A COMPUTATION OF PARTIAL ISOMORPHISM RANK ON ORDINAL STRUCTURES

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We compute the partial isomorphism rank, in the sense Scott and Karp, of a pair of ordinal structures using an Ehrenfeucht-Fraisse game. A complete formula is proven by induction given any two arbitrary ordinals written in Cantor normal form.

CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Back-and-forth arguments date back to Cantor. The standard proof that two countable dense linear orders without endpoints are isomorphic can be found in Hausdorff's [9]. Langford in [11] relaxed the condition of isomorphism ($\cong$) and used the back-and-forth method to get that any two dense linear orders without endpoints of any cardinality are elementarily equivalent ($\equiv$). At the November 1948 meeting of the American Mathematical Society at UCLA, Tarski presented a preliminary report [13] of work that he and Mostowski completed in 1941. Inspired partly by Langford's results, they were able to show using an elimination of quantifiers argument that two ordinal structures $(\alpha, <)$ and $(\beta, <)$ are elementarily equivalent iff they are congruent $\pmod{\omega^\omega}$. As a corollary, they showed

$$(\mathbf{ON}, <) \equiv (\omega^\omega, <)$$

(Here, **ON** is the class of all ordinals. Modular arithmetic on **ON** is extended in the natural way. See II of [5].) Furthermore, Tarski conjectured that $(\mathbf{ON}, <, +) \equiv \omega^{\omega^\omega}$ and $(\mathbf{ON}, <, +, \cdot) \equiv \omega^{\omega^{\omega^\omega}}$, but it was known that standard elimination of quantifier methods were insufficient. New techniques were needed.

In 1952, Fraïssé announced in [6] to the Colloque de logique mathématique in Paris that he had developed new purely algebraic definitions and techniques that gave a new proof of Tarski and Mostowski's results without the elimination of quantifiers arguments. This gave rise to his thesis [8] and finally [7]. But, it was Ehrenfeucht's recasting of Fraïssé's work into the language of a game, which now bears both of their names, that broke through at last, and in [5] Ehrenfeucht was able to reprove the original Tarski and Mostowski results as well as both of Tarski's conjectures. Finally, Karp's [10] and Scott's [14] infinitary logic reformulated all of Ehrenfeucht's and Fraïssé's work into the form is exists today.

Virtually all of this historical background can be found in (4.1) of Dickmann's [3] and §§1, 2 of Vaught's [15]. The author takes no credit for their diligent and thorough treatments.

Fraïssé's standard notion is that of a partial isomorphism existing between two structures. That is, given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, and each ordinal α, define $(\mathcal{M}, \bar{a}) \cong_\alpha (\mathcal{N}, \bar{b})$ by induction where $\bar{a} \in M^n$ and $\bar{b} \in N^n$, for $n = 0, 1, 2, \ldots$. $(\mathcal{M}, \bar{a}) \cong_0 (\mathcal{N}, \bar{b})$ if $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{b})$ for all atomic $\mathcal{L}$-formulas. For all ordinals α, $(\mathcal{M}, \bar{a}) \cong_{\alpha+1} (\mathcal{N}, \bar{b})$ if for all $c \in M$ there is a $d \in N$ such that $(\mathcal{M}, \bar{a}, c) \cong_\alpha (\mathcal{N}, \bar{b}, d)$ (the *forth property*) and for all $d \in N$ there is a $c \in M$ such that $(\mathcal{M}, \bar{a}, c) \cong_\alpha (\mathcal{N}, \bar{b}, d)$ (the *back property*). For all limit ordinals λ, $(\mathcal{M}, \bar{a}) \cong_\lambda (\mathcal{N}, \bar{b})$ iff $(\mathcal{M}, \bar{a}) \cong_\alpha (\mathcal{N}, \bar{b})$ for all $\alpha < \lambda$. If $(\mathcal{M}, \bar{a}) \cong_\alpha (\mathcal{N}, \bar{b})$, then $\mathcal{M}$ and $\mathcal{N}$ are said to be partially isomorphic, sometimes denoted $\mathcal{M} \cong^p_\alpha \mathcal{N}$. When $\mathcal{M} \cong^p_\alpha \mathcal{N}$, both $\mathcal{M}$ and $\mathcal{N}$ will agree on $\mathcal{L}$-sentences of quantifier rank α where the quantifier rank $\mathrm{qr}(\phi)$ of an $\mathcal{L}$-sentence ϕ is defined inductively

$$\begin{aligned}
\mathrm{qr}(\phi) = 0 \quad &\text{iff} \quad \phi \text{ is quantifier-free} \\
\mathrm{qr}(\neg\phi) \quad &= \quad \mathrm{qr}(\phi) \\
\mathrm{qr}(\phi \wedge \psi) \quad &= \quad \mathrm{qr}(\phi \vee \psi) = \max\{\mathrm{qr}(\phi), \mathrm{qr}(\psi)\} \\
\mathrm{qr}(\exists v \phi) \quad &= \quad \mathrm{qr}(\phi) + 1
\end{aligned}$$

With these definitions it can be shown that $\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \cong_\omega \mathcal{N}$.

In the next chapter, we describe the Ehrenfeucht-Fraïssé game (sometimes called the back-and-forth game) and how it captures this notion of partial isomorphism between two ordinal structures with the single binary relation $<$. Our goal is to explicitly compute the rank α of partial isomorphism between the two ordinals. That is, given ordinals α_1, α_2, compute α such that $\alpha_1 \cong_\alpha \alpha_2$ and $\alpha_1 \not\cong_{\alpha+1} \alpha_2$. This is accomplished by analyzing the Cantor Normal Forms (CNF) of α_1, α_2, as Ehrenfeucht used in Theorem 14 of [5] (a paper unknown to the author until recently.)

Our general strategy for computing α is as follows: first write α_1, α_2 in CNF and look for the least power in which they disagree. Compute an ordinal term for each block that they have in common and one for the rest of the uncommon part. If a given block is the same in both ordinals we assign ∞ to that term. α is then the minimum of these ordinal terms. Our proof is by induction and begins with analyzing the simple case when the ordinals are finite (Ch. 3). Optimal play in this case is straightforward; both players play their respective midpoints until the game is over so that the rank is approximately $\log_2$ of the smaller ordinal, truncating the fractional part, of course. This simple strategy actually occurs in the formula for the general case. We then proceed to simple transfinite cases when one or both of the ordinals are infinite isolating key concepts that generalize to the general transfinite case in the last chapter.

The intuition behind each ordinal term is as follows: player I moves in one of the common blocks of the CNF or in the uncommon block of one ordinal and Player II must respond in the other ordinal. The ordinal term then corresponds to computing what is the best that I can hold II to when he moves in that block. In most cases, it is in II's best interest to follow I's play in the same block. In some small cases, however, a better move for II exists in some block to the left or right of the one in which I played. This ability for II to run to the left or right produces some interesting and unexpected phenomena in the final formula which we will describe completely in the last theorem. In general, each ordinal term is approximately twice the power of that block plus a $\log_2$ term similar to the one from the game on finite ordinals.

CHAPTER 2

PRELIMINARIES

We briefly review the basic notions of the Ehrenfeucht-Fraïssé game which can also be found in [12] (p. 52ff). A treatment that emphasizes the model theoretic aspects can be found in [4] and [3].

Let α, β, γ be ordinals and define a two-player game $G(\alpha, \beta, \gamma)$ as follows:

$$\begin{array}{lllll} \text{I} & (a_1, \gamma_1) & (a_2, \gamma_2) & \cdots & (a_n, \gamma_n) \\ \\ \text{II} & \quad b_1 & \quad b_2 & \cdots & \quad b_n \end{array}$$

Players alternate playing ordinals in either α or β which we view as two disjoint copies. (Fig. 2.1.) Neither player is allowed to replay previous moves in the same ordinal. Call these moves $a_1, a_2, \ldots$ for I and $b_1, b_2, \ldots$ for II. Player I can freely move in either α or β, but Player II must always respond to I's move in the ordinal which II did not move. We will say that I plays a_n **in** α or **in** β to identify on which board I makes his move. We call a_i, b_i the **ordinal moves** for I and II, respectively. In addition to each of I's ordinal moves a_i, I must play an ordinal γ_i, called the **counter**, such that $\gamma > \gamma_1 > \gamma_2 > \cdots$. When the context is clear for γ, we simply denote the game on α and β by $G(\alpha, \beta)$. Furthermore, II must always respond order isomorphically to I's move. For example, the $\times$ move in Figure 2.1 is a forbidden response for II to I's move a_n.

The game ends when either player can no longer move and the last player to move is declared the winner. That is, if II has responded to all of I's challenges, and I can no longer lower the counter, II wins. On the other hand, if II can no longer respond order isomorphically to I's ordinal play, I wins.

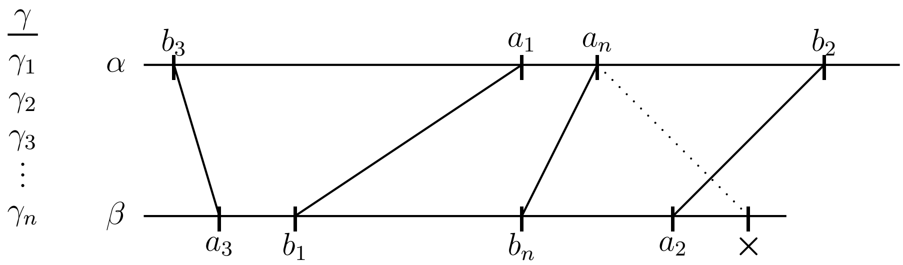

FIGURE 2.1. The game $G(\alpha, \beta, \gamma)$

For every α, β, γ the tree of legal positions of $G(\alpha, \beta, \gamma)$ is necessarily well-founded, because I must decrease the counter in each of his moves. Thus, $G(\alpha, \beta, \gamma)$ is a clopen game, and therefore, it is determined. If $\alpha = \beta$, then II has a winning strategy in $G(\alpha, \beta, \gamma)$: II copies I's moves. If II has a winning strategy in $G(\alpha, \beta, \gamma)$, then II has a winning strategy in $G(\beta, \alpha, \gamma)$, namely, turn the game upside-down. If II has winning strategies in both $G(\alpha, \beta, \gamma)$ and $G(\beta, \delta, \gamma)$ for some ordinal δ, then II can compose these winning strategies to get a winning strategy in $G(\alpha, \delta, \gamma)$. Thus, a winning strategy for II defines an equivalence relation on pairs of ordinals, and we write

$$\text{I has a winning strategy in } G(\alpha, \beta, \gamma) \Leftrightarrow \alpha \nsim_\gamma \beta$$

$$\text{II has a winning strategy in } G(\alpha, \beta, \gamma) \Leftrightarrow \alpha \sim_\gamma \beta$$

In the case $\alpha = \beta$ we write $\alpha \sim_\infty \beta$. When γ is a limit ordinal and we write $\alpha \sim_\gamma \beta$, we mean that for all $\delta < \gamma(\alpha \sim_\delta \beta)$.

For every pair of ordinals $\alpha \neq \beta$ we claim that there is a unique γ for which $\alpha \sim_\gamma \beta$ and $\alpha \nsim_{\gamma+1} \beta$, which we denote $\gamma(\alpha, \beta)$. Clearly, when it exists, $\gamma(\alpha, \beta) = \gamma(\beta, \alpha)$. For $\alpha > \beta > 0$, it follows from the order isomorphic restrictions on II's play that $\alpha \nsim_{\beta+1} \beta$. Moreover, we will prove in Lemma 1 that $\alpha \sim_1 \beta$ for $\alpha > \beta > 0$. Furthermore, suppose $\alpha \sim_\gamma \beta$ and $\gamma' < \gamma$ is any smaller counter. Then, a winning strategy for II in $\alpha \sim_\gamma \beta$ is also winning in $G(\alpha, \beta, \gamma')$, and thus $\alpha \sim_{\gamma'} \beta$. Similarly, if $\alpha \nsim_\gamma \beta$ and $\gamma' > \gamma$, then $\alpha \nsim_{\gamma'} \beta$. So it follows that the ordinal $\gamma(\alpha,\beta)$ exists for all $\alpha \neq \beta$. A formula which computes $\gamma(\alpha,\beta)$ from α and β will be proven by induction.

The computation of $\gamma(\alpha,\beta)$ is done by comparing the Cantor Normal Forms of α, β and looking at the least disagreement in their CNFs. I plays some a_1 based on this comparison and II responds with b_1. The game $G(\alpha,\beta,\gamma)$ is now split into two games: one on the left and one on the right, which we denote $G^{a_1,b_1}_{\mathsf{LHS}}$ and $G^{a_1,b_1}_{\mathsf{RHS}}$. (Fig. 2.2.) We inductively compute a value of γ for each new subgame on the left and right which we denote $\gamma^{a_1,b_1}_{\mathsf{LHS}}$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}}$.

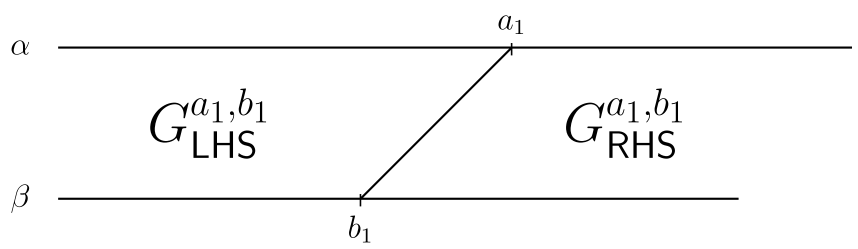

FIGURE 2.2. The games $G^{a_1,b_1}_{\mathsf{LHS}}$,$G^{a_1,b_1}_{\mathsf{RHS}}$

Each induction is divided into two parts: a computation of an upper bound, $\gamma(\alpha,\beta) \leq \theta$; and then the lower bound, $\gamma(\alpha,\beta) \geq \theta$ for some θ. Suppose that θ is a successor. To prove the upper bound, we show that there is a legal ordinal move a_1 for I such that for all legal responses b_1 for II either $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \theta - 1$ or $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq \theta - 1$. It then follows that $\gamma(\alpha,\beta) \leq \theta$ because I can then lower the counter θ by one and move a_1. Regardless of II's response, I can always choose to play out the rest of $G(\alpha,\beta,\gamma)$ on the side with the smaller γ. For the lower bound, the situation is reversed. We show that for any ordinal move a_1, there is some response for II b_1 such that both $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \theta - 1$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \theta - 1$. Then it follows that $\gamma(\alpha,\beta) \geq \theta$ because regardless of both I's ordinal move a_1 and the smallest lowering of the counter he can affect $\theta - 1$, II always has a response b_1 that insures that II can survive on whichever side, left or right, I chooses to play out the rest of $G(\alpha,\beta,\gamma)$. In other words, when θ is a successor

$$\gamma(\alpha,\beta) \leq \theta \Leftrightarrow \exists a_1 \forall b_1 (\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \theta - 1 \vee \gamma^{a_1,b_1}_{\mathsf{RHS}} \leq \theta - 1)$$

$$\gamma(\alpha,\beta) \geq \theta \Leftrightarrow \forall a_1 \exists b_1 (\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \theta - 1 \wedge \gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \theta - 1)$$

The case when θ is limit generally follows from the successor case.

$$\gamma(\alpha,\beta) \leq \theta \Leftrightarrow \exists a_1 \forall b_1 \exists \theta' < \theta (\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \theta' \vee \gamma^{a_1,b_1}_{\mathsf{RHS}} \leq \theta')$$

$$\gamma(\alpha,\beta) \geq \theta \Leftrightarrow \forall a_1 \exists b_1 \forall \theta' < \theta (\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \theta' \wedge \gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \theta')$$

CHAPTER 3

THE RANK OF FINITE GAMES

We first compute $\gamma(\alpha, \beta)$ when both $\alpha, \beta < \omega$. It should be clear that $\gamma(\alpha, \beta) \geq 0$ for all $\alpha \neq \beta$. Our first lemma computes $\gamma(\alpha, \beta)$ when $\beta = 0, 1, 2$ for any value of α.

Lemma 1. For all $\alpha \in \mathbf{ON}$,

(1) if $\alpha > 0$, then $\gamma(\alpha, 0) = 0$,

(2) if $\alpha > 1$, then $\gamma(\alpha, 1) = 1$,

(3) if $\alpha > 2$, then $\gamma(\alpha, 2) = 1$.

Proof. (1) is immediate. I simply plays arbitrarily on the nonempty side. (2) should also be clear as $\alpha \not\sim_2 \beta$ follows by I playing twice in α. (3) is similar to (2) except that in his first move, I cannot move either the left-hand endpoint in α or, if it exists, the right-hand endpoint in α (for otherwise II simply copies I's move.)

$\square$

We are now ready to compute $\gamma(k, l)$ for all integers k, l. If $k = l$, then $\gamma(k, l) = \infty$. It remains to compute $\gamma(k, l)$ for $k \neq l$. By the symmetry in the game it is enough to compute $\gamma(k, l)$ for $k > l$. Note that $\lfloor x \rfloor$ denotes the integer floor function, the greatest integer below x.

Theorem 1. For all integers $k > l$, $\gamma(k, l) = \lfloor \log_2 (l + 1) \rfloor$.

Proof. Let $k > l$ be integers. We prove $\gamma(k, l) = \lfloor \log_2 (l + 1) \rfloor$ by induction on l. Lemma 1 shows the formula holds for $l = 0, 1, 2$. Let $l \geq 3$ and assume that for all $l' < l$ and $k' > l'$ that $\gamma(k, l) = \lfloor \log_2 (l' + 1) \rfloor$. First, we show $\gamma(k, l) \leq \lfloor \log_2 (l + 1) \rfloor$ and then we show $\gamma(k, l) \geq \lfloor \log_2 (l + 1) \rfloor$.

Upper Bound. $\gamma(k,l) \le \lfloor \log_2(l+1) \rfloor$

I plays $a_1 = \left\lfloor \frac{k}{2} \right\rfloor$ in k and II responds with some $b_1 = l'$ in l where $0 \le l' \le l-1$.

Case 1. $l' < \left\lfloor \frac{l}{2} \right\rfloor$

Observe first that $l' < \left\lfloor \frac{k}{2} \right\rfloor$. So, by induction, we have $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \lfloor \log_2(l'+1) \rfloor \le \left\lfloor \log_2\left(\left\lfloor \frac{l}{2} \right\rfloor + 1\right) \right\rfloor$. Write $l = 2^{\lfloor \log_2 l \rfloor + 1} - j$ where $1 \le j \le 2^{\lfloor \log_2 l \rfloor}$. We have two subcases depending on the value of j.

Subcase 1.1. $j = 1$

First this means that l is odd so that $\left\lfloor \frac{l}{2} \right\rfloor = \left\lfloor \frac{l-1}{2} \right\rfloor = \frac{l-1}{2}$. So,

$$\begin{aligned}
\gamma_{\mathsf{LHS}}^{a_1,b_1} &\le \left\lfloor \log_2\left(\left\lfloor \frac{l}{2} \right\rfloor + 1\right) \right\rfloor \\
&= \left\lfloor \log_2\left(\frac{l-1}{2} + 1\right) \right\rfloor \\
&= \left\lfloor \log_2\left(\frac{l+1}{2}\right) \right\rfloor \\
&= \lfloor \log_2(l+1) \rfloor - 1
\end{aligned}$$

Thus, when $j = 1$, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \le \lfloor \log_2(l'+1) \rfloor \le \lfloor \log_2(l+1) \rfloor - 1$.

Subcase 1.2. $2 \le j \le 2^{\lfloor \log_2 l \rfloor}$

First observe in this case that $l \ge 4$. Now we have a similar computation as before.

$$\begin{aligned}
\gamma_{\mathsf{LHS}}^{a_1,b_1} &\le \left\lfloor \log_2\left(\left\lfloor \frac{l}{2} \right\rfloor + 1\right) \right\rfloor \\
&= \left\lfloor \log_2\left(\left\lfloor \frac{2^{\lfloor \log_2 l \rfloor + 1} - j}{2} \right\rfloor + 1\right) \right\rfloor \\
&\le \left\lfloor \log_2\left(\left\lfloor \frac{2^{\lfloor \log_2 l \rfloor + 1} - 2^{\lfloor \log_2 l \rfloor}}{2} \right\rfloor + 1\right) \right\rfloor \\
&= \left\lfloor \log_2\left(2^{\lfloor \log_2 l \rfloor - 1} + 1\right) \right\rfloor \\
&= \left\lfloor \log_2\left(2^{\lfloor \log_2 l \rfloor - 1}\right) \right\rfloor \qquad (l \ge 4)
\end{aligned}$$

$$= \lfloor \log_2 l \rfloor - 1$$

$$= \lfloor \log_2 (l+1) \rfloor - 1 \qquad (l = 2^{\lfloor \log_2 l \rfloor + 1} - j \text{ and } j \geq 2)$$

Thus, when II response is $b_1 = l' < \lfloor \frac{l}{2} \rfloor$, we have $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \lfloor \log_2 (l+1) \rfloor - 1$. Thus, $\gamma(k,l) \leq \lfloor \log_2 (l+1) \rfloor$.

Case 2. $l' = \lfloor \frac{l}{2} \rfloor$ and $k > l+1$

We still have $\lfloor \frac{l}{2} \rfloor < \lfloor \frac{k}{2} \rfloor$. So, by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \lfloor \log_2 \left(\lfloor \frac{l}{2} \rfloor + 1 \right) \rfloor$. The same computation as above shows that $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \lfloor \log_2 (l+1) \rfloor - 1$. Thus, $\gamma(k,l) \leq \lfloor \log_2 (l+1) \rfloor$.

Case 3. $l' = \lfloor \frac{l}{2} \rfloor$ and $k = l+1$ or $l' > \lfloor \frac{l}{2} \rfloor$

In either of these two cases we now have $l - l' < \lfloor \frac{k}{2} \rfloor$. So, by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \lfloor \log_2 (l - l' + 1) \rfloor \leq \lfloor \log_2 \left(\lfloor \frac{l}{2} \rfloor + 1 \right) \rfloor$. The same computation as above now shows that $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq \lfloor \log_2 (l+1) \rfloor - 1$. Thus, $\gamma(k,l) \leq \lfloor \log_2 (l+1) \rfloor$.

So when I plays $a_1 = \lfloor \frac{k}{2} \rfloor$ in k, for every response for II b_1 in l, we have $\gamma(k,l) \leq \lfloor \log_2 (l+1) \rfloor$.

Lower Bound. $\gamma(k,l) \geq \lfloor \log_2 (l+1) \rfloor$

Case 1. I plays $a_1 = l'$ in l

II response depends on the location of a_1 with respect to the midpoint of l.

Subcase 1.1. $a_1 = l' \leq \lfloor \frac{l}{2} \rfloor$

Then II responds with $b_1 = l'$ in k. On the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$. On the right, by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \lfloor \log_2 (l - l' + 1) \rfloor$. Now $l - l' \geq \lfloor \frac{l}{2} \rfloor$. So $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \lfloor \log_2 \left(\lfloor \frac{l}{2} \rfloor + 1 \right) \rfloor$ and the same computation as above shows that $\lfloor \log_2 \left(\lfloor \frac{l}{2} \rfloor + 1 \right) \rfloor = \lfloor \log_2 (l+1) \rfloor - 1$. Thus, $\gamma(k,l) \geq \gamma^{a_1,b_1}_{\mathsf{RHS}} + 1 = \lfloor \log_2 (l+1) \rfloor$.

Remark 1. This strategy for II will be used in future arguments. Whenever II responds with a move $a_1 = b_1$ that gives an $\infty-$game on the left, we will simply say that II **copies from below** (*See* Figure 3.1.).

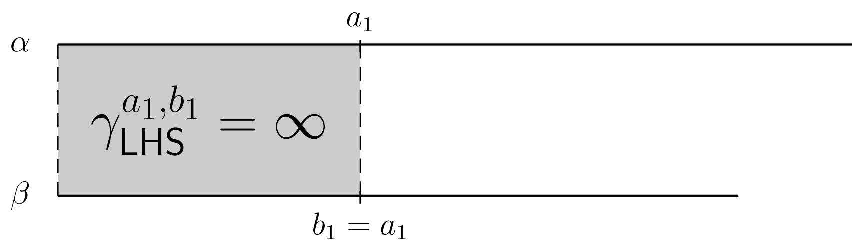

Figure 3.1. II copies from below

Subcase 1.2. $a_1 = l' > \left\lfloor \frac{l}{2} \right\rfloor$

Then II responds with $b_1 = k - (l - l')$ in k. Now on the right $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \lfloor \log_2 (l' + 1) \rfloor$. Since $l' > \left\lfloor \frac{l}{2} \right\rfloor$, we have $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(\left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \right\rfloor = \lfloor \log_2 (l + 1) \rfloor - 1$. Thus, $\gamma(k, l) \geq \gamma_{\mathsf{LHS}}^{a_1,b_1} + 1 = \lfloor \log_2 (l + 1) \rfloor$.

Remark 2. This strategy for II will also be used in future arguments. Whenever II responds with some b_1 so that the game on the right is an $\infty-$game, we will simply say that II **copies from above**. (*See* Figure 3.2.)

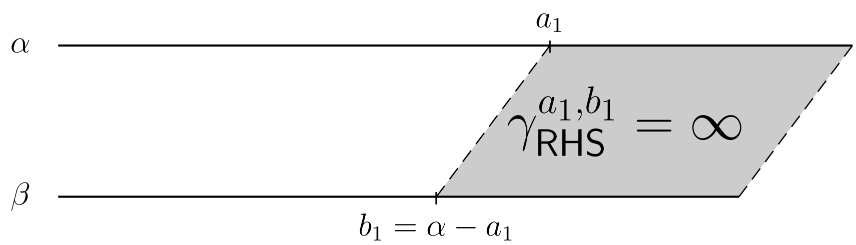

Figure 3.2. II copies from above

So if I plays any a_1 in l, II has a response b_1 in k that insures $\gamma(k,l) \geq \lfloor \log_2 (l+1) \rfloor$.

Case 2. I plays $a_1 = k'$ in k

Now II's response depends on the location of $a_1 = k'$ within k.

Subcase 2.1. $a_1 = k' \leq \left\lfloor \frac{l}{2} \right\rfloor$

Then II responds by copying from below playing $b_1 = a_1$ in l. The argument is the same as above when I played $a_1 = l' \leq \left\lfloor \frac{l}{2} \right\rfloor$ in l.

Subcase 2.2. $a_1 = k' \geq k - \left\lfloor \frac{l}{2} \right\rfloor$

Then II responds by copying from above playing $b_1 = l - (k - k')$ in l. The argument is the same as above when I played $a_1 = l' > \left\lfloor \frac{l}{2} \right\rfloor$ in l.

Subcase 2.3. $\left\lfloor \frac{l}{2} \right\rfloor < a_1 < k - \left\lfloor \frac{l}{2} \right\rfloor$

Then II plays $b_1 = \left\lfloor \frac{l}{2} \right\rfloor$, the midpoint of l. Both $\gamma_{\mathsf{LHS}}^{a_1,b_1}$ and $\gamma_{\mathsf{RHS}}^{a_1,b_1}$ are computed by induction and the same computations show that both $\gamma_{\mathsf{LHS}}^{a_1,b_1}, \gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \lfloor \log_2 (l+1) \rfloor - 1$. Thus, $\gamma(k,l) \geq \lfloor \log_2 (l+1) \rfloor$.

So if I plays any a_1 in k, II has a response b_1 in l that insures $\gamma(k,l) \geq \lfloor \log_2 (l+1) \rfloor$.

$\square$

Remark 3. From the proof of Theorem 1, an optimal strategy for playing the integer game $G(k,l)$ emerges. Namely, both players play their respective midpoints with I always choosing the longer side first. For future reference, we denote this method of play for either player as the **midpoint strategy**.

CHAPTER 4

THE RANK OF GENERAL TRANSFINITE GAMES

Having computed the $\gamma(\alpha, \beta)$ for finite values of both α, β, we are ready to compute $\gamma(\alpha, \beta)$ when at least one of $\alpha, \beta \geq \omega$.

4.1. Trivial Transfinite Games

Our first lemma computes $\gamma(\alpha, \beta)$ whenever exactly one of either α or β is finite or whenever one of either α or β has a finite part that the other does not.

LEMMA 2. Suppose λ, λ' are limit ordinals and that $n, m \in \omega$. Then

(1) $\gamma(\lambda, n) = 1$ for $n > 0$

(2) $\gamma(\lambda + n, \lambda') = 1$ for $n > 0$

(3) $\gamma(\lambda + n, m) = 2$ for $n > 0$ and $m > 2$

PROOF. Refer to Figure 4.1. For (1), I plays $n - 1$ in n. II must respond with some b_1 in λ where $b_1 < \lambda$. In his second move I plays $b_1 + 1$ in λ. II cannot respond and loses. A similar argument for (2) shows that after I plays $\lambda + (n - 1)$ in $\lambda + n$ in his first move and II responds with b_1 in λ', I defeats II by playing $b_1 + 1$ in λ' in his second move. For (3), I plays λ in $\lambda + n$. II must respond with some b_1 in m. If $b_1 = 0$ or $b_1 = m - 1$, then II loses immediately. Otherwise, if $0 < b_1 < m - 1$, then I plays $a_2 = b_1 - 1$ and II repsonds with some $b_2 < \lambda$. Then $a_3 = b_2 + 1$ is a win for I. $\square$

We will refer to the games (1) and (2) from Lemma 2 as **trivially separated**, and a game like (3) as **trivially unbalanced**. Generalizing these notions will prove useful in the sequel. We can summarize Lemma 2 by observing that when $G(\alpha, \beta)$ is trivially separated, $\gamma(\alpha, \beta) = 1$ and when $G(\alpha, \beta)$ is trivially unbalanced, $\gamma(\alpha, \beta) = 2$.

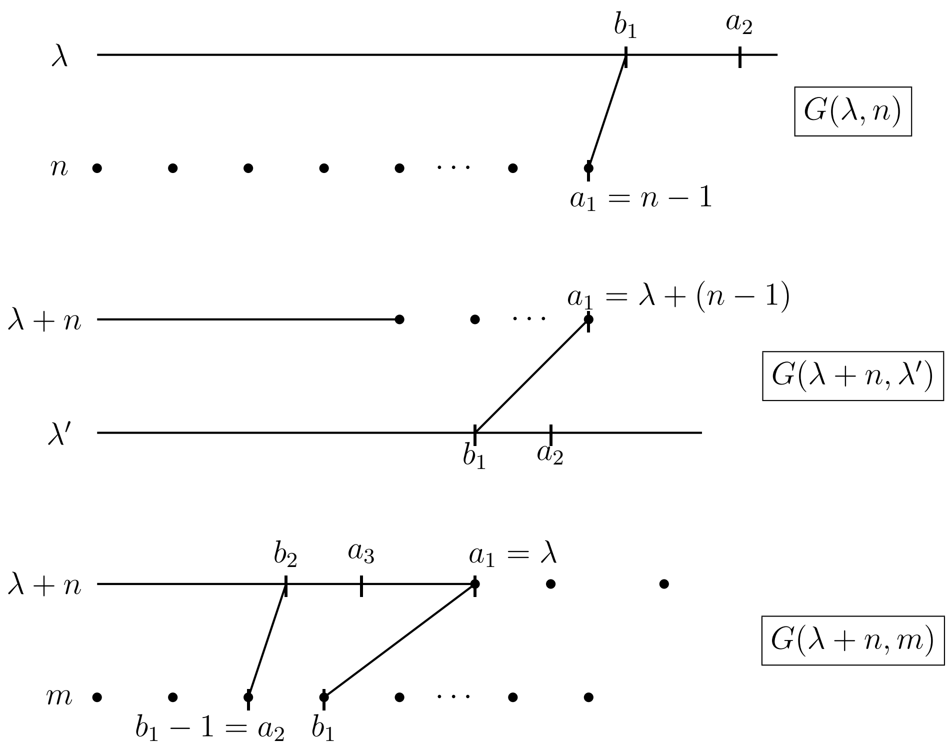

FIGURE 4.1. Trivial transfinite games

When both α and β are infinite and have a nonempty nonequal finite part, we can compute an upper bound for $\gamma(\alpha, \beta)$. The reader should note that the computation is similar to the proof of the upper bound in the proof of the finite formula for $\gamma(k, l)$.

LEMMA 3. Suppose λ, λ' are limit ordinals and that $n > m > 0$. Then

$$\gamma(\lambda + n, \lambda' + m) \leq \lfloor \log_2 (m+4) \rfloor$$

PROOF. Let $\alpha = \lambda + n$, $\beta = \lambda' + m$ where $n > m > 0$. We prove the upper bound holds by induction on m. Since $\gamma(\alpha, \beta) \leq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\}$, we must show that there is a move a_1 for I such that for every response b_1 for II either $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq \lfloor \log_2 (m+4) \rfloor - 1$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq \lfloor \log_2 (m+4) \rfloor - 1$. We argue the cases $m = 1, 2, 3$ individually.

I plays the same move $a_1 = \lambda'$ in β for $m = 1, 2, 3$, and II responds with some b_1 in α.

Suppose $m = 1$. If $b_1 = \lambda + (n-1)$, then $G^{a_1,b_1}_{\mathsf{LHS}} = G(\lambda + (n-1), \lambda)$ is trivially separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 1$. If $b_1 < \lambda + (n-1)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha', 0)$ for some $1 \leq \alpha' \leq \alpha$. By Lemma 1, $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 0$. In all cases for II's response b_1, we have $\gamma(\alpha, \beta) \leq 2 = \lfloor \log_2 (1+4) \rfloor$.

Suppose $m = 2$. If $b_1 = \lambda + (n-1)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(1, 0)$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 0$ again by Lemma 1. If $b_1 = \lambda + (n-2)$, then $G^{a_1,b_1}_{\mathsf{LHS}} = G(\lambda + (n-2), \lambda)$ is trivially separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 1$. If $b_1 < \lambda + (n-2)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha', 1)$ for some $2 \leq \alpha' \leq \alpha$ so that $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 1$, again by Lemma 1. In all cases, we have $\gamma(\alpha, \beta) \leq 2 = \lfloor \log_2 (2+4) \rfloor$.

Suppose $m = 3$. If $b_1 = \lambda + (n-1)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(2, 0)$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 0$ as before. If $b_1 = \lambda + (n-2)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(2, 1)$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 1$ by Lemma 1. If $b_1 = \lambda + (n-3)$, then $G^{a_1,b_1}_{\mathsf{LHS}} = G(b_1, \lambda')$ is trivially separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 1$. If $b_1 < \lambda + (n-3)$, then $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha', 2)$ for some $3 \leq \alpha' \leq \alpha$ again by Lemma 1. In all cases, we have $\gamma(\alpha, \beta) \leq 2 = \lfloor \log_2 (3+4) \rfloor$.

For $m \geq 4$, assume that for all $m' < m$ and all $n' > m'$ that $\gamma(\lambda + n', \lambda' + m') \leq \lfloor \log_2 (m'+4) \rfloor$. I plays $\lambda + (m - 2^{\lfloor \log_2 m \rfloor}) + 1$ in α and II responds with some b_1 in β. If $b_1 < \lambda'$, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is trivially unbalanced and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2$. Thus, $\gamma(\alpha, \beta) \leq 3 \leq \lfloor \log_2 (m+4) \rfloor$. If $b_1 = \lambda'$, then $G^{a_1,b_1}_{\mathsf{RHS}}$ is trivially separated. Thus, $\gamma(\alpha, \beta) \leq 2 < \lfloor \log_2 (m+4) \rfloor$. Now suppose $b_1 = \lambda' + m'$ for some $1 \leq m' < m$. There are two cases:

(1) $1 \leq m' \leq m - 2^{\lfloor \log_2 m \rfloor}$ or

(2) $m - 2^{\lfloor \log_2 m \rfloor} + 1 \leq m' < m$

In the first case, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \lfloor \log_2 (m'+4) \rfloor$ by induction. We claim that

$$\lfloor \log_2 (m'+4) \rfloor \leq \lfloor \log_2 (m+4) \rfloor - 1$$

Assuming the claim holds, we then have in this first case $\gamma(\alpha, \beta) \leq \gamma^{a_1,b_1}_{\mathsf{LHS}} + 1 \leq \lfloor \log_2 (m+4) \rfloor$.

Proof (claim). Write $m = 2^{\lfloor \log_2 m \rfloor + 1} - j$ where $0 < j \leq 2^{\lfloor \log_2 m \rfloor}$. By hypothesis,

$$m' \leq m - 2^{\lfloor \log_2 m \rfloor} = 2^{\lfloor \log_2 m \rfloor + 1} - j - 2^{\lfloor \log_2 m \rfloor} = 2^{\lfloor \log_2 m \rfloor} - j$$

and hence

$$m' + 4 \le 2^{\lfloor \log_2 m \rfloor} + (4 - j)$$

Now if $1 \le j \le 4$, then

$$\lfloor \log_2 (m' + 4) \rfloor \le \lfloor \log_2 m \rfloor = \lfloor \log_2 (m + 4) \rfloor - 1$$

On the other hand, if $4 < j \le 2^{\lfloor \log_2 m \rfloor}$, then

$$\lfloor \log_2 (m' + 4) \rfloor \le \lfloor \log_2 m \rfloor - 1 = \lfloor \log_2 (m + 4) \rfloor - 1$$

This proves the claim.

Now suppose that $m - 2^{\lfloor \log_2 m \rfloor} + 1 \le m' < m$. Then $G^{a_1,b_1}_{\mathsf{RHS}}$ is a finite versus finite game. By the finite game formula,

$$\gamma^{a_1,b_1}_{\mathsf{RHS}} = \lfloor \log_2 (m - m') \rfloor \le \left\lfloor \log_2 \left(2^{\lfloor \log_2 m \rfloor} - 1 \right) \right\rfloor = \lfloor \log_2 m \rfloor - 1$$

Thus, $\gamma(\alpha, \beta) \le \gamma^{a_1,b_1}_{\mathsf{RHS}} + 1 \le \lfloor \log_2 m \rfloor \le \lfloor \log_2 (m + 4) \rfloor$.

□

4.2. The Separated CNF Game

Recall that for every ordinal α there are unique ordinals $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ and unique nonzero integers $k_1, \ldots, k_n$ such that

$$\alpha = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$$

This unique decomposition is called the **Cantor Normal Form (CNF)** of α. We will refer to each term of the CNF of α as the α_i-block, or if the power is clear, simply the i^{th} block. If α has only one term in its CNF, i.e. $n = 1$, then α is a **monomial**. A monomial having a coefficient of 1 is **monic**. We say that α_n, the least power in the CNF of an ordinal, is the **terminal** power of α.

We fix the following terminology and notation for any ordinal α written in CNF as above. For $1 \le i \le n$ define

$$\Phi^{\alpha}_i = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_i} \cdot k_i$$

the sum of the first i blocks of the CNF of α. Consider a single α_i-block for $\alpha_i > 0$. We refer to the endpoints of a given block as the left and right **fences** of the i^{th}-block and the multiples of $\omega^{\alpha_i} \cdot k'$ as the **holes**. (*See* Figure 4.2.) We do not consider the left fence in Φ_1^α a true fence since this equals zero.

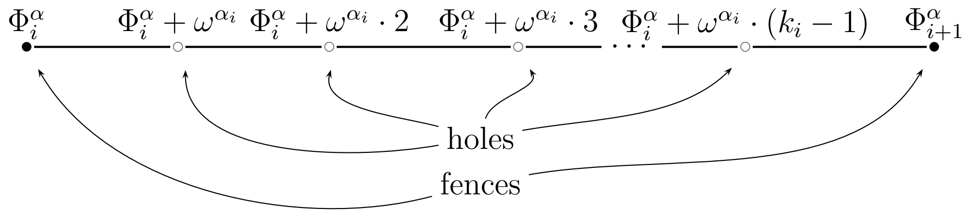

FIGURE 4.2. Fences and holes in the i^{th}-block

Let $\alpha = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$ and $\beta = \omega^{\beta_1} \cdot l_1 + \cdots + \omega^{\beta_m} \cdot l_m$ be written in CNF. If $\alpha_n \neq \beta_m$, then we say α and β are **separated**. THe next theorem can be viewed as a generalization of parts (1) and (2) of Lemma 2.

THEOREM 2 (The Separated Game formula). Let $\alpha = \omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$ and $\beta = \omega^{\beta_1} \cdot l_1 + \cdots + \omega^{\beta_m} \cdot l_m$ be written in CNF and $\alpha_n > \beta_m$. Then we have

$$\gamma(\alpha, \beta) = \begin{cases} 2\beta_1 & \text{if } \beta \text{ is a monic monomial} \\ 2\beta_m + 1 & \text{otherwise} \end{cases}$$

A symmetric formula holds for $\alpha_n < \beta_m$.

PROOF. Let α, β be as above. We prove the result by induction on the CNF of β.

CASE 3. β is a monic monomial, i.e., $m = 1, l_1 = 1$

UPPER BOUND. $\gamma(\alpha, \beta) \leq 2\beta_1$

I plays β in α and II responds with some b_1 in β. Now $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated. By induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\beta' + 1$ where β' is the terminal power of b_1. If β_1 is a successor, then $\beta' \leq \beta_1 - 1$;

otherwise, if β_1 is limit, then $\beta' < \beta_1$. Either way, $\gamma(\alpha, \beta) \leq \gamma^{a_1,b_1}_{\mathsf{LHS}} + 1 = 2\beta' + 2 \leq 2(\beta_1 - 1) + 2 = 2\beta_1$.

Lower Bound. $\gamma(\alpha, \beta) \geq 2\beta_1$

If I plays a_1 in β or a_1 in α for some $a_1 < \beta$, then II copies from below. This gives $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \gamma(\alpha, \beta)$, and this move for I does not gain anything for I.

Remark 4. For future reference, whenever a_1 is such that there is a b_1 such that either $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \gamma(\alpha, \beta)$ or vice versa, then we say that a_1 is a **stalling move** for I.

So assume I plays $a_1 \geq \beta$ in α and let the CNF of $a_1 = \omega^{\delta_1} \cdot p_1 + \cdots + \omega^{\delta_r} \cdot p_r$. Among the $\{\delta_i\}_{1 \leq i \leq r}$, identify all of the powers greater than or equal to β_1 as $\delta_1^* = \delta_1, \ldots, \delta_i^* = \delta_i$ for some $1 < i \leq r$. That is, i is the largest index such that $\delta_i \geq \beta_1$. Assuming for the moment that β_1 is a successor, II responds to a_1 in α with b_1 in β where

$$b_1 = \omega^{\beta_1 - 1} \cdot p_1' + \cdots + \omega^{\beta_1 - 1} \cdot p_i' + \omega^{\delta_{i+1}} \cdot p_{i+1} + \cdots + \omega^{\delta_r} \cdot p_r$$

where for $1 \leq j \leq i$, $p_j' = 2$ if $p_j = 1$ and $p_j' = p_j$ otherwise. (*See* Figure 4.3.) Thus, II copies

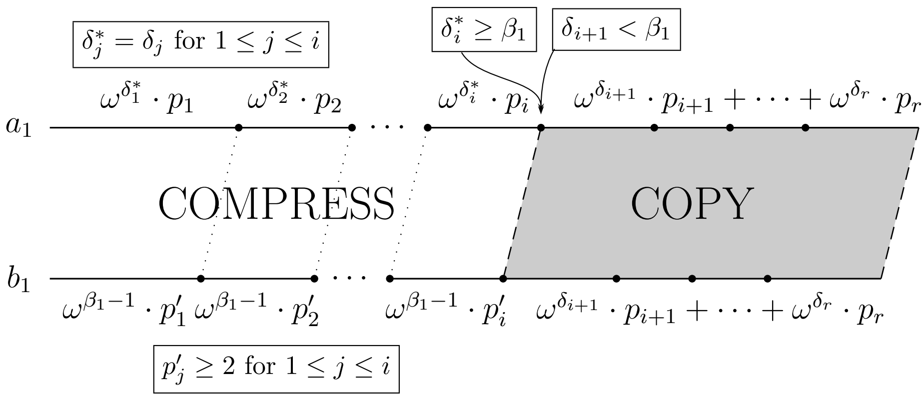

Figure 4.3. A $(\beta_1 - 1)$-compressed copy of a_1

what parts of the CNF of a_1 that he can, namely all of the powers of a_1 which are $\beta_1 - 1$ or less. Note that for a_1 with large ($\geq \beta_1$) terminal power, there is no copied part. On the rest of a_1, II *compresses* the data in all of the δ_i^*, the powers larger than $\beta_1 - 1$, into a number of blocks (at least 2) of the highest power that he has, $\beta_1 - 1$.

Now we have two games: $G_{\mathsf{LHS}}^{a_1,b_1} = G(a_1, b_1)$ and $G_{\mathsf{RHS}}^{a_1,b_1} = G(\alpha - a_1, \beta - b_1)$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is at worst $G(\alpha', \beta)$ where $\alpha' \geq \omega^{\alpha_n}$. This is still a separated game and thus $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \gamma(\alpha, \beta)$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is comprised of r-many subgames each one corresponding to a block in the CNF of a_1. On the blocks $\omega^{\delta_{i+1}}, \ldots, \omega^{\delta_r}$, $\gamma = \infty$ since each is a copying move. On the $\omega^{\delta_1^*}, \ldots, \omega^{\delta_i^*}$ blocks, these games are all separated and the limiting factor in the separated formula is II's response: $\omega^{\beta_1 - 1} \cdot p_i'$. Since II played at least two copies of $\omega^{\beta_1 - 1}$ in each block, II can last at least $2(\beta_1 - 1) + 1$ many moves in each of these subgames by induction. Therefore, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2(\beta_1 - 1) + 1$ and thus, $\gamma(\alpha, \beta) \geq 2\beta_1$.

If β_1 is a limit, we must show that for any $\gamma' < 2\beta_1$, $\gamma(\alpha, \beta) \geq \gamma'$. This is easily accomplished by a similar argument as above, except that in the compressed part of II's response b_1, the $\beta_1 - 1$ are replaced by some sufficiently large $\beta' < \beta_1$. This ends Case 1.

Remark 5. For future reference, we will call this strategy by II **data compression** (Fig. 4.3), where II responds with b_1 to I's move a_1 by playing a number of copies of II's highest power followed by some copied blocks of lower powers, depending on the CNF of a_1. If we want to emphasize that largest power η of b_1, we call b_1 an **η-compressed copy** of a_1. So in the previous argument when β_1 is a successor, b_1 is a $(\beta_1 - 1)$-compressed copy of a_1.

Case 4. β is not a monic monomial

So in this case, we have $\beta = \omega^{\beta_1} \cdot l_1 + \cdots + \omega^{\beta_m} \cdot l_m$ where either $m = 1$ and $l_1 = l_m > 1$ or $m > 1$. In either case the argument is the same.

Upper Bound. $\gamma(\alpha, \beta) \leq 2\beta_m + 1$, where β_m is terminal

I plays the last hole in β. That is, if β is a monomial, I plays $\omega^{\beta_1} \cdot (l_1 - 1)$ in β. If β is not a monomial, I plays $\omega^{\beta_1} \cdot l_1 + \cdots + \omega^{\beta_m} \cdot (l_m - 1)$ in β. In either case, II must respond with some b_1 in α. Now $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha', \omega^{\beta_m})$ where at worst $\alpha' \geq \omega^{\alpha_n}$. (Fig. 4.4) This game is separated on the right, and by induction, $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\beta_m$ and hence $\gamma(\alpha, \beta) \leq 2\beta_m + 1$.

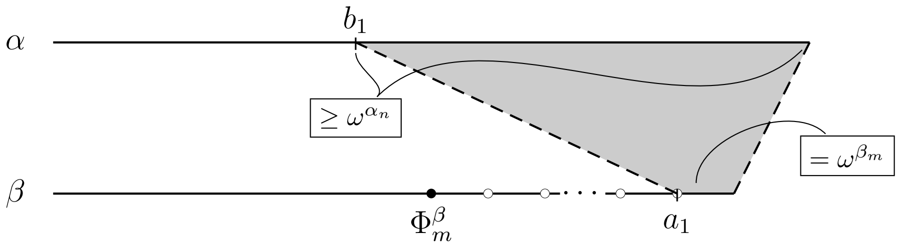

FIGURE 4.4. Pinching off a block

REMARK 6. For future reference, we will call this strategy for I **pinching off a block** where I plays the largest possible move that leaves a single block on the right. Note that, however, when I pinches off a block, it is not necessary that the resulting game be separated.

LOWER BOUND. $\gamma(\alpha, \beta) \geq 2\beta_m + 1$, where β_m is terminal

If I opens with either a_1 in β or $a_1 < \beta$ in α, then II copies from below and I has made a stalling move. Otherwise, $a_1 \geq \beta$ in α with $a_1 = \omega^{\delta_1} \cdot p_1 + \cdots + \omega^{\delta_r} \cdot p_r \geq \beta$ and II responds with b_1 in β where b_1 is a β_m-compressed copy of a_1:

$$b_1 = \omega^{\beta_m} \cdot l_1 + \cdots + \omega^{\beta_m} \cdot (l_m - 1) + \omega^{\delta_{i+1}} \cdot p_{i+1} + \cdots + \omega^{\delta_r} \cdot p_r$$

We let be $i+1$ the smallest index so that $\delta_{i+1} < \beta_m$ and thus $\omega^{\delta_{i+1}} \cdot p_{i+1} + \cdots + \omega^{\delta_r} \cdot p_r < \omega^{\beta_m}$. This makes $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha - a_1, \omega^{\beta_m})$. Since this game is separated, by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq 2\beta_m$. On the left, we have $(r-i)+1$-many subgames. Each game corresponding to the CNF of a_1 is an ∞-game while the game on the far left is separated, and hence covered by the induction hypothesis, $\gamma \geq 2\beta_m$. In all cases we have $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_m$. So, $\gamma(\alpha, \beta) \geq 2\beta_m + 1$.

□

Henceforth, we assume that α, β are not separated so that the terminal powers of α and β are equal.

4.3. The Pure Monomial Game

As a generalization of the finite game $G(k, l) = G(\omega^0 \cdot k, \omega^0 \cdot l)$, consider $G(\omega^\delta \cdot k, \omega^\delta \cdot l)$ when $\delta > 0$ and k, l are nonzero integers. We identify this particular game as the **pure monomial game**. In the pure monomial game, we view the holes of $G(\omega^\delta \cdot k, \omega^\delta \cdot l)$ as the points in $G(k-1, l-1)$.

Lemma 4. Let $\alpha = \omega^\delta \cdot k$ and $\beta = \omega^\delta \cdot l$ where $\delta > 0$ and $k \neq l$ are nonzero integers. Then

$$\gamma(\alpha, \beta) = 2\delta + \lfloor \log_2 (k \wedge l) \rfloor$$

Proof. Let $\alpha = \omega^\delta \cdot k$ and $\beta = \omega^\delta \cdot l$ be as above. Clearly the formula is symmetric in k and l, so without loss of generality assume $k > l$. We prove the result by induction on l.

Upper Bound. $l = 1$

We show that $\gamma(\alpha, \beta) \leq 2\delta + \lfloor \log_2 l \rfloor = 2\delta$. Observe that for $k > l = 1$, α has at least one hole, but β has none. So I plays $a_1 = \omega^\delta \cdot 1$ the first hole in α and II responds with some b_1 in β. Now the terminal power of b_1 is $< \delta$, so $G^{a_1,b_1}_{\mathsf{LHS}}$ is necessarily separated. If δ is a successor, then $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2(\delta - 1) + 1$ by the Separated Game formula (Lemma 2) so that $\gamma(\alpha, \beta) \leq 2(\delta - 1) + 2 = 2\delta$. If δ is a limit, then $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\delta' + 1$ for some $\delta' < \delta$, again by the Separated Game formula. Thus, $\gamma(\alpha, \beta) \leq 2\delta$.

Remark 7. This situation occurs often, and we make the follwing definition. Suppose in some $G(\alpha, \beta)$, I plays a_1 which has terminal power some η. We call an **η-descent** any response b_1 for II such that the terminal power of b_1 is some $\eta' < \eta$. It follows that $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and by the Separated Game formula, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\eta' + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\eta' + 2 \leq 2\eta$. So in the above case $l = 1$, every response b_1 for II is a δ-descent.

Lower Bound. $l = 1$

We show that for $l = 1$, $\gamma(\alpha, \beta) \geq 2\delta + \lfloor \log_2 l \rfloor = 2\delta$. If a_1 is in β, then II responds by copying from below with $b_1 = a_1$ in α. On the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$. On the right, $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha, \beta)$

so this a_1 is a stalling move for I. Similarly, if I plays $a_1 < \omega^\delta \cdot 1$ in α, this is stalling for I. So suppose I plays $a_1 \geq \omega^\delta \cdot 1$ in α. II responds by playing b_1 in β, a δ'-compression of a_1 where, depending on whether or not δ is a limit or successor, $\delta' < \delta$ is as in the proof of the lower bound of the Separated Game formula. In either case, using an identical argument from Lemma 2, $\gamma(\alpha, \beta) \geq 2\delta$. Thus, we have for $l = 1$, $\gamma(\alpha, \beta) = 2\delta = 2\delta + \lfloor \log_2 l \rfloor$.

Now let $l > 1$ and assume for all $l' < l$ and $k > l'$

$$\gamma(\omega^\delta \cdot k, \omega^\delta \cdot l') = 2\delta + \lfloor \log_2 l' \rfloor$$

Upper Bound. $l > 1$

Notice that $G(\alpha, \beta) = G(\omega^\delta \cdot k, \omega^\delta \cdot l)$ looks like the finite game $G(k-1, l-1)$ and we argue similarly as in the proof of the Finite Game formula (Lemma 1). First, we show $\gamma(\alpha, \beta) \leq 2\delta + \lfloor \log_2 l \rfloor$. I plays the "midpoint" hole $\omega^\delta \cdot \left\lfloor \frac{k}{2} \right\rfloor$ and II responds with some b_1 in β. Observe that any b_1 that is not a hole in β is a δ-descent and thus, for such b_1, $\gamma(\alpha, \beta) \leq 2\delta \leq 2\delta + \lfloor \log_2 l \rfloor$. So suppose $b_1 = \omega^\delta \cdot l'$ is a hole in β where $1 \leq l' < l$. This b_1 then splits β into l' many copies of ω^δ on the left and $l - l'$ many copies on the right:

$$\beta = \omega^\delta \cdot l = \omega^\delta \cdot l' + \omega^\delta \cdot (l - l')$$

Let $\hat{l} = \min\{l', l - l'\}$. If $l' < l - l'$, then we have $\left\lfloor \frac{k}{2} \right\rfloor > l'$. Thus, by induction,

$$\begin{aligned} \gamma_{\mathsf{LHS}}^{a_1, b_1} &= 2\delta + \left\lfloor \log_2 \hat{l} \right\rfloor \\ &\leq 2\delta + \left\lfloor \log_2 \left\lfloor \frac{l}{2} \right\rfloor \right\rfloor \\ &= 2\delta + (\lfloor \log_2 l \rfloor - 1) \end{aligned}$$

So $\gamma(\alpha, \beta) \leq 2\delta + \lfloor \log_2 l \rfloor$. If $l' = l - l'$ and $\left\lfloor \frac{k}{2} \right\rfloor \neq l'$, then $\gamma_{\mathsf{LHS}}^{a_1, b_1}$ computes the same, and we again have $\gamma(\alpha, \beta) \leq 2\delta + \lfloor \log_2 l \rfloor$. If $l' = l - l'$ and $\left\lfloor \frac{k}{2} \right\rfloor = l'$ (which can only occur when k is odd and $k = l + 1$) or if $l - l' > l'$, then by induction and a similar computation as above $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\delta + (\lfloor \log_2 l \rfloor - 1)$. So, $\gamma(\alpha, \beta) \leq 2\delta + \lfloor \log_2 l \rfloor$.

Lower Bound. $l > 1$

We show that $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. First suppose that I plays a hole in β, $a_1 = \omega^\delta \cdot l'$ for some $1 \leq l' \leq l-1$. If $1 \leq l' \leq \left\lfloor \frac{l}{2} \right\rfloor$, then II copies from below and plays $b_1 = \omega^\delta \cdot l'$ in α. On the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$. On the right, by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\delta + \lfloor \log_2 (l-l') \rfloor \geq 2\delta + \left\lfloor \log_2 \left\lfloor \frac{l}{2} \right\rfloor \right\rfloor = 2\delta + \lfloor \log_2 l \rfloor - 1$. So $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. If on the other hand $\left\lfloor \frac{l}{2} \right\rfloor < l' \leq l-1$, then II copies from above playing $b_1 = \omega^\delta \cdot (k-(l-l'))$. Now on the right $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \infty$. On the left, by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\delta + \lfloor \log_2 l' \rfloor \geq 2\delta + \left\lfloor \log_2 \left\lfloor \frac{l}{2} \right\rfloor \right\rfloor = 2\delta + \lfloor \log_2 l \rfloor - 1$. So $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. Now suppose I plays a hole in α, $a_1 = \omega^\delta \cdot k'$ for some $1 \leq k' \leq k-1$. If $1 \leq k' \leq \left\lfloor \frac{l}{2} \right\rfloor$, then II copies from below playing $b_1 = \omega^\delta \cdot k'$ in β. The computation is the same as above and it follows that $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. If $k - \left\lfloor \frac{l}{2} \right\rfloor \leq k' \leq k-1$, then II copies from above playing $b_1 = \omega^\delta \cdot (l-(k-k'))$. Now on the right $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \infty$. On the left, by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\delta + \lfloor \log_2 (l-(k-k')) \rfloor \geq 2\delta + \left\lfloor \log_2 \left\lfloor \frac{l}{2} \right\rfloor \right\rfloor = 2\delta + (\lfloor \log_2 l \rfloor - 1)$. So, $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. If $\left\lfloor \frac{l}{2} \right\rfloor < k' < k - \left\lfloor \frac{l}{2} \right\rfloor$, then II plays the "midpoint" hole in β, $b_1 = \omega^\delta \cdot \left\lfloor \frac{l}{2} \right\rfloor$. Now on the left by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\delta + \left\lfloor \log_2 \left\lfloor \frac{l}{2} \right\rfloor \right\rfloor = 2\delta + (\lfloor \log_2 l \rfloor - 1)$. On the right, if $k - k' = l - \left\lfloor \frac{l}{2} \right\rfloor$, then $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \infty$. Otherwise, by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\delta + \left\lfloor \log_2 \left(l - \left\lfloor \frac{l}{2} \right\rfloor\right) \right\rfloor \geq 2\delta + (\lfloor \log_2 l \rfloor - 1)$. In any case, $\gamma(\alpha,\beta) \geq 2\delta + \lfloor \log_2 l \rfloor$. This exhausts all possibilities for I playing a_1 that is a hole in either α or β.

Now suppose a_1 is not a hole in either α or β. If $a_1 < \omega^\delta \cdot 1$ in either α or β, then II copies from below playing $b_1 = a_1$. This a_1 is then easily seen to be a stalling move for I. So a_1 is of the form $\omega^\delta \cdot p + \eta$ where $\eta < \omega^\delta$ and p is some integer less than k or l depending on what side I plays. II responds by playing $b_1 = \omega^\delta \cdot p' + \eta$ where is the same hole that he would have in the previous paragraph plus a copy of the small tail η. We claim that the presence of the tail η does not decrease the lower bound.

Proof (claim). Let $\overline{a_1}, \overline{b_1}$ be the untailed versions of the above moves a_1, b_1, respectively. On the left, using a compression-type argument as in the Separated Game formula, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \gamma^{\overline{a_1},\overline{b_1}}_{\mathsf{LHS}}$. On the right, $G^{a_1,b_1}_{\mathsf{RHS}} = G(\alpha - a_1, \beta - b_1) = G(\alpha - \overline{a_1}, \beta - \overline{b_1})$ so that $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \gamma^{\overline{a_1},\overline{b_1}}_{\mathsf{RHS}}$.

□

4.4. The Common CNF Game

Toward the final formula for those α, β which are not separated, we identify the **common part** of their CNFs as

$$\alpha = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_n} \cdot k_n$$

$$\beta = \Phi_0^\beta + \omega^{\gamma_1} \cdot l_1 + \cdots + \omega^{\gamma_n} \cdot l_n$$

where the CNFs of $\Phi_0^\alpha, \Phi_0^\beta$ are separated. We allow the possibility that one or both of $\Phi_0^\alpha, \Phi_0^\beta$ may be empty. In case they are not empty, identify the terminal terms of Φ_0^α and Φ_0^β as $\omega^{\alpha_0} \cdot k_0$ and $\omega^{\beta_0} \cdot l_0$, respectively. We will ultimately prove that $\gamma(\alpha, \beta)$ is the minimum of finitely many ordinal terms τ_i, $0 \leq i \leq n$ where each τ_i corresponds to a block in the common CNF of α and β (Fig. 4.5), as follows:

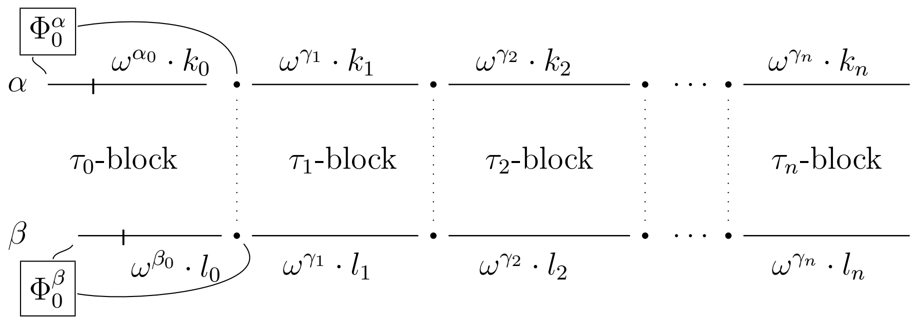

Figure 4.5. Common Cantor Normal Form

4.4.1. $n = 1$

To simplify the exposition, we first consider the case where $n = 1$. That is, the common CNFs of α, β have one block of the same power and one separated block on the left. As we said before, one of the $\Phi_0^\alpha, \Phi_0^\beta$ may be empty (if both are empty, this is just the pure monomial game). Our next lemma computes $\gamma(\alpha, \beta)$ whenever exactly one of Φ_0^α or Φ_0^β are nonempty. For future reference we will call this game the **unbalanced game**. This lemma can be viewed as a generalization of part (3) of Lemma 2.

LEMMA 5 (The Unbalanced Game formula). Let $\alpha > \beta$ be written in common CNF: $\alpha = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1$, $\beta = \omega^{\gamma_1} \cdot l_1$ where $\gamma_1 > 0$. Then

$$\gamma(\alpha, \beta) = \begin{cases} 2\gamma_1 & l_1 = 1 \\ 2\gamma_1 + 1 & l_1 = 2, 3 \\ 2\gamma_1 + 2 & l_1 \geq 4 \end{cases}$$

PROOF. Let α, β be as above. We prove the result by induction on l_1 and we argue the cases $l_1 = 1, 2, 3$ individually. Note that when we identify a game as either separated or pure monomial, we expect the reader to understand that we are using the formulas from Theorem 2 and Lemma 4.

CASE 1. $l_1 = 1$

UPPER BOUND. $\gamma(\alpha, \beta) \leq 2\gamma_1$

I plays Φ_0^α in α and II responds with some b_1 in β. Observe that the terminal power of b_1 must be $< \gamma_1$, and thus is a descending move for II. As we have argued before, whether γ_1 is limit or successor, $G_{\mathsf{LHS}}^{a_1, b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1, b_1} \leq 2\gamma' + 1$ for some $\gamma' < \gamma$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1$.

LOWER BOUND. $\gamma(\alpha, \beta) \geq 2\gamma_1$

Suppose I plays a_1. Any a_1 in β or any $a_1 < \omega^{\gamma_1}$ in α is easily seen to be a stalling move for I: II copies from below playing $b_1 = a_1$ in α. So suppose I plays $a_1 \geq \omega^{\gamma_1}$ in α. Then II plays b_1 in β, a γ'-compression of a_1 where, depending on whether or not γ_1 is a limit or successor, $\gamma' < \gamma$ is as in the proof of the lower bound of the Separated Game formula. In either case, $\gamma(\alpha, \beta) \geq 2\gamma_1$. Thus, for $l_1 = 1$, $\gamma(\alpha, \beta) = 2\gamma_1$.

CASE 2. $l_1 = 2$

UPPER BOUND. $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$

I plays Φ_0^α in α and II responds with some b_1 in β. Any b_1 that is not the hole $\omega^{\gamma_1} \cdot 1$ is a descent. Suppose $b_1 = \omega^{\gamma_1} \cdot 1$. If $k_1 = 1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated on one copy of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$. If $k_1 > 1$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial on one copy of ω^{γ_1} and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. Thus, regardless of the value of k_1, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$.

Lower Bound. $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$

Suppose I plays a_1. Any a_1 in β or any $a_1 < \omega^{\gamma_1} \cdot 2$ in α is stalling for I: II copies from below playing $b_1 = a_1$ in α. So suppose I plays $a_1 \geq \omega^{\gamma_1}$ in α. If the terminal power of a_1 is $\geq \gamma_1$, then II plays $b_1 = \omega^{\gamma_1}$, the hole in β. When the terminal power of a_1 is $> \gamma_1$, on the left $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated on one copy of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, if $a_1 = \Phi_0^\alpha$, then either $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$ (when $k_1 = 1$) or $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. If $a_1 < \Phi_0^\alpha$ still with terminal power $> \gamma_1$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ case above and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. In any case, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1$ so that $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. Now, if the terminal power of $a_1 = \gamma_1$, then on the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the above $l_1 = 1$ case so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right either $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \neq \infty$ and $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial on one copy of ω^{γ_1} and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. In either case, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. Now if the terminal power of a_1 is $< \gamma_1$, then II plays $b_1 = \omega^{\gamma_1} \cdot 1 + \eta$ where η is the small tail of a_1 that II copies. Using the same argument at the end of the proof of the lower bound of the Pure Monomial formula (Lemma 4) the presence of the tail does not decrease the lower bound. So $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. Thus, for $l_1 = 2$, $\gamma(\alpha, \beta) = 2\gamma_1 + 1$.

Case 3. $l_1 = 3$

Upper Bound. $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$

If $k_1 = 1$, then I plays $a_1 = \omega^{\gamma_1} \cdot 2$ in α and II responds with some b_1 in β. Any b_1 that is not a hole is a descent. If b_1 is the first hole in β, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. If b_1 is the second hole in β, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ case above so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. In either case, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$. If $k_1 > 1$, then I plays $a_1 = \Phi_0^\alpha$ in α. Again,

b_1 that is not a hole in β is a descent. If b_1 is the first hole in β, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1$. If b_1 is the second hole in β, then $G^{a_1,b_1}_{\mathsf{RHS}}$ is pure monomial and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\gamma_1$. In either case, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$.

Lower Bound. $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$

Suppose I plays a_1. If a_1 is in β or if $a_1 < \omega^{\gamma_1} \cdot 3$ in α, II copies from below so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}}$ is as in the $l_1 = 1$ case above so that $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. So suppose $a_1 \geq \omega^{\gamma_1} \cdot 3$ in α. Then the argument is almost identical to the $l_1 = 2$ case except that II plays the second hole in β instead of the first hole. Suppose that the terminal power of a_1 is $\geq \gamma_1$. On the left, if the terminal power of a_1 is $> \gamma_1$, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated on two copies of ω^{γ_1} and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1 + 1$. If the terminal power of a_1 is γ_1, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is the $l_1 = 2$ case so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1 + 1$. On the right, there is only one copy of ω^{γ_1} on the bottom so the argument is the same as the $l_1 = 2$ case. If the terminal power of a_1 is $< \gamma_1$, the argument is the same on both sides: II plays the second hole and copies the small tail of a_1.

Case 4. $l_1 \geq 4$

Upper Bound. $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2$

I plays Φ^{α}_0 in α and II responds with some b_1 in β. Any b_1 that is not a hole is a descent. Suppose b_1 is some hole in β. On the left, $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 1$. So, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2$.

Lower Bound. $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$

We prove the formula by induction on l_1. Suppose that for all $l'_1 < l_1$ the formula holds and suppose I plays a_1. First we consider a_1 in β. If $a_1 < \omega^{\gamma_1} \cdot (l_1 - 1)$ in β, then II copies from below playing $b_1 = a_1$ in α. On the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$ and on the right $G^{a_1,b_1}_{\mathsf{RHS}}$ is as in the $l_1 = 2$ case so that $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\gamma_1 + 1$. If $a_1 = \omega^{\gamma_1} \cdot (l_1 - 1)$ the last hole in β, then II copies from above playing $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$ in α. On the right $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, there are two possibilities for $G_{\mathsf{LHS}}^{a_1,b_1}$. If, on one hand, $b_1 = \Phi_0^\alpha$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. If, on the other hand, $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for some $k' \geq 1$, by induction we have $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$ since $l_1 - 1 \geq 3$. If $a_1 > \omega^{\gamma_1} \cdot (l_1 - 1)$ in β, then II plays the last hole $b_1 = \omega^{\gamma_1} \cdot (k_1 - 1) + \eta$ copying the small tail of a_1. Again, the presence of the tail does not decrease the lower bound. So for all possible a_1 in β, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$.

Now suppose a_1 is in α. If $a_1 < \omega^{\gamma_1} \cdot (l_1 - 1)$, then II copies from below. On the left $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$ and on the right $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 2$ case so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Suppose $a_1 \geq \omega^{\gamma_1} \cdot (l_1 - 1)$. If the terminal power of a_1 is $> \gamma_1$, then II plays $\omega^{\gamma_1} \cdot 2$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated on two copies of ω^{γ_1} and by the Separated Game formula, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If the terminal power of a_1 is γ_1, then II plays $b_1 = \omega^{\gamma_1} \cdot \left\lfloor \frac{l_1}{2} \right\rfloor$, the "midpoint" hole in β. On the left, there are two possiblities: either $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial or it is not. If $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial, then by the Pure Monomial formula $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$ and on the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$ so that $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If $G_{\mathsf{LHS}}^{a_1,b_1}$ is not pure monomial, then by induction, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. In this case, on the right either $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \neq \infty$ and either by the Pure Monomial forumla or by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. In any case, when the terminal power of a_1 is γ_1, we have $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Finally, if the terminal power of a_1 is $< \gamma_1$, II copies the small tail of a_1 on top of playing the same b_1 he would have if a_1 had no tail. The presence of the tail does not decrease the lower bound. $\square$

To complete the case for $n = 1$, we consider the case where α, β are written in common CNF and both $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$. First, we fix the following notation. Let α, β have common CNFs:

$$\begin{aligned} \alpha &= \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1 \\ \beta &= \Phi_0^\beta + \omega^{\gamma_1} \cdot l_1 \end{aligned}$$

where the CNFs of $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ are separated. As before, identify the terminal terms of $\Phi_0^\alpha, \Phi_0^\beta$ as $\omega^{\alpha_0} \cdot k_0$ and $\omega^{\beta_0} \cdot l_0$, respectively. We define the ordinals τ_i for $i = 0, 1$ as follows:

Term τ_0: Suppose $\alpha_0 > \beta_0$. If $\beta_0 > \gamma_1 + 1$, then

$$\tau_0 = \begin{cases} 2\beta_0 & \text{if } \Phi_0^\beta \text{ is a monic monomial} \\ 2\beta_0 + 1 & \text{otherwise} \end{cases}$$

If $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0}$, then

$$\tau_0 = \begin{cases} 2\beta_0 & \text{if } l_1 \leq 3 \\ 2\beta_0 + 1 & \text{if } l_1 \geq 4 \end{cases}$$

If $\Phi_0^\beta = \omega^{\beta_0} \cdot 2$, then

$$\tau_0 = 2\beta_0 + 1$$

If $\Phi_0^\beta \geq \omega^{\beta_0} \cdot 3$ and has terminal power β_0, then

$$\tau_0 = \begin{cases} 2\beta_0 + 1 & \text{if } l_1 \leq 3 \\ 2\beta_0 + 2 & \text{if } l_1 \geq 4 \end{cases}$$

A symmetric formula for τ_0 holds for $\alpha_0 < \beta_0$.

Term τ_1: If $k_1 = l_1$, $\tau_1 = \infty$. Suppose $k_1 > l_1$. If $\gamma_1 = 0$, then

$$\tau_1 = \lfloor \log_2 (l_1 + 4) \rfloor$$

Suppose $\gamma_1 > 0$. If $l_1 = 1$, then

$$\tau_1 = \begin{cases} 2\gamma_1 + 1 & \text{if } k_1 = 2 \\ 2\gamma_1 + 2 & \text{if } k_1 \geq 3 \end{cases}$$

If $l_1 = 2, 3$, then

$$\tau_1 = 2\gamma_1 + 2$$

If $l_1 = 4$

$$\tau_1 = \begin{cases} 2\gamma_1 + 2 & \text{if } k_1 = 5 \\ 2\gamma_1 + 3 & \text{if } k_1 \geq 6 \end{cases}$$

If $l_1 \geq 5$,

$$\tau_1 = 2\gamma_1 + \lfloor \log_2 (l_1 + 3) \rfloor$$

A symmetric formula holds for $k_1 < l_1$.

THEOREM 3 (The Common CNF Game, $n = 1$). Let $\alpha = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1$ and $\beta = \Phi_0^\beta + \omega^{\gamma_1} \cdot l_1$ be written in common CNF where $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$. Then if τ_0, τ_1 are defined as above

$$\gamma(\alpha, \beta) = \min\{\tau_0, \tau_1\}$$

PROOF. Let α, β be as above. We first prove that $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1\}$.

UPPER BOUND. $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1\}$

Observe that I's choice of his first move depends on which of τ_0, τ_1 is smaller. So we break up the proof of the upper bound into cases: either $\tau_0 \leq \tau_1$ or $\tau_1 < \tau_0$.

CASE 1. $\tau_0 \leq \tau_1$.

We will show that $\gamma(\alpha, \beta) \leq \tau_0$. We assume, for this $\tau_0 \leq \tau_1$ case, without loss of generality that $\alpha_0 > \beta_0$. For if $\alpha_0 < \beta_0$, reverse the labels on α and β and the labels on the coefficients in the τ_1-block. We adopt the notational convention that the k_1 coefficient remains with α and the l_1 coefficient remains with β.

SUBCASE 1.1. $\beta_0 > \gamma_1 + 1$ and Φ_0^β is a monic monomial

I plays $a_1 = \omega^{\beta_0}$ in α and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$ and $\beta' < \beta_0$ is the terminal power of b_1, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\beta' + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\beta' + 2 \leq 2(\beta' + 1) \leq 2\beta_0 = \tau_0$. (Recall that we refer to this kind of response for II as a β_0-descent, because it holds II to at most $2\beta_0$.) If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2$.

Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3 = 2(\gamma_1 + 1) + 1 < 2\beta_0 = \tau_0$. If $b_1 > \Phi_0^\beta$, then b_1 is a β_0-descent since $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 < 2\beta_0 = \tau_0$.

Subcase 1.2. $\beta_0 > \gamma_1 + 1$ and Φ_0^β is not a monic monomial

I pinches off a block of ω^{β_0} in β (Recall Fig. 4.4) by playing a_1, the last β_0 hole in the τ_0-block of β. II responds with some b_1 in α. If $b_1 < \Phi_0^\alpha$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the previous case where Φ_0^β in $G_{\mathsf{RHS}}^{a_1,b_1}$ is a monic monomial. So, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\beta_0$ and thus $\gamma(\alpha, \beta) \leq 2\beta_0 + 1 = \tau_0$. If $b_1 = \Phi_0^\alpha$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3 < 2\beta_0 + 1 = \tau_0$. If $b_1 > \Phi_0^\alpha$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 < 2\beta_0 + 1 = \tau_0$.

Subcase 1.3. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0}$

I plays ω^{β_0} in α and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then b_1 is a β_0-descent and $\gamma(\alpha, \beta) \leq 2\beta_0 \leq \tau_0$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and there are two possibilities: either $l_1 \leq 3$ or $l_1 \geq 4$. If $l_1 \leq 3$, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$ so that $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = 2(\gamma_1 + 1) = 2\beta_0 = \tau_0$. If $l_1 \geq 4$, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 2$ so that $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3 = 2\beta_0 + 1 = \tau_0$. If $b_1 > \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = 2\beta_0 \leq \tau_0$.

Subcase 1.4. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0} \cdot 2$

I plays $\omega^{\beta_0} \cdot 2$ in α and II responds with some b_1 in β. If $b_1 = \omega^{\beta_0}$, the hole in the Φ_0^β-block, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0$. Thus, $\gamma(\alpha, \beta) \leq 2\beta_0 + 1 = \tau_0$. If $b_1 < \Phi_0^\beta$ and b_1 is not the hole in the Φ_0^β-block, then b_1 is a β_0-descent and $\gamma(\alpha, \beta) \leq 2\beta_0 < \tau_0$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2 = 2\beta_0$. Thus, $\gamma(\alpha, \beta) \leq 2\beta_0 + 1 = \tau_0$. If $b_1 > \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and b_1 is a β_0-descent. So, $\gamma(\alpha, \beta) \leq 2\beta_0 < \tau_0$.

Subcase 1.5. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta \geq \omega^{\beta_0} \cdot 3$ and has terminal power β_0

Write $\Phi_0^\beta = \Phi_{-1}^\beta + \omega^{\beta_0} \cdot l_0$ where Φ_{-1}^β (possibly empty) has terminal power $> \beta_0$ in its CNF and l_0 is a nonzero integer. Then I plays $\Phi_{-1}^\beta + \omega^{\beta_0} \cdot (l_0 - 1)$ in β, pinching off a block of ω^{β_0} in the Φ_0^β-block of β and II responds with some b_1 in α. If $b_1 < \Phi_0^\alpha$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $\Phi_0^\beta = \omega^{\beta_0}$ case. If $l_1 \leq 3$, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\beta_0$ so that $\gamma(\alpha, \beta) \leq 2\beta_0 + 1 = \tau_0$. If $l_1 \geq 4$, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\beta_0 + 1$ so that $\gamma(\alpha, \beta) \leq 2\beta_0 + 2 = \tau_0$. If $b_1 = \Phi_0^\alpha$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2 = 2\beta_0$. Thus, $\gamma(\alpha, \beta) \leq 2\beta_0 + 1 \leq \tau_0$. If $b_1 > \Phi_0^\alpha$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = 2\beta_0 < \tau_0$.

This ends the case when $\tau_0 \leq \tau_1$.

Case 2. $\tau_1 < \tau_0$

We show that $\gamma(\alpha, \beta) \leq \tau_1$. First, it cannot be the case that $k_1 = l_1$ since $\tau_1 < \tau_0 \neq \infty$. So, suppose that $k_1 > l_1$. If $\gamma_1 = 0$, then $\gamma(\alpha, \beta) \leq \lfloor \log_2 (l_1 + 4) \rfloor = \tau_1$ by Lemma 3. For the remainder of this case, suppose $\gamma_1 > 0$.

Subcase 2.1. $l_1 = 1$ and $k_1 = 2$

I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1}$ the hole in the γ_1-block of α and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 > \Phi_0^\beta$ and has terminal power $\gamma' < \gamma_1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is again separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma' + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma' + 2 \leq 2\gamma_1 < \tau_1$.

Subcase 2.2. $l_1 = 1$ and $k_1 \geq 3$

I again plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$ the last hole in the γ_1-block of α and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 > \Phi_0^\beta$, then b_1 is a γ_1-descent so that $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_1$.

Subcase 2.3. $l_1 = 2$

I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 2)$ the next to last hole in the γ_1-block in α and II repsonds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus,

$\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1}$ the hole in the γ_1-block in β, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 > \Phi_0^\beta$ and is not the hole in the γ_1-block, this b_1 is a γ_1-descent and thus $\gamma(\alpha,\beta) \leq 2\gamma_1 < \tau_1$.

Subcase 2.4. $l_1 = 3$

I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 3)$ the third hole from the end in the γ_1-block in α and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If b_1 is either hole in the γ_1-block of β, $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. Any $b_1 > \Phi_0^\beta$ that is not a hole is a γ_1-descent so that $\gamma(\alpha,\beta) \leq 2\gamma_1 < \tau_1$.

Subcase 2.5. $l_1 = 4$ and $k_1 = 5$

I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 3)$ the third hole from the end in the γ_1-block and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1}$ the first hole in the γ_1-block, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ and $k_1 = 2$ case above so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 > \Phi_0^\beta$ is any other hole, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 = \tau_1$. Any $b_1 > \Phi_0^\beta$ that is not a hole is a γ_1-descent and thus $\gamma(\alpha,\beta) \leq 2\gamma_1 < \tau_1$.

Subcase 2.6. $l_1 = 4$ and $k_1 \geq 6$

I again plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 3)$ the third hole from the end in the γ_1-block and II responds with some b_1 in β. If $b_1 < \Phi_0^\beta$, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 < \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 < \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1}$ the first hole in the γ_1-block, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ and $k_1 \geq 3$ case above so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 2$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 3 = \tau_1$. If b_1 is either of the two other holes in the γ_1-block, then $G^{a_1,b_1}_{\mathsf{RHS}}$ is pure monomial and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1+1$. Any $b_1 > \Phi^{\beta}_0$ that is not a hole is a γ_1-descent and thus $\gamma(\alpha,\beta) \leq 2\gamma_1 < \tau_1$.

Subcase 2.7. $l_1 \geq 5$

We show by induction that $\gamma(\alpha,\beta) \leq \tau_1 = 2\gamma_1 + \lfloor \log_2 (l_1+3) \rfloor$. Assume that the formula for τ_1 holds for all $l' < l_1$. The reader should recall the argument from Lemma 3. I plays $a_1 = \Phi^{\alpha}_0 + \omega^{\gamma_1} \cdot (l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1)$ and II responds with some b_1 in β. If $b_1 < \Phi^{\beta}_0$, then $G^{a_1,b_1}_{\mathsf{RHS}}$ is unbalanced and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1 + 2$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 3 \leq 2\gamma_1 + \lfloor \log_2 (l_1+3) \rfloor = \tau_1$. If $b_1 = \Phi^{\beta}_0$, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha,\beta) \leq 2\gamma_1 + 2 < \tau_1$. Any $b_1 > \Phi^{\beta}_0$ that is not a hole is a γ_1-descent so that $\gamma(\alpha,\beta) \leq 2\gamma_1 < \tau_1$. So suppose $b_1 = \Phi^{\beta}_0 + \omega^{\gamma_1} \cdot l'$ is some hole in the γ_1-block. There are two cases:

(1) $1 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$ or

(2) $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq l' < l_1$

Suppose first that $1 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$. Then $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + \lfloor \log_2 (l'+3) \rfloor$ either by induction or by the formula when $l' = 1,2,3,4$, except possibly when $l' = 4$ and $k' = l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \geq 6$. We claim that this anomalous case does not adversely affect the proof.

Proof (claim). When $5 \leq l_1 \leq 11$, we have $l' \leq 3$ since we are in the case where $l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$. Thus, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 2 = 2\gamma_1 + \lfloor \log_2 (l_1+3) \rfloor - 1$. Now when $l_1 = 12$, we have $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 = 5$ so that if $l' = 4$, we are not in the anomalous case and the formula computes $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 2 = 2\gamma_1 + \lfloor \log_2 (l_1+3) \rfloor - 1$. For $l_1 \geq 13$, we have $\lfloor \log_2 (l_1+3) \rfloor \geq 4$ and the $l' = 4$, $k' = 6$ case is not detrimental.

Now, we claim that

$$\lfloor \log_2 (l'+3) \rfloor \leq \lfloor \log_2 (l_1+3) \rfloor - 1$$

Proof (claim). The case for $5 \leq l_1 \leq 12$ is covered by the above claim. So suppose $l_1 \geq 13$ and write $l_1 = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j$ where $1 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. By hypothesis,

$$l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor} = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j - 2^{\lfloor \log_2 l_1 \rfloor} = 2^{\lfloor \log_2 l_1 \rfloor} - j$$

and hence

$$l' + 3 \leq 2^{\lfloor \log_2 l_1 \rfloor} + (3 - j)$$

Now if $1 \leq j \leq 3$, then

$$\lfloor \log_2 (l' + 3) \rfloor \leq \lfloor \log_2 l_1 \rfloor = \lfloor \log_2 (l_1 + 3) \rfloor - 1$$

On the other hand, if $3 < j \leq 2^{\lfloor \log_2 l_1 \rfloor}$, then

$$\lfloor \log_2 (l' + 3) \rfloor \leq \lfloor \log_2 l_1 \rfloor - 1 = \lfloor \log_2 (l_1 + 3) \rfloor - 1$$

This proves the claim.

Thus, when $1 \leq l' l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, we have $\gamma(\alpha, \beta) \leq 2\gamma_1 + \lfloor \log_2 (l_1 + 3) \rfloor = \tau_1$.

Now suppose that $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq l' < l_1$. Then $G^{a_1, b_1}_{\mathsf{RHS}}$ is pure monomial and

$$\gamma^{a_1, b_1}_{\mathsf{RHS}} = 2\gamma_1 + \lfloor \log_2 (l_1 - l') \rfloor \leq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} - 1 \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$$

Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + \lfloor \log_2 l_1 \rfloor \leq 2\gamma_1 + \lfloor \log_2 (l_1 + 3) \rfloor = \tau_1$.

This ends the case $l_1 \geq 5$ and this exhausts all of the cases of the formula when $k_1 > l_1$. If $k_1 < l_1$, then the argument is symmetric using the obvious changes to the formula for τ_1.

This ends the case when $\tau_1 < \tau_0$. Thus, $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1\}$.

Lower Bound. $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1\}$

Now we show that for every instance of the formula and every move for I a_1 there is a response for II b_1 such that either $\gamma(\alpha, \beta) \geq \tau_0$ or $\gamma(\alpha, \beta) \geq \tau_1$. We break up the cases first depending on the location of I's move a_1: either I moves in the τ_0-block or I moves in the τ_1-block. Note that we will adopt the convention that both fence moves Φ^α_0 and Φ^β_0 are in the τ_1-block.

Case 3. I plays a_1 in the τ_0-block

In this case, I plays either $a_1 < \Phi^\alpha_0$ in α or $a_1 < \Phi^\beta_0$ in β. Suppose $\alpha_0 > \beta_0$.

Subcase 3.1. $\beta_0 > \gamma_1 + 1$ and $\Phi^\beta_0 = \omega^{\beta_0} \cdot 1$ is a monic monomial

If a_1 is in β or if $a_1 < \Phi_0^\beta$ is in α, then II copies from below playing $b_1 = a_1$ in either α or β, respectively. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1} = G(\alpha,\beta)$ and this a_1 is stalling for I. Suppose $a_1 \geq \omega^\beta$ in α. If β_0 is a successor, then II plays b_1 in β a $(\beta_0 - 1)$-compression of a_1. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2(\beta_0 - 1) + 1$ by a compression argument. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is almost identical to $G(\alpha,\beta)$ and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \tau_1$. Thus, $\gamma(\alpha,\beta) \geq 2\beta_0$. If β_0 is limit, let $\beta' < \beta_0$. Then II plays b_1 a β'-compression of a_1 in β. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta' + 1$ by a compression argument. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \tau_1$. Thus, for any $\beta' < \beta_0$, $\gamma(\alpha,\beta) \geq 2\beta' + 2$. So we have $\gamma(\alpha,\beta) \geq 2\beta_0$.

Subcase 3.2. $\beta_0 > \gamma_1 + 1$ and Φ_0^β is not a monic monomial

Suppose a_1 is in β. If $a_1 < \Phi_0^\alpha$, then II copies from below playing $b_1 = a_1$ in α and this a_1 is stalling for I. Note this case is vacuous for small Φ_0^β. If $\Phi_0^\alpha \leq a_1 \leq \Phi_0^\beta$ (or just $a_1 < \Phi_0^\beta$ when Φ_0^β is small) and the terminal power of a_1 is $\geq \beta_0$, then II plays b_1 in α to pinch off a block of ω^{α_0}. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$ using the τ_0 term of $G_{\mathsf{RHS}}^{a_1,b_1}$. Thus, $\gamma(\alpha,\beta) \geq 2\beta_0 + 1 = \tau_0$. If $\Phi_0^\alpha \leq a_1 \leq \Phi_0^\beta$ (or just $a_1 < \Phi_0^\beta$ when Φ_0^β is small) and the terminal power of a_1 is $< \beta_0$, then II plays the same b_1 he would have played on the untailed version of a_1, plus II copies a tail. The presence of the tail does not decrease the lower bound. Now suppose a_1 is in α. If $a_1 < \Phi_0^\beta$, then II copies from below and everything is as above. If $\Phi_0^\beta \leq a_1 \leq \Phi_0^\alpha$ and the terminal power of a_1 is $\geq \beta_0$. Then II plays b_1 to pinch off a block of ω^{β_0} in β. On the left, whenever the terminal power of a_1 is $> \beta_0$, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0$. When the terminal power of a_1 is β_0 and $l_0 = 1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is again separated. When the terminal power of a_1 is β_0 and $l_0 > 1$, then the β_0-block of $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and in that block the γ is at least $2\beta_0$. So, on the left, we have in all cases $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the monic monomial case, so by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$. Thus, $\gamma(\alpha,\beta) \geq 2\beta_0 + 1$.

Subcase 3.3. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0}$

Any a_1 in β or any $a_1 < \Phi_0^\beta$ in α is stalling for I. Suppose $a_1 \geq \Phi_0^\beta$ in α. If $l_1 \leq 3$, then II plays exactly the same as in the above $\beta_0 > \gamma_1 + 1$ case so that $\gamma(\alpha, \beta) \geq 2\beta_0$. If $l_1 \geq 4$ and if the terminal power of a_1 is $> \beta_0$, then II plays $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 2 = 2\beta_0$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$. If the terminal power of a_1 is β_0, then II plays b_1 a γ_1-compress of a_1 in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2 = 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is almost identical to $G(\alpha, \beta)$ and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \tau_1$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$. If the terminal power of a_1 is γ_1, then II plays $\omega^{\gamma_1} \cdot 4$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2 = 2\beta_0$. On the right, using the τ_0 term of $G_{\mathsf{RHS}}^{a_1,b_1}$, we have by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$. If the terminal power of a_1 is $< \gamma_1$, then II plays the same b_1 as if the terminal power of a_1 were equal γ_1 plus copying the small tail of a_1. The presence of the small tail does not decrease the lower bound. Thus, when $l_1 \geq 4$, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$.

Subcase 3.4. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0} \cdot 2$

Any a_1 in β or any $a_1 < \Phi_0^\beta$ in α is stalling for I. Suppose $a_1 \geq \Phi_0^\beta$ in α. If the terminal power of a_1 is $> \beta_0$, then II plays ω^{β_0} in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the above $\Phi_0^\beta = \omega^{\beta_0}$ case and thus $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$. If the terminal power of a_1 is β_0, then II still plays ω^{β_0}. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the above $\Phi_0^\beta = \omega^{\beta_0}$ case and thus $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$. Thus $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$. If the terminal power of a_1 is $< \beta_0$, then II plays $b_1 = \omega^{\beta_0} + \eta$ where η is the small tail of a_1. The presence of the small tail does not decrease the lower bound. Thus, when $\Phi_0^\beta = \omega^{\beta_0} \cdot 2$, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$.

Subcase 3.5. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta \geq \omega^{\beta_0} \cdot 3$ with terminal power β_0

Suppose a_1 is in β. If $a_1 < \Phi_0^\alpha$, then II copies from below playing $b_1 = a_1$ in α and this a_1 is stalling for I. Note this case is vacuous for small Φ_0^β. If $\Phi_0^\alpha \leq a_1 \leq \Phi_0^\beta$ (or just $a_1 \leq \Phi_0^\beta$ for small Φ_0^β) and the terminal power of a_1 is $> \beta_0$, then II plays b_1 in α to pinch off a block of α_0. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 + 2$. On the right, using the τ_0 term of $G_{\mathsf{RHS}}^{a_1,b_1}$, by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$. If $\Phi_0^\alpha \leq a_1 \leq \Phi_0^\beta$ and the terminal power of a_1 is β_0, then II plays a β_0-compression of a_1. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is at worst as in the $\Phi_0^\beta = \omega^{\beta_0}$ case and thus $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 1$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 2$ depending on the number of copies of ω^{γ_1}. Thus, $\gamma(\alpha, \beta) \geq \tau_0$. Now suppose a_1 is in α. If $a_1 < \Phi_0^\beta$, then II copies from below and everything is as above. Suppose $a_1 \geq \Phi_0^\beta$ in α. If the terminal power of a_1 is $> \beta_0$, then II plays b_1 to pinch off a block of ω^{β_0}. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $\Phi_0^\beta = \omega^{\beta_0}$ case so that either $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 1$ depending on whether or not $l_1 \leq 3$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$ or $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$ depending or whether or not $l_1 \leq 3$. If the terminal power of a_1 is β_0, then II plays again to pinch off a block of ω^{β_0}. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 + 1$. On the right $G_{\mathsf{RHS}}^{a_1,b_1}$ is again as in the above case when $\Phi_0^\beta = \omega^{\beta_0}$. So $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$ or $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$ depending on whether or not $l_1 \leq 3$. As before, if the terminal power of a_1 is $< \beta_0$, II plays to pinch off a block of ω^{β_0} and copies the small tail of a_1. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$ or $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$ depending on whether or not $l_1 \leq 3$.

So it follows if I plays a_1 in the τ_0-block, $\gamma(\alpha, \beta) \geq \tau_0 \geq \min\{\tau_0, \tau_1\}$.

Case 4. I plays a_1 in the τ_1-block

In this case we suppose that I plays either $a_1 \geq \Phi_0^\alpha$ in α or $a_1 \geq \Phi_0^\beta$ in β. Moreover, assume without loss of generality $\alpha_0 > \beta_0$.

Suppose $k_1 = l_1$. In this case, $\tau_1 = \infty$ and II responds in the same way to I's a_1: If $a_1 = \Phi_0^\alpha + \eta$ in α where $0 \leq \eta < \omega^{\gamma_1} \cdot k_1$ or if $a_1 = \Phi_0^\beta + \eta$ in β where $0 \leq \eta < \omega^{\gamma_1} \cdot l_1$, then II responds with the corresponding copying move $b_1 = \Phi_0^\beta + \eta$ in β or $b_1 = \Phi_0^\alpha + \eta$ in α, respectively. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$ in all cases. So, it is enough to analyze $G_{\mathsf{LHS}}^{a_1,b_1}$ to show that $\gamma(\alpha, \beta) \geq \tau_0$ as follows.

Subcase 4.1. $\beta_0 > \gamma_1 + 1$

We show that $\gamma(\alpha, \beta) \geq 2\beta_0$ or $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$ depending on whether or not Φ_0^β is a monic monomial. On the left, observe that $G_{\mathsf{LHS}}^{a_1,b_1}$ is a separated game (when a_1 is a fence move) or $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated with an ∞-game on the right. As before, the ∞-game does not decrease the lower bound. $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_0$ if Φ^β_0 is a monic monomial and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_0+1$ if Φ^β_0 is not a monic monomial. In either case, $\gamma(\alpha,\beta) \geq \tau_0$.

Subcase 4.2. $\beta_0 = \gamma_1 + 1$

Suppose also that $\Phi^\beta_0 = \omega^{\beta_0}$. Observe that in the last case we did not use the fact that $\beta_0 > \gamma_1 + 1$. Thus, the same argument shows that $\gamma(\alpha,\beta) \geq \tau_0$. Suppose $\Phi^\beta_0 = \omega^{\beta_0} \cdot 2$. Then, $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated or separated followed by an ∞-game. In either case, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_0$. Thus, $\gamma(\alpha,\beta) \geq 2\beta_0 + 1 = \tau_0$. Finally, if $\Phi^\beta_0 \geq \omega^{\beta_0} \cdot 3$ and has terminal power β_0. Then, $G^{a_1,b_1}_{\mathsf{LHS}}$ is either separated or separated followed by an ∞-game. Moreover, the separated game is on more than one copy of ω^{β_0} so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_0 + 1$. Thus, $\gamma(\alpha,\beta) \geq 2\beta_0 + 2 \geq \tau_0$.

When $\alpha_0 < \beta_0$, a symmetric argument shows $\gamma(\alpha,\beta) \geq \tau_0$. This ends the case $k_1 = l_1$.

Now suppose $k_1 > l_1 \geq 1$. Furthermore, suppose for the moment $\gamma_1 = 0$. Note that we are no longer necessarily assuming $\alpha_0 > \beta_0$. We show that if $a_1 \geq \Phi^\alpha_0$ in α or $a_1 \geq \Phi^\beta_0$ in β, then $\gamma(\alpha,\beta) \geq \lfloor \log_2(l_1+4) \rfloor = \tau_1$. We argue the first few cases $l_1 = 1,2,3,4$ individually and then $l_1 \geq 5$ in general.

Subcase 4.3. $l_1 = 1$

If I plays Φ^α_0 in α, then II plays $b_1 = 2$ in β. On the left, $G^{a_1,b_1}_{\mathsf{LHS}} = G(\Phi^\alpha_0, 2)$ so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 1$ by Lemma 2. On the right, $G^{a_1,b_1}_{\mathsf{RHS}}$ is as in Lemma 2 so that $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \lfloor \log_2(l_1+4) \rfloor$. Similarly, if I plays $a_1 > \Phi^\alpha_0$ in α, say $a_1 = \Phi^\alpha_0 + k'$, then II responds again with $b_1 = 2$ in β. Both left and right games are trivially separated (recall Lemma 2) so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 1$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \tau_1$. If I plays $a_1 = \Phi^\alpha_0 + (k_1 - 1)$ in α, then II responds with $b_1 = \Phi^\beta_0$ in β. On the right, $G^{a_1,b_1}_{\mathsf{RHS}}$ is empty. On the left, $G^{a_1,b_1}_{\mathsf{LHS}}$ is trivially separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \tau_1$. Similarly, if $a_1 = \Phi^\beta_0$ in β, then II responds with $b_1 = \Phi^\alpha_0 + (k_1 - 1)$ in α. The argument is the same and $\gamma(\alpha,\beta) \geq 2 = \tau_1$.

Subcase 4.4. $l_1 = 2$

If I plays either endpoint $a_1 = \Phi_0^\alpha + (k_1 - 1)$ in α or $a_1 = \Phi_0^\beta + 1$ in β, then II responds with the other corresponding endpoint. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is empty. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the previous $l_1 = 1$ case so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2$. Thus, $\gamma(\alpha,\beta) \geq 3 > \lfloor \log_2(l_1+4) \rfloor = \tau_1$. If I plays $a_1 = \Phi_0^\alpha + k'$ where $0 \leq k' < k_1 - 1$, then II plays $b_1 = 2$ in α. On both the left and right $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 1$ and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \tau_1$. If I plays $a_1 = \Phi_0^\beta$ in β, then II responds with Φ_0^α in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0 \geq 2 \cdot 1 = 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2(l_1+1) \rfloor = 1$ by Lemma 1. Thus, $\gamma(\alpha,\beta) \geq \lfloor \log_2(l_1+4) \rfloor$.

Subcase 4.5. $l_1 = 3$

If I plays either endpoint, II responds with the other corresponding endpoint and the argument is the same as above. If I plays $a_1 = \Phi_0^\alpha + (k_1 - 2)$ in α or $a_1 = \Phi_0^\beta + 1$ in β, then II copies from above. On the right $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the case $l_1 = 1$ above so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \tau_1$. If I plays $a_1 = \Phi_0^\alpha + k'$ in α where $0 \leq k' < k_1 - 2$, then II plays $b_1 = 2$ in β. On both the left and right $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 1$ and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 1$. Thus, $\gamma(\alpha,\beta) \geq 2 = \tau_1$. If I plays $b_1 = \Phi_0^\beta$ in β, then II responds with $b_1 = \Phi_0^\alpha$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 = 2 \cdot 1 = 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2(l_1+1) \rfloor = 2$. Thus, $\gamma(\alpha,\beta) \geq 3 > \tau_1$.

Subcase 4.6. $l_1 = 4$

If I plays $a_1 \geq \Phi_0^\alpha + (k_1 - 3)$ in α or $a_1 \geq \Phi_0^\beta + 1$, then II copies from above. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is either empty or an ∞-game. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $l_1 = 1,2,3$ case so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2$. Thus, $\gamma(\alpha,\beta) \geq 3 = \lfloor \log_2(l_1+4) \rfloor = \tau_1$. If $a_1 = \Phi_0^\alpha + k'$ in α where $0 \leq k' < k_1 - 3$, then II plays $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 \geq 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2(l_1+1) \rfloor = 2$. Thus, $\gamma(\alpha,\beta) \geq 3 = \lfloor \log_2(l_1+4) \rfloor = \tau_1$. If I plays $a_1 = \Phi_0^\beta$ in β, then II plays $b_1 = \Phi_0^\alpha$ in α and the argument is the same.

Thus, for $l_1 = 1, 2, 3, 4$, we have $\gamma(\alpha, \beta) \geq \lfloor \log_2 (l_1 + 4) \rfloor$.

Subcase 4.7. $l_1 \geq 5$

We prove the result by induction and assume that for all $l' < l_1$, $\gamma(\alpha, \beta) \geq \lfloor \log_2 (l' + 4) \rfloor$. First, write $l_1 = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j$ where $1 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. We divide first into two cases: $j = 1, 2, 3, 4$ and $5 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$.

Suppose $j = 1, 2, 3, 4$. If I plays a_1 in β, say $\Phi_0^\beta + l'$ where $0 \leq l' < l_1$, then II copies from either below or above, depending on the value of l'. If $0 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then II copies from below playing $b_1 = \Phi_0^\alpha + l'$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is either separated (i.e., $l' = 0$) or $G_{\mathsf{LHS}}^{a_1,b_1}$ is a separated game followed by an ∞-game (i.e., $l' > 0$). Suppose $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated. If $\beta_0 > \gamma_1$, then $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \tau_0$ since the τ_0-term of $G(\alpha, \beta)$ is the same as the separated game formula. If $\beta_0 = \gamma_1 + 1$, then by inspection of the formula $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \tau_0 - 1$. In either case, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \tau_0 - 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2 (l_1 - l') \rfloor \geq \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$. On the other hand, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq l' \leq l_1 - 1$, then II copies from above playing $b_1 = \Phi_0^\alpha + (k_1 - (l_1 - l'))$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \lfloor \log_2 (l' + 4) \rfloor \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (5 - j) \right) \right\rfloor$. Since $j \leq 4$, this means $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + 1 \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$. Now suppose I plays a_1 in α, say $a_1 = \Phi_0^\alpha + k'$ where $0 \leq k' < k_1$. If $0 \leq k' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then as above, II copies from below playing $a_1 = \Phi_0^\beta + k'$ and the argument is the same as before. If $k_1 - (2^{\lfloor \log_2 l_1 \rfloor} - 1) \leq k' \leq k_1 - 1$, then as before, II copies from above playing $b_1 = \Phi_0^\beta + (l_1 - (k_1 - k'))$ and the argument is the same as before. Finally, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq k' \leq k_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then II plays $b_1 = \Phi_0^\beta + l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 4 \right) \right\rfloor = \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (4 - j) \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor$ since $j \leq 4$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite, and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$.

Suppose $5 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. If I plays a_1 in β, say $\Phi_0^\beta + l'$ where $0 \leq l' \leq l_1 - 1$, then again II either copies from below or above, depending on the value of l'. If $0 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then II copies from below playing $b_1 = \Phi_0^\alpha + l'$ in α. Using the same reasoning as above, on the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \tau_0 - 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2 (l_1 - l') \rfloor \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor - 1} \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$ since $5 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. If $l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 1 \leq l' \leq l_1 - 1$, then II copies from above playing $b_1 = \Phi_0^\alpha + (k_1 - (l_1 - l'))$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \lfloor \log_2 (l' + 4) \rfloor \geq \left\lfloor \log_2 \left(l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 4 \right) \right\rfloor \geq \left\lfloor \log_2 \left\lfloor \frac{l_1}{2} \right\rfloor \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$. On the other hand, suppose I plays a_1 in α, say $\Phi_0^\alpha + k'$ where $0 \leq k' \leq k_1 - 1$. If $0 \leq k' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then as before, II copies from below playing $b_1 = \Phi_0^\beta + k'$ in β and the argument is the same as before. If $k_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} \leq k' \leq k_1 - 1$, then as before, II copies from above playing $b_1 = \Phi_0^\beta + (l_1 - (k_1 - k'))$ and the argument is the same as before. Finally, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} \leq k' \leq k_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then II plays $b_1 = \Phi_0^\beta + (l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1})$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 4 \right) \right\rfloor \geq \left\lfloor \log_2 \left\lfloor \frac{l_1}{2} \right\rfloor \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor - 1} \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$ since $5 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. This ends the case for $\gamma_1 = 0$.

Now suppose $\gamma_1 > 0$.

Subcase 4.8. $l_1 = 1$ and $k_1 = 2$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 \geq 2\gamma_1 + 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. The argument is similar if I plays $a_1 = \Phi_0^\beta$ in β. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1}$, the hole in α, then II plays $b_1 = \Phi_0^\beta$ the fence in β. On the right $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. If I plays any a_1 in the γ_1-block with a small tail, II can copy a tail and keep $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$.

Subcase 4.9. $l_1 = 1$ and $k_1 \geq 3$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced on at least 3 copies and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If I plays $a_1 = \Phi_0^\beta$ in β, then II copies from above playing $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$ in α, then II copies from above playing $b_1 = \Phi_0^\beta$ in β and the argument is identical. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ in α where $1 \leq k' < k_1 - 1$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced on at least 2 copies and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If I plays any a_1 with a small tail in either α or β, then II responds by playing his response to the untailed a_1 along with copying the small tail. The presence of the tail does not decrease the lower bound.

Subcase 4.10. $l_1 = 2$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 \geq 2(\gamma_1 + 1) = 2\gamma_1 + 2$. On the left, $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + \lfloor \log_2 2 \rfloor = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\beta$ in β, then II responds with $b_1 = \Phi_0^\alpha$ in α and the argument is the same. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$ in α, then II copies from above playing $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 1$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 1$, II copies from above and the argument is the same. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ in α where $1 \leq k' < k_1 - 1$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced on at least 2 copies of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced on 2 copies of ω^{γ_1} and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If I plays any a_1 with a small tail in either α or β, then II responds by playing his response to the untailed a_1 along with copying the small tail. The presence of the tail does not decrease the lower bound.

Subcase 4.11. $l_1 = 3$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 \geq 2\gamma_1 + 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + \lfloor \log_2 3 \rfloor = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\beta$ in β, then II responds with

$b_1 = \Phi_0^\beta$ in β and the argument is the same. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ in α where $k' = k_1 - 1$ or $k_1 - 2$, then II copies from above playing $b_1 = \Phi_0^\beta + \omega^{\gamma_1} + l'$ where $l' = 2$ or 1, respectively. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the above $l_1 = 2$ or $l_1 = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 1$ or $\Phi_0^\beta + \omega^{\gamma_1} \cdot 2$ in β, II plays $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 1)$ and the argument is the same. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ in α where $1 \leq k' < k_1 - 2$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced on 2 copies of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is unbalanced on at least 3 copies of ω^{γ_1} and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If I plays any a_1 with a small tail in either α or β, then II responds by playing his response to the untailed a_1 along with copying the small tail. The presence of the tail does not decrease the lower bound.

Subcase 4.12. $l_1 = 4$ and $k_1 = 5$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 \geq 2\gamma_1 + 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{RHS}}^{a_1,b_1} = 2\gamma_1 + \lfloor \log_2 4 \rfloor = 2\gamma_1 = 2$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 3 > 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\beta$ in β, then II responds with $b_1 = \Phi_0^\alpha$ in α and the argument is the same. Now suppose I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$. If $k' = k_1 - 1, k_1 - 2$, or $k_1 - 3$, then II copies from above playing $b_1 = \Phi_0^\beta + \omega^{\gamma \cdot} l'$ where $l' = 3, 2$, or 1, respectively. In all three cases, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $l_1 = 3, 2$, or 1 cases, respectively. When $l_1 = 3$ or 2, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$ and when $l_1 = 1$, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. Similarly, if I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ in β where $l' = 1, 2, 3$. Now when $1 \leq k' < k_1 - 3$, II responds with $b_1 = \omega^{\gamma_1} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated on 4 copies of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is separated on at least 4 copies of ω^{γ_1} and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If I plays any a_1 with a small tail in either α or β, then II responds by playing his response to the untailed a_1 along with copying the small tail. The presence of the tail does not decrease the lower bound.

Subcase 4.13. $l_1 = 4$ and $k_1 \geq 6$

Observe from the $k_1 = 5$ case that we actually have $\gamma(\alpha, \beta) \geq 2\gamma_1 + 3$ except when I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_1 - 3)$ in α and II responds with $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 1$. Under our current hypothesis of $k_1 \geq 6$, this now puts $G_{\mathsf{LHS}}^{a_1,b_1}$ in the $l_1 = 1$ and $k_1 = 3$ case so that $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. All other cases when $k_1 \geq 6$ are argued the same as in the $k_1 = 5$ case.

Subcase 4.14. $l_1 \geq 5$

The argument is similar to the $\gamma_1 = 0$ case except that we consider the holes in the γ_1-block in the same way we did the points in the $\gamma_1 = 0$ case. We prove the result by induction and assume that for all $l' < l_1$, $\gamma(\alpha, \beta) \geq 2\gamma_1 + \lfloor \log_2 (l' + 3) \rfloor$. First, write $l_1 = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j$ where $1 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. We divide first into two cases: $j = 1, 2, 3$ and $4 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$.

Suppose $j = 1, 2, 3$. If I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ in α where $0 \leq l' l_1$, then II copies either from below or above, depending on the value of l'. If $0 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then II copies from below playing $b_1 = \Phi_0^\alpha + l'$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is either separated (i.e., $l' = 0$) or $G_{\mathsf{LHS}}^{a_1,b_1}$ is a separated game followed by an ∞-game (i.e., $l' > 0$). Suppose $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated. If $\beta_0 > \gamma_1$, then $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \tau_0$ since the τ_0-term of $G(\alpha, \beta)$ is the same as the separated game formula. If $\beta_0 = \gamma_1 + 1$, then by inspection of the formula $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \tau_0 - 1$. In either case, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \tau_0 - 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is finite so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \lfloor \log_2 (l_1 - l') \rfloor \geq \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 3) \rfloor\}$. On the other hand, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq l' \leq l_1 - 1$, then II copies from above playing $b_1 = \Phi_0^\alpha + (k_1 - (l_1 - l'))$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} = \infty$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \lfloor \log_2 (l' + 4) \rfloor \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (4 - j)\right) \right\rfloor$. Now $j \leq 3$ means $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + 1\right) \right\rfloor = \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 3) \rfloor\}$. Now suppose I plays a_1 in α, say $a_1 = \Phi_0^\alpha + k'$ where $0 \leq k' < k_1$. If $0 \leq k' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then as above, II copies from below playing $a_1 = \Phi_0^\beta + k'$ and the argument is the same as before. If $k_1 - (2^{\lfloor \log_2 l_1 \rfloor} - 1) \leq k' \leq k_1 - 1$, then as before, II copies from above playing $b_1 = \Phi_0^\beta + (l_1 - (k_1 - k'))$ and the argument is the same as before. Finally, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1 \leq k' \leq k_1 - 2^{\lfloor \log_2 l_1 \rfloor}$, then II plays $b_1 = \Phi_0^\beta + l_1 - 2^{\lfloor \log_2 l_1 \rfloor}$. On the left, by induction $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq \left\lfloor \log_2 \left(l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 3\right) \right\rfloor = \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (3 - j)\right) \right\rfloor = \lfloor \log_2 l_1 \rfloor$

since $j \le 3$. On the right, $G^{a_1,b_1}_{\mathsf{RHS}}$ is finite, and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \lfloor \log_2 (2^{\lfloor \log_2 l_1 \rfloor}) \rfloor = \lfloor \log_2 l_1 \rfloor$. Thus, $\gamma(\alpha, \beta) \ge \min\{\tau_0, \lfloor \log_2 l_1 \rfloor + 1\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 3) \rfloor\}$.

Suppose $4 \le j \le 2^{\lfloor \log_2 l_1 \rfloor}$. If I plays a_1 in β, say $\Phi^\beta_0 + l'$ where $0 \le l' \le l_1 - 1$, then again II either copies from below or above, depending on the value of l'. If $0 \le l' \le l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then II copies from below playing $b_1 = \Phi^\alpha_0 + l'$ in α. Using the same reasoning as above, on the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} \ge \tau_0 - 1$. On the right, $G^{a_1,b_1}_{\mathsf{RHS}}$ is finite and $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \lfloor \log_2 (l_1 - l') \rfloor \ge \lfloor \log_2 (2^{\lfloor \log_2 l_1 \rfloor - 1}) \rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \ge \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 3) \rfloor\}$ since $4 \le j \le 2^{\lfloor \log_2 l_1 \rfloor}$. If $l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 1 \le l' \le l_1 - 1$, then II copies from above playing $b_1 = \Phi^\alpha_0 + (k_1 - (l_1 - l'))$. On the right, $\gamma^{a_1,b_1}_{\mathsf{RHS}} = \infty$. On the left, by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} \ge \lfloor \log_2 (l' + 3) \rfloor \ge \lfloor \log_2 (l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 3) \rfloor \ge \lfloor \log_2 \lfloor \frac{l_1}{2} \rfloor \rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \ge \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 3) \rfloor\}$. On the other hand, suppose I plays a_1 in α, say $\Phi^\alpha_0 + k'$ where $0 \le k' \le k_1 - 1$. If $0 \le k' \le l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then as before, II copies from below playing $b_1 = \Phi^\beta_0 + k'$ in β and the argument is the same as before. If $k_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} \le k' \le k_1 - 1$, then as before, II copies from above playing $b_1 = \Phi^\beta_0 + (l_1 - (k_1 - k'))$ and the argument is the same as before. Finally, if $l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} \le k' \le k_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1}$, then II plays $b_1 = \Phi^\beta_0 + (l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1})$. On the left, by induction $\gamma^{a_1,b_1}_{\mathsf{LHS}} \ge \lfloor \log_2 (l_1 - 2^{\lfloor \log_2 l_1 \rfloor - 1} + 3) \rfloor \ge \lfloor \log_2 \lfloor \frac{l_1}{2} \rfloor \rfloor = \lfloor \log_2 l_1 \rfloor - 1$. On the right, $G^{a_1,b_1}_{\mathsf{RHS}}$ is finite and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \ge \lfloor \log_2 (2^{\lfloor \log_2 l_1 \rfloor - 1}) \rfloor = \lfloor \log_2 l_1 \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \ge \min\{\tau_0, \lfloor \log_2 l_1 \rfloor\} = \min\{\tau_0, \lfloor \log_2 (l_1 + 4) \rfloor\}$ since $4 \le j \le 2^{\lfloor \log_2 l_1 \rfloor}$. This ends the case for $\gamma_1 > 0$. A symmetric formula holds if $k_1 < l_1$. Thus, in all cases $\gamma(\alpha, \beta) \ge \min\{\tau_0, \tau_1\}$. $\square$

4.4.2. $n > 1$

Our final theorem computes $\gamma(\alpha, \beta)$ for the Common CNF game when $n > 1$. The theorem will be the culmination of all of the formulas we have proven thus far with one new twist. We begin by defining a formula that we will henceforth refer to as the **recursive condition**. This formula checks whether or not a suitable condition exists for player II to exploit a small advantage. Let $\varphi(c, s, t)$ be the formula

$$s \le t < n \land (c_{t+1} \ne 3 \lor \gamma_t > \gamma_{t+1} + 1) \land \forall s \le j < t(c_{j+1} = 3 \land \gamma_j = \gamma_{j+1} + 1)$$

The variable c stands for coefficient and, according to our notational conventions, c will always be either k or l.

We reset our notation. Let α, β have common CNFs:

$$\begin{aligned}\alpha &= \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_n} \cdot k_n \\ \beta &= \Phi_0^\beta + \omega^{\gamma_1} \cdot l_1 + \cdots + \omega^{\gamma_m} \cdot l_m\end{aligned}$$

where the CNFs of $\Phi_0^\alpha, \Phi_0^\beta$ are separated and $n > 1$. When $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$, identify the terminal terms of $\Phi_0^\alpha, \Phi_0^\beta$ as $\omega^{\alpha_0} \cdot k_0$ and $\omega^{\beta_0} \cdot l_0$. We define the ordinal terms τ_i for $0 \leq i \leq n$ as follows:

Term τ_0: If both $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$, then $\tau_0 = \infty$. Henceforth in this case, assume that not both $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$.

Suppose $\Phi_0^\alpha \neq \emptyset$ and $\Phi_0^\beta = \emptyset$. If $l_1 = 1$,

$$\tau_0 = \begin{cases} 2\gamma_1 + 1 & \exists 1 \leq t < n(\varphi(l,1,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge l_{t+1} > 3) \\ 2\gamma_1 & \text{otherwise} \end{cases}$$

If $l_1 = 2$, then $\tau_0 = 2\gamma_1 + 1$.

If $l_1 = 3$, then

$$\tau_0 = \begin{cases} 2\gamma_1 + 2 & \exists 1 \leq t < n(\varphi(l,1,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge l_{t+1} > 3) \\ 2\gamma_1 + 1 & \text{otherwise} \end{cases}$$

If $l_1 \geq 4$, then $\tau_0 = 2\gamma_1 + 2$. The formula for $\Phi_0^\alpha = \emptyset$, $\Phi_0^\beta \neq \emptyset$ is symmetric.

Now suppose $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $\alpha_0 > \beta_0$. The formula for $\alpha_0 < \beta_0$ is symmetric. For the sake of the $\varphi(c, s, t)$ formula, let $\gamma_0 = \beta_0$. First, suppose $\beta_0 > \gamma_1 + 1$. Then,

$$\tau_0 = \begin{cases} 2\beta_0 & \Phi_0^\beta \text{ is a monic monomial} \\ 2\beta_0 + 1 & \text{otherwise} \end{cases}$$

Now suppose $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0} \cdot l_0$ is a monomial.

If $l_0 = 1$,

$$\tau_0 = \begin{cases} 2\beta_0 + 1 & \exists 0 \le t < n(\varphi(l,0,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge l_{t+1} > 3) \\ 2\beta_0 & \text{otherwise} \end{cases}$$

If $l_0 = 2$, then $\tau_0 = 2\beta_0 + 1$.

If $l_0 = 3$, then

$$\tau_0 = \begin{cases} 2\beta_0 + 2 & \exists 0 \le t < n(\varphi(l,0,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge l_{t+1} > 3) \\ 2\beta_0 + 1 & \text{otherwise} \end{cases}$$

If $l_0 \ge 4$, then $\tau_0 = 2\beta_0 + 2$.

If Φ_0^β not a monomial, then

$$\tau_0 = \begin{cases} 2\beta_0 + 2 & \exists 0 \le t < n(\varphi(l,0,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge l_{t+1} > 3) \\ 2\beta_0 + 1 & \text{otherwise} \end{cases}$$

Terms τ_i, for $1 \le i \le n$: For any $1 \le i \le n$, if $k_i = l_i$, then $\tau_i = \infty$.

Suppose $k_i > l_i$. First define for $1 \le i < n$ and $c = k$ or $c = l$

$$\mathsf{R}_i^c = \begin{cases} 1 & \exists i \le t < n(\varphi(c,i,t) \wedge \gamma_t = \gamma_{t+1} + 1 \wedge c_{t+1} > 3) \\ 0 & \text{otherwise} \end{cases}$$

This is a flag which essentially says whether or not the recursive condition holds for the i^{th} block on the k or l side.

Let $i = 1$.

If $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$, then $\tau_1 = 2\gamma_1 + \left\lfloor \log_2\left(l_1 + \mathsf{R}_1^l\right)\right\rfloor$.

If $\Phi_0^\alpha \ne \emptyset$ and $\Phi_0^\beta = \emptyset$, then $\tau_1 = \tau_0$.

If $\Phi_0^\alpha = \emptyset$ and $\Phi_0^\beta \ne \emptyset$, then

$$\tau_1 = \begin{cases} 2\gamma_1 + 1 & \text{if either } l_1 = 1 \text{ and } k_1 = 2 \\ & \quad \text{or } l_1 = 2 \text{ and } k_1 = 3 \text{ and } \mathsf{R}_1^k = 0 \\ 2\gamma_1 + 2 & \text{otherwise} \end{cases}$$

Now let $1 \leq i \leq n$.

If $i = 1$ and $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ or if $1 < i < n$, then if $l_i = 1$ and $k_i = 2$, then

$$\tau_i = \begin{cases} 2\gamma_i + 2 & \text{if } \mathsf{R}_i^k = \mathsf{R}_i^l = 1 \\ 2\gamma_i + 1 & \text{otherwise} \end{cases}$$

If $l_i = 1$ and $k_i \geq 3$, then $\tau_i = 2\gamma_i + 2$.

If $l_i = 2$, then $\tau_i = 2\gamma_i + 2$.

If $l_i = 3$ and $k_i = 4$, then $\tau_i = 2\gamma_i + 2$.

If $l_i = 3$ and $k_i \geq 5$ or if $l_1 = 4$ and $k_1 = 5$

$$\tau_i = \begin{cases} 2\gamma_i + 3 & \mathsf{R}_i^k = \mathsf{R}_i^l = 1 \\ 2\gamma_i + 2 & \text{otherwise} \end{cases}$$

If $l_i = 4$ and $k_1 \geq 6$, then $\tau_i = 2\gamma_1 + 3$.

If $l_i \geq 5$, $\tau_i = 2\gamma_i + \left\lfloor \log_2 \left(l_i + 3 + \mathsf{R}_i^l \right) \right\rfloor$. A symmetric formula holds for $k_i < l_i$.

For $i = n$, we have simply the τ_1-term from the $n = 1$ case:

If $\gamma_n = 0$, then

$$\tau_n = \lfloor \log_2 (l_n + 4) \rfloor$$

Suppose $\gamma_n > 0$. If $l_n = 1$, then

$$\tau_n = \begin{cases} 2\gamma_n + 1 & \text{if } k_n = 2 \\ 2\gamma_n + 2 & \text{if } k_n \geq 3 \end{cases}$$

If $l_n = 2, 3$, then $\tau_n = 2\gamma_n + 2$. If $l_n = 4$

$$\tau_n = \begin{cases} 2\gamma_n + 2 & \text{if } k_n = 5 \\ 2\gamma_n + 3 & \text{if } k_n \geq 6 \end{cases}$$

If $l_n \geq 5$, $\tau_n = 2\gamma_n + \lfloor \log_2 (l_n + 3) \rfloor$. A symmetric formula holds for $k_n < l_n$.

THEOREM 4 (The Common CNF Game, $n > 1$). Let $\alpha = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k_1 + \cdots + \omega^{\gamma_n} \cdot k_n$ and $\beta = \Phi_0^\beta + \omega^{\gamma_1} \cdot l_1 + \cdots + \omega^{\gamma_n} \cdot l_n$ be written in common CNF where $\Phi_0^\alpha, \Phi_0^\beta$ are separated. Then if $\tau_0, \tau_1, \ldots, \tau_n$ are defined as above

$$\gamma(\alpha, \beta) = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$$

PROOF. Let α, β be as above. We prove first that $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

UPPER BOUND. $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$

We prove the bound holds by induction. Assume that the formula holds for smaller games.

Observe that I's choice of his first move depends on which of $\tau_0, \tau_1, \ldots, \tau_n$ is smallest. So we break up the proof into cases as in the $n = 1$ case. We note here that at least one of the τ_i, $0 \leq i \leq n$ must $\neq \infty$.

CASE 1. $\tau_0 \leq \tau_1, \ldots, \tau_n$

Immediately we have that $\tau_0 \neq \infty$, so it cannot be the case that both $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$.

SUBCASE 1.1. $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$, and $l_1 = 1$

I plays $\omega^{\gamma_1} \cdot 1$ in α. If II responds with any b_1 in β having terminal power $< \gamma_1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated. Thus, any such b_1 is easily seen to be a γ_1-descent so that $\gamma(\alpha, \beta) \leq 2\gamma_1 \leq \tau_0$. The only response that is not immediately a descent is $b_1 = \Phi_1^\beta$ in β. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$, so it is enough to show that the bound in τ_0 formula holds on the right. Suppose the recursive condition in the τ_0 formula holds. Then by induction, using the τ_0 term from $G_{\mathsf{RHS}}^{a_1,b_1}$, we have

$\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2 = 2(\gamma_2 + 1) = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_0$. Now suppose that the recursive condition in the τ_0 formula does not hold. If $\gamma_1 > \gamma_2 + 1$, then again by induction, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_2 + 3 = 2(\gamma_2 + 1) + 1 < 2\gamma_1$. If $\gamma_1 = \gamma_2 + 1$, then it must be the case that $l_2 \leq 3$, for otherwise we would contradict our assumption that the recursive condition holds. If $l_2 = 1, 2$, then by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_2 + 2 = 2\gamma_1$. If $l_2 = 3$, then it cannot be the case that the recursive condition in the τ_0 term of $G_{\mathsf{RHS}}^{a_1,b_1}$ holds, for that would contradict our assumption that it does not hold. So when $l_2 = 3$, we must have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_2 + 2 = 2\gamma_1$.

Subcase 1.2. $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$, and $l_1 = 2$

I plays $\omega^{\gamma_1} \cdot 2$ in α. Any response for II b_1 in β that has terminal power $< \gamma_1$ is a γ_1-descent and $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_0$. If $b_1 = \omega^{\gamma_1} \cdot 1$ in β, the hole in the γ_1-block, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + \lfloor \log_2 1 \rfloor = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_0$. If $b_1 = \Phi_1^\beta$, the fence on the γ_1, γ_2-blocks, then by induction, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2 = 2(\gamma_2 + 1) \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$.

Subcase 1.3. $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$, and $l_1 = 3$

I plays $\omega^{\gamma_1} \cdot 3$ in α. Any response for II b_1 in β that has terminal power $< \gamma_1$ is a γ_1-descent and $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_0$. If $b_1 = \omega^{\gamma_1} \cdot 1$ in β, the hole in the γ_1-block, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + \lfloor \log_2 1 \rfloor = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 \leq \tau_0$. If $b_1 = \omega^{\gamma_1} \cdot 2$ in β, the second hole in the γ_1-block, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 1$ case above. If the recusive condition holds, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$ and thus $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_0$. If the the recursive condition does not hold, then $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1$ and thus $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_0$. If $b_1 = \Phi_1^\beta$, then by induction, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 \leq \tau_0$.

Subcase 1.4. $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$, and $l_1 \geq 4$

I plays Φ_0^α in α. Any response for II b_1 must have terminal power $\leq \gamma_1$. Since $\alpha_0 > \gamma_1$, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma 1 + 2$.

Thus, when $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$, $\gamma(\alpha, \beta) \leq \tau_0$.

Now assume $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and assume $\alpha_0 > \beta_0$.

Subcase 1.5. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$, $\beta_0 > \gamma_1 + 1$ and Φ_0^β is a monic monomial

The argument is similar to the same subcases in the proof of the upper bound in Theorem 3 with one exception. Instead of $G_{\mathsf{RHS}}^{a_1,b_1}$ being unbalanced so that we use the Unbalanced Game formula to get $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2$, we are using induction and the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ to get the same inequality.

Subcase 1.6. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$, $\beta_0 > \gamma_1 + 1$ and Φ_0^β is not a monic monomial

The same comments from the previous subcase apply here as well.

Subcase 1.7. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$, $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0} \cdot l_0$

The argument is identical to the $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$ subcase above, replacing γ_1 in that argument with β_0.

Subcase 1.8. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$, $\beta_0 = \gamma_1 + 1$ and Φ_0^β not a monomial

The argument is similar to the case $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta \geq \omega^{\beta_0} \cdot 3$ subcase from Theorem 3. I plays $\Phi_{-1}^\beta + \omega^\beta \cdot (l_0 - 1)$ in β to pinch off a block of ω^{β_0}. As before, where $G_{\mathsf{RHS}}^{a_1,b_1}$ was unbalanced in that argument, we invoke induction to get the same bound.

This ends the case when $\tau_0 = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

Case 2. $\tau_1 = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$

It cannot be the case that $k_1 = l_1$, so suppose $k_1 > l_1$.

Subcase 2.1. $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$

If $\mathsf{R}_1 = 0$, then $\tau_1 = \lfloor \log_2 l_1 \rfloor$ and the argument is identical to the proof of the pure monomial formula. So suppose $\mathsf{R}_1 = 1$.

Suppose $l_1 = 1$. I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_0 - 1)$ in α, the last hole in the γ_1-block. Any b_1 in β having terminal power $< \gamma_1$ is a γ_1-descent and $\gamma(\alpha, \beta) \leq 2\gamma_1 < 2\gamma_1 + \lfloor \log_2 (l_1 + \mathsf{R}_1) \rfloor$. Suppose $b_1 = \Phi_1^\beta$ in β. Now since $\mathsf{R}_1 = 1$, we must have $\gamma_1 = \gamma_2 + 1$ and $l_2 \geq 3$. Inspecting the τ_0-term of $G_{\mathsf{RHS}}^{a_1, b_1}$, we see that $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_2 + 2 = 2\gamma_1$. For, if $l_2 \geq 4$, then $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_2 + 2$, and in the case $l_2 = 3$, observe that the recursive condition must still hold so that $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_2 + 2$. In either case $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_1$ so that $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$.

Now suppose $l_1 > 1$. I plays exactly as in the pure monomial game playing the midpoint hole of the γ_1-block. Any response for II having terminal power $< \gamma_1$ is a γ_1-descent. The only remaining b_1 that are not descents are the holes in the γ_1-block in β. From this point, an argument similar to the pure monomial game shows that $\gamma(\alpha, \beta) \leq 2\gamma_1 + \lfloor \log_2 (l_1 + 1) \rfloor$.

Subcase 2.2. $\Phi_0^\alpha \neq \emptyset$ and $\Phi_0^\beta = \emptyset$

I plays $a_1 = \Phi_0^\alpha$ in α. All of the cases of the $\tau_1 = \tau_0$ formula check the same way they did in the first case.

Subcase 2.3. $\Phi_0^\alpha = \emptyset$, $\Phi_0^\beta \neq \emptyset$, $l_1 = 1$ and $k_1 = 2$

I plays $a_1 = \Phi_0^\beta$ in β. If II plays any b_1 having terminal power $< \gamma_1$, then $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_1$. If II responds with ω^{γ_1} in α, then $G_{\mathsf{LHS}}^{a_1, b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1, b_1} \leq 2\gamma_1$. If II responds with $\omega^{\gamma_1} \cdot 2$, then using the τ_0 term of $G_{\mathsf{RHS}}^{a_1, b_1}$, we have $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_1$ by induction. For either response for II, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$.

Subcase 2.4. $\Phi_0^\alpha = \emptyset$, $\Phi_0^\beta \neq \emptyset$, $l_1 = 2$, $k_1 = 3$, and $\mathsf{R}_1^k = 0$

I plays $a_1 = \Phi_0^\beta$ in β. If II plays any b_1 having terminal power $< \gamma_1$, then $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_1$. If II responds with ω^{γ_1} in α, then $G_{\mathsf{LHS}}^{a_1, b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1, b_1} \leq 2\gamma_1$. If II responds with $\omega^{\gamma_1} \cdot 2$, then using the τ_1 term of $G_{\mathsf{RHS}}^{a_1, b_1}$, we have $\gamma_{\mathsf{RHS}}^{a_1, b_1} \leq 2\gamma_1 + \left\lfloor \log_2 \left(1 + \mathsf{R}_1^k\right) \right\rfloor = 2\gamma_1$

by induction. If II responds with $\omega^{\gamma_1} \cdot 3$, then using the τ_0 term of $G^{a_1,b_1}_{\mathsf{RHS}}$, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. For any of these responses for II, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$.

Subcase 2.5. $\Phi^\alpha_0 = \emptyset$, $\Phi^\beta_0 \neq \emptyset$, and neither of the two previous conditions hold

I plays $a_1 = \Phi^\beta_0$ in β. The key observation in this case is that $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated so that $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

Subcase 2.6. $\Phi^\alpha_0, \Phi^\beta_0 \neq \emptyset$ and $l_1 = 1$ and $k_1 = 2$

Suppose $\mathsf{R}^k_1 = \mathsf{R}^l_1 = 1$, that is, the recursive condition holds on both sides. Then I plays $a_1 = \Phi^\beta_0$ in α. Any b_1 in β having terminal power $< \gamma_1$ is a descent so that $\gamma(\alpha, \beta) \leq 2\gamma_1 < \tau_1$. If $b_1 = \Phi^\beta_1$ in β, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 = \Phi^\beta_0$ in β, then by induction using the τ_1-term of $G^{a_1,b_1}_{\mathsf{RHS}}$, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi^\beta_0$ and has terminal power $\geq \gamma_1$, then by induction using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2$.

Now suppose at least one of $\mathsf{R}^k_1, \mathsf{R}^l_1$ is zero. If $\mathsf{R}^l_1 = 0$, then I plays $a_1 = \Phi^\beta_0$ in β and II responds with some b_1 in α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi^\alpha_1$ in α, then using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$ by induction we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 = \Phi^\alpha_0 + \omega^{\gamma_1}$, the hole in the γ_1-block of α, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 = \Phi^\alpha_0$ in α, then using the τ_1-term of $G^{a_1,b_1}_{\mathsf{RHS}}$, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1$ since $\mathsf{R}^l_1 = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 < \Phi^\alpha_0$ in α has terminal power $\geq \gamma_1$, then using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1$ again, since $\mathsf{R}^l_1 = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1$.

If $\mathsf{R}^l_1 = 1$ and $\mathsf{R}^k_1 = 0$, then I plays $a_1 = \Phi^\alpha_0 + \omega^{\gamma_1}$ in α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi^\beta_1$ in β, then using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$ by induction we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 = \Phi^\beta_0$ in β, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$. If $b_1 < \Phi^\beta_0$ in β has terminal power $\geq \gamma_1$, then using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$ by induction we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1$ since $\mathsf{R}^k_1 = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 = \tau_1$.

SUBCASE 2.7. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 1$ and $k_1 \geq 3$

I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot (k_0 - 1)$ the last hole in the γ_1-block in α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi_1^\beta$ in β, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 = \Phi_0^\beta$ in β, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated on at least 2 copies of ω^{γ_1} and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi_0^\beta$ in β has terminal power $\geq \gamma_1$, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

SUBCASE 2.8. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 2$

I plays $\Phi_0^\beta + \omega^{\gamma_1} \cdot (k_0 - 1)$ the last hole in the γ_1-block of α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi_1^\beta$ in β, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1}$ the hole in the γ_1-block of β, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case (Theorem 3) and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta$ in β, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi_0^\beta$ in β has terminal power $\geq \gamma_1$, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

SUBCASE 2.9. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 3$ and $k_1 = 4$

I plays $\Phi_0^\beta + \omega^{\gamma_1} \cdot 2$ the middle hole in the γ_1-block of α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi_1^\beta$ in β, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_2 + 2 \leq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 1 < \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 2$ the last hole in the γ_1-block of β, then using the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta + \omega^{\gamma_1}$ the first hole in the γ_1-block of β, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\beta$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi_0^\beta$ in β has terminal power $\geq \gamma_1$, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

SUBCASE 2.10. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 3$ and $k_1 \geq 5$

Suppose $\mathsf{R}_1^l = 0$. I plays $a_1 = \Phi_0^\beta$ in β. Any b_1 in α having terminal power $< \gamma_1$ is a descent. If II plays $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for $1 \leq k' \leq k_1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 = \Phi_0^\alpha$, then using the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$ since $\mathsf{R}_1^l = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi_0^\alpha$, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$ since $\mathsf{R}_1^l = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

Now suppose $\mathsf{R}_1^l = 1$. I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 2$ in α. Any b_1 in β having terminal power $< \gamma_1$ is a descent. If II plays $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ for $1 \leq l' \leq 3$, then using the τ_1-term of $G_{\mathsf{LHS}}^{a_1,b_1}$ (or the τ_0-term when $l' = 3$) by induction we have $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 \leq \tau_1$. If $b_1 = \Phi_0^\beta$ in β, then $G_{\mathsf{RHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $b_1 < \Phi_0^\beta$ in β, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$ if $\mathsf{R}_1^k = 0$ or $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 2$ if $\mathsf{R}_1^k = 1$. In either case $\gamma(\alpha, \beta) \leq \tau_1$.

SUBCASE 2.11. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 4$ and $k_1 = 5$

Suppose $\mathsf{R}_1^l = 0$. I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1}$ in β. Any b_1 in α having terminal power $< \gamma_1$ is a descent. Suppose II plays $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for $0 \leq k' \leq 5$. If $k' = 3, 4, 5$, then using the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $k' = 2$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $k' = 0, 1$, then using the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + \lfloor \log_2 (3 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 1$ since $\mathsf{R}_1^l = 0$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$. If $a_1 < \Phi_0^\alpha$, then using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + \lfloor \log_2 (3 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 = \tau_1$.

Now suppose $\mathsf{R}_1^l = 1$. I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 2$ in α. Any b_1 in β having terminal power $< \gamma_1$ is a descent. Suppose II plays $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ for $0 \leq l' 4$. If $l' = 2, 3, 4$, then using the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 \leq \tau_1$. If $l' = 1$, then $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 \leq \tau_1$.

If $l' = 0$, then using the τ_1-term of $G^{a_1,b_1}_{\mathsf{RHS}}$ by induction $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1 + \left\lfloor \log_2 \left(3 + \mathsf{R}^k_1\right) \right\rfloor$. Thus, if $\mathsf{R}^k_1 = 1$ or $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3 = \tau_1$ and if $\mathsf{R}^k_1 = 0$, then $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2$.

Subcase 2.12. $\Phi^\alpha_0, \Phi^\beta_0 \neq \emptyset$ and $l_1 = 4$ and $k_1 \geq 6$

I plays $a_1 = \Phi^\alpha_0 + \omega^{\gamma_1} \cdot (k_0 - 3)$ in α. Any b_1 having terminal power $< \gamma_1$ is a descent. If $b_1 = \Phi^\beta_1$ or $b_1 = \Phi^\beta_0 + \omega^{\gamma_1} \cdot 3$ or $b_1 = \Phi^\beta_0 + \omega^{\gamma_1} \cdot 2$, then $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1 + 1$ by induction using the same argument as before. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 < \tau_1$. If $b_1 = \Phi^\beta_0 + \omega^{\gamma_1}$, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is as in the $n = 1$ case and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3$. If $b_1 = \Phi^\beta_0$, then $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 2 < \tau_1$. If $b_1 < \Phi^\beta_0$ has terminal power $\geq \gamma_1$, then $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\gamma_1 + 2$ by induction using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$. Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + 3 = \tau_1$.

Subcase 2.13. $\Phi^\alpha_0, \Phi^\beta_0 \neq \emptyset$ and $l_1 \geq 5$ and $\mathsf{R}^l_1 = 0$

This argument is identical to the same subcase in the $n = 1$ case (Theorem 3).

Subcase 2.14. $\Phi^\alpha_0, \Phi^\beta_0 \neq \emptyset$ and $l_1 \geq 5$ and $\mathsf{R}^l_1 = 1$

This argument is similar to the previous case except that I moves his play one hole to the right. I plays $a_1 = \Phi^\alpha_0 + \omega^{\gamma_1} \cdot (l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 2)$ in α. As in the previous case, we need only check that the formula holds when II responds with some hole in the γ_1-block in β since any other move easily holds the bound. Suppose II responds with $b_1 = \Phi^\beta_0 + \omega^{\gamma_1} \cdot l'$. There are two cases:

(1) $1 \leq l' \leq l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1$ or
(2) $l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 2 \leq l' \leq l_1 - 1$

In the first case, $G^{a_1,b_1}_{\mathsf{LHS}}$ is as in the $n = 1$ case and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq 2\gamma_1 + \lfloor \log_2 (l' + 3) \rfloor$. Now $\lfloor \log_2 (l' + 3) \rfloor \leq \lfloor \log_2 (l' + 4) \rfloor$. As we have shown in previous arguments, we claim $\gamma^{a_1,b_1}_{\mathsf{LHS}} \leq \lfloor \log_2 (l' + 3) \rfloor \leq \lfloor \log_2 (l_1 + 4) \rfloor - 1 = \left\lfloor \log_2 \left(l_1 + 3 + \mathsf{R}^l_1\right) \right\rfloor - 1$. Assuming that the claim holds, then we have $\gamma(\alpha, \beta) \leq 2\gamma_1 + \left\lfloor \log_2 \left(l_1 + 3 + \mathsf{R}^l_1\right) \right\rfloor$.

Proof (claim). Write $l_1 = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j$ where $1 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$. Now

$$l' + 3 \leq (l_1 - 2^{\lfloor \log_2 l_1 \rfloor} + 1) + 3 = 2^{\lfloor \log_2 l_1 \rfloor} + (4 - j)$$

If $j = 1, 2, 3, 4$, then

$$\lfloor \log_2 (l' + 3) \rfloor \leq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (4 - j) \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor = \lfloor \log_2 (l_1 + 4) \rfloor - 1$$

If $5 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$, then

$$\lfloor \log_2 (l' + 3) \rfloor \leq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} + (4 - j) \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1 = \lfloor \log_2 (l_1 + 4) \rfloor - 1$$

In the second case, using the τ_1-term of $G^{a_1, b_1}_{\mathsf{RHS}}$ by induction we have

$$\gamma^{a_1, b_1}_{\mathsf{RHS}} \leq 2\gamma_1 + \left\lfloor \log_2 \left(l_1 - l' + \mathsf{R}^l_1 \right) \right\rfloor \leq \left\lfloor \log_2 \left(2^{\lfloor \log_2 l_1 \rfloor} - 1 \right) \right\rfloor = \lfloor \log_2 l_1 \rfloor - 1$$

Thus, $\gamma(\alpha, \beta) \leq 2\gamma_1 + \lfloor \log_2 l_1 \rfloor \leq 2\gamma_1 + \left\lfloor \log_2 \left(l_1 + 3 + \mathsf{R}^l_1 \right) \right\rfloor$. In both cases, we have $\gamma(\alpha, \beta) \leq \tau_1$.

This ends the case when $\tau_1 = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

Case 3. $\tau_i = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$ for $1 < i < n$

The formula for τ_i is the same as the τ_1 formula when $\Phi^\beta_0 \neq \emptyset$. The argument is the same.

Case 4. $\tau_n = \min\{\tau_0, \tau_1 \ldots, \tau_n\}$

The formula for τ_n is the same as the formula for τ_1 when $n = 1$. The argument is the same.

In all cases, we have $\gamma(\alpha, \beta) \leq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

Lower Bound. $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$

We show that for every instance of the formula and every move a_1 for I there is a response b_1 for II such that $\gamma(\alpha, \beta) \geq \tau_i$ for some $0 \leq i \leq n$. We break up the cases according to the location of I's move. As before, we adopt the convention that we will treat fence moves $\Phi_i^\alpha, \Phi_i^\beta$ in the γ_{i+1}-block.

CASE 5. I plays in the τ_0-block

First, this means that at least one of $\Phi_0^\alpha, \Phi_0^\beta$ are nonempty. Suppose $\Phi_0^\alpha \neq \emptyset$ and $\Phi_0^\beta = \emptyset$. Now suppose I plays $a_1 < \Phi_0^\alpha$ in α.

SUBCASE 5.1. $l_1 = 1$

Suppose first that the recursive condition holds. If $a_1 < \Phi_1^\beta$ in α, then II copies from below and this is a stalling move for I. Suppose $\Phi_1^\beta \leq a_1 < \Phi_0^\alpha$ in α. If a_1 has terminal power $> \gamma_1$, then II plays $b_1 = \Phi_1^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_2 + 2 = 2\gamma_1$ since the recursive condition holds. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1 = \tau_0$. If now the terminal power of a_1 equals γ_1, then II again plays $b_1 = \Phi_1^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma(\alpha, \beta) \geq 2\gamma_1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is the same as before. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1 = \tau_0$. If the terminal power of a_1 is $\gamma_2 = \gamma_1 + 1$, then II plays $\omega^{\gamma_2} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_2 + 2 = 2\gamma_1$. On the right, usin the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1 = \tau_0$. If the terminal power of a_1 is $< \gamma_2$, then II plays the same b_1 he would have against the untailed version of a_1 plus copying the tail. The presence of the small tail does not decrease the lower bound.

Now suppose the recursive condition fails. Observe that all of the above argument is the same except the case when I plays $\Phi_1^\beta \leq a_1 < \Phi_0^\alpha$ in α. II then plays a γ'-compression of a_1 where depending on whether or not γ_1 is a limit or a successor. The argument proceeds as in the proof of the Separated Game formula.

SUBCASE 5.2. $l_1 = 2$

This case is identical to the $l_1 = 2$ case in the Unbalanced Game formula, except that $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1$ is now computed by induction.

SUBCASE 5.3. $l_1 = 3$

Suppose first that the recursive condition holds. If $a_1 < \Phi_1^\beta$ in α, then II copies from below and this is a stalling move for I. Suppose $\Phi_1^\beta \leq a_1 < \Phi_0^\alpha$ in α. If a_1 has terminal power $> \gamma_1$, then II plays $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the above $l_1 = 1$ case and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2$. If a_1 has terminal power equal γ_1, then II again plays $b_1 \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is again as in the $l_1 = 1$ case and $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2 = \tau_0$. If a_1 has terminal power $< \gamma_1$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2 + \eta$ where η is the small tail of a_1. The presence of the small tail does not decrease the lower bound.

Now suppose the recursive condition fails. Then II plays the same as before. Since the recursive condition fails, it also fails in the $l_1 = 1$ case so that now $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1$. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$.

SUBCASE 5.4. $l_1 \geq 4$

The argument is by induction on l_1 and is the same as in the $l_1 = 4$ case of the Unbalanced Game formula.

This ends the case for $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$. A symmetrics argument shows that the lower bound holds for $\Phi_0^\alpha = \emptyset, \Phi_0^\beta \neq \emptyset$.

Now suppose that both $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$. Moreover, suppose $\alpha_0 > \beta_0$.

SUBCASE 5.5. $\beta_0 > \gamma_1 + 1$

Whether or not Φ_0^β is a monic monomial or not, this argument is the same as in the same subcase in the proof of the lower bound of the $n = 1$ case.

SUBCASE 5.6. $\beta_0 = \gamma_1 + 1$ and $\Phi_0^\beta = \omega^{\beta_0} \cdot l_0$ and $l_0 = 1$

Suppose the recursive condition holds. If I plays $a_1 < \Phi_0^\beta$ in α or β, then II copies from below and these moves are stalling for I. Suppose $\Phi_0^\beta \leq a_1 < \Phi_0^\alpha$ in α. If the terminal power of a_1 is $> \beta_0$, then II plays $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0$. On the right, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 2 = 2\beta_0$ since the recursive condition holds. Thus, $\gamma(\alpha, \beta) \geq 2\gamma_1 + 1$. If the terminal power of a_1 equals β_0, then again II responds with $b_1 = \Phi_0^\beta$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0$. On the right, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 2 = 2\beta_0$. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 1$. If the terminal power of a_1 is $< \beta_0$, then II copies a tail.

Now suppose the recursive condition fails. Any $a_1 < \Phi_0^\beta$ in α or β is stalling for I. If $\Phi_0^\beta \leq a_1 < \Phi_0^\alpha$ in α. Then II plays as in the monic monomial case of the Separated Game formula by playing a compression of a_1.

Subcase 5.7. $\beta_0 = \gamma_1 + 1$ and $l_0 \geq 2$

All of these instances of the formula are proven similarly to the $\Phi_0^\alpha \neq \emptyset, \Phi_0^\beta = \emptyset$ cases.

Subcase 5.8. $\beta_0 = \gamma_1 + 1$ and Φ_0^β is not a monomial

Suppose the recursive condition holds. If I plays $a_1 < \Phi_0^\beta$ in β having terminal power $\geq \beta_0$, then II plays $\omega^{\beta_0} \cdot 4$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is either separated, in which case $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0 + 1$, or $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced, in which case $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\beta_0 + 2$. On the right, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 1$ since the recursive condition holds. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$. If $a_1 < \Phi_0^\beta$ in β and the terminal power is $< \beta_0$, then II plays the same b_1 in α plus copies a tail. If I plays $a_1 < \Phi_0^\beta$ in α, then II copies from below and this a_1 is stalling for I. Suppose I plays $a_1 \geq \Phi_0^\beta$ in α. If the terminal power of a_1 is $> \beta_0$, then II responds with b_1 pinching off a block of ω^{β_0} in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\beta_0 + 1$. On the right, using the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\beta_0 + 1$ since the recursive condition holds. Thus, $\gamma(\alpha, \beta) \geq 2\beta_0 + 2$. If the terminal power of a_1 is β_0, then II again plays to pinch off a block of ω^{β_0} in β. As before,

$\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\beta_0 + 1$ and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq 2\beta_0 + 1$ since the recursive condition holds. If the terminal power of a_1 is $< \beta_0$, then II plays to copy tail.

If the recursive condition fails, then just as in the $\Phi^\alpha_0 \neq \emptyset, \Phi^\beta_0 = \emptyset$ and $l_1 = 3$ subcase, II plays just as before and $\gamma^{a_1,b_1}_{\mathsf{RHS}} \leq 2\beta_0$ since the recursive condition fails.

This ends the case when I plays in the τ_0-block.

Case 6. I plays in the τ_1-block

This case deals with a_1 in α where $\Phi^\alpha_0 \leq a_1 < \Phi^\alpha_1$ or a_1 in β where $\Phi^\beta_0 \leq a_1 < \Phi^\beta_1$. Either Φ^α_0 or Φ^β_0 may be empty.

First suppose $k_1 = l_1$. If I plays any $a_1 = \Phi^\alpha_0 + \eta$ in α where $0 \leq \eta < \omega^{\gamma_1} \cdot k_1$, then II copies playing $b_1 = \Phi^\beta_0 + \eta$ in β, and vice versa. By inspection of the formula, it should be clear on the left that $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \tau_0 - 1$ when τ_0 is a successor and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq \tau_0$ when τ_0 is limit. This is because if a_1 changed the recursive condition from true to false, the overall formula only goes down by 1, and this cost Player I a move to do this. Similarly, on the right, $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{\tau_2, \ldots, \tau_n\}$ since no move in the γ_1-block can change the value of any of the terms in blocks to the right of the γ_1-block. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$ for whatever values the τ_i terms take. So any move in an ∞-block is a stalling move for I. For the rest of this case, assume that $k_1 \neq l_1$, and by the symmetry of the formula, in fact, assume $k_1 > l_1$.

We introduce the following notation. When we need to distinguish between the terms of the original game $G(\alpha, \beta)$ and the terms of a left or right game, we will use a superscript RHS or LHS. Terms without a superscript refer to the original $G(\alpha, \beta)$.

Subcase 6.1. $\Phi^\alpha_0, \Phi^\beta_0 = \emptyset$ and $\mathsf{R}^l_1 = 0$

In this instance of the formula where $\tau_1 = 2\gamma_1 + \left\lfloor \log_2 \left(l_1 + \mathsf{R}^l_1 \right) \right\rfloor = 2\gamma_1 + \lfloor \log_2 l_1 \rfloor$, we prove $\gamma(\alpha, \beta) \geq \min\{\tau_1, \ldots, \tau_n\}$ by induction on l_1.

Suppose $l_1 = 1$. If I plays any a_1 hole in α, then II responds by playing b_1 in β a γ'-compression of a_1 where, as usual, $\gamma' < \gamma_1$ is appropriate to whether γ_1 is a successor or limit. On the left, $G^{a_1,b_1}_{\mathsf{LHS}}$ is separated and $\gamma^{a_1,b_1}_{\mathsf{LHS}} \geq 2\gamma' + 1$. On the right, by induction

$\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{\tau^{\mathsf{RHS}}_1, \ldots, \tau^{\mathsf{RHS}}_n\}$. Now it should be clear by inspection of the formula that this a_1, b_1 does not disturb the formula in blocks to the right so that for each $2 \leq i \leq n$, $\tau^{\mathsf{RHS}}_i = \tau_i$. For τ^{RHS}_1, if $\tau^{\mathsf{RHS}}_1 \neq \infty$, II can last at least as long as he does in the pure monomial game so that $\tau^{\mathsf{RHS}}_1 \geq 2\gamma_1 + \lfloor \log_2 (l_1 + \mathsf{R}^l_1) \rfloor = 2\gamma_1$. Thus, we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{2\gamma_1, \tau_2, \ldots, \tau_n\}$ and thus,

$$\begin{aligned}
\gamma(\alpha, \beta) &= \min\{\gamma^{a_1,b_1}_{\mathsf{LHS}} + 1, \gamma^{a_1,b_1}_{\mathsf{RHS}} + 1\} \\
&\geq \min\{2\gamma' + 2, 2\gamma_1 + 1, \tau_2 + 1, \ldots, \tau_n + 1\} \\
&\geq \min\{2\gamma_1, \tau_2, \ldots, \tau_n\} \\
&= \min\{\tau_1, \ldots, \tau_n\}
\end{aligned}$$

If a_1 is any nonhole move in α, then II again compresses. If a_1 is in β, then II copies from below playing $b_1 = a_1$ in α. On the left, $\gamma^{a_1,b_1}_{\mathsf{LHS}} = \infty$ and on the right $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{2\gamma_1, \tau_2, \ldots, \tau_n\}$. Thus, reasoning similarly as above $\gamma(\alpha, \beta) = \min\{\gamma^{a_1,b_1}_{\mathsf{LHS}} + 1, \gamma^{a_1,b_1}_{\mathsf{RHS}} + 1\} \geq \min\{2\gamma_1 + 1, \tau_2, \ldots, \tau_n\} \geq \min\{\tau_1, \ldots, \tau_n\}$.

If $l_1 > 1$, then II responds to I as in a pure monomial game. The argument is the same.

Subcase 6.2. $\Phi^\alpha_0, \Phi^\beta_0 = \emptyset$ and $\mathsf{R}^l_1 = 1$.

In this instance of the formula where $\tau_1 = 2\gamma_1 + \lfloor \log_2 (l_1 + \mathsf{R}^l_1) \rfloor = 2\gamma_1 + \lfloor \log_2 l_1 \rfloor$, we prove $\gamma(\alpha, \beta) \geq \min\{\tau_1, \ldots, \tau_n\}$ by induction on l_1.

Suppose $l_1 = 1$. If I plays any hole in α, then II responds with $b_1 = \Phi^\beta_1$ in β. On the left, $G^{a_1,b_1}_{\mathsf{LHS}}$ is pure monomial and $\gamma^{a_1,b_1}_{\mathsf{LHS}} = 2\gamma_1 + \lfloor \log_2 l_1 \rfloor = 2\gamma_1$. On the right, using the τ_0-term of $G^{a_1,b_1}_{\mathsf{RHS}}$ by induction we have $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{2\gamma_2 + 2, \tau^{\mathsf{RHS}}_1, \ldots, \tau^{\mathsf{RHS}}_n\}$ and since $\mathsf{R}^l_1 = 1$ we have $2\gamma_2 + 2 = 2\gamma_1$. For the τ^{RHS}_1 term we observe that since $\mathsf{R}^l_1 = 1$, we must have $l_2 \geq 3$ (note that l_2 in $G^{a_1,b_1}_{\mathsf{RHS}}$ now corresponds to the τ^{RHS}_1 term). So even though that now in $G^{a_1,b_1}_{\mathsf{RHS}}$ in the τ^{RHS}_1 term (which is the τ_2 term in $G(\alpha, \beta)$) II can no longer run to the left, the formula for the γ_1 block has only decreased by one. That is, $\tau^{\mathsf{RHS}}_1 = \tau_2 - 1$. Thus, by arguments similar to those given above, $\gamma^{a_1,b_1}_{\mathsf{RHS}} \geq \min\{2\gamma_1, \tau_2 - 1, \tau_3, \ldots, \tau_n\}$, and thus,

$\gamma(\alpha,\beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1}+1, \gamma_{\mathsf{RHS}}^{a_1,b_1}+1\} \geq \min\{2\gamma_1+1, \tau_2, \tau_3+1, \ldots, \tau_n+1\} \geq \min\{\tau_1, \ldots, \tau_n\}$. If a_1 in α is not a hole or fence, then II copies a tail. The presence of the tail does not decrease the lower bound. If $a_1 < \Phi_1^{\beta}$ is in β, then II copies from below and this move is stalling for I.

Suppose $l_1 > 1$ and for all $l' < l_1$ the formula holds. Suppose for the moment that I plays a hole in β, say $a_1 = \omega^{\gamma_1} \cdot l'$ where $1 \leq l' \leq l_1 - 1$. There are two cases when I plays a hole in β:

(1) $1 \leq l' \leq \left\lfloor \frac{l_1+1}{2} \right\rfloor$

(2) $\left\lfloor \frac{l_1+1}{2} \right\rfloor + 1 \leq l' \leq l_1 - 1$

If $1 \leq l' \leq \left\lfloor \frac{l_1+1}{2} \right\rfloor$, then II copies from below playing $b_1 = a_1$ in α. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$. On the right, by induction

$$\begin{aligned} \gamma_{\mathsf{RHS}}^{a_1,b_1} &\geq \min\{2\gamma_1 + \left\lfloor \log_2\left(l_1 - l' + \mathsf{R}_1^{l}\right)\right\rfloor, \tau_2^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\} \\ &\geq \min\{2\gamma_1 + \left\lfloor \log_2\left(\left\lfloor \frac{l_1}{2} \right\rfloor + 1\right)\right\rfloor, \tau_2, \ldots, \tau_n\} \end{aligned}$$

We claim that $\left\lfloor \log_2\left(\left\lfloor \frac{l_1}{2} \right\rfloor + 1\right)\right\rfloor \geq \lfloor \log_2(l_1+1)\rfloor - 1$. From this it follows that $\gamma(\alpha,\beta) \geq \min\{\tau_1, \ldots, \tau_n\}$.

Proof (claim). Write $l_1 = 2^{\lfloor \log_2 l_1 \rfloor + 1} - j$ where $1 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$.

If $j = 1$, then

$$\begin{aligned} \left\lfloor \log_2\left(\left\lfloor \frac{l_1}{2} \right\rfloor + 1\right)\right\rfloor &= \left\lfloor \log_2\left(\frac{l_1 - 1}{2} + 1\right)\right\rfloor \\ &= \left\lfloor \log_2\left(\frac{l_1+1}{2}\right)\right\rfloor \\ &= \lfloor \log_2(l_1+1)\rfloor - 1 \end{aligned}$$

If $2 \leq j \leq 2^{\lfloor \log_2 l_1 \rfloor}$, then

$$\begin{aligned} \left\lfloor \log_2\left(\left\lfloor \frac{l_1}{2} \right\rfloor + 1\right)\right\rfloor &\geq \left\lfloor \log_2\left(\left\lfloor \frac{l_1}{2} \right\rfloor\right)\right\rfloor \\ &= \lfloor \log_2 l_1 \rfloor - 1 \end{aligned}$$

$$= \quad \lfloor \log_2 (l_1+1) \rfloor - 1$$

Now suppose $\lfloor \frac{l_1+1}{2} \rfloor + 1 \leq l' \leq l_1 - 1$. Then II copies from above (in the γ_1 block) playing $b_1 = \omega^{\gamma_1} \cdot (k_1 - (l_1 - l'))$. On the right, $\tau_1^{\mathsf{RHS}} = \infty$ and none of the terms to the right are disturbed from the original $G(\alpha, \beta)$ so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2+1, \ldots, \tau_n+1\}$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is pure monomial and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + \lfloor \log_2 l' \rfloor \geq 2\gamma_1 + \lfloor \log_2 \lfloor \frac{l_1+1}{2} \rfloor \rfloor = 2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor - 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\} = \min\{2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_1, \ldots, \tau_n\}$.

If I plays a hole in α, then there are three cases:

(1) $1 \leq l' \leq \lfloor \frac{l_1+1}{2} \rfloor$

(2) $\lfloor \frac{l_1+1}{2} \rfloor + 1 \leq l' \leq k_1 - (l_1 - \lfloor \frac{l_1}{2} \rfloor) - 1$

(3) $k_1 - (l_1 - \lfloor \frac{l_1}{2} \rfloor) \leq l' \leq k_1 - 1$

Now if I plays a_1 to be in either cases (1) or (3), then II plays vice versa to when I played in β and the argument is the same as above. So suppose we are in case (2). Note that when $k_1 = l_1 + 1$, case (2) is empty. If $k_1 > l_1 + 1$, then II plays $b_1 = \omega^{\gamma_1} \cdot \lfloor \frac{l_1+1}{2} \rfloor$. On the left, by the same argument as above, $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor - 1$. On the right, by induction $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 \left(\lfloor \frac{l_1}{2} \rfloor + \mathsf{R}_1^l \right) \rfloor = 2\gamma_1 + \lfloor \log_2 \left(\lfloor \frac{l_1}{2} \rfloor + 1 \right) \rfloor$. The same claim above shows that $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor - 1$. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor - 1, \tau_2, \ldots, \tau_n\}$ and thus, $\gamma(\alpha, \beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\} \geq \min\{2\gamma_1 + \lfloor \log_2 (l_1+1) \rfloor, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_1, \ldots, \tau_n\}$.

If I plays any nonhole in either α or β, then II plays as above plus copies a tail. The presence of the tail does not decrease the lower bound.

Subcase 6.3. $\Phi_0^\alpha \neq \emptyset$, $\Phi_0^\beta = \emptyset$

Observe first that in all cases for $l_1 \geq 1$, if I plays $a_1 = \Phi_0^\alpha$ in α, all of the same arguments from the case when I played in τ_0 still hold. So it is enough to show that II holds the lower bound when $\Phi_0^\alpha < a_1 < \Phi_1^\alpha$ in α or $a_1 < \Phi_1^\beta$ in β.

Suppose $l_1 = 1$. If I plays $a_1 < \Phi_1^\beta$ in β, then II copies from below. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$. On the right, all of the terms in $\gamma_{\mathsf{RHS}}^{a_1,b_1}$ are the same as in $G(\alpha,\beta)$. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \gamma(\alpha,\beta)$, and this a_1 is stalling for I.

Now suppose I plays $\Phi_0^\alpha < a_1 < \Phi_1^\alpha$ in α. Also, suppose the recursive condition holds, $\mathsf{R}_1^l = 1$. If I plays any hole in the τ_1 block of α, then II plays $b_1 = \Phi_1^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_0^{\mathsf{RHS}}, \tau_1^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\}$. Now, since $\mathsf{R}_1^l = 1$, we have $\tau_0^{\mathsf{RHS}} \geq 2\gamma_2 + 2 = 2\gamma_1$. As in the case above when $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$, $\tau_1^{\mathsf{RHS}} = \tau_2 - 1$. Moreover, the remaining terms in $G_{\mathsf{RHS}}^{a_1,b_1}$ are undisturbed so that $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1, \tau_2 - 1, \tau_3, \ldots, \tau_n\}$. Thus, $\gamma(\alpha,\beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\} \geq \min\{\tau_0 + 1, 2\gamma_1 + 1, \tau_2, \tau_3 + 1 \ldots, \tau_n + 1\} = \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays any nonhole in the τ_1 block of α, then II copies a tail, the presence of which does not decrease the lower bound.

Now suppose the recursive condition fails, $\mathsf{R}_1^l = 0$. If I plays any hole in the τ_1 block of α, then II responds with a γ'-compression of a_1 where $\gamma' < \gamma_1$ is appropriate to whether γ_1 is a limit or a successor. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma' + 1$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_1^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\}$. Now by induction $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2\left(l_1 + \mathsf{R}_1^l\right)\right\rfloor = 2\gamma_1$. Moreover, the all of the other terms to the right in $G_{\mathsf{RHS}}^{a_1,b_1}$ are undisturbed by this move. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1, \tau_2, \ldots, \tau_n\}$, and thus, $\gamma(\alpha,\beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\} \geq \min\{\tau_0, 2\gamma_1, 2\gamma_1 + 1, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays any nonhole in the τ_1 block of α, then II copies a tail, the presence of which does not decrease the lower bound.

Suppose $l_1 = 2$. If I plays any $a_1 < \Phi_1^\beta$ in β, then II copies from below, and the argument is the same as above. If I plays any hole in the τ_1 block of α, then II plays $b_1 = \omega^{\gamma_1}$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + \left\lfloor \log_2\left(l_1 + \mathsf{R}_1^l\right)\right\rfloor \geq 2\gamma_1$ using the $l_1 = 1$ case when $\Phi_0^\alpha, \Phi_0^\beta = \emptyset$. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 1, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays any nonhole in the τ_1 block of α, then II copies a tail, the presence of which does not decrease the lower bound.

Suppose $l_1 = 3$. If I plays any $a_1 < \omega^{\gamma_1} \cdot 2$ in β, then II copies from below and the argument is the same as before. If $\omega^{\gamma_1} \cdot 2 \leq a_1 < \Phi_1^{\beta}$, then II plays to pinch off the last block of ω^{γ_1} in the τ_1 block in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $\tau_1^{\mathsf{RHS}} = \infty$ and all of the other terms are undisturbed. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

Now suppose I plays a hole in the τ_1 block of α. If I plays $a_1 = \Phi_0^{\alpha} + \omega^{\gamma_1} \cdot k'$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, by induction, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + \left\lfloor \log_2\left(1 + \mathsf{R}_1^l\right) \right\rfloor, \tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + \left\lfloor \log_2\left(3 + \mathsf{R}_1^l\right) \right\rfloor, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays any nonhole in the τ_1 block of α, then II copies a tail, the presence of which does not decrease the lower bound.

Suppose $l_1 \geq 4$. This is by induction on l_1, but it proceeds as it has before. All of the cases where I plays in the τ_1 block of β are as before. If I plays any hole in the τ_1-block of α except the last, then II responds with $\omega^{\gamma_1} \cdot 2$ in β. If I plays the last hole $a_1 = \Phi_0^{\alpha} + \omega^{\gamma_1} \cdot (k_1 - 1)$, then II plays $b_1 = \omega^{\gamma_1} \cdot 3$ in β. In each case, our previous arguments have shown that $\gamma(\alpha, \beta) \geq \min\{\tau_0, \ldots, \tau_n\}$.

Subcase 6.4. $\Phi_0^{\alpha} = \emptyset$, $\Phi_0^{\beta} \neq \emptyset$

Suppose $l_1 = 1$ and $k_1 = 2$. If I plays Φ_0^{β} in β, then II responds with ω^{γ_1} in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, from what we have said above it should be clear that $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 1, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays a hole in α, then II plays vice versa and the argument is the same. If I plays a nonhole in α or β then II copies a tail.

Suppose $l_1 = 2$, $k_1 = 3$, and $\mathsf{R}_1^k = 0$. If I plays Φ_1^{β} in β, then II plays ω^{γ_1} in α and the argument is almost identical to the $l_1 = 1$, $k_1 = 2$ case. Vice versa if I plays ω^{γ_1} in α. If I plays $\Phi_1^{\beta} + \omega^{\gamma_1}$ in β, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in α and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$

is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is identical to the $l_1 = 1$, $k_1 = 2$ case. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_+ 2, \tau_2, \ldots, \tau_n\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays a nonhole in α or β then II copies a tail.

Now the τ_1 term is $2\gamma_1 + 2$ in all of the rest of the cases when $\Phi_0^\alpha = \emptyset$ and $\Phi_0^\beta \neq \emptyset$.

Suppose $l_1 = 1$ and $k_1 \geq 3$. If I plays Φ_1^β in β, then II responds with $\omega^{\gamma_1} \cdot 2$ in α and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as before so that the terms to the right are undisturbed. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays ω^{γ_1} in α, then II runs to the left playing a copying move $b_1 = \omega^{\gamma_1}$. On the left, $\gamma_{\mathsf{LHS}}^{a_1,b_1} = \infty$. On the right, $G_{\mathsf{RHS}}^{a_1,b_1}$ is as in the $l_1 = 1$, $k_1 = 2$ case. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays a nonhole in α or β then II copies a tail.

Suppose $l_1 = 2$, $k_1 = 3$ and $\mathsf{R}_1^k = 1$. If I plays Φ_1^β in β, then II responds with $\omega^{\gamma_1} \cdot 2$ in α and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_1^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\}$. Now $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(1 + \mathsf{R}_1^k\right) \right\rfloor = 2\gamma_1 + 1$ and each of the remaining terms of $G_{\mathsf{RHS}}^{a_1,b_1}$ are undisturbed. Thus, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + 1, \tau_2, \ldots, \tau_n\}$. So we have $\gamma(\alpha, \beta) \geq \min\{\gamma_{\mathsf{LHS}}^{a_1,b_1} + 1, \gamma_{\mathsf{RHS}}^{a_1,b_1} + 1\} \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $\Phi_1^\beta + \omega^{\gamma_1}$ in β, then II responds with $\omega^{\gamma_1} \cdot 2$ in α. Now the argument is the same as the $l_1 = 2$, $k_1 = 3$, and $\mathsf{R}_1^k = 0$ case when I played ω^{γ_1} in α so that $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1 \ldots, \tau_n\}$. If I plays ω^{γ_1} in α, then II runs to the left playing a copying move $b_1 = \omega^{\gamma_1}$ in β. The same argument above shows that II holds the bound. If I plays a nonhole in α or β then II copies a tail.

The rest of the arguments in this repeat previous ones. We will simply identify I's move and II's response that holds the bound when I plays some hole in α or β.

Suppose $l_1 = 2$ and $k_1 \geq 4$. If I plays the first hole in α, $a_1 = \omega^{\gamma_1}$, then II responds by running to the left and copying $b_1 = \omega^{\gamma_1}$ in β. Observe that now $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(3 + l_1 + \mathsf{R}_1^l\right) \right\rfloor \geq 2\gamma_1 + 2$, so that II easily holds the lower bound. If I plays the last hole in the γ_1-block of α, then II responds with $b_1 = \Phi_1^\beta + \omega^{\gamma_1}$ in β and vice versa. If I plays any other hole in α, then II responds with $b_1 = \Phi_1^\beta$ in β.

Suppose $l_1 \geq 3$. If I plays ω^{γ_1} in α, II copies $b_1 = a_1$ in β. If I plays Φ_1^β in β, then II responds with $b_1 = \omega^{\gamma_1} \cdot 2$ in α and vice versa. If I plays any hole in β, then II copies from above the same number of holes from the right in the γ_1 block in β and vice versa. If I plays any hole in α not covered by the previous cases, II responds with $b_1 = \Phi_1^\beta$ in β.

This ends the case when $\Phi_0^\alpha = \emptyset$ and $\Phi_0^\beta \neq \emptyset$.

Subcase 6.5. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 1$ and $k_1 = 2$

Suppose first that $\mathsf{R}_1^k = \mathsf{R}_1^l = 1$. If I plays $a_1 = \Phi_0^\alpha$ in α, then II responds with $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_0$ where $\gamma_0 = \min\{\alpha_0, \beta_0\}$ and $2\gamma_0 \geq 2\gamma_1 + 2$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + 1, \tau_2, \ldots, \tau_n\}$ since $\mathsf{R}_1^l = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. Similarly, if I plays $a_1 = \Phi_0^\beta$ in β, II responds with $b_1 = \Phi_0^\alpha$ in α. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1}$, the hole in the γ_1-block of α, then II plays $\omega^{\gamma_1} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + 1, \tau_2, \ldots, \tau_n\}$ since $\mathsf{R}_1^k = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If a_1 is not a hole or fence in the τ_1-block, then II copies a tail.

If either of the recursive conditions fails, II still plays the same as he did before. If I plays Φ_0^β in β, then II responds with ω^{γ_1} in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$ since the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ is ∞. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 1, \tau_2 + 1, \ldots, \tau_n\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. In the other possibilities for I's move, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1$ now because the recursive condition does not hold on one side or the other.

Subcase 6.6. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 1$ and $k_1 \geq 3$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II plays $b_1 = \omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_0^{\mathsf{RHS}}, \tau_1^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\}$.

Now $\tau_0^{\mathsf{RHS}} \geq 2\gamma_1+1$. And, $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(l_1 + 3 + \mathsf{R}_1^l \right) \right\rfloor \geq 2\gamma_1+1$. The remaining terms are undisturbed. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0+1, 2\gamma_1+2, \tau_2+1, \ldots, \tau_n\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays Φ_0^β in β, then II plays $\Phi_0^\alpha + \omega^{\gamma_1} \cdot 2$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1+1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1}$ in α, the first hole in the γ_1-block, then II plays $\omega^{\gamma_1} \cdot 2$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1+1$. On the right, by induction we have $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_0^{\mathsf{RHS}}, \tau_1^{\mathsf{RHS}}, \ldots, \tau_n^{\mathsf{RHS}}\}$. Now $\tau_0^{\mathsf{RHS}} = 2\gamma_1+1$. And, $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(l_1 + 3 + \mathsf{R}_1^l \right) \right\rfloor \geq 2\gamma_1+1$. The remaining terms are undisturbed. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0+1, 2\gamma_1+2, \tau_2+1, \ldots, \tau_n\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 2$ in α, then II plays $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1+1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha,\beta) \geq 2\gamma_1+2$. If I plays any a_1 in α or β that is not a fence or hole, then II copies a tail.

Subcase 6.7. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 2$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II plays $b_1 = \Phi_0^\beta$ in β and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1+2$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1+1, \tau_2, \ldots, \tau_n\}$ since $l_1 = 2$. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0+1, 2\gamma_1+2, \tau_2+1, \ldots, \tau_n+1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1}$ in β, then II plays $b_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 2$ in α and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n=1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1+1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha,\beta) \geq 2\gamma_1+2$. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1}$ in α, then II plays $\omega^{\gamma_1} \cdot 2$ in β. If I plays a_1 that is not a hole or fence, then II copies a tail.

Subcase 6.8. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 3$ and $k_1 = 4$

If I plays $a_1 = \Phi_0^\alpha$ in α, then II plays $b_1 = \Phi_0^\beta$ and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1+2$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1+1, \tau_2, \ldots, \tau_n\}$ since $l_1 = 3$. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0+1, 2\gamma_1+2, \tau_2+1, \ldots, \tau_n+1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $\Phi_0^\alpha + \omega^{\gamma_1}$ in α, then II plays $\omega^{\gamma_1} \cdot 2$ in β. The argument then proceed as in previous cases when II runs to the left and copies. Thus, $\gamma(\alpha,\beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays

$a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 2$ in α, then II plays $\Phi_1^\beta + \omega^{\gamma_1} \cdot 1$ and vice versa. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 1$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot 3$ in α, then II repsonds with $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot 2$ and vice versa. The argument is the same as when I plays one hole to the left. If I plays any nonhole or nonfence, II copies a tail.

Subcase 6.9. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 3$ and $k_1 \geq 5$

Suppose first that $\mathsf{R}_1^k = \mathsf{R}_1^l = 1$. First suppose I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for $0 \leq k' \leq k_1$ in α. For $k' = k_1 - 1, k_1 - 2$, then II plays $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ where $l' = 1, 2$, respectively. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If $0 \leq k' \leq k_1 - 3$, then II responds with $b_1 = \omega^{\gamma_1} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced (or separated when $k' = 0$) and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. On the right, by induction $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{2\gamma_1 + 1, \tau_2, \ldots, \tau_n\}$ since $\mathsf{R}_1^k = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0 + 1, 2\gamma_1 + 2, \tau_2 + 1, \ldots, \tau_n + 1\} \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. Now suppose I plays $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ for $0 \leq l'2$. If $l' = 1, 2$, then I plays vice verse as above. If $l' = 0$, then II plays $b_1 = \Phi_0^\alpha$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. On the right, the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_1^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(3 + \mathsf{R}_1^l\right) \right\rfloor = 2\gamma_1 + 2$ since $\mathsf{R}_1^l = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If a_1 is not a hole or fence, then II copies a tail.

Now suppose either $\mathsf{R}_1^k = 0$ or $\mathsf{R}_1^l = 0$. Then all of II's responses above show that $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

Subcase 6.10. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 4$ and $k_1 = 5$

Suppose first that $\mathsf{R}_1^k = \mathsf{R}_1^l = 1$. First suppose I plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for $0 \leq k' \leq 4$ in α. For $k' = 3, 4$, then II plays $b_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ where $l' = 2, 3$, respectively. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If $k' = 1, 2$, then II plays $\omega^{\gamma_1} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, the τ_0-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_0^{\mathsf{RHS}} = 2\gamma_1 + \left\lfloor \log_2 \left(3 + \mathsf{R}_1^k\right) \right\rfloor = 2\gamma_1 + 2$ since $\mathsf{R}_1^k = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$.

If $k' = 0$, then II plays $b_1 = \Phi_0^\beta$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_0 \geq 2\gamma_1 + 2$ where $\gamma_0 = \min\{\alpha_0, \beta_0\}$. On the right, the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 (4 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. Now suppose I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ in β for $0 \leq l' \leq 3$. If $l' = 2, 3$, then II plays b_1 vice versa in α as above. If $l' = 1$, then II plays $b_1 = \Phi_0^\alpha + \omega^{\gamma_1}$ in α. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_0 \geq 2\gamma_1 + 2$. On the right, the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 (3 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 2$ since $\mathsf{R}_1^l = 1$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If $l' = 0$, then II plays $b_1 = \Phi_0^\alpha$. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_0 \geq 2\gamma_1 + 2$. On the right, the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 (4 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If a_1 is not a hole or fence, then II copies a tail.

Now suppose either $\mathsf{R}_1^k = 0$ or $\mathsf{R}_1^l = 0$. Then all of II's responses above show that $\gamma(\alpha, \beta) \geq 2\gamma_1 + 2 = \tau_1$.

Subcase 6.11. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 = 4$ and $k_1 \geq 6$

First suppose plays $a_1 = \Phi_0^\alpha + \omega^{\gamma_1} \cdot k'$ for $0 \leq k' \leq k_1 - 1$ in α. For $k' = k_1 - 1, k_1 - 2, k_1 - 3$, then II plays $\Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ in β where $l' = 3, 2, 1$, respectively. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is as in the $n = 1$ case and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. On the right, $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq \min\{\tau_2, \ldots, \tau_n\}$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If $1 \leq k' \leq k_1 - 4$, then II plays $\omega^{\gamma_1} \cdot 4$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is unbalanced and $\gamma_{\mathsf{LHS}}^{a_1,b_1} = 2\gamma_1 + 2$. On the right, both the τ_0 and τ_1 terms of $G_{\mathsf{RHS}}^{a_1,b_1}$ give $\gamma_{\mathsf{RHS}}^{a_1,b_1} \geq 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. If $k' = 0$, then II plays $b_1 = \Phi_0^\beta$ in β. On the left, $G_{\mathsf{LHS}}^{a_1,b_1}$ is separated and $\gamma_{\mathsf{LHS}}^{a_1,b_1} \geq 2\gamma_0 \geq 2\gamma_1 + 2$. On the right, the τ_1-term of $G_{\mathsf{RHS}}^{a_1,b_1}$ by induction is $\tau_1^{\mathsf{RHS}} \geq 2\gamma_1 + \lfloor \log_2 (4 + \mathsf{R}_1^l) \rfloor = 2\gamma_1 + 2$. Thus, $\gamma(\alpha, \beta) \geq \min\{\tau_0, \tau_1, \ldots, \tau_n\}$. Now suppose I plays $a_1 = \Phi_0^\beta + \omega^{\gamma_1} \cdot l'$ in β where $0 \leq l' \leq 3$. If $l' = 1, 2, 3$, then II plays vice verse in α as before. If $l' = 0$, then II plays $b_1 = \Phi_0^\alpha$ in α as before. If a_1 is not a hole or fence, then II copies a tail.

Subcase 6.12. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 \geq 5$ and $\mathsf{R}_1^l = 0$

This case is identical to the same subcase in the $n = 1$ case.

Subcase 6.13. $\Phi_0^\alpha, \Phi_0^\beta \neq \emptyset$ and $l_1 \geq 5$ and $\mathsf{R}_1^l = 1$

The argument is by induction and the computational details are identical to those in the proof of the the finite game.

This ends the case when I plays in the τ_1-block.

Case 7. I plays in the τ_i-block, $1 < i < n$.

The formula is the same as the τ_1 formula and the argument proceeds by induction similarly to the τ_1 case.

Case 8. I plays in the τ_n-block

The formula is the same as in τ_1 formula in the $n = 1$ case and the argument proceeds by induction similarly.

$\square$

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