TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 617

STABILITY OF STRUCTURAL MEMBERS UNDER AXIAL LOAD

By Eugene E. Lundquist
Langley Memorial Aeronautical Laboratory

Washington
October 1937

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE NO. 617

STABILITY OF STRUCTURAL MEMBERS UNDER AXIAL LOAD

By Eugene E. Lundquist

SUMMARY

The principles of the Cross method of moment distribution are used to check the stability of structural members under axial load. A brief theoretical treatment of the subject, together with an illustrative problem, is included as well as a discussion of the reduced modulus at high stresses and a set of tables to aid in the solution of practical problems.

INTRODUCTION

One of the problems in the design of structures is to make certain that the compression members are stable under the loads to be carried. For example, it is assumed that the usual column formulas give the critical stress at which a compression member becomes unstable in bending. In order to use these formulas, however, the value of the restraint coefficient c must be known.

For a structure built with the members joined to each other by frictionless pins at each end, $c = 1$. For a structure built with the members continuous at the joints, however, the value of c for any compression member is dependent upon the size of all members in the structure and the axial loads in them. The design of the compression members for a structure continuous at the joints is therefore a problem in trial-and-error calculation. The procedure recommended for design is, first, to proportion the compression members on the basis of assumed restraint coefficients and, second, to check the stability of the system of members by a simple calculation. If the system of members is found to be unstable, new values of the restraint coefficient must be assumed, new sizes for the members selected, and another check of the stability made.

The suggestions and comment of Dr. William R. Osgood of the National Bureau of Standards on the subject matter of this report are greatly appreciated, particularly his suggestions regarding the evaluation of the effective modulus at stresses above the elastic range.

DEFINITIONS AND SYMBOLS

The following definitions of stiffness and carry-over factor parallel those given in references 1 and 2 with some changes in wording:

Stiffness.- If a member is on unyielding supports at each end, the moment at one end necessary to produce a rotation of 1/4 radian of that end is called the "stiffness." The stiffness of a member will depend upon the amount of restraint at the far end. In the derivation of the criterion for stability, three types of restraint at the far end are considered. The symbols used to designate the stiffness for the different types of restraint are

S, far end fixed.

S', far end elastically restrained.

S'', far end pinned.

Carry-over factor.- If a member is on unyielding supports at each end and a moment is applied at the near end, the ratio of the moment developed at the far end to the moment applied at the near end is called the "carry-over factor." As in the case of stiffness, the carry-over factor will depend upon the degree of restraint at the far end of the member. The symbols used to designate the carry-over factor for the different types of restraint considered in this report are

C, far end fixed.

C', far end elastically restrained.

$C'' = 0$, far end pinned.

The stiffness of a member computed according to the foregoing definition is 1/4 that computed according to the definition given in references 1 and 2. In the Cross method the relative stiffness of the members is of importance and not the absolute value. The foregoing definition was selected so that the stiffness of a member of constant cross section with no axial load and fixed at the far end would be $\bar{E}I/L$ instead of $4\bar{E}I/L$.

Sign convention.- The sign convention used in this report is the same as that used by James in reference 2. A clockwise moment acting on the end of a member is positive. A counterclockwise moment acting on a joint is positive. An external moment applied at a joint is considered to act on the joint.

Symbols.-

Σ, summation.

E, modulus of elasticity.

$\bar{E}$, effective modulus of elasticity.

I, moment of inertia of cross section about a centroidal axis normal to the plane of bending.

L, length of member.

P, axial load (absolute value).

A, area of cross section.

c, restraint coefficient in the usual column formula.

$\rho = \sqrt{\frac{I}{A}}$, radius of gyration

$$\left.\begin{aligned}\alpha &= 6\,\frac{\frac{L}{j}\operatorname{cosec}\frac{L}{j} - 1}{\left(\frac{L}{j}\right)^2}\\[2ex] \beta &= 3\,\frac{1 - \frac{L}{j}\cot\frac{L}{j}}{\left(\frac{L}{j}\right)^2}\end{aligned}\right\}\quad \text{For compression members}$$

$$\left.\begin{aligned}\alpha &= 6\,\frac{\frac{L}{j}\operatorname{csch}\frac{L}{j} - 1}{-\left(\frac{L}{j}\right)^2}\\[2ex] \beta &= 3\,\frac{1 - \frac{L}{j}\coth\frac{L}{j}}{-\left(\frac{L}{j}\right)^2}\end{aligned}\right\}\quad \text{For tension members}$$

$$j = \sqrt{\frac{EI}{P}}$$

$$\frac{L}{j} = L\sqrt{\frac{P}{EI}}$$

$$\left(\frac{L}{j}\right)_{eff} = L\sqrt{\frac{P}{\overline{E}I}}$$

Effective values of α and β are obtained by substitution of $(L/j)_{eff}$ for L/j.

CRITERION FOR STABILITY

The joints of the structure are assumed to be held rigidly in space but are free to rotate under the elastic restraint of the interconnecting members. This assumption is also basic in the Cross method of moment distribution (reference 1).

The method used to check the stability of the structure is based upon the principles of moment distribution. In this method either of two criterions may be used.

Stiffness criterion for stability.- From a structure of many members consider the section comprising one joint shown in figure 1. Apply a unit external moment at joint b.

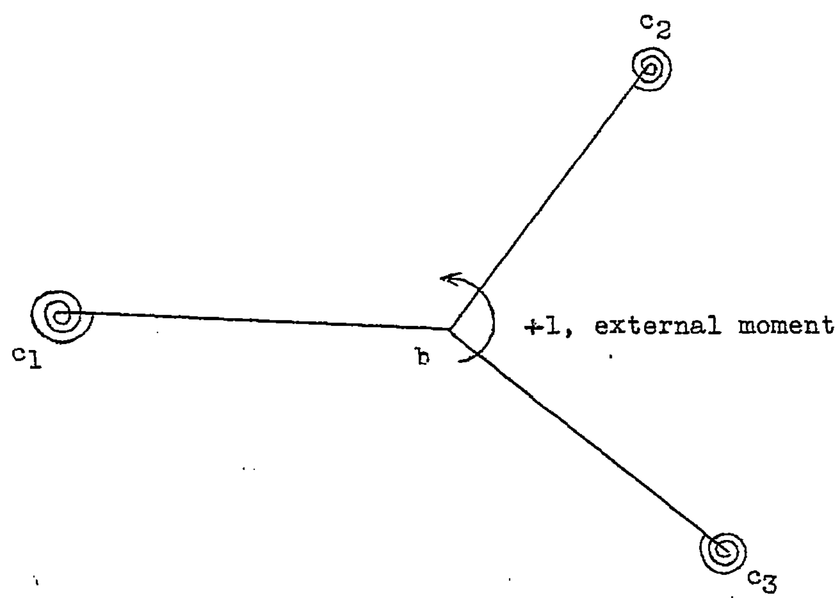


Figure 1.

By the Cross method, the moment of -1 added to balance joint b is divided between members bc in proportion to their stiffnesses. Because there are other members beyond joints c, the far end of members bc will be elastically restrained as indicated in figure 1 by coiled springs at c_1, c_2, and c_3. It is possible, theoretically, to calculate the restraint at joints c and the stiffness of members bc when they are elastically restrained at their far ends. Thus, if the stiffnesses of members bc are determined with the far ends c elastically restrained, the moment of -1 added to balance joint b is distributed

$-\dfrac{S'_{bc_1}}{\Sigma S'_{bc}}$ to member bc_1

$-\dfrac{S'_{bc_2}}{\Sigma S'_{bc}}$ to member bc_2

etc.

The moments carried over to the far ends of members bc are

$-\dfrac{S'_{bc_1} C'_{bc_1}}{\Sigma S'_{bc}}$ to far end of member bc_1

$-\dfrac{S'_{bc_2} C'_{bc_2}}{\Sigma S'_{bc}}$ to far end of member bc_2

etc.

The moments carried over to the far ends of members bc will be absorbed by all the members beyond joints c. Thus, the moment at each end of every member in the structure will be some quantity divided by $\Sigma S'_{bc}$.

Before the structure is loaded, the stiffness of each member of the structure is positive (no axial load in the members) making $\Sigma S'_{bc}$ positive. As the structure is loaded, the effects of axial tension and compression will cause the stiffness of some members to increase and the stiffness of other members to decrease. For stability, the moment at each end of every member must be finite. Therefore, the stiffness criterion for stability is

$$\Sigma\ S'_{bc} > 0 \tag{1}$$

It is desirable to emphasize that, if the stiffness criterion for stability is satisfied, not only is the stability of members bc in figure 1 checked but the stability of every member in the structure is proved.

The condition of neutral stability gives the critical buckling load for the structure and is obtained by setting the stiffness stability factor $\Sigma S'_{bc}$ equal to zero, or

$$\Sigma S'_{bc} = 0 \tag{2}$$

<u>Series criterion for stability</u>.- From a structure of many members consider the section comprising two joints shown in figure 2. Apply a unit external moment at joint b. By the Cross method, the moment of -1 added to balance joint b is divided between member bc and members ba in proportion to their stiffnesses. Because there are other members to the left of joints a, the left end of each member ba will be elastically restrained as indicated in figure 2 by a coiled spring at a_1, a_2, and a_3.

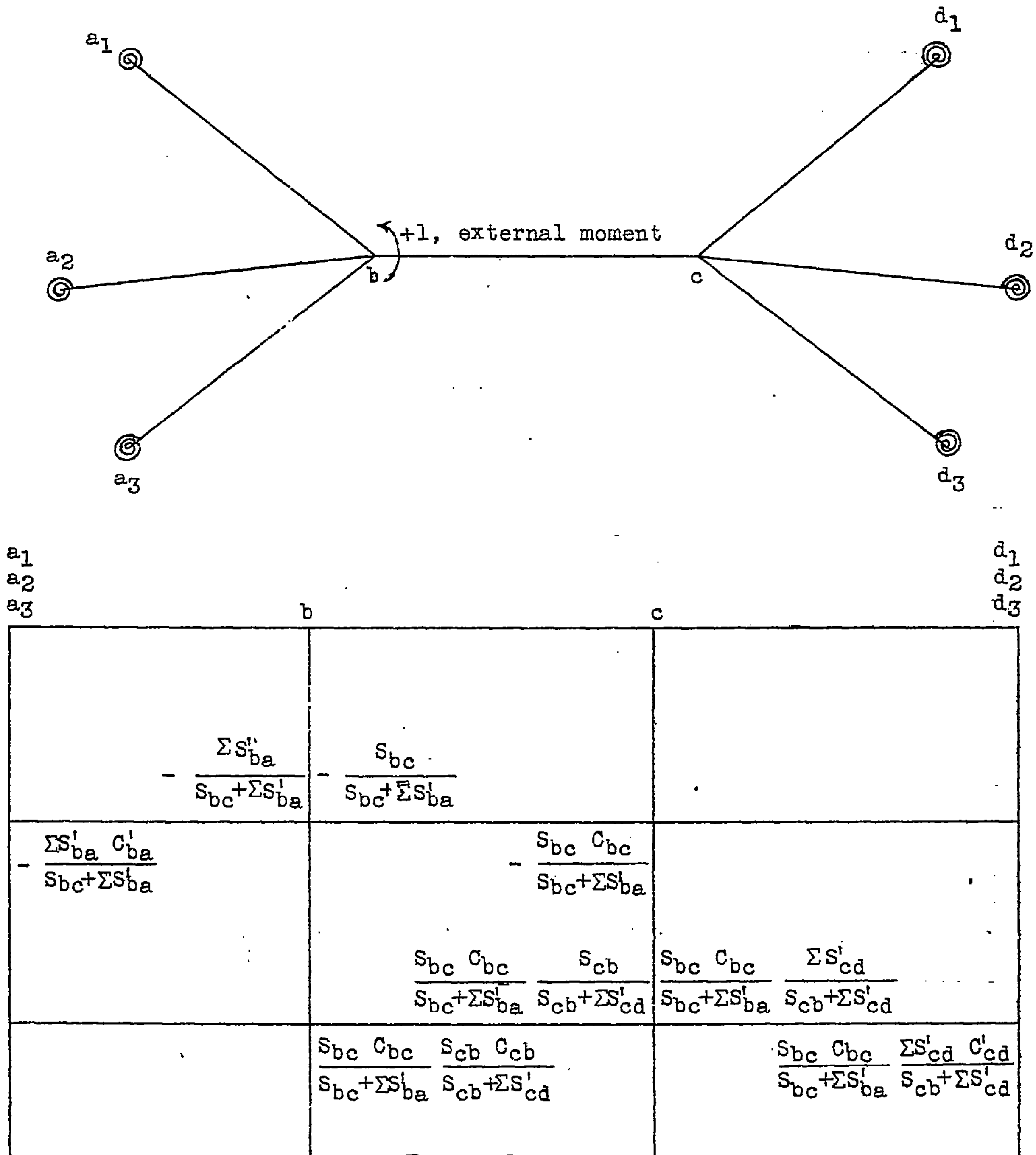

a_1
a_2
a_3
d_1
d_2
d_3
+1, external moment
b
c
$-\frac{\Sigma S'_{ba}}{S_{bc}+\Sigma S'_{ba}}$
$-\frac{S_{bc}}{S_{bc}+\Sigma S'_{ba}}$
$-\frac{\Sigma S'_{ba}\ C'_{ba}}{S_{bc}+\Sigma S'_{ba}}$
$-\frac{S_{bc}\ C_{bc}}{S_{bc}+\Sigma S'_{ba}}$
$\frac{S_{bc}\ C_{bc}}{S_{bc}+\Sigma S'_{ba}}\ \frac{S_{cb}}{S_{cb}+\Sigma S'_{cd}}$
$\frac{S_{bc}\ C_{bc}}{S_{bc}+\Sigma S'_{ba}}\ \frac{\Sigma S'_{cd}}{S_{cb}+\Sigma S'_{cd}}$
$\frac{S_{bc}\ C_{bc}}{S_{bc}+\Sigma S'_{ba}}\ \frac{S_{cb}\ C_{cb}}{S_{cb}+\Sigma S'_{cd}}$
$\frac{S_{bc}\ C_{bc}}{S_{bc}+\Sigma S'_{ba}}\ \frac{\Sigma S'_{cd}\ C'_{cd}}{S_{cb}+\Sigma S'_{cd}}$

Figure 2

The stiffness of span bc is calculated on the assumption that joint c is fixed. Thus, if S_{bc} is the stiffness of member bc fixed at c and $\Sigma S'_{ba}$ is the sum of the stiffnesses of members ba elastically restrained at joints a, the moment of −1 added to balance the external moment of +1 at joint b is distributed:

$$-\frac{S_{bc}}{S_{bc} + \Sigma S'_{ba}}$$

to member bc, and

$$-\frac{\Sigma S'_{ba}}{S_{bc} + \Sigma S'_{ba}}$$

to members ba. These moments, together with the moments carried over to joint c and joints a, are set down in the table of figure 2.

Because the stiffness and carry-over factor for members ba take proper account of the elastic restraint at joints a, the moments carried over to joints a are absorbed by those portions of the structure to the left of these joints. Thus, there is no unbalanced moment at any joint a.

It was assumed that joint c was fixed when in reality it was elastically restrained. The moment

$$-\frac{S_{bc}\ C_{bc}}{S_{bc} + \Sigma S'_{ba}}$$

carried over to this joint has therefore caused it to be out of balance. Accordingly, joint c is balanced and the proper moments are carried over to joint b and joints d. (See table of fig. 2.) Because the stiffness and carry-over factor for members cd take proper account of the elastic restraint at joints d, the moments carried over to joints d are absorbed by those portions of the structure to the right of these joints. Hence, the only unbalanced joint is b and the unbalanced moment at this joint is r where

$$r = \frac{S_{bc}\ C_{bc}}{S_{bc} + \Sigma S'_{ba}}\ \frac{S_{cb}\ C_{cb}}{S_{cb} + \Sigma S'_{cd}} \qquad (3)$$

Joint b was the starting point with an unbalanced moment of unity. Therefore, if the present unbalanced moment of r at joint b is distributed in the manner described for the initial unbalanced moment of unity, another set of entries for the table of figure 2 will be obtained that are exactly r times those already made. It will then be found that the unbalanced moment at joint b is r^2. Distribution of this unbalanced moment will give a third set of entries in the table of figure 2 that are r^2 times the first set. Thus the nth set of entries in the table of figure 2 will be r^{n-1} times the first set of entries.

According to the Cross method, the moment at the end of any member is obtained by the addition of the entries in the corresponding column of the table of figure 2. For any member, this moment is some quantity times the infinite series

$$1 + r + r^2 + r^3 + \dots\dots$$

For stability, the moment at the end of each member must be finite. Thus, for stability, the sum of the infinite series must be finite. This condition is satisfied when the value of r lies between -1 and $+1$.

It will now be provedthat r cannot have a value between -1 and 0 without first having a value greater than $+1$. The product of stiffness and carry-over factor for any member is positive for any condition of restraint at the far end. Therefore r can be negative only if the denominator on the right side of equation (3) is negative. Before the structure is loaded, the stiffness of each member of the structure is positive (no axial load in the members), making the denominator positive. As the structure is loaded, the effects of axial tension and compression cause the stiffness of some members to increase and the stiffness of other members to decrease. Thus, as the load on the structure is increased, the denominator on the right side of equation (3) cannot be negative without passing through zero. When the denominator is zero, r is infinite, which means that the structure is unstable. Therefore, the criterion for stability is

$$0 < r < 1 \tag{4}$$

If the series criterion for stability is satisfied, not only is the stability of member bc in figure 2 checked but the stability of every member in the structure is proved. If the cross section and axial load vary along the length of any member, the effect of these variations is included in the evaluation of the stiffness and carry-over factor for that member regardless of which criterion for stability is used. If desired, the effect of shear can also be included.

The condition of neutral stability gives the critical buckling load for the structure and is obtained by setting the series stability factor r equal to unity, or

$$r = \frac{S_{bc}\ C_{bc}}{S_{bc} + \Sigma S'_{ba}} \quad \frac{S_{cb}\ C_{cb}}{S_{cb} + \Sigma S'_{cd}} = 1 \qquad (5)$$

CARRY-OVER FACTOR AND STIFFNESS

In order to calculate the critical buckling load in actual problems, it is necessary to have suitable expressions for the stiffness and carry-over factor. Before these expressions are summarized, however, equations will first be derived for the carry-over factor and stiffness of a member elastically restrained at its far end.

Consider the member ij shown in figure 3, simply supported at i and elastically restrained at j by members jk. The members jk are also elastically restrained at their far ends k. Apply an external moment −M at support i. The moment of +M added to balance this joint is all distributed to member ij. On the assumption that joint j is fixed, the moment carried over to the far end j is MC_{ij}. The moment $-MC_{ij}$ added to balance joint j is then distributed between member ji and members jk in proportion to their stiffnesses as shown in the table of figure 3:

$$-MC_{ij}\ \frac{S''_{ji}}{S''_{ji} + \Sigma S'_{jk}}$$

to member ji, and

$$-MC_{ij}\ \frac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}}$$

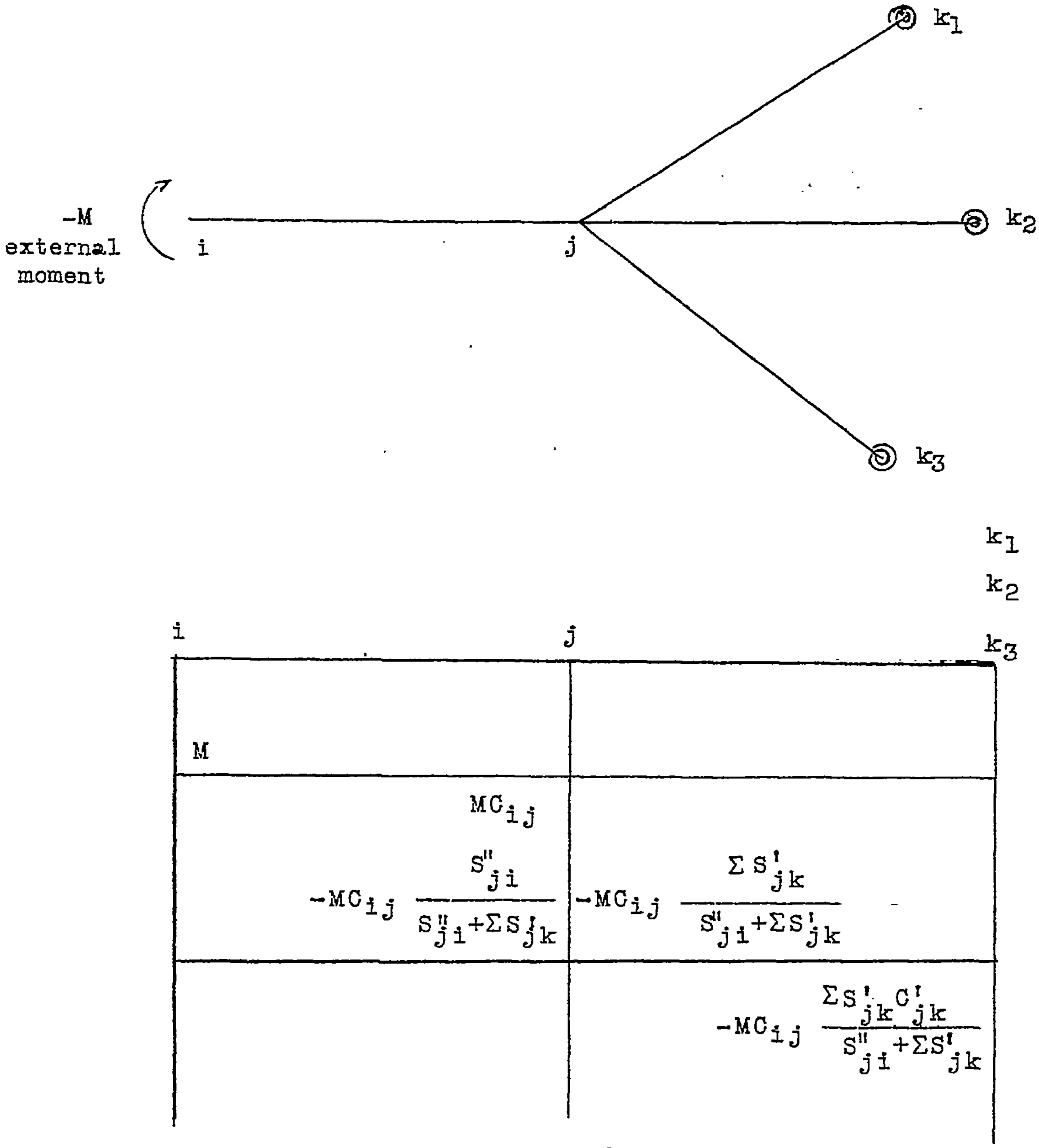

-M
external
moment
i
j
k_1
k_2
k_3
M
MC_{ij}
$-MC_{ij} \frac{S''_{ji}}{S''_{ji}+\Sigma S'_{jk}}$
$-MC_{ij} \frac{\Sigma S'_{jk}}{S''_{ji}+\Sigma S'_{jk}}$
$-MC_{ij} \frac{\Sigma S'_{jk} C'_{jk}}{S''_{ji}+\Sigma S'_{jk}}$

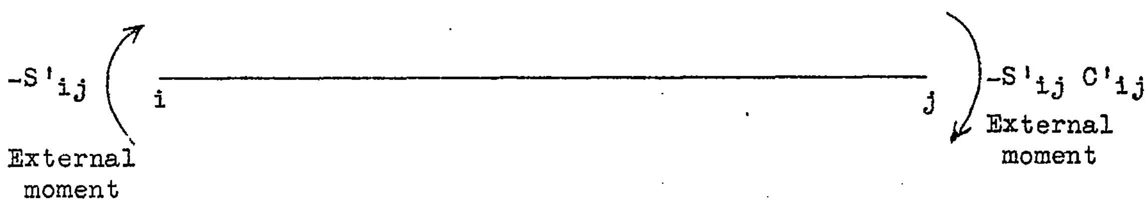
Figure 3

to members jk. Because the stiffness S''_{ji} of span ji takes proper account of the pin end at i, no moment is carried over to i. The stiffness and carry-over factor for members jk take proper account of the elastic restraint at joints k. Therefore the moments carried over to joints k will be absorbed by the structure to the right of these joints and the moment distribution analysis is complete so far as moments in member ij are concerned. Thus the moments at the ends of member ij are:

At end i, $\quad M$

At end j, $\quad MC_{ij} \dfrac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}}$

By definition, the carry-over factor C'_{ij} for member ij elastically restrained at j is the ratio of the moment



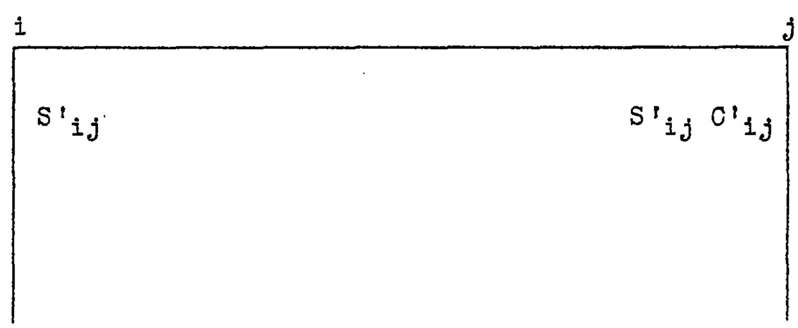


Figure 4

at end j to the moment at end i, or

$$C'_{ij} = C_{ij} \frac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}} \qquad (6)$$

In order to derive an equation for the stiffness S'_{ij} of member ij elastically restrained at the far end j, assume that $-M$ (fig. 3) has the value $-S'_{ij}$. Then member ij will have the end moments shown in the table of figure 4 and the tangent at i will have been rotated through 1/4 radian. Now consider a duplicate of member ij pinned at each end (fig. 5). Apply an external moment $-S''_{ij}$ at i. The moment of $+S''_{ij}$ added to balance this joint is distributed to member ij. If the far end j is assumed to be pinned, the tangent at i will have been rotated through 1/4 radian. At this stage apply an external moment of $-S'_{ij}\ C'_{ij}$ at j. The moment of $+S'_{ij}\ C'_{ij}$

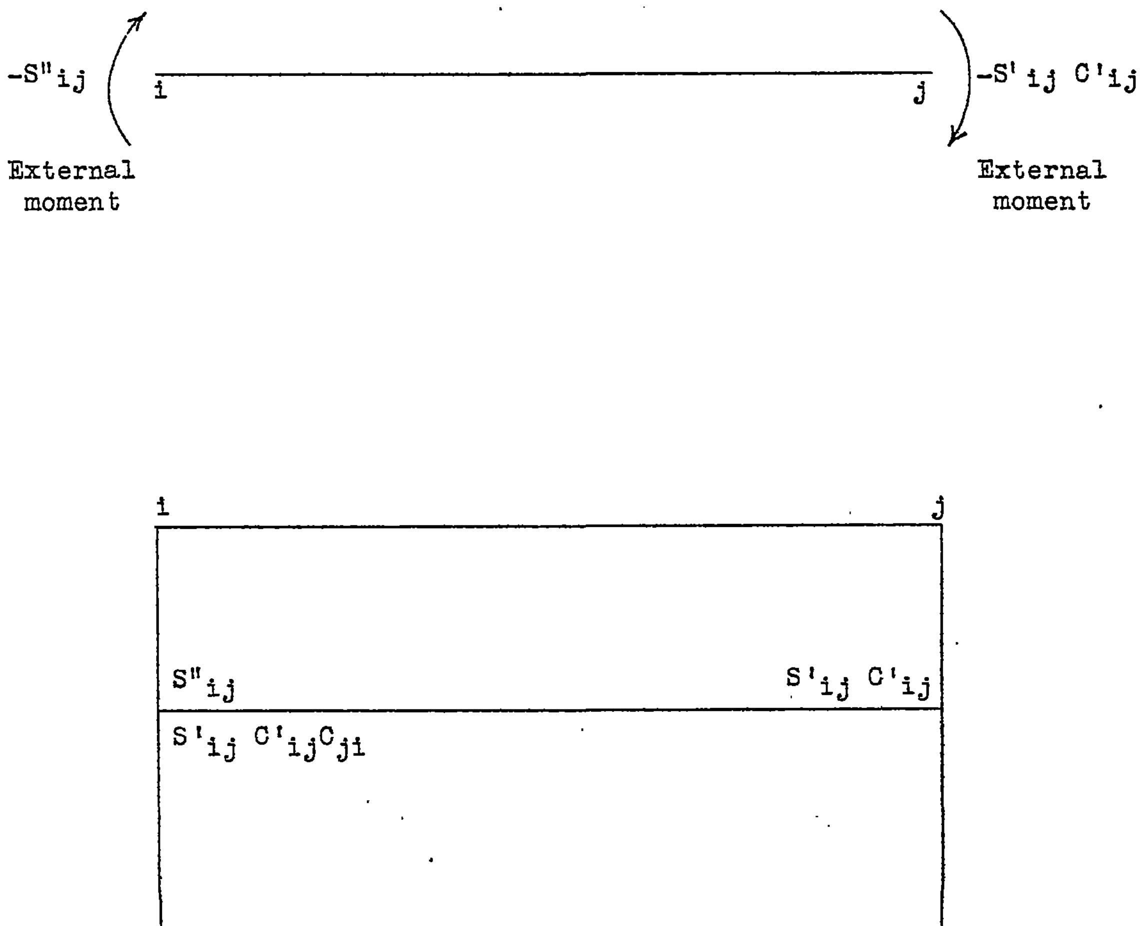


Figure 5.

added to balance this joint is distributed to member ji. On the assumption that the far end i is fixed, the moment carried over to joint i is $+S'_{ij}\ C'_{ij}\ C_{ji}$. In this condition the moment at j and the rotation of the tangent at i are the same for the original member ij (fig. 4) and the duplicate member ij (fig. 5). It therefore follows that the moments at i in the original and duplicate member must also be equal. Therefore

$$S'_{ij} = S''_{ij} + S'_{ij}\ C'_{ij}\ C_{ji}$$

from which

$$S'_{ij} = \frac{S''_{ij}}{1 - C_{ji}\ C'_{ij}} \qquad (7)$$

Substitution of the value of C'_{ij} as given by equation (6) gives for the stiffness of a member ij elastically restrained at the far end j by other members jk, also elastically restrained at their far ends,

$$S'_{ij} = \frac{S''_{ij}}{1 - C_{ji}\ C_{ij}\ \dfrac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}}} \qquad (8)$$

For member ij, the limiting values of the carry-over factor and stiffness given by equations (6) and (8), respectively, are obtained as follows. When the far end j is pinned, there is no elastic restraint at j and $\Sigma S'_{jk} = 0$. For this limiting condition, the carry-over factor $C'_{ij} = C''_{ij} = 0$ and the stiffness $S'_{ij} = S''_{ij}$. When the far end j is fixed, there is complete restraint at j and $\Sigma S'_{jk} = \infty$. For this limiting condition, the carry-over factor $C'_{ij} = C_{ij}$ and the stiffness $S'_{ij} = S_{ij}$ where

$$S_{ij} = \frac{S''_{ij}}{1 - C_{ji}\ C_{ij}} \qquad (9)$$

Up to this point, all the equations in this report on stability are general. In nearly all cases encountered in practice, however, the cross section and axial load do not vary along the length of each member. For this special case,

$C_{ij} = C_{ji}$, $S''_{ij} = S''_{ji}$, $S_{ij} = S_{ji}$, and the carry-over factor of any member ij, fixed at the far end, is (see reference 2)

$$C_{ij} = \frac{\alpha_{ij}}{2\beta_{ij}} \tag{10}$$

also, the stiffness of any such member ij is:

Far end j pinned (see reference 2)

$$S''_{ij} = \frac{EI}{L}\left[\frac{3}{4\beta_{ij}}\right] \tag{11}$$

Far end j elastically restrained by members jk,

$$S'_{ij} = \frac{S''_{ij}}{1 - C^2_{ij}\,\dfrac{\Sigma S'_{jk}}{S''_{ij} + \Sigma S'_{jk}}} \tag{12}$$

Far end j fixed,

$$S_{ij} = \frac{S''_{ij}}{1 - C^2_{ij}} \tag{13}$$

$$= \frac{EI}{L}\left[\frac{\dfrac{3}{4\beta_{ij}}}{1 - \left(\dfrac{\alpha_{ij}}{2\beta_{ij}}\right)^2}\right] \tag{14}$$

When the cross section and axial load do not vary throughout the length of each member, the series stability factor as given by equation (3) becomes (see fig. 2)

$$r = \frac{(S_{bc}\,C_{bc})^2}{(S_{bc} + \Sigma S'_{ba})\,(S_{bc} + \Sigma S'_{cd})} \tag{15}$$

The values of the quantities that appear in this expression are obtained by the use of equations (10) through (14). It is more convenient, however, to tabulate certain of these quantities as has been done in tables I and II.

THE EFFECTIVE MODULUS

If equations (10) to (14), inclusive, are to be applicable in the short-column range, an effective modulus $\overline{E}$ must be substituted for Young's modulus E. This substitution requires that an effective value of L/j be used to evaluate α and β in all equations of this report, where

$$\left(\frac{L}{j}\right)_{eff} = L\sqrt{\frac{P}{\overline{E}I}} \qquad (16)$$

As noted in the list of symbols, the formulas used in the evaluation of α and β differ for tension and compression members.

For compression members in the elastic or Euler range, $\overline{E} = E$. For the short-column range, $\overline{E} < E$. In order that the calculated critical load for a structure shall be consistent with the usual column formulas based upon tests, it is recommended that $\overline{E}$ for compression members be determined in the following manner:

1. Solve for the effective slenderness ratio $L/\rho\sqrt{c}$ in the accepted column formula for the material under consideration.

2. Substitute this value of $L/\rho\sqrt{c}$ in the equation

$$\overline{E} = \frac{1}{\pi^2}\frac{P}{A}\left(\frac{L}{\rho\sqrt{c}}\right)^2 \qquad (17)$$

The result will be an equation that gives $\overline{E}$ as a function of the stress P/A in the member.

3. If desired, this value of $\overline{E}$ may be corrected for small differences caused by changes in the cross-sectional shape from that used in the tests on which the column formula is based; but this correction is usually neglected.

If it is inconvenient to solve for $L/\rho\sqrt{c}$ in the accepted column formula, the procedure outlined in reference 3 can be used and a curve of $\overline{E}$ against P/A be drawn.

The variation of $\bar{E}$ with stress for tension members can be established, theoretically, by the use of the double-modulus theory of bending and of the stress-strain curve for the material. (See references 4, 5, and 6.) For such calculations, however, the stress-strain curve must be accurately drawn to a suitable scale. In the absence of a known or calculated variation of $\bar{E}$ with stress, the following approximate method can be used to establish $\bar{E}$ for tension members:

1. When the stress is less than the maximum allowed for a column of the same material, use the same values of $\bar{E}$ for tension as for compression at the same stress.

2. When the stress is greater than the maximum allowed for a column of the same material, assume that $\bar{E} = 0$.

The values of $\bar{E}$ for tension members obtained by this method will be conservative. Whether or not they are too conservative is a matter to be settled by tests. Certainly in the regions of yield point and of maximum tensile strength the flatness of the stress-strain curve will cause $\bar{E}$ to approach zero. Because the maximum stress allowed in columns is closely associated with the yield point, this method offers a convenient solution of $\bar{E}$ for tension members.

Axial load in pounds; T, tension; C, compression

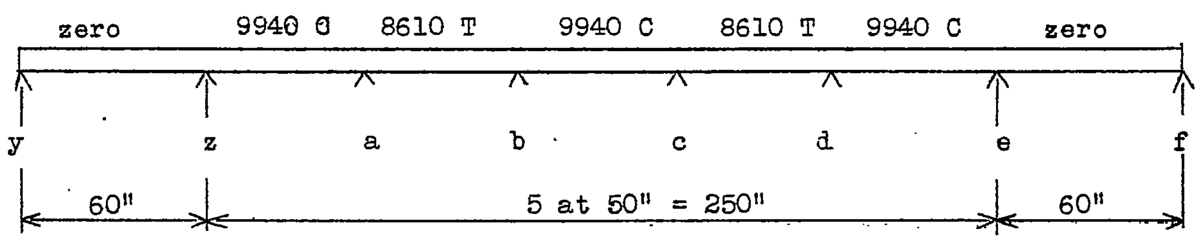


Figure 6.

PROBLEM

Design a continuous member of 1025 steel to carry the loads shown in figure 6. For simplicity, the same cross section will be used in all spans, even though only three of the spans are under axial compression.

The usual column formulas for 1025-steel tubes are:

For $\frac{L}{\rho} < 124$,

$$\frac{P}{A} = 36,000 - 1.172 \frac{1}{c} \left(\frac{L}{\rho}\right)^2 \tag{18}$$

For $\frac{L}{\rho} > 124$,

$$\frac{P}{A} = \frac{276 \times 10^6}{\frac{1}{c} \left(\frac{L}{\rho}\right)^2} \tag{19}$$

It is desired that L/ρ be less than 124. Therefore, equation (18) is used and, on the assumption that $c = 2$, a tube of the following dimensions is selected as a trial design for compression members za, bc, and de.

Diameter, d	1.625 in.
Wall thickness, t	0.065 in.
Area, A	0.3186 sq. in.
Moment of inertia, I	0.09707 in.^{4}

According to the problem, this tube is used as a continuous member from y to f (fig. 6).

In order to check the stability of the tube selected in the trial design, the critical buckling load will be calculated and compared with the loads given in figure 6. It is assumed that the axial load in the tension spans is always 8610/9940 or 0.866 times the axial load in the compression spans. This assumption conforms to the condition that the forces in all members increase in the same ratio as the load on the structure.

Both the dimensions and loading of the member shown in figure 6 are symmetrical about span bc. It is therefore convenient to determine the critical buckling load by use of the series criterion for stability. Imagine the unit external moment to be applied at joint b. Then the series stability factor is given by equation (15) with the summation signs omitted. If the symmetry about span bc is considered, the series stability factor becomes

$$r = \frac{(S_{bc}\ C_{bc})^2}{(S_{bc} + S'_{cd})^2} \qquad (20)$$

where

$$S'_{cd} = \frac{S''_{cd}}{1 - C^2_{cd}\ \dfrac{S'_{de}}{S''_{cd} + S'_{de}}}$$

$$S'_{de} = \frac{S''_{de}}{1 - C^2_{de}\ \dfrac{S''_{ef}}{S''_{de} + S''_{ef}}}$$

In the equation for S'_{de} it is assumed that the ends at y and f are pinned.

The detailed procedure of calculating the critical buckling load is as follows:

1. Assume a series of values for the axial load in one of the members. In order that reasonable loads will be assumed, a compression member should always be selected and the axial loads for this member computed from the column formula using a series of values of c. In this problem, compression member bc is selected and the column formula is equation (18).

2. For each assumed axial load in the selected member, calculate the corresponding axial load in every other member. In this problem the axial load in all compression members is the same and the axial load in the tension members is 0.866 times the axial load in the compression members.

3. For each load in each of the members, calculate P/A, $\bar{E}$, and $(L/j)_{eff}$. In this problem, $\bar{E}$

is obtained from equations (17) and (18), as previously outlined, or

$$\overline{E} = \frac{1}{\pi^2}\frac{P}{A}\left[\frac{36000 - \frac{P}{A}}{1.172}\right]$$

4. For each load in each of the members, determine the value of the terms required to evaluate equation (20), using tables I and II.

5. The assumed load that gives $r = 1$ is the critical buckling load.

The results of this procedure as applied to the problem of figure 6 are given in table III. The values of c in the first column of table III are given for reference only. As stated in paragraph 1 of the foregoing procedure, these values were assumed so that a series of reasonable values for the axial load P in the compression member bc could be obtained. In the last column of table III are given the values of r corresponding to the assumed values of c. It will be noted that, as the value of c increases from 1.4 to 2.6, the value of r increases from 0.133 to 1.63. If the data of table III are plotted in curve form, it is found that when $r = 1$ the lowest critical buckling loads for the trial design are

za, bc, and de . . . 10,260 compression

ab and cd . . . . . 8,890 tension

These critical loads are greater than the loads to which the respective members are subjected. (See fig. 6.) The tube selected for the trial design is therefore stable and the margin of safety for the system is

$$\left[\frac{10260}{9940} - 1\right] = \left[\frac{8890}{8610} - 1\right] = 0.03$$

This margin of safety is obtained regardless of which member is used for its calculation. The reason for a single margin of safety for the whole system is that, when the critical load is reached, all members deflect. Some members deflect more than others, however, with the result that ultimate failure is concentrated in one or more members.

It will be noted in table III that, as the loads P increase, the stability factor r increases to a value greater than 1, then falls to a value less than 1, and finally again rises to a value greater than 1. The reason for this result is that, theoretically, more than one type of instability is possible. For each type of instability there is a corresponding critical load. In design, however, the lowest critical load is the only one of interest. Therefore, when the stability of the trial design is checked, the lowest critical load should be calculated and compared with the loads given in the problem.

It will be further noted in table III that, between $c = 1.4$ and 1.5, the value of S'_{de} changes from positive to negative. According to the stiffness criterion for stability, this change of sign means that members de and ef, considered alone, have changed from stable to unstable. It is also noted that S'_{cd} changes from positive to negative between $c = 2.6$ and 2.7, which means that members cd, de, and ef, considered alone, have changed from stable to unstable but at a much higher load. As previously discussed, the change from stable to unstable for all members occurs between $c = 2.5$ and 2.6 where $r = 1$.

Many short cuts can be made in the solution of special problems. Much can also be said concerning the application of the method to the best advantage in a given problem. These points, as well as other points relating to the practical application of the method, are beyond the scope of this report.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., September 1, 1937.

REFERENCES

1. Cross, Hardy: Analysis of Continuous Frames by Distributing Fixed-End Moments. A.S.C.E. Trans., vol. 96, 1932, pp. 1-10.

2. James, Benjamin Wylie: Principal Effects of Axial Load on Moment-Distribution Analysis of Rigid Structures. T.N. No. 534, N.A.C.A., 1935.

3. Lundquist, Eugene E., and Fligg, Claude M.: A Theory for Primary Failure of Straight Centrally Loaded Columns. T.R. No. 582, N.A.C.A., 1937.

4. Timoshenko, S.: Strength of Materials, Part II. D. van Nostrand Co., Inc., 1930, p. 572, art. 37.

5. Osgood, William R.: Column Curves and Stress-Strain Diagrams. Research Paper No. 492, Bur. Standards Jour. Res., vol. 9, Oct. 1932, pp. 571-582.

6. Osgood, William R.: The Double-Modulus Theory of Column Action. Civil Engineering, vol. 5, no. 3, March 1935, pp. 173-175.

TABLE I

Functions for Compression Members of Constant Cross Section

$\left(\frac{L}{j}\right)_{eff}$	C	$\frac{S''}{\left(\frac{EI}{L}\right)}$	$\frac{S}{\left(\frac{EI}{L}\right)}$	C^2	$\frac{S^2 C^2}{\left(\frac{EI}{L}\right)^2}$
0	0.5000	0.7500	1.000	0.2500	0.2500
.1	.5002	.7495	.9997	.2502	.2501
.2	.5010	.7480	.9987	.2510	.2503
.3	.5023	.7455	.9970	.2523	.2508
.4	.5040	.7420	.9947	.2541	.2513
.5	.5063	.7374	.9916	.2564	.2521
.6	.5092	.7318	.9879	.2593	.2530
.7	.5126	.7251	.9836	.2627	.2542
.8	.5166	.7174	.9785	.2668	.2555
.9	.5211	.7085	.9727	.2716	.2570
1.0	.5264	.6985	.9662	.2771	.2587
1.1	.5323	.6873	.9590	.2833	.2606
1.2	.5389	.6748	.9511	.2904	.2627
1.3	.5463	.6611	.9424	.2985	.2651
1.4	.5546	.6460	.9329	.3076	.2677
1.5	.5637	.6295	.9227	.3178	.2705
1.6	.5739	.6114	.9116	.3293	.2737
1.7	.5851	.5918	.8998	.3423	.2771
1.8	.5974	.5704	.8871	.3569	.2809
1.9	.6111	.5473	.8735	.3735	.2850
2.0	.6263	.5221	.8590	.3922	.2894
2.1	.6430	.4948	.8436	.4135	.2943
2.2	.6616	.4651	.8273	.4377	.2996
2.3	.6823	.4329	.8099	.4655	.3053
2.4	.7053	.3978	.7915	.4974	.3116
2.5	.7310	.3595	.7720	.5343	.3184
2.6	.7598	.3176	.7513	.5773	.3259
2.7	.7923	.2715	.7295	.6277	.3340
2.8	.8291	.2208	.7064	.6874	.3429
2.9	.8709	.1647	.6819	.7585	.3527
3.0	.9189	.1021	.6560	.8444	.3634
3.1	.9744	.03183	.6287	.9494	.3752
π	1.000	0	.6169	1.000	.3805
3.2	1.039	-.04765	.5997	1.080	.3883
3.3	1.115	-.1385	.5691	1.243	.4027
3.4	1.206	-.2436	.5366	1.454	.4186
3.5	1.316	-.3670	.5021	1.731	.4364
3.6	1.451	-.5147	.4655	2.106	.4562
3.7	1.622	-.6953	.4265	2.630	.4784
3.8	1.843	-.9227	.3850	3.397	.5035

TABLE I (cont.)

Functions for Compression Members of Constant Cross Section

$\left(\frac{L}{j}\right)_{eff}$	C	$\frac{S''}{\left(\frac{EI}{L}\right)}$	$\frac{S}{\left(\frac{EI}{L}\right)}$	C^2	$\frac{S^2 C^2}{\left(\frac{EI}{L}\right)^2}$
3.9	2.140	-1.220	0.3407	4.582	0.5317
4.0	2.560	-1.629	.2933	6.556	.5639
4.1	3.197	-2.235	.2424	10.22	.6007
4.2	4.271	-3.237	.1878	18.24	.6430
4.3	6.461	-5.246	.1287	41.75	.6919
4.31	6.812	-5.566	.1226	46.41	.6972
4.32	7.204	-5.922	.1164	51.89	.7026
4.33	7.643	-6.322	.1101	58.42	.7081
4.34	8.140	-6.773	.1038	66.26	.7137
4.35	8.706	-7.287	.09742	75.79	.7194
4.36	9.357	-7.877	.09100	87.55	.7251
4.37	10.11	-8.562	.08453	102.3	.7310
4.38	11.00	-9.368	.07801	121.1	.7369
4.39	12.07	-10.33	.07143	145.6	.7429
4.40	13.36	-11.50	.06480	178.4	.7491
4.41	14.96	-12.94	.05811	223.7	.7553
4.42	16.99	-14.78	.05136	288.7	.7616
4.43	19.67	-17.19	.04455	386.9	.7681
4.44	23.35	-20.52	.03769	545.3	.7746
4.45	28.73	-25.36	.03077	825.4	.7813
4.46	37.33	-33.11	.02378	1393.0	.7880
4.47	53.27	-47.48	.01674	2838.0	.7949
4.48	93.00	-83.27	.009629	8648.0	.8019
4.49	365.8	-329.0	.002459	13380.0	.8090
4.50	-188.7	170.5	-.004788	35600.0	.8163
4.51	-75.17	68.20	-.01207	5650.0	.8237
4.52	-46.90	42.74	-.01944	2200.0	.8312
4.53	-34.08	31.19	-.02687	1162.0	.8388
4.54	-26.77	24.60	-.03437	716.6	.8466
4.55	-22.04	20.33	-.04194	485.8	.8545
4.56	-18.73	17.35	-.04958	350.9	.8625
4.57	-16.29	15.14	-.05729	265.3	.8707
4.58	-14.41	13.44	-.06507	207.6	.8790
4.59	-12.92	12.10	-.07293	166.9	.8875
4.60	-11.71	11.00	-.08086	137.1	.8961
4.61	-10.70	10.09	-.08887	114.6	.9049
4.62	-9.861	9.330	-.09695	97.23	.9139
4.63	-9.140	8.676	-.1051	83.55	.9230
4.64	-8.518	8.112	-.1134	72.56	.9323
4.65	-7.976	7.619	-.1217	63.62	.9418

TABLE I (cont.)

Functions for Compression Members of Constant Cross Section

$\left(\frac{L}{j}\right)_{eff}$	C	$\frac{S''}{\left(\frac{\overline{E}I}{L}\right)}$	$\frac{S}{\left(\frac{\overline{E}I}{L}\right)}$	C^2	$\frac{S^2 C^2}{\left(\frac{\overline{E}I}{L}\right)^2}$
4.66	-7.499	7.184	-0.1301	56.23	0.9515
4.67	-7.076	6.799	-.1386	50.07	.9613
4.68	-6.698	6.454	-.1471	44.86	.9713
4.69	-6.359	6.144	-.1558	40.44	.9816
4.70	-6.053	5.864	-.1645	36.64	.9920
4.8	-4.093	4.052	-.2572	16.75	1.108
4.9	-3.102	3.110	-.3607	9.622	1.252
5.0	-2.507	2.521	-.4772	6.283	1.431
5.1	-2.112	2.110	-.6099	4.459	1.659
5.2	-1.833	1.799	-.7629	3.358	1.954
5.3	-1.626	1.550	-.9422	2.645	2.348
5.4	-1.470	1.341	-1.156	2.160	2.888
5.5	-1.348	1.159	-1.418	1.817	3.655
5.6	-1.253	.9949	-1.748	1.569	4.795
5.7	-1.177	.8426	-2.180	1.386	6.591
5.8	-1.119	.6977	-2.778	1.251	9.653
5.9	-1.073	.5566	-3.668	1.152	15.49
6.0	-1.040	.4163	-5.159	1.081	28.77
6.1	-1.017	.2742	-8.234	1.033	70.05
6.2	-1.003	.1275	-18.59	1.007	348.0
2π	-1.000	0	$-\infty$	1.000	∞

TABLE II

Functions for Tension Members of Constant Cross Section

$\left(\frac{L}{j}\right)_{eff}$	C	$\frac{S''}{\left(\frac{\overline{EI}}{L}\right)}$	$\frac{S}{\left(\frac{\overline{EI}}{L}\right)}$	C^2	$\frac{S^2 C^2}{\left(\frac{\overline{EI}}{L}\right)^2}$
0	0.5000	0.7500	1.000	0.2500	0.2500
.1	.4998	.7505	1.000	.2498	.2499
.2	.4990	.7520	1.001	.2490	.2497
.3	.4978	.7545	1.003	.2478	.2493
.4	.4960	.7580	1.005	.2460	.2487
.5	.4938	.7624	1.008	.2439	.2479
.6	.4912	.7678	1.012	.2412	.2470
.7	.4881	.7742	1.016	.2382	.2460
.8	.4845	.7814	1.021	.2348	.2448
.9	.4806	.7896	1.027	.2310	.2435
1.0	.4762	.7986	1.033	.2268	.2420
1.1	.4716	.8085	1.040	.2224	.2404
1.2	.4665	.8192	1.047	.2177	.2386
1.3	.4612	.8307	1.055	.2127	.2368
1.4	.4556	.8429	1.064	.2075	.2348
1.5	.4497	.8559	1.073	.2022	.2328
1.6	.4436	.8696	1.083	.1968	.2306
1.7	.4373	.8839	1.093	.1912	.2284
1.8	.4308	.8989	1.104	.1856	.2261
1.9	.4242	.9144	1.115	.1799	.2237
2.0	.4174	.9306	1.127	.1742	.2213
2.1	.4105	.9472	1.139	.1685	.2188
2.2	.4036	.9644	1.152	.1629	.2162
2.3	.3966	.9820	1.165	.1573	.2136
2.4	.3896	1.000	1.179	.1518	.2110
2.5	.3825	1.019	1.193	.1463	.2084
2.6	.3755	1.038	1.208	.1410	.2057
2.7	.3685	1.057	1.223	.1358	.2030
2.8	.3615	1.076	1.238	.1307	.2004
2.9	.3546	1.096	1.254	.1257	.1977
3.0	.3477	1.117	1.270	.1209	.1950
3.1	.3409	1.137	1.287	.1162	.1924
3.2	.3341	1.158	1.304	.1117	.1897
3.3	.3275	1.179	1.321	.1073	.1871
3.4	.3210	1.200	1.338	.1030	.1845
3.5	.3146	1.222	1.356	.09895	.1820
3.6	.3083	1.244	1.374	.09502	.1794
3.7	.3021	1.265	1.393	.09124	.1769
3.8	.2960	1.288	1.411	.08761	.1745
3.9	.2900	1.310	1.430	.08412	.1720

TABLE II (cont.)

Functions for Tension Members of Constant Cross Section

$\left(\frac{L}{j}\right)_{eff}$	C	$\frac{S''}{\left(\frac{EI}{L}\right)}$	$\frac{S}{\left(\frac{EI}{L}\right)}$	C^2	$\frac{S^2 C^2}{\left(\frac{EI}{L}\right)^2}$
4.0	0.2842	1.332	1.449	0.08078	0.1697
5.0	.2231	1.562	1.652	.05435	.1483
6.0	.1940	1.800	1.870	.03765	.1317
7.0	.1645	2.042	2.098	.02707	.1192
8.0	.1421	2.286	2.333	.02019	.1099
9.0	.1247	2.531	2.571	.01556	.1028
10.0	.1110	2.778	2.812	.01232	.09747
11.0	.09996	3.025	3.056	.009993	.09329
12.0	.09090	3.273	3.300	.008262	.08997
13.0	.08333	3.521	3.545	.006944	.08728
14.0	.07692	3.769	3.792	.005917	.08507
15.0	.07143	4.018	4.038	.005102	.08321
16.0	.06667	4.267	4.285	.004444	.08163
17.0	.06250	4.516	4.533	.003906	.08028
18.0	.05882	4.765	4.781	.003460	.07910
19.0	.05556	5.014	5.029	.003086	.07807
20.0	.05263	5.263	5.278	.002770	.07716
25.0	.04167	6.510	6.522	.001736	.07384
30.0	.03448	7.759	7.768	.001189	.07175
35.0	.02941	9.007	9.015	.0008651	.07031
40.0	.02564	10.26	10.26	.0006575	.06925
45.0	.02273	11.51	11.51	.0005165	.06845
50.0	.02041	12.76	12.76	.0004165	.06782

TABLE III

Calculated Results for Solution of Problem

	Members bc and de				Member cd			
c	P (lb.)	$\frac{P}{A}$ (lb./sq. in.)	$\bar{E}$ (lb./sq. in.)	$\left(\frac{L}{j}\right)_{eff}$	P (lb.)	$\frac{P}{A}$ (lb./sq. in.)	$\bar{E}$ (lb./sq. in.)	$\left(\frac{L}{j}\right)_{eff}$
1.4	9280	29130	17.30×10^6	3.72	8040	25230	23.49×10^6	2.97
1.5	9430	29590	16.39	3.85	8170	25620	22.98	3.03
1.6	9550	29990	15.59	3.97	8270	25970	22.52	3.08
1.7	9670	30340	14.84	4.10	8370	26270	22.09	3.12
1.8	9770	30660	14.16	4.22	8460	26550	21.69	3.17
1.9	9860	30940	13.52	4.33	8540	26790	21.32	3.21
2.0	9940	31190	12.98	4.44	8610	27010	20.99	3.25
2.1	10010	31420	12.44	4.55	8670	27210	20.68	3.29
2.2	10080	31630	11.96	4.66	8730	27390	20.38	3.32
2.3	10140	31820	11.49	4.77	8780	27560	20.12	3.35
2.4	10190	31990	11.10	4.86	8820	27700	19.89	3.38
2.5	10240	32150	10.71	4.96	8870	27840	19.63	3.41
2.6	10290	32300	10.34	5.06	8910	27970	19.41	3.44
2.7	10340	32440	9.99	5.16	8950	28090	19.21	3.46
2.8	10380	32570	9.66	5.26	8990	28210	18.99	3.49
2.9	10410	32680	9.38	5.35	9020	28300	18.85	3.51
3.0	10450	32790	9.10	5.44	9050	28400	18.66	3.53

Note.- For member ef, $P = 0$, $\frac{P}{A} = 0$, $\bar{E} = 28 \times 10^6$ lb. per sq. in., $\left(\frac{L}{j}\right)_{eff} = 0$, $S^{II}_{ef} = 3.397 \times 10^4$ lb.-in.

TABLE III (Cont'd.)

Calculated Results for Solution of Problem II

c	Member bc S_{bc} (lb.-in.)	Member bc $S^a_{bc}\,C^a_{bc}$ (lb.^{2}- in.^{2})	Member cd C^a_{cd}	Member cd S^{II}_{cd} (lb.-in.)	Member de C^a_{de}	Member de S^{II}_{de} (lb.-in.)	S^I_{de} (lb.-in.)	S^I_{cd} (lb.-in.)	r
1.4	1.404 X 10^4	5.45 X 10^8	0.1224	5.07 X 10^4	2.783	-2.49 X 10^4	2643	51010	0.133
1.5	1.155	5.24	.1197	5.01	3.989	-3.41	-32.69	50100	.138
1.6	.930	5.07	.1174	4.95	5.964	-4.56	-2476	49200	.148
1.7	.699	4.99	.1151	4.90	10.22	-6.44	-5189	48320	.163
1.8	.484	4.94	.1130	4.85	22.94	-10.01	-7833	47460	.181
1.9	.289	4.88	.1112	4.80	58.42	-16.59	-10340	46560	.200
2.0	.095	4.92	.1096	4.76	545.3	-51.71	-13140	45680	.226
2.1	-.101	4.99	.1077	4.72	485.8	49.12	-16150	44700	.261
2.2	-.302	5.13	.1064	4.68	56.23	16.68	-19590	43450	.314
2.3	-.512	5.34	.1051	4.65	22.72	10.25	-22020	42470	.383
2.4	-.688	5.54	.1040	4.62	12.47	7.51	-26050	40700	.484
2.5	-.896	5.88	.1026	4.58	7.619	5.73	-31210	37540	.720
2.6	-1.118	6.33	.1014	4.56	5.189	4.57	-37610	30870	1.63
2.7	-1.361	6.91	.1006	4.53	3.798	3.73	-46050	-8750	1.38
2.8	-1.633	7.71	.0994	4.50	2.930	3.095	-58070	80560	.187
2.9	-1.911	8.69	.0986	4.48	2.402	2.633	-74590	59480	.533
3.0	-2.227	9.96	.0978	4.45	2.023	2.239	-102200	53820	1.00

[illegible] 3, Table III.

Note.- For member ef, $P = 0$, $\frac{P}{A} = 0$, $\overline{E} = 28 \times 10^6$ lb. per sq. in., $\left(\frac{L}{j}\right)_{eff} = 0$,

$S^{II}_{ef} = -3.397 \times 10^4$ lb.-in.