicates the existence of a torsional instability which may arise for a wing having only one degree of freedom. This instability is due to the wing being negatively damped in torsion and is associated with the vanishing (and
change in sign) of the torsional damping coefficient $M_4$ (equation (26)).

Certain considerations for the case of slow oscillations made by Possio (reference 1) and further discussed by Tomole and Jahn serve to bring out the main results. Thus from equation (20), for slow oscillations,

$$f_\lambda(M, k) \approx \frac{1}{\lambda + 1} - i \frac{2kM^2}{M^2 - 1} \frac{1}{\lambda + 2}$$

and

$$M_4 = \frac{1}{\sqrt{M^2 - 1}} \left[ 4 - 9x_0 + 6x_0^2 + \frac{M^2}{M^2 - 1} (2 - 3x_0) \right]$$

The condition $M_4(M, x_0) = 0$ is shown plotted in figure 21, where the shaded area is the region in which the instability is possible (negative $M_4$). The maximum ranges for the parameters $x_0$ and $M$ in this region are $x_0$ less than $2/3$ and $M$ less than $\sqrt{2.5}$ (and greater than unity).

(It may be appropriate to mention that a similar torsional instability is theoretically indicated even in the subsonic (incompressible) case for positions of the axis of rotation between the leading edge and the quarter-chord point. However, the combination of parameters required for this indicated instability is practically unattainable.)

The torsional instability may be studied more fully in the general case. It is found that the range of instability for the parameters $x_0$ and $M$ remains essentially as in the simple case (large $1/k$) but more information may be obtained regarding the critical speed and frequency. The moment equation is equivalent to $\ddot{X} + \alpha = 0$, or to the two equations

$$\Omega_\alpha X - \mu \omega_\alpha^2 + M_2(M, x_0) = 0$$

$$M_4(M, x_0) + \omega_\alpha \Omega_\alpha X = 0$$
where the structural damping coefficient in torsion $\xi_{\alpha}$ has been introduced as in reference 6. The critical speed and frequency may be studied as functions of the parameters $x_0$, $M$, $\xi_{\alpha}$ and the product combination $\mu r_{\alpha}^2$. Results of a few selected calculations are shown plotted in figure 22. Since instabilities are indicated for the range of near-sonic values ($1 < M < 1.58$) it would seem that a more comprehensive investigation of this problem is very desirable.

It may be remarked that a similar analysis for pure bending exhibits no instability while the case of the aileron alone does exhibit a range where such instability may occur. This range for an aileron hinged at its leading edge is $1 < M \leq \sqrt{2}$.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., May 29, 1946
APPENDIX

SYMBOLS

\( \varnothing \) disturbance velocity potential
\( t \) time at which disturbance influence is felt
\( T \) time at which disturbance is created
\( \tau = t - T \)
\( p \) pressure
\( p' \) pressure difference
\( \rho \) density
\( \gamma \) adiabatic index (for air, \( \gamma \approx 1.4 \))
\( v \) velocity of main stream (supersonic)
\( c \) velocity of sound in undisturbed medium
\( M \) Mach number (v/c)
\( x \) coordinate measured in direction of main stream
\( y \) ordinate
\( x_0 \) abscissa of axis of rotation of wing section (elastic axis)
\( x_1 \) abscissa of aileron hinge
\( \xi, \eta \) abscissa and ordinate of point of disturbance
\( b \) one-half chord

After equation (12) the quantities \( x, y, x_0, x_1, \) and \( \xi \) are employed nondimensionally and are referred to the chord 2b as reference length.

\( w(x, t) \) vertical velocity at position \( x \) on chord and at time \( t \)
vertical displacement of axis of rotation
angular displacement about axis of rotation
angular displacement of aileron; measured with respect to \( \alpha \)
angular frequency of oscillation
reduced frequency \((\omega_b/v)\)
frequency parameter \(\bar{\omega} = \left(\frac{2kM^2}{M^2 - 1}\right)\)
function given in equations (12) and (12')
Bessel function of order \(n\)

The following additional symbols, employed in the flutter equations, conform to the notation used in references 4 and 6, in which the half-chord \(b\) is the unit reference length.

- **\(M\)**: mass of wing per unit span
- **\(S_\alpha\)**: static moment of wing-aileron combination per unit span referred to the elastic axis
- **\(S_\beta\)**: static moment of aileron per unit span referred to aileron hinge
- **\(I_\alpha\)**: moment of inertia of wing-aileron combination about elastic axis per unit span
- **\(I_\beta\)**: moment of inertia of aileron about its hinge per unit span
- **\(a\)**: coordinate of elastic axis measured from mid-chord \((2x_0 - 1)\)
- **\(c\)**: coordinate of aileron hinge axis measured from the midchord \((2x_1 - 1)\)
- **\(x_u\)**: location of center of gravity of wing-aileron system measured from elastic axis \(S_\alpha/M_b\); location of center of gravity in percent total chord measured from leading edge is \(100 \frac{1 + a + x_\alpha}{2} = 100\left(x_0 + \frac{x_\alpha}{2}\right)\)
$x_b$  reduced location of center of gravity of aileron referred to $c$ $(\bar{c}_b/M_b)$

$r_a$  radius of gyration of wing-aileron combination referred to $a$ $\left(\sqrt{\frac{I_a}{M_b^2}}\right)$

$r_\beta$  reduced radius of gyration of aileron referred to $c$ $\left(\sqrt{\frac{I_\beta}{M_b^2}}\right)$

$C_\alpha$  torsional stiffness of wing around $a$ per unit span

$C_\beta$  torsional stiffness of aileron system around $c$ per unit span

$C_h$  stiffness of wing in deflection

$\omega_\alpha$  natural angular frequency of torsional vibrations about elastic axis $\left(\sqrt{\frac{C_\alpha}{I_\alpha}}\right)$; $\omega_\alpha = 2\pi f_\alpha$, where $f_\alpha$ is in cycles per second

$\omega_\beta$  natural angular frequency of torsional vibrations of aileron around $c$ $\left(\sqrt{\frac{C_\beta}{I_\beta}}\right)$

$\omega_h$  natural angular frequency of wing in deflection $\left(\sqrt{\frac{C_h}{M}}\right)$

$\mu = \frac{\pi}{4} \frac{1}{\kappa}$  wing density parameter, where $\kappa = \frac{\pi pb^2}{M}$

is the ratio of a mass of cylinder of air of a diameter equal to the chord of the wing to the mass of the wing, both taken for equal length along the span; this ratio may be expressed as $\kappa = 0.24(b^2/w)(\rho/\rho_o)$ where $w$ is the weight in pounds per foot span, $b$ is in feet and $\rho/\rho_o$ is the ratio of air density at altitude to that for normal standard air

$\left(\mu = \frac{M}{4\rho b^2} = \frac{\pi}{4} \frac{1}{\kappa}\right)$
\( \varepsilon_\alpha, \varepsilon_\beta, \varepsilon_n \) structural damping coefficients (see reference 6)

\( L_1, L_2, L_3, L_4, M_1, M_2, M_3, M_4 \) quantities defined in table II and by equations (26) and (28)

\( v/\omega \alpha \) flutter coefficient; that is, flutter speed divided by reference speed \( b\omega \alpha \)

\[ \Omega_\alpha X = \mu r_\alpha^2 \left( \frac{\omega_\alpha}{\omega} \right)^2 \]

\[ \Omega_\beta X = \mu r_\beta^2 \left( \frac{\omega_\beta}{\omega} \right)^2 \]

\[ \Omega_n X = \mu \left( \frac{\omega_n}{\omega} \right)^2 \]

where \( \omega \) is the angular (flutter) frequency and

\[ X = \mu r_\alpha^2 \left( \frac{\omega_\alpha}{\omega} \right)^2 \]

for case of bending-torsion (Note that in the incompressible case (references 4 and 6) \( \mu \) is replaced by \( 1/\kappa \).)
REFERENCES


\[
\begin{align*}
\text{TABLE I. VALUES OF } f_0(M, \varpi) &= (f_0)_R + i(f_0)_I \\
(f_0)_R &= \frac{1}{\varpi} \int_{0}^{\varpi} J_0(\frac{u}{M}) \cos u \, du \\
(f_0)_I &= -\frac{1}{\varpi} \int_{0}^{\varpi} J_0(\frac{u}{M}) \sin u \, du
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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
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TABLE II.—VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS

The expressions employed in the calculation of this table are:

\[
L_1 = \frac{1}{bW - 1} \left\{ \frac{2}{5} \left[ g(x) + \frac{2}{3} f(x) \sin W + \frac{4}{3} f(x) \cos W \right] \right\}
\]

\[
L_2 = \frac{1}{bW - 1} \left\{ \frac{2}{5} \left[ g(x) + \frac{2}{3} f(x) \cos W - \frac{4}{3} f(x) \sin W \right] \right\}
\]

\[
L_3 = L_1 + \frac{1}{3} L_2 + A_1
\]

\[
L_4 = L_1 - \frac{1}{3} L_2 + A_2
\]

\[
M_1 = L_1 - A_1
\]

\[
M_2 = L_2 - A_2
\]

\[
M_3 = \frac{3}{2} (A_1 - B_1) + \frac{1}{2} (c_1 - A_2)
\]

\[
M_4 = \frac{3}{2} (A_1 - B_1) - \frac{1}{2} (A_2 - B_2)
\]

where

\[
A_1 = \frac{2}{bW - 1} \frac{1}{5} \left[ \frac{2}{3} g(x) + \frac{2}{3} f(x) \cos W + \frac{4}{3} f(x) \sin W \right]
\]

\[
A_2 = \frac{2}{bW - 1} \frac{1}{5} \left[ \frac{2}{3} g(x) + \frac{2}{3} f(x) \sin W - \frac{4}{3} f(x) \cos W \right]
\]

\[
B_1 = \frac{2}{bW - 1} \frac{1}{5} \left[ \frac{2}{3} g(x) + \frac{2}{3} f(x) \cos W + \frac{4}{3} f(x) \sin W \right]
\]

\[
B_2 = \frac{2}{bW - 1} \frac{1}{5} \left[ \frac{2}{3} g(x) + \frac{2}{3} f(x) \sin W - \frac{4}{3} f(x) \cos W \right]
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RATIONAL AERODYNAMIC COMMISSION FOR AEROSPACE
### Table II - Values of Functions Used in Flutter Calculations

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**NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS**
Figure 1. - Mach angle $\mu$. The disturbance at point $(\xi, \eta)$ moving forward with supersonic velocity $v$ influences the angular region having half vertex angle $\mu = \sin^{-1} \phi$.

Figure 2. - Influence of impulse created at point $(\xi_0, 0)$ at time $t = T$ on a point $(x, y)$ fixed relative to $(\xi_0, 0)$ and moving with supersonic velocity $v$. (Observe that the disturbance influences the point $(x, y)$ only during the time interval $t_2 - t_1$.)
Figure 3.- Sketch showing that only disturbances created forward of the Mach angle region with vertex at $\mathbf{X}_0$, can affect $(x, y)$.

Figure 4.- Sketch illustrating the three degrees of freedom $h, \alpha, \beta$ of the oscillating airfoil.
Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

Figure 5. The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{9}$; $\mu = 3.927$.
Figure 6. - The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. \( M = \frac{10}{g}; \ \mu = 7.854 \).
Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

Figure 7a.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{5}$; $\mu = 15.708$. 
Figure 8. - The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 3.927$. 

Center-of-gravity location, percent chord

(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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Figure 9.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 7.854$. 
Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

Figure 10. - The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 15.708$. 
Figure 11.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 2; \mu = 3.927$. 

Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.
Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

Figure 12.— The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 2; \nu = 7.854$. 

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.
Figure 13. - The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 2; \mu = 15.708$.
Figure 14.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 5; \mu = 3.927$. 

Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
Figure 15.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. \( M = 5; \mu = 7.854 \).
Figure 16. The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = b; \mu = 15.706$. 

Center-of-gravity location, percent chord

(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.
Figure 17.— The flutter coefficient against Mach number for two locations of the center of gravity. Other parameters are $\frac{\omega_b}{\omega_c} = 0.707; \alpha = 0; 
\mu = 7.864.$
Figure 18. - The flutter coefficient against Mach number for several values of $\mu$. Other parameters are $\frac{\delta_{\infty}}{\delta} = 0$; $x_{\alpha} = 0.3$; $\alpha = 0$. 
Figure 19. - The flutter coefficient against frequency ratio for several values of M. Other parameters are a = 0; x_0 = 0.2; μ = 7.854.
Figure 20. - The flutter coefficient against frequency ratio for three values of $x_0$. Other parameters are $M = \frac{10}{7}$; $a = 0$; $\mu = 7.854$. 

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Figure 21.- Plot of $M_{a}(M_{x_{0}}) = 0$. 

Stable region:
(a) Flutter coefficient against axis-of-rotation position for several values of $M$ ($\mu = 15.708$). Note that the range of $x_0$ narrows with increase in $M$ and disappears at $M = 1.58$ and $x_0 = 0.33$.

(b) Flutter coefficient against torsional damping coefficient for two values of $x_0$ ($M = \frac{10}{9}$; $\mu = 15.708$). Negative damping values are shown dashed and have no physical existence.

(c) Flutter coefficient against wing-density parameter $\mu$ for several values of $x_0$ ($M = \frac{10}{9}$). The straight-line curve shown corresponds to $M_3 = 0$ ($x_0 = 0.327$).

Figure 22.—Curves for one-degree-of-freedom torsional instability.