CHARACTERIZATIONS OF CONTINUA OF FINITE DEGREE

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Thesis Prepared for the Degree of

MASTER OF ARTS

UNIVERSITY OF NORTH TEXAS

August 2006

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Irwin, Shana, Characterizations of Continua of Finite Degree. Master of Arts (Mathematics), August 2006. 25 pp., references, 10 titles.

In this thesis, some characterizations of continua of finite degree are given. It turns out that being of finite degree (by formal definition) can be described by saying there exists an equivalent metric in which the $\mathcal{H}^1$ or Hausdorff linear measure of the continuum is finite. I discuss this result in detail.

ACKNOWLEDGEMENTS

I would like to thank my advisor, R. Daniel Mauldin for discerning my interests better than I could; and for showing great patience in helping me pursue them.

I would like to thank William Cherry for all the time he spent proofreading and the questions that helped me see where there were blanks to fill; and I would like to thank my committee members, William Cherry, Paul Lewis, and Su Gao.

Additionally, I would like to thank Ed Tymchatyn for personal communication Regarding background, updates and notation.

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CHAPTER 1

BACKGROUND AND NOTATION

Throughout this paper, X and Y will be compact metrizable spaces. Specifically, we are studying continua of finite degree. These continua are sometimes referred to as 'totally regular.' The notation 'totally regular' was introduced by Nikiel [9]. Totally regular is weaker than completely regular. In completely regular continua, all proper non-degenerate continua have interior.

The formal definition of a point being of finite degree is as follows:

Definition 1.1. A point, x, is of finite degree in a compact metric space, X, if, for each $\epsilon > 0$, there is an uncountable family of open sets, $\{U_\lambda\}_{\lambda \in \Lambda}$, so that $\forall \lambda, \beta \in \Lambda$,

i) $x \in U_\lambda$
ii) $| \overline{U}_\lambda \backslash U_\lambda | < \omega = \omega_0$
iii) $\overline{U}_\lambda \subset U_\beta$ or $\overline{U}_\beta \subset U_\lambda$ and
iv) $\mathrm{diam}(U_\lambda) < \varepsilon$.

And, most naturally, a continuum is of finite degree means each point is of finite degree.

The definition does not depend on a metric, and can be stated by saying for each neighborhood U containing x there is an uncountable (strongly monotone) family of neighborhoods all lying in U, each with finite boundary and each containing x. However, this thesis deals only with metric spaces, so we will use the definition above. A space X is considered to be of finite degree provided all of its points are of finite degree.

We recall that a strongly monotone family differs from a "monotone" family in that for any two sets in the family, the CLOSURE of one is contained in the INTERIOR of the other.

Let us note for future use that there is an equivalent definition using closed sets, where we simply replace $\overline{U}$ by U and U by U^o, the interior of U. We note also that if a continuum X is of finite degree, then it is a *regular curve*. A space X is a regular curve if there is a basis of open sets, each with finite boundary [6]. However, the converse is far from being true. For example, the Sierpinski gasket is a regular curve; but is not of finite degree, as shown by Kuratowski [6].

Continua of finite degree were characterized almost up to our current knowledge by Samuel Eilenberg and O.G. Harrold, Jr. [2]. That paper represents the accumulation of several years of research on this topic by various authors and it is the focus of this thesis to present parts of the proof of the theorem in that paper in greater detail. We want to mention one additional theorem by D.H. Fremlin which answers a question left open in the Eilenberg-Harrold paper. While Eilenberg and Harrold showed that a continuum of finite degree can be embedded into the Hilbert space, ℓ_2, so that the image has finite linear measure, the question of whether ℓ_2 could be replaced by $[0,1]^n$ for some n, with the usual metric, was left open. Fremlin later showed that, in fact, a continuum of finite degree can be embedded into $[0,1]^3$ so that its image has finite linear measure [4]. This, of course, also makes existence of a homeomorphic image of finite linear measure in $[0,1]^\infty$ imply a homeomorphic image of finite linear measure exists in $[0,1]^3$. This result is the best possible, since by a classic result of Kuratowski, neither the complete graph on five vertices, K_5 nor the complete bipartite graph, $K_{3,3}$ can be embedded into the plane, and each is of finite degree [5].

We recall the basic definitions of Hausdorff measures and dimension. We will be using various properties of Hausdorff measures and dimension, which may be found in books by Falconer, Matilla, and Edgar.[4, 7, 1]

Definition 1.2. Let ρ be a metric on a space X. For each $\alpha \geq 0$, $\mathcal{H}^\alpha$, the Hausdorff α-dimensional measure (with respect to ρ) is defined on subsets A of X as follows:

$$\mathcal{H}^\alpha(A) := \lim_{\delta \to 0}\big(\mathcal{H}^\alpha_\delta(A)\big) \text{ where } \mathcal{H}^\alpha_\delta(A) := inf\{\sum_{G \in F}(\text{diam}(G))^\alpha : F \text{ is a } \delta\text{-mesh cover of } A\}.$$

F is a $\delta - mesh$ cover of A means that F is a family of sets whose union contains A and the diameter of each set in F is no more than δ.

By way of explanation, we recall that $\mathcal{H}^\alpha$ is a metric outer measure on X. This implies that all Borel subsets of X are $\mathcal{H}^\alpha$ measurable. Also, we note that $\mathcal{H}^0$ is counting measure and $\mathcal{H}^1$ is linear Hausdorff measure.

Definition 1.3. A space X has Hausdorff dimension α provided that for each $\beta < \alpha$, $\mathcal{H}^{\beta}(X) = \infty$ and for each $\lambda > \alpha$, $\mathcal{H}^{\lambda}(X) = 0$.

Definition 1.4. A continuum X is *rectifiable* means $\exists$ a metric ρ and a continuous surjection

$$f : [0, 1] \to X$$

such that f has finite arclength.

Definition 1.5. A point, p is a separating point of a connected set, U means that $U \setminus \{p\} = A \cup B$ where A and B are both open in U.

Definition 1.6. A point, p in a locally compact separable metric space, X, is a local separating point of X provided $\exists U$, open in X so that p is a separating point of the component, C, of p in U. [10]

Definition 1.7. We need to define, for ease of reference, the following:

For any function, $f : X \to Y$, let $F(f) := \{y \in Y :| f^{-1}(y) |< \omega\}$

Thus, $F(f)$ consists of those points whose preimage is a finite set. Note that $F(f)$ is a $G_{\delta\sigma}$ for a continuous function on a compact set. This will be shown in Appendix B.

CHAPTER 2

STATEMENT OF THEOREM, COMMENTS AND APPLICATIONS

Here is a slightly reformulated version of the main theorem in the paper by Eilenberg and Harrold [2].

THEOREM 2.1. *For any continuum, X, the following are equivalent:*

I) *X is of finite degree.*

II) *Given any disjoint closed subsets, X_0, X_1 in X, there is a continuous mapping $f(X) = I = [0,1]$ so that:*

a$_2$) *$f(x) = i$ when $x \in X_i$ and $i = 0, 1$.*

b$_2$) $|F(f)| \geq \omega_1$.

III) *Given any disjoint closed subsets, X_0, X_1 in X, there is a continuous mapping $f(X) = I$ so that:*

a$_3$) *$f(x) = i$ when $x \in X_i$ and $i = 0, 1$.*

b$_3$) $I \setminus \mathbb{Q} \subseteq F(f)$

IV) *X can be embedded into the Hilbert space, ℓ_2, so as to have finite linear measure.*

V) *X has a homeomorphic image of finite linear measure.*

VI) *Every subcontinuum of X contains uncountably many local separating points of X.*

VII) *X is locally connected and for every pair of closed, disjoint subsets X_0, X_1 in X, there is a finite collection of disjoint perfect sets $N_1, ...N_k$ so that any continuum in X intersecting both X_0 and X_1 contains some N_i.*

VIII) *Given a sequence $X_0, X_1, X_2...$ of subcontinua so that $lim(X_i) = X$ there is an integer n for which $\bigcap_{i=n}^{\infty} X_i$ is uncountable.*

Being of finite degree is certainly a valuable property to recognize in general. The results contained here are useful in that they give alternate ways to check this. Let us give an example. The dimension of the limit set of Apollonian packing, or the Curvilinear Sierpinski Gasket, was a long open question [3]. If one is not familiar with the curvilinear gasket, one can consider that it is homeomorphic to the usual gasket. It was clear that its dimension,

α, satisfies $1 \leq \alpha \leq 2$; but to make the inequalities strict was surprisingly difficult. Strict inequality was first shown by Hirst in 1965; but Mauldin and Urbanski showed how the question could be answered with relative ease [8]. Using their theory of conformal iterated function systems, they showed that if A had dimension 1, then $\mathcal{H}^1(A) < \infty$. Since A is a continuum, by 2.1 part VI, A would have uncountably many local separating points. However, Kuratowski showed that the Gasket has only countably many local separating points [6]. Therefore, the gasket has Hausdorff dimension strictly greater than 1.

For an example of a continuum of finite degree, one might consider the graph of the Takagi function on the unit interval, $[0, 1]$. This is a famous example of a continuous function which is nowhere differentiable. It certainly has infinite linear measure in the usual metric. In fact, the graph of a continuous function on a closed interval has finite linear Hausdorff measure in the usual metric if and only if the function is of bounded variation. In that case, the linear measure is equal to the arclength of the graph [3]. However, the projection map is a homeomorphism of the graph to the closed interval. Hence, the graph of the Takagi function is a continuum of finite degree, while not having finite linear Hausdorff measure in the usual metric.

One of the beautiful things about theorem 2.1 is that parts of it are purely topological, as the definition itself, whereas other characterizations are highly analytic in nature.

CHAPTER 3

PROOF

Let us begin by showing the equivalence of the first five statements by showing $I \to III \longleftrightarrow II$ and $III \to IV \to V \to I$. We note that II and III tell us that stronger versions of Urysohn's Lemma hold for continua of finite degree.

3.1. I → III

Let X be of finite degree by the formal definition.

Now, for the purpose of showing property III, let X_0, X_1 be disjoint, closed subsets in X. Identify all points of X_0 with some point x_0 of X_0. Define $X^* = \big(X \setminus X_0\big) \cup \{x_0\}$ So, $\phi : X \longrightarrow X^*$ is defined by the rule:

$$\phi(x) = x_0 \text{ if } x \in X_0,\ x \text{ otherwise.}$$

Then, under the quotient topology on X^*, defined in the natural way by: V is open in X^* if and only if $\phi^{-1}\big(V\big)$ is open in X, X^* is a continuous image of a continuum, and is therefore a continuum itself, by construction.

X^* is clearly of finite degree at every point except perhaps x_0. The fact that X^* is of finite degree at x_0 is an easy corollary of a theorem of Whyburn's [10]. Then, since X^* is of finite degree, there exists $\{V_\lambda\}_{\lambda\in\Lambda}$, an uncountable family of closed sets in X^* so that $\forall\lambda,\beta \in \Lambda$, $x_0 \in V_\lambda$, $\mid V_\lambda \backslash V_\lambda^o \mid < \omega$ and $V_\lambda \subset V_\beta^o$ or $V_\beta \subset V_\lambda^o$. We use the equivalent definition mentioned earlier for clearer argument later.

Now, for each $\lambda \in \Lambda$, define $K_\lambda := \phi^{-1}\big(V_\lambda\big)$. Then, since ϕ is continuous, $\{K_\lambda\}_{\lambda\in\Lambda}$ is an uncountable family of closed sets so that $\forall\lambda,\beta \in \Lambda$, $X_0 \subset K_\lambda$, $\mid K_\lambda \backslash K_\lambda^o \mid < \omega$ and $K_\lambda \subset K_\beta^o$ or $K_\beta \subset K_\lambda^o$.

Next we want to apply the following theorem of Kuratowski (proven in Appendix A).

THEOREM 3.1. *Given any two disjoint closed subsets, X_0 and X_1, of a compact metric space X, if $\{K_\lambda\}_{\lambda\in I\backslash\mathbb{Q}}$ is a family of closed subsets of X so that*

1) $\lambda < \beta \iff K_\lambda \subset K_\beta^o$

2) *For all but countably many* λ, $\overline{\bigcup_{\beta<\lambda} K_\beta} = K_\lambda$

and $\bigcup_{\beta<\lambda} K_\beta = K_\lambda^o$

3) *For all but countably many* λ, $\bigcap_{\lambda<\beta} K_\beta = K_\lambda$

4) *For all* λ, $X_0 \subset K_\lambda$ *and* $X_1 \cap K_\lambda = \emptyset$

5) $\exists\ N$ *so that* $\forall\ \lambda$, $\mid K_\lambda \setminus K_\lambda^o \mid \leq N$,

there is a continuous function, $f : X \to I$ *so that* $f(X_0) = \{0\}$, $f(X_1) = \{1\}$ *and for all* $t \in I \setminus \mathbb{Q}$, $\mid \{f^{-1}(t)\} \mid \leq N$.

We need to show that the hypotheses hold. The following will show the hypotheses that are not automatic, through reindexing our sets in a reasonable way.

Suppose, instead of the initial Kuratowski hypotheses, we have $\{K_\lambda\}_{\lambda\in\Lambda}$, an uncountable strongly monotone family of closed sets in a continuum X so that $X_0 \subset K_\lambda$ for all λ and $X_1 \cap K_\lambda = \emptyset$ for all λ and for each λ, $K_\lambda \setminus K_\lambda^o$ is finite.

By taking a suitable uncountable subfamily, if necessary, we can assume there is a positive integer N such that, for each λ, $\mid K_\lambda \setminus K_\lambda^o \mid < N$.

Let $\{B_n\}_{n=1}^\infty$ be a base for the topology of X and define $\phi : \Lambda \longrightarrow I$ by:

$$\phi(\lambda) = \sum_{B_n \subset K_\lambda} \frac{1}{2^n}$$

.

Then ϕ is certainly injective.

For each $\lambda, \beta \in \Lambda$, either $K_\lambda \subset K_\beta$ or $K_\beta \subset K_\lambda$. Without loss of generality, suppose $K_\lambda \subset K_\beta$. Then for each n so that $B_n \subset K_\lambda$, automatically, $B_n \subset K_\beta$. Therefore, $\phi\big(\lambda\big) \leq \phi\big(\beta\big)$. But, there is some point x so that $x \in K_\beta^o \setminus K_\lambda$ and there is some n so that $x \in B_n \subset K_\beta^o \setminus K_\lambda$ as $K_\beta^o \setminus K_\lambda$ is open. Thus, $\phi\big(\lambda\big) < \phi\big(\beta\big)$. Hence $\phi(\lambda) < \phi(\beta) \iff K_\lambda \subset K_\beta^o$.

Since $\phi(\Lambda)$ is an uncountable subset of I, $\phi(\Lambda) = D \cup C$ where C is countable and every point of D is a condensation point of D. Let $P = \overline{D}$. Then P is perfect.

Define an "above" function, $A : P \longrightarrow \mathcal{K}(X) = \{M \subset X \mid M \text{ is compact}\}$ by

$$A(p) = \bigcap \{K_\lambda : p < \phi(\lambda)\}$$

and define a "below" function, $B : P \longrightarrow \mathcal{K}(X)$ by

$$B(p) = \overline{\bigcup\{K_\lambda : p > \phi(\lambda)\}}.$$

Then, for all but possibly countably many $\phi(\lambda) \in D$, A, $B(\phi(\lambda)) = K_\lambda$, as all but countably many points in D must be limit points from both the right and the left.

Claim: $A(p) = B(p)$ for all but countably many $p \in P$.

Suppose not. Then there is an uncountable set $\tilde{P}$ so that $A(p) \neq B(p)$ for any $p \in \tilde{P}$. Clearly $B(p) \subset A(p)$ for all $p \in P$. So, it must be that for all $p \in \tilde{P}$, $A(p) \not\subset B(p)$.

But $B(p)$ and $A(p)$ are closed for each $p \in P$. Then, if there is $x \in A(p) \setminus B(p)$, there also is n_p so that $x \in B_{n_p} \subset X \setminus B(p)$.

So, for each $p \in \tilde{P}$, there is an n_p so that $B_{n_p} \cap A(p) \neq \emptyset$, but $B_{n_p} \cap B(p) = \emptyset$.

Since there are only countably many B_n, there must be an n so that for uncountably many $p \in \tilde{P}$, $B_n \cap A(p) \neq \emptyset$. But, there can only be countably many pairs, p_1, p_2, in P so that there is not a point of D between them (which gives uncountably many points of D between them), else we have uncountably many disjoint intervals in I.

Then, for all but countably many pairs, p_1, p_2, in $\tilde{P}$, if $p_1 < p_2$, then $A(p_1) \subset B(p_2)$. Hence, for all but countably many pairs, p_1, p_2, in $\tilde{P}$, if $p_1 < p_2$, and $B_n \subset A(p_1)$, then $B_n \subset B(p_2)$. This is a contradiction.

We now need to show that for all but countably many $p \in P$, $\mid Fr(B(p)) \mid \leq N$. This will allow us to simply add $\{A(p)\}_{p \in P}$ into our family of sets.

Suppose not. Suppose $\{x_1, \ldots\ x_{N+1}\} \subset Fr(B(p))$. Let $V_1, \ldots\ V_{N+1}$ be connected neighborhoods of $\{x_1, \ldots\ x_{N+1}\}$, respectively, so that $V_i \cap V_j = \emptyset$, $i \neq j$. Then for each i, there is a $p_i < p$ so that $K_{\beta_i} \cap V_i \neq \emptyset$ and $\phi(\beta_i) = p_i$. Let $m = max\{p_i\}$, and choose β so that $\phi(\beta) = m$. Then for each i, $K_\beta \cap V_i \neq \emptyset$. But, $K_\beta \subset B(p)$, so for each i, $Fr(K_\beta) \cap V_i \neq \emptyset$. This is a contradiction to the cardinality of the frontier of K_β.

So, since P contains a homeomorphic copy of Cantor's middle third set, C, we can consider that we have $\{K_c\}_{c \in C}$, a strongly monotone family of closed subsets of X, so that

$X_0 \subset K_c$ for all $c \in C$ and $X_1 \cap K_c = \emptyset$ for all $c \in C$ and for each $c \in C$, $K_c \setminus K_c^o$ is finite. Then, with the Cantor singular function, these sets can be reindexed with a set which contains the irrationals. Then we can throw out the rational-indexed sets, and we have a continuous function, $f : X \to I$ so that $f(X_0) = \{0\}$, $f(X_1) = \{1\}$, and for all but countably many $t \in I \setminus \mathbb{Q}$, $\mid f^{-1}(t) \mid < \omega$.

Therefore, by the theorem of Kuratowski, there is a continuous function $f : X \longrightarrow I$ so that $f\big(X_0\big) = 0$, $f\big(X_1\big) = 1$, and for all $t \in I \setminus \mathbb{Q}$, $\mid f^{-1}(t) \mid < \omega$.

3.2. II $\to$ III

Let X_0, X_1 be disjoint, closed subsets in X. Suppose there is a continuous function, $f : X \longrightarrow I$ so that $f(X_0) = 0$ and $f(X_1) = 1$ and $\mid F(f) \mid \geq \omega_1$. Since $F(f)$ is an uncountable Borel set of real numbers, it contains a homeomorphic copy, N, of Cantor's middle third set. Let h be a continuous, injective map of $[inf(N), sup(N)]$ onto I so that h maps N onto C homeomorphically. Then let $\tilde{h} : I \longrightarrow I$ extend h so that $\tilde{h}$ restricted to $[inf(N), sup(N)]$ is equal to h and $\tilde{h}[[0, h^{-1}(0))] = \{0\}$ and $\tilde{h}[(h^{-1}(1), 1]] = \{1\}$.

Let $g : I \longrightarrow I$ be the continuous singular function of Cantor. Thus, g is constant on every complimentary interval of C, and for every irrational, t, $g^{-1}(t) = \{c\}, c \in C$ and $g(0) = 0$ and $g(1) = 1$.

Now define $f_1 = g \circ \tilde{h} \circ f : X \longrightarrow I$. Then $f_1^{-1}(t) = (g \circ \tilde{h} \circ f)^{-1}(t) = f^{-1}(\tilde{h}^{-1}(g^{-1}(t)))$.

Suppose that $t \in I \setminus \mathbb{Q}$. Then $g^{-1}(t) \in C$, so $\tilde{h}^{-1}(g^{-1}(t)) \in N \subset F(f)$, and, consequently, $\mid f_1^{-1}(t) \mid = \mid f^{-1}(\tilde{h}^{-1}(g^{-1}(t))) \mid < \omega$.

3.3. III $\to$ II

This implication is trivial.

3.4. III $\to$ IV

(We note that this is the first equivalence which is more analytic.)

Suppose that for any two, X_0, X_1, disjoint closed subsets of X, there is a continuous function, $f : X \longrightarrow I$ so that

i) $f(X_0) = 0$ and $f(X_1) = 1$ and
ii) $I \setminus \mathbb{Q} \subset F(f)$

Claim: There also exists, for each pair, X_0, X_1, of disjoint closed subsets of X, a continuous map $g : X \longrightarrow I$ so that

i) $g(X_0) = 0$ and $g(x) > 0$, $x \in X_1$ and

ii) for any regular decomposition, $\{F_i\}$ of X, (that is F_i is a continuum for all i and $F_i \cap F_j$ is finite when $i \neq j$),

$\sum_i \operatorname{diam}(g(F_i)) \leq 1.$

Proof of Claim:

Let X_0, X_1, disjoint closed subsets of X, and $f : X \longrightarrow I$ be as above. Also, define $K : I \longrightarrow \mathbb{N}^*$ by $K(t) = \mid f^{-1}(t) \mid$. Define $g : X \longrightarrow I$ by $g(x) = \int_0^{f(x)} \frac{dt}{K(t)}$.

If $x \in X_0$, $f(x) = 0$, so $g(x) = 0$. If $x \in X_1$, $f(x) = 1$, $g(x) = \int_0^1 \frac{dt}{K(t)}$. Since $f(X) = I$, $K(t) \neq 0$ for any t, and $K(t) < \infty$ on a set of positive measure, so $g(x) > 0$.

Now suppose that $\{F_i\}$ is a regular decomposition. Since for each i, F_i is a continuum, $f(F_i) = [a_i, b_i]$ where $a_i \leq b_i$. So, since g is non-decreasing, for each i, $\operatorname{diam}[g(F_i)] = \int_{a_i}^{b_i} \frac{dt}{K(t)}$. Then, $\sum_i \operatorname{diam}[g(F_i)] = \sum_i \int_{a_i}^{b_i} \frac{dt}{K(t)}$.

Now, arrange $\{a_i\} \cup \{b_i\}$ in the usual order and relabel them $\{c_j\}_{j=1}^r$ so that $c_j < c_{j+1} \forall j$. Now, for each i, $[a_i, b_i] = \bigcup_{l=0}^{m} [c_{j+l}, c_{j+l+1}]$ for some j and some m.

So, $\sum_i \operatorname{diam}[g(F_i)] = \sum_{j=1}^{r-1} K_j \int_{c_j}^{c_{j+1}} \frac{dt}{K(t)}$ where K_j is the number of $[a_i, b_i]$ for which $[c_j, c_{j+1}] \subset [a_i, b_i]$. It follows that $[c_j, c_{j+1}] \subset f(F_i)$ for K_j many i. Now, since $\mid F_i \cap F_j \mid < \infty$ for all $i \neq j$, $\mid f^{-1}(t) \mid \geq K_j$ for all but possibly finitely many $t \in [c_j, c_{j+1}]$.

So, $K_j \leq K(t)$ almost everywhere in $[c_j, c_{j+1}]$

Therefore, $K_j \int_{c_j}^{c_{j+1}} \frac{dt}{K(t)} = \int_{c_j}^{c_{j+1}} \frac{K_j}{K(t)} dt \leq \int_{c_j}^{c_{j+1}} dt = c_{j+1} - c_j$.

So, $\sum_i \operatorname{diam}[g(F_i)] = \sum_{j=1}^{r-1} K_j \int_{c_j}^{c_{j+1}} \frac{dt}{K(t)} \leq \sum_{j=1}^{r-1} (c_{j+1} - c_j) = 1$.

End Claim.

Let $\{U_i\}$ be a countable base. For each $n \in \mathbb{N}$, let $\{X_{0,n}, X_{1,n}\}$ be so that $X_{0,n} = \overline{U}_i$ for some i and $X_{1,n} = X \setminus U_j$ for some j where $\overline{U}_i \subset U_j$. We may easily choose these so that whenever $n \neq m$, $X_{0,n} \neq X_{0,m}$ or $X_{1,n} \neq X_{1,m}$ and so that each U_i appears in some pair. So without loss of generality, we shall do so.

Now, for each n, let $g_n : X \longrightarrow I$ be so that $g_n(X_{0,n}) = 0$ and $g_n(X_{1,n}) > 0$ and for any regular decomposition $\{F_i\}$, of X, $\sum_i \operatorname{diam}(g_n(F_i)) \leq 1$.

Let $h : X \longrightarrow [0,1]^\omega$ be defined by:

$$h(x) = (\frac{1}{n^2} g_n(x))_{n \in \omega}.$$

For all x, $h(x) \in \ell_2$ and h is clearly continuous and bijective. Therefore, X is homeomorphic to $h(X) \subset \ell_2$. Let ℓ_2 have the metric induced by the usual norm. Then,

$$\rho((x_n),(y_n)) = \Big(\sum_{n=1}^{\infty} (x_n - y_n)^2\Big)^{\frac{1}{2}}$$

.

So, $\rho(h(x_0), h(x_1)) = \Big(\sum_{n=1}^{\infty} (\frac{1}{n^2} g_n(x_0) - \frac{1}{n^2} g_n(x_1))^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{\infty} (\frac{1}{n^4})(g_n(x_0) - g_n(x_1))^2\Big)^{\frac{1}{2}}$.

Then, if $A \subset X$, $\operatorname{diam}[h(A)] \le \Big(\sum_{n=1}^{\infty} \frac{1}{n^4} (\operatorname{diam}(g_n(A)))^2\Big)^{\frac{1}{2}} \le \sum_{n=1}^{\infty} \Big(\frac{1}{n^2} (\operatorname{diam}(g_n(A)))\Big)$.

Now, let $\{F_i\}$ be a regular decomposition of X. Then

$$\sum_i \operatorname{diam}(h(F_i)) \le \sum_i \sum_{n=1}^{\infty} \frac{1}{n^2} (\operatorname{diam}(g_n(F_i))) \le \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_i \operatorname{diam}(g_n(F_i)) \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

So, for all regular decompositions, $\sum_i \operatorname{diam}(h(F_i)) \le \frac{\pi^2}{6}$. Hence, if for any $\delta > 0$, there is a regular decomposition of X which is a δ-mesh cover, then $\mathcal{H}^1(h(X)) \le \frac{\pi^2}{6} < \infty$. We prove this next.

Claim: For $\delta > 0$, there exists a regular decomposition, $\{F_i\}$ of X so that $\operatorname{diam}(F_i) < 2\delta$ for all i.

Let $\delta > 0$. Temporarily fix $x \in X$. Then, there exists a continuous function $f_x : X \longrightarrow I$ so that $f_x(x) = 0$ and $f_x(X \setminus B(x,\delta)) = 1$ and for all $t \in I \setminus \mathbb{Q}$, $| f_x^{-1}(t) | < \omega$.

Unfix x. For each $x \in X$, let $U_x = f_x^{-1}\big([0, \frac{\pi}{4})\big)$. Then, for each x, $x \in U_x \subset f^{-1}([0,1)) \subset B(x,\delta)$. But for each x, $f_x^{-1}(\frac{\pi}{4}) = f_x^{-1}\big([0, \frac{\pi}{4}]\big) \setminus f_x^{-1}\big([0, \frac{\pi}{4})\big)$ and $f_x^{-1}\big([0, \frac{\pi}{4}]\big)$ is a closed set which contains $f_x^{-1}\big([0, \frac{\pi}{4})\big)$.

Therefore, $f_x^{-1}(\frac{\pi}{4}) = f_x^{-1}\big([0, \frac{\pi}{4}]\big) \setminus f_x^{-1}\big([0, \frac{\pi}{4})\big)$ contains $\overline{f_x^{-1}\big([0, \frac{\pi}{4})\big)} \setminus f_x^{-1}\big([0, \frac{\pi}{4})\big)$ which is the boundary of U_x. So, $\{U_x\}_{x \in X}$ is an open cover of X so that the boundary of each open set is finite, since $\frac{\pi}{4} \in F(f)$.

Choose $\{U_i\}$, a finite subcover. We would like to "almost" disjointify them, so we shall do the following iterative action:

Let $V_1 = U_1$. For each $i > 1$, Let $V_i = U_i \setminus \bigcup_{k=1}^{i-1} U_k$. Since $Fr(V_i) = Fr(U_i \setminus \bigcup_{k=1}^{i-1} U_k) \subset \bigcup_{k=1}^{i} Fr(U_k)$, for each i, $Fr(V_i)$ is finite.

Temporarily fix i, and let $\{b_1, b_2 ... b_j\}$ be the boundary points of V_i.

There are only finitely many components of V_i which have more than one b_i as a limit point. Otherwise, as there are only finitely many distinct subsets of $\{b_1, b_2 ... b_j\}$, there is at least a pair, b_x, b_y so that infinitely many components have both of them as limit points. Let $\Delta = \rho(b_1, b_2)$. Then any open set containing b_1 with diameter less than Δ must have a point from each of these components in its boundary. Then b_1 wouldn't be of finite degree. This is a contradiction.

Let $D_1, ... D_m$ be this finite collection. For each boundary point, let D_{b_i} be the union of the components which have only b_i as a boundary point. Then, $\{\overline{D_k} : k \in \{1, ...m\}$ or $k \in \{b_1, ...b_j\}$ is a finite set of subcontinua so that the intersection of any two of them is finite and each has diameter less than 2δ. Hence, there is a regular decomposition, $\{F_i\}$ of X so that $\text{diam}(F_i) < 2\delta$ for all i. Therefore, $\mathcal{H}^1(h(X)) \leq \frac{\pi^2}{6} < \infty$.

3.5. IV $\rightarrow$ V

This is clear.

3.6. V $\rightarrow$ I

Suppose X has a homeomorphic image with finite linear measure. Let $x_0 \in X$ be arbitrary and $h(x_0) = x$ where $h(X)$ has finite linear measure, with metric ρ. Define $f : h(X) \rightarrow \mathbb{R}$ by $f(y) = \rho(x, y)$ for our fixed x.

Then $\mid f(x_1) - f(x_2) \mid = \mid \rho(x_1, x) - \rho(x_2, x) \mid$. Without loss of generality, $\rho(x_1, x) \geq \rho(x_2, x)$. So, $\mid \rho(x_1, x) - \rho(x_2, x) \mid = \rho(x_1, x) - \rho(x_2, x) \leq (\rho(x_1, x_2) + \rho(x_2, x)) - \rho(x_2, x) = \rho(x_1, x_2)$.

Thus, f is nonexpansive.

We now apply the following theorem of Eilenberg concerning non-expansive maps [1]:

Given a metric space, X, and a real valued function, f, on X which is nonexpansive,

$$\int_{-\infty}^{\infty} \mathcal{H}^{\alpha}[f^{-1}(t)]dt \leq \mathcal{H}^{\alpha+1}(X).$$

This is proven in Appendix C.

By our lemma, $\int_0^\infty \mathcal{H}^0[f^{-1}(t)]dt \leq \mathcal{H}^1(h(X)) < \infty$. So, $\mathcal{H}^0[f^{-1}(t)]$ must be finite for all t eexcept for a set of measure zero. Noting that for any real number t, $f^{-1}(t) = Fr(B(x,t))$, it follows that X is of finite degree.

Now, for the remaining equivalences, we shall give a less rigorous, more intuitive proof. The reason for this is that the proof of each of these envolves a multitude of theorems. To avoid confusion, it is preferable to simply reference most of the theorems quoted in the proofs.

3.7. I $\rightarrow$ VI

Let X be a continuum of finite degree and M a proper, nondegenerate subcontinuum. Choose $p \in M$. Because p is of finite degree, there is an uncountable family, $\{U_\lambda\}_{\lambda\in\Lambda}$ of strongly monotone open sets containing p so that for all λ, the diameter of U_λ is less than the diameter of M and the boundary of each U_λ is finite. Temporarilty fix λ and let $\{b_1, \ldots, b_n\}$ be the points on the boundary of U_λ.

Now each b_i is an isolated point of a closed cutting set $(Fr(U_\lambda))$ and there is at least one i so that $b_i \in M$ and b_i is a limit point of both $U_\lambda \cap M$ and $M \setminus U_\lambda$. Hence, b_i is a limit point of U_λ and also of $X \setminus U_\lambda$. We now apply the following theorem of Whyburn [10].

If a point, p, of a continuum, X, is an isolated point of a closed cutting set, K, and is a limit point of both X_1 and X_2, where $X_1 \cup X_2$ is the separation of $X \setminus K$, then p is a local separating point of X. $({}_1)$

Proof:

Choose open sets G and R so that $p \in G \subset \overline{G} \subset R$ and $R \cap K = \{p\}$. Set $R_i = R \cap X_i$. Then $R \setminus \{p\} = R_1 \cup R_2$ is a separation of R. Let N be the component of p in R. Now, since p is a limit point of M_1, it is also a limit point of R_1. If p is a limit point of $N \cap R_1$, then $N \cap R_1 \neq \emptyset$. Suppose p is not a limit point of $N \cap R_1$. Then, there is a sequence, $J_1, J_2, \ldots$ of components in $\overline{G} \cap R_1$ converging to a limit continuum, J containing p.

In the latter case, $J \setminus \{p\} \subset R_1$ and $J \setminus \{p\} \subset N$ since N contains the component of p in $\overline{G}$. Hence, $J \setminus \{p\} \subset R_1 \cap N$. Note, $J \setminus \{p\} \neq \emptyset$ since it contains at least one point on the frontier of G. So, in either case, $N \cap R_1 \neq \emptyset$. Similarly, $N \cap R_2 \neq \emptyset$. Therefore, p is a local separating point of X.

End Claim.

Unfix λ. For each λ, there is at least one point, p_λ, so that $p_\lambda \in Fr(U_\lambda) \cap M$ and p_λ is a local separating point of X. Therefore, $\{p_\lambda\}_{\lambda \in \Lambda}$ is an uncountable set of local separating points of X contained in M.

3.8. VI $\rightarrow$ I

Suppose that each subcontinuum, M, of X contains uncountably many local separating points of X, but that X is not of finite degree.

Then there is a point, p of X so that p is not of finite degree. Then, by a theorem of Whyburn's, there is a non-degenerate subcontinuum, N of X containing p so that no point of N is of finite degree. But, N must contain uncountably many local separating points of X. Also by a theorem of Whyburn's, only countably many local separating points of any continuum are of degree greater than two. So, it must be that N contains at least one point of degree two. This is a contradiction.

3.9. I $\rightarrow$ VII

Suppose X is of finite degree, with metric ρ, and let X_0, X_1 be disjoint, closed subsets of X. Then, by the method used in section 3.1, we can get an uncountable family of strongly monotone closed sets, $\{K_\lambda\}_{\lambda \in \Lambda}$, so that the boundary of each is of finite cardinality and for each K_λ, $X_0 \subset K_\lambda$ and $X_1 \cap K_\lambda = \emptyset$. Then, certainly, the boundaries of the K_λs are cutting sets; and similarly to the argument in a previous section, each point on the boundary is a local separating point of X. Then, since X has only a countable number of local separating points with degree greater than 2, all but possibly countably many of the K_λs have boundaries consisting only of local separating points of degree 2. So, we can choose a K_λ whose boundary consists only of local separating points of degree 2. Then X_0 and X_1 are separated by $x_0, ... x_n$ which are all local separating points of degree 2. Then set $d = min(\rho(x_i, x_j), i \neq j, \rho(x_i, X_0 \cup X_1))$. BUT, each x_i is of degree 2, so there is an uncountable family for each x_i, $\{K^i_\lambda\}_{\lambda \in \Lambda_i}$, so that $diam(K^i_\lambda) < \frac{d}{2}$ $\forall i, \lambda$ and for each i, $\{K^i_\lambda\}_{\lambda \in \Lambda_i}$ is a strongly monotone family of closed sets, each of which has a boundary containing only two points.

For each i, let $F_i = \overline{\bigcup_{\lambda \in \Lambda_i} Fr\{K^i_\lambda\}}$. Then each F_i is the closure of a countable set, along with a set of condensation points of itself. So, each F_i contains a perfect set. For each i, let N_i be a perfect subset of F_i.

Now let M be any subcontinuum of X intersecting both X_0 and X_1. M must contain some x_i. Since X is locally connected, M contains an arc which also intersects X_0 and X_1, and contains x_i. The arc, then, must contain N_i.

3.10. VII $\rightarrow$ VI

Suppose X, with metric ρ, is locally connected and for every pair of closed, disjoint subsets X_0, X_1 in X, there is a finite collection of disjoint perfect sets $N_1, ...N_k$ so that any continuum in X intersecting both X_0 and X_1 contains some N_i.

Let M be a subcontinuum of X and $x_o, x_1 \in M$. Then let $X_0 = \{x_0\}$ and $X_1 = \{x_1\}$. Then there are disjoint perfect sets $N_1, ...N_k$ so that Mcontains some N_i, say N_1. Let L be the set of local separating points of X. If $M \cap L$ is uncountable, we're done. So, suppose not. Then there is a point $n_1 \in N_1$ so that $n_1 \notin L$. Let

$$d = \frac{\rho(N_1, \{\{x_0\} \cup \{x_1\} \cup \bigcup_{j=2}^{k} N_j\}}{4}.$$

Let $V = B(n_1, d) \cap X$. Then $d_1 = \rho(n_1, Fr(V) \leq d$. Let $W = B(n_1, \frac{d_1}{2}) \cap X$. Let Y_0 and Y_1 be the components of $M \setminus W$ containing x_0, x_1, respectively. Since Y_0 and Y_1 each contain a point of V (X connected) and $n_1 \in X \setminus L$, there must be an arc, s which is in $V \setminus \{n_1\}$ so that $Y = Y_0 \cup Y_1 \cup s$ is connected (and thus a continuum). Since $n_1 \notin Y$, certainly $N_1 \not\subset Y$. Y is a subcontinuum of X containing x_0 and x_1. Then, without loss of generality, suppose that Y contains N_2. A similar argument can produce a subcontinuum of Y which (using the same d) does not contain N_2. With finitely many (fewer than k) steps, we will construct a subcontinuum of X containing x_0 and x_1 and not containing any N_i. This is a contradiction to the hypothesis.

3.11. I $\rightarrow$ VIII

Suppose X is of finite degree. X is thus locally connected. Suppose also that $\{K_i\}_{i \in \omega}$ is a sequence of nondegenerate continua in X with $lim\{K_i\} = K$ and K nondegenerate. Since X is locally connected, we can take $\{K_i\}$ to be arcs. Then, for each i, let a_i, b_i be the endpoints of K_i so that $lim\{a_i\} = a$ and $lim\{b_i\} = b$ Since K is nondegenerate, $a \neq b$. Since every subcontinuum of X has uncountably many points of degree 2, we can choose $z \in K \setminus \{a, b\}$ so that z is of degree 2. Let A, B, Z be neighborhoods of a, b, z, respectively so that $\overline{A} \cap \overline{B} = \overline{A} \cap \overline{Z} = \overline{B} \cap \overline{Z} = \emptyset$. There is then a strongly monotone, uncountable family of open sets, $\{U_\lambda\}_{\lambda \in \Lambda}$, so that for each λ, $z \in U_\lambda \subset \overline{U_\lambda} \subset Z$ and $\mid Fr(U_\lambda) \mid \leq 2$ for all λ. So choose β so that $V = \overline{U_\beta} \subset U_\lambda$ for uncountably many λ. Since $z \in lim K_i$, there is N so that whenever $n \geq N$, $K_n \cap V \neq \emptyset$. We can choose $m \geq N$ large enough so that for all $n \geq m$, $a_n \in A$ and $b_n \in B$ also. Then for each U_λ that contains V, of which there are uncountably many, every K_n with $n \geq m$ must pass through two boundary points. Then, since the boundaries of each of the U_λ only contain two points, they must be the same points for each K_n, but distinct for each λ. So, $\bigcap_{i=n}^{\infty} K_i$ is uncountable.

3.12. VIII $\rightarrow$ I

By contrapositive, assume that X is a continuum with metric ρ which is not of finite degree. If X is not locally connected, one can easily construct a sequence of subcontinua converging to a subcontinuum K so that they do not intersect at all. So, we can limit our attention to a continuum which is not of finite degree but IS locally connected. Since X is not of finite degree, there must be a nondegenerate subcontinuum, M of X so that M only contains countably many local separating points of X. Without loss of generality, let M be an arc with endpoints a and b. For each n, let $W_n = B(M, \frac{1}{n})$. Let M_n be the component of W_n containing M. Define L_n as the set of points in M_n which separate a and b in M_n. Then there exists arcs, $s_n, t_n \subset M_n$ from a to b so that $s_n \cap t_n = \{a\} \cup \{b\} \cup L_n$, which is countable. Then let $K_{2n-1} = s_n$ and $K_{2n} = t_n$. Then $lim\{K_n\} = M$, but for any two consecutive sets, K_i, K_{i+1}, their intersection is at most countable. Then, $\bigcap_{i=n}^{\infty} K_i \leq \omega$ despite the choice of n.

APPENDIX A

KURATOWSKI'S THEOREM

THEOREM .2 (Kuratowski). *Given any two disjoint closed subsets, X_0 and X_1, of a compact metric space X, if $\{ K_\lambda \mid \lambda \in I \setminus \mathbb{Q} \}$ is a family of closed subsets of X so that*

(1) $\lambda < \beta \iff K_\lambda \subset K^o_\beta$,

(2) *for all but countably many λ, $\overline{\bigcup_{\beta<\lambda} K_\beta} = K_\lambda$ and $\bigcup_{\beta<\lambda} K_\beta = K^o_\lambda$,*

(3) *for all but countably many λ, $\bigcap_{\lambda<\beta} K_\beta = K_\lambda$,*

(4) *for all λ, $X_0 \subset K_\lambda$ and $X_1 \cap K_\lambda = \emptyset$, and*

(5) *there is an N so that $\mid K_\lambda \setminus K^o_\lambda \mid \leq N$ for all λ,*

then there is a continuous function, $f\colon X \to I$ so that $f(X_0) = \{0\}$, $f(X_1) = \{1\}$ and $\mid \{f^{-1}(t)\} \mid \leq N$ for all $t \in I \setminus \mathbb{Q}$, .

PROOF. Note, it suffices to show this for all but countably many $t \in I \setminus \mathbb{Q}$, as there would be a continuous order preserving map onto all of $I \setminus \mathbb{Q}$, whereby we could throw out the bad sets.

Suppose X_0, X_1, and $\{ K_\lambda \mid \lambda \in I \setminus \mathbb{Q} \}$ are as above. Then define $f\colon X \to I$ as follows:

$$f(x) = \begin{cases} \inf\{ \lambda \mid x \in K_\lambda \} & \text{if such a } \lambda \text{ exists} \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $f(X_0) = \{0\}$ and $f(X_1) = \{1\}$. Also, if $x \in Fr(K_\lambda)$ for some λ, then certainly $f(x) = \lambda$. [What is "*Fr*?" If it's some kind of math symbol, the the correct syntax for that symbol should be used. One guess: *Fr* means "boundary" (that is, *frontier*), in which case Fr would be correct.]

Conversely, suppose $f(x) = \lambda$ for some $\lambda \in I \setminus \mathbb{Q}$. For all but countably many $\lambda \in I \setminus \mathbb{Q}$, $\overline{\bigcup_{\beta<\lambda} K_\beta} = \bigcap_{\lambda<\beta} K_\beta = K_\lambda$. Also, if $f(x) = \lambda$, $x \in \bigcap_{\lambda<\beta} K_\beta$, but $x \notin \bigcup_{\beta<\lambda} K_\beta$. Otherwise, there is a $\beta < \lambda$ so that $x \in K_\beta$ making $f(x) \leq \beta < \lambda$. So, if $f(x) = \lambda$, $x \in \bigcap_{\lambda<\beta} K_\beta \setminus \bigcup_{\beta<\lambda} K_\beta$. However, for all but countably many $\lambda \in I \setminus \mathbb{Q}$, $\bigcap_{\lambda<\beta} K_\beta \setminus \bigcup_{\beta<\lambda} K_\beta = K_\lambda \setminus K^o_\lambda$. Then, for all but countably many $\lambda \in I \setminus \mathbb{Q}$, $f^{-1}(\lambda) = Fr(K_\lambda)$. Now, we need to show f is continuous.

Suppose $(y-\epsilon, y+\epsilon) \subset I$, and that $x \in f^{-1}(y-\epsilon, y+\epsilon)$. Then there is a $\tilde{y} \in (y-\epsilon, y+\epsilon)$ so that $f(x) = \tilde{y}$. Also, there exists γ, $\alpha \in I \setminus \mathbb{Q}$ so that $y - \epsilon < \gamma < \tilde{y} < \alpha < y + \epsilon$.

Then $x \notin K_\gamma$ and $x \in K^o_\alpha$.

If $\tilde{x} \in K^o_\alpha \setminus K_\gamma$, then $f(\tilde{x}) \in [\gamma, \alpha] \subset (y-\epsilon, y+\epsilon)$.

Continuity in an open set U is easier to show if $1 \in U$ or $0 \in U$. $\square$

APPENDIX B

BOREL NATURE OF F(f)

THEOREM .3. *If $f : X \to Y$ is continuous and X is compact metric, then $F(f)$ is a $G_{\delta,\sigma}$.*

PROOF. For each $m, n \in \mathbb{N}$, define $D_{m,n}$ as follows:

$D_{m,n} = \{y \in Y : \exists\, x_1, ..., x_n \in X$ so that $\rho(x_i, x_j) \geq \frac{1}{m}$ when $i \neq j$ and $f(x_i) = y\ \forall i\}$.

Claim: $D_{m,n}$ is closed for all $m, n \in \mathbb{N}$.

Let y be a limit point of $D_{m,n}$ for arbitrary $m, n \in \mathbb{N}$. Then, there is a sequence $\{y_p\}$ in $D_{m,n}$ so that $\{y_p\} \to y$. Now, for each y_p, let $x_{p,1}, ...x_{p,n}$ be the pre-image points that witness that $y_p \in D_{m,n}$.

For each $1 \leq j \leq n$, $\{x_{p,j}\}$ is a sequence of points in a compact metric space. Therefore, $\{x_{p,j}\}$ has a convergent subsequence. Without loss of generality, taking subsequences of $\{y_p\}$ possibly n times, assume for each $1 \leq j \leq n$, $\{x_{p,j}\} \to x_j$.

However, since f is continuous, $f(x_j) = y$ for $1 \leq j \leq n$. Suppose, for the purpose of a contradiction, that $\rho(x_j, x_q) < \frac{1}{m}$ for some j, q. Then, there is $\epsilon > 0$ so that $\rho(x_j, x_q) < \frac{1}{m} - 2\epsilon$. Consider $B(x_j, \epsilon)$ and $B(x_q, \epsilon)$. If $z_j \in B(x_j, \epsilon)$ and $z_q \in B(x_q, \epsilon)$ then $\rho(z_q, z_j) \leq \rho(z_q, x_q) + \rho(x_q, x_j) + \rho(x_j, z_j) < \epsilon + \frac{1}{m} - 2\epsilon + \epsilon = \frac{1}{m}$. But, this contradicts that $\{x_{p,j}\} \to x_j$ and $\{x_{p,q}\} \to x_q$. Hence, $y \in D_{m,n}$.

End Claim.

Now, suppose that $y \notin F(f)$. Then for each n, there is some m so that $y \in D_{m,n}$. So, for each n, $y \in \bigcup_{m=1}^{\infty} D_{m,n}$.

Hence, $X \setminus F(f) = \bigcap_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} D_{m,n})$, which is an $F_{\sigma,\delta}$. So, $F(f)$ is a $G_{\delta,\sigma}$.

$\square$

APPENDIX C

EILENBERG'S EQUATION

THEOREM .4. *Given a metric space, X, and a real valued function, f, on X which is non-expansive,*

$$\int_{-\infty}^{\infty} \mathcal{H}^{\alpha}[f^{-1}(t)]dt \leq \mathcal{H}^{\alpha+1}(X).$$

PROOF. For each $n \in \mathbb{N}$, let $\{A_{n,i}\}_{n=1}^{\infty}$ be a decomposition of X so that $\operatorname{diam}(A_{n,i}) < \frac{1}{n}\ \forall\ i$. These can clearly be chosen so that $A_{n,i} \cap A_{n,j} = \emptyset\ \forall\ i \neq j$. Now, for each n, given the definition of $\mathcal{H}^{\alpha+1}_{\frac{1}{n}}(X)$ there exists $\{A_{n,i}\}_{i=1}^{\infty}$, a $\frac{1}{n}$-cover of X so that

$$\sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i}))^{\alpha+1} < \mathcal{H}^{\alpha+1}_{\frac{1}{n}}(X) + \frac{1}{n}.$$

Since $\mathcal{H}^{\alpha+1}(X) = \lim_{\delta\to 0} \mathcal{H}^{\alpha+1}_{\delta}(X)$, we would have chosen the $\{A_{n,i}\}_{i=1}^{\infty}$ so that $\mathcal{H}^{\alpha+1}(X) = \lim_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i}))^{\alpha+1}$. Now, define $g(f(X)) \to \mathbb{R}$ by

$$g(t) = \liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i}))^{\alpha}\chi_{\overline{f(A_{n,i})}}(t).$$

Then, g is measurable. Since $\{A_{n,i} \cap f^{-1}(t)\}_{n=1}^{\infty}$ is a $\frac{1}{n}$-mesh cover of $f^{-1}(t)$ for all n, $\mathcal{H}^{\alpha}[f^{-1}(t)] \leq \liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i} \cap f^{-1}(t)))^{\alpha}$. Also, if $\operatorname{diam}(A_{n,i} \cap f^{-1}(t)) > 0$, then $t \in f(A_{n,i}) \subset \overline{f(A_{n,i})}$, so $(\operatorname{diam}(A_{n,i} \cap f^{-1}(t)))^{\alpha} = (\operatorname{diam}(A_{n,i}))^{\alpha}\chi_{\overline{f(A_{n,i})}}(t)$ for all i.

If, on the other hand, $\operatorname{diam}(A_{n,i} \cap f^{-1}(t)) = 0$, $(\operatorname{diam}(A_{n,i}))^{\alpha}\chi_{\overline{f(A_{n,i})}}(t) = 0$ for all i.
So, $\mathcal{H}^{\alpha}[f^{-1}(t)] \leq \liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i} \cap f^{-1}(t)))^{\alpha}\chi_{\overline{f(A_{n,i})}}(t) \leq g(t)$.
Now, $\int_{-\infty}^{\infty} g(t)dt = \int_{-\infty}^{\infty} (\liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i} \cap f^{-1}(t)))^{\alpha}\chi_{\overline{f(A_{n,i})}}(t))dt$,

which, by Fatou's Lemma,
$\leq \liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i}))^{\alpha} \int_{-\infty}^{\infty} \chi_{\overline{f(A_{n,i})}}(t))dt$.
But, $\int_{-\infty}^{\infty} \chi_{\overline{f(A_{n,i})}}(t))dt = m(\overline{f(A_{n,i})} \leq \operatorname{diam}(f(A_{n,i})) \leq \operatorname{diam}(A_{n,i})$.
So, $\int_{-\infty}^{\infty} g(t)dt \leq \liminf_{n\to\infty} \sum_{i=1}^{\infty}(\operatorname{diam}(A_{n,i}))^{\alpha+1}$, and $\int_{-\infty}^{\infty} g(t)dt \leq \mathcal{H}^{\alpha+1}(X)$, giving that

$$\mathcal{H}^{\alpha}[f^{-1}(t)] \leq g(t).$$

Therefore, $\int_{-\infty}^{\infty} \mathcal{H}^{\alpha}[f^{-1}(t)] \leq \int_{-\infty}^{\infty} g(t)dt \leq \mathcal{H}^{\alpha+1}(X)$.

$\square$

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