OBSERVATIONS ON THE MAXIMUM AVERAGE STRESS OF FLAT PLATES BUCKLED IN EDGE COMPRESSION

By E. H. Schuette
The Dow Chemical Company
Observations on the Maximum Average Stress of Flat Plates Buckled in Edge Compression

By E. H. Schuette

SUMMARY

Compressive-strength data on flat plates for a number of aircraft structural materials were subjected to graphical study. It was found that the maximum average stress, in the region where it differs appreciably from the critical stress, could be expressed as a constant times the one-fourth power of the critical stress times three-fourths power of the compressive yield stress. Good fits to all data obtained from tests of extrusions having flat-plate elements, with width-thickness ratios up to about 25, were obtained with the constant equal to 0.80. The data for specimens formed from flat sheet indicated that the constant varied in some inverse fashion with the width-thickness ratio and that not all this variation could be ascribed to the increased compressive yield stress of the material in the formed corners. Other possible causes of the variation are discussed and possible profitable lines of future research are suggested.

INTRODUCTION

The theory of elastic buckling of flat rectangular plates in edge compression has been well covered in the literature; references 1 to 3 are examples. The problem of buckling of such plates beyond the elastic range has had extensive empirical treatment, particularly in references 4 to 12. References 4 to 10 have shown, moreover, that in the high-stress region the maximum average stress attained after buckling averages about 4 percent above the critical or buckling stress. Thus at the present time, the phase of this problem presenting the greatest research interest seems to be the determination of the maximum average stress attained after buckling in the low-stress region, that is, where this maximum stress differs appreciably from the critical stress.

Figure 1 shows a type of plot that has been used for presenting test data on plate buckling and also indicates a major difference that has been evidenced between data for specimens formed from flat sheet and data from tests of extrusions with flat-plate elements. Whereas for extruded material a single curve of maximum average stress
in the region D (fig. 1(b)) includes all the width–thickness ratios tested, the location of this same part of the curve for sheet material (fig. 1(a)) varies with the width–thickness ratio. The various phases of the general problem are defined by the four regions shown in figure 1(b) (the regions apply to both sheet and extruded material). In region A, buckling is elastic. Region B is the range of buckling beyond the elastic range. Throughout region C, the maximum average stress can be taken as 1.04 times the critical stress. The present report is concerned solely with region D and presents some empirical studies of the available test data.

SYMBOLS

*K, R, n* constants

*b* plate width

*b*\(_{\text{eff}}\) effective plate width

*b*\(^{'}\) hypothetical, constant, added effective plate width

*b*\(_{w}\) web width

t plate thickness

*t*\(_{w}\) web thickness

*\(\sigma_{cr}\)* critical, or buckling, stress

*\(\sigma_{cy}\)* compressive yield stress

*\(\sigma_{e}\)* edge stress after buckling

*\(\bar{\sigma}\)* average stress

*\(\bar{\sigma}_{\text{max}}\)* maximum average stress (average stress at maximum load)

GRAPHICAL STUDY

Method of plotting.— In making a general study of test data on a number of different materials, it is desirable first to arrive at nondimensional parameters that will enable all the data to be combined in a single plot. One such parameter *\(\frac{\sigma_{cr}}{\bar{\sigma}_{\text{max}}}\)* is already listed in the
tables of test data in references 4 to 10. In the references, values of this parameter were used in plots of $\sigma_{cr}$ against $\sigma_{cr}/\sigma_{max}$. Such a plot can be nondimensionalized simply by dividing the critical-stress values by the compressive yield stress. Because $\sigma_{max}$ is the quantity of major interest, the present study employs plots of $\sigma_{max}/\sigma_{cr}$ against $\sigma_{cr}/\sigma_{cy}$. Inspection of the original curves (references 4 to 10) indicated that a simple power function might link $\sigma_{max}$ and $\sigma_{cr}$. A logarithmic scale was therefore selected for the plots.

Study of extruded materials.—The plots of $\sigma_{max}/\sigma_{cr}$ against $\sigma_{cr}/\sigma_{cy}$ for extruded material are presented in figures 2 to 7. The data were taken from references 4 to 8 and only those data which definitely applied to the part of the curve below the point where $\sigma_{max}$ becomes equal to about $1.04\sigma_{cr}$ were plotted. Figure 2 shows all the applicable data for all five materials. A single straight line corresponding to the equation

$$\frac{\sigma_{max}}{\sigma_{cy}} = 0.80\left(\frac{\sigma_{cr}}{\sigma_{cy}}\right)^{-0.75}$$

has been fitted to the data. Other, possibly more useful, forms of the same equation are

$$\frac{\sigma_{max}}{\sigma_{cy}} = 0.80\left(\frac{\sigma_{cr}}{\sigma_{cy}}\right)^{0.25}$$

and

$$\sigma_{max} = 0.80\sigma_{cr}^{0.25}\sigma_{cy}^{-0.75}$$

Figures 3 to 7 present separate comparisons of the curve with data for each of the materials. The slight difference that occurs frequently between H-section data and Z- and channel-section data is undoubtedly due to the use of the same yield stress, regardless of section. As pointed out in the references, the flanges generally had higher compressive yield strengths than the webs, so that machining off flanges to produce Z- and channel sections would give a section of slightly lowered average properties. The average curve was therefore felt to be the generally applicable one.

The data for 14S-T aluminum alloy (fig. 5) are substantially above the line.\(^1\) But here again the discrepancy can be very readily explained by the supposition that the compressive yield stresses used in converting

\(^1\)New Alcoa temper designations: 14S-T now 14S-T6, 75S-T now 75S-T6, and 24S-T now 24S-T4 in case of extrusions; 24S-T now 24S-T3 in case of sheet.
the data were perhaps not as representative of the specimens tested as they were for the other materials.

There is little evidence that different curves should be ascribed to different values of \( b_W/t_W \). The data, however, are too limited as regards variation in \( b_W/t_W \) to justify such a general conclusion. It will be seen subsequently that such a variation definitely exists for formed-sheets specimens and that values of \( \frac{\overline{\sigma}_{\text{max}}}{\sigma_{cr}} \) are lowered by increases in \( b_W/t \). A safe conclusion from the extruded data would therefore be that the equation

\[
\overline{\sigma}_{\text{max}} = 0.80(\sigma_{cr})^{0.25}(\sigma_{cy})^{0.75}
\]

is satisfactory for design of extruded plates with \( b/t < 25 \) and fillets having a radius equal to the plate thickness at each edge (test conditions).

Quite possibly, \(-0.75\) is not the mathematically best slope to ascribe to the line, but it was chosen because it gives a reasonable fit and is well suited to slide-rule calculations.

Study of sheet materials.—The available data for sheet materials embrace only two aluminum alloys, but for 246-T sheet they cover a greater range than do the extrusion data. The plots of \( \frac{\overline{\sigma}_{\text{max}}}{\sigma_{cr}} \) against \( \frac{\sigma_{cr}}{\sigma_{cy}} \) for sheet material are presented in figures 8 to 10. The data were taken from references 9 and 10 and, as before, only those data which definitely applied to the part of the curve below the point where \( \overline{\sigma}_{\text{max}} \) becomes equal to about 1.04\( \sigma_{cr} \) were plotted.

As in the case of extrusions, good fits to the test data can be obtained with a straight line having a slope of \(-0.75\). For formed sheet, however, a very definite variation of intercept values is noted for different values of \( b_W/t \). The intercept value, which is the constant \( K \) in the general equation

\[
\frac{\overline{\sigma}_{\text{max}}}{\sigma_{cr}} = K\left(\frac{\sigma_{cr}}{\sigma_{cy}}\right)^{-0.75}
\]

is decreased for increasing values of \( b_W/t \). One possible explanation (item (1) in the following discussion) has already been advanced for this variation in reference 9. There are other possible explanations and the most reasonable ones seem to be the following:

(1) "The stress-strain curves have indicated . . . that the bending of the material in the forming of the columns raises the compressive yield stress for the material in the corners . . . . An examination of the
data showed that as \( \frac{b_w}{t} \) is reduced for a given thickness \( t \) and a given value of \( \frac{\sigma}{\sigma_{cr}} \), the total cross-sectional area is also reduced; the fixed area of high-strength bent material in the corners therefore becomes a higher percentage of the total area. This fact may account for the higher values of \( \frac{\bar{\sigma}}{\sigma_{cr}} \) obtained at given values of \( \frac{b_w}{t} \). (From reference 9, p. 8.)

(2) The curved material in the corners may tend to remain more fully effective after buckling of the webs and flanges than does the flat material. As explained in item (1), reductions in \( \frac{b_w}{t} \), other conditions fixed, produce a section with an increased percentage of curved material. Effective width would therefore be greater at any given value of \( \frac{\sigma}{\sigma_{cr}} \), the ratio of edge stress to critical stress. Then, for a given edge stress, the value of average stress \( \sigma = \sigma_e \left( \frac{b_{off}}{b} \right) \) would be increased. The result would probably be an ability to reach a higher average stress before failure takes place.

(3) The higher percentage of curved material may in some way make possible the attainment of a higher edge stress before failure takes place.

These three points will be taken up again in the DISCUSSION. For present purposes, it may definitely be said that no effect of the curved corner would be apparent at \( b_w/t = \infty \) (percentage of curved-corner material goes to zero). An extrapolation of the constant \( K \) to \( b_w/t = \infty \) is possible by plotting the values of \( K \), picked from the curves of figures 8 to 10, against \( t/b_w \), since \( b_w/t = \infty \) when \( t/b_w = 0 \). Such a plot is given in figure 11.

Figure 11 also includes a curve faired through the experimental values and extrapolated to \( t/b_w = 0 \) \( (b_w/t = \infty) \). So great an extrapolation is admittedly stretching the data in the extreme. As the data do not cover any greater range, however, figure 11 will have to be considered simply as one possible guess as to where the line might go. For use in the equation

\[
\frac{\bar{\sigma}_{max}}{\sigma_{cr}} = K \left( \frac{\sigma_{cr}}{\sigma_{cy}} \right)^{-0.75}
\]

or

\[
\frac{\bar{\sigma}_{max}}{\sigma_{cy}} = K \left( \frac{\sigma_{cr}}{\sigma_{cy}} \right)^{0.25}
\]
the following values of $K$ were picked from figure 11:

\[
\begin{array}{ll}
  & K \\
 b_w/t & \\
 20 & 0.90 \\
 30 & 0.82 \\
 40 & 0.75 \\
 50 & 0.71 \\
 100 & 0.63 \\
 \infty & 0.55 \\
\end{array}
\]

These values apply to plates formed at their edges to an inside radius of 3t, in which increases in compressive yield strength comparable with those found for 248-T (reference 9) are encountered as a result of forming. Because an infinitely wide plate will display no effect of the edge forming, the curve for $b_w/t = \infty$ should give accurate or conservative results for any width-thickness ratio if no edge forming has been done. The various curves of $\bar{\sigma}_{\text{max}}/\sigma_{\text{cy}}$ against $\sigma_{\text{cr}}/\sigma_{\text{cy}}$, corresponding to the constants tabulated, are shown in figure 12, which also includes for comparison the curve previously arrived at for extrusions.

DISCUSSION

An effective-width hypothesis.— The fact that all the data follow so well curves corresponding to the general equation

\[
\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = K \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75}
\]

immediately leads to an attempt to find an explanation for the phenomenon. A relatively simple hypothesis can be developed from consideration of effective width. Suppose that the effective width is given by an equation of the type

\[
\frac{b_{\text{eff}}}{b} = \left(\frac{\sigma_{\text{cr}}}{\sigma_{e}}\right)^n
\]

Then the average stress at any given value of edge stress after buckling is given by
\[ \bar{\sigma} = \sigma_e \frac{b_{\text{eff}}}{b} \]

\[ = \sigma_e \left( 1 - n \right)^n \]

Now if it is assumed that failure occurs when the edge stress reaches some fixed proportion of the compressive yield stress \( R_{cy} \), then an equation for \( \bar{\sigma}_{\text{max}} \) can be obtained simply by substituting \( R_{cy} \) for \( \sigma_e \) in the equation for \( \bar{\sigma} \). Thus,

\[ \bar{\sigma}_{\text{max}} = \left( R_{cy} \right)^{1-n} \sigma_{cr} \]

\[ = R \sigma_{cr} \]

This equation corresponds to the general equation obtained from the graphical studies of this report:

\[ \bar{\sigma}_{\text{max}} = K \left( \sigma_{cr} \right)^{0.25} \left( \sigma_{cy} \right)^{0.75} \]

Discrepancies in the hypothesis.—Consideration of past experience with problems of a similar nature makes the hypothesis seem almost too simple to be accurate. At any rate, it is by no means a complete explanation. It would not be presented at all, were it not for the hope that its presentation may stimulate further and more productive thought on the part of other researchers.

The discrepancies arise largely out of considerations of the test results obtained from formed-sheet specimens. If the differences in the curves of \( \bar{\sigma}_{\text{max}}/\sigma_{cy} \) against \( \sigma_{cr}/\sigma_{cy} \) (fig. 12) were wholly attributable to the increased compressive yield stress in the corners, then the curve for \( b_w/t = \infty \) should be applicable to any unformed sheet and should correspond exactly to results obtained from extrusions, since the latter showed no variation with width-thickness ratio. Figure 12 shows that such a condition definitely does not exist. It is immediately tempting to believe that the fillets at the edges of the extruded plates raise the extrusion curve. If this were so, however, then an extrusion curve for \( b_w/t_w = \infty \) should correspond to this lowest sheet curve and, in order to make such correspondence possible, a very definite difference

\[ ^2 \text{A value for the "failing edge stress" can be arrived at by equating to zero the slope of an analytical curve of average stress against edge strain, but the value so obtained fails by a wide margin to check the experimental results.} \]
in the equation constant should have shown up in the data for extrusions of different $b_W/t_W$ values. Such a difference was completely lacking.

Moreover, in considering the compressive-yield-strength explanation, the method was tried of assigning other values between the flat- and curved-material yield strengths, such as weighted averages, to the compressive-yield-strength values used to compute $\sigma_{cr}/\sigma_0$. No logical procedure could be found to produce a shift of one curve, sufficient in magnitude and sufficiently different from the corresponding shift produced in the other curves, to bring the several curves into coincidence.

It was the failure of this attempt that led to the second explanation, given in the section entitled "Study of sheet materials," that the presence of the curved material adds a constant amount to the effective width of the section. This explanation, however, immediately presents difficulties, as the proper effective-width equation would then have to be of some such form as

$$\frac{b_{eff}}{b} = \frac{b' + b - b'/(\sigma_{cr}/\sigma_0)^N}{b}$$

where $b'$ is the constant added amount of effective width. Complex equations of this type do not yield analytically the simple relationship between $\bar{\sigma}_{max}$ and $\sigma_{cr}$ which was found to give such an excellent fit to the test data. Moreover, such a phenomenon would not explain all the effects of the curved corner, unless the idea of a constant edge stress at failure is abandoned. For if this idea is taken in conjunction with the preceding equation — or, for that matter, with the equation $b_{eff}/b = (\sigma_{cr}/\sigma_0)^N$ — then all curves should reach $\bar{\sigma}_{max}/\sigma_{cr} = 1$ at the same value of $\sigma_{cr}/\sigma_0$. As previously pointed out, no method was found for prescribing yield-stress values, such that the curves could be brought together at any point.

The foregoing considerations lead to the idea, previously mentioned, that the curved corner in some way makes possible the development of a higher edge stress before failure when higher percentages of curved material are present. How such an effect might be brought about, however, is not readily explained.

Undoubtedly the total effect produced by the curved corner is a compound of a number of different effects. It is to be hoped that some of the discussion in the present report may lead to a complete solution from another source.

Productive lines of future research.—A number of lines of future experimentation, which would yield valuable information, suggest themselves. Some of the more promising ones are listed as follows:
(1) Testing of specimens constructed of a sheet material having substantially the same properties in the flat and formed conditions to isolate the effect of increased corner compressive yield strengths on specimen properties.

(2) Testing of specimens constructed of sheet material with a very small corner radius. Presumably, these specimens would be formed in a soft condition and subsequently heat-treated. The result would be an independent evaluation of the radius of bend as it affects the specimen properties.

(3) Testing of specimens, both of sheet and of extrusions, with very high width-thickness ratios, to provide for more accurate extrapolation to an infinite ratio.

(4) Testing of extrusions with no fillets in the corners to isolate the strengthening effect of the fillet.

Role of the flange width-thickness ratio.—In all the studies made to date, the width-thickness ratio of the web has been used as one of the independent variables. Such a procedure is quite convenient, because the test programs of references 4 to 10 were laid out for a series of fixed values of this ratio. Strength variations for a given ratio were obtained by variations in the ratio of flange width to web width. In order to prevent Euler column failures in the test specimens and to insure the most accurate results possible, however, values of this flange-to-web ratio were so chosen that almost all the test specimens were of proportions for which the flange is primarily responsible for failure (reference 13).

If the flange is the element most active in causing buckling, it would seem that the logical approach to an analysis of test results would be through considerations of the flange width-thickness ratio. Unfortunately, however, the test results do not correspond to a series of fixed values of this ratio; the values vary in a more or less random fashion. It was for this reason that the present study was made with web, rather than flange, width-thickness ratio as an independent variable.

Comparison of present results with other work.—Shortly after the completion of the analyses contained in the present report, reference 14 came to the author's attention. The latter paper presents formulas for the maximum average stress of extruded sections that are strikingly similar to the formulas presented herein. Put into the same form, the sets of formulas compare as follows:
In reference 14, for Z- and channel sections

\[ \frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.769 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.80} \]

for H-sections

\[ \frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.794 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.80} \]

In the present report, average for Z-, channel, and H-sections

\[ \frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.800 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75} \]

Within the range covered by test data, the maximum variation between any two of these formulas is at \( \sigma_{\text{cr}}/\sigma_{\text{cy}} = 0.25 \), where the formula of the present paper gives a value of \( \bar{\sigma}_{\text{max}}/\sigma_{\text{cr}} \) that is 6.1 percent lower than that given by the formula for H-sections in reference 14. This difference is substantially less than the maximum scatter of test data (about 18 percent).

The reasons for choosing a single formula for all three types of section have already been explained in the paragraphs on extruded materials, included in the section GRAPHICAL STUDY. It is of interest that an exponent of \(-0.80\) was first selected, as given in reference 14, but subsequently a better fit to data for sheet material was obtained with an exponent of \(-0.75\). For the sake of uniformity and because of better adaptability to slide-rule calculations, the value of \(-0.75\) was then adopted for extrusions also. The quality of fit to the test data was not materially impaired by this choice.

CONCLUDING REMARKS

Empirical studies of compressive-strength data on flat plates for a number of aircraft structural materials led to the description of the relationship between maximum average stress and critical stress, in the region where they are appreciably different, by equations of the type in which the maximum average stress equals a constant times the one-fourth power of the critical stress times the three-fourths power of the compressive yield stress. Good fits to all data for extrusions with flat-plate elements (with width-thickness ratios up to
about 25) were obtained with the constant equal to 0.80. The data for specimens formed from flat sheet indicated that the constant varied in some inverse fashion with the width-thickness ratio and not all this variation could be ascribed to the increased compressive yield stress of the material in the formed corners. A discussion of possible causes of the variation is given and possible profitable lines of future research are suggested.

The Dow Chemical Company
Midland, Mich., October 13, 1947
REFERENCES


Figure 1.- Types of curves obtained for experimental stresses against calculated elastic critical stresses for flat plates in edge compression.

(a) Sheet material.  
(b) Extruded material.
Figure 2. - Relationship of $\bar{\sigma}_{\text{max}}$ and $\sigma_{\text{cr}}$ for all extruded materials included in references 4 to 8.

$$\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.80 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75}.$$  

Figure 3. - Relationship of $\bar{\sigma}_{\text{max}}$ and $\sigma_{\text{cr}}$ for extruded 75S-T aluminum alloy. Data from reference 4.

$$\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.80 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75}.$$
Figure 4. - Relationship of $\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}}$ and $\sigma_{\text{cr}}$ for extruded R303-T aluminum alloy. Data from reference 5.

$$\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}} = 0.80 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75}.$$ 

Figure 5. - Relationship of $\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}}$ and $\sigma_{\text{cr}}$ for extruded 14S-T aluminum alloy. Data from reference 6.

$$\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}} = 0.80 \left(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}}\right)^{-0.75}.$$
Figure 6.- Relationship of $\bar{\sigma}_{\text{max}}$ and $\sigma_{\text{cr}}$ for extruded 24S-T aluminum alloy. Data from reference 7.

$$\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = 0.80 \left( \frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}} \right)^{-0.75}.$$
Figure 8.- Concluded.
Figure 9.—Relationship of $\sigma_{\text{max}}$ and $\sigma_{\text{cr}}$ for channel sections formed from 24S-T aluminum-alloy sheet. Data from reference 9. $\frac{\sigma_{\text{max}}}{\sigma_{\text{cr}}} = K(\frac{\sigma_{\text{cr}}}{\sigma_{\text{cy}}})^{-0.75}$. 
Figure 9.- Concluded.
Figure 10. - Relationship of $\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}}$ and $\sigma_{\text{cr}}$ for Z- and channel sections formed from 17S-T aluminum-alloy sheet. Data from reference 10. $\frac{\bar{\sigma}_{\text{max}}}{\sigma_{\text{cr}}} = \frac{K}{\sigma_{\text{cr}}/\sigma_{\text{cy}}}^{0.75}$.

Figure 11. - Curve for extrapolation of values of $K$ for values of $\frac{b_w}{t}$ greater than 43.
Figure 12.- Summary curves for relationship between $\bar{\sigma}_{\text{max}}$ and $\sigma_{cr}$ for sheet and extrusions.