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EFFECT OF FINITE SPAN ON THE AIRLOAD DISTRIBUTIONS FOR OSCILLATING WINGS

II - METHODS OF CALCULATION AND EXAMPLES OF APPLICATION

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SUMMARY

Lift and moment distributions are calculated for oscillating wings of finite span on the basis of the three-dimensional theory of part I of this report. The results obtained are compared with the corresponding results of the two-dimensional theory.

Rectangular and elliptical wings of aspect ratios 3 and 6 are considered and the range of values of the reduced frequency k extends from 0 to about 1.5. The calculations are made for various bending, torsion, and aileron deflection functions and the results are given in tabular and in graphical form. It is found that for a given wing and given reduced-frequency value the effect of finite span depends appreciably on the shape of the wing deflection functions. It is also found that for a given wing and given deflection function the finite-span effect decreases as the reduced frequency increases. For wings of aspect ratio 3 an appreciable three-dimensional effect occurs for values of k up to about 1.0 and for wings of aspect ratio 6, for values of k up to about 0.5.

A practical scheme of calculations is described and auxiliary tables are given for the numerical analysis of additional examples. Formulas are included which allow direct incorporation of the three-dimensional results in flutter determinants of the kind described in AAF Technical Report No. 4798. Examples of flutter calculations on the basis of these formulas are given in an appendix to this report. Change of the two-dimensional into the three-dimensional air forces appears to be responsible for flutter-speed changes of from 10 to 20 percent for wings, with the possibility of larger corrections in tail flutter.

While in the examples analyzed so far the aerodynamic-span effect increases the theoretical flutter speed, the possibility of the effect being in the opposite direction in other examples has to be considered.

INTRODUCTION

The magnitude of the aerodynamic-span effect in wing theory is one of the problems for which no complete answer has yet been found. With attention restricted to wings of aspect ratio that is not too small (say, $AR \geqq 3$) it may be said that, for wings in uniform motion, lifting-line theory represents a satisfactory solution of the problem. For wings in nonuniform motion the problem is considerably more difficult and most attempts to analyze the aerodynamic-span effect appear to be either incomplete or in their applicability restricted to special cases of the problem.

In part I of the present report an aerodynamic theory of the oscillating wing of finite span has been given which is considered to be as inclusive as lifting-line theory is for the wing in uniform motion. The final results of this theory were formulas for the spanwise distribution of air forces and moments for the four basic types of motion of flexible wings, namely bending, torsion, aileron, and tab deflection. In these formulas the effect of three-dimensionality appears as a correction term σ to the basic function $C(k)$ of the two-dimensional theory. The calculation of σ necessitates the solution of an integral equation which is similar to, but less simple than, the integral equation of lifting-line theory for the lift (and circulation) distribution on wings in uniform motion.

With a method thus established for the systematic calculations of finite-span corrections for the customary two-dimensional theory it becomes possible to arrive at statements with regard to the quantitative importance of the aerodynamic-span effect. It has in the past been held by a number of investigators that this effect, while significant for wings in uniform motion, is no longer so for oscillating wings. Physical consideration of the wake pattern indicates that for a given wing the actual three-dimensional flow approaches more and more nearly the two-dimensional pattern as the frequency of oscillation increases. It is thus permissible to say that for sufficiently high frequencies the aerodynamic-span effect is of an insignificant magnitude. This, however, leaves the question as to what constitutes a sufficiently high frequency and what is the magnitude of the effect if the frequency is not sufficiently high. It was therefore considered desirable to establish, at least roughly, the range of frequencies for which there is an appreciable aerodynamic-span effect and to indicate the nature of the effect in this range. The variables which are mainly involved are aspect ratio, wing deflection form, and reduced frequency $k = \omega b/U$, where ω is the circular frequency of oscillation, b the semichord of the wing, and U the velocity of flight.

The calculations in this report are in part designed to permit a rapid estimation of the magnitude of the aerodynamic-span effect by providing for consideration the results of an appreciable number of sample cases for wings of aspect ratios 3 and 6 and rectangular or elliptic plan form and for a range of values of the reduced frequency k.

It is felt that a point of further interest of this work should be the presentation of a scheme for incorporating the three-dimensional air forces in flutter calculations, for those cases in which such an incorporation is deemed worth while. Although the authors agree that such a refinement will be unnecessary in a good many cases for which great accuracy is not required, they also feel that if it is desired to obtain theoretical flutter speeds which are within, say, 10 percent of the actual flutter speeds, then in most cases it will be found necessary to incorporate the aerodynamic-span effect in the analysis. Appreciably larger corrections will often be obtained, particularly in problems of tail flutter.

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SYMBOLS

U	velocity of flight
x, y	Cartesian coordinates in plane of projection of wing surface
ρ	density
$x_l(y)$	coordinate of wing leading edge
$x_t(y)$	coordinate of wing trailing edge
b	semichord
b_o	semichord at midspan
s	ratio of span to chord at midspan
ω	circular frequency
$\bar{\Omega}$	amplitude of three-dimensional circulation function defined by equations (8) and (13)
y^*, η^*	dimensionless spanwise coordinates; $y^* = y/b_o$, $\eta^* = \eta/b_o$
z_m	dimensionless coordinate of midchord line; $z_m = (x_l + x_t)/2b_o$

k	reduced frequency; $k = \omega b/U$
k_o	reduced frequency at midspan
k_m =	$k_o z_m$
μ	function defined by equation (10)
$\overline{\Omega}^{(s)}$	amplitude of two-dimensional circulation function given by equation (7)
λ	variable of integration
S_n	function defined by equation (16)
$C(k)$	function defined by Theodorsen
σ	correction function defined by equation (6)
a	location of elastic axis in units of the semichord b
c	location of aileron leading edge in units of b
e	location of aileron hinge line in units of b
d	location of tab leading edge in units of b
f	location of tab hinge line in units of b
l, m	aileron and tab overhang in units of b; $l = e - c$, $m = f - d$
h	bending deflection of wings; $h = \overline{h}(y)e^{i\omega t}$
α	angle of attack of wings; $\alpha = \overline{\alpha}(y)e^{i\omega t}$
β, γ	aileron and tab deflection angles; $\beta = \overline{\beta}\, e^{i\omega t}$
L	lift per unit of span; $L = \overline{L}\, e^{i\omega t}$
M_a	moment about elastic axis per unit of span
M_β, M_γ	aileron and tab hinge moments per unit of span
F	function defined by equation (9)
J, Y	Bessel functions in customary notation

H	Hankel functions in customary notation
$\bar{C}_L$	lift coefficient defined by equation (12a)
$\bar{C}_R$	rolling-moment coefficient defined by equation (12b)
$\bar{P}$, $\bar{Q}$	functions defined by equations (32) and (33)
$\phi_j(y)$	phase angle functions $(j = h, \alpha, \beta, \gamma)$

LIFT AND MOMENT FUNCTIONS OF THE THREE-DIMENSIONAL THEORY

According to part I of the present report the following formulas hold for the distribution of lift $\bar{L}$, moment $\bar{M}_\alpha$, aileron hinge moment $\bar{M}_\beta$, and tab hinge moment $\bar{M}_\gamma$ due to a bending deflection $\bar{h}$, a torsional angular deflection $\bar{\alpha}$, an aileron angular deflection $\bar{\beta}$, and a tab angular deflection $\bar{\gamma}$:

$$\frac{\bar{L}}{2\rho U^2 b_o} = \pi \left\{ -\frac{k^2}{2} + ik[C(k) + \sigma_h] \right\} \frac{\bar{h}}{b_o}$$

$$+ \pi \left\{ \frac{1}{2} [ik + k^2 a] + \left[1 + ik \left(\frac{1}{2} - a \right) \right] [C(k) + \sigma_\alpha] \right\} \bar{\alpha} \frac{b}{b_o}$$

$$+ \left\{ \frac{\pi}{2} [ikC_{\beta_2} - k^2 C_{\beta_1}] + \left[E_1 + \frac{ik}{2} E_2 \right] [C(k) + \sigma_\beta] \right\} \bar{\beta} \frac{b}{b_o}$$

$$+ \left\{ \frac{\pi}{2} [ikC_{\gamma_2} - k^2 C_{\gamma_1}] + \left[E_1(d) + \frac{ik}{2} E_2(d) \right] [C(k) + \sigma_\gamma] \right\} \bar{\gamma} \frac{b}{b_o} \quad (1)$$

$$\frac{\bar{M}_a}{2\rho U^2 b_o^2} = \frac{b}{b_o}\pi\left\{\frac{ak^2}{2} - \left(\frac{1}{2}+a\right) ik[C(k)+\sigma_h]\right\}\frac{\bar{h}}{b_o} + \left(\frac{b}{b_o}\right)^2\pi$$

$$\times\left\{\frac{1}{2}[ikA_{\alpha_2} - k^2A_{\alpha_1}] - \left(\frac{1}{2}+a\right)\left[1 + ik\left(\frac{1}{2}-a\right)\right][C(k)+\sigma_\alpha]\right\}\bar{\alpha}$$

$$+\left(\frac{b}{b_o}\right)^2\left\{\frac{\pi}{2}[A_{\beta_3} + ikA_{\beta_2} - k^2A_{\beta_1}]\right.$$

$$\left.-\left(\frac{1}{2}+a\right)\left[E_1 + \frac{ik}{2}E_2\right][C(k)+\sigma_\beta]\right\}\bar{\beta} + \left(\frac{b}{b_o}\right)^2$$

$$\times\left\{\frac{\pi}{2}[A_{\gamma_3} + ikA_{\gamma_2} - k^2A_{\gamma_1}] - \left(\frac{1}{2}+a\right)\left[E_1(d) + \frac{ik}{2}E_2(d)\right][C(k)\right.$$

$$\left.+\sigma_\gamma]\right\}\bar{\gamma} \qquad (2)$$

$$\frac{\bar{M}_\beta}{2\rho U^2 b_o^2} = \frac{b}{b_o}\left\{-\frac{\pi k^2}{2}B_{h_1} + \frac{1}{2}E_3\, ik[C(k)+\sigma_h]\right\}\frac{\bar{h}}{b_o} + \left(\frac{b}{b_o}\right)^2$$

$$\times\left\{\frac{\pi}{2}[ikB_{\alpha_2} - k^2B_{\alpha_1}] + \frac{1}{2}E_3\left[1 + ik\left(\frac{1}{2}-a\right)\right][C(k)+\sigma_\alpha]\right\}\bar{\alpha}$$

$$+\left(\frac{b}{b_o}\right)^2\left\{\frac{\pi}{2}[B_{\beta_3} + ikB_{\beta_2} - k^2B_{\beta_1}] + \frac{1}{2\pi}E_3\left[E_1 + \frac{ik}{2}E_2\right][C(k)+\sigma_\beta]\right\}\bar{\beta}$$

$$+\left(\frac{b}{b_o}\right)^2\left\{\frac{\pi}{2}[B_{\gamma_3} + ikB_{\gamma_2} - k^2B_{\gamma_1}] + \frac{1}{2\pi}E_3\left[E_1(d) + \frac{ik}{2}E_2(d)\right]\right.$$

$$\left.\times[C(k)+\sigma_\gamma]\right\}\bar{\gamma} \qquad (3)$$

$$\frac{\bar{M}_\gamma}{2\rho U^2 b_o^2} = \frac{b}{b_o}\left\{-\frac{\pi k^2}{2} D_{h_1} + \frac{1}{2} E_3(d) ik[C(k) + \sigma_h]\right\}\frac{\bar{h}}{b_o} + \left(\frac{b}{b_o}\right)^2$$

$$\times\left\{\frac{\pi}{2}[ikD_{\alpha_2} - k^2 D_{\alpha_1}] + \frac{1}{2} E_3(d)\left[1 + ik\left(\frac{1}{2} - a\right)\right][C(k) + \sigma_\alpha]\right\}\bar{\alpha}$$

$$+ \left(\frac{b}{b_o}\right)^2 \left\{\frac{\pi}{2}[D_{\beta_3} + ikD_{\beta_2} - k^2 D_{\beta_1}] + \frac{1}{2} E_3(d)\left[E_1 + \frac{ik}{2} E_2\right][C(k) + \sigma_\beta]\right\}$$

$$\times \bar{\beta} + \left(\frac{b}{b_o}\right)^2 \left\{\frac{\pi}{2}[D_{\gamma_3} + ikD_{\gamma_2} - k^2 D_{\gamma_1}] + \frac{1}{2\pi} E_3(d)\left[E_1(d)\right.\right.$$

$$\left.\left. + \frac{ik}{2} E_2(d)\right][C(k) + \sigma_\gamma]\right\}\bar{\gamma} \qquad (4)$$

Equations (1) to (4) employ the notation of reference 1 and the terms A, B, C, and D are defined there. The values of the terms E are

$$\left.\begin{aligned} E_1 &= T_{10} - lT_{21} & E_1(d) &= T_{10}(d) - mT_{21}(d) \\ E_2 &= T_{11} - 2lT_{10} & E_2(d) &= T_{11}(d) - 2mT_{10}(d) \\ E_3 &= T_{12} - 2lT_{20} & E_3(d) &= T_{12}(d) - 2mT_{20}(d) \end{aligned}\right\} \qquad (5)$$

and the terms T are also defined in reference 1.

The dimensions a, c, d, e, f, l, and m (all as defined in reference 1 and in fig. 1) are in units of the local semichord b ($b = b_o$ at midspan) and $k = \omega b/U$ stands for the local reduced frequency.

The terms $\sigma_j (j = h, \alpha, \beta, \gamma)$ which alone represent the effect of finite span are given by

$$\sigma_j = \left[C(k) + \frac{iJ_1(k)}{J_o(k) - iJ_1(k)} \right] \left(\frac{\bar{\Omega}_j}{\bar{\Omega}_j^{(2)}} - 1 \right) \tag{6}$$

The two-dimensional circulation functions $\bar{\Omega}_j^{(2)}$ in equation (6) are defined by

$$\left.\begin{aligned}
\bar{\Omega}_h^{(2)} &= \frac{4iC(k)}{kH_1^{(2)}(k)} e^{ik_m} ik \frac{\bar{h}(y)}{b_o} \\
\bar{\Omega}_\alpha^{(2)} &= \frac{4iC(k)}{kH_1^{(2)}(k)} e^{ik_m} \left[1 + ik \left(\frac{1}{2} - a \right) \right] \frac{b}{b_o} \bar{\alpha}(y) \\
\bar{\Omega}_\beta^{(2)} &= \frac{4iC(k)}{\pi kH_1^{(2)}(k)} e^{ik_m} \left[E_1 + \frac{ik}{2} E_2 \right] \frac{b}{b_o} \bar{\beta}(y) \\
\bar{\Omega}_\gamma^{(2)} &= \frac{4iC(k)}{\pi kH_1^{(2)}(k)} e^{ik_m} \left[E_1(d) + \frac{ik}{2} E_2(d) \right] \frac{b}{b_o} \bar{\gamma}(y)
\end{aligned}\right\} \tag{7}$$

In order to take into account the possibility of phase variation in span direction the function $\bar{h}(y)$, $\bar{\alpha}(y)$, $\bar{\beta}(y)$, and $\bar{\gamma}(y)$ in equations (1) to (4) and (7) must be multiplied by terms $e^{-i\phi_h(y)}$, $e^{-i\phi_\alpha(y)}$, $e^{-i\phi_\beta(y)}$, and $e^{-i\phi_\gamma(y)}$, respectively. In equations (7) the quantity $k_m = k(x_t + x_l)/2b_o$ represents the effect of sweep of the midchord line of the wing.

The three-dimensional circulation functions $\bar{\Omega}_j$ are solutions of the following integral equation:

$$\bar{\Omega}_j(y^*) + \frac{b}{b_o}\mu(k)\left[\oint_{-s}^{s} \frac{d\bar{\Omega}_j}{y^* - \eta^*}\right.$$

$$\left. - ik_o \int_{-s}^{s} \frac{y^* - \eta^*}{|y^* - \eta^*|} F(k_o|y^* - \eta^*|)d\,\bar{\Omega}_j\right] = \bar{\Omega}_j^{(2)}(y^*) \qquad (8)$$

In equation (8) y^* indicates the location of points along the span in units of the semichord at midspan b_o.

The function F in the second integral is defined as

$$F(x) = \int_0^\infty e^{-i\lambda}\left[\frac{1}{x} + \frac{1}{\lambda} - \frac{\sqrt{\lambda^2 + x^2}}{x\lambda}\right]d\,\lambda \qquad (9)$$

The function μ is defined in terms of Bessel functions as

$$\mu(k) = \frac{J_o(k) - iJ_1(k)}{\pi k\left\{\left[J_o(k) - Y_1(k)\right] - i\left[J_1(k) + Y_o(k)\right]\right\}} \qquad (10)$$

Numerical values of the functions $C(k)$, $\mu(k)$,

$$C(k) + \frac{iJ_1(k)}{J_o(k) - iJ_1(k)}, \quad \frac{iC(k)}{kH_1^{(2)}(k)},$$

and $F(x)$, which are of importance for the evaluation of the general formulas, are given in tables I to V.

Lift and moments at points of no wing deflection.- At points of no wing deflection equations (1) to (4) cannot be used directly because one or more of the σ_j terms becomes infinite, while the corresponding deflection is zero. In view of equations (6) and (7) there may be written for a point where a deflection function $j(b, \alpha, \beta, \gamma)$ becomes zero,

$$\left.\begin{aligned}
\frac{\bar{L}_j}{2\rho U^2 b_o} &= \pi \frac{C(k) + \dfrac{iJ_1(k)}{J_o(k) - iJ_1(k)}}{4\dfrac{iC(k)}{kH_1^{(2)}(k)} e^{ik_m}} \bar{\Omega}_j \\
\frac{\bar{M}_{a,j}}{2\rho U^2 b_o^2} &= \pi \frac{b}{b_o}\left[-\left(\frac{1}{2} + a\right)\right] \frac{C(k) + \dfrac{iJ_1(k)}{J_o(k) - iJ_1(k)}}{4\dfrac{iC(k)}{kH_1^{(2)}(k)} e^{ik_m}} \bar{\Omega}_j \\
\frac{\bar{M}_{\beta,j}}{2\rho U^2 b_o^2} &= \frac{E_3}{2} \frac{b}{b_o} \frac{C(k) + \dfrac{iJ_1(k)}{J_o(k) - iJ_1(k)}}{4\dfrac{iC(k)}{kH_1^{(2)}(k)} e^{ik_m}} \bar{\Omega}_j \\
\frac{\bar{M}_{\gamma,j}}{2\rho U^2 b_o^2} &= \frac{E_3(i)}{2} \frac{b}{b_o} \frac{C(k) + \dfrac{iJ_1(k)}{J_o(k) - iJ_1(k)}}{4\dfrac{iC(k)}{kH_1^{(2)}(k)} e^{ik_m}} \bar{\Omega}_j
\end{aligned}\right\} \qquad (11)$$

Expressions for lift and rolling-moment coefficients.- The coefficient of wing lift may be defined as

$$\bar{C}_L = \frac{\int_{-sb_o}^{sb_o} \bar{L}dy}{\frac{1}{2}\rho U^2 S_w} = \frac{8sb_o^2}{S_w}\int_0^1 \frac{\bar{L}}{2\rho U^2 b_o} d\left(\frac{y}{sb_o}\right) \qquad (12a)$$

For a rectangular wing

$$\frac{8sb_o^2}{S_w} = 2$$

For an elliptical wing

$$\frac{8sb_o^2}{S_w} = \frac{8}{\pi}$$

The wing rolling-moment coefficient may be defined as

$$\bar{C}_R = \frac{\int_{-sb_o}^{sb_o} \bar{L}\, y\, dy}{\frac{1}{2}\rho U^2 (2sb_o) S_w} = \frac{4sb_o^2}{S_w} \int_0^1 \frac{\bar{L}}{2\rho U^2 b_o} \left(\frac{y}{sb_o}\right) d\left(\frac{y}{sb_o}\right) \quad (12b)$$

Corresponding formulas may be written down for a pitching-moment coefficient and for control-surface hinge-moment coefficients. In the present paper, however, use is made only of the definitions of $\bar{C}_L$ and $\bar{C}_R$.

SOLUTION OF THE INTEGRAL EQUATION FOR THE CIRCULATION FUNCTION

For the solution of equation (8) a procedure is chosen which is analogous to one of the known methods for the solution of the lifting-line equation for uniform motion.

Define new variables ϕ and θ by the relations,

$$y^* = s \cos \phi, \quad \eta^* = s \cos \theta$$

and write

$$\bar{\Omega}_{j_m}(y^*) = \sum_n K_{nj} \frac{\sin n \phi_m}{n} \quad (13)$$

In terms of the new variables equation (8) becomes

$$\sum K_{nj} \frac{\sin n\phi}{n} + \frac{b}{b_o} \mu(k) \left[- \oint_0^\pi \frac{\sum K_{nj} \cos n\theta}{s(\cos\phi - \cos\theta)} d\theta \right.$$

$$\left. + ik_o \int_0^\pi \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} F(k_o s|\cos\phi - \cos\phi|) \sum K_{nj} \cos n\theta d\theta \right]$$

$$= \bar{\Omega}_j^{(2)}(s \cos\phi) \tag{14}$$

Introduce the known formula,

$$\oint_0^\pi \frac{\cos n\theta d\theta}{\cos\phi - \cos\theta} = -\pi \frac{\sin n\phi}{\sin\phi}$$

and write equation (14) in the form,

$$\sum K_{nj} \left\{ \frac{\sin n\phi}{n} + \frac{b}{b_o} \frac{\pi}{s} \mu \left[\frac{\sin n\phi}{\sin\phi} \right. \right.$$

$$\left. \left. + \frac{ik_o s}{\pi} \int_0^\pi \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} F(k_o s|\cos\phi - \cos\theta|) \cos n\theta d\theta \right] \right\}$$

$$= \bar{\Omega}_j^{(2)}(s \cos\phi) \tag{15}$$

Now, define a set of functions S_n by the relation,

$$S_n(k_o s,\phi) = \frac{\sin n\phi}{\sin\phi}$$

$$+ i\frac{k_o s}{\pi}\int_0^\pi \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} F(k_o s|\cos\phi - \cos\theta|)\cos n\theta d\theta \tag{16}$$

Equation (15) may then be written

$$\sum K_{nj}\left\{\frac{\sin n\phi}{n} + \frac{\pi}{s}\frac{b}{b_o}\mu(k)S_n(k_o s,\phi)\right\} = \bar{\Omega}_j^{(2)}(s\cos\phi) \tag{17}$$

Equation (17) is solved approximately by satisfying it at as many suitably chosen points ϕ_m as there are coefficients K_n. Write

$$A_{nm} = \frac{\sin n\phi}{n} + \frac{\pi}{s}\frac{b}{b_o}\mu(k)S_n(k_o s,\phi_m) \tag{18}$$

and

$$\bar{\Omega}_j^{(2)}(s\cos\phi_m) = \bar{\Omega}_{j,m}^{(2)} \tag{19}$$

Then the system of equations to be solved is

$$\sum_n K_{nj}A_{nm} = \bar{\Omega}_{j,m}^{(2)} \tag{20}$$

Note, that if the wing deflection is symmetrical about midspan, only odd values of n occur, while for antisymmetrical deflections only even values of n occur. If this is taken into account, all points ϕ_m may be chosen within a semispan. If the number of points ϕ_m which are selected is N, then equation (20) is a system of N simultaneous complex linear equations to be solved for the N complex quantities K_{nj}. Note further that the left-hand side of equation (20) depends only on wing plan form and on reduced frequency, but not on the form of the deflection function which determines the right-hand side.

The functions $S_n(k_os,\phi_m)$ which occur in equation (18) have been calculated for a number of values of n, k_os, and ϕ_m and the results are contained in table VI. The functions $\sin n\phi_m/n$ are also calculated for the same values of n and ϕ_m and the results are given in table VII. Figure 2 contains a series of plots of S_n against k_os, so that also for intermediate values of k_os the values of S_n are available.

Calculation of the functions $\overline{S_n}$ according to equation (16) involves the evaluation of an integral containing the function F defined by equation (9). With tabular values of F available, the integration can be carried out graphically or by approximating F in the range of interest of the variable x by a function which permits explicit integration. Both these procedures have been used. (See appendix A.) Use has also been made of calculations carried out in England (reference 2) in connection with a theory of the aerodynamic-span effect which differs from the one evaluated here.

SUMMARY OF PROCEDURE FOR CALCULATING THREE-DIMENSIONAL EFFECTS

(1) Choose spanwise stations at which integral equation (8) is to be satisfied. (Present range of calculations of auxiliary functions restricts choice to the six stations,

$$\cos \phi_m = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ \text{and}\ 1.0$$

in units of the semispan.)

(2) Tabulate the values of b/b_o, k, k_m, a, and, if needed, of c, d, e, f, l and m at these stations. Take from table IV the values of the function $4iC/kH_1^{(2)}$ at these stations. Calculate, according to equations (7) the values $\bar{\Omega}_{j,m}^{(2)}$ at these stations.

(3) Calculate by means of tables II, VI, VII the values of A_{nm} of of equation (18).

(4) Solve the system of equation (20) for the coefficients K_{nj}, preferably by the Crout method for the solution of systems of simultaneous equations with complex coefficients.

(5) Calculate the values of $\bar{\Omega}_j$ at the selected stations by means of equation (13). Make use of table VII for factors $\sin n\phi_m/n$.

(6) Calculate the values of $\bar{\Omega}_j/\bar{\Omega}_j^{(2)}$ at the selected stations.

(7) Calculate terms σ_j from equation (6) making use of table III.

(8) Substitute terms σ_j in equations (1) to (4) for lift and moments.

It is this procedure that has been used to obtain the results which are discussed in the following section of this report.

COMPARISON OF TWO-DIMENSIONAL AND THREE-DIMENSIONAL LIFT DISTRIBUTIONS

In order to judge the effect of the three-dimensional correction terms in the expressions for lift and moments, a number of cases have been analyzed numerically and the results are shown in table VIII, and by means of figures 3 to 21 which contain the two-dimensional as well as the three-dimensional distributions.

Calculations have been made for wings of elliptical plan form and for wings of rectangular plan form. In both cases the aspect ratios chosen are 3 and 6, and a range of values of the reduced frequency k_o is covered. As wing deflections the following were chosen:

(1) Translation of rigid wing $\left(\frac{h}{b_o} = 1\right)$

(2) Pitching of rigid wing about midchord $(\alpha = 1, \ a = 0)$

(3) Rolling of rigid wing $\left(\frac{\bar{h}}{b_o} = y/sb_o\right)$

(4) Linear antisymmetrical torsion $(\bar{\alpha} = y/sb_o, \ a = 0)$

(5) Linear symmetrical torsion $(\bar{\alpha} = |y|/sb_o, \ a = 0)$

(6) Linear symmetrical bending $\left(\frac{\bar{h}}{b_o} = |y|/sb_o\right)$

(7) Parabolic symmetrical bending $\left(\frac{\bar{h}}{b_o} = (y/sb_o)^2\right)$

(8) Deflection of full-span aileron $(\bar{\beta} = 1)$

(9) Deflection of partial-span aileron
$(|y| < \lambda sb_o : \beta = 0; \ |y| > \lambda sb_o : \bar{\beta} = 1)$

Figures 3 to 21 contain the results of these calculations. (Some of the results were first obtained in reference 3.) The points at which the integral equation for the circulation has been satisfied are indicated in the figures; the number of these points determines the number of simultaneous equations to be solved. The following conclusions may be stated:

The aerodynamic-span effect for a given aspect ratio and given wing deflection decreases noticeably as the values of the reduced frequency k_o increase. There is substantial agreement between the results of the three-dimensional theory and the results of the two-dimensional theory when the value of k_o is sufficiently large. The rate at which this approach between the results of the two theories takes place depends on aspect ratio and wing deflection function. The lower the aspect ratio, the slower the approach. The finite-span effect persists to values of k_o which are higher for linear spanwise variation of deflection than they are for no spanwise variation of deflection, and higher for parabolic variation than for linear variation. While these facts are what would be expected, the present calculations should furnish the quantitative information which is needed in order to know when to neglect the aerodynamic-span effect and when not to neglect it. Roughly, it may be stated then that when AR = 6, there is no need to consider the aerodynamic-span effect when $k_o > 1$; and when AR = 3, there is no need to consider the effect when $k_o > 2$. For smaller values of k_o the the effect should be given greater consideration. Important corrections certainly do occur when $k_o < 0.5$ and AR = 6 and when $k_o < 1$ and AR = 3. Corresponding conclusions will hold for values of aspect ratio between 3 and 6 and for values beyond these limits. Altogether the numerical results obtained may also serve as an indication of the magnitude of the effect to be expected in cases which are somewhat different from those analyzed.

It appears from the present calculations that the aerodynamic-span effect is appreciably less for hinge-moment distributions. It is thought that this fact was not known previously.

In connection with present results for partial-span control surfaces, it should be said that greater accuracy than has been obtained is desirable. The discontinuous spanwise variation of deflection should be taken into account either by using more terms in the series for $\bar{\Omega}$ or by using a special function $\bar{\Omega}_{disc}$; which accommodates the discontinuities in a manner analogous to that in uniform-motion wing theory.

It is noted that for the rectangular wings the lift distribution is not zero at the tip as would be expected. This may be explained as follows: While in an exact lifting-surface theory the chordwise pressure distribution at the tip would vanish, this condition cannot be satisfied in the present approximate theory. In the approximate theory only an average condition can be prescribed and the natural choice of this average condition is to make the circulation vanish at the tip. For not too large values of k this circulation condition is effectively equivalent to the condition of vanishing tip-lift intensity. With increasing value of k these two conditions become less and less similar. The same observation can be made regarding the moment intensities at the tip for rectangular wings. In an exact theory the moment functions would vanish at the tip while the present theory is unable to ensure this.

It must be said that to the extent that the tip conditions are not completely taken care of, the present theory does not fully account for the effect of finite span. However, even when this difficulty occurs, corrections are obtained which are in the right direction and which therefore may be applied. It should be emphasized that no such difficulties occur for wings with zero tip chord because for such wings lift and moment intensities approach zero near the tip in the two-dimensional theory.

In addition to the diagrams showing spanwise lift distributions there are given in figures 15, 16, and 17 curves showing the variation of wing lift coefficient for rigid translation and pitching, and of wing rolling-moment coefficients for rigid rolling and linear antisymmetrical torsion. These figures show again how with the increasing k_o the results of the two-dimensional and of the three-dimensional theory approach each other.

AERODYNAMIC-SPAN CORRECTIONS IN FLUTTER CALCULATIONS

The use of the finite-span corrections of this report in flutter calculations may be explained by use of the procedure described in reference 4.

In the case of combined torsion and bending aileron flutter with geared tab, flutter speed and frequency are determined from a determinantal equation of the form

$$\begin{vmatrix} \underline{\underline{A}} & \underline{\underline{B}} & \underline{\underline{C}} \\ \underline{\underline{D}} & \underline{\underline{E}} & \underline{\underline{F}} \\ \underline{\underline{G}} & \underline{\underline{H}} & \underline{\underline{I}} \end{vmatrix} = 0 \tag{21}$$

The terms of this determinant involve structural components and also aerodynamic components which are designated in the following calculations by the subscript A. The aerodynamic components only are considered here. According to and using the notation of reference 4, with a dimensionless spanwise coordinate z, defined by

$$Z = \frac{y}{b_o s}$$

$$
\left.
\begin{aligned}
\underline{\underline{A}}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^2 L_h \left[f_h(z)\right]^2 d(z) \\
\underline{\underline{B}}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left[L_\alpha - L_h\left(\frac{1}{2} + a\right)\right] \left[f_h(z)\right] \left[f_\alpha(z)\right] d(z) \\
\underline{\underline{C}}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left[L_\beta - L_z(c - e)\right] \left[f_h(z)\right] \left[f_\beta(z)\right] d(z) \\
&\quad + n\int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^3 \left[L_\delta\right] \left[f_h(z)\right] \left[f_\beta(z)\right] d(z) \\
\underline{\underline{D}}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left[\frac{1}{2} - L_h\left(\frac{1}{2} + a\right)\right] \left[f_h(z)\right] \left[f_\alpha(z)\right] d(z) \\
\underline{\underline{E}}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left[-\frac{1}{2}\left(\frac{1}{2} + a\right) + M_\alpha - L_\alpha\left(\frac{1}{2} + a\right)\right. \\
&\quad \left. + L_h\left(\frac{1}{2} + a\right)^2\right] \left[f_\alpha(z)\right]^2 d(z) \\
\underline{\underline{F}}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left[M_\beta - L_\beta\left(\frac{1}{2} + a\right) - M_z(c - e) + L(c - e)\right. \\
&\quad \left. \times \left(\frac{1}{2} + a\right)\right] \left[f_\alpha(z)\right] \left[f_\beta(z)\right] d(z) + n\int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^4 \left[M_\delta - L_\delta\right. \\
&\quad \left. \times \left(\frac{1}{2} + a\right)\right] \left[f_\alpha(z)\right] \left[f_\beta(z)\right] d(z) \\
\underline{\underline{G}}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left[T_h - P_h(c - e)\right] \left[f_h(z)\right] \left[f_\beta(z)\right] d(z) \\
\underline{\underline{H}}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left[T_\alpha - T_h\left(\frac{1}{2} + a\right) - P_\alpha(c - e) + P_h(c - e)\right. \\
&\quad \left. \times \left(\frac{1}{2} + a\right)\right] \left[f_\alpha(z)\right] \left[f_\beta(z)\right] d(z) \\
\underline{\underline{I}}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left[T_\beta + P_z(c - e)^2 - (P_\beta + T_z)(c - e)\right] \\
&\quad \times \left[f_\beta(z)\right]^2 d(z) + n\int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^4 \left[T_\delta - P_\delta(c - e) + Q_\beta - Q_z\right. \\
&\quad \left. \times (c - e)\right] \left[f_\beta(z)\right]^2 d(z)
\end{aligned}
\right\} \qquad (22)
$$

Introducing for the L, M, T, P, and Q functions their equivalent values in the notation of the present report, the aerodynamic determinant terms become:

$$
\left.
\begin{aligned}
\underline{A}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^2 \left[1 - \frac{2i}{k} C(k)\right] [f_h(z)]^2 \, d(z) \\
\underline{B}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left[-\left(\frac{i}{k} + a\right) - \left(\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right) C(k)\right] [f_h(z)] \\
&\quad \times [f_\alpha(z)] \, d(z) \\
\underline{C}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left\{-\left(\frac{i}{k} C_{\beta_2} - C_{\beta_1}\right) - \frac{1}{\pi}\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right] C(k)\right\} \\
&\quad \times [f_h(z)][f_\beta(z)] \, d(z) + n \int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^3 \left\{-\left(\frac{i}{k} C_{\gamma_2} - C_{\gamma_1}\right)\right. \\
&\quad \left. - \frac{1}{\pi}\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right] C(k)\right\} [f_h(z)] [f_\beta(z)] \, d(z) \\
\underline{D}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left[-a + \left(\frac{1}{2} + a\right) \frac{2i}{k} C(k)\right] [f_h(z)][f_\alpha(z)] \, d(z) \\
\underline{E}_A &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left\{-\left(\frac{i}{k} A_{\alpha_2} - A_{\alpha_1}\right) + \left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\right.\right. \\
&\quad \left.\left. \times \left(\frac{1}{2} - a\right)\right] C(k)\right\} [f_\alpha(z)]^2 \, d(z) \\
\underline{F}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\left(\frac{A_{\beta_3}}{k^2} + \frac{iA_{\beta_2}}{k} - A_{\beta_1}\right) + \left(\frac{1}{2} + a\right)\left(\frac{1}{\pi}\right)\left[\frac{2E_1}{k^2}\right.\right. \\
&\quad \left.\left. + \frac{iE_2}{k}\right] C(k)\right\} [f_\alpha(z)] [f_\beta(z)] \, d(z) + n \int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^4 \left\{-\left(\frac{A_{\gamma_3}}{k^2}\right.\right. \\
&\quad \left.\left. + \frac{iA_{\gamma_2}}{k} - A_{\gamma_1}\right) + \left(\frac{1}{2} + a\right)\left(\frac{1}{\pi}\right)\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right] C(k)\right\} \\
&\quad \times [f_\alpha(z)] [f_\beta(z)] \, d(z) \\
\underline{G}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left[Bh_1 - \frac{E_3}{\pi} \frac{1}{k} C(k)\right] [f_h(z)] [f_\beta(z)] \, d(z) \\
\underline{H}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\left(\frac{i}{k} B_{\alpha_2} - B_{\alpha_1}\right) - \frac{E_3}{\pi}\left[\frac{1}{k^2} + \frac{i}{k}\left(\frac{1}{2} - a\right)\right]\right. \\
&\quad \left. \times C(k)\right\} [f_\alpha(z)][f_\beta(z)] \, d(z) \\
\underline{I}_A &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\left[\frac{B_{\beta_3}}{k^2} + \frac{iB_{\beta_2}}{k} - B_{\beta_1}\right] - \frac{E_3}{2\pi^2}\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right] C(k)\right\} \\
&\quad \times [f_\beta(z)]^2 \, d(z) + n \int_{z_3}^{z_4} \left(\frac{b}{b_r}\right)^4 \left\{-\left[\frac{B_{\gamma_3}}{k^2} + \frac{iB_{\gamma_2}}{k} - B_{\gamma_1}\right]\right. \\
&\quad \left. - \frac{E_3}{2\pi^2}\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right] C(k)\right\} [f_\beta(z)]^2 \, d(z)
\end{aligned}
\right\} \quad (23)
$$

According to equations (1) to (4) the finite-span corrections appear as corrections to the Theodorsen function $C(k)$ in such a manner that $C(k)$ becomes $C(k) + \sigma_j$. By substituting $C(k) + \sigma_j$ for $C(k)$ in the determinant terms, the determinant terms developed using two-dimensional aerodynamic considerations become three-dimensional terms. With the three-dimensional aerodynamic series indicated by a superscript 3 there may be written:

$$
\left.
\begin{aligned}
\underline{\underline{A}}_A^{(3)} &= \underline{\underline{A}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^2 \left[-\frac{2i}{k}\sigma_h\right][f_h(z)]^2\,dz \\
\underline{\underline{B}}_A^{(3)} &= \underline{\underline{B}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha\right\} \\
&\quad \times [f_h(z)][f_\alpha(z)]\,d(z) \\
\underline{\underline{C}}_A^{(3)} &= \underline{\underline{C}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\frac{1}{\pi}\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta\right\} \\
&\quad \times [f_h(z)][f_\beta(z)]\,d(z) + n\int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\frac{1}{\pi}\left[\frac{2E_1(d)}{k^2}\right.\right. \\
&\quad \left.\left. + \frac{iE_2(d)}{k}\right]\sigma_\gamma\right\}[f_h(z)][f_\beta(z)]\,dz \\
\underline{\underline{D}}_A^{(3)} &= \underline{\underline{D}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left[\left(\frac{1}{2} + a\right)\frac{2i}{k}\sigma_h\right][f_h(z)][f_\alpha(z)]\,d(z) \\
\underline{\underline{E}}_A^{(3)} &= \underline{\underline{E}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left\{\left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha\right\} \\
&\quad \times [f_\alpha(z)]^2\,d(z) \\
\underline{\underline{F}}_A^{(3)} &= \underline{\underline{F}}_A + \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left\{\left(\frac{1}{2} + a\right)\left(\frac{1}{\pi}\right)\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta\right\} \\
&\quad \times [f_\alpha(z)][f_\beta(z)]\,d(z) + n\int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left\{\left(\frac{1}{2} + a\right)\right. \\
&\quad \left. \times \left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right]\sigma_\gamma\right\}[f_\alpha(z)][f_\beta(z)]\,d(z) \\
\underline{\underline{G}}_A^{(3)} &= \underline{\underline{G}}_A + \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left[-\frac{E_3}{\pi}\,\frac{i}{k}\sigma_h\right][f_h(z)][f_\beta(z)]\,d(z) \\
\underline{\underline{H}}_A^{(3)} &= \underline{\underline{H}}_A + \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{\pi}\left[\frac{1}{k^2} + \frac{i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha\right\} \\
&\quad \times [f_\alpha(z)][f_\beta(z)]\,d(z) \\
\underline{\underline{I}}_A^{(3)} &= \underline{\underline{I}}_A + \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{2\pi^2}\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta\right\}[f_\beta(z)]^2\,d(z) \\
&\quad + n\int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{2\pi^2}\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right]\sigma_\gamma\right\}[f_\beta(z)]^2\,d(z)
\end{aligned}
\right\} \quad (24)
$$

By rewriting the finite-span correction terms and indicating them by a Δ prefix, the finite-span corrections for the bending-torsion-aileron flutter determinant of reference 4 are then as follows:

$$
\left.
\begin{aligned}
\Delta \underline{\underline{A}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^2 \left[-\frac{2i}{k}\sigma_h\right] [f_h(z)]^2 d(z) \\
\Delta \underline{\underline{B}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha\right\} \\
&\quad \times [f_h(z)][f_\alpha(z)] d(z) \\
\Delta \underline{\underline{C}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\frac{1}{\pi}\left[2\frac{E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta\right\} [f_h(z)][f_\beta(z)] d(z) \\
&\quad + n \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left\{-\frac{1}{\pi}\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right]\sigma_\gamma\right\} [f_h(z)] \\
&\quad \times [f_\beta(z)] d(z) \\
\Delta \underline{\underline{D}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^3 \left(\frac{1}{2} + a\right)\frac{2i}{k}\sigma_h [f_h(z)][f_\alpha(z)] d(z) \\
\Delta \underline{\underline{E}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha [f_\alpha(z)]^2 d(z) \\
\Delta \underline{\underline{F}} &= \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left(\frac{1}{2} + a\right)\left(\frac{1}{\pi}\right)\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta [f_\alpha(z)] \\
&\quad \times [f_\beta(z)] d(z) + n \int_0^{1.0} \left(\frac{b}{b_r}\right)^4 \left(\frac{1}{2} + a\right)\left(\frac{1}{\pi}\right)\left[\frac{2E_1(d)}{k^2}\right. \\
&\quad \left. + \frac{iE_2(d)}{k}\right]\sigma_\gamma [f_\alpha(z)][f_\beta(Z)] d(z) \\
\Delta \underline{\underline{G}} &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^3 \left[-\frac{E_3}{\pi}\,\frac{i}{k}\sigma_h\right][f_h(z)][f_\beta(z)] d(z) \\
\Delta \underline{\underline{H}} &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{\pi}\left[\frac{1}{k^2} + \frac{i}{k}\left(\frac{1}{2} - a\right)\right]\sigma_\alpha\right\} \\
&\quad \times [f_\alpha(z)][f_\beta(z)] d(z) \\
\Delta \underline{\underline{I}} &= \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{2\pi^2}\left[\frac{2E_1}{k^2} + \frac{iE_2}{k}\right]\sigma_\beta\right\} [f_\beta(z)]^2 d(z) \\
&\quad + n \int_{z_1}^{z_2} \left(\frac{b}{b_r}\right)^4 \left\{-\frac{E_3}{2\pi^2}\left[\frac{2E_1(d)}{k^2} + \frac{iE_2(d)}{k}\right]\sigma_\gamma\right\} \\
&\quad \times [f_\beta(z)]^2 d(z)
\end{aligned}
\right\} \quad (25)
$$

These integrals for the finite-span corrections for the terms of the flutter determinant cannot in general be integrated explicitly. It is necessary in all but very special cases to integrate these corrections graphically or by numerical means.

Bending and torsion of rectangular wing.- This case may be considered individually as an example in which explicit integration is possible. By taking the values of σ_h and σ_α from equation (6), there may first be written

$$\left.\begin{aligned}
\Delta \underline{\underline{A}} &= -\frac{2i}{k}\left[C + \frac{iJ_1}{J_o - iJ_1}\right]\int_0^1 \left(\frac{\bar{\Omega}_h}{\bar{\Omega}_h^{(2)}} - 1\right)\left[f_h(z)\right]^2 dz \\
\Delta \underline{\underline{B}} &= -\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right] \\
&\quad \times \left[C + \frac{iJ_1}{J_o - iJ_1}\right]\int_0^1 \left(\frac{\bar{\Omega}_\alpha}{\bar{\Omega}_\alpha^{(2)}} - 1\right) f_h(z)f_\alpha(z)dz \\
\Delta \underline{\underline{D}} &= \left(\frac{1}{2} + a\right)\frac{2i}{k}\left[C + \frac{iJ_1}{J_o - iJ_1}\right]\int_0^1 \left(\frac{\bar{\Omega}_h}{\bar{\Omega}_h^{(2)}} - 1\right) \\
&\quad \times f_h(z)f_\alpha(z)dz \\
\Delta \underline{\underline{E}} &= \left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right] \\
&\quad \times \left[C + \frac{iJ_1}{J_o - iJ_1}\right]\int_0^1 \left(\frac{\bar{\Omega}_\alpha}{\bar{\Omega}_\alpha^{(2)}} - 1\right)\left[f_\alpha(z)\right]^2 dz
\end{aligned}\right\} \tag{26}$$

In view of the constancy of k for rectangular wings and the fact that the circulation functions occur as quotients only, it is permissible to normalize the two-dimensional circulations in such a way that

$$\bar{\Omega}_h^{(2)} = f_h, \qquad \bar{\Omega}_\alpha^{(2)} = f_\alpha \tag{27}$$

and to indicate a corresponding change in the three-dimensional circulation functions by writing

$$\overline{\Omega}'_j = \sum K'_{nj} \frac{\sin n\phi}{n} \tag{28}$$

Introduction of equations (27) and (28) into equation (26) gives

$$\left.\begin{aligned}
\Delta \underline{\underline{A}} &= -\frac{2i}{k}\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\int_0^1 \left[\left\{\sum K'_{nh}\frac{\sin n\phi}{n}\right\} f_h(z) \right. \\
&\quad \left. - [f_h(z)]^2\right] dz \\
\Delta \underline{\underline{B}} &= -\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\int_0^1 \left\{\left(\sum K'_{n\alpha}\right.\right. \\
&\quad \left.\left. \times \frac{\sin n\phi}{n}\right) f_h(z) - f_h(z) f_\alpha(z)\right\} dz \\
\Delta \underline{\underline{D}} &= \left(\frac{1}{2} + a\right)\frac{2i}{k}\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\int_0^1 \left[\left\{\sum K'_{nh}\frac{\sin n\phi}{n}\right\}\right. \\
&\quad \left. \times f_\alpha(z) - f_\alpha(z) f_h(z)\right] dz \\
\Delta \underline{\underline{E}} &= \left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\left[C\right. \\
&\quad \left. + \frac{iJ_1}{J_0 - iJ_1}\right]\int_0^1 \left\{\left(\sum K'_{n\alpha}\frac{\sin n\phi}{n}\right) f_\alpha(z)\right. \\
&\quad \left. - [f_\alpha(z)]^2\right\} dz
\end{aligned}\right\} \tag{29}$$

By assuming linear torsion and parabolic bending, that is,

$$f_{\alpha}(z) = z, \qquad f_h(z) = z^2 \tag{30}$$

the remaining integrations in equations (29) may be carried out and the result is

$$\left.\begin{aligned}
\Delta \underline{\underline{A}} &= -\frac{2i}{k}\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\frac{\frac{5\pi}{16}\left(K'_{1h} + \frac{1}{3}K'_{3h}\right) - 1}{5} \\
\Delta \underline{\underline{B}} &= -\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\left[C + \frac{iJ_1}{J_0 - iJ_1}\right] \\
&\quad \times \frac{\frac{\pi}{4}\left(K'_{1\alpha} + \frac{1}{3}K'_{3\alpha}\right) - 1}{4} \\
\Delta \underline{\underline{D}} &= \left(\frac{1}{2} + a\right)\frac{2i}{k}\left[C + \frac{iJ_1}{J_0 - iJ_1}\right] \\
&\quad \times \frac{\frac{4}{3}\left(K'_{1h} + \frac{1}{5}K'_{3h} - \frac{1}{35}K'_{5h} + \frac{1}{105}K'_{7h} - \ldots\right) - 1}{4} \\
\Delta \underline{\underline{E}} &= \left(\frac{1}{2} + a\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\left[C + \frac{iJ_1}{J_0 - iJ_1}\right] \\
&\quad \times \frac{\left(K'_{1\alpha} + \frac{1}{5}K'_{3\alpha} - \frac{1}{35}K'_{5\alpha} + \frac{1}{105}K'_{7\alpha} - \ldots\right) - 1}{3}
\end{aligned}\right\} \tag{31}$$

Equations (31) will be applied in appendix B of this report for the flutter analysis of a rectangular uniform wing. Equations (25) will be applied for the analysis of a tail-flutter problem.

THE RESULTS OF BIOT AND BOEHNLEIN

In reference 5 theoretical expressions were given for lift and moment at midspan of a rigid elliptical wing with motion consisting of translation and pitching about the midchord line. The results of reference 5 may be compared with corresponding results obtained on the basis of the formulas of the present report.

The relevant expressions of reference 5 may be written in the form

$$\frac{\bar{L}(o)}{2\rho U^2 b_o} = \pi \left\{ -\frac{k_o^2}{2} + ik_o \bar{P}_{AR}(k_o) \right\} \frac{\bar{h}}{b_o} + \pi \left\{ \frac{ik_o}{2} + \left(1 + \frac{ik_o}{2}\right) \bar{P}_{AR}(k_o) \right\} \bar{\alpha} \tag{32}$$

$$\frac{\bar{M}_o(o)}{2\rho U^2 b_o^2} = \pi \left\{ -\frac{ik_o}{2} \bar{Q}_{AR}(k_o) \right\} \frac{\bar{h}}{b_o} + \pi \left\{ \frac{1}{2} \left(ik_o A_{\alpha_2} - k_o^2 A_{\alpha_1}\right) - \frac{1}{2}\left(1 + \frac{ik_o}{2}\right) \bar{Q}_{AR}(k_o) \right\} \bar{\alpha} \tag{33}$$

It may be seen that the effect of three-dimensionality in these formulas is responsible for the change of the function $C(k_o)$ of the two-dimensional theory into the functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ which are tabulated in reference 5 for a range of aspect ratio and of values of k_o.

Comparison with equations (1) and (2) of the present report shows the following relations between the results of reference 5 and the present results:

$$\left.\begin{aligned} \bar{P}_{AR}(k_o) &\longleftrightarrow C(k_o) + \sigma_h(k_o) \\ \bar{P}_{AR}(k_o) &\longleftrightarrow C(k_o) + \sigma_\alpha(k_o) \\ \bar{Q}_{AR}(k_o) &\longleftrightarrow C(k_o) + \sigma_h(k_o) \\ \bar{Q}_{AR}(k_o) &\longleftrightarrow C(k_o) + \sigma_\alpha(k_o) \end{aligned}\right\} \tag{34}$$

Calculations show, that for the uniform deflection functions which are considered here,

$$\sigma_h(k_o) = \sigma_\alpha(k_o) \quad \text{(approx.)} \tag{35}$$

As a formal difference between the two sets of results there remains the fact that in the present work the correction terms for lift and moment are the same, while in the work of reference 5 this is not so. The authors of the present paper are not able to decide which one of the two sets of results is nearer to the actual facts in this respect. It may, however, be noted that the differences between the $\overline{P}$- and $\overline{Q}$-functions are quite small. The order of magnitude of the differences between the two appears to be no greater than the order of magnitude of the deviations of the approximate results from the results of an as yet not existing exact solution of the problem.

A numerical comparison between the $\overline{P}$- and $\overline{Q}$-functions and the function $C + \sigma$ has been made for the four values of aspect ratio, 6, 3, 2, and 1.5. Figure 22 reproduces the real and imaginary parts of the various functions, together with the function C of the two-dimensional theory. It is seen that agreement between the two kinds of three-dimensional corrections is quite close. A theoretically important difference is the fact that for increasing k_o the results of the present work converge toward the result of the two-dimensional theory while, for aspect ratios 3, 2, and 1.5, the results of reference 5 do not converge toward the result of the two-dimensional theory. As k_o approaches zero the results of the present work approach the results of lifting-line theory. It is noted that the same is very nearly true for the function $\overline{Q}$, while the function $\overline{P}$ is consistently somewhat smaller than would follow from lifting-line theory.

Determination of terms σ.- It was found that, for the two forms of motion considered previously, sufficiently accurate results were obtained from a one-term approximation for the circulation function Ω. By taking

$$\overline{\Omega} = K_1 \sin \phi \tag{36}$$

and determining the coefficient K_1 by satisfying the integral equation (8) at midspan $\left(\phi_1 = \frac{\pi}{2}\right)$ there follows from equation (17):

$$K_1 \left[1 + \frac{\pi}{s} \mu(k_o) S_1\left(k_o s, \frac{\pi}{2}\right)\right] = \overline{\Omega}^{(2)}(0)$$

and hence

$$\frac{\bar{\Omega}(o)}{\bar{\Omega}^{(2)}(o)} = \frac{1}{1 + \frac{\pi}{s}\mu(k_o)S_1\left(k_o s, \frac{\pi}{2}\right)} \tag{37}$$

Then, according to equation (6) there follows for the correction terms

$$\sigma_h(k_o) = \sigma_\alpha(k_o) = \left[C(k_o) + \frac{iJ_1(k_o)}{J_o(k_o) - iJ_1(k_o)}\right] \times \left[\frac{1}{1 + \frac{\pi}{s}\mu(k_o)S_1\left(k_o s, \frac{\pi}{2}\right)} - 1\right] \tag{38}$$

Equation (38) is evaluated by means of tables II, III, and IV. Numerical values of the functions σ so obtained are listed in table IX. It may be noted that equation (38), in addition to giving the correction term for translation and pitching, also gives the proper correction for the deflection of a full-span aileron on an elliptical wing as long as the dimensionless aileron-hinge-line coordinate e and leading-edge coordinate $\bar{c}$ are constant along the span.

As a conclusion to this discussion it may be stated that from a practical point of view there seems to be little to choose between the results of reference 5 and the present results in those cases where the results of reference 5 apply. A practical disadvantage of the results of reference 5 appears to be that no provision is made to obtain corresponding results when other than uniform deflection functions are to be taken into consideration.

The present authors would have liked to include also a comparison of their results with the results of W. P. Jones in reference 2; however, notational difficulties could not be overcome in time.

Massachusetts Institute of Technology,
Cambridge, Mass., January 4, 1946.

APPENDIX A

NOTES ON THE EVALUATION OF THE FUNCTION S_n

According to equation (16)

$$S_n(k_o s, \phi) = \frac{\sin n\phi}{\sin \phi} + \frac{ik_o s}{\pi}\int_0^{\pi} \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|}$$

$$\times F(k_o s|\cos\phi - \cos\theta|)\cos n\theta d\theta$$

and according to equation (9)

$$F(x) = \int_0^{\infty} e^{-i\lambda}\left[\frac{1}{x} + \frac{1}{\lambda} - \frac{\sqrt{x^2 + \lambda^2}}{x\lambda}\right] d\lambda = R(x) - iI(x)$$

The function F is given numerically in table V. It can be shown that it possesses the following limiting behavior:

$$x \rightarrow \infty : F(x) \approx \frac{1}{2x^2} - \frac{i}{x}$$

$$x \rightarrow 0 : F(x) \approx -\ln x - 0.39 \ln 2 - \frac{i\pi}{2}$$

It can further be shown (reference 6) that the following representation holds:

$$F(x) = i\int_x^{\infty} \frac{G(u) - 1}{u^2} du \qquad \text{(a)}$$

where $G(u)$ satisfies the differential equation,

$$\frac{d^2G}{du^2} - \frac{1}{u}\frac{dG}{du} - G = \frac{i}{u} \qquad \text{(b)}$$

with the initial conditions,

$$G(0) = 1, \quad G'(0) = -i \tag{c}$$

Equations (a) to (c) are listed here for future reference. So far, it has not been found possible to derive computational advantages from them.

Graphical integration.– The integral,

$$\int_0^\pi \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} I(k_o s|\cos\phi - \cos\theta|)\cos n\theta d\theta$$

may readily ~~be~~ evaluated by graphical or numerical (Simpson's rule) means as the integrand remains finite throughout.

The integral,

$$\int_0^\pi \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} R(k_o s|\cos\phi - \cos\theta|)\cos n\theta d\theta$$

is subject to the difficulty that the function R is logarithmically infinite when $\cos\phi = \cos\theta$. There may be written, for $\epsilon \leq \phi \leq \pi-\epsilon$,

$$\int_0^\pi = \int_0^{\phi-\epsilon} + \int_{\phi-\epsilon}^{\phi+\epsilon} + \int_{\phi+\epsilon}^{\pi}$$

and the first and third integrals evaluated graphically. When $\phi = 0$, write instead

$$\int_0^\pi = \int_0^\epsilon + \int_\epsilon^\pi$$

The second integral may, for sufficiently small ϵ, be estimated analytically as follows:

$$R(k_o s|\cos\phi - \cos\theta|) \approx \ln\left[2^{0.39}k_o s|\cos\phi - \cos\theta|\right]$$

$$\left.\begin{aligned} \theta &= \phi + \eta \qquad d\theta = d\eta \\ \cos\phi - \cos\theta &\approx \eta \sin\phi \\ \cos n\theta &\approx \cos n\phi - \eta n \sin n\phi \end{aligned}\right\} \qquad (d)$$

Hence

$$\int_{\phi-\epsilon}^{\phi+\epsilon} \approx \int_{-\epsilon}^{\epsilon} \frac{\eta}{|\eta|} \ln\left(2^{0.39} k_o s \sin\phi |\eta|\right) \left(\cos n\phi - \eta n \sin n\phi\right) d\eta = -2n \sin n\phi \int_{\epsilon}^{\epsilon} \eta \ln(2^{0.39} k_o s \eta \sin\phi) d\eta$$

$$= -\epsilon^2 (n \sin n\phi) \left[\ln \epsilon + \ln(2^{0.39} k_o s \sin\phi) - \frac{1}{2}\right]$$

This approximation is satisfactory when $\sin\phi\, k_o s \epsilon < 0.05$.

In this manner the functions S_n have been calculated for $k_o s = 0.5$ and various check calculations have been carried out for the values of S_n obtained from reference 2.

Approximate analytical integration.— Write S_n in the form

$$S_n = \frac{k_o s}{\pi} \int_0^{\pi} \left\{\frac{1}{x} - \frac{ix}{|x|} F(|x|)\right\} \cos n\theta d\theta \qquad (e)$$

where

$$x = k_o s(\cos\theta - \cos\phi)$$

The range of integration in x is $|x| \leq 2k_o s$. In this range there may be written approximately:

$$\left.\begin{aligned} R_a(x) &= -\ln x + \sum_{n=o}^{N} A_n x^n \\ I_a(x) &= \sum_{n=o}^{M} B_h x^n \end{aligned}\right\} \qquad (f)$$

For a given $k_o s$ an arbitrarily close approximation for S_n may be be obtained by making the number of terms, N and M, large enough.

For $k_o s \leq 2$, $(x < 4)$, the following approximations have been used:

$$\left.\begin{aligned} R_a(x) &= -\ln x - 0.270 + 0.764x - 0.129x^2 + 0.011x^3 \\ I_a(x) &= 1.319 - 0.757x + 0.202x^2 - 0.020x^3 \end{aligned}\right\} \qquad (g)$$

The coefficients of these expressions were obtained by making R_a and I_a agree with R and I, respectively, in four points of the interval $0 < x \leq 4$.

By introducing equations (g) into equation (e) the following integrals occur:

$$\int_0^\pi \frac{\cos\theta - \cos\phi}{|\cos\theta - \cos\phi|}\cos n\theta\,d\theta = 2\,\frac{\sin n\phi}{n}$$

$$\int_0^\pi (\cos\theta - \cos\phi)\cos n\theta\,d\theta = \begin{cases} \frac{\pi}{2}, & n = 1 \\ 0, & n = 2, 3, \ldots \end{cases}$$

$$\int_0^\pi \frac{(\cos\theta - \cos\phi)^3}{|\cos\theta - \cos\phi|}\cos n\theta\,d\theta = \begin{cases} \frac{4}{3}\sin^3\phi + 2\cos\phi\left(\frac{\pi}{2} - \phi + \frac{\sin 2\phi}{2}\right), \\ \quad n = 1 \\ \left(\frac{\phi}{2} - \frac{\pi}{4}\right) - \frac{\sin 4\phi}{8} - \frac{2\sin 2\phi\sin^2\phi}{3}, \\ \quad n = 2 \\ \left\{\frac{\sin(n-2)\phi}{n(n-1)(n-2)} - \frac{2\sin n\phi}{n(n^2-1)} \right. \\ \quad \left. + \frac{\sin(n+2)\phi}{n(n+1)(n+2)}\right\}, n = 3, 4, \ldots \end{cases}$$

$$\int_0^\pi (\cos\theta - \cos\phi)^3\cos n\theta\,d\theta = \begin{cases} \frac{3\pi}{8}(1 + 4\cos^2\phi), & n = 1 \\ -\frac{3}{4}\pi\cos\phi, & n = 2 \\ \frac{\pi}{8}, & n = 3 \\ 0, & n = 4, 5, \ldots \end{cases}$$

A somewhat less simple calculation leads to a recursion formula for the integral,

$$f_n = \int_0^{\pi} \frac{\cos\theta - \cos\phi}{|\cos\theta - \cos\phi|} \left\{ \ln |\cos\theta - \cos\phi| \right\} \cos n\theta d\theta$$

The recursion formula is

$$\left(1 - \frac{1}{n}\right) f_{n-1} - 2\cos\phi\, f_n + \left(1 + \frac{1}{n}\right) f_{n+1}$$

$$= \frac{2}{n}\left[\frac{\sin (n-1)\phi}{n-1} - \frac{\sin (n+1)\phi}{n+1}\right], \qquad n = 2, 3, \ldots$$

With the initial conditions,

$$f_1 = 2\sin\phi \left[2 \ln |\sin\phi| + \ln 2 - 1\right] - 2\cos\phi\left(\phi - \frac{\pi}{2}\right)$$

$$f_2 = \sin 2\phi \left[2 \ln |\sin\phi| + \ln 2 - 1\right] - \frac{\sin 2\phi}{2} - \cos 2\phi\left(\phi - \frac{\pi}{2}\right)$$

On the basis of these formulas a number of values of the functions S_n have been calculated and the results compared with the results of the graphical solution. It was found that there was very good agreement for the imaginary parts of S_n, differences between the two values obtained by the two methods occurring only in the third or fourth decimal. Somewhat less satisfactory agreement was found for the real parts of S_n, with deviations of from 1 to 3 percent. While for practical purposes an uncertainty of this magnitude with regard to the values of S_n is of no importance, it is believed that the results of the graphical analysis are the more reliable. It still appears to be desirable to set up an analytical scheme by means of which the values of S_n may be calculated accurate to, say, three or four decimals.

A transformation of $S_1(k_o s, \pi/2)$.— It may be shown that this function can be written as a single integral when use is made of complete elliptical integrals. From equation (16)

$$S_1\left(k_0 s, \frac{\pi}{2}\right) = 1 - ik_0 s \frac{2}{\pi} \int_0^{\pi/2} \cos\theta \, F(k_0 s \cos\theta) d\theta$$

with

$$\int_0^{\pi/2} \cos\theta \, F(k_0 s \cos\theta) d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left\{ \int_0^{\infty} e^{-i\lambda}\left(\frac{1}{k_0 s \cos\theta} + \frac{1}{\lambda} - \frac{\sqrt{(k_0 s \cos\theta)^2 + \lambda^2}}{k_0 s (\cos\theta)\lambda}\right) d\lambda \right\} d\theta$$

$$= \int_0^{\infty} e^{-i\lambda} \left\{ \int_0^{\pi/2} \left(\frac{1}{k_0 s} + \frac{\cos\theta}{\lambda} - \frac{\sqrt{(k_0 s \cos\theta)^2 + \lambda^2}}{k_0 s \lambda}\right) d\theta \right\} d\lambda$$

$$= \int_0^{\infty} e^{-i\lambda} \left\{ \frac{\frac{\pi}{2}}{k_0 s} + \frac{1}{\lambda} - \frac{\sqrt{\lambda^2 + (k_0 s)^2}}{k_0 s \lambda} \int_0^{\pi/2} \sqrt{1 - \frac{(k_0 s)^2}{\lambda^2 + (k_0 s)^2} \sin^2\theta} \, d\theta \right\} d\lambda$$

$$= \int_0^{\infty} e^{-ik_0 s\lambda} \left\{ \frac{\pi}{2} + \frac{1}{\lambda} - \frac{\sqrt{1 + \lambda^2}}{\lambda} E\left(\frac{1}{\sqrt{1 + \lambda^2}}\right) \right\} d\lambda$$

there follows then

$$S_1\left(k_0 s, \frac{\pi}{2}\right) = 1 - \frac{2ik_0 s}{\pi} \int_0^{\infty} e^{-ik_0 s\lambda} \left\{ \frac{\pi}{2} + \frac{1}{\lambda} \left[1 - \sqrt{1 + \lambda^2} \, E\left(\frac{1}{\sqrt{1 + \lambda^2}}\right) \right] \right\} d\lambda$$

where E designates the complete elliptical integral of the second kind. It appears that this representation might serve as the starting point of further analytical work. It is probable that corresponding, less simple expressions may be obtained for $S_n(k_0 s, \phi)$ also when $n \neq 1$ and $\phi \neq \frac{\pi}{2}$.

APPENDIX B

APPLICATION OF THE THEORY TO WING FLUTTER PROBLEMS

DISCUSSION OF THE THEORY

It was shown in reference 4 that the equations of motion of a wing in bending and torsion aileron flutter with a geared tab can be written in the form of a determinant as

$$\begin{vmatrix} \underline{\underline{A}} & \underline{\underline{B}} & \underline{\underline{C}} \\ \underline{\underline{D}} & \underline{\underline{E}} & \underline{\underline{F}} \\ \underline{\underline{G}} & \underline{\underline{H}} & \underline{\underline{I}} \end{vmatrix} = 0 \qquad \text{(B1)}$$

In the present report the elements of the flutter determinant of reference 4 are represented as the sum of a structural term, a two-dimensional aerodynamic term, and a three-dimensional correction term as follows:

$$\left.\begin{aligned} \underline{\underline{A}} &= \underline{\underline{A}}_s + \underline{\underline{A}}_A + \Delta \underline{\underline{A}} \\ \underline{\underline{B}} &= \underline{\underline{B}}_s + \underline{\underline{B}}_A + \Delta \underline{\underline{B}} \\ \underline{\underline{C}} &= \underline{\underline{C}}_s + \underline{\underline{C}}_A + \Delta \underline{\underline{C}} \\ \underline{\underline{D}} &= \underline{\underline{D}}_s + \underline{\underline{D}}_A + \Delta \underline{\underline{D}} \\ \underline{\underline{E}} &= \underline{\underline{E}}_s + \underline{\underline{E}}_A + \Delta \underline{\underline{E}} \\ \underline{\underline{F}} &= \underline{\underline{F}}_s + \underline{\underline{F}}_A + \Delta \underline{\underline{F}} \\ \underline{\underline{G}} &= \underline{\underline{G}}_s + \underline{\underline{G}}_A + \Delta \underline{\underline{G}} \\ \underline{\underline{H}} &= \underline{\underline{H}}_s + \underline{\underline{H}}_A + \Delta \underline{\underline{H}} \\ \underline{\underline{I}} &= \underline{\underline{I}}_s + \underline{\underline{I}}_A + \Delta \underline{\underline{I}} \end{aligned}\right\} \qquad \text{(B2)}$$

In these terms the subscript s refers to the structural terms; the subscript A refers to the aerodynamic terms calculated by conventional two-dimensional methods integrated along the span; and the prefix Δ refers to the finite-span correction to the aerodynamic parts of the elements of the determinant.

The aerodynamic parts of equation (B2) are given by equations (23) and (25) of this report. The structural terms are listed below in the notation of reference 4.

$$\left.\begin{aligned}
\underline{\underline{A}}_s &= (\mu)_r\left[1-\left(\frac{\omega_\alpha}{\omega}\right)^2\left(\frac{\omega_h}{\omega_\alpha}\right)^2\left(1+ig_h\right)\right]\int_0^{1.0}\frac{M}{M_r}\,[f_h(z)]^2\,dz\\
\underline{\underline{B}}_s &= (\mu x_\alpha)_r\int_0^{1.0}\frac{S_\alpha}{(S_\alpha)_r}\,[f_h(z)]\,[f_\alpha(z)]\,dz\\
\underline{\underline{C}}_s &= (\mu x_\beta)_r\int_{z_1}^{z_2}\frac{S_\beta}{(S_\beta)_r}\,[f_h(z)]\,[f_\beta(z)]\,dz\\
&\quad + n(\mu x_\delta)_r\int_{z_3}^{z_4}\frac{S_\delta}{(S_\delta)_r}\,[f_h(z)]\,[f_\beta(z)]\,dz\\
\underline{\underline{D}}_s &= (\mu x_\alpha)_r\int_0^{1.0}\frac{S_\alpha}{(S_\alpha)_r}\,[f_h(z)]\,[f_\alpha(z)]\,dz\\
\underline{\underline{E}}_s &= (\mu r_\alpha^2)_r\left[1-\left(\frac{\omega_\alpha}{\omega}\right)^2(1+ig_\alpha)\right]\int_0^{1.0}\frac{I_\alpha}{(I_\alpha)_r}\,[f_\alpha(z)]^2\,dz\\
\underline{\underline{F}}_s &= \left\{\mu\left[r_\beta^2+(e-a)x_\beta\right]\right\}_r\int_{z_1}^{z_2}\frac{I_\beta+(e-a)bS_\beta}{[I_\beta+(e-a)bS_\beta]_r}\,[f_\alpha(z)]\,[f_\beta(z)]\,dz\\
&\quad + n\left\{\mu\left[r_\delta^2+(f-a)x_\delta\right]\right\}_r\int_{z_3}^{z_4}\frac{I_\delta+(e-a)bS_\delta}{[I_\delta+(f-a)bS_\delta]_r}\,[f_\alpha(z)]\,[f_\beta(z)]\,dz\\
\underline{\underline{G}}_s &= (\mu x_\beta)_r\int_{z_1}^{z_2}\frac{S_\beta}{(S_\beta)_r}\,[f_h(z)]\,[f_\beta(z)]\,dz + n(\mu x_\delta)_r\int_{z_3}^{z_4}\frac{S_\delta}{(S_\delta)_r}\,[f_h(z)]\,[f_\beta(z)]\,dz\\
\underline{\underline{H}}_s &= \left\{\mu\left[r_\beta^2+(e-a)x_\beta\right]\right\}_r\int_{z_1}^{z_2}\frac{I_\beta+(e-a)bS_\beta}{[I_\beta+(e-a)bS_\beta]_r}\,[f_\alpha(z)]\,[f_\beta(z)]\,dz\\
&\quad + n\left\{\mu\left[r_\delta^2+(f-a)x_\delta\right]\right\}_r\int_{z_3}^{z_4}\frac{I_\delta+(f-a)bS_\delta}{[I_\delta+(f-a)bS_\delta]_r}\,[f_\alpha(z)]\,[f_\beta(z)]\,dz\\
\underline{\underline{I}}_s &= (\mu r_\beta^2)_r\left[1-\left(\frac{\omega}{\omega_\alpha}\right)^2\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2(1+ig_\beta)\right]\int_{z_1}^{z_2}\frac{I_\beta}{(I_\beta)_r}\,[f_\beta(z)]^2\,dz\\
&\quad + n\left\{\mu\left[r_\delta^2(2+n)+2(f-e)x_\delta\right]\right\}_r\int_{z_3}^{z_4}\frac{I_\delta(2+n)+2(f-e)bS_\delta}{[I_\delta(2+n)+2(f-e)bS_\delta]_r}\\
&\quad \times[f_\beta(z)]^2\,dz
\end{aligned}\right\}\quad\text{(B3)}$$

In addition to the general structural terms of equation (B3), the structural terms for the special case of the uniform rectangular wing in bending and torsion flutter with a parabolic bending and a linear torsion mode are listed separately as follows. The corresponding aerodynamic terms are given in equations (23) and (31).

$$\left.\begin{aligned}
\underline{\underline{A}}_s &= \frac{M}{5\pi\rho b_o}\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2\left(\frac{\omega_h}{\omega_\alpha}\right)^2\left(1 + ig_h\right)\right] \\
\underline{\underline{B}}_s &= \frac{S_\alpha}{4\pi\rho b_o{}^3} \\
\underline{\underline{D}}_s &= \frac{S_\alpha}{4\pi\rho b_o{}^3} \\
\underline{\underline{E}}_s &= \frac{I_\alpha}{3\pi\rho b_o{}^4}\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]
\end{aligned}\right\} \quad \text{(B4)}$$

where

M mass of the wing per unit span

S_α static moment of wing per unit span about the elastic axis

I_α moment of inertia of the wing per unit span about the elastic axis

In order to apply finite-span corrections to flutter analysis a procedure should be outlined. A possible method is to analyze first the problem in question by using the conventional two-dimensional aerodynamic terms integrated along the span. If the resulting speed occurs in a range deemed critical, the finite-span corrections may be introduced.

Since the analysis using the two-dimensional values of the aerodynamic parts of the terms of the flutter determinant gives the approximate range of frequency parameters in which flutter may be expected, the range of frequency parameters which must be investigated for an analysis using finite-span corrections is minimized. In general, the number of frequency parameters investigated by using three-dimensional aerodynamic considerations may be limited to three values when this method is used. In the analysis conducted it has been found that the calculated flutter speed occurs at a somewhat lower value of the frequency parameter when the finite-span corrections are applied. This should be kept in mind when conducting flutter analysis which takes into account the effect of a finite span.

The procedure outlined will be used in the two examples presented in this appendix.

EXAMPLE I

This example is presented primarily to illustrate the method of applying finite-span corrections to a simple wing in bending and torsion flutter. The wing selected for the analysis was the N-75 wing of reference 7. In order to supply more accurate data on the characteristics of the wing, it was again tested in the M.I.T. flutter laboratory with equipment which was not available when the original tests were conducted.

The N-75 wing is a rectangular wing of aspect ratio 6 with the following characteristics:

$2sb_o = 60$ in.

$2b_o = 10$ in.

$a = -0.30$

$M = 0.0086$ slug/ft

$S_\alpha = 0.00068$ slug-ft/ft

$I_\alpha = 0.00059$ slug-ft^2/ft

ω_α (measured uncoupled ground frequency in torsion) $= 8.9$ cps

ω_h (measured uncoupled ground frequency in bending) $= 3.9$ cps

$g_h = 0.068$

$g_\alpha = 0.070$

Since all experimental tests were conducted at approximately standard sea-level conditions, the analysis will be conducted using the standard sea-level value for the density,

$$\rho_o = 0.002378 \text{ slug/ft}^3$$

No data on the true flutter mode shapes were provided; so the analysis will be carried out with a linear torsion and a parabolic bending mode assumed. The terms of the flutter determinant for this special case have been integrated explicitly and are provided in equations (23), (31), and (B4). The analysis will be performed and these integrated flutter terms used.

Flutter Analysis with Two-Dimensional Values

for the Aerodynamic Parts of the Determinant

When the supplied data is incorporated in the structural parts of the determinant terms from equation (B4), these terms become,

$$\underline{\underline{A}}_s = \frac{M}{5\pi\rho b_o^2}\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 \left(\frac{\omega_h}{\omega_\alpha}\right)^2 (1 + ig_h)\right]$$

$$= 1.3331\left[1 - 0.1920\left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$$

$$\underline{\underline{B}}_s = \underline{\underline{D}}_s = \frac{S_\alpha}{4\pi\rho b_o^3} = 0.3154$$

$$\underline{\underline{E}}_s = \frac{I_\alpha}{3\pi\rho b_o^4}\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$$

$$= 0.8789\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$$

The corresponding aerodynamic parts of the determinant terms $\underline{\underline{A}}_A$, $\underline{\underline{B}}_A$, $\underline{\underline{D}}_A$, and $\underline{\underline{E}}_A$ from equations (23) from two-dimensional considerations may be calculated directly for several values of the frequency parameter by the use of table I.

The two-dimensional aerodynamic terms for the rectangular wing with a linear torsion mode and a parabolic bending mode are:

$$\underline{\underline{A}}_A = \frac{1}{5}\left[1 - \frac{2iC(k)}{k}\right]$$

$$\underline{\underline{B}}_A = \frac{1}{4}\left\{-\left(\frac{1}{k} + a\right) - C(k)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\right\}$$

$$\underline{\underline{D}}_A = \frac{1}{4}\left\{-a + \frac{2i}{k}\left(\frac{1}{2} + a\right)C(k)\right\}$$

$$\underline{\underline{E}}_A = \frac{1}{3}\left\{-\frac{i}{k}\left(\frac{1}{2} - a\right) + \left(\frac{1}{8} + a^2\right) + \left(a + \frac{1}{2}\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\right.$$

$$\left. \times C(k)\right\}$$

Since the aerodynamic terms of the flutter determinant for this special case may be calculated by simple formula substitution of the values of $C(k)$, a, and k, and since $C(k)$ is tabulated as a function of k in table I, the actual calculation of the aerodynamic terms is not performed here. The determinant terms however, are tabulated in table X as a function of the frequency parameter k.

The flutter determinant thus evaluated and tabulated in table X will be solved by the method outlined in reference 4. This method will not be illustrated, but the two solutions to each of the resultant complex quadratic equations are tabulated as a function of the frequency parameter k_o in the following table:

ROOTS OF THE FLUTTER EQUATION

k	First solution		Second solution	
	V_1 (mph)	g_1	V_2 (mph)	g_2
0.4	17.4	-0.639	29.0	0.081
.5	13.7	- .439	25.1	- .030
.6	11.3	- .330	22.3	- .081

The flutter speed by this method is determined graphically in figure 23, and is equal to 28.6 mph.

Flutter Analysis with Finite-Span Corrections

Since the primary purpose of the report is to illustrate a means of incorporating finite-span corrections in flutter analysis, the means of calculating the correction functions $\sigma_j(h, \alpha, \beta, \gamma)$ will be illustrated and these functions applied in this simple flutter problem.

It has been shown in equation (31) for this special case that the finite-span corrections may be expressed in the following form:

$$f_h(z) = z^2; \qquad f_\alpha(z) = z$$

$$\Delta \underline{A} = -\frac{2i}{k}\int_0^{1.0} \sigma_h z^4 dz = -\frac{2i}{5k}\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\left[\frac{5\pi}{16}\left(K'_{1h} + \frac{K'_{3h}}{3}\right) - 1\right]$$

$$\Delta \underline{B} = -\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\int_0^{1.0} \sigma_\alpha z^3 dz$$

$$= -\frac{1}{2}\left[\frac{1}{k^2} + \frac{i}{k}\left(\frac{1}{2} - a\right)\right]\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\left[\frac{\pi}{4}\left(K'_{1\alpha} + \frac{K'_{3\alpha}}{3}\right) - 1\right]$$

$$\Delta \underline{D} = \frac{2i}{k}\left(\frac{1}{2} + a\right)\int_0^{1.0} \sigma_h z^3 dz = \frac{i}{2k}\left(\frac{1}{2} + a\right)\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]$$

$$\times\left[\frac{4}{3}K'_{1h} + \frac{4}{15}K'_{3h} - \frac{4}{105}K'_{5h} + \frac{4}{315}K'_{7h} - \ldots - 1\right]$$

$$\Delta \underline{E} = \left(a + \frac{1}{2}\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\int_0^{1.0} \sigma_\alpha z^2 dz$$

$$= \frac{1}{3}\left(a + \frac{1}{2}\right)\left[\frac{2}{k^2} + \frac{2i}{k}\left(\frac{1}{2} - a\right)\right]\left[C + \frac{iJ_1}{J_0 - iJ_1}\right]$$

$$\times\left[K'_{1\alpha} + \frac{1}{5}K'_{3\alpha} - \frac{1}{35}K'_{5\alpha} + \frac{1}{105}K'_{7\alpha} - \ldots - 1\right]$$

Since the algebraic terms developed for this special example cannot be developed easily for the general case, the correction terms will usually have to be determined by graphical or numerical integration. For this example, then, the corrections will be determined by graphical integration and by formula substitution in the relations listed previously.

It can be shown from equation (27) that the correction function σ_j may be written for the rectangular wing as,

$$\sigma_j = \left[C + \frac{iJ_1}{J_0 - iJ_1}\right]\left(\frac{w}{f_j(y)} - 1\right)$$

where $f_j(y)$ is the deflection at the station under consideration and w is a function which is proportional to the circulation and may be expressed as,

$$w = \sum K'_{nj} \frac{\sin n\phi}{n}$$

The K'_{nj} terms of this Fourier series may be found from the set of equations,

$$\sum K'_{nj} A_{nm} = f_j(y)_m$$

The relation,

$$\frac{\bar{\Omega}_j}{\bar{\Omega}_j^{(2)}} = \frac{w}{f_j(y)}$$

holds only to the special case of the rectangular wing, but the general method of finding w is the same as that for finding $\bar{\Omega}_j$.

In order to illustrate the method the correction terms will be calculated for $k_o = 0.4$. Since the method is similar for any value of k_o, the procedure will not be repeated for the other values of the frequency parameter investigated. The results, however, will be tabulated for other values of k_o.

The parameters and tabulated functions necessary for the determination of σ_j for the selected value of the frequency parameter, $k_o = 0.4$, are

$$k_o = k = 0.4$$

$$\frac{b}{b_o} = 1$$

$$s = 6$$

$$k_o s = 2.4$$

$$\mu(k) = \mu(k_o) = 0.2644 - 0.0964i$$

$$C + \frac{iJ_1}{J_o - iJ_1} = 0.5850 + 0.0309i$$

The $S_n(\phi, k_o s)$ functions listed in the following table are obtained from the plots of $S_n(\phi, k_o s)$ against $k_o s$ in figure 2:

$\frac{y}{sb_o}$	n	$\frac{\sin n\phi}{n}$	$S_n(\phi, ks)$
0	1	1	0.230 − 0.295i
	3	−0.3333	−.493 + .354i
	5	.2000	.630 − .337i
	7	−.1429	−.715 + .310i
0.4	1	0.9165	0.280 − 0.300i
	3	−.1099	−.204 + .115i
	5	−.0936	−.326 + .174i
	7	.1380	.772 − .315i
0.8	1	0.600	0.588 − 0.280i
	3	.3120	.985 − .535i
	5	−.0151	−.110 + .137i
	7	−.1393	−1.330 + .360i
1.0	1	0	1.665 + 1.310i
	3	0	3.115 + .772i
	5	0	5.050 + .485i
	7	0	7.025 + .344i

When these functions have been tabulated, the integral equation must be solved to fix the circulation distribution along the span. It is thought that the circulation can be approximated satisfactorily for a rectangular wing by the use of a four-term Fourier series approximation. For this example the integral equation will be satisfied at $y/sb_o = 0$, 0.4, 0.8, and 1.0.

The integral equation,

$$\sum K'_{nj}A_{nm} = f_j(y)_m$$

then becomes a set of four simultaneous linear equations to be solved for the K_{nj} coefficients. Since the modes assumed are symmetrical, only odd values of n will occur.

The A_{nm} functions are equal to

$$A_{nm} = \frac{\sin n\phi}{n} + \frac{\pi b}{sb_o} \mu(k) S_n(\phi,\ ks)$$

and when these terms are evaluated for $n = 1, 3, 5, 7$ at $y/sb_o = 0$, 0.4, 0.8, 1.0, the A_{nm} functions become, for $y/sb_o = 0$,

$$A_{11} = 1 + \frac{\pi(0.2644 - 0.0964i)}{6}(0.230 - 0.295i)$$

$$= 1 + (0.1384 - 0.0505i)(0.230 - 0.295i)$$

$$= 1.0169 - 0.0524i$$

$$A_{31} = -0.3333 + (0.1384 - 0.0505i)(-0.493 + 0.354i)$$

$$= -0.3837 + 0.0739i$$

$$A_{51} = 0.2000 + (0.1384 - 0.0505i)(0.630 - 0.337i)$$

$$= 0.2702 - 0.0785i$$

$$A_{71} = -0.1429 + (0.1384 - 0.0505i)(-0.715 + 0.310i)$$
$$= -0.2262 + 0.0790i$$

for $y/sb_o = 0.4$,

$$A_{12} = 0.9165 + (0.1384 - 0.0505i)(0.280 - 0.300i)$$
$$= 0.9401 - 0.0557i$$

$$A_{32} = -0.1099 + (0.1384 - 0.0505i)(-0.204 + 0.115i)$$
$$= -0.1323 + 0.0262i$$

$$A_{52} = -0.0936 + (0.1384 - 0.0505i)(-0.326 + 0.174i)$$
$$= -0.1299 + 0.0405i$$

$$A_{72} = 0.1380 + (0.1384 - 0.0505i)(0.772 - 0.315i)$$
$$= 0.2289 - 0.0826i$$

for $y/sb_o = 0.8$,

$$A_{13} = 0.600 + (0.1384 - 0.0505i)(0.588 - 0.280i)$$
$$= 0.6672 - 0.0684i$$

$$A_{33} = 0.3120 + (0.1384 - 0.0505i)(0.985 - 0.535i)$$
$$= 0.4213 - 0.1238i$$

$$A_{53} = -0.0151 + (0.1384 - 0.0505i)(-0.110 + 0.137i)$$
$$= -0.0234 + 0.0245i$$

$$A_{73} = -0.1398 + (0.1384 - 0.0505i)(-1.330 + 0.360i)$$
$$= -0.3057 + 0.1170i$$

and for $y/sb_o = 1.0$,

$$A_{14} = (0.1384 - 0.0505i)(1.665 + 1.310i)$$
$$= 0.2966 + 0.0972i$$

$$A_{34} = (0.1384 - 0.0505i)(3.115 + 0.772i)$$
$$= 0.4701 - 0.0505i$$

$$A_{54} = (0.1384 - 0.0505i)(5.050 + 0.485i)$$
$$= 0.7234 - 0.1879i$$

$$A_{74} = (0.1384 - 0.0505i)(7.025 + 0.344i)$$
$$= 0.9896 - 0.3072i$$

With the values of A_{nm} determined, and since the modes in question are known, the set of simultaneous equations,

$$\sum K'_{nj} A_{nm} = f_j(y)_m$$

may be solved. This solution is carried out in table XI by the use of the Crout method of reference 8.

When the K'_{nj} terms have been calculated, the function w which is proportional to the circulation is established and the correction functions $\sigma_j(j = \alpha, h)$ may be calculated. The values of w at various stations along the span are calculated in table XII and the $\sigma_j(j = \alpha, h)$ functions are calculated in table XIII.

By using the correction function, the corrections to the determinant terms may be evaluated graphically. In table XIV the integrands of the following pertinent integrals are tabulated for this graphical integration:

$$\int_0^{1.0} \sigma_h z^4 dz , \quad \int_0^{1.0} \sigma_\alpha z^3 dz , \quad \int_0^{1.0} \sigma_h z^3 dz , \quad \int_0^{1.0} \sigma_\alpha z^2 dz$$

These integrals are evaluated graphically in figure 24, (a), (b), (c), and (d).

For this special case it is possible to compare the values of the graphical integration with those obtained by substituting values in the algebraic equations of equation (31) as is done in the following table:

For $k_0 = 0.4$

Term	By graphical integration	By direct integration
$\int_0^{1.0} \sigma_h z^4 dz$	$-0.040 + 0.010i$	$-0.039 + 0.011i$
$\int_0^{1.0} \sigma_\alpha z^3 dz$	$-.042 + .014i$	$-.042 + .014i$
$\int_0^{1.0} \sigma_h z^3 dz$	$-.043 + .012i$	$-.043 + .013i$
$\int_0^{1.0} \sigma_\alpha z^2 dz$	$-.048 + .017i$	$-.047 + .018i$

The aerodynamic corrections themselves are tabulated for a range of k_0 in table XV along with the calculated determinant terms corrected for finite-span effects.

The flutter determinants corrected for finite-span effects, the terms of which are listed in table XV, can be expanded and solved. This was done, and the resulting complex quadratic equation was solved by use of the method of reference 4. These results are tabulated as a function of k_0 in the following table:

ROOTS OF THE FLUTTER EQUATION

k_o	First solution		Second solution	
	V_1 (mph)	g_1	V_2 (mph)	g_2
0.333	20.8	−0.565	33.3	0.035
.40	17.2	−.428	29.7	−.058
.50	13.5	−.316	25.8	−.120

The actual determination of the calculated flutter speed is carried out graphically in figure 23 and the flutter speed is equal to 34.2 mph.

Discussion of the results of example I.- When the N-75 wing of reference 7 with a 10-inch chord was tested for flutter, the wing was observed to flutter at about 34 mph for low angles of attack. The same wing was again tested by the staff of the M.I.T. flutter laboratory, when this wing was selected for analysis in the present report, to provide the measured uncoupled ground frequencies in bending and torsion and to determine the decay curves of the uncoupled bending and torsion modes so the damping coefficients could be calculated. In addition, the wing was again tested in flutter and was observed to flutter at 34.2 mph.

Since the calculated value of the flutter speed with finite-span corrections was found to be 34.2 mph for a value of the frequency parameter k_o of about 0.3, the check between the theoretical and measured values is remarkable. It should be noted, however, that this close check was entirely unexpected since the flutter modes used in the analysis were not found by conducting a ground-vibration-mode analysis, nor were they observed in actual flutter tests. It is believed, however, that the modes used were reasonable.

As an afterthought, the M.I.T. flutter laboratory installed end plates in the wind tunnel as near the tips of the wing model as was practicable (the gap was about 1/16 in.) in order to approximate two-dimensional flow conditions as nearly as possible. When this was done, the measured flutter speed was 30.7 mph. The calculated two-dimensional value was 28.6 mph for a value of k_o of about 0.41. This too is a remarkable check, considering that a truly two-dimensional flow was not possible under the test conditions because a slight gap was left between the wing tips and the end plates. From this analysis it seems that the extra work of a three-dimensional analysis is justified.

EXAMPLE II

In this example the tail-flutter problem of Example No. 7 of reference 4 will be analyzed with the benefit of finite-span corrections. The tail in question flutters in fuselage torsion and fuselage bending. For the sake of the analysis the surfaces themselves are assumed to be rigid.

When this example was analyzed in reference 4, a method of equivalent aerodynamic chords was used to calculate the aerodynamic coefficients. This method is unsuitable for more exact analyses and is replaced in this problem by a two-dimensional method of calculating the aerodynamic coefficients which depends on the plan form of the surfaces to be analyzed and in which the two-dimensional aerodynamic coefficients are integrated along the span. In this problem the structural parts of the determinant terms used will be those of Example No. 7 (reference 4).

The vertical and horizontal tail surfaces used in this problem are shown in figures 25 and 26. Various other physical characteristics of the surfaces are listed in tables XVI and XVII. The values of the uncoupled frequencies to be used in the analysis are

ω_α(assumed uncoupled fuselage side-bending frequency) = 46.1 radians/sec

ω_h(assumed uncoupled fuselage torsion frequency) = 65.0 radians/sec

ω_β(free-rudder frequency) = 0 radian/sec

$g = g_h = g_\alpha = 0.038$

From Example No. 7 of reference 4, the structural terms used are

$$\underline{A}_s = 67.27 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 1.988(1 + ig_h)\right] \text{ft}$$

$$\underline{B}_s = 100.1 \text{ ft}$$

$$\underline{C}_s = 1.731 \text{ ft}$$

$$\underline{D}_s = 100.1 \text{ ft}$$

$$\underline{E}_s = 1891 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right] \text{ft}$$

$$\underline{F}_s = 23.86 \text{ ft}$$

$$\underline{G}_s = 1.737 \text{ ft}$$

$$\underline{H}_s = 23.916 \text{ ft}$$

$$\underline{I}_s = 5.992 \text{ ft}$$

The fuselage torsion and side-bending modes assumed in this example give the tail surfaces the following modes: The horizontal tail will flutter with a rigid rolling mode. The vertical tail will have a rigid rolling mode and a uniform pitching mode about the fuselage bending axis. The rudder will have a uniform deflection mode and since a geared tab is provided on the rudder, the tab will also have a uniform deflection mode. It is for these modes that the aerodynamic parts of the flutter determinant must be calculated. Because of the difference in the root chords, the value of k_o for the vertical tail will be 1.16 times the k_o of the horizontal tail.

Flutter Analysis with Two-Dimensional Values

for the Aerodynamic Parts of the Determinant

By rearranging the relations for the two-dimensional aerodynamic terms in equation (23) to fit the special conditions of this example (where the H subscript indicates values pertaining to the horizontal tail and the V subscripts, to the vertical tail), the terms become

$$\underline{A}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33}\right)^2 \left[1 - \frac{2i}{k_V} C(k_V)\right] z^2 dz$$

$$+ 2(sb_o)_H \int_0^{1.0} \left(\frac{b_H}{3.33}\right)^2 \left[1 - \frac{2i}{k_H} C(k_H)\right] z^2 dz$$

$$\underline{B}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33}\right)^3 \left\{ 1 - \left(\frac{i}{k_V} + a\right) - \left[\frac{2}{k_V^2}\right.\right.$$

$$\left.\left. + \frac{2i}{k_V}\left(\frac{1}{2} - a\right)\right] C(k_V) \right\} z dz$$

$$\underline{C}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33}\right)^3 \left\{ - \left[\frac{i}{k_V} C_{\beta_2} - C_{\beta_1}\right]\right.$$

$$\left. - \frac{1}{\pi}\left[\frac{2E_1}{k_V^2} + \frac{iE_2}{k}\right] C(k_V) \right\} z dz$$

$$+ n(sb_o)_V \int_{z_3}^{z_4} \left(\frac{b_V}{3.33}\right)^3 \left\{ - \left[\frac{i}{k_V} C_{\gamma_2} - C_{\gamma_1}\right]\right.$$

$$\left. - \frac{1}{\pi}\left[\frac{2E_1(d)}{k_V^2} + \frac{iE_2(d)}{k_V}\right] C(k_V) \right\} z dz$$

$$\underline{D}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33} \right)^3 \left\{ -a + \left(\frac{1}{2} + a \right) \frac{2i}{k_V} C(k_V) \right\} z dz$$

$$\underline{E}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ -\left[\frac{iA_{\alpha 2}}{k_V} - A_{\alpha_1} \right] + \left(\frac{1}{2} + a \right) \left[\frac{2}{k^2_V} \right. \right.$$

$$\left. \left. + \frac{2i}{k_V} \left(\frac{1}{2} - a \right) \right] C(k_V) \right\} dz$$

$$\underline{F}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ - \left[\frac{A_{\beta_3}}{k^2_V} + \frac{iA_{\beta_2}}{k_V} - A_{\beta_1} \right] \right.$$

$$\left. + \left(\frac{1}{2} + a \right) \left(\frac{1}{\pi} \right) \left[\frac{2E_1}{{k_V}^2} + \frac{iE_2}{k_V} \right] C(k_V) \right\} dz$$

$$+ n(sb_o)_V \int_{z_3}^{z_4} \left(\frac{b_V}{3.33} \right)^4 \left\{ - \left[\frac{A_{\gamma 3}}{k_V^2} + \frac{iA_{\gamma_2}}{k_V} - A_{\alpha_1} \right] \right.$$

$$\left. + \left(\frac{1}{2} + a \right) \left(\frac{1}{\pi} \right) \left[\frac{2E_1(d)}{k_V^2} + \frac{iE_2(d)}{k_V} \right] C(k_V) \right\} dz$$

$$\underline{G}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33} \right)^3 \left[B_{1h} - \frac{E_3}{\pi} \frac{i}{k_V} C(k_V) \right] z dz$$

$$\underline{H}_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ - \left[\frac{i}{k} B_{\alpha_2} - B_{\alpha_1} \right] \right.$$

$$\left. - \frac{E_3}{\pi} \left[\frac{1}{k_V^2} + \frac{i}{k_V} \left(\frac{1}{2} - a \right) \right] C(k_V) \right\} dz$$

$$I_A = (sb_o)_V \int_0^{1.0} \left(\frac{b_V}{3.33}\right)^4 \left\{-\left[\frac{B_{\beta_3}}{k_V^2} + \frac{iB_{\beta_2}}{k_V} - B_{\beta_1}\right]\right.$$

$$\left. - \frac{E_3}{2\pi^2}\left[\frac{2E_1}{k_V^2} + \frac{iE_2}{k_V}\right] C(k_V)\right\} dz$$

$$+ n(sb_o)_V \int_{z_3}^{z_4} \left(\frac{b_V}{3.33}\right)^4 \left\{-\left[\frac{B_{\gamma_3}}{k_V^2} + \frac{iB_{\gamma_2}}{k_V} - B_{\gamma_1}\right]\right.$$

$$\left. - \frac{E_3}{2\pi^2}\left[\frac{2E_1(d)}{k_V^2} + \frac{iE_2(d)}{k_V}\right] G(k_V)\right\} dz$$

These integrals may be evaluated graphically for a given value of the frequency parameter. By noting that the frequency parameters are different for the vertical and horizontal tail surfaces, the function $C(k)$ can be found from the tables. The functions A, B, and C can be found in reference 1. The function E is defined in equation (5). There is, then, no obstacle to this graphical integration.

The results of evaluating these integrals for a range of k_o are tabulated in table XVIII along with the elements of the determinant. The determinants themselves are solved by the method of reference 4 and are tabulated in the following table:

ROOTS OF FLUTTER DETERMINANT FOR VARIOUS k_o

k_{oH}	k_{oV}	V_1 (mph)	g_1	V_2 (mph)	g_2
0.686	0.795	172	0.004	268	−0.176
.600	.696	194	.011	314	−.248
.436	.506	257	.052	431	−.303

The calculated flutter speed is determined graphically in figure 27.

The true airspeed calculated by this method is 243 mph.

Flutter Analysis with Aerodynamic Terms Corrected for Finite-Span Effects

The relations for the finite-span corrections from equation (25) modified for this example become

$$\Delta \underline{A} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33}\right)^2 \left[-\frac{2i}{k_V}\sigma_{hV}\right] z^2 dz$$

$$+ 2(sb_o)_H \int_o^{1.0} \left(\frac{b_H}{3.33}\right)^2 \left[-\frac{2i}{k_H}\sigma_{hH}\right] z^2 dz$$

$$\Delta \underline{B} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33}\right)^3 \left\{-\left[\frac{2}{k_V^2} + \frac{2i}{k_V}\left(\frac{1}{2} - a\right)\right] \sigma_{\alpha V}\right\} z dz$$

$$\Delta \underline{C} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33}\right)^3 \left\{-\frac{1}{\pi}\left[\frac{2E_1}{k_V^2} + \frac{iE_2}{k_V}\right]\sigma_{\beta_V}\right\} z dz$$

$$+ n(sb_o)_V \int_{z_3}^{z_4} \left(\frac{b_V}{3.33}\right)^3 \left\{-\frac{1}{\pi}\left[\frac{2E_1(d)}{{k_V}^2} + \frac{iE_2(d)}{k_V}\right]\sigma_{\gamma_V}\right\} z dz$$

$$\Delta \underline{D} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33}\right)^3 \left\{\left(\frac{1}{2} + a\right)\frac{2i}{k_V}\sigma_{hV}\right\} z dz$$

$$\Delta \underline{E} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33}\right)^4 \left\{\left(\frac{1}{2} + a\right)\left[\frac{2}{{k^2}_V} + \frac{2i}{k_V}\left(\frac{1}{2} - a\right)\right]\sigma_{\alpha_V}\right\} dz$$

$$\Delta \underline{F} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ \left(\frac{1}{2} + a \right) \left(\frac{1}{\pi} \right) \left[\frac{2E_1}{k_V^2} + \frac{1E_2}{k_V} \right] \sigma_{\beta_V} \right\} dz$$

$$+ n(sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ \left(\frac{1}{2} + a \right) \left(\frac{1}{\pi} \right) \left[\frac{2E_1(d)}{k_V^2} + \frac{1E_2(d)}{k_V} \right] \right.$$

$$\left. \times \sigma_{\beta_V} \right\} dz$$

$$\Delta \underline{G} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^3 \left[-\frac{E_3}{\pi} \frac{1}{k_V} \sigma_{hV} \right] zdz$$

$$\Delta \underline{H} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ -\frac{E_3}{\pi} \left[\frac{1}{k_V^2} + \frac{1}{k_V} \left(\frac{1}{2} - a \right) \right] \sigma_{\alpha V} \right\} dz$$

$$\Delta \underline{I} = (sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ -\frac{E_3}{2\pi^2} \left[\frac{2E_1}{k_V^2} + \frac{1E_2}{k_V} \right] \sigma_{\beta_V} \right\} dz$$

$$+ n(sb_o)_V \int_o^{1.0} \left(\frac{b_V}{3.33} \right)^4 \left\{ -\frac{E_3}{2\pi^2} \left[\frac{2E_1(d)}{k_V^2} + \frac{1E_2(d)}{k_V} \right] \sigma_{\gamma_V} \right\} dz$$

When these terms are investigated, it is observed that five types of σ_j appear in the different relations. This means that five sets of simultaneous equations must be solved to determine the circulation distributions over these surfaces and thus the correction functions σ_j for each value of ω/U selected.

For the surfaces provided in this problem, it is felt that a three-term Fourier series approximation for the circulation will be satisfactory. This means that five sets of three simultaneous equations must be solved.

In this problem there are three basic types of solution in each of which the A_{nm} terms are the same: the rolling of the horizontal tail surfaces, the rolling of the vertical tail surfaces, and the pitching of the vertical tail surfaces, rudder deflection, and tab deflection. Of the five solutions to be conducted, only the solution for the tab deflection is problematical, because of the discontinuous angles of attack at the tab extremities which make it impossible to obtain a satisfactory circulation distribution with a finite number of terms in the Fourier series approximation. The stand is taken that some correction is better than none though; so the three-term approximation for the circulation is used for this mode too.

For the two rolling modes the integral equation will be satisfied at $y/sb_o = 0.4$, 0.6, and 0.8, and for the three symmetrical modes $y/sb_o = 0$, 0.4, and 0.8. For the rolling modes, $n = 2, 4, 6$, and for the symmetrical modes $n = 1, 3, 5$.

In the analysis the vertical tail is handled as if it were half of a wing with a full-span aileron. This is assumed to be satisfactory since the horizontal tail acts as an end plate at the root chord.

In order to illustrate the procedure a value of ω/U will be selected which will give for the vertical tail a value of $k_o = 0.506$ and for the horizontal tail a value of $k_o = 0.436$.

For the rolling of the horizontal tail the correction functions q_h may be calculated as follows:

The necessary parameters and functions for the solutions are first listed in the following tables:

$(k_o = 0.436)$

$\frac{y}{sb_o}$	k	$\frac{b}{b_o}$	$\frac{iC(k)}{kH_1^{(2)}(k)}$	$C(k)+\frac{iJ_1(k)}{J_o(k)-iJ_1(k)}$	$\mu(k)$
0.4	0.375	0.860	0.9126 − 0.1496i	0.5988 + 0.0154i	0.2463 − 0.0873i
.6	.336	.770	.9451 − .1747i	.6205 − .0088i	.2834 − .1038i
.8	.286	.655	.9953 − .2054i	.6523 − .0394i	.3007 − .1089i

$k_o s = 1.101, \quad y/sb_o = 0.4$

n	$\frac{\sin n\phi}{n}$	$S_n(\phi, k_o s)$
2	0.3667	0.494 − 0.278i
4	−.2493	−.850 + .293i
6	.1132	.575 − .150i
	$y/sb_o = 0.6$	
2	0.4800	0.780 − 0.390i
4	−.1344	−.547 + .160i
6	−.1098	−.710 + .178i
	$y/sb_o = 0.8$	
2	0.4800	0.788 − 0.433i
4	.1344	.525 − .170i
6	−.1098	−.710 + .203i

When this is done, the A_{nm} coefficients of the integral equation,

$$\sum K_{nj} A_{nm} = \bar{\Omega}_{j\,m}^{(2)}$$

may be evaluated.

By taking the necessary values from the tables and noting that

$$A_{nm} = \frac{\sin n\phi}{n} + \frac{\pi b}{sb_o} \mu(k) S_n(\phi, k_o s)$$

the A_{nm} coefficients become,

for $y/sb_o = 0.4$,

$$A_{21} = 0.3667 + \frac{\pi(0.860)}{2.525}(0.2463 - 0.0873i)(0.494 - 0.278i)$$

$$= 0.3667 + (0.2635 - 0.0934i)(0.494 - 0.278i)$$

$$= 0.4709 - 0.1194i$$

$$A_{41} = -0.2493 + (0.2635 - 0.0934i)(-0.850 + 0.293i)$$

$$= -0.4459 + 0.1566i$$

$$A_{61} = 0.1132 + (0.2635 - 0.0934i)(0.575 - 0.150i)$$

$$= 0.2507 - 0.0932i$$

for $y/sb_o = 0.6$,

$$A_{22} = 0.4800 + \frac{\pi(0.770)}{2.525}(0.2834 - 0.1038i)(0.790 - 0.390i)$$

$$= 0.4800 + (0.2715 - 0.0994i)(0.790 - 0.390i)$$

$$= 0.6559 - 0.1844i$$

$$A_{42} = -0.1344 + (0.2715 - 0.0994i)(-0.547 + 0.160i)$$

$$= -0.2670 + 0.0978i$$

$$A_{62} = -0.1098 + (0.2715 - 0.0994i)(-0.710 + 0.178i)$$

$$= -0.2849 + 0.1111i$$

for $y/sb_o = 0.8$,

$$A_{2s} = 0.4800 + \frac{\pi(0.655)}{2.525}(0.3007 - 0.10891)(0.788 - 0.4331)$$

$$= 0.4800 + (0.2450 - 0.08871)(0.788 - 0.4331)$$

$$= 0.6347 - 0.1761$$

$$A_{4s} = 0.1344 + (0.2450 - 0.08871)(0.525 - 0.1701)$$

$$= 0.2479 - 0.08821$$

$$A_{6s} = -0.1098 + (0.2450 - 0.08871)(-0.710 + 0.2031)$$

$$= -0.2657 + 0.11271$$

When the A_{nm} coefficients have been calculated, the two-dimensional circulation functions $\bar{\Omega}_{jm}^{(2)}$ must be evaluated. For the rolling mode,

$$\bar{\Omega}_h^{(2)} = 4\left[\frac{iC(k)}{k\,H_1^{(2)}(k)}\right]e^{ikm}\,ik\,f_h(y)$$

where in this case,

$$f_h(y) = \frac{y}{sb_o}$$

In this example the sweepback function e^{ikm} appears as there is some sweep in the surfaces. The trailing and leading-edge coordinates of the surfaces can be found in figures 25 and 26. By using the previously tabulated parameters and functions and introducing the sweep functions, the values of $\bar{\Omega}_h^{(2)}$ may be determined as follows:

for $y/sb_o = 0.4$,

$$z_m = \frac{x_l + x_t}{2b_o} = \frac{-0.733 + 0.988}{2} = 0.128$$

$$k_m = k_o z_m = 0.0558$$

$$e^{ikm} = (0.9985 + 0.0558i)$$

$$\bar{\Omega}_h^{(2)} = 4(0.9126 - 0.1496i)(0.9985 + 0.0558i)(0.3750i)(0.4)$$

$$= 0.0591 + 0.5518i$$

for $y/sb_o = 0.6$,

$$z_m = \frac{-0.600 + 0.930}{2} = 0.165$$

$$k_m = 0.0835$$

$$e^{ikm} = (0.9965 + 0.834i)$$

$$\bar{\Omega}_h^{(2)} = 4(0.9451 - 0.1747i)(0.9965 + 0.0834i)(i\ 0.336)(0.6)$$

$$= 0.0768 + 0.7709i$$

for $y/sb_o = 0.8$,

$$z_m = \frac{-0.475 + 0.855}{2} = 0.190$$

$$k_m = 0.0961$$

$$e^{ikm} = (0.9954 + 0.0960i)$$

$$\bar{\Omega}_h^{(2)} = 4(0.9953 - 0.2054i)(0.9954 + 0.0960i)(0.286i)(0.8)$$
$$= 0.0996 + 0.9245i$$

The set of equations,

$$\sum K_{nj} A_{nm} = \bar{\Omega}_{hm}^{(2)}$$

may now be solved. This is done in table XIX by the Crout method.

When the values of K_{nh} have been determined, the values of

$$\bar{\Omega}_h = \sum K_{nh} \frac{\sin n\phi}{n}$$

can be found and this is done in table XX.

The values of the correction functions can now be calculated with it noted that

$$\sigma_j = \left[C(k) + \frac{iJ_1(k)}{J_o(k) - iJ_1(k)}\right] \left(\frac{\bar{\Omega}_j}{\bar{\Omega}_j^{(2)}} - 1\right)$$

This is done as follows,

for $y/sb_o = 0.4$,

$$\sigma_{hH} = (0.5988 + 0.0154i) \left[\frac{-0.0594 + 0.4342i}{0.0591 + 0.5518i} - 1\right]$$

$$= -0.1427 + 0.1101i$$

for $y/sb_o = 0.6$

$$\sigma_{hH} = (0.6205 - 0.0088i) \left[\frac{-0.0895 + 0.5804i}{0.0768 + 0.7709i} - 1\right]$$

for $y/sb_o = 0.8$

$$\sigma_{hH} = (0.6523 - 0.0394i)\left[\frac{-0.1057 + 0.6686i}{0.0996 + 0.9245i} - 1\right]$$

An exactly similar procedure must be followed to find the corrections, σ_{hV}, $\sigma_{\alpha V}$, $\sigma_{\beta V}$, and $\sigma_{\gamma V}$ for a given value of k_{oV}. It should be remembered though that the value of k_o is different for the vertical tail.

When all the correction functions have been evaluated, the values of the finite-span corrections can be found by a graphical or numerical integration of the integrals listed in the beginning of this section of Example II. This integration was carried out, and the values of the finite-span corrections found are listed in table XXI. The flutter determinant terms calculated with use of two-dimensional aerodynamic considerations and those corrected for finite-span effects are also listed in this table.

The determinants can now be solved. For this problem the solutions are found by the method of reference 4 and the results are tabulated in the following table:

k_{oH}	k_{oV}	First solution		Second solution	
		V_1 (mph)	g_1	V_2 (mph)	g_2
0.436	0.506	267	0.030	423	−0.212
.600	.696	196	.012	308	−.151
.686	.795	173	.006	233	−.071

The actual flutter speed is determined graphically in figure 27. The calculated value of the true airspeed at which flutter occurs when the finite-span corrections are used is 299 mph.

<u>Discussion of Example II</u>.– Example No. 7 of reference 4 indicates that the observed flutter speed for the tail analyzed in this problem fluttered experimentally at about 262 mph true airspeed. The fact that the theoretical analysis, however, does not check the observed flutter speed is not considered significant because the observed flutter speed was reported by a pilot some time ago and may not be accurate and the

analysis is approximate in that no attempt was made to take into account the effective inertia due to wing motion or the elastic restraint and damping in the rudder control system.

It is considered significant that the three theoretical analyses give such widely varying values for the calculated flutter speed. These values of the true airspeed are 220 mph for flutter speed calculated by the method of equivalent chords of reference 4; 243 mph for a value of k_o for the flutter speed, of about 0.55, when the two-dimensional aerodynamic terms are integrated along the span; and 299 mph for a value of k_o of about 0.45, when the aerodynamic terms are corrected for the effects of a finite span.

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TABLE I

VALUES OF FUNCTION C = F + iG

k	C
0.00	1.0000 - 0.0000i
0.01	0.9824 - 0.0482i
0.025	0.9545 - 0.0872i
0.04	0.9267 - 0.1160i
0.05	0.9090 - 0.1305i
0.06	0.8920 - 0.1426i
0.08	0.8604 - 0.1604i
0.10	0.8320 - 0.1723i
0.12	0.8063 - 0.1801i
0.16	0.7628 - 0.1876i
0.20	0.7276 - 0.1886i
0.24	0.6989 - 0.1862i
0.30	0.6650 - 0.1793i
0.34	0.6469 - 0.1738i
0.40	0.6250 - 0.1650i
0.44	0.6130 - 0.1592i
0.50	0.5979 - 0.1507i
0.56	0.5857 - 0.1428i
0.60	0.5788 - 0.1378i
0.65	0.5713 - 0.1312i
0.66	0.5699 - 0.1308i
0.70	0.5647 - 0.1265i
0.75	0.5591 - 0.1212i
0.80	0.5541 - 0.1165i
0.85	0.5507 - 0.1120i
0.90	0.5460 - 0.1078i
0.95	0.5425 - 0.1040i
1.00	0.5394 - 0.1003i
1.20	0.5300 - 0.0877i
1.50	0.5210 - 0.0736i
2.00	0.5129 - 0.0577i
3.00	0.5063 - 0.0400i
4.00	0.5037 - 0.0305i
6.00	0.5017 - 0.0206i
10.00	0.5006 - 0.0124i
∞	0.5000 - 0.0000i

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TABLE II

VALUES OF FUNCTION $\mu(k)$

k	μ
0.00	0.5000 - 0.0000i
0.02	0.4810 - 0.0422i
0.04	0.4607 - 0.0665i
0.06	0.4410 - 0.0829i
0.08	0.4226 - 0.0942i
0.10	0.4051 - 0.1018i
0.125	0.3857 - 0.1082i
0.15	0.3690 - 0.1120i
0.175	0.3522 - 0.1130i
0.20	0.3393 - 0.1139i
0.225	0.3268 - 0.1132i
0.25	0.3154 - 0.1116i
0.275	0.3049 - 0.1099i
0.30	0.2955 - 0.1076i
0.35	0.2787 - 0.1023i
0.40	0.2644 - 0.0964i
0.45	0.2519 - 0.0903i
0.50	0.2408 - 0.0842i
0.55	0.2289 - 0.0775i
0.60	0.2220 - 0.0722i
0.65	0.2139 - 0.0665i
0.70	0.2062 - 0.0610i
0.75	0.1991 - 0.0557i
0.80	0.1924 - 0.0507i
0.90	0.1801 - 0.0413i
1.00	0.1688 - 0.0329i
1.25	0.1436 - 0.0159i
1.50	0.1218 - 0.0042i
1.75	0.1027 + 0.0030i
2.00	0.0864 + 0.0066i
2.10	0.0807 + 0.0070i
2.15	0.0780 + 0.0072i
2.20	0.0754 + 0.0072i
2.25	0.0730 + 0.0072i
2.30	0.0706 + 0.0070i
2.35	0.0684 + 0.0069i
2.40	0.0663 + 0.0066i
2.48	0.0632 + [illegible]
2.54	0.0610 + 0.0057i
2.60	0.0591 + 0.0052i

TABLE III

VALUES OF FUNCTION

$$C(k) + \frac{iJ_1(k)}{J_o(k)-iJ_1(k)}$$

k	$C + \frac{iJ_1}{J_o - iJ_1}$
0.00	1.0000 - 0.0000i
0.01	0.9824 - 0.0432i
0.025	0.9543 - 0.0747i
0.040	0.9263 - 0.0960i
0.05	0.9084 - 0.1055i
0.06	0.8911 - 0.1126i
0.08	0.8588 - 0.1204i
0.10	0.8295 - 0.1224i
0.12	0.8027 - 0.1202i
0.16	0.7564 - 0.1079i
0.20	0.7176 - 0.0891i
0.24	0.6845 - 0.0671i
0.30	0.6425 - 0.0310i
0.34	0.6180 - 0.0063i
0.40	0.5850 + 0.0310i
0.44	0.5646 + 0.0554i
0.50	0.5354 + 0.0913i
0.56	0.5073 + 0.1259i
0.60	0.4889 + 0.1483i
0.65	0.4658 + 0.1753i
0.70	0.4424 + 0.2012i
0.75	0.4187 + 0.2262i
0.80	0.3945 + 0.2498i
0.85	0.3706 + 0.2723i
0.90	0.3443 + 0.2935i
0.95	0.3179 + 0.3133i
1.00	0.2909 + 0.3319i
1.20	0.1747 + 0.3909i
1.50	-0.0220 + 0.4245i
2.00	-0.3550 + 0.2797i
3.00	-0.1233 - 0.5229i

TABLE IV

VALUES OF FUNCTION

$$\frac{iC(k)}{kH_1^{(2)}(k)}$$

k	$\frac{iC}{k H_1^{(2)}}$
0.00	1.5708 - 0.0000i
0.01	1.5420 - 0.0760i
0.04	1.4515 - 0.1800i
0.05	1.4218 - 0.2012i
0.06	1.3932 - 0.2188i
0.08	1.3393 - 0.2429i
0.10	1.2901 - 0.2567i
0.12	1.2451 - 0.2644i
0.16	1.1671 - 0.2631i
0.20	1.1020 - 0.2507i
0.24	1.0473 - 0.2319i
0.30	0.9795 - 0.1973i
0.34	0.9413 - 0.1722i
0.40	0.8921 - 0.1334i
0.44	0.8632 - 0.1075i
0.50	0.8241 - 0.0691i
0.56	0.7889 - 0.0319i
0.60	0.7671 - 0.0077i
0.65	0.7411 + 0.0215i
0.66	0.7362 + 0.0273i
0.70	0.7164 + 0.0499i
0.75	0.6921 + 0.0772i
0.80	0.6691 + 0.1034i
0.85	0.6471 + 0.1289i
0.90	0.6238 + 0.1528i
0.95	0.6013 + 0.1758i
1.00	0.5791 + 0.1978i
1.20	0.4901 + 0.2755i
1.50	0.3544 + 0.3606i
2.00	0.1281 + 0.4209i
3.00	- 0.2279 + 0.2799i

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TABLE V

VALUES OF FUNCTION $F(x) = \int_0^\infty e^{-i\lambda}\left(\frac{1}{x} + \frac{1}{\lambda} - \frac{\sqrt{x^2 + \lambda^2}}{x\lambda}\right)d\lambda$

x	F
0.00	∞ - 1.571i
.05	2.750 - 1.468i
.10	2.109 - 1.375i
.20	1.490 - 1.248i
.30	1.155 - 1.146i
.40	0.935 - 1.060i
.50	0.778 - 0.987i
.60	0.658 - 0.922i
.70	0.564 - 0.865i
.80	0.489 - 0.813i
.90	0.427 - 0.768i
1.00	0.376 - 0.726i
1.10	0.333 - 0.689i
1.20	0.297 - 0.654i
1.30	0.265 - 0.624i
1.40	0.238 - 0.593i
1.50	0.214 - 0.567i
1.60	0.194 - 0.540i
1.70	0.176 - 0.517i
1.80	0.160 - 0.496i
1.90	0.146 - 0.475i
2.00	0.134 - 0.458i
2.2	0.113 - 0.425i
2.4	0.097 - 0.395i
2.6	0.083 - 0.369i
2.8	0.072 - 0.345i
3.0	0.063 - 0.324i
3.2	0.055 - 0.305i
3.4	0.048 - 0.289i
3.6	0.043 - 0.274i
3.8	0.039 - 0.260i
4.0	0.035 - 0.248i
4.2	0.031 - 0.237i
4.4	0.028 - 0.226i
4.6	0.026 - 0.217i
4.8	0.024 - 0.208i
5.0	0.022 - 0.200i
5.2	0.020 - 0.192i
5.4	0.018 - 0.185i
5.6	0.017 - 0.179i
5.8	0.016 - 0.172i
6.0	0.015 - 0.167i
	$1/2x^2$ $1/x$

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TABLE VII

VALUES OF FUNCTIONS $\frac{\sin n\phi}{n}$ AGAINST $\cos\phi = \frac{y^*}{s}$

$\frac{y^*}{s} = \cos\phi$	0	0.2	0.4	0.5	0.6	0.8	0.9	1.00
$\frac{\sin\phi}{1}$	1	0.9798	0.9165	0.8660	0.800	0.600	0.4355	0
$\frac{\sin 2\phi}{2}$	0	0.1959	0.3667	0.4330	0.4800	0.4800	0.3921	0
$\frac{\sin 3\phi}{3}$	-0.3333	-0.2744	-0.1099	0	0.1173	0.3120	0.3255	0
$\frac{\sin 4\phi}{4}$	0	-0.1803	-0.2493	-0.2165	-0.1344	0.1344	0.2433	0
$\frac{\sin 5\phi}{5}$	0.2000	0.1070	-0.0936	-0.1732	-0.1994	-0.0151	0.1550	0
$\frac{\sin 6\phi}{6}$	0	0.1558	0.1132	0	-0.1098	-0.1098	0.0707	0
$\frac{\sin 7\phi}{7}$	-0.1429	-0.0472	0.1380	0.1238	0.0295	-0.1398	-0.0022	0
$\frac{\sin 8\phi}{8}$	0	-0.1249	0.0188	0.1083	0.1134	-0.1134	-0.0558	0
$\frac{\sin 9\phi}{9}$	0.1111	-0.0246	-0.0940	0	0.0980	-0.0525	-0.0882	0

TABLE VI

VALUES OF FUNCTION

$$S_n(\phi, k_o s) = \frac{\sin n\phi}{\sin\phi} + i\,\frac{k_o s}{\pi}\int_0^{\pi} \frac{\cos\phi - \cos\theta}{|\cos\phi - \cos\theta|} F \cos n\theta \, d\theta$$

FOR VARIOUS VALUES OF n, $k_o s$, and $y^*/s = \cos\phi$

$k_o s = 0$

n \ cos ϕ	0	0.2	0.4	0.6	0.8	1.0
1	1	1	1	1	1	1
2	0		0.8001	1.2000	1.6000	2
3	-1	-0.8401	-0.3593	0.4399	1.5601	3
4	0		-1.0879	-0.6722	0.8963	4
5	1	0.5461	-0.5105	-1.2464	-0.1259	5
6	0		0.6792	-0.8234	-1.3723	6
7	-1	-0.3373	1.0542	0.2578	-1.6307	7
8	0		0.1644	1.1335	-1.5113	8
9	1	-0.2257	-0.9232	1.1020	-0.7876	9

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$k_o s = 0.5$

n \ cos ϕ	0		0.4		0.8
1	0.665 -0.314i		0.701 -0.293i		0.828 -0.181i
3	-0.851 +0.215i		-0.311 0.063i		1.420 -0.254i
5	0.908 -0.161i		-0.465 0.079i		-0.114 +0.030i

$k_o s = 1$

n \ $\cos\phi$	0	0.2	0.4	0.6	0.8	1.0
1.	0.468 -0.356i	0.479 -0.352i	0.516 -0.340i	0.589 -0.309i	0.729 -0.223i	1.250 0.727i
2.	0.000 0.000i		0.516 -0.274i	0.823 -0.382i	1.2204 -0.419i	2.068 0.476i
3	-0.725 0.301i	-0.615 0.249i	-0.274 0.094i	0.331 -0.140i	1.280 -0.384i	3.021 0.331i
4	0.000 0.000i		-0.869 0.282i	-0.556 0.153i	0.7610 -0.162i	4.009 0.249i
5	0.820 -0.244i	0.451 -0.130i	-0.424 0.123i	-1.063 0.270i	-0.116 0.063i	5.003 0.201i
6	0.000 0.000i		0.584 -0.141i	-0.720 0.167i	-1.269 0.189i	6.003 0.166i
7	-0.868 0.205	-0.316 0.032i	0.925 -0.206i	0.231 -0.042i	-1.498 0.226i	7.002 0.143i
8	0.000 0.000i		0.146 -0.032i	1.026 -0.187i	-1.402 0.175i	8.001 0.125i
9	0.895 -0.177i	-0.201 0.066i	-0.834 0.154i	1.008 -0.171i	-0.736 0.067i	9.001 0.111i

$k_o s = 2$

n \ $\cos\phi$	0	0.2	0.4	0.6	0.8	1.0
1	0.271 -0.318i	0.284 -0.320i	0.327 -0.321i	0.421 -0.318i	0.622 -0.269i	1.551 1.173i
2	0.000 0.000i		0.351 -0.296i	0.595 -0.429i	0.979 -0.504i	2.210 0.877i
3	-0.545 0.355i	-0.468 0.294i	-0.221 0.113i	0.245 -0.175i	1.062 -0.505i	3.083 0.647i
4	0.000 0.000		-0.704 0.361i	-0.472 0.200i	0.651 -0.203i	4.038 0.496i
5	0.679 -0.321i	0.376 -0.172i	-0.352 0.166i	-0.912 0.370i	-0.112 0.115i	5.014 0.403i
6	0.000 0.000i		0.505 -0.195i	-0.630 0.238i	-1.179 0.282i	6.012 0.332i
7	-0.755 0.288i	-0.299 0.044i	0.814 -0.294i	0.208 -0.058i	-1.377 0.331i	7.006 0.285i
8	0.000 0.000i		0.130 -0.047i	0.931 -0.275i	-1.301 0.252i	8.004 0.250i
9	0.803 -0.259i	-0.178 0.108i	-0.755 0.227i	0.923 -0.258i	-0.687 0.088i	9.003 0.222i

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$k_o s = 4$

n \ cos φ	0	0.2	0.4	0.6	0.8	1.0
1	0.132 -0.234i	0.140 -0.236i	0.164 -0.248i	0.256 -0.282i	0.484 -0.315i	2.055 1.782i
2	0.000 0.000i		0.193 -0.254i	0.390 -0.390i	0.689 -0.525i	2.540 1.499i
3	-0.341 0.337i	-0.300 0.284i	-0.160 0.120i	0.139 -0.170i	0.766 -0.572i	3.274 1.204i
4	0.000 0.000i		-0.488 0.384i	-0.359 0.232i	0.468 -0.184i	4.143 0.963i
5	0.486 -0.352i	0.273 -0.191i	-0.250 0.184i	-0.688 0.435i	-0.114 0.222i	5.065 0.804i
6	0.000 0.000i		0.389 -0.232i	-0.487 0.296i	-1.037 0.388i	6.046 0.658i
7	-0.584 0.351i	-0.273 0.054i	0.640 -0.367i	0.176 -0.071i	-1.172 0.431i	7.023 0.565i
8	0.000 0.000i		0.101 -0.063i	0.772 -0.366i	-1.120 0.314i	8.014 0.497i
9	0.651 -0.337i	-0.140 0.173i	-0.623 0.302i	0.774 -0.353i	-0.596 0.088i	9.008 0.442i

$k_o s = 6.0$

n \ cos φ	0	0.2	0.4	0.6	0.8	1.0
1	0.082 -0.189i	0.086 -0.187i	0.109 -0.~~178~~i 240	0.179 -0.240i	0.387 -0.334i	2.474 2.237i
2	-0.000 0.000i		0.126 -0.217i	0.242 -0.333i	0.511 -0.497i	2.862 1.989i
3	-0.240 0.300i	-0.215 0.256i	-0.123 0.118i	0.089 -0.146i	0.579 -0.569i	3.502 1.672i
4	0.000 0.000		-0.364 0.360i	-0.284 0.260i	0.341 -0.080i	4.293 1.386i
5	0.371 -0.332i	0.210 -0.185i	-0.188 0.169i	-0.536 0.435i	-0.119 0.339i	5.148 1.190i
6	0.000 0.000i		0.308 -0.236i	-0.383 0.306i	-0.928 0.448i	6.099 0.975i
7	-0.463 0.359i	-0.254 0.056i	0.511 -0.384i	0.154 -0.075i	-1.003 0.462i	7.046 0.839i
8	0.000 0.000		0.077 -0.068i	0.645 -0.404i	-0.959 0.316i	8.024 0.740i
9	0.531 -0.367i	-0.109 0.225i	-0.517 0.333i	0.649 -0.400i	-0.512 0.057i	9.022 0.659i

TABLE VIII

UNIFORM TRANSLATION OF RECTANGULAR WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(a)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.167	0	0.757-0.188i	-0.105+0.129i	0.055+0.397i	-0.013+0.342i
0.167	0.4	0.757-0.188i	-0.137+0.125i	0.055+0.397i	-0.011+0.325i
0.167	0.8	0.757-0.188i	-0.285+0.115i	0.055+0.397i	-0.006+0.247i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	-0.043+0.078i	0.008+0.681i	-0.073+0.636i
0.333	0.4	0.650-0.175i	-0.069+0.083i	0.008+0.681i	-0.079+0.608i
0.333	0.8	0.650-0.175i	-0.196+0.083i	0.008+0.681i	-0.079+0.475i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.003+0.029i
0.667	0	0.569-0.130i	-0.018+0.030i	-0.426+1.191i	-0.489+1.155i
0.667	0.4	0.569-0.130i	-0.036+0.033i	-0.426+1.191i	-0.495+1.116i
0.667	0.8	0.569-0.130i	-0.127+0.009i	-0.426+1.191i	-0.444+0.927i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.191i	-0.042+0.230i

UNIFORM TRANSLATION OF RECTANGULAR WING OF AR = 6

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(a)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.167	0	0.757-0.188i	-0.020+0.059i	0.055+0.397i	0.024+0.384i
0.167	0.4	0.757-0.188i	-0.040+0.070i	0.055+0.397i	0.018+0.375i
0.167	0.8	0.757-0.188i	-0.169+0.104i	0.055+0.397i	0.003+0.314i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	0.001+0.027i	0.008+0.681i	-0.020+0.681i
0.333	0.4	0.650-0.175i	-0.013+0.039i	0.008+0.681i	-0.032+0.667i
0.333	0.8	0.650-0.175i	-0.100+0.065i	0.008+0.681i	-0.060+0.576i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.002+0.029i
0.667	0	0.569-0.130i	0.000+0.007i	-0.426+1.191i	-0.441+1.193i
0.667	0.4	0.569-0.130i	-0.006+0.014i	-0.426+1.191i	-0.455+1.180i
0.667	0.8	0.569-0.130i	-0.058+0.017i	-0.426+1.191i	-0.461+1.071i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.191i	-0.042+0.230i

UNIFORM PITCHING OF RECTANGULAR WING OF AR = 3 (a=0)

k	$\frac{y}{b_o s}$	C	σ_α	$\frac{\bar{L}^{(a)}}{2\rho U^2 b_o \bar{\alpha}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{\alpha}}$
0	0	1	-0.313	3.142	2.158
0	0.4	1	-0.348	3.142	2.049
0	0.8	1	-0.505	3.142	1.554
0	1.0	1	-1	3.142	0
0.167	0	0.757-0.188i	-0.105+0.129i	2.427-0.130i	2.063+0.248i
0.167	0.4	0.757-0.188i	-0.137+0.125i	2.427-0.130i	1.963+0.227i
0.167	0.8	0.757-0.188i	-0.285+0.115i	2.427-0.130i	1.501+0.157i
0.167	1.0	0.757-0.188i	-0.750+0.105i	2.427-0.130i	0.044+0.003i
0.333	0	0.650-0.175i	-0.043+0.078i	2.133+0.315i	1.957+0.536i
0.333	0.4	0.650-0.175i	-0.069+0.083i	2.133+0.315i	1.873+0.541i
0.333	0.8	0.650-0.175i	-0.196+0.083i	2.133+0.315i	1.473+0.474i
0.333	1.0	0.650-0.175i	-0.622+0.010i	2.133+0.315i	0.173+0.022i
0.667	0	0.569-0.130i	-0.018+0.030i	1.924+1.234i	1.837+1.310i
0.667	0.4	0.569-0.130i	-0.036+0.033i	1.924+1.234i	1.783+1.301i
0.667	0.8	0.569-0.130i	-0.127+0.009i	1.924+1.234i	1.517+1.129i
0.667	1.0	0.569-0.130i	-0.459-0.183i	1.924+1.234i	0.673+0.178i

TABLE VIII (continued)

UNIFORM PITCHING OF RECTANGULAR WING OF AR = 6 (a = 0)

0	0	1	-0.1743	3.142	2.597
0	0.4	1	-0.2037	3.142	2.502
0	0.8	1	-0.3467	3.142	2.052
0	1.0	1	-1	3.142	0
0.167	0	0.757-0.188i	-0.025+0.059i	2.427-0.130i	2.334+0.043i
0.167	0.4	0.757-0.188i	-0.041+0.070i	2.427-0.130i	2.279+0.078i
0.167	0.8	0.757-0.188i	-0.158+0.099i	2.427-0.130i	1.905+0.139i
0.167	1.0	0.757-0.188i	-0.750+0.105i	2.427-0.130i	0.045+0.003i
0.333	0	0.650-0.175i	0.001+0.027i	2.133+0.315i	2.121+0.401i
0.333	0.4	0.650-0.175i	-0.013+0.039i	2.133+0.315i	2.073+0.430i
0.333	0.8	0.650-0.175i	-0.100+0.065i	2.133+0.315i	1.785+0.468i
0.333	1.0	0.650-0.175i	-0.622+0.104i	2.133+0.315i	0.173+0.022i
0.667	0	0.569-0.130i	0.000+0.007i	1.924+1.234i	1.918+1.258i
0.667	0.4	0.569-0.130i	-0.006+0.014i	1.924+1.234i	1.891+1.272i
0.667	0.8	0.569-0.130i	-0.058+0.017i	1.924+1.234i	1.725+1.226i
0.667	1.0	0.569-0.130i	-0.459-0.183i	1.924+1.234i	0.673+0.1774i

LINEAR SYMMETRICAL TORSION OF RECTANGULAR WING OF AR = 3 (a = 0)

k	$\frac{y}{b_0 s}$	C	σ_α	$\frac{\overline{L}^{(2)}}{2\rho U^2 b_0 \overline{\alpha}}$	$\frac{\overline{L}}{2\rho U^2 b_0 \overline{\alpha}}$
0	0	1	∞	0	0.453
0	0.4	1	-0.352	1.257	0.814
0	0.8	1	-0.596	2.513	1.016
0	1.0	1	-1	3.142	0
0.167	0	0.757-0.188i	∞-∞i	0	0.406-0.020i
0.167	0.4	0.757-0.188i	-0.144+0.120i	0.971-0.052i	0.777+0.083i
0.167	0.8	0.757-0.188i	-0.363+0.141i	1.942-0.104i	1.001+0.174i
0.167	1.0	0.757-0.188i	-0.750+0.105i	2.427-0.130i	0.044+0.003i
0.333	0	0.650-0.175i	∞-∞i	0	0.340+0.005i
0.333	0.4	0.650-0.175i	-0.076+0.082i	0.353+0.126i	0.741+0.213i
0.333	0.8	0.650-0.175i	-0.262+0.098i	1.707+0.252i	1.007+0.388i
0.333	1.0	0.650-0.175i	-0.622+0.010i	2.133+0.315i	0.173+0.022i
0.667	0	0.569-0.130i	∞-∞i	0	0.235+0.086i
0.667	0.4	0.569-0.130i	-0.044+0.036i	0.770+0.494i	0.699+0.521i
0.667	0.8	0.569-0.130i	-0.173+0.012i	1.539+0.987i	1.095+0.874i
0.667	1.0	0.569-0.130i	-0.459-0.183i	1.924+1.234i	0.673+0.178i

LINEAR SYMMETRICAL TORSION OF RECTANGULAR WING OF AR = 6 (a = 0)

0	0	1	∞	0	0.423
0	0.4	1	-0.2087	1.257	0.994
0	0.8	1	-0.4290	2.513	1.435
0	1.0	1	-1	3.142	0
0.167	0	0.757-0.188i	∞-∞i	0	0.292-0.102i
0.167	0.4	0.757-0.188i	-0.049+0.069i	0.971-0.052i	0.902+0.030i
0.167	0.8	0.757-0.188i	-0.215+0.128i	1.942-0.104i	1.373+0.173i
0.167	1.0	0.757-0.188i	-0.750+0.105i	2.427-0.130i	0.045+0.003i
0.333	0	0.650-0.175i	∞-∞i	0	0.230-0.082i
0.333	0.4	0.650-0.175i	-0.033+0.040i	0.853+0.126i	0.304+0.169i
0.333	0.8	0.650-0.175i	-0.137+0.093i	1.707+0.252i	1.325+0.429i
0.333	1.0	0.650-0.175i	-0.622+0.010i	2.133+0.315i	0.173+0.022i
0.667	0	0.569-0.130i	∞+∞i	0	0.138-0.007i
0.667	0.4	0.569-0.130i	-0.012+0.015i	0.770+0.494i	0.841+0.518i
0.667	0.8	0.569-0.130i	-0.079+0.031i	1.539+0.987i	1.301+0.983i
0.667	1.0	0.569-0.130i	-0.459-0.183i	1.924+1.234i	0.673+0.177i

LINEAR SYMMETRICAL BENDING FOR RECTANGULAR WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.167	0	0.757-0.188i	∞-∞i	0	0.009+0.067i
0.167	0.4	0.757-0.188i	-0.144+0.120i	0.022+0.159i	-0.003+0.128i
0.167	0.8	0.757-0.188i	-0.363+0.141i	0.044+0.317i	-0.015+0.165i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	∞-∞i	0	0.017+0.1106i
0.333	0.4	0.650-0.175i	-0.076+0.082i	0.003+0.272i	-0.031+0.238i
0.333	0.8	0.650-0.175i	-0.262+0.098i	0.007+0.544i	-0.075+0.325i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.003+0.029i
0.667	0	0.569-0.130i	∞-∞i	0	-0.005+0.158i
0.667	0.4	0.569-0.130i	-0.044+0.036i	-0.170+0.477i	-0.201+0.440i
0.667	0.8	0.569-0.130i	-0.173+0.012i	-0.341+0.954i	-0.361+0.664i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.192i	-0.031+0.230i

LINEAR SYMMETRICAL BENDING FOR RECTANGULAR WING OF AR = 6

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.167	0	0.757-0.188i	∞-∞i	0	0.021+0.047i
0.167	0.4	0.757-0.188i	-0.049+0.069i	0.022+0.159i	0.007+0.148i
0.167	0.8	0.757-0.188i	-0.215+0.128i	0.044+0.317i	-0.010+0.227i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	∞-∞i	0	0.039+0.070i
0.333	0.4	0.650-0.175i	-0.033+0.040i	0.003+0.272i	-0.013+0.259i
0.333	0.8	0.650-0.175i	-0.137+0.093i	0.007+0.544i	-0.071+0.430i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.002+0.029i
0.667	0	0.569-0.130i	∞-∞i	0	0.032+0.082i
0.667	0.4	0.569-0.130i	-0.012+0.015i	-0.170+0.477i	-0.183+0.467i
0.667	0.8	0.569-0.130i	-0.079+0.031i	-0.341+0.954i	-0.393+0.822i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.192i	-0.042+0.230i

PARABOLIC SYMMETRICAL BENDING FOR RECTANGULAR WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.167	0	0.757-0.188i	∞-∞i	0	0.006+0.036i
0.167	0.4	0.757-0.188i	0.035+0.106i	0.009+0.063i	0.000+0.066i
0.167	0.8	0.757-0.188i	-0.392+0.143i	0.035+0.254i	-0.013+0.123i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	∞-∞i	0	0.012+0.059i
0.333	0.4	0.650-0.175i	0.079+0.072i	0.001+0.109i	-0.011+0.122i
0.333	0.8	0.650-0.175i	-0.287+0.097i	0.005+0.436i	-0.060+0.243i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.003+0.029i
0.667	0	0.569-0.130i	∞-∞i	0	0.005+0.081i
0.667	0.4	0.569-0.130i	0.069+0.050i	-0.068+0.191i	-0.085+0.214i
0.667	0.8	0.569-0.130i	-0.188+0.006i	-0.272+0.763i	-0.280+0.511i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.192i	-0.031+0.230i

PARABOLIC SYMMETRICAL BENDING FOR RECTANGULAR WING OF AR = 6

0.167	0	0.757-0.188i	∞ -∞i	0	0.019+0.023i
0.167	0.4	0.757-0.188i	0.105+0.020i	0.009+0.063i	0.007+0.072i
0.167	0.8	0.757-0.188i	-0.239+0.133i	0.035+0.254i	-0.010+0.174i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0	0.650-0.175i	∞ -∞i	0	0.023+0.033i
0.333	0.4	0.650-0.175i	0.091-0.005i	0.001+0.109i	0.002+0.124i
0.333	0.8	0.650-0.175i	-0.153+0.095i	0.005+0.436i	-0.059+0.333i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.003+0.631i	-0.0002+0.029i
0.667	0	0.569-0.130i	∞ +∞i	0	0.020+0.036i
0.667	0.4	0.569-0.130i	0.057+0.001i	-0.068+0.191i	-0.068+0.210i
0.667	0.8	0.569-0.130i	-0.086+0.027i	-0.272+0.763i	-0.309+0.648i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.192i	-0.042+0.230i

LINEAR ANTISYMMETRICAL TORSION OF RECTANGULAR WING AR = 3

k	$\frac{y}{b_o s}$	C	σ_j	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{J}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{J}}$
0	0.4	1	-0.490	1.257	0.641
0	0.6	1	-0.528	1.885	0.890
0	0.8	1	-0.584	2.513	1.947
0	1.0	1	-1	3.142	0
0.167	0.4	0.757-0.188i	-0.253+0.165i	0.971-0.052i	0.636+0.129i
0.167	0.6	0.757-0.188i	-0.289+0.162i	1.456-0.078i	0.887+0.182i
0.167	0.8	0.757-0.188i	-0.342+0.159i	1.942-0.104i	1.049+0.224i
0.167	1.0	0.757-0.188i	-0.750+0.105i	2.427-0.130i	0.044+0.003i
0.333	0.4	0.650-0.175i	-0.149+0.130i	0.853+0.126i	0.639+0.259i
0.333	0.6	0.650-0.175i	-0.183+0.128i	1.280+0.189i	0.895+0.374i
0.333	0.8	0.650-0.175i	-0.233+0.122i	1.707+0.252i	1.069+0.461i
0.333	1.0	0.650-0.175i	-0.622+0.010i	2.133+0.315i	0.173+0.022i
0.667	0.4	0.569-0.130i	-0.072+0.061i	0.770+0.494i	0.645+0.537i
0.667	0.6	0.569-0.130i	-0.108+0.054i	1.155+0.741i	0.917+0.774i
0.667	0.8	0.569-0.130i	-0.146+0.040i	1.539+0.987i	1.138+0.966i
0.667	1.0	0.569-0.130i	-0.459-0.183i	1.924+1.234i	0.673+0.178i

RIGID ROLLING OF RECTANGULAR WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_j	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{J}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{J}}$
0.167	0.4	0.757-0.188i	-0.253+0.165i	0.022+0.159i	-0.013+0.106i
0.167	0.6	0.757-0.188i	-0.289+0.162i	0.033+0.238i	-0.018+0.147i
0.167	0.8	0.757-0.188i	-0.342+0.159i	0.044+0.317i	-0.023+0.174i
0.167	1.0	0.757-0.188i	-0.750+0.105i	0.055+0.397i	-0.0002+0.004i
0.333	0.4	0.650-0.175i	-0.149+0.130i	0.003+0.272i	-0.051+0.210i
0.333	0.6	0.650-0.175i	-0.183+0.128i	0.005+0.408i	-0.076+0.293i
0.333	0.8	0.650-0.175i	-0.233+0.122i	0.007+0.544i	-0.094+0.351i
0.333	1.0	0.650-0.175i	-0.622+0.010i	0.008+0.681i	-0.003+0.029i
0.667	0.4	0.569-0.130i	-0.078+0.061i	-0.170+0.477i	-0.222+0.411i
0.667	0.6	0.569-0.130i	-0.108+0.054i	-0.255+0.715i	-0.323+0.579i
0.667	0.8	0.569-0.130i	-0.146+0.040i	-0.341+0.854i	-0.408+0.708i
0.667	1.0	0.569-0.130i	-0.459-0.183i	-0.426+1.192i	-0.042+0.230i
1.333	0.4	0.526-0.081i	-0.034-0.001i	-0.981+0.881i	-0.979+0.824i
1.333	0.6	0.526-0.081i	-0.042-0.015i	-1.471+1.322i	-1.434+1.215i
1.333	0.8	0.526-0.081i	-0.053-0.032i	-1.961+1.763i	-1.853+1.586i
1.333	1.0	0.526-0.081i	-0.090+0.414i	-2.452+2.203i	-0.716+1.825i

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LINEAR SYMMETRICAL TORSION OF ELLIPTICAL WING OF AR = 3 (a = 0)

k	$\frac{y}{b_o s}$	C	σ_α	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{\alpha}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{\alpha}}$
0	0	1	∞	0	0.391
0	0.4	1	-0.419	1.152	0.669
0	0.8	1	-0.535	1.508	0.702
0.212	0	0.719-0.188i	∞ - ∞i	0	0.349-0.012i
0.194	0.4	0.733-0.189i	-0.181+0.114i	0.865-0.023i	0.644+0.088i
0.127	0.8	0.799-0.181i	-0.347+0.141i	1.222-0.102i	0.685+0.078i
0.424	0	0.618-0.162i	∞ - ∞i	0	0.294+0.037i
0.388	0.4	0.629-0.167i	0.029+0.054i	0.762+0.172i	0.630+0.230i
0.254	0.8	0.691-0.185i	-0.274+0.124i	1.077+0.046i	0.642+0.183i
0.847	0	0.551-0.113i	∞ - ∞i	0	0.175+0.114i
0.776	0.4	0.556-0.119i	-0.060-0.011i	0.694+0.559i	0.630+0.520i
0.508	0.8	0.596-0.150i	-0.177+0.066i	0.956+0.386i	0.664+0.418i
LINEAR SYMMETRICAL TORSION OF ELLIPTICAL WING OF AR = 6 (a = 0)					
0	0	1	+ ∞	0	0.388
0	0.4	1	-0.264	1.152	0.849
0	0.8	1	-0.376	1.508	0.941
0.212	0	0.719-0.188i	∞ - ∞i	0	0.294-0.071i
0.194	0.4	0.733-0.189i	-0.080+0.076i	0.865-0.023i	0.764+0.056i
0.127	0.8	0.799-0.181i	-0.209+0.124i	1.222-0.102i	0.893+0.065i
0.424	0	0.618-0.162i	∞ - ∞i	0	0.212-0.044i
0.388	0.4	0.629-0.167i	-0.046+0.043i	0.762+0.172i	0.700+0.213i
0.254	0.8	0.691-0.185	-0.145+0.112i	1.077+0.046i	0.837+0.187i
0.847	0	0.551-0.113i	∞ - ∞i	0	0.100+0.023i
0.776	0.4	0.556-0.119i	-0.026-0.004i	0.694+0.559i	0.680+0.544i
0.508	0.8	0.596-0.150i	-0.088+0.064i	0.956+0.336i	0.800+0.441i

LINEAR SYMMETRICAL BENDING OF ELLIPTICAL WING AR = 3

k	$\frac{y}{b_o s}$	C	σ_h	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.212	0	0.719-0.188i	- i	0	0.006+0.074i
0.194	0.4	0.733-0.189i	-0.180+0.119i	0.022+0.179i	-0.007+0.135i
0.127	0.8	0.799-0.181i	-0.347+0.139i	0.038+0.255i	-0.007+0.144i
0.424	0	0.613-0.162i	- i	0	0.016+0.070i
0.388	0.4	0.629-0.167i	-0.115+0.075i	-0.013+0.307i	-0.050+0.251i
0.254	0.8	0.691-0.185i	-0.277+0.116i	0.037+0.442i	-0.037+0.264i
0.847	0	0.551-0.113i	- i	0	-0.039+0.165i
0.776	0.4	0.556-0.119i	-0.057+0.001i	-0.263+0.543i	-0.279+0.497i
0.508	0.8	0.596-0.150i	-0.177+0.063i	-0.134+0.762i	-0.214+0.536i

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LINEAR SYMMETRICAL BENDING OF ELLIPTICAL WING OF AR = 6

0.212	0	0.719-0.188i	- i	0	0.021+0.061i
0.194	0.4	0.733-0.189i	-0.078+0.080i	0.022+0.179i	0.003+0.160i
0.127	0.8	0.799-0.181i	-0.211+0.123i	0.038+0.255i	-0.002+0.188i
0.424	0	0.618-0.162i	- i	0	0.034+0.085i
0.388	0.4	0.629-0.167i	-0.042+0.041i	-0.013+0.307i	-0.034+0.287i
0.254	0.8	0.691-0.185i	-0.146+0.110i	0.037+0.442i	-0.034+0.349i
0.847	0	0.551-0.113i	- i	0	0.011+0.096i
0.776	0.4	0.556-0.119i	-0.021+0.002i	-0.263+0.543i	-0.266+0.524i
0.508	0.8	0.596-0.150i	-0.087+0.063i	-0.134+0.762i	-0.216+0.652i

PARABOLIC SYMMETRICAL BENDING OF ELLIPTICAL WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{h}/b_o}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{h}/b_o}$
0.212	0	0.718-0.188i	∞ - ∞i	0	0.005+0.039i
0.194	0.4	0.733-0.189i	-0.030+0.105i	0.009+0.071i	-0.001+0.069i
0.127	0.8	0.799-0.181i	-0.372+0.144i	0.030+0.204i	-0.007+0.109i
0.424	0	0.618-0.162i	∞ - ∞i	0	0.003+0.065i
0.388	0.4	0.629-0.167i	0.029+0.054i	-0.005+0.123i	-0.016+0.128i
0.254	0.8	0.691-0.185i	-0.274+0.124i	0.029+0.353i	-0.034+0.213i
0.847	0	0.551-0.113i	∞ - ∞i	0	-0.010+0.085i
0.776	0.4	0.556-0.119i	0.039+0.012i	-0.105+0.217i	-0.116+0.255i
0.508	0.8	0.596-0.150i	-0.187+0.061i	-0.107+0.610i	-0.169+0.418i

PARABOLIC SYMMETRICAL BENDING OF ELLIPTICAL WING OF AR = 6

0.212	0	0.719-0.188i	∞ - ∞i	0	0.012+0.030i
0.194	0.4	0.733-0.189i	0.049+0.038i	0.009+0.072i	0.005+0.076i
0.127	0.8	0.799-0.181i	-0.228+0.128i	0.030+0.204i	-0.003+0.146i
0.424	0	0.618-0.162i	∞ - ∞i	0	0.021+0.039i
0.388	0.4	0.629-0.167i	0.049+0.003i	-0.005+0.123i	-0.006+0.133i
0.254	0.8	0.691-0.185i	-0.157+0.113i	0.029+0.353i	-0.028+0.273i
0.847	0	0.551-0.113i	∞ - ∞i	0	0.012+0.042i
0.776	0.4	0.556-0.119i	0.038-0.013i	-0.105+0.217i	-0.101+0.232i
0.508	0.8	0.596-0.150i	-0.087+0.059i	-0.107+0.610i	-0.145+0.521i

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LINEAR ANTISYMMETRICAL TORSION OF ELLIPTICAL WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_J	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{J}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{J}}$
		k_o = 0			
0	0.4	1	-0.571	1.152	0.494
0	0.6	1	-0.571	1.508	0.647
0	0.8	1	-0.571	1.508	0.647
		k_o = 0.212			
0.194	0.4	0.733-0.189i	-0.289+0.152i	0.865-0.023i	0.515+0.119i
0.170	0.6	0.754-0.188i	-0.304+0.155i	1.162-0.059i	0.683+0.136i
0.127	0.8	0.799-0.181i	-0.305+0.162i	1.222-0.101i	0.747+0.114i
		k_o = 0.424			
0.388	0.4	0.629-0.167i	-0.183+0.096i	0.762+0.172i	0.530+0.242i
0.339	0.6	0.647-0.174i	-0.197+0.114i	1.021+0.159i	0.695+0.280i
0.254	0.8	0.691-0.185i	-0.200+0.150i	1.077+0.046i	0.747+0.233i
		k_o = 0.847			
0.777	0.4	0.562-0.124i	-0.099+0.004i	0.703+0.556i	0.587+0.517i
0.678	0.6	0.569-0.130i	-0.114+0.030i	0.924+0.607i	0.738+0.593i
0.509	0.8	0.569-0.150i	-0.121+0.093i	0.957+0.386i	0.738+0.480i
		RIGID ROLLING OF ELLIPTICAL WING OF AR = 3			
		k_o = 0.212			
0.194	0.4	0.733-0.189i	-0.289+0.156i	0.022+0.179i	-0.016+0.108i
0.170	0.6	0.754-0.188i	-0.304+0.158i	0.033+0.241i	-0.018+0.144i
0.127	0.8	0.799-0.181i	-0.304+0.162i	0.038+0.255i	-0.014+0.158i
		k_o = 0.424			
0.388	0.4	0.629-0.167i	-0.182+0.104i	-0.013+0.307i	-0.064+0.218i
0.339	0.6	0.647-0.174i	-0.196+0.118i	0.003+0.414i	-0.073+0.288i
0.254	0.8	0.691-0.185i	-0.200+0.148i	0.037+0.441i	-0.058+0.313i
		k_o = 0.847			
0.777	0.4	0.562-0.124i	-0.098+0.016i	-0.258+0.549i	-0.274+0.454i
0.678	0.6	0.569-0.130i	-0.113+0.037i	-0.268+0.727i	-0.315+0.583i
0.507	0.8	0.596-0.150i	-0.120+0.089i	-0.134+0.762i	-0.248+0.608i
		k_o = 1.695			
1.553	0.4	0.520-0.072i	-0.005-0.037i	-1.376+1.015i	-1.304+1.006i
1.356	0.6	0.525-0.080i	-0.030-0.035i	-1.527+1.343i	-1.438+1.265i
1.017	0.8	0.539-0.099i	-0.056+0.026i	-1.046+1.377i	-1.113+1.233i

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UNIFORM PITCHING OF ELLIPTICAL WING OF AR = 3

k	$\frac{y}{b_o s}$	C	σ_J	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{J}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{J}}$
0	0	1	-0.400	3.142	1.885
0	0.4	1	-0.400	2.878	1.727
0	0.8	1	-0.400	1.727	1.036
0.212	0	0.719-0.188i	-0.148+0.118i	2.322+0.006i	1.816+0.327i
0.194	0.4	0.733-0.189i	-0.162+0.122i	2.161-0.059i	1.659+0.248i
0.127	0.8	0.799-0.181i	-0.227+0.131i	1.527-0.126i	1.084+0.093i
0.424	0	0.618-0.162i	-0.078+0.063i	2.049+0.569i	1.760+0.716i
0.388	0.4	0.629-0.167i	-0.083+0.081i	1.904+0.430i	1.620+0.618i
0.254	0.8	0.691-0.185i	-0.151+0.118i	1.347+0.057i	1.034+0.244i
0.847	0	0.551-0.113i	-0.037+0.001i	1.880+1.710i	1.761+1.663i
0.776	0.4	0.557-0.119i	-0.049+0.013i	1.735+1.397i	1.579+1.380i
0.508	0.8	0.596-0.150i	-0.101+0.079i	1.195+0.483i	0.967+0.583i
1.695	0	0.518-0.067i	0.005-0.018i	1.861+3.829i	1.925+3.787i
1.553	0.4	0.520-0.072i	-0.006-0.022i	1.657+3.189i	1.687+3.113i
1.017	0.8	0.539-0.099i	-0.081+0.020i	1.110+1.288i	0.939+1.247i
UNIFORM TRANSLATION OF ELLIPTICAL WING AR = 3					
0.212	0	0.719-0.188i	-0.148+0.121i	0.055+0.478i	-0.026+0.380i
0.194	0.4	0.733-0.189i	-0.162+0.124i	0.056+0.447i	-0.020+0.348i
0.127	0.8	0.799-0.181i	-0.228+0.125i	0.047+0.319i	-0.003+0.228i
0.424	0	0.618-0.162i	-0.076+0.070i	-0.067+0.823i	-0.160+0.721i
0.388	0.4	0.629-0.167i	-0.082+0.036i	-0.033+0.767i	-0.138+0.667i
0.254	0.8	0.691-0.185i	-0.154+0.107i	0.046+0.552i	-0.040+0.429i
0.847	0	0.551-0.113i	-0.036+0.008i	-0.828+1.466i	-0.849+1.369i
0.776	0.4	0.557-0.119i	-0.047+0.017i	-0.657+1.358i	-0.698+1.243i
0.508	0.8	0.596-0.150i	-0.103+0.065i	-0.167+0.952i	-0.271+0.787i
1.695	0	0.518-0.067i	0.003-0.012i	-4.154+2.757i	-4.088+2.773i
1.553	0.4	0.520-0.072i	-0.006-0.016i	-3.436+2.537i	-3.357+2.506i
1.017	0.8	0.539-0.099i	-0.075+0.008i	-1.308+1.721i	-1.333+1.481i
LINEAR ANTISYMMETRICAL TORSION OF AN ELLIPTICAL WING OF AR = 3 FOR HIGHER VALUES OF k_o					
k_o = 1.695					
1.553	0.4	0.520-0.072i	-0.001-0.049i	0.663+1.277i	0.706+1.220i
1.356	0.6	0.525-0.080i	-0.030-0.045i	0.874+1.438i	0.876+1.340i
1.017	0.8	0.539-0.099i	-0.075+0.016i	0.888+1.030i	0.763+0.998i
k_o = 2.542					
2.330	0.4	0.511-0.052i	0.039-0.000i	0.648+1.967i	0.693+2.020i
2.034	0.6	0.513-0.057i	0.031-0.031i	0.861+2.234i	0.954+2.234i
1.525	0.8	0.521-0.073i	-0.037-0.030i	0.869+1.639i	0.847+1.552i

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UNIFORM SPANWISE AILERON DEFLECTION OF A FULL-SPAN AILERON OF ELLIPTICAL WING OF AR = 3 ($e = 0$, $l = 0.2\ b/b_o$)

k	$\frac{y}{b_o s}$	C	σ_β	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{\beta}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{\beta}}$
0	0	1	-0.400	2.589	1.652
0	0.4	1	-0.400	2.373	1.514
0	0.8	1	-0.400	1.553	0.991
0.424	0	0.618-0.162i	-0.080+0.062i	1.664+0.379i	1.411+0.481i
0.389	0.4	0.629-0.167i	-0.090+0.074i	1.554+0.282i	1.294+0.400i
0.254	0.8	0.691-0.185i	-0.150+0.121i	1.111+0.020i	0.846+0.167i
0.847	0	0.551-0.113i	-0.041+0.001i	1.369+1.200i	1.264+1.141i
0.776	0.4	0.556-0.119i	-0.050+0.009i	1.317+0.974i	1.186+0.932i
0.508	0.3	0.596-0.150i	-0.098+0.083i	0.958+0.329i	0.761+0.406i

UNIFORM SPANWISE AILERON DEFLECTION OF A FULL-SPAN AILERON OF ELLIPTICAL WING OF AR = 3 ($e = 0$, $l = 0.2\ b/b_o$)

k	$\frac{y}{b_o s}$	C	σ_β	$\frac{\bar{M}_\beta^{(2)}(e)}{2\rho U^2 b_o^2 \bar{\beta}}$	$\frac{\bar{M}_\beta(e)}{2\rho U^2 b_o^2 \bar{\beta}}$
0	0	1	-0.400	0.187	0.124
0	0.4	1	-0.400	0.157	0.104
0	0.8	1	-0.400	0.067	0.045
0.424	0	0.618-0.162i	-0.080+0.062i	0.097+0.191i	0.081+0.197i
0.389	0.4	0.629-0.167i	-0.090+0.074i	0.087+0.155i	0.073+0.161i
0.254	0.8	0.691-0.185i	-0.150+0.121i	0.047+0.037i	0.037+0.042i
0.847	0	0.551-0.113i	-0.041+0.001i	-0.023+0.408i	-0.029+0.406i
0.776	0.4	0.556-0.119i	-0.050+0.009i	0.003+0.312i	-0.005+0.310i
0.508	0.8	0.596-0.150i	-0.098+0.083i	0.023+0.084i	0.021+0.087i

UNIFORM AILERON HINGE DEFLECTION OF A HALF-SPAN AILERON OF AN ELLIPTICAL WING OF AR = 6 $\left(e = \frac{1}{2},\ l = 0.1\ \frac{b}{b_o}\right)$

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k	$\frac{y}{b_o s}$	C	σ_β	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o \bar{\beta}}$	$\frac{\bar{L}}{2\rho U^2 b_o \bar{\beta}}$
0	0	1	∞	0	0.080
0	0.4	1	∞	0	0.255
0	0.6	1	-0.494	1.554	0.786
0	0.8	1	-0.360	1.154	0.738
0	0.9	1	-0.239	0.744	0.566
0.212	0	0.719-0.188i	∞ - ∞i	0	0.036-0.056i
0.194	0.4	0.733-0.189i	∞ - ∞i	0	0.175-0.121i
0.170	0.6	0.754-0.188i	-0.273+0.126i	1.187-0.182i	0.752-0.010i
0.127	0.8	0.799-0.181i	-0.188+0.130i	0.930-0.148i	0.707-0.007i
0.092	0.9	0.843-0.168i	-0.116+0.091i	0.629-0.102i	0.541-0.036i
0.424	0	0.618-0.162i	∞ - ∞i	0	0.006-0.057i
0.389	0.4	0.629-0.167i	∞ - ∞i	0	0.108-0.026i
0.339	0.6	0.648-0.174i	-0.187+0.087i	1.032-0.070i	0.726+0.031i
0.254	0.8	0.691-0.185i	-0.107+0.127i	0.813-0.100i	0.677+0.036i
0.185	0.9	0.741-0.188i	-0.044+0.148i	0.557-0.099i	0.519+0.010i

UNIFORM AILERON HINGE DEFLECTION OF A HALF-SPAN AILERON OF AN ELLIPTICAL WING

OF AR = 6 $\left(e = \frac{1}{2},\ l = 0.1\frac{b}{b_o}\right)$

k	$\frac{y}{b_o s}$	C	σ_β	$\frac{\bar{M}_\beta^{(2)}(e)}{2\rho U^2 b_o^{\,2}\bar{\beta}}$	$\frac{\bar{M}_\beta(e)}{2\rho U^2 b_o^{\,2}\bar{\beta}}$
		$k_o = 0$			
0	0.6	1	-0.494	0.030	0.023
0	0.8	1	-0.360	0.015	0.013
0	0.9	1	-0.239	0.003	0.003
		$k_o = 0.212$			
0.170	0.6	0.754-0.188i	-0.273+0.126i	0.026+0.005i	0.022+0.007i
0.127	0.8	0.799-0.181i	-0.188+0.130i	0.014+0.002i	0.012+0.003i
0.092	0.9	0.843-0.168i	-0.116+0.091i	0.003+0.000i	0.003+0.001i
		$k_o = 0.424$			
0.339	0.6	0.648-0.174i	-0.187+0.087i	0.024+0.013i	0.021+0.014i
0.254	0.8	0.691-0.185i	-0.107+0.127i	0.013+0.005i	0.012+0.006i
0.185	0.9	0.741-0.188i	-0.044+0.148i	0.003+0.001i	0.003+0.001i

ROLLING-MOMENT COEFFICIENTS $\bar{C}_R$ FOR A RECTANGULAR WING OF AR = 3

k_o	Due to rigid rolling		Due to linear antisymmetrical torsion	
	$\bar{C}_R/\bar{h}$	$\bar{C}_R^{(2)}/\bar{h}$	$\bar{C}_R/\bar{\alpha}$	$\bar{C}_R^{(2)}/\bar{\alpha}$
0	0	0	0.372	1.047
0.167	-0.008+0.062i	0.018+0.132i	0.375+0.078i	0.809-0.044i
0.333	-0.033+0.126i	0.003+0.227i	0.388+0.161i	0.711+0.105i
0.667	-0.142+0.262i	-0.142+0.397i	0.439+0.345i	0.641+0.411i
1.333	-0.676+0.656i	-0.817+0.734i	0.620+0.835i	0.608+0.980i

ROLLING-MOMENT COEFFICIENTS $\bar{C}_R$ FOR AN ELLIPTICAL WING OF AR = 3

k_o	Due to rigid rolling		Due to linear antisymmetrical torsion	
	$\pi\bar{C}_R/4\bar{h}$	$\pi\bar{C}_R^{(2)}/4\bar{h}$	$\pi\bar{C}_R/4\bar{\alpha}$	$\pi\bar{C}_R^{(2)}/4\bar{\alpha}$
0	0	0	0.250	0.502
0.212	-0.006+0.058i	0.013+0.095i	0.276+0.049i	0.458-0.029i
0.424	-0.026+0.116i	0.006+0.164i	0.279+0.109i	0.404+0.044i
0.847	-0.111+0.232i	-0.082+0.287i	0.288+0.211i	0.363+0.199i
1.695	-0.511+0.486i	-0.513+0.524i	0.319+0.469i	0.340+0.491i
2.542	-1.231+0.753i	-1.256+0.764i	0.341+0.756i	0.333+0.768i

LIFT COEFFICIENTS $\bar{C}_L$ FOR AN ELLIPTICAL WING OF AR = 3

k_o	Due to uniform translation		Due to uniform pitching	
	$\pi\bar{C}_L/8\bar{h}$	$\pi C_L^{(2)}/8\bar{h}$	$\pi\bar{C}_L/8\bar{\alpha}$	$\pi\bar{C}_L^{(2)}/8\bar{\alpha}$
0	0	0	1.480	2.467
0.212	-0.001+0.294i	0.048+0.389i	1.420+0.212i	1.867-0.069i
0.424	-0.103+0.566i	-0.013+0.672i	1.356+0.456i	1.665+0.297i
0.847	-0.554+1.056i	-0.494+1.172i	1.337+1.089i	1.495+1.075i
1.695	-2.637+2.080i	-2.665+2.163i	1.385+2.421i	1.431+2.468i

RECTANGULAR WING OF AR = 13.5 WITH INDICATED MODES (k_o = 0.097)

Mode y/b_os	Torsion	First bending	Second bending
0	0	0	0
0.4	0.586	0.230	0.683
0.6	0.809	0.461	0.589
0.8	0.951	0.725	-0.068
1.0	1.0	1.0	-1.0

k	y/b_os	C	σ_j	$\frac{\bar{L}^{(2)}}{2\rho U^2 b_o\bar{j}}$	$\frac{\bar{L}}{2\rho U^2 b_o\bar{j}}$
Torsion mode					
0.097	0	0.837-0.170i	∞ - ∞ i	0	0.302-0.140i
0.097	0.4	0.837-0.170i	-0.045+0.048i	1.563-0.151i	1.472-0.070i
0.097	0.6	0.837-0.170i	-0.069+0.071i	2.159-0.209i	1.968-0.045i
0.097	0.8	0.837-0.170i	-0.109+0.089i	2-538-0.245i	2.190-0.008i
0.097	1.0	0.837-0.170i	-0.835+0.122i	2.669-0.258i	0.013-0.102i
FIRST BENDING					
0.097	0	0.837-0.170i	∞ - ∞i	0	0.009+0.013i
0.097	0.4	0.837-0.170i	0.056+0.001i	+0.009+0.058i	0.008+0.062i
0.097	0.6	0.837-0.170i	-0.046+0.064i	+0.017+0.117i	0.008+0.111i
0.097	0.8	0.837-0.170i	-0.124+0.096i	0.027+0.184i	0.006+0.157i
0.097	1.0	0.837-0.170i	-0.835+0.122i	0.037+0.254i	+0.000+0.001i
SECOND BENDING					
0.097	0	0.837-0.170i	- i	0	.0.011+0.026i
0.097	0.4	0.837-0.170i	-0.153+0.101i	0.025+0.173i	0.004+0.142i
0.097	0.6	0.837-0.170i	-0.176+0.103i	0.022+0.150i	0.003+0.118i
0.097	0.8	0.837-0.170i	-0.619+0.309i	-0.003-0.017i	0.004-0.005i
0.097	1.0	0.837-0.170i	-0.835+0.122i	-0.037-0.254i	0.000-0.001i

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TABLE IX.- SPAN CORRECTIONS AT MIDSPAN FOR RIGID ELLIPTICAL WING IN TRANSLATION AND PITCHING. (ONE-POINT APPROXIMATION)

k_0	$k_0 s$	C	$\sigma_h = \sigma_\alpha = \sigma$	C + σ	AR
0	0	1 + 0i	−0.571	0.429	1.5
0.424	0.5	0.618 − 0.162i	−0.196 + 0.092i	0.422 − 0.070i	
0.847	1.0	0.551 − 0.113i	−0.114 + 0.008i	0.437 − 0.105i	
1.695	2.0	0.518 − 0.066i	−0.018 − 0.044i	0.500 − 0.110i	
3.390	4.0	0.505 − 0.036i	+0.005 + 0.007i	0.511 − 0.029i	
0	0	1 + 0i	−0.500	0.500	2
0.318	0.5	0.657 − 0.177i	−0.176 + 0.116i	0.481 − 0.061i	
0.637	1.0	0.573 − 0.134i	−0.103 + 0.048i	0.470 − 0.086i	
1.273	2.0	0.528 − 0.084i	−0.042 − 0.018i	0.486 − 0.102i	
2.546	4.0	0.509 − 0.048i	+0.004 + 0.017i	0.513 − 0.031i	
0	0	1 + 0i	−0.400	0.600	3
0.212	0.5	0.719 − 0.188i	−0.154 + 0.125i	0.565 − 0.063i	
0.424	1.0	0.618 − 0.162i	−0.079 + 0.075i	0.539 − 0.087i	
0.847	2.0	0.551 − 0.113i	−0.043 + 0.018i	0.507 − 0.095i	
1.695	4.0	0.518 − 0.067i	−0.010 − 0.013i	0.508 − 0.080i	
2.542	6.0	0.509 − 0.047i	+0.007 − 0.005i	0.516 − 0.052i	
0	0	1 − 0i	−0.250	0.750	6
0.106	0.5	0.824 − 0.175i	−0.108 + 0.096i	0.716 − 0.078i	
0.212	1.0	0.719 − 0.189i	−0.052 + 0.077i	0.669 − 0.111i	
0.424	2.0	0.618 − 0.162i	−0.021 + 0.037i	0.597 − 0.124i	
0.847	4.0	0.551 − 0.113i	−0.013 + 0.009i	0.538 − 0.103i	
1.273	6.0	0.528 − 0.084i	−0.008 − 0.000i	0.520 − 0.084i	

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TABLE X.– DETERMINANT TERMS AS A FUNCTION OF k FOR EXAMPLE I

Term	Structural part of term	Aero. part of term (two-dim.)	Determinant terms (two-dim.) $\left(g = g_h = g_\alpha \text{ and } \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig) = \Omega\right)$
k = 0.333			
A	$1.3331 \left[1 - 0.1920 \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	$-0.0096 - 0.7799i$	$1.3235 - 0.7799i - 0.2560\Omega$
B	0.3154	$-3.0592 - 0.7437i$	$-2.7438 - 0.7437$
D	0.3154	$0.1274 + 0.1950i$	$0.4428 + 0.1950i$
E	$0.8789 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$	$0.9075 - 0.8017i$	$1.7864 - 0.8017 - 0.8789\Omega$
k = 0.4			
A	$1.3331 \left[1 - 0.1920 \left(\frac{\omega}{\omega_\alpha}\right)^2 (1 + ig_h)\right]$	$0.0350 - 0.6250i$	$1.3681 - 0.6250i - 0.2560\Omega$
B	0.3154	$-2.0431 - 0.7344i$	$-1.7277 - 0.7344i$
D	0.3154	$0.1163 + 0.1563i$	$0.4317 + 0.1563i$
E	$0.8789 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$	$0.6365 - 0.6375i$	$1.5154 - 0.6375i - 0.8789\Omega$
k = 0.5			
A	$1.3331 \left[1 - 0.1920 \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	$0.0794 - 0.4783i$	$1.4125 - 0.4783i - 0.2560\Omega$
B	0.3154	$-1.2414 - 0.6769i$	$-0.9260 - 0.6769i$
D	0.3154	$0.1051 + 0.1196i$	$0.4205 + 0.1196i$
E	$0.8789 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$	$0.4227 - 0.4862i$	$1.3016 - 0.4862i - 0.8789\Omega$
k = 0.6			
A	$1.3331 \left[1 - 0.1920 \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	$0.1081 - 0.3859i$	$1.4412 - 0.3859i - 0.2560\Omega$
B	0.3154	$-0.8208 - 0.6112i$	$-0.5054 - 0.6112i$
D	0.3154	$0.0980 + 0.0965i$	$0.4134 + 0.0961i$
E	$0.8789 \left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + g_\alpha)\right]$	$0.3535 - 0.4927i$	$1.2513 - 0.4927 - 0.8789\Omega$

TABLE XI

SOLUTION OF SIMULTANEOUS EQUATIONS BY USE OF THE CROUT METHOD OF REFERENCE 8 (EXAMPLE I)

				Linear torsion		Parabolic bending	
				$F_\alpha(y)$	Check	$F_h(y)$	Check
Given Matrix							
1.0169 −0.0524i	−0.3837 0.0739i	0.2702 −0.0785i	−0.2262 0.0790i	0	0.6772 0.0220i	0	0.6772
.9401 −.0557i	−.1323 .0262i	−.1299 .0405i	.2289 −.0826i	0.4	1.3068 −.0716i	0.16	1.0668 −.0507i
.6672 −.0684i	.4213 −.1238i	−.0234 .0245i	−.3057 .1170i	.8	1.5594 −.0507i	.64	1.3994 −.0507i
.2966 .0972i	.4701 −.0505i	.7234 −.1879i	.9896 −.3072i	1.0	3.4797 −.4484i	1.0	3.4797i .4484i
Supplementary Matrix							
0.9808 .0505i	−0.3837 .0739i	0.2702 −.0785i	−0.2262 .0790i	0	0.6772 .0220i	0	0.6772 .0220i
4.3261 .8746i		−.3793 .1150i	.4375 −.1573i	0.4	.6803 −.0871i	0.16	.4903 −.0871i
.9353 .3470i			−1.4673 .6334i	−.4264 .0856i	−.9840 .3701i	.1494 .0343i	−.3782 .3187i
.3681 .1118i				.7055 −.1898i	3.1925 −.9454i	.3483 −.1570i	2.8352 −.9122i
Auxiliary Matrix							
1.0169 −.0524i	−0.3801 .0531i	0.2690 −.0633i	−0.2258 .0661i	0	0.6631 .0558i	0	0.6631 .0558i
.9401 −.0557i	.2221 −.0449i	−1.7415 .1658i	2.0302 −.2979i	1.7304 .3498i	3.0192 .2182i	0.6922 .1399i	1.9810 .0003i
.6672 −.0684i	.6713 −.1852i	.9398 −.3487i	−1.5922 .0833i	−.4285 −.0679i	−1.0207 .0151i	.1278 .0839i	−.4643 .1668i
.2966 .0972i	.5880 −.0293i	1.6566 −.3488i	2.4870 −.7556i	.2809 .0090i	1.2808 .0090i	.1458 −.0189i	1.1456 −.0188i
				$K'_{1\alpha}$ = 0.5264 .0485i	1.5264 .0484i	K'_{1h} = 0.3377 .0320i	1.3378 .0319i
				$K'_{3\alpha}$ = 1.1786 .2779i	2.1785 .2779i	K'_{3h} = 1.0329 .2349i	2.0330 .2347i
				$K'_{5\alpha}$ = 0.0195 −.0770i	1.0193 −.0773i	K'_{5h} = 0.3584 .0417i	1.3582 .0414i
				$K'_{7\alpha}$ = 0.2809 .0090i	1.2808 .0090i	K'_{7h} = 0.1458 −.0189i	1.1456 −.0188i

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TABLE XII.– VALUES OF w AT VARIOUS STATIONS ALONG THE SPAN (EXAMPLE I)

$\frac{y}{b_o s}$	n	$\frac{\sin n\phi}{n}$	$K_{nh} \frac{\sin n\phi}{n}$	$K_{n\alpha} \frac{\sin n\phi}{n}$
0.4	1	0.9165	0.3095 + 0.0293i	0.4824 + 0.0445i
	3	-0.1099	-0.1135 - 0.0258i	-0.1295 - 0.0305i
	5	-0.0936	-0.0335 - 0.0039i	-0.0018 + 0.0072i
	7	0.1380	0.0201 - 0.0026i	0.0388 + 0.0012i
		$\bar{w}$;	0.1826 - 0.0030i	0.3899 + 0.0224i
0.6	1	0.800	0.2702 + 0.0256i	0.4211 + 0.0388i
	3	0.1173	0.1212 + 0.0276i	0.1382 + 0.0326i
	5	-0.1994	-0.0715 - 0.0083i	-0.0039 + 0.0154i
	7	0.0295	0.0043 - 0.0006i	0.0083 + 0.0003i
		$\bar{w}$;	0.3242 + 0.0443i	0.5637 + 0.0871i
0.8	1	0.600	0.2026 + 0.0192i	0.3158 + 0.0291i
	3	0.3120	0.3223 + 0.0733i	0.3677 + 0.0867i
	5	-0.0151	-0.0054 - 0.0006i	-0.0003 + 0.0012i
	7	-0.1398	-0.0204 + 0.0026i	-0.0393 - 0.0013i
		$\bar{w}$;	0.4991 + 0.0945i	0.6439 + 0.1157i
0.9	1	0.4359	0.1472 + 0.0139i	0.2295 + 0.0211i
	3	0.3255	0.3362 + 0.0765i	0.3836 + 0.0905i
	5	0.1550	0.0556 + 0.0065i	0.0030 - 0.0119i
	7	-0.0022	-0.0003 + 0.0000i	-0.0006 - 0.0000i
		$\bar{w}$;	0.5387 + 0.0969i	0.6155 + 0.0997i

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TABLE XIII.– CALCULATION OF σ_j ($j = \alpha$, h) FUNCTIONS

(EXAMPLE I)

PARABOLIC BENDING

$\frac{y}{b_o s}$	w_j	$f_j(y)$	$\frac{\overline{w}_j}{f_j(y)} - 1$	$C + \frac{iJ}{J_o - iJ}$	σ_j
0	————	0	$\infty - \infty i$	0.5850 + 0.0309i	$\infty - \infty i$
0.4	0.1326 – 0.0030i	0.16	0.1415 – 0.0188i	0.5850 + 0.0309i	0.0832 – 0.0066i
0.6	0.3242 + 0.0445i	0.36	–0.0994 + 0.1251i	0.5850 + 0.0309i	–0.0620 + 0.0689i
0.8	0.4991 + 0.0945i	0.64	–0.2202 + 0.1477i	0.5850 + 0.0309i	–0.1334 + 0.0796i
0.9	0.5387 + 0.0969i	0.81	–0.3349 + 0.1196i	0.5850 + 0.0309i	–0.1996 + 0.0596i
1.0	0	1.0	– 1	0.5850 + 0.0309i	–0.5850 – 0.0309i
LINEAR TORSION					
0	————	0	$\infty - \infty i$	0.5850 + 0.0309i	$\infty - \infty i$
0.4	0.3899 + 0.0224i	0.4	–0.0252 + 0.0560i	0.5850 + 0.0309i	–0.0165 + 0.0320i
0.6	0.5637 + 0.0871i	0.6	–0.0605 + 0.1452i	0.5850 + 0.0309i	–0.0399 + 0.0851i
0.8	0.6439 + 0.1157i	0.8	–0.1951 + 0.1446i	0.5850 + 0.0309i	–0.1186 + 0.0786i
0.9	0.6155 + 0.0997i	0.9	–0.3161 + 0.1108i	0.5850 + 0.0309i	–0.1885 + 0.0551i
1.0	0	1.0	– 1	0.5850 + 0.0309i	–0.5850 – 0.0309i

TABLE XIV.– VALUES OF INTEGRANDS FOR USE IN GRAPHICAL INTEGRATION

(EXAMPLE I)

z	z^3	z^4	σ_h	$\sigma_h z^4$	$\sigma_h z^3$
0.4	0.064	0.0256	0.0832 – 0.0066i	0.0021 – 0.0002i	0.0053 – 0.0004i
0.6	0.216	0.1296	–0.0620 + 0.0689i	–0.0080 + 0.0103i	–0.0134 + 0.0149i
0.8	0.512	0.4096	–0.1334 + 0.0796i	–0.0546 + 0.0326i	–0.0683 + 0.0408i
0.9	0.729	0.6561	–0.1996 + 0.0596i	–0.1310 + 0.0391i	–0.1455 – 0.0309i
1.0	1.0	1.0	–0.5850 – 0.0309i	–0.5850 – 0.0309i	–0.5850 + 0.0309i
z	z^2	z^3	σ_α	$\sigma_\alpha z^3$	$\sigma_\alpha z^2$
0.4	0.16	0.064	–0.0165 + 0.0320i	–0.0011 + 0.0020i	–0.0026 + 0.0051i
0.6	0.36	0.216	–0.0399 + 0.0851i	–0.0086 + 0.0179i	–0.0144 + 0.0299i
0.8	0.64	0.512	–0.0016 + 0.0786i	–0.0607 + 0.0402i	–0.0759 + 0.0503i
0.9	0.81	0.729	–0.1885 + 0.0551i	–0.1375 + 0.0402i	–0.1525 + 0.0446i
1.0	1.0	1.0	–0.5850 – 0.0309i	–0.5850 – 0.0309i	–0.5850 – 0.0309i

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TABLE XV

AERODYNAMIC CORRECTIONS AND CALCULATED DETERMINANT TERMS CORRECTED FOR FINITE-SPAN EFFECTS (EXAMPLE I)

$$\left[\Omega = \left(\frac{\omega_\alpha}{\omega} \right)^2 (1 + ig) \quad \text{where } g = g_\alpha = g_h \right]$$

Term	Structural part of term	Aerodynamic part of term (two-dim.)	Finite-span corrections	Determinant term corrected for the effect of a finite span
k = 0.333				
A	$1.3331\left[1 - 0.1920\left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	−0.0096 − 0.7799i	0.0876 + 0.2628i	1.4111 − 0.5171i − 0.2560 Ω
B	0.3154	−3.0592 − 0.7437i	0.9096 − 0.1098i	−1.8180 − 0.8535i
D	0.3154	0.1274 + 0.1950i	−0.0210 − 0.0571i	0.4218 + 0.1379i
E	$0.8789\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$	0.9075 − 0.8017i	−0.2127 0.0343i	1.5737 − 0.7674i − 0.8789 Ω
k = 0.40				
A	$1.3331\left[1 - 0.1920\left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	0.0350 − 0.6250i	0.0530 + 0.1965i	1.4211 − 0.4285i − 0.2560 Ω
B	0.3154	−2.0431 − 0.7344i	0.5752 − 0.0103i	−1.1525 − 0.7447i
D	0.3154	0.1163 + 0.1563i	−0.0129 − 0.0427i	0.4188 + 0.1136i
E	$0.8789\left[1 - \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_\alpha)\right]$	0.6365 − 0.6375i	−0.1324 + 0.0081i	1.3830 − 0.6294i − 0.8789 Ω
k = 0.50				
A	$1.3331\left[1 - 0.1920\left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig_h)\right]$	0.0794 − 0.4783i	0.0208 + 0.1312i	1.4333 − 0.4285i − 0.2560 Ω
B	0.3154	−1.2414 − 0.6769i	0.3014 + 0.0445i	−0.6246 − 0.6324i
D	0.3154	0.1051 + 0.1196i	−0.0054 − 0.0282i	0.4151 + 0.0914i
E	$0.8789\left[1 - \left(\frac{\omega_\alpha}{\omega}\right) (1 + ig_\alpha)\right]$	0.4227 − 0.4862i	−0.0692 − 0.0065i	1.2324 − 0.4927i − 0.8789 Ω

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TABLE XVI

CHARACTERISTICS ALONG SPAN OF TAIL SURFACES USED IN EXAMPLE II

$\frac{y}{sb_o}$	Horizontal tail		Vertical tail					
	b ft	b/b_o	b ft	b/b_o	a	c	e	f
0	4	1	4.64	1	−1.909	−0.032	0.246	1
.3	—	—	3.95	.851	−2.413			.828
.4	3.44	0.860	3.69	.797	−2.634	−.230	.098	.835
.[illegible]13			3.33	.718	−2.970			.877
.6	3.08	.770	3.00	.648	−3.300	−.340	.060	1
.8	2.62	.655	2.18	.471	−4.587	−.404	.037	1
1.0	0	0	0	0	—	—	0	1

TABLE XVII

GENERAL CHARACTERISTICS OF TAIL SURFACES USED IN EXAMPLE II

	Horizontal tail	Vertical tail
s	2.525	1.800
b_o	4 ft	4.64 ft
b_r	3.33 ft	3.33 ft
ab	—	−10.08 ft
n	—	− .40(lag)

TABLE XVIII.– EVALUATION OF INTEGRALS FOR EXAMPLE II

$$\left[\Omega = \left(\frac{\omega_\alpha}{\omega}\right)^2 (1 + ig) \quad \text{where} \quad g = g_\alpha = g_h\right]$$

Determinant term	Structural part (ft)	Two-dimensional aerodynamic part (ft)	Determinant term two-dimensional aerodynamic part (ft)
	(k_{oH} = 0.686 k_{oV} = 0.795)		
A	67.27 [1 − 1.988 Ω]	2.1228 − 14.6648i	69.393 − 14.6648i − 133.73 Ω
B	100.1	−9.0606 − 24.9334i	91.039 − 24.9334i
C	1.731	−11.2670 − 3.6993i	−9.5360 − 3.6993i
D	100.1	4.4524 − 17.8319i	104.552 − 17.8319i
E	1891 [1 − Ω]	−27.7895 − 167.093i	1863.2 − 167.093i − 1891 Ω
F	23.862	−67.1873 − 38.6370i	−93.3253 − 38.6370i
G	1.737	0.3202 − 0.1728i	2.0572 − 0.1728i
H	23.96	0.2137 − 4.716i	24.1297 − 4.716i
I	5.992	−0.1613 − 3.2510i	5.8307 − 3.2510i
	(k_{oH} = 0.600 k_{oV} = 0.696)		
A	67.27 [1 − 1.988 Ω]	1.5917 − 17.3940i	68.862 − 17.3940i − 133.73 Ω
B	100.1	−13.9834 − 28.8686i	86.117 − 28.8686i
C	1.731	−14.8607 − 3.5703i	−13.1297 − 3.5703i
D	100.1	3.7185 − 20.1290i	103.8185 − 20.1290i
E	1891 [1 − Ω]	−54.9543 − 184.636i	1836 − 184.631 − 1891 Ω
F	23.862	−88.6384 − 39.5645i	−64.7764 − 39.5645i
G	1.737	0.3108 − 0.2044i	2.0478 − 0.2044i
H	23.916	2.0841 − 5.3128i	26.0000 − 5.3128i
I	5.992	−0.4069 − 3.5080i	5.5851 − 3.5080i
	(k_{oH} = 0.436 k_{oV} = 0.506)		
A	67.27 [1 − 1.988 Ω]	−0.6677 − 25.8568i	66.6023 − 25.8568 − 133.73 Ω
B	100.1	−34.9401 − 39.1934i	65.1599 − 39.1934i
C	1.731	−30.2362 − 2.9483i	−28.5052 − 2.9483i
D	100.1	0.7482 − 29.5701i	100.8482 − 29.5701i
E	1891 [1 − Ω]	−166.9957 − 255.1697i	1724 − 255.1697i − 1891 Ω
F	23.862	−177.2376 − 43.3112i	−153.3756 − 43.3112i
G	1.737	0.2795 − 0.2897i	2.0165 − 0.2897i
H	23.916	0.7526 − 7.2411i	24.6686 − 7.2411i
I	5.992	−1.1038 − 4.7278i	4.8882 − 4.7278i

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TABLE XIX.– SOLUTION OF SET OF EQUATIONS $\Sigma K_{nh} A_{nm} = \bar{\Omega}_{hm}^{(2)}$ BY THE CROUT METHOD (EXAMPLE II)

				$\bar{\Omega}_h^{(2)}$	Check
Given Matrix					
	0.4709 -0.1194i	-0.4459 0.1566i	0.2507 -0.0932i	0.0591 0.5518i	0.3348 0.4958i
	0.6559 -0.1844i	-0.2670 0.0978i	-0.2849 0.1111i	0.0768 0.7709i	0.1808 0.7954i
	0.6347 -0.1760i	0.2479 -0.0882i	-0.2657 0.1127i	0.0996 0.9245i	0.7165 0.7730i
Supplementary Matrix					
1.9953 0.5059i		-0.4459 0.1566i	0.2507 -0.0932i	0.0591 0.5518i	0.3348 0.4958i
2.4599i 0.9623i			-0.6330 0.2508i	-0.0260 -0.0006i	-0.3065 0.1124i
0.9673 0.3416i				0.0649 0.1809i	0.9843 -0.1439i
Auxiliary Matrix					
	0.4709 -0.1194i	-0.9689 0.0869i	0.5474 -0.0591i	-0.1612 1.1309i	0.4172 1.1586i
	0.6559 -0.1844i	0.3525 -0.1379i	-1.7985 0.0078i	-0.0634 -0.0265i	-0.8621i -0.0185i
	0.6347 -0.1760	0.8476 -0.3139i	0.9192 -0.3246i	0.0010 0.1972i	1.0013 0.1970i
			k_{2h}	-0.2031 1.3462i	0.7970 1.3460i
			k_{4h}	-0.0601 0.3282i	0.9403 0.3280i
			k_{6h}	0.0010 0.1972i	1.0013 0.1970i

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TABLE XX.- VALUES OF $\bar{\Omega}_n$ FOR EXAMPLE II

$\frac{y}{b_o s}$	n	$\frac{\sin n\phi}{n}$	$K_{nh} \frac{\sin n\phi}{n}$
0.4	2	0.3667	-0.0745+0.4937i
	4	-0.2493	0.0150-0.0818i
	6	+0.1132	+0.0001+0.0223i
		$\bar{\Omega}_{\bar{h}}$	-0.0594+0.4342i
0.6	2	0.4800	-0.0975+0.6462i
	4	-0.1344	0.0081-0.0441i
	6	-0.1098	-0.0001-0.0217i
		$\bar{\Omega}_{\bar{h}}$	-0.0895+0.5804i
0.8	2	0.4800	-0.0975+0.6462i
	4	0.1344	-0.0081+0.0441i
	6	-0.1098	-0.0001-0.0217i
		$\bar{\Omega}_{\bar{h}}$	-0.1057+0.6686i

TABLE XXI.- AERODYNAMIC CORRECTIONS AND CALCULATED DETERMINANT TERMS CORRECTED FOR FINITE-SPAN EFFECTS (EXAMPLE II)

Term	k_{oH} = 0.436	k_{oV} = 0.506	
	Det. term with two-dim. aero. part.	Finite-span corrections	Det. term corrected for finite-span effects
A	66.6023-25.8568i -133.73 Ω	5.0885+6.9884i	71.6908-18.8684i -133.73 Ω
B	65.1599-39.1934i	11.5520+2.8680i	76.7119-36.3254i
C	-28.5052-2.9483i	6.5593-3.2360i	-21.9459-6.1843i
D	100.8482-29.5701i	6.3861+9.0492i	107.2343-20.5209i
E	1724-255.1697i -1891 Ω	65.4194+16.1656i	1789.4-239.004i -1891 Ω
F	-153.3756-43.3112i	34.6350-17.1113i	-118.7415-60.4225i
G	2.0165-0.2897i	0.0572+0.0946i	2.0737-0.1951i
H	24.6686-7.2411i	-4.7027+1.7546i	19.9659-5.4865i
I	4.8882-4.7278i	0.3885-0.2012i	5.2767-4.9290i

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TABLE XXI.– Concluded

$k_{oH} = 0.600$ $k_{oV} = 0.696$

A	68.8617-17.3940i -133.73 Ω	2.8478+3.9822i	71.7095-13.4118i -133.73 Ω
B	86.117-28.8686i	5.8026+2.4983i	91.9196-26.3703i
C	-13.1297-3.5703i	2.8707-1.1077i	-10.2590-4.6780i
D	103.819-20.1290i	3.0639+5.6423i	106.8829-14.4867i
E	1836-184.631i -1891 Ω	29.5923+15.0188i	1865.6-169.617i -1891 Ω
F	-64.7764-39.5645i	14.5931-5.2815i	-50.1833-44.8460i
G	2.0478-0.2044i	0.0264+0.0561i	2.0742-0.1483i
H	26.0001-5.3128i	-1.9233+0.3250i	24.0768-4.9878i
I	5.5851-3.5080i	0.1624-0.0418i	5.7475-3.5498i

$k_{oH} = 0.686$ $k_{oV} = 0.795$

A	69.3928-14.6648i -133.73 Ω	1.9813+3.0673i	71.3741-11.5975i -133.73 Ω
B	91.0394-24.9334i	4.0837+2.1955i	95.1231-22.7379i
C	-9.5360-3.6993i	1.9336-0.5926i	-7.6024-4.2919i
D	104.5524-17.8319i	2.2890+4.2618i	106.8414-13.5701i
E	1863.2-167.093i -1891 Ω	20.5217+13.1200i	1883.7-153.973i -1891 Ω
F	-43.3253-38.6370i	9.8602-2.5080i	33.4651-41.1450i
G	2.0572-0.1728i	0.0182+0.0416i	2.0754-0.1312i
H	26.053-4.716i	-1.2893+0.0935i	24.7637-4.6225i
I	5.8307-3.2510i	0.1060-0.0172i	5.9367-3.2682i

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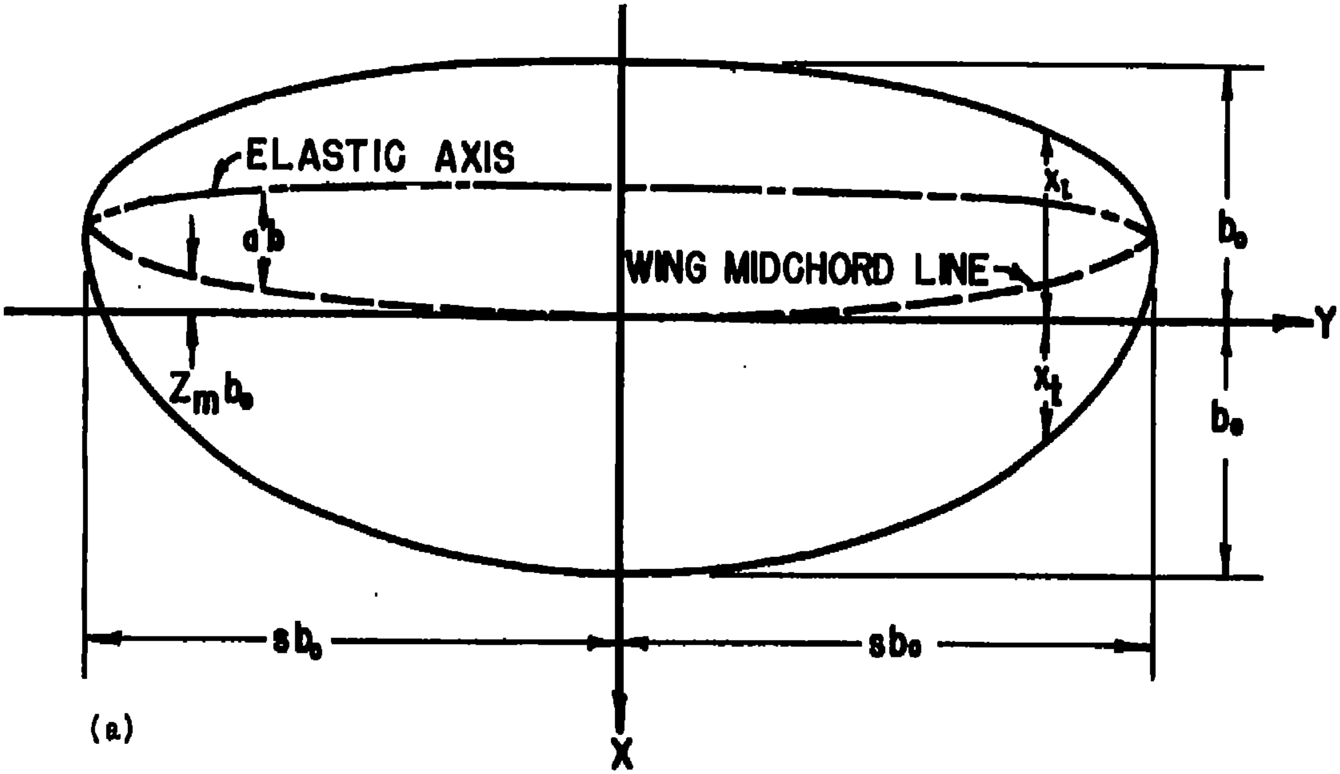
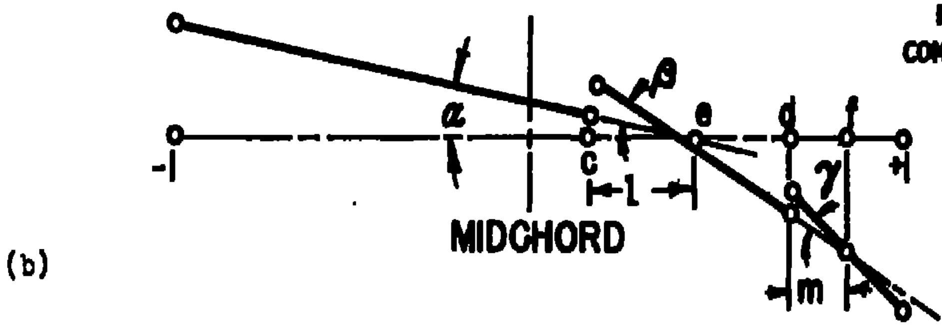


Figure 1.- (a), Plan form of wing, showing location of elastic axis and midchord line.
(b), Wing section with aileron and tab, showing main parameters in units of local semichord.



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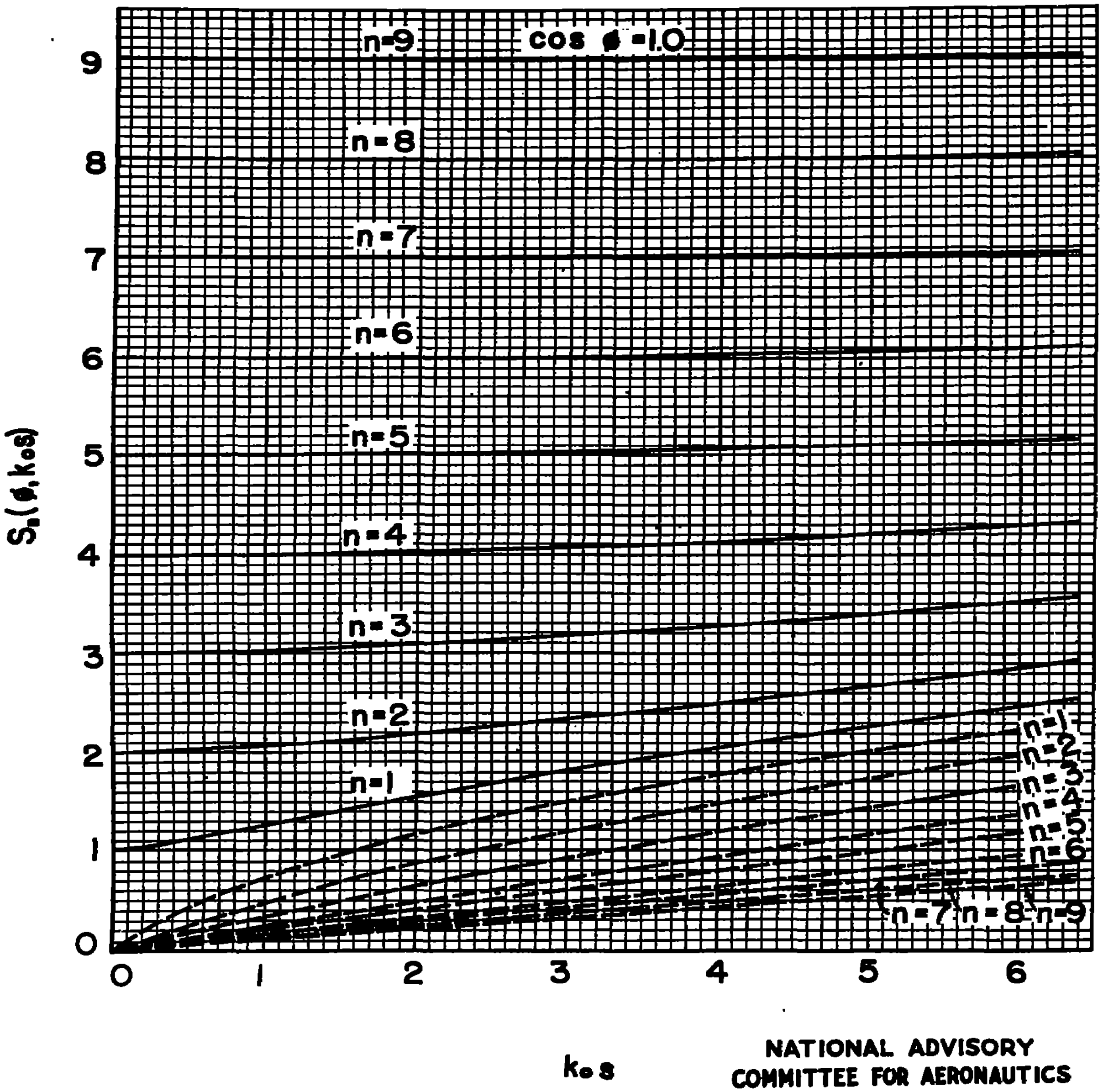
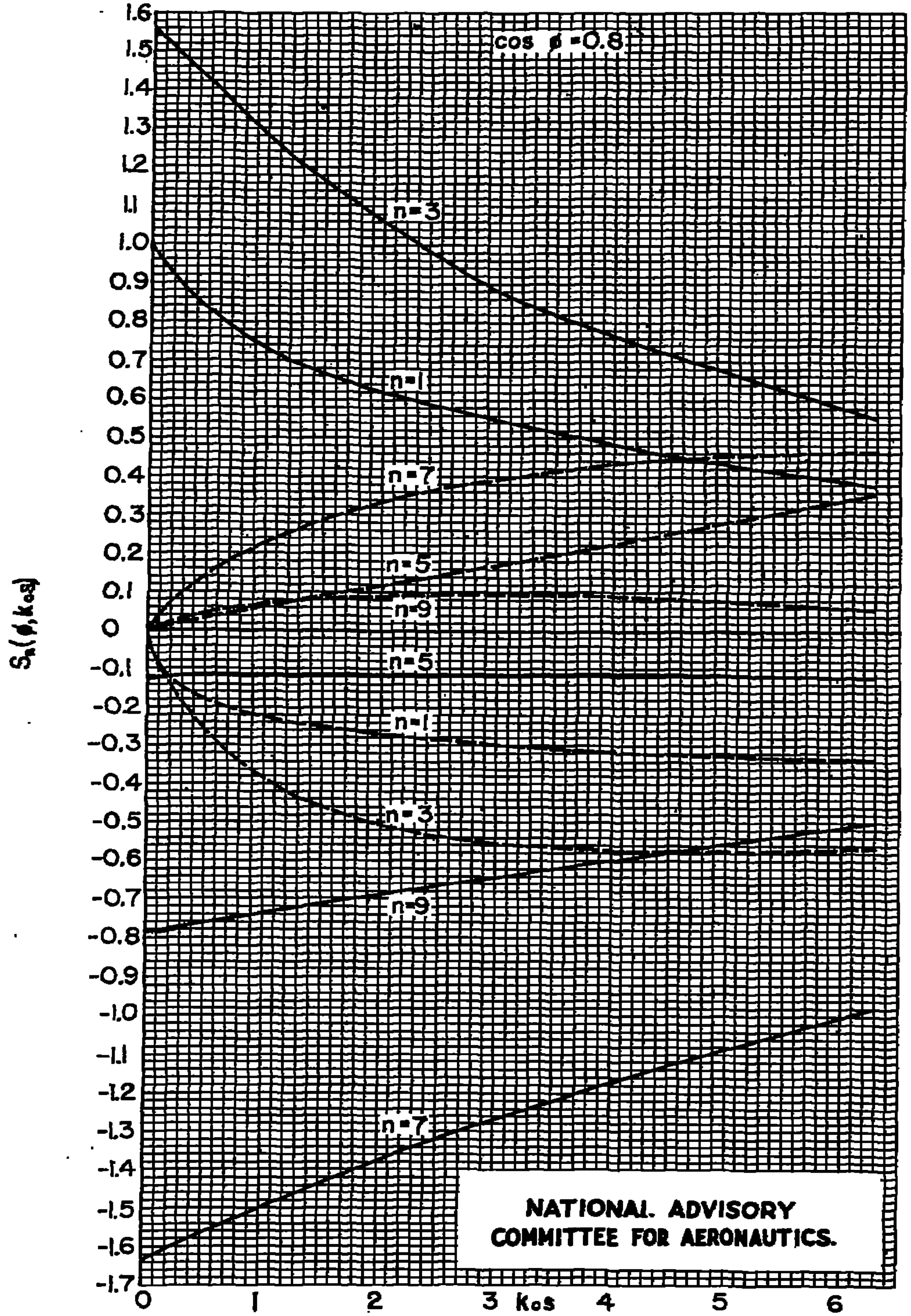


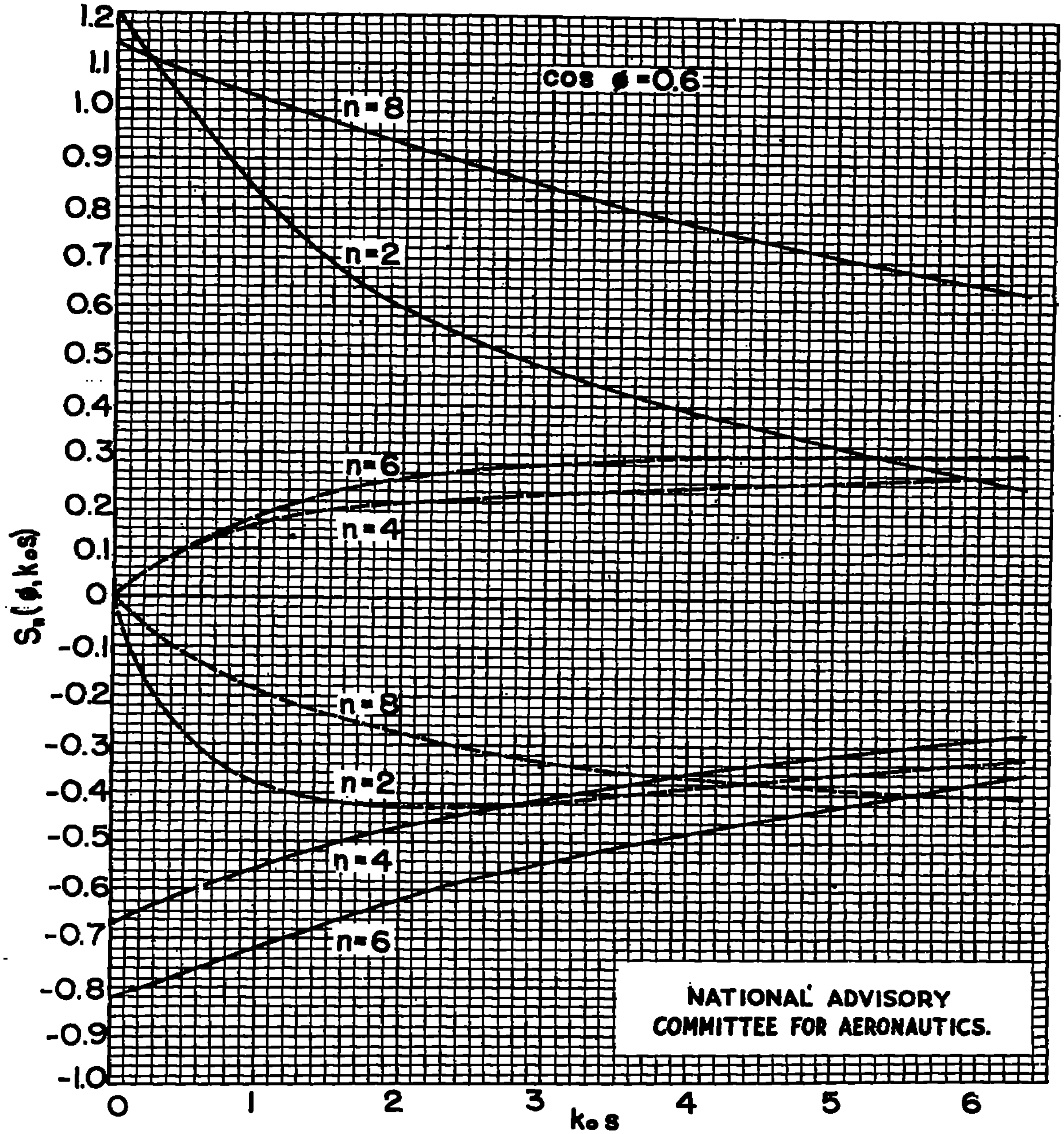
(a) Cos ϕ = 1; n = 1, 2, 3, 4, 5, 6, 7, 8, 9.

Figure 2 (a to h).- Function $S_n(\phi, k_o s)$ against $k_o s$ for various values of n and of cos ϕ. (Dash lines are for



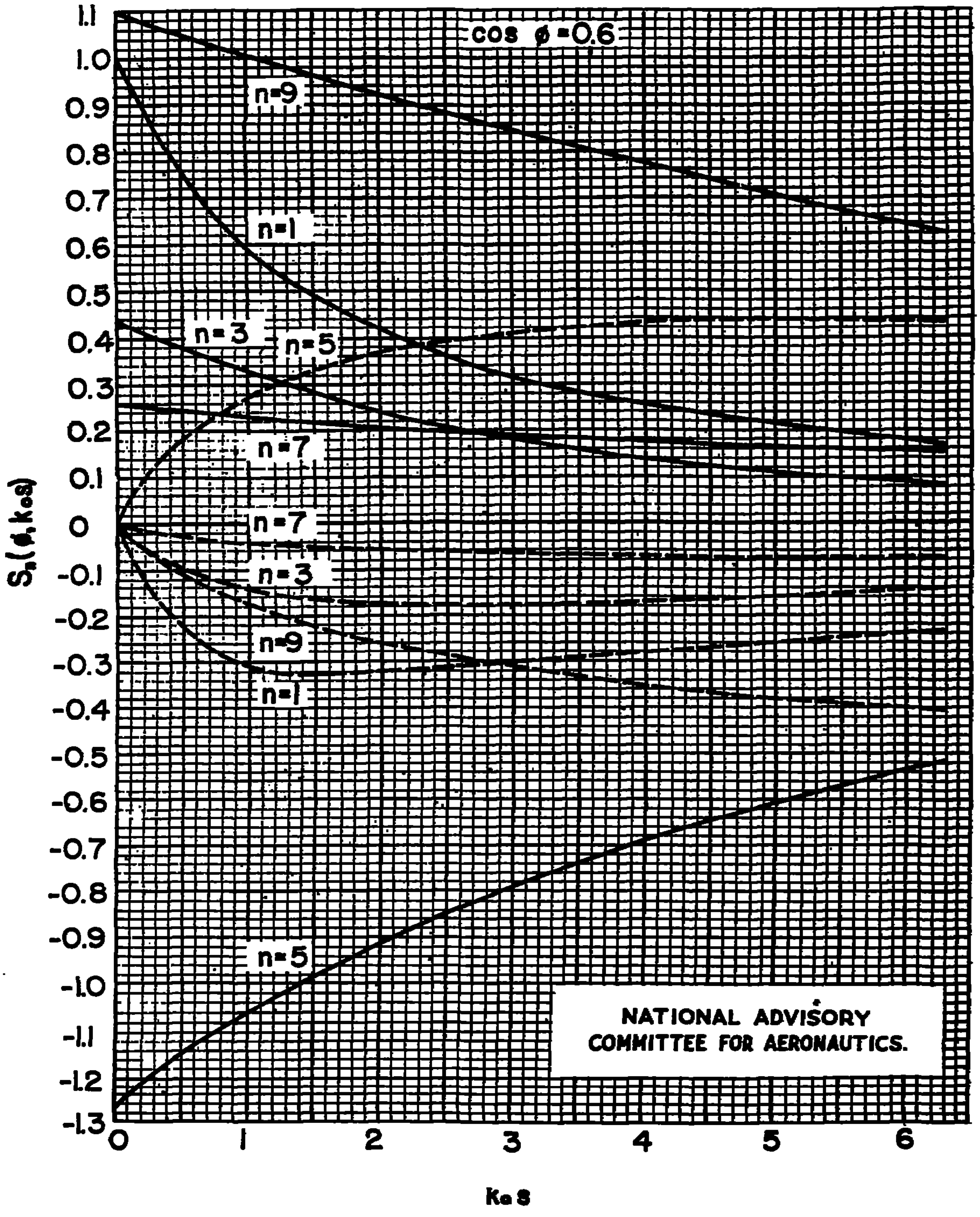
(b) Cos ϕ = 0.8; n = 1, 3, 5, 7, 9.

Figure 2.- Continued.



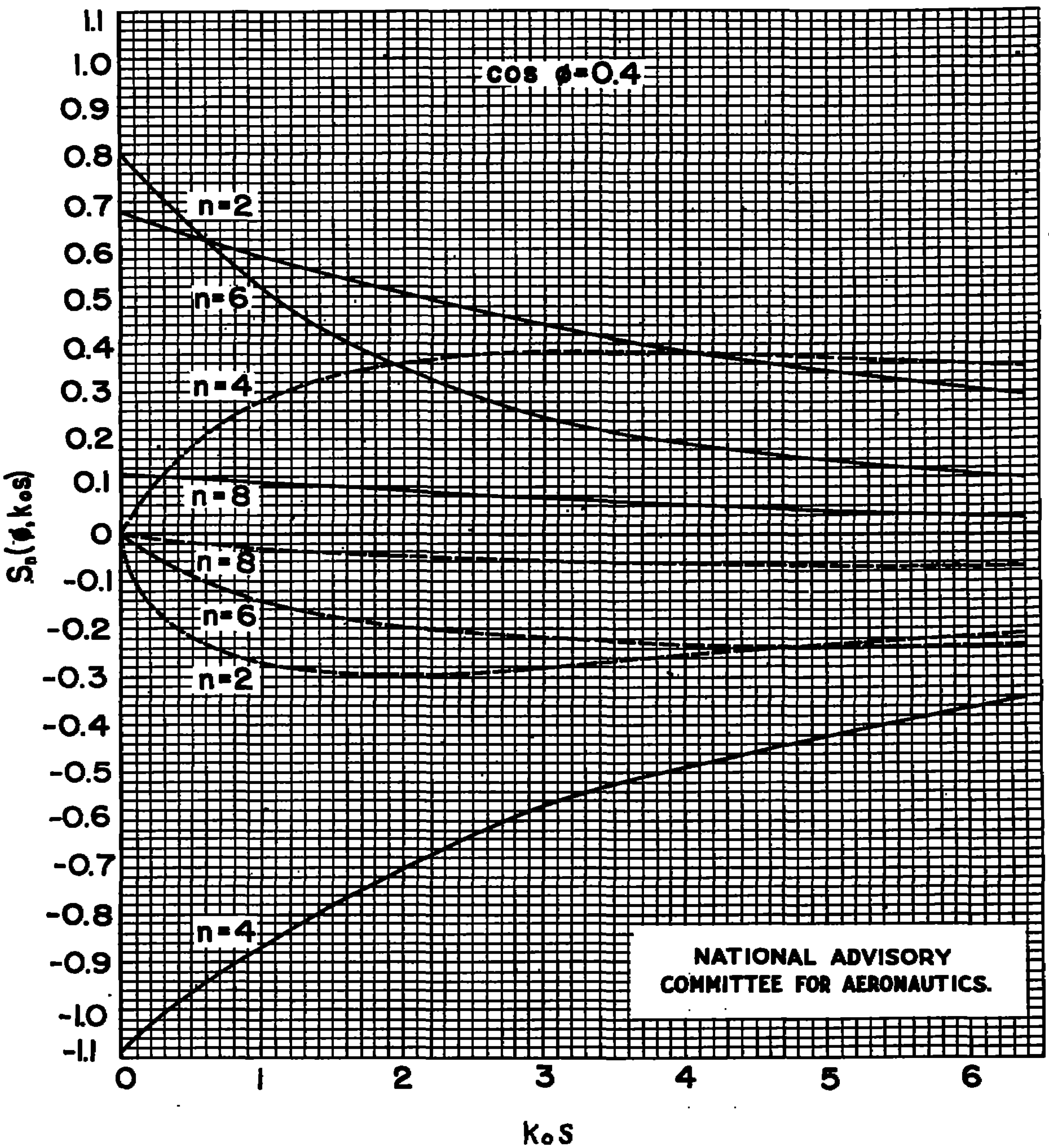
(c) Cos ϕ = 0.8; $\underline{n}$ = 2, 4, 6, 8.

Figure 2.- Continued.



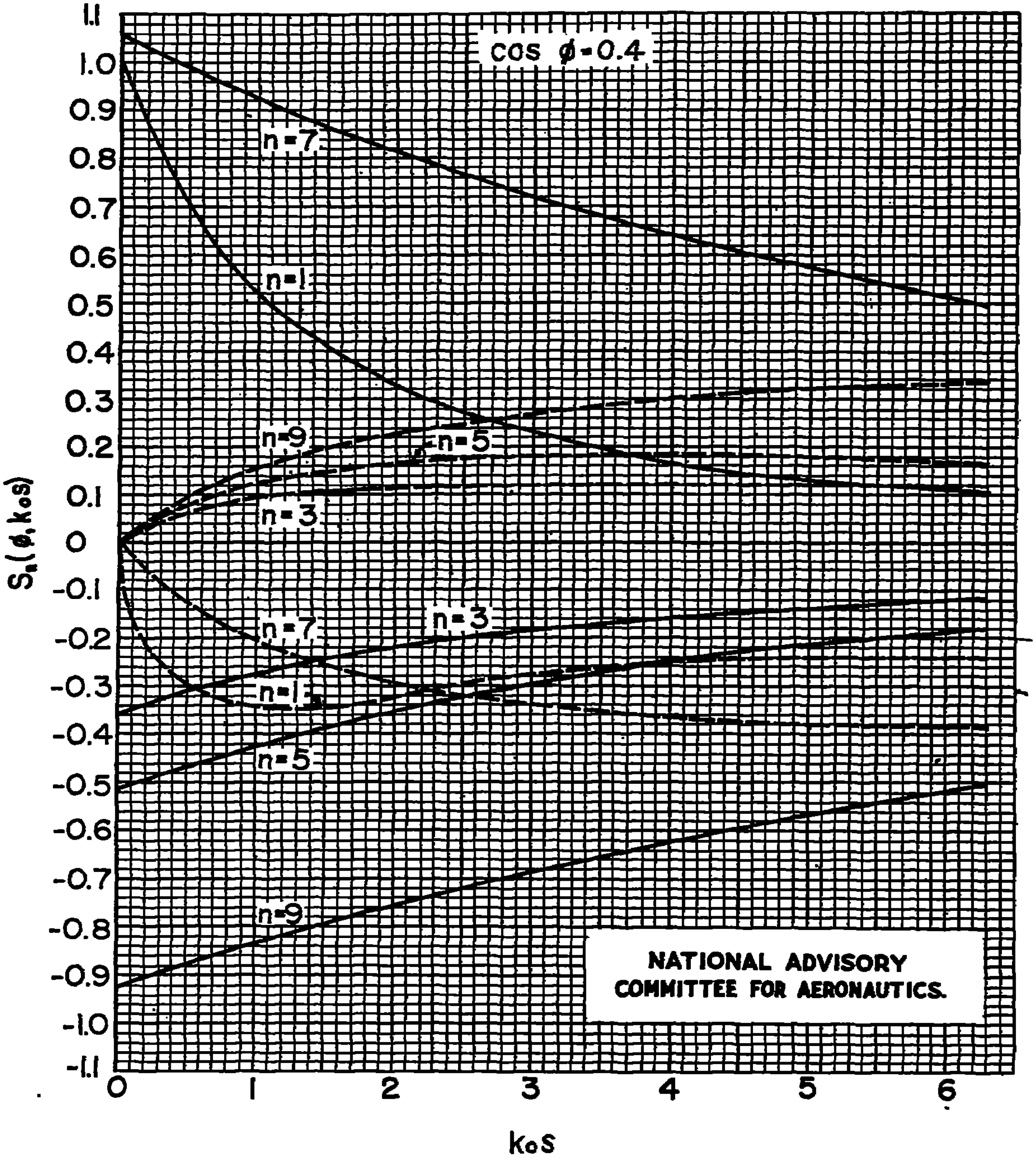
(d) Cos ϕ = 0.6; n = 1, 3, 5, 7, 9.

Figure 2.- Continued.



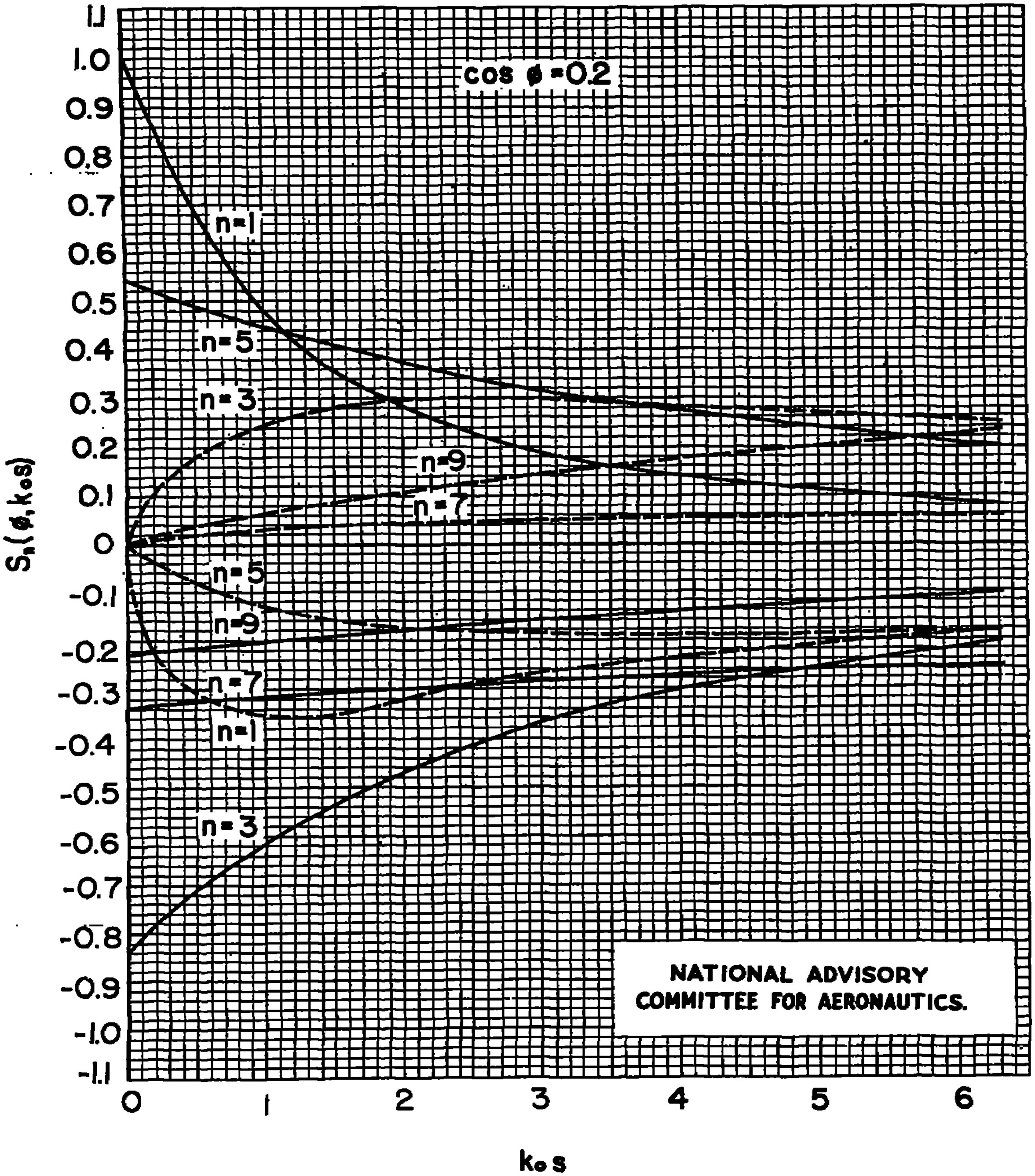
(e) Cos ϕ = 0.4; n = 2, 4, 6, 8.

Figure 2.- Continued.



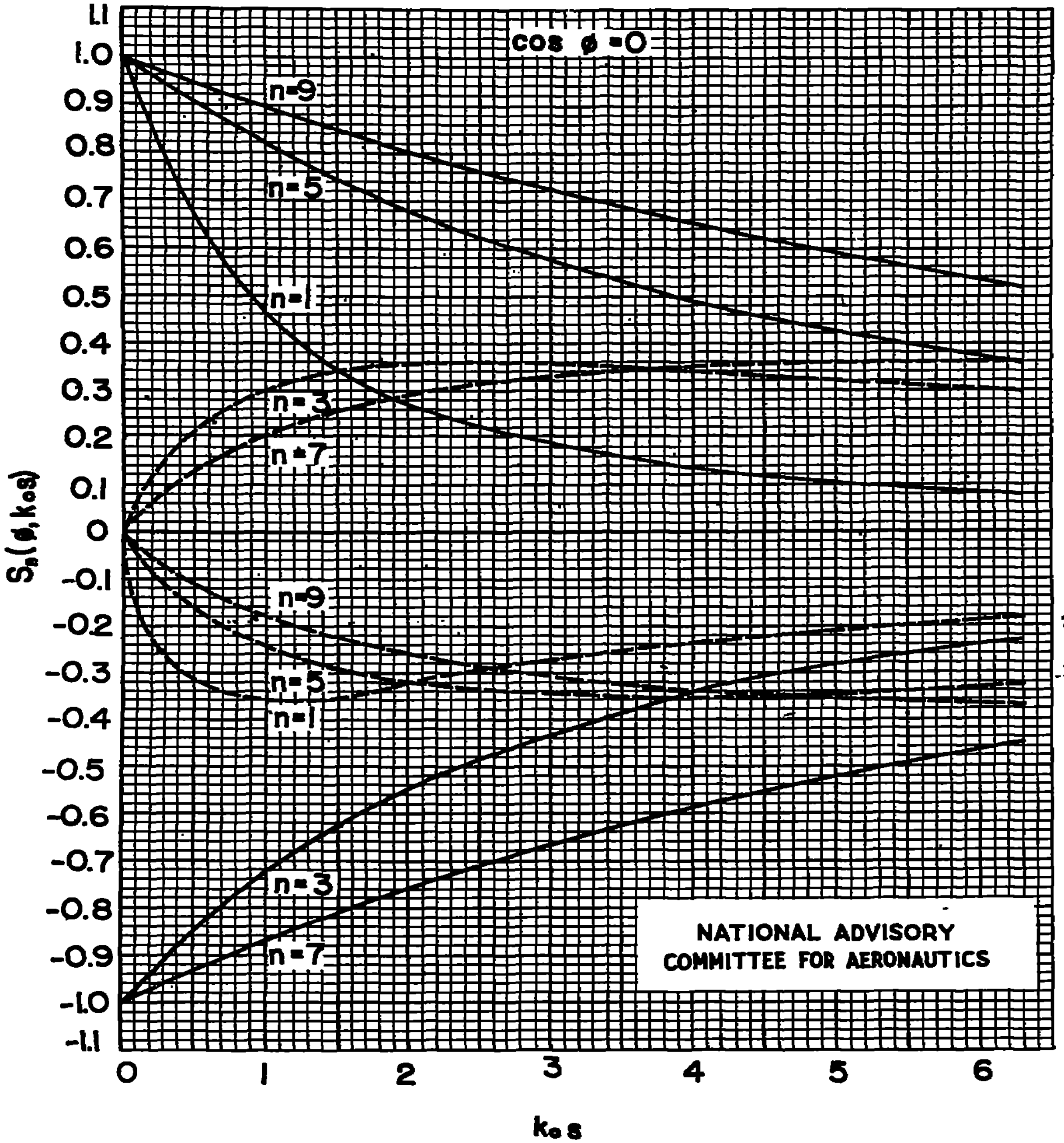
(f) Cos ϕ = 0.4; n = 1, 3, 5, 7, 9.

Figure 2.- Continued.



(g) Cos ϕ = 0.2; n = 1, 3, 5, 7, 9.

Figure 2.- Continued.



(h) Cos ϕ = 0; n = 1, 3, 5, 7, 9.

Figure 2.- Concluded.

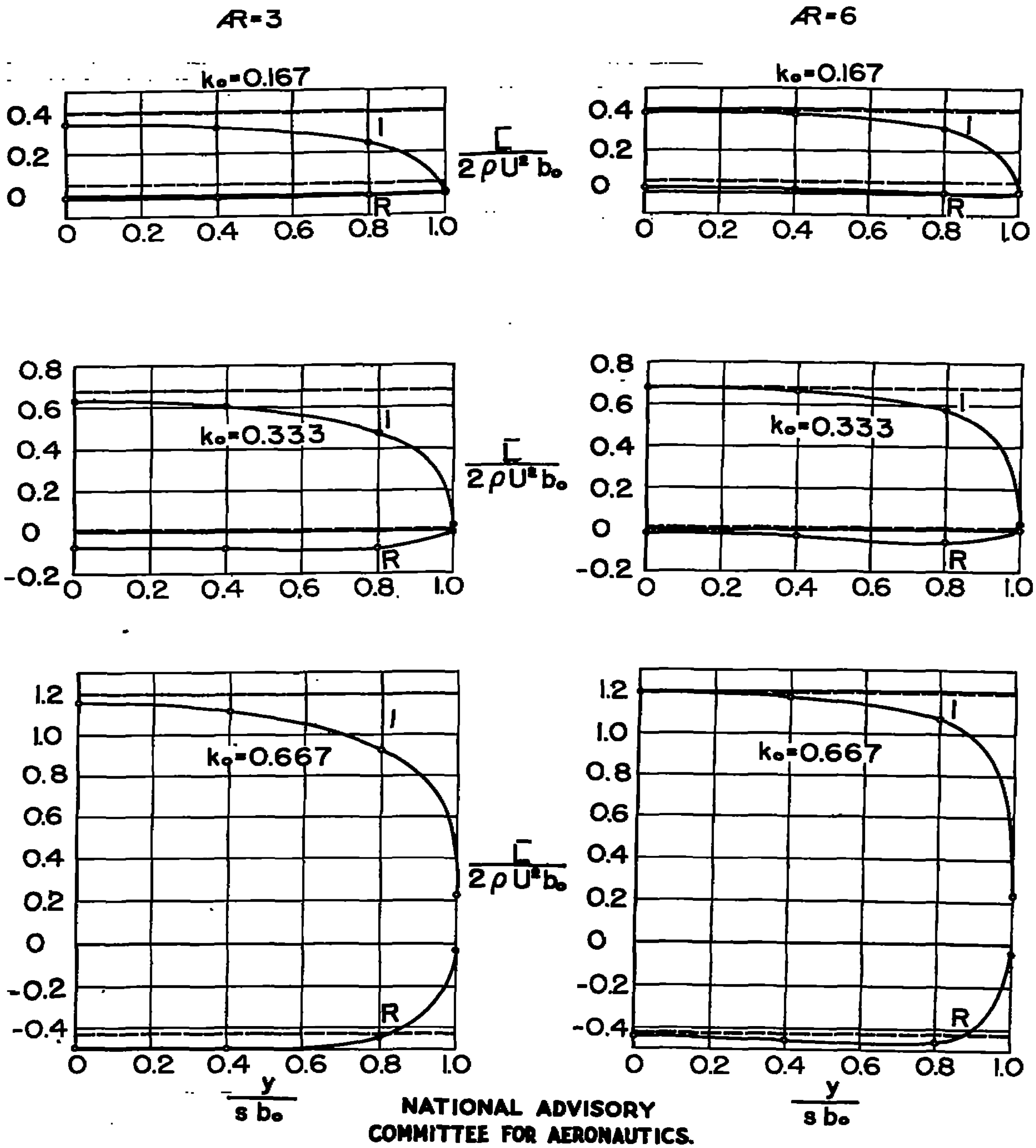


Figure 3.- Lift distributions for uniform translation of rectangular wings of aspect ratios 3 and 6 for k_o = 0.167, 0.333, 0.667.

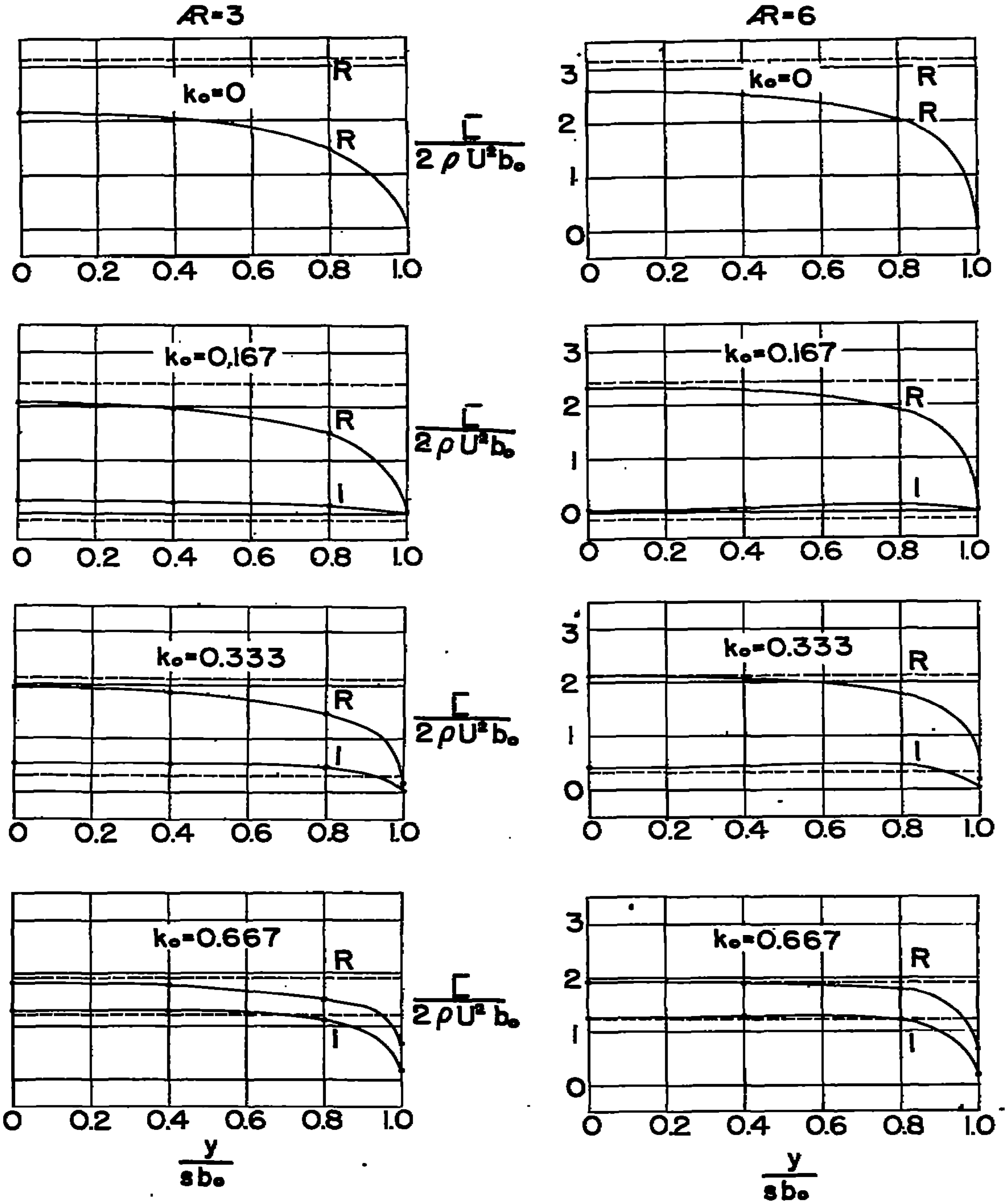


Figure 4.- Lift distributions for uniform pitching of rectangular wings of aspect ratios 3 and 6 for k_o = 0, 0.167, 0.333, 0.667.

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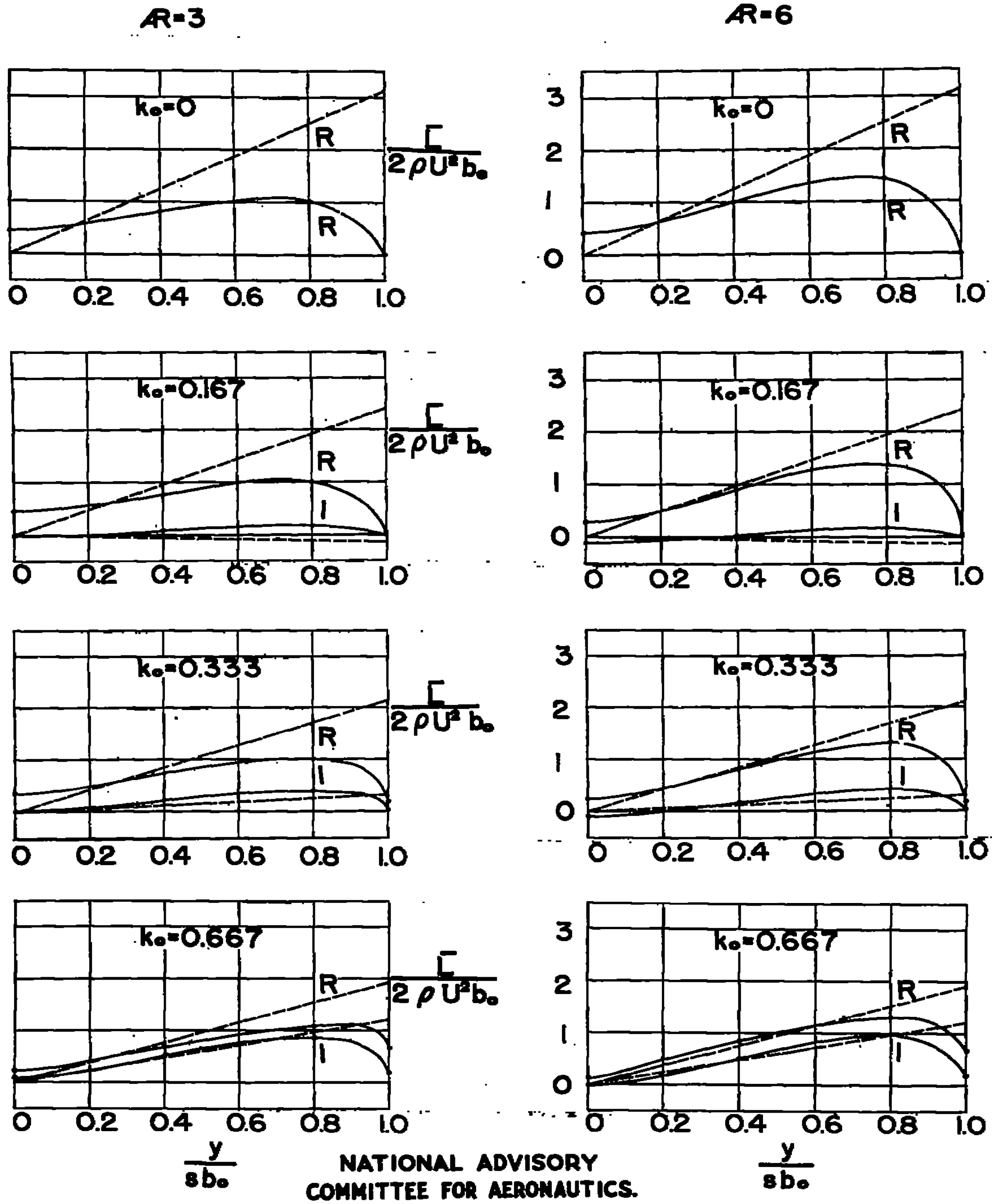

Figure 5.- Lift distributions for linear symmetrical torsion of rectangular wings of aspect ratios 3 and 6 for k_0 = 0, 0.167, 0.333, 0.667.

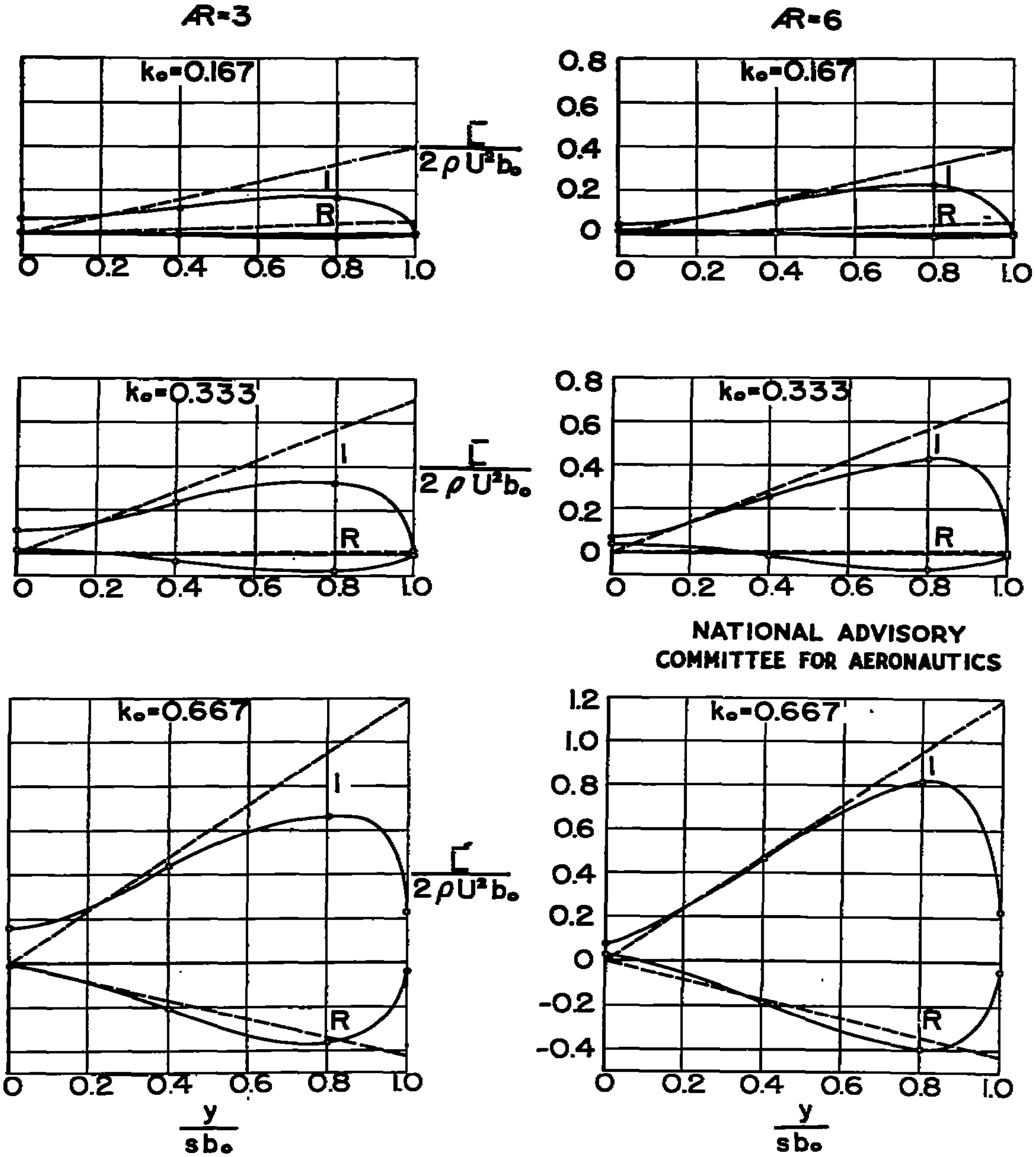

Figure 6.- Lift distributions for linear symmetrical bending of rectangular wings of aspect ratios 3 and 6 for k_0 = 0.167, 0.333, 0.667.

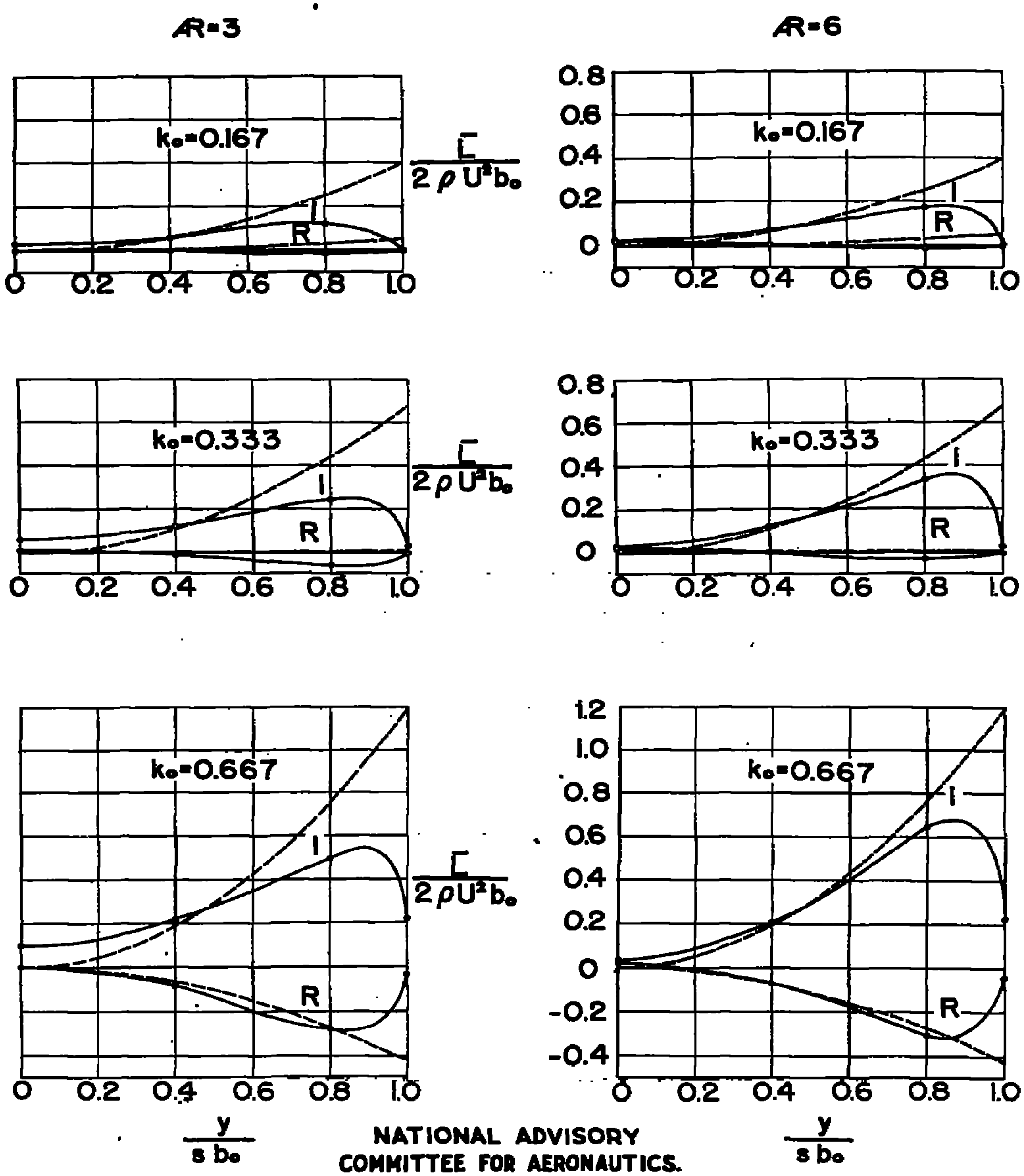

Figure 7.- Lift distributions for parabolic symmetrical bending of rectangular wings of aspect ratios 3 and 6 for k_0 = 0.167, 0.333, 0.667.

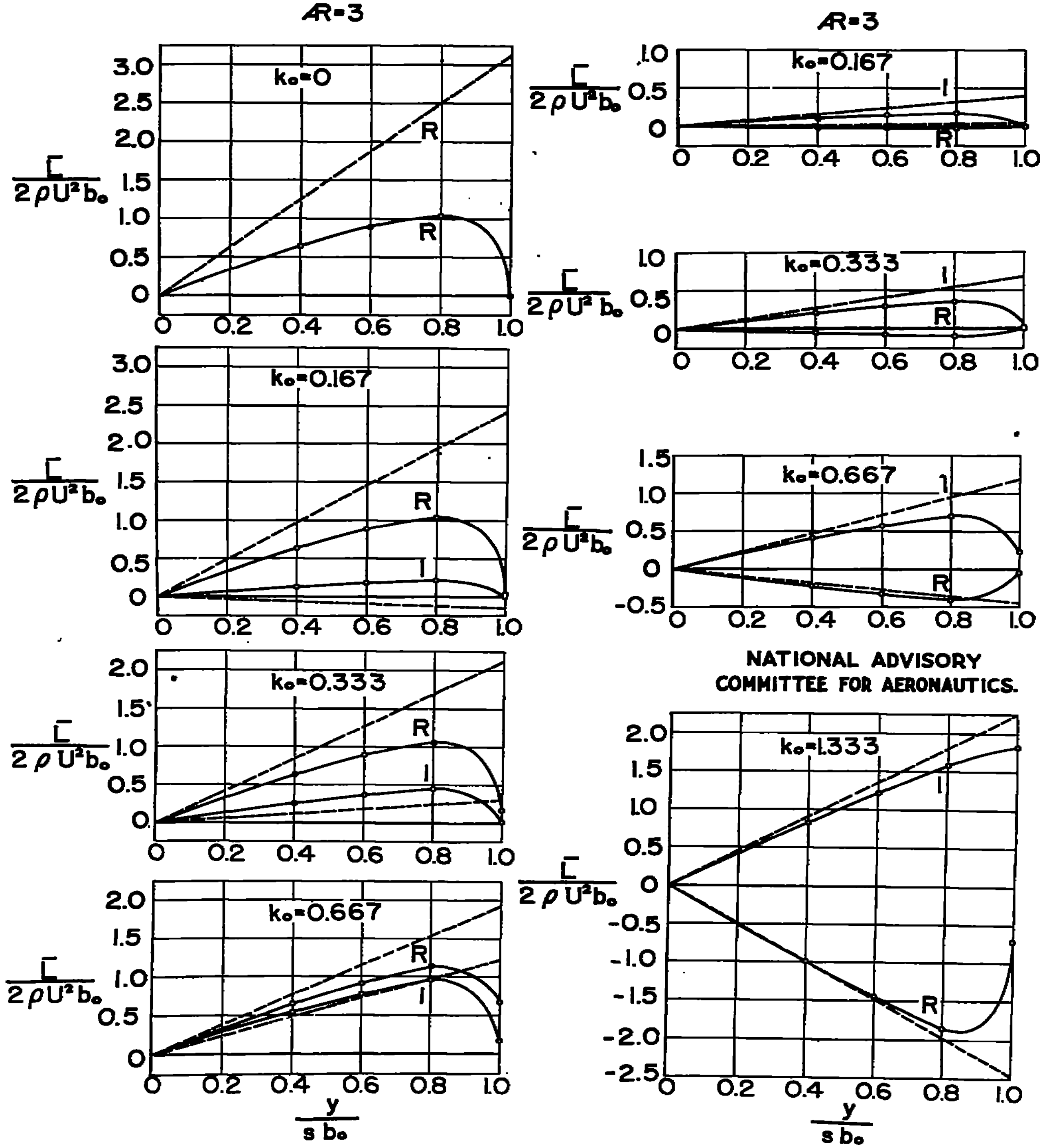

Figure 8.- Lift distributions for linear antisymmetrical torsion of a rectangular wing of aspect ratio 3 with k_o = 0, 0.167, 0.333, 0.667 (left column), and for rigid rolling (right column) of a rectangular wing of aspect ratio 3 with k_o = 0.167, 0.333, 0.667, 1.333.

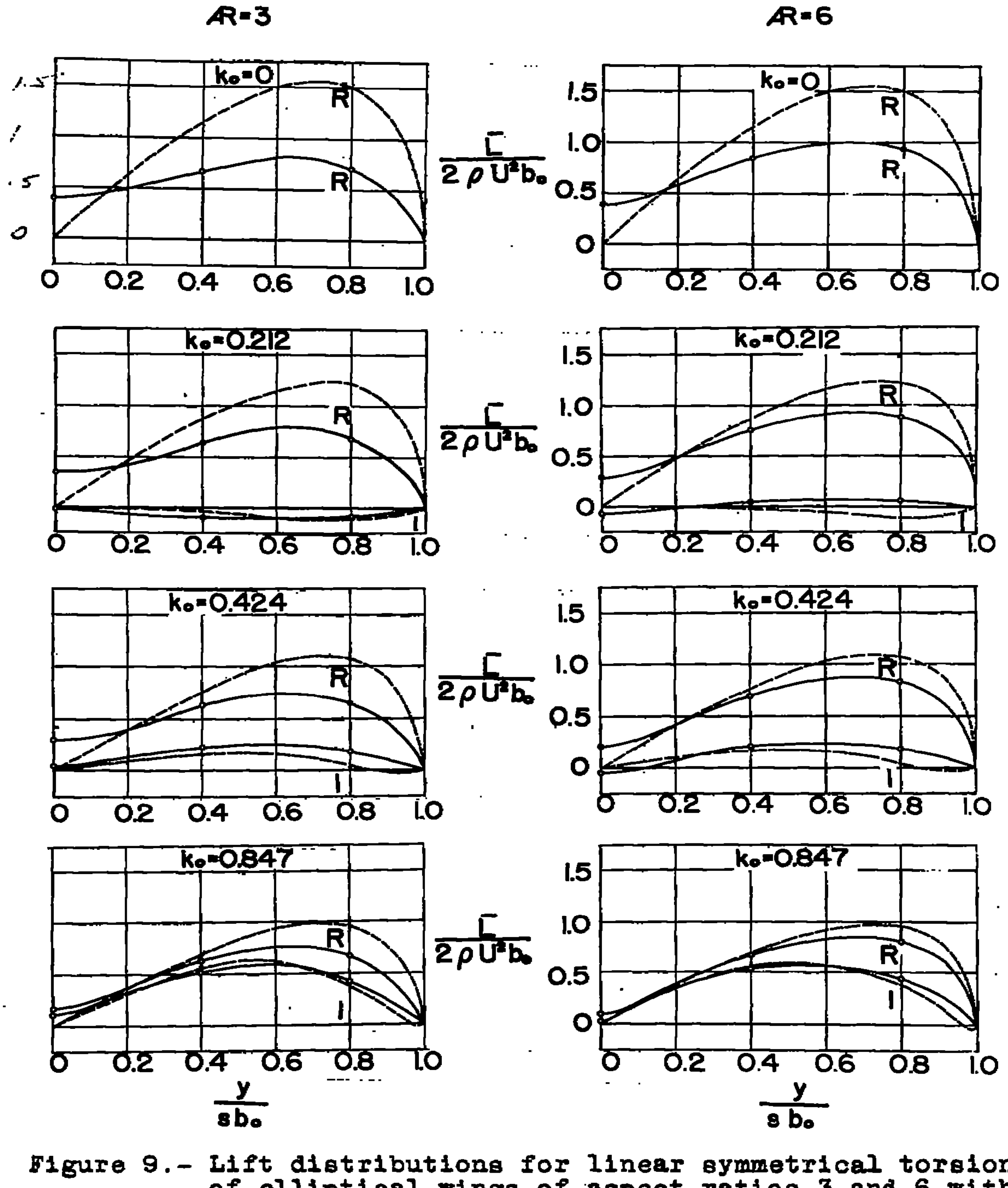

Figure 9.- Lift distributions for linear symmetrical torsion of elliptical wings of aspect ratios 3 and 6 with k_0 = 0, 0.212, 0.424, 0.847.

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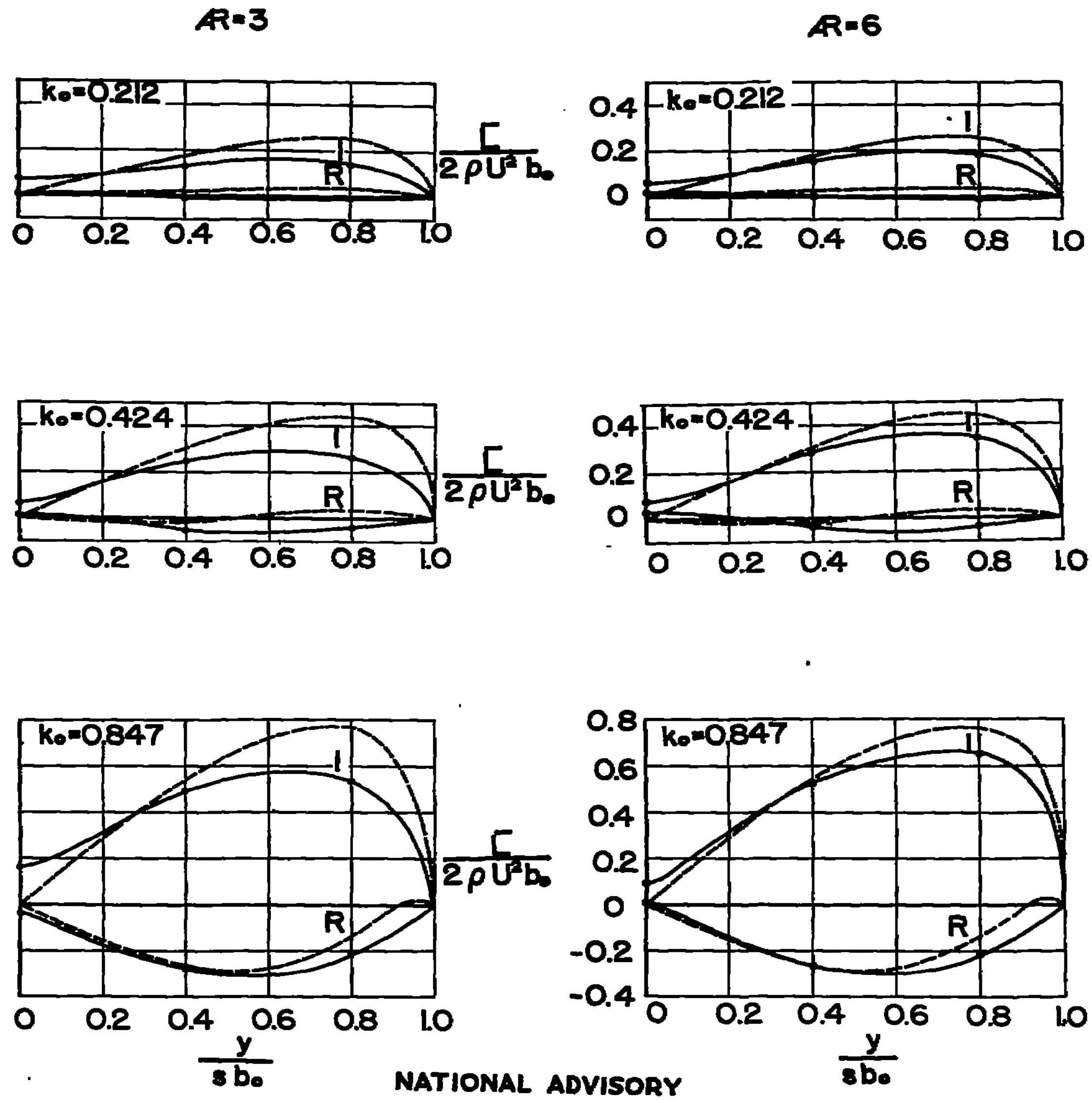

Figure 10.- Lift distributions for linear symmetrical bending of elliptical wings of aspect ratios 3 and 6 with k_0 = 0.212, 0.424, 0.847.

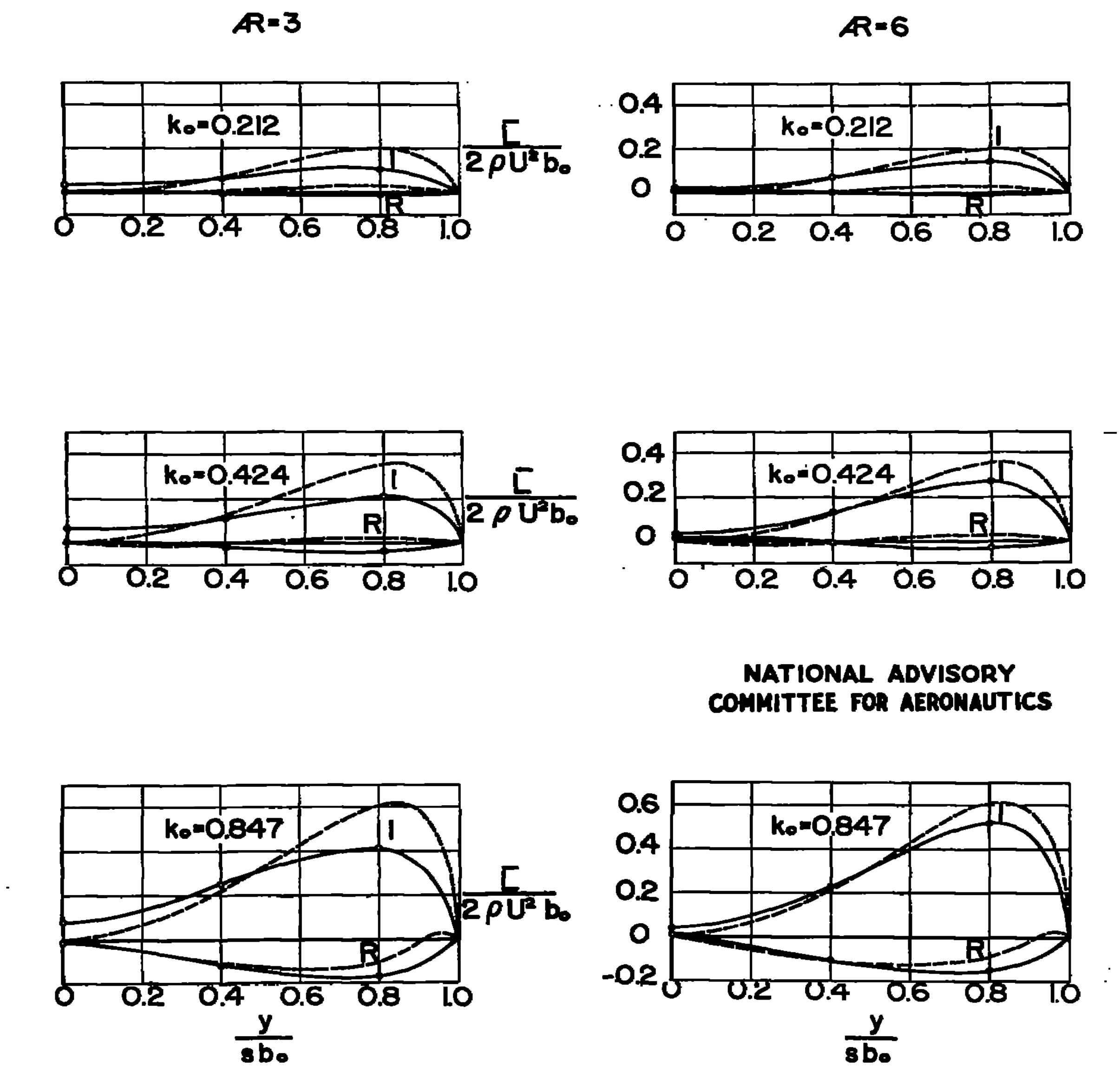

Figure 11.- Lift distributions for parabolic symmetrical bending of elliptical wings of aspect ratios 3 and 6 with k_0 = 0.212, 0.424, 0.847.

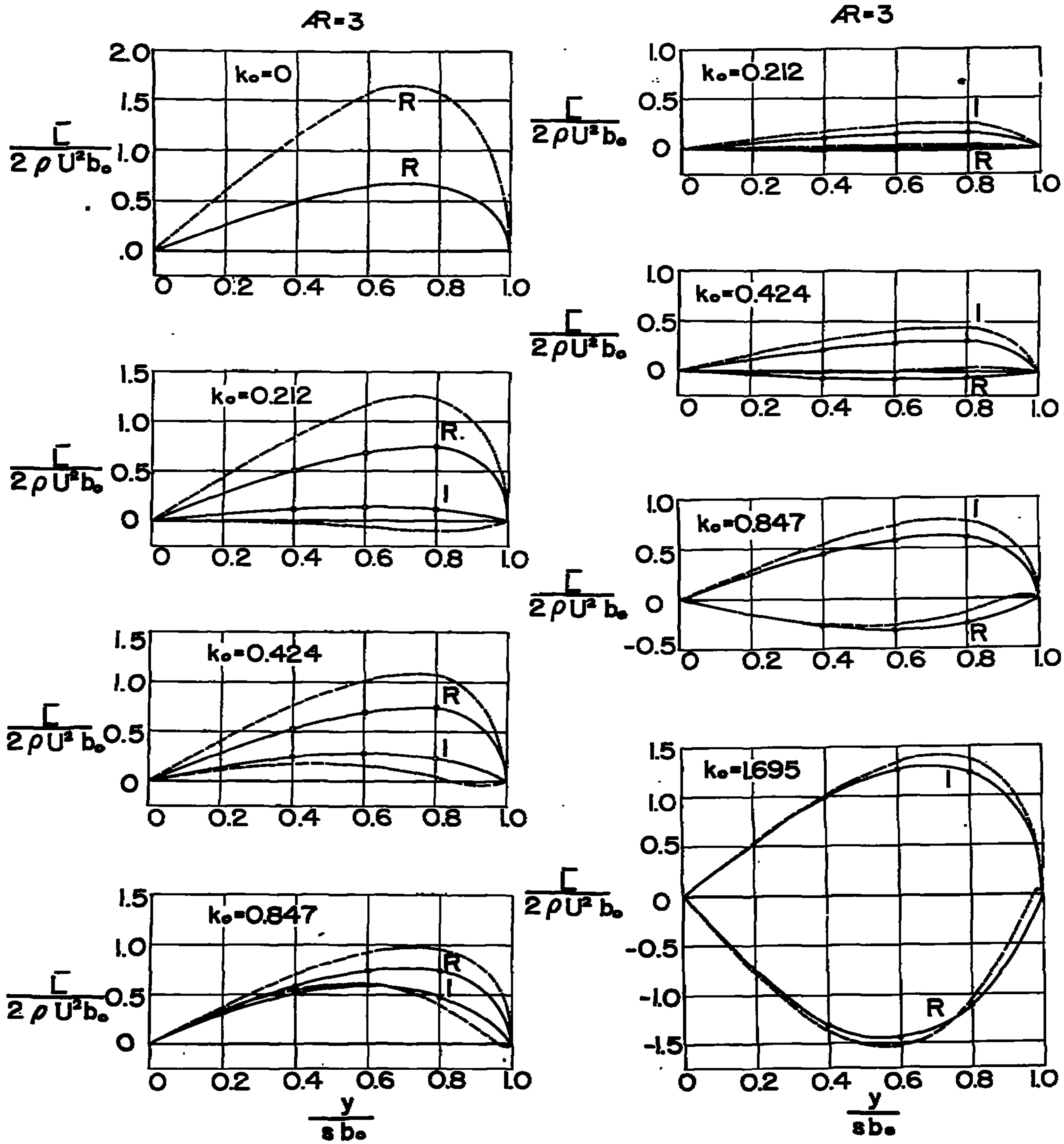

Figure 12.- Lift distributions for linear antisymmetrical torsion of an elliptical wing of aspect ratio 3 with k_0 = 0, 0.212, 0.424, 0.847 (left column) and for rigid rolling of an elliptical wing of aspect ratio 3 with k_0 = 0.212, 0.424, 0.847, 1.695 (right column).

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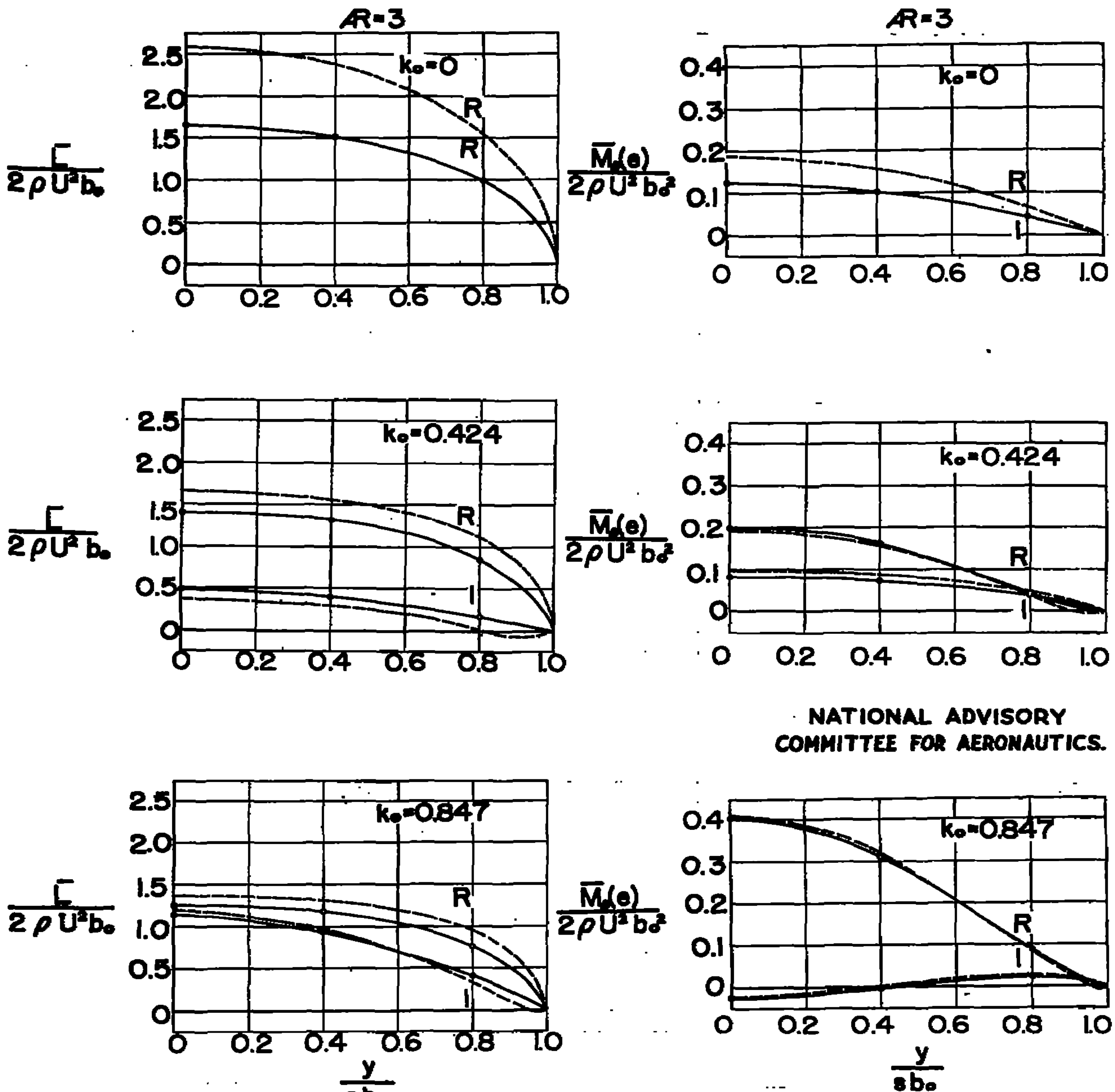

Figure 13.- Lift distributions and hinge-moment distributions for full-span aileron with uniform spanwise deflection for an elliptical wing of aspect ratio 3 ($e = 0$, $l = 0.2b/b_o$) for $k_o = 0$, 0.424, 0.847.

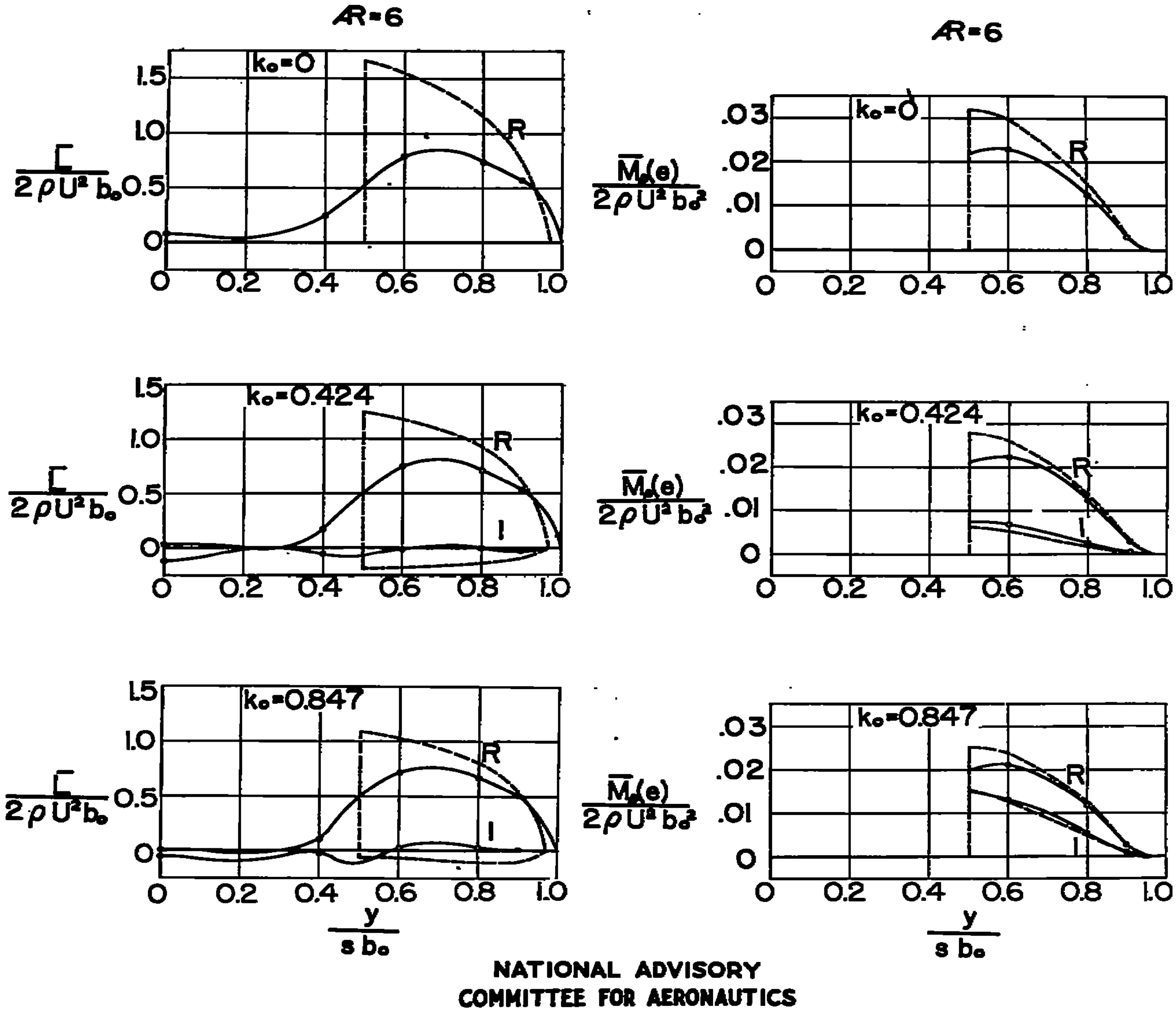

Figure 14.- Lift and hinge-moment distributions for an elliptical wing of aspect ratio 6 with half-span ailerons of approximately one-quarter the wing chord and with $k_o = 0$, 0.424, 0.847 ($l = 0.1b/b_o$).

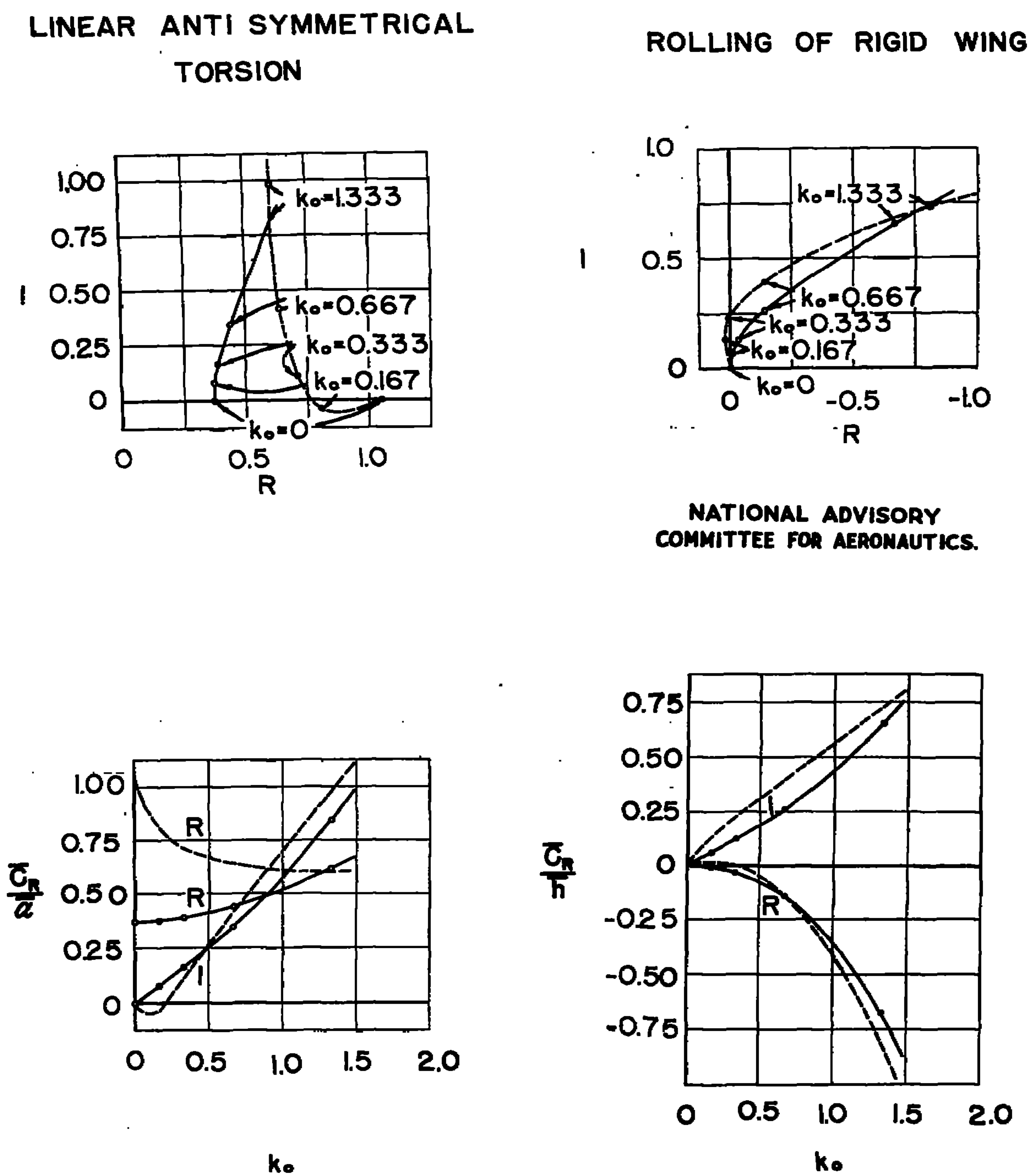

Figure 15.- Rolling-moment coefficients for a rectangular wing of aspect ratio 3 in rigid rolling and for a rectangular wing of aspect ratio 3 in linear antisymmetrical torsion plotted as vector diagrams and as functions of k_0.

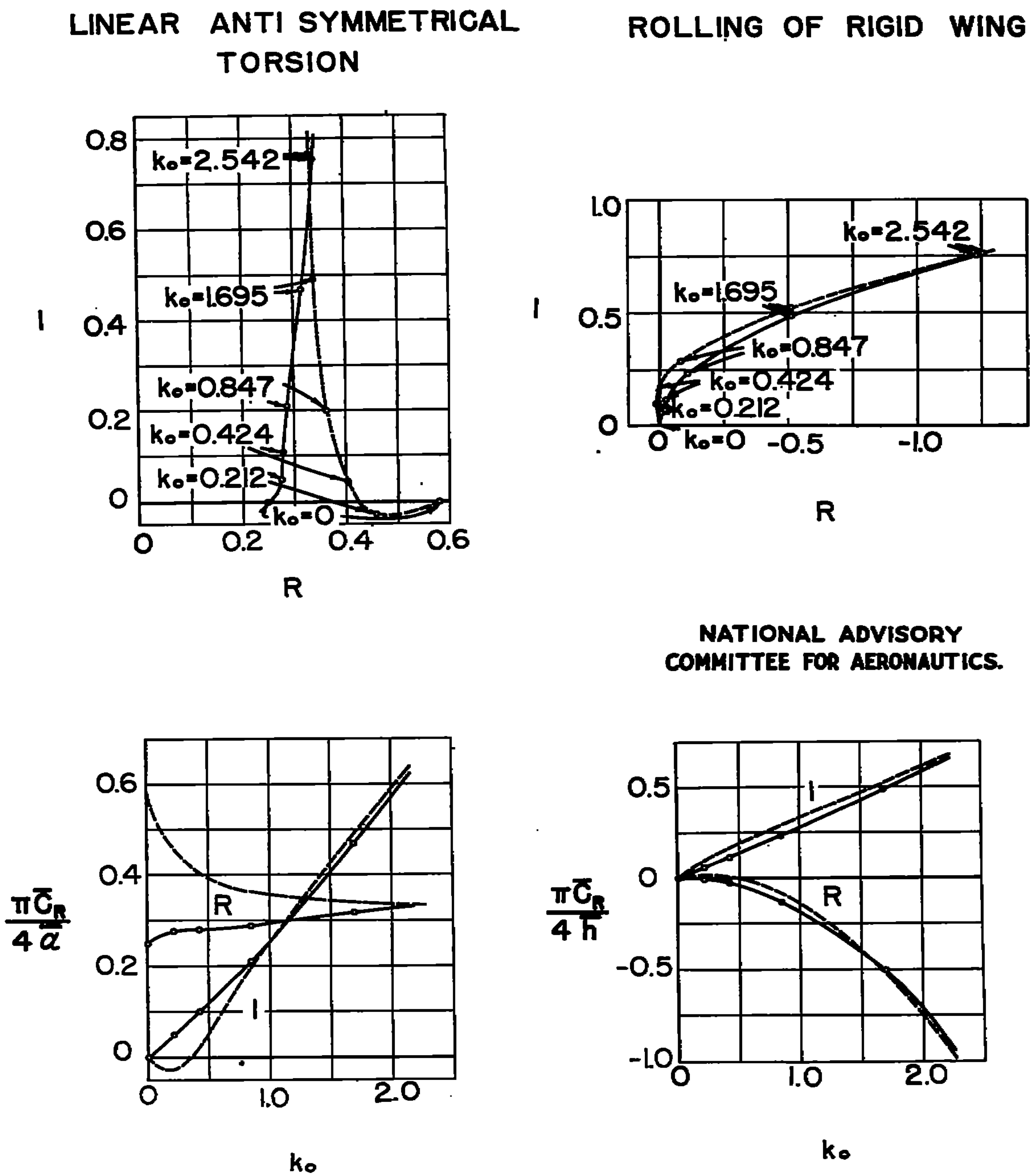

Figure 16.- Rolling-moment coefficients for an elliptical wing of aspect ratio 3 in rigid rolling and for an elliptical wing of aspect ratio 3 in linear antisymmetrical torsion plotted as vector diagrams and as functions of k_o.

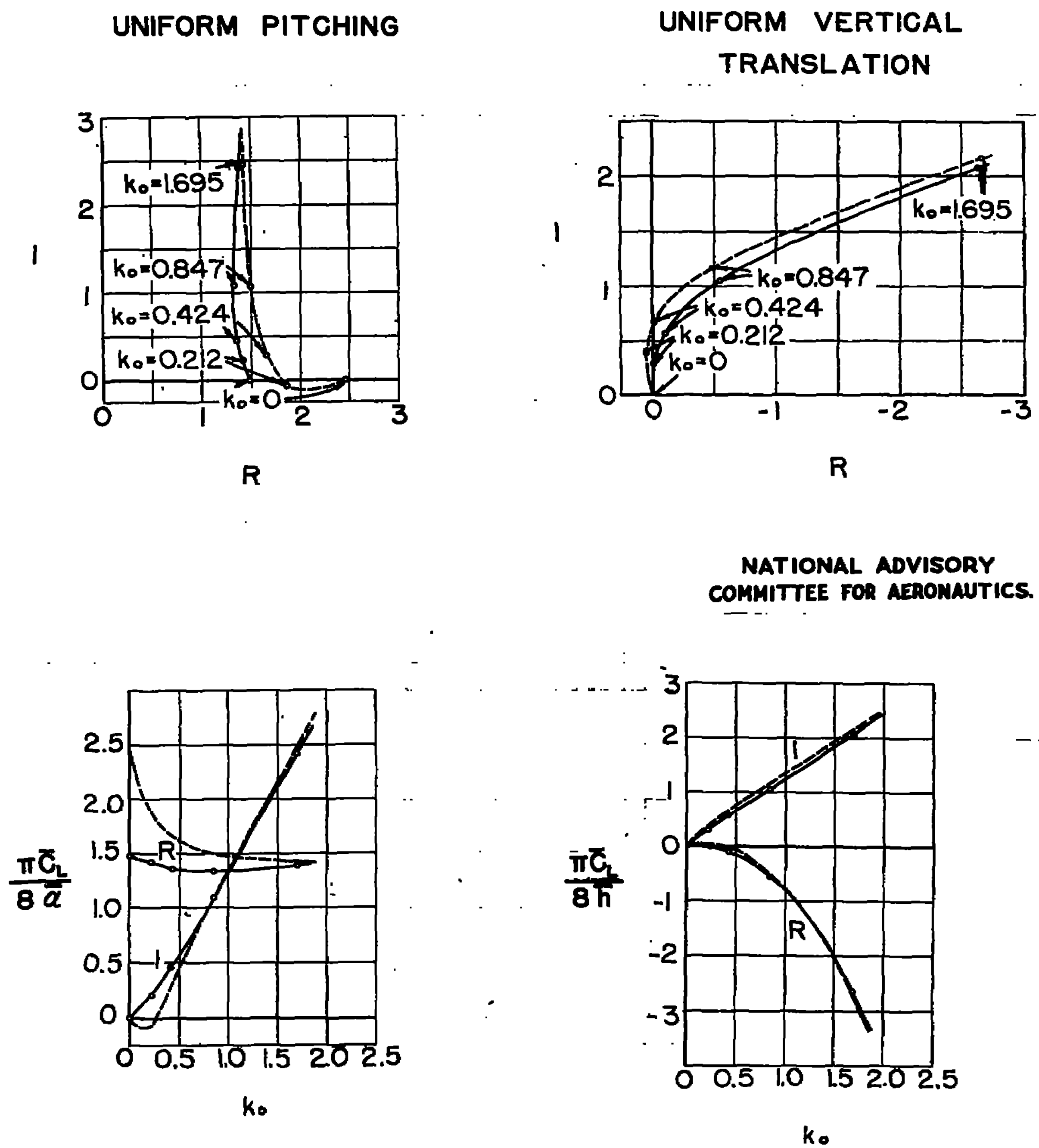

Figure 17.- Lift coefficients for an elliptical wing of aspect ratio 3 in uniform pitching and for an elliptical wing in uniform vertical translation plotted as vector diagrams and as functions of k_0.

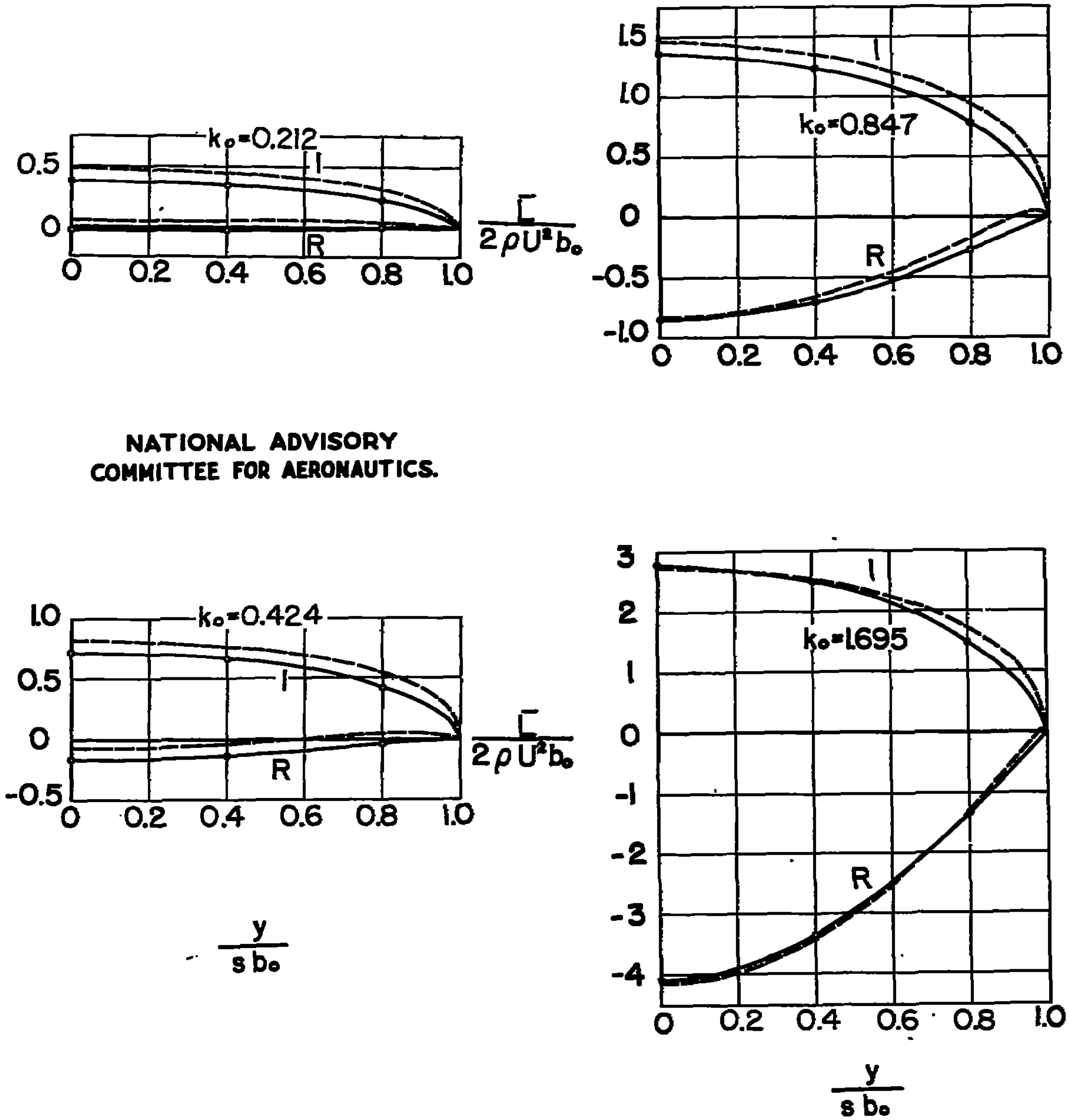

Figure 18.- Lift distributions for uniform translation of an elliptical wing of aspect ratio 3 for k_0 = 0.212, 0.424, 0.847, 1.695.

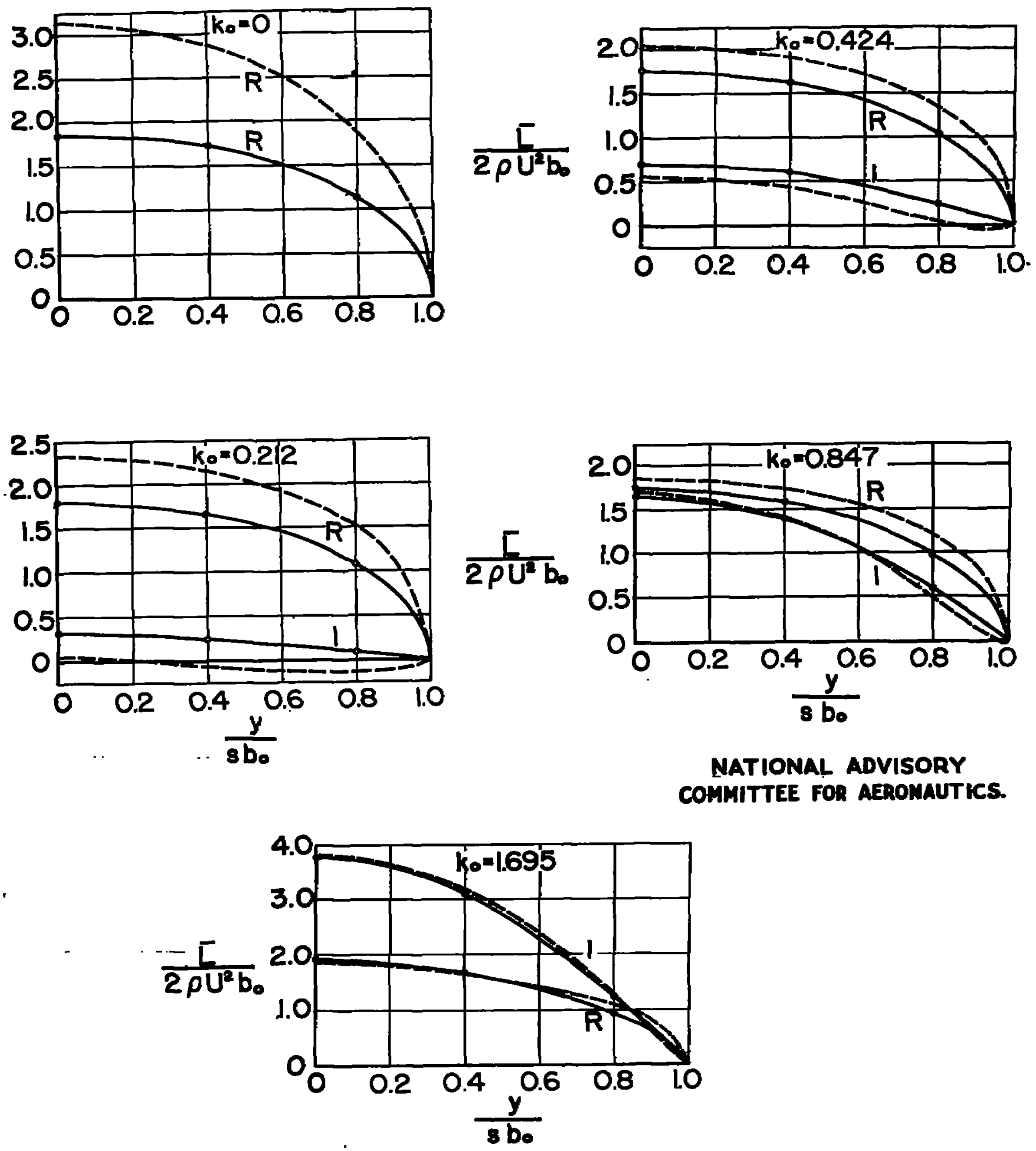

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Figure 19.- Lift distributions for uniform pitching of an elliptical wing of aspect ratio 3 for k_o = 0, 0.212, 0.424, 0.847, 1.695.

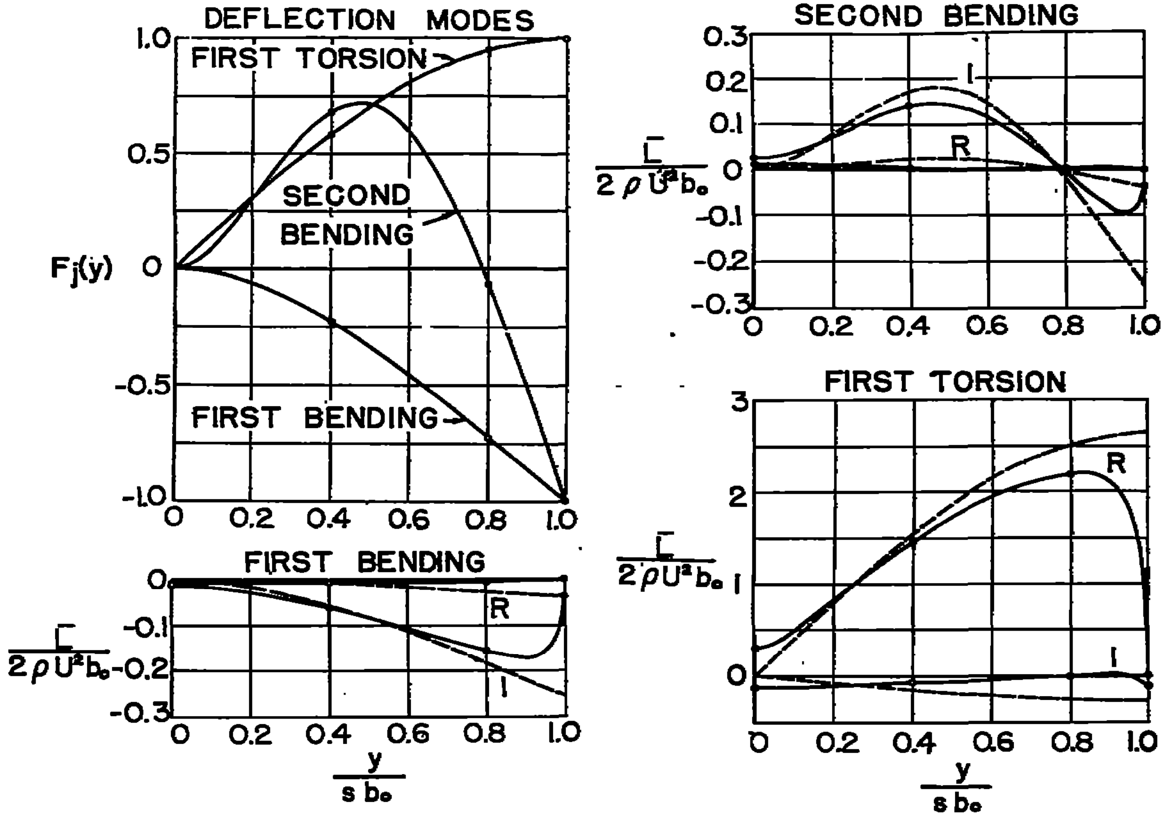


Figure 20.- First-torsion, first-bending, and second-bending modes for a rectangular wing of aspect ratio 13.5 and lift distributions for these modes for $k_o = 0.0965$.

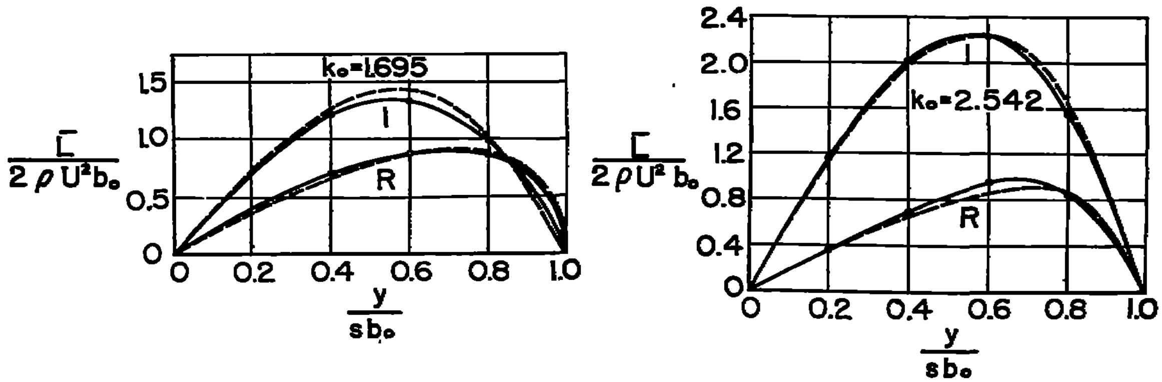
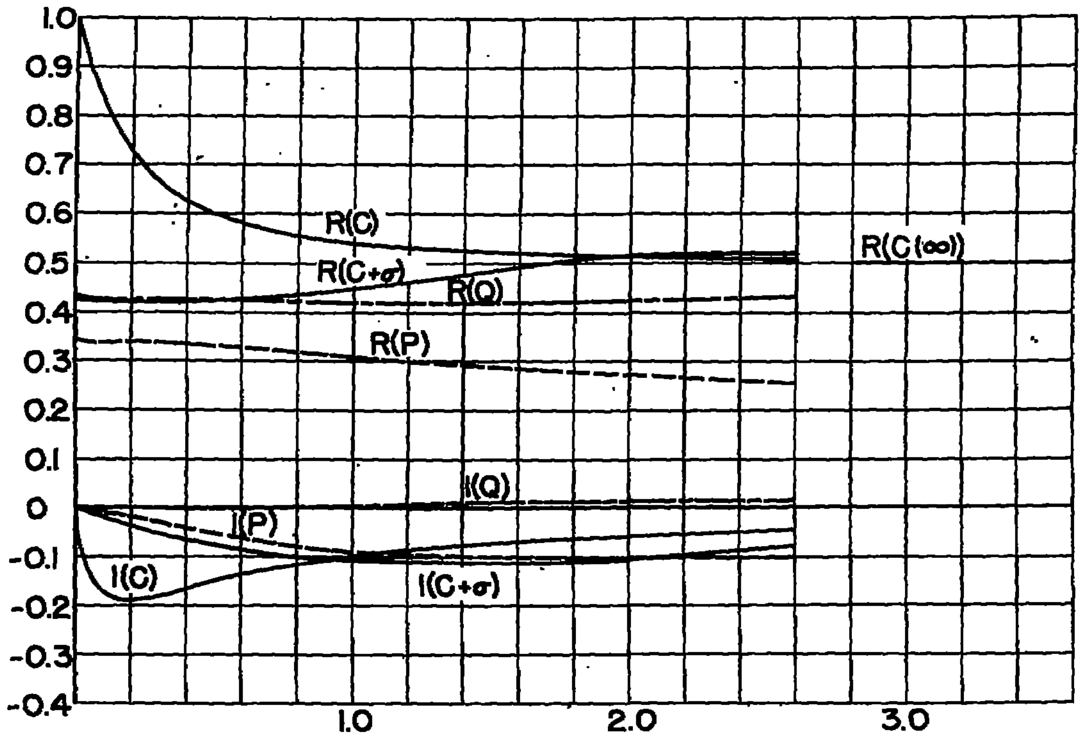
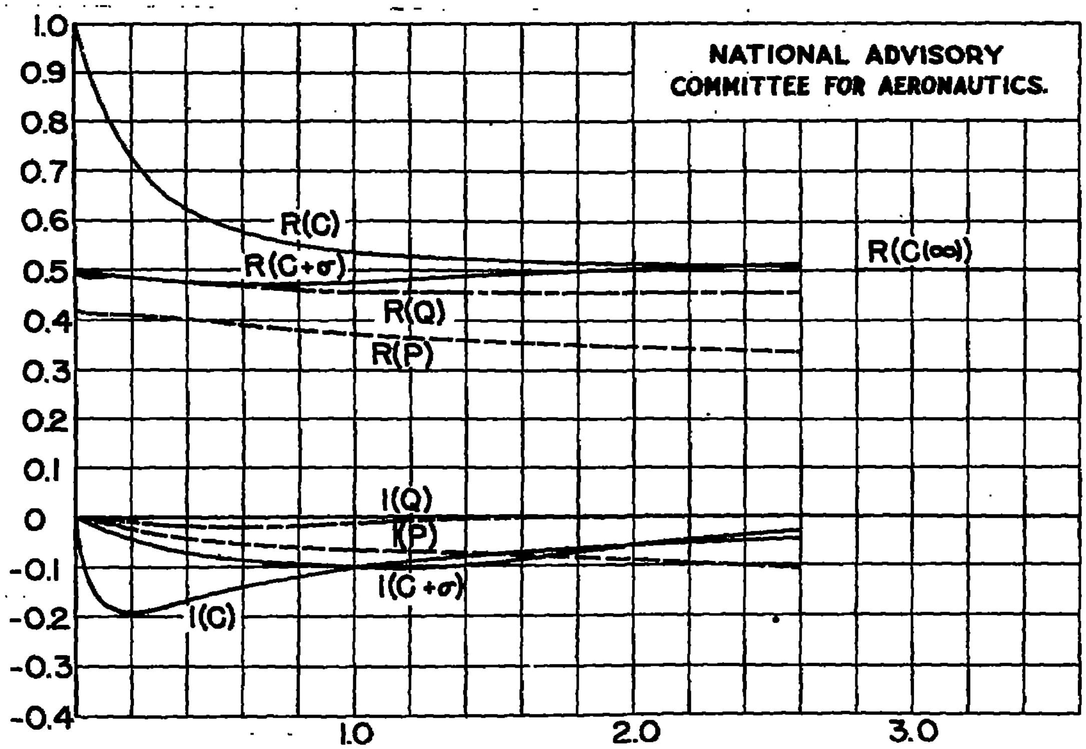


Figure 21.- Lift distributions for linear antisymmetrical torsion of an elliptical wing of aspect ratio 3 for $k_o = 1.695$ and 2.542.



(a) AR = 1.5

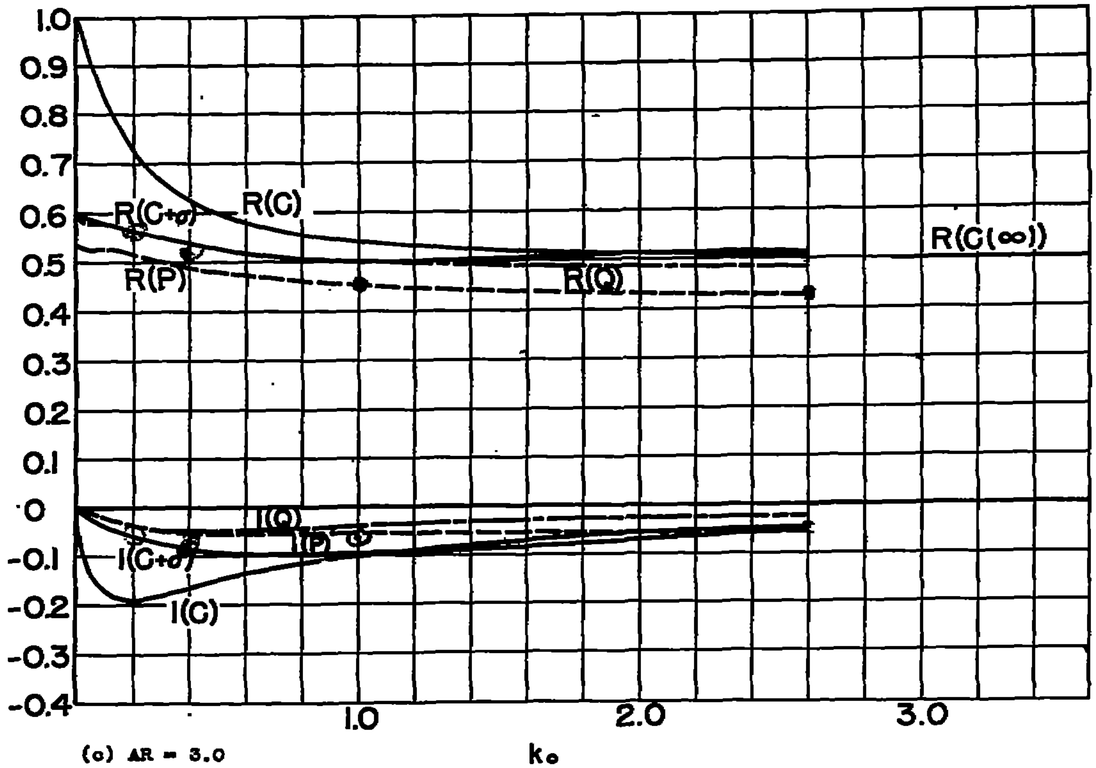
FIGURE 22a



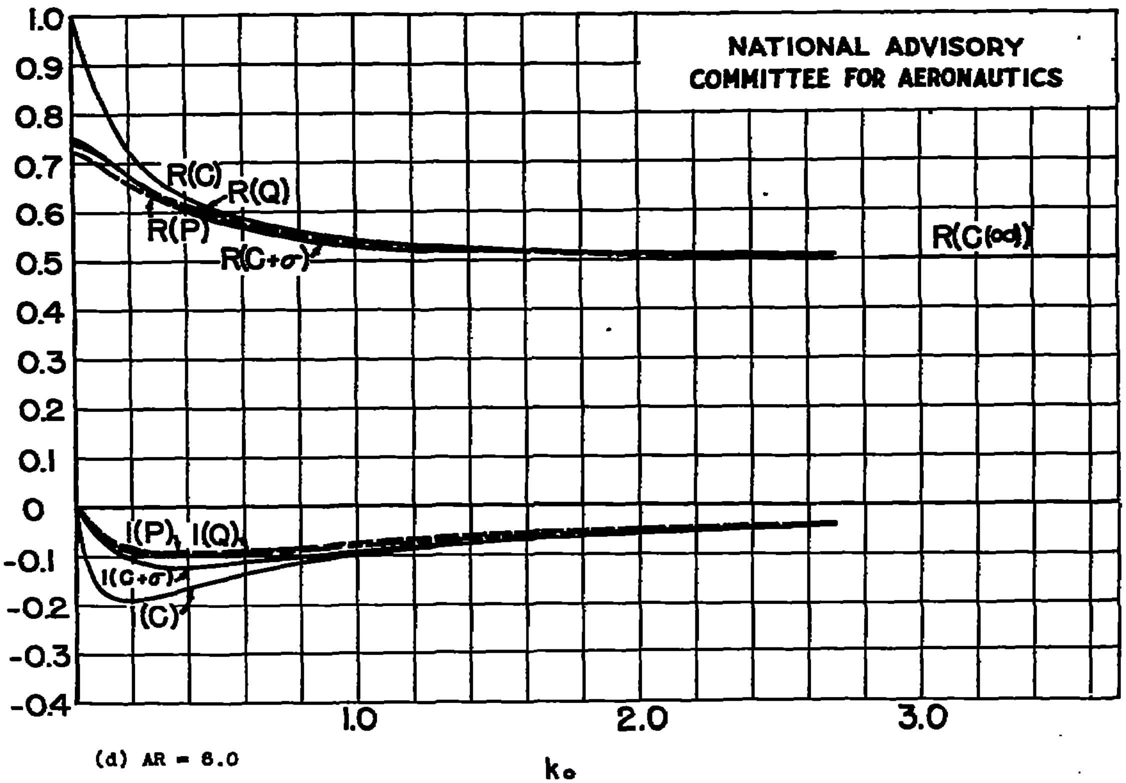
(b) AR = 2.0

FIGURE 22b

Figure 22 (a to d).- Uniform pitching and translation of rigid elliptical wing (one-point collocation). Comparison of functions $\bar{C}$, C + σ with functions F and $\bar{Q}$ of reference 5.



(c) AR = 3.0



(d) AR = 6.0

Figure 22.- Concluded.

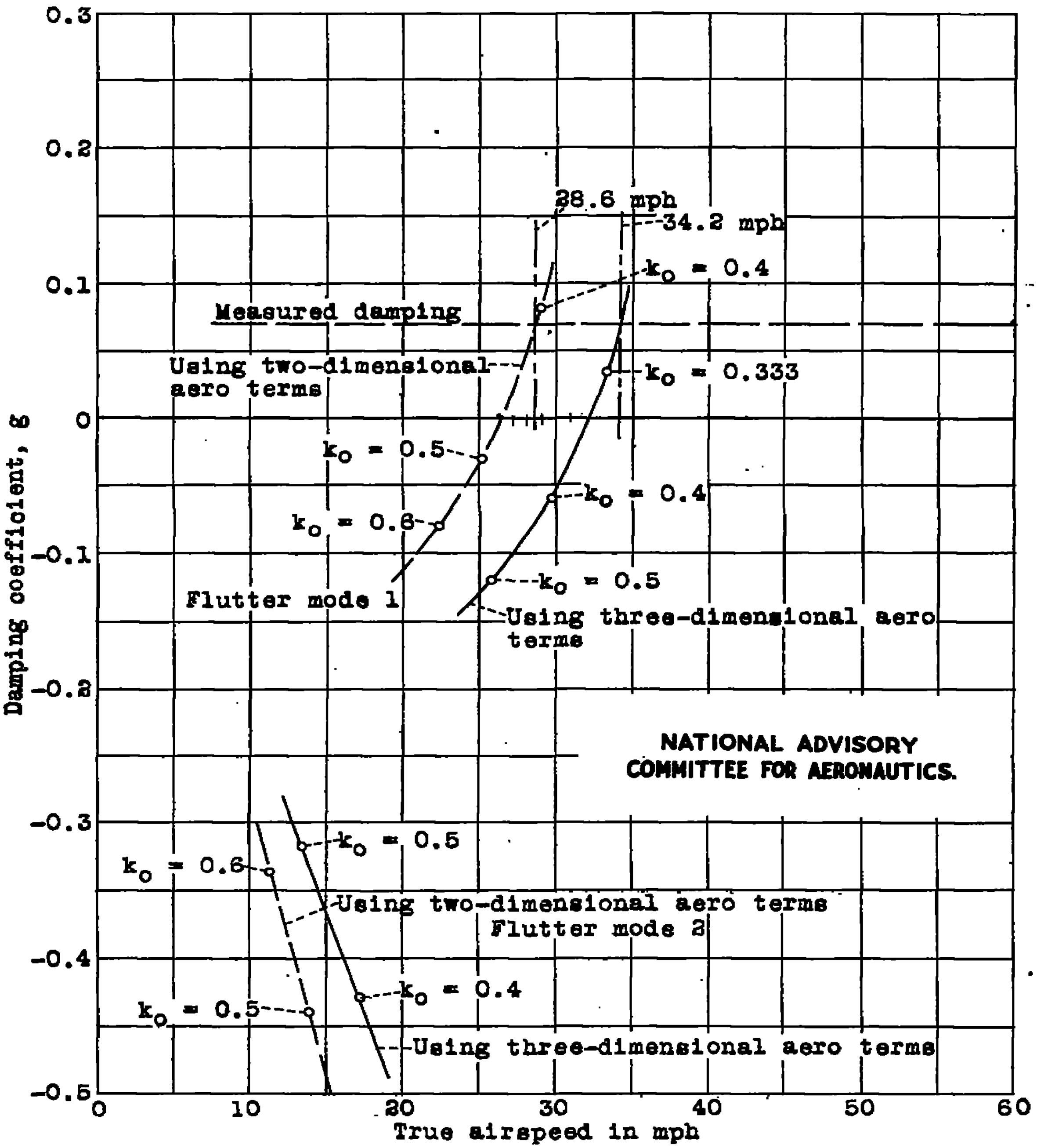


Figure 23.- Damping coefficient against true airspeed for a rectangular wing of aspect ratio 6, b_o = 10 inches; g = 0.070. (Analysis by method of reference 4.)

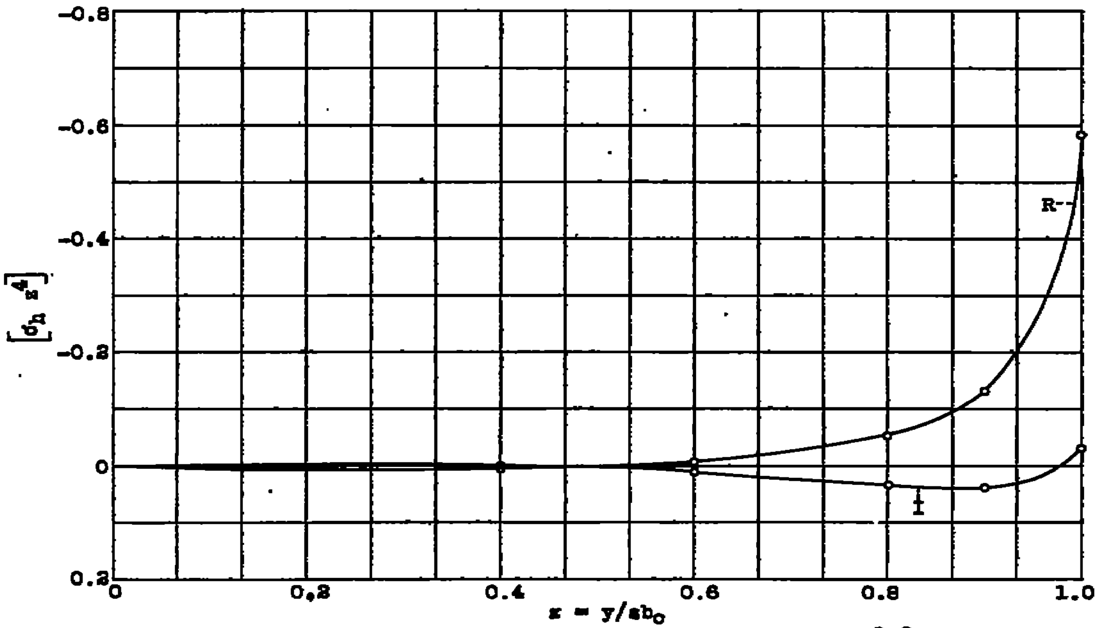


Figure 24 (a to d).- Graphical integration. (a) For $\Delta\underline{A}$: $\int_0^{1.0} \sigma_h z^4 \, dz = -0.040 + 0.010i$; $k_o = 0.4$.

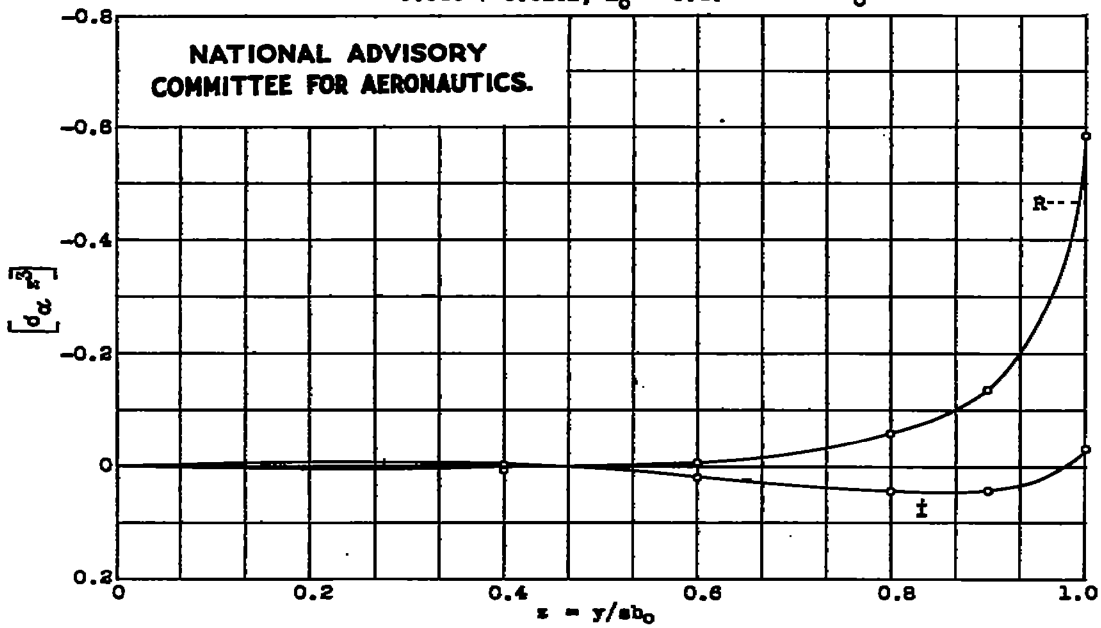


Figure 24.- Continued. (b) For $\Delta\underline{B}$: $\int_0^{1.0} \sigma_\alpha z^3 \, dz = -0.042 + 0.014i$; $k_o = 0.4$.

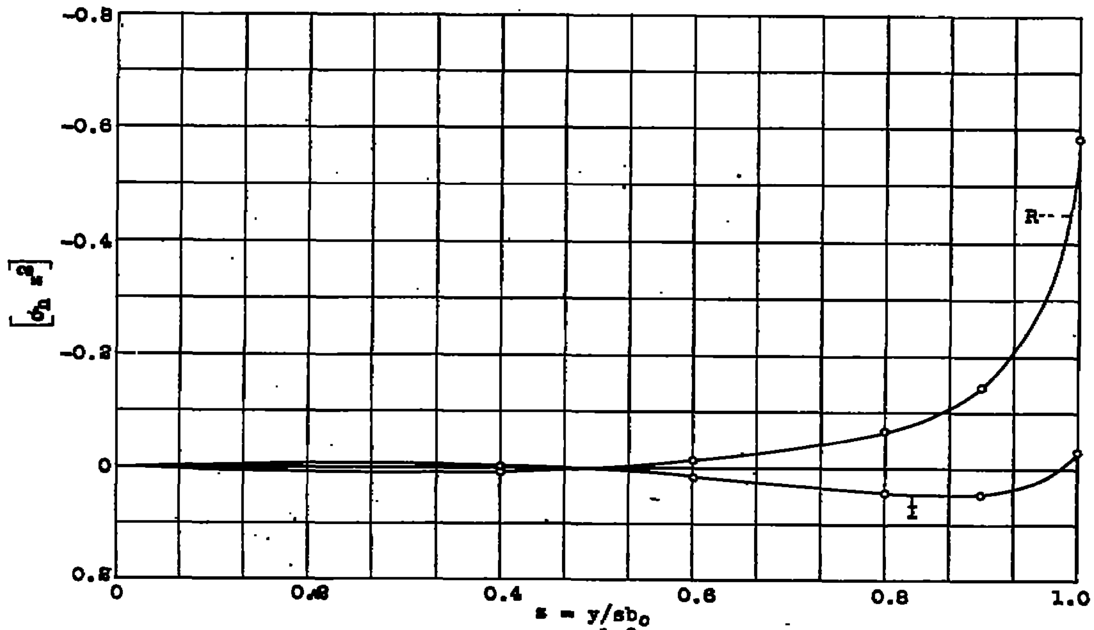


Figure 24.- Continued. (c) For $\Delta \frac{D}{m}$: $\int_0^{1.0} \sigma_h z^3 dz = -0.043 + 0.012i$; $k_o = 0.4$.

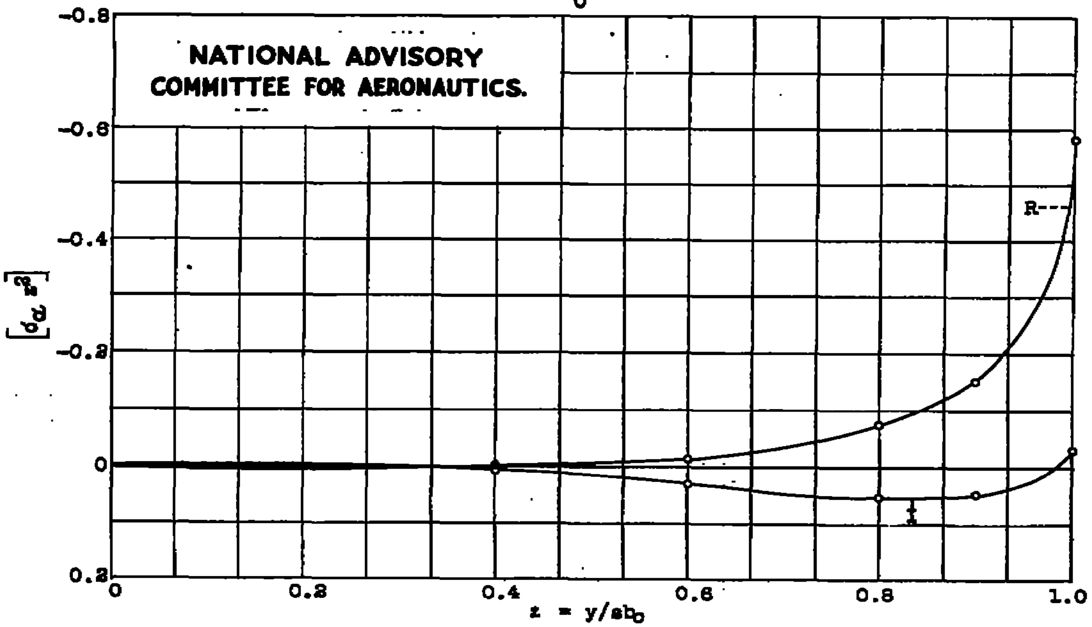


Figure 24.- Concluded. (d) For $\Delta \underline{E}$: $\int_0^{1.0} \sigma_\alpha z^2 dz = -0.048 + 0.017i$; $k_o = 0.4$.

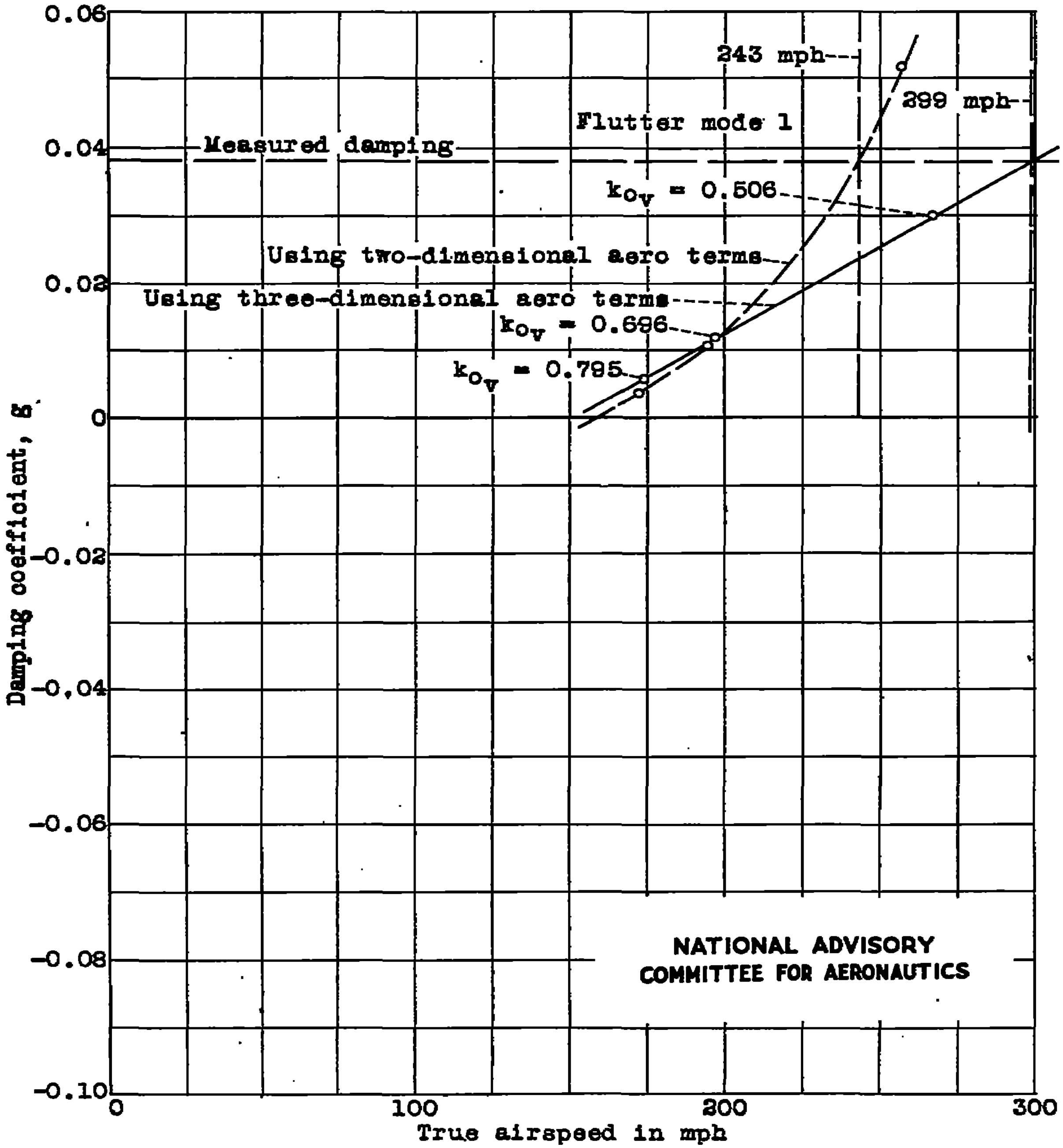


Figure 27.- Damping coefficient g against true airspeed for the tail-flutter problem of example II; g = 0.038. Note: Flutter determinants solved by method of reference 4.

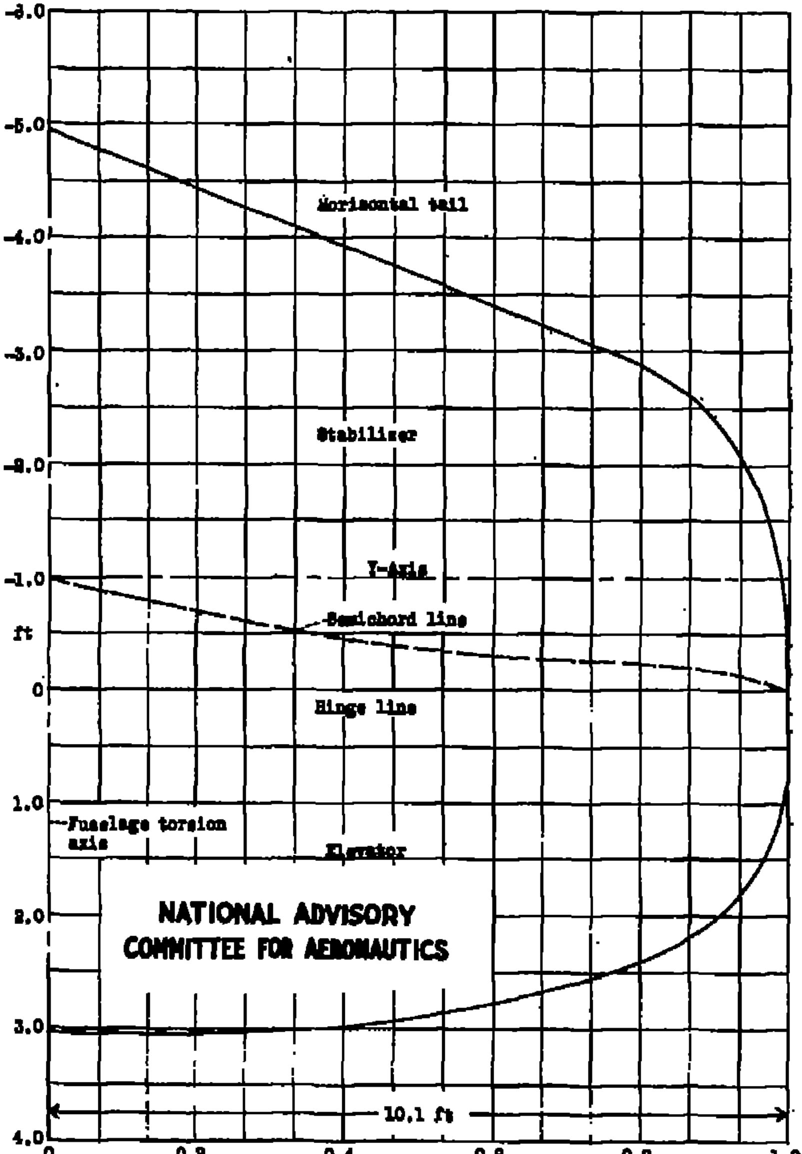


Figure 25.- Horizontal tail used in example II.

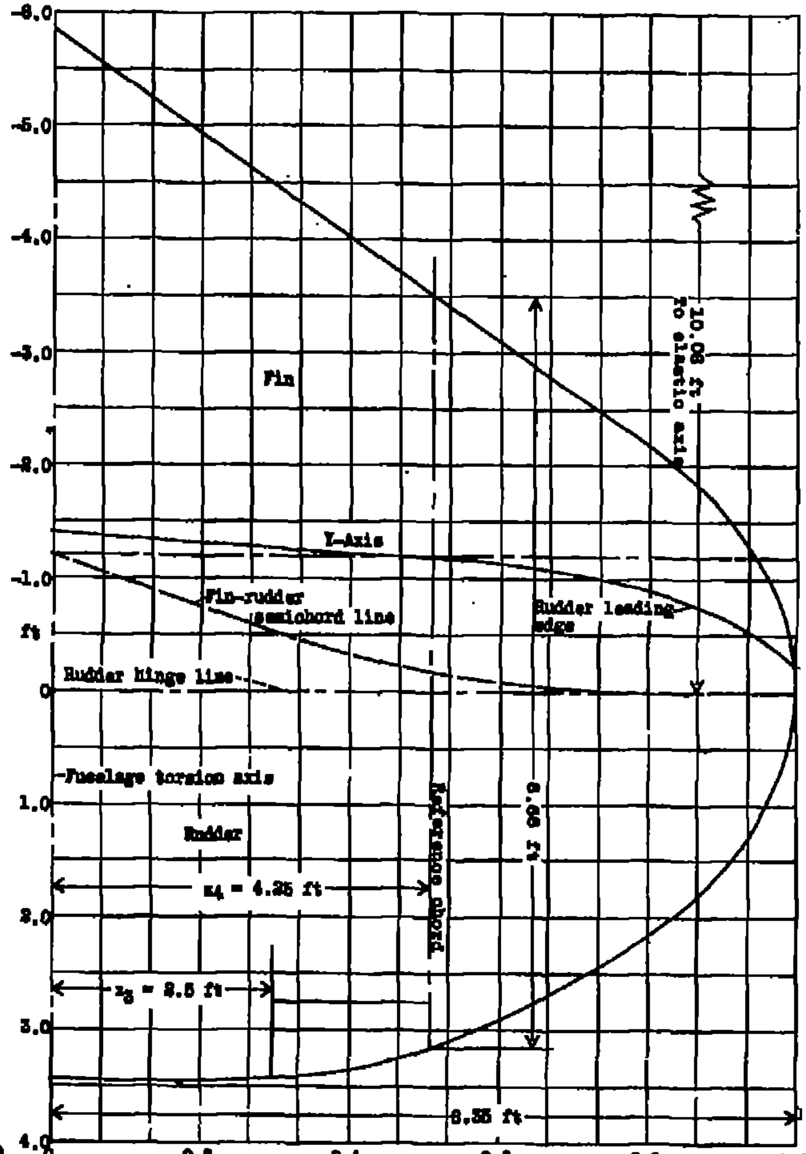


Figure 26.- Fin-rudder combination used in example II.