COMPLETE ORDERED FIELDS

THESIS

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By

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The purpose of this thesis is to study the concept of completeness in an ordered field. Several conditions which are necessary and sufficient for completeness in an ordered field are examined.

In Chapter I the definitions of a field and an ordered field are presented and several properties of fields and ordered fields are noted. Chapter II defines an Archimedean field and presents several conditions equivalent to the Archimedean property. Definitions of a complete ordered field (in terms of a least upper bound) and the set of real numbers are also stated. Chapter III presents eight conditions which are equivalent to completeness in an ordered field. These conditions include the concepts of nested intervals, Dedekind cuts, bounded monotonic sequences, convergent subsequences, open coverings, cluster points, Cauchy sequences, and continuous functions.
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CHAPTER I

ORDERED FIELDS

The purpose of this thesis is to study the concept of completeness in an ordered field. In particular, an attempt is made to examine several conditions which are necessary and sufficient for completeness in an ordered field.

In Chapter I the definitions of a field and an ordered field are presented and numerous fundamental properties of fields and ordered fields are noted. The chapter also includes a proof of the Principle of Mathematical Induction and the Well Ordering Principle.

Chapter II defines an Archimedean field and presents several conditions equivalent to the Archimedean property. A definition of a complete ordered field (in terms of the least upper bound principle) and the set of real numbers is presented so that several properties of the real numbers can be examined.

Chapter III presents eight different conditions which are necessary and sufficient for completeness in an ordered field. These conditions include the concepts of nested intervals, Dedekind cuts, bounded monotonic sequences, convergent subsequences, open coverings, cluster points, Cauchy sequences, and continuous functions.
1.1. **Definition.** A binary operation in a set $F$ is a function from $F \times F$ into $F$.

1.2. **Definition.** A field consists of a set $F$ together with two binary operations in $F$, usually denoted by "+" and ":", satisfying the following properties:

1. If $a, b \in F$, then $a + b = b + a$.
2. If $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.
3. There exists an element $0$ in $F$ such that for each $a$ in $F$, $a + 0 = a$.
4. If $a \in F$, then there exists an element $(−a)$ in $F$ such that $a + (−a) = 0$.
5. If $a, b \in F$, then $a \cdot b = b \cdot a$.
6. If $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
7. There exists an element $1 \in F, 1 \neq 0$, such that for each $a$ in $F$, $a \cdot 1 = a$.
8. If $a \in F, a \neq 0$, then there exists an element $a^{-1} \in F$ such that $a \cdot (a^{-1}) = 1$.
9. If $a, b, c \in F$, then $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

1.3. **Example.** The set of real numbers, denoted by $R$, with the usual operations of addition and multiplication, is a field.

1.4. **Example.** The set of rational numbers, denoted by $Q$, with the usual operations of addition and multiplication, is a field.
1.5. **Example.** The set \( \{0, 1, 2\} \) with the operations "+" and "." defined in Figure 1 is a field.

\[
\begin{array}{c|ccc}
  + & 0 & 1 & 2 \\
  \hline
  0 & 0 & 1 & 2 \\
  1 & 1 & 2 & 0 \\
  2 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
  \cdot & 0 & 1 & 2 \\
  \hline
  0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 2 \\
  2 & 0 & 2 & 1 \\
\end{array}
\]

Fig. 1--Definition of the operations of "+" and "." on the set \( \{0, 1, 2\} \).

1.6. **Remark.** The usual properties of a field are assumed. For instance, if \( F \) is a field the following properties hold:

1. If \( a \in F \), \( a \cdot 0 = 0 \).
2. If \( a \in F \), \((-a) = a \).
3. If \( a, b \in F \), then \((-a) \cdot (b) = -(a \cdot b) \).
4. If \( a, b \in F \), then \((-a) \cdot (-b) = a \cdot b \).
5. If \( a \in F \) and \( a \neq 0 \), then \((a^{-1})^{-1} = a \).
6. If \( a, b \in F \) and \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

1.7. **Example.** Consider the set \( S = \{0, 1, 2, 3\} \) and the operations defined in Figure 2. Note that this example is similar to the field defined in Example 1.5. However, in this case, \( 2 \cdot 2 = 0 \). Since \( 2 \neq 0 \), by Remark 1.6 \( S \) is not a field.
Fig. 2—Definition of the operations of "+" and "\cdot" on the set \{0,1,2,3\}.

1.8. Definition. Suppose \(F\) is a field with binary operations "+" and "\cdot". Then the nonempty set \(S\) is called a subfield of \(F\) iff:

1. \(S \subseteq F\).
2. The set \(S\) with the binary operations "+" and "\cdot" is a field.

1.9. Theorem. A subset \(S\) of a field \(F\) is a subfield of \(F\) iff:

1. \(0,1 \in S\).
2. If \(a, b \in S\), then \(a + b \in S\).
3. If \(a, b \in S\), then \(a \cdot b \in S\).
4. If \(a \in S\), then \(-a \in S\).
5. If \(a \in S\), \(a \neq 0\), then \(a^{-1} \in S\).

A proof is omitted.

1.10. Example. From Example 1.4, the set of rational numbers is a field. Since \(Q \subseteq \mathbb{R}\), \(Q\) is a subfield of \(\mathbb{R}\).

1.11. Example. Consider the set \(S = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}\). Using the properties of \(\mathbb{R}\), it can be shown that addition and
multiplication are binary operations in $S$ and that $S$ is a field. Since $S \subseteq R$, $S$ is a subfield of $R$.

1.12. **Definition.** A relation in a set $S$ is a subset of $S \times S$.

1.13. **Definition.** A linear order in a set $S$ is a relation "$\leq$" in $S$ such that:

1. If $a, b \in S$, exactly one of the following is true: $a \leq b$, $b \leq a$, $a = b$.
2. If $a, b, c \in S$ and if $a \leq b$ and $b \leq c$, then $a \leq c$.

1.14. **Definition.** A partial order in a set $S$ is a relation "$\preceq$" in $S$ such that:

1. If $a \in S$, then $a \preceq a$.
2. If $a, b, c \in S$ and if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.
3. If $a, b \in S$ and if $a \preceq b$ and $b \preceq a$, then $a = b$.

1.15. **Definition.** A total order in a set $S$ is a relation "$\preceq$" in $S$ such that:

1. "$\preceq$" is a partial order in $S$.
2. If $a, b \in S$, then $a \preceq b$ or $b \preceq a$.

If "$\preceq$" is a linear order in $S$, it is said that "$\preceq$" linearly orders $S$. Similarly, if "$\preceq$" is a partial order in $S$, it can be said that "$\preceq$" partially orders $S$.

1.16. **Theorem.** Suppose $S$ is a set and suppose the relation "$\preceq$" linearly orders $S$. Define the relation "$\preceq$" as follows: if $a, b \in S$, $a \preceq b$ iff $a \leq b$ or $a = b$. Then "$\preceq$" is a total order in $S$. 
Proof. Suppose $S$ is a set and suppose the relation "≤" linearly orders $S$. Let $a \in S$. Since "≤" is a linear order in $S$, exactly one of the following is true: $a \leq a$ or $a = a$. By definition of "≤", $a \leq a$.

Suppose $a, b, c \in S$ and suppose $a \leq b$ and $b \leq c$. If $a \leq b$ and $b \leq c$, $a \leq c$ by Definition 1.13. If $a \leq b$ and $b = c$, then $a \leq c$. If $a = b$ and $b \leq c$, then $a \leq c$. And if $a = b$ and $b = c$, then $a = c$. Thus in all four cases, either $a \leq c$ or $a = c$. Therefore, $a \leq c$.

Suppose $a, b \in S$ and suppose $a \leq b$ and $b \leq a$. Then the following cases result: (1) $a \leq b$ and $b \leq a$; (2) $a \leq b$ and $b = a$; (3) $a = b$ and $b \leq a$; or (4) $a = b$ and $b = a$. Since each of the first three cases contradicts the trichotomy property of a linear order, only case four is true. Thus, $a = b$.

Suppose $a, b \in S$. Then exactly one of the following is true: (1) $a \leq b$; (2) $b \leq a$; or (3) $a = b$. If $a \leq b$, then $a \leq b$. If $b \leq a$, then $b \leq a$. And if $a = b$, then $a \leq b$ and $b \leq a$. So, in all cases, either $a \leq b$ or $b \leq a$.

Therefore, by definition, "≤" is a total order in $S$.

1.17. Theorem. Suppose $S$ is a set and suppose the relation "≤" totally orders $S$. Define the relation "≤" as follows: if $a, b \in S$, $a \leq b$ iff $a \leq b$ and $a \neq b$. Then, "≤" is a linear order in $S$.

Proof. Suppose $S$ is a set and suppose the relation "≤" totally orders $S$. Let $a$ and $b$ be elements of $S$. By Definition
1.15 either \( a \leq b \) or \( b \leq a \). In other words, \( a \leq b \) or \( a = b \) or \( b \leq a \). By way of contradiction first assume \( a \leq b \) and \( a = b \). This assumption implies that \( a \leq a \) which contradicts the definition of "\( \leq \)". Thus, only one of the following can be true: \( a \leq b \) or \( a = b \). Secondly, assume \( a \leq b \) and \( b \leq a \). Then \( a \leq b \) and \( b \leq a \), which implies that \( a = b \). Thus, \( a \leq b \) and \( a = b \). Since this contradicts an earlier conclusion, only one of the following can be true: \( a \leq b \) or \( b \leq a \). Therefore, exactly one of the following is true: \( a \leq b \) or \( b = a \) or \( b \leq a \).

Suppose \( a, b, c \in S \) and suppose \( a \leq b \) and \( b \leq a \). By definition, \( a \leq b \), \( a \neq b \), \( b \leq c \), and \( b \neq c \); and since "\( \leq \)" is a total order, \( a \leq c \). Suppose \( a = c \). Then \( c \leq b \) and \( b \leq c \). But as was shown above, this conclusion is untenable. Therefore, \( a \neq c \) so that \( a \not\leq c \).

1.18. Definition. An order relation "\( \leq \)" in a field \( F \) is a binary relation in \( F \) which satisfies the following properties:

1. If \( a, b \in F \), then exactly one of the following is true: \( a = b \), \( a \leq b \), or \( b \leq a \).
2. If \( a, b \in F \) and if \( a \leq b \) and \( b \leq c \), then \( a \leq c \).
3. If \( a, b, c \in F \) and if \( a \leq b \), then \( a + c \leq b + c \).
4. If \( a, b, c \in F \), \( a \leq b \), and \( 0 \leq c \), then \( ac \leq bc \).

1.19. Definition. An ordered field consists of a field \( F \) and an order relation "\( \leq \)" in \( F \).
If $F$ is an ordered field and $a, b \in F$, then $a > b$ means $b \not< a$, $a \not< b$ means $a < b$ or $a = b$, and $a \geq b$ means $b \not> a$.

1.20. **Definition.** A positive class $P$ in a field $F$ is a nonempty subset of $F$ such that:

1. If $a, b \in P$, then $(a + b) \in P$.
2. If $a, b \in P$, then $(a \cdot b) \in P$.
3. If $a \in F$, then exactly one of the following is true: $a \in P$, $-a \in P$, or $a = 0$.

1.21. **Theorem.** If $F$ is a field and "<" is an order relation in $F$, then the set $P = \{x \in F \mid x > 0\}$ is a positive class in $F$. Conversely, if $F$ is a field and $P$ is a positive class in $F$, then the relation "<" defined in $F$ by $a < b$ iff $(b - a) \in P$ is an order relation in $F$.

**Proof.** Suppose "<" is an order relation in a field $F$ and let $P = \{x \in F \mid x > 0\}$. By Definition 1.2, $0 \in F$, $1 \in F$ and $0 \neq 1$. From Definition 1.18 either $0 < 1$ or $1 < 0$. Thus, either $1 > 0$ or $-1 > 0$, and consequently, either $1 \in P$ or $-1 \in P$. Therefore, $P$ is a nonempty subset of $F$.

Suppose $x, y \in P$. Then $0 < x$ and $0 < y$. By Definition 1.18, $x = 0 + x < y + x$, so that $0 < x < x + y$. Therefore, $(x + y) \in P$.

Again suppose $x, y \in P$. Then $0 < x$ and $0 < y$. By Definition 1.18, $(0 \cdot y) < (x \cdot y)$. Thus, $0 < x \cdot y$ and $(x \cdot y) \in P$. 
Suppose $a \in F$. Since $0 \in F$, exactly one of the following is true: $a = 0$, $0 \leq a$ or $a \leq 0$. By the definition of $P$, $0 \leq a$ iff $a \in P$. Note that $a \leq 0$ iff $-a > 0$, and, thus, $a < 0$ iff $(-a) \in P$. Therefore, exactly one of the following is true: $a \in P$, $(-a) \in P$ or $a = 0$. Hence, by Definition 1.20, $P$ is a positive class in $F$.

Conversely assume $P$ is a positive class in an ordered field $F$. Define the relation "$\leq$" as follows: if $a, b \in F$, then $a \leq b$ iff $(b - a) \in P$.

Suppose $a, b \in F$. Then $(b - a) \in F$. Since $P$ is a positive class, exactly one of the following is true:

$(b - a) \in P$, $(a - b) \in P$ or $(b - a) = 0$. By definition of "$\leq$", $(b - a) \in P$ iff $b > a$. Similarly, $(a - b) \in P$ iff $a > b$. In addition, $(b - a) = 0$ iff $b = a$. Hence, exactly one of the following is true: $b > a$, $a > b$ or $b = a$.

Suppose $a, b, c \in F$, $a \leq b$ and $b \leq c$. Then $(b - a) \in P$ and $(c - b) \in P$. Since $P$ is a positive class, $(b - a) + (c - b) = (c - a) \in P$. Thus, $a \leq c$.

Suppose $a, b, c \in F$ and suppose $a \leq b$. Then $(b - a) \in P$. Since $(b - a) = (b + c) - (a + c)$, $(b + c) - (a + c) \in P$. Therefore, $(a + c) \leq (b + c)$.

Suppose $a, b, c \in F$ and suppose $a \leq b$ and $0 \leq c$. Then $(b - a) \in P$ and $c \in P$. Since $P$ is a positive class, $(b - a) \cdot c = (bc - ac) \in P$. Therefore, $ac \leq bc$.

Thus, by Definition 1.18, "$\leq$" is an order relation in $F$. 
If \( x \) is an element of an ordered field, the statement that \( x \) is positive means that \( x > 0 \). Similarly, \( x \) is negative iff \( -x > 0 \).

1.22. **Corollary.** There exists an order relation in a field \( F \) iff \( F \) contains a positive class.

1.23. **Example.** Consider the set \( S = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \) and note that \( S \) together with the usual operations of addition and multiplication is a field and, also, is a subfield of \( \mathbb{R} \). Let \( P_1 = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \text{ and } a + b\sqrt{2} > 0 \} \) and let \( P_2 = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \text{ and } a - b\sqrt{2} > 0 \} \). Using the properties of \( \mathbb{R} \) it can be verified that each of \( P_1 \) and \( P_2 \) is a positive class in \( S \). There exist \( a, b \in \mathbb{Q} \) such that \( 0 < a < b\sqrt{2} \). Since \( a + b\sqrt{2} > 0 \), \( a + b\sqrt{2} \in P_1 \). However, since \( a - b\sqrt{2} < 0 \), \( a + b\sqrt{2} \notin P_2 \). Thus, \( P_1 \neq P_2 \) and \( S \) is an example of a field that can be ordered in more than one way.

1.24. **Example.** The field of real numbers with the usual properties of "+", "\cdot", and "\leq" is an ordered field.

1.25. **Example.** Let \( \mathbb{Q}(t) \) be the field of rational functions. In other words, an element of \( \mathbb{Q}(t) \) is of the form \( \frac{p(t)}{q(t)} \), where each of \( p \) and \( q \) is a polynomial and not all the coefficients of \( q \) are zero. Define the subset \( P \) as follows: \( P = \{ \frac{p}{q} \in \mathbb{Q}(t) \mid \text{the leading coefficients of } p \text{ and } q \text{ are both positive or both negative} \} \). The following is a verification that \( P \) is a positive class in \( \mathbb{Q}(t) \).
Suppose \( \frac{p}{q}, \frac{r}{s} \in P \). Without loss of generality, assume the leading coefficient of each of \( p \) and \( q \) is positive. If the leading coefficients of \( r \) and \( s \) are positive, then the leading coefficients of \( (ps + qr) \), \( pr \), and \( qs \) are also positive. If the leading coefficients of \( r \) and \( s \) are negative, then the leading coefficients of \( (ps + qr) \), \( pr \), and \( qs \) are negative. Thus, in either case, \( \frac{(ps + qr)}{qs} = \frac{p}{q} + \frac{r}{s} \in P \) and 
\[
\frac{pr}{qs} = \left( \frac{p}{q} \right) \left( \frac{r}{s} \right) \in P.
\]
Suppose \( \frac{p}{q} \in Q(t) \), and without loss of generality assume the leading coefficient of \( q \) is positive. Then, the leading coefficient of \( p \) is positive iff \( \frac{p}{q} \in P \), the leading coefficient of \( p \) is negative iff \( -\frac{p}{q} \in P \), and \( p = 0 \) iff \( \frac{p}{q} = 0 \). Thus, \( P \) is a positive class in \( Q(t) \) so that by Theorem 1.21, \( Q(t) \) is an ordered field.

Many properties of an ordered field are assumed. Some of these properties are stated without proof in Theorem 1.26.

1.26. Theorem. Suppose \( F \) is an ordered field. Then the following statements are true:

(1) \( 0 \preceq 1 \).
(2) If \( a, b \in F \) and if \( ab > 0 \), then either \( a > 0 \) and \( b > 0 \) or \( a < 0 \) and \( b < 0 \).
(3) If \( a, b \in F \), then \( a \preceq b \) iff \( -b \preceq -a \).
(4) If \( a \in F \) and \( a \neq 0 \), then \( a^2 = a \cdot a > 0 \).
(5) If \( a \in F \), then \( a > 0 \) iff \( -a < 0 \).
(6) If \( a, b \in F \), then \( 0 \preceq a \preceq b \) iff \( 0 \preceq \frac{1}{b} \preceq \frac{1}{a} \).
(7) If \( a, b, c, d \in F \), \( a \preceq b \) and \( c \preceq d \), then \( a + c \preceq b + d \).
1.27. **Theorem.** If $a$ and $b$ are elements of an ordered field such that $a > 0$ and $b > 0$, then $a \angle b$ iff $a^2 \angle b^2$.

**Proof.** Suppose $a$ and $b$ are elements of an ordered field such that $a > 0$, $b > 0$ and $a \angle b$. Then $a^2 \angle ab$ and $ab \angle b^2$. By the transitive property, $a^2 \angle b^2$.

Now suppose $a$ and $b$ are elements of an ordered field such that $a > 0$, $b > 0$ and $a^2 \angle b^2$. Then, $0 \angle b^2 - a^2 = (b - a)(b + a)$. By Theorem 1.26(2), either $(b - a) > 0$ and $(b + a) > 0$ or $(b - a) \angle 0$ and $(b + a) \angle 0$. Now by Theorem 1.26(7), $a + b > 0$, and the law of trichotomy implies $a + b \angle 0$ is false. Thus, $b - a > 0$ and $b > a$.

1.28. **Definition.** A set $S$ is said to be **finite** iff $S$ is empty or there exists a one-to-one function from $S$ onto a set $\{1, 2, 3, \ldots, n\}$, where $n$ is some natural number.

1.29. **Definition.** A set is **infinite** iff it is not finite.

1.30. **Theorem.** A set $S$ is infinite iff $S$ contains a subset $T$ such that there exists a one-to-one function from $T$ onto the set of natural numbers.

A proof is omitted.

1.31. **Definition.** If $x$ and $y$ are elements of a field $F$ such that $y = x^2$, then $x$ is called a **square root** of $y$.

1.32. **Lemma.** If $F$ is a field and $x$ is a square root of $y$, then $(-x)$ is a square root of $y$.

**Proof.** Suppose $F$ is a field and suppose $x$ is a square root of $y$. Then, by 1.6(4), $(-x)^2 = (-x)(-x) = x \cdot x = x^2 = y$. Thus, $(-x)$ is a square root of $y$. 
1.33. **Example.** The set of real numbers is an example of a field in which some elements have no square root. Theorem 1.26(4) states that if \( a \in \mathbb{R} \) and \( a \neq 0 \), then \( a^2 > 0 \). Thus, any negative element of \( \mathbb{R} \) has no square root.

1.34. **Example.** Consider the field consisting of the set \( \{0,1\} \) and the operations "\(+" \) and "\( \cdot " \) defined in Figure 3. Note that 1 is the only square root of 1. Here, \(-1 = 1\). Hence, in some fields an element other than 0 has only one square root.

<table>
<thead>
<tr>
<th>+</th>
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<table>
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Fig. 3--Definition of the operations of "\( + " \) and "\( \cdot " \) on the set \( \{0,1\} \).

1.36. **Theorem.** If \( F \) is a field and \( y \in F \), then \( y \) has at most two square roots.

**Proof.** Suppose \( F \) is a field and \( x, y \) and \( z \) are elements of \( F \) such that \( x \) and \( z \) are square roots of \( y \). Since \( y = z^2 \) and \( y = x^2 \), by the transitive property, \( x^2 = z^2 \). Hence, \((x^2 - z^2) = 0 \) which implies that \((x - z)(x + z) = 0 \). By Remark 1.6(6) either \((x - z) = 0 \) or \((x + z) = 0 \). Thus either \( z = x \) or \( z = (-x) \) which implies that \( y \) has at most two square roots. Note that if \( x \) is a square root of \( y \), \( \{x, -x\} \) is the set of all square roots of \( y \).
1.36. **Theorem.** If $F$ is an ordered field, then $F$ contains infinitely many elements.

**Proof.** By way of contradiction, suppose $F$ is an ordered field with only a finite number of elements. Then there exists an element $y$ in $F$ such that if $x \in F$, $y \leq x$.

From Theorem 1.26(1), $0 \leq 1$. Thus, $x = x + 0 \leq x + 1$. Since this inequality contradicts the fact that $x$ is the greatest element in $F$, the assumption that $F$ is a finite set must be false. Therefore, $F$ contains infinitely many elements.

1.37. **Definition.** If $G$ is a subset of an ordered field $F$, then $G$ is **dense in** $F$ iff for all $a, b \in F$ such that $a \leq b$, there exists some element $c$ in $G$ such that $a \leq c \leq b$.

1.38. **Theorem.** If $F$ is an ordered field, then $F$ is dense in $F$.

**Proof.** Suppose $F$ is an ordered field. Let $a, b \in F$ such that $a \leq b$. Since $1 \in F$, $(1 + 1) = 2 \in F$ which implies that $2a, 2b \in F$. In addition, $(a + b) \in F$. Since $a \leq b$, $2a = a + a \leq b + a$ and $a + b \leq b + b = 2b$. By the transitive property, $2a \leq a + b \leq 2b$. Also $2^{-1} \in F$ and $2^{-1} > 0$. By Definition 1.18, $(2^{-1})(2a) \leq (2^{-1})(a + b) \leq (2^{-1})(2b)$. Thus, $a \leq (2^{-1})(a + b) \leq b$ where $(2^{-1})(a + b) \in F$. Therefore, $F$ is dense in $F$.

1.39. **Definition.** Suppose $x$ is an element of an ordered field $F$. Then the **absolute value of** $x$, denoted
by $|x|$, is defined as follows:

$$
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
-x, & \text{if } x < 0.
\end{cases}
$$

From the definition of absolute value, it follows readily that for every $x$ in $F$, $x \leq |x|$. Several additional properties of absolute value are given in Theorem 1.40.

1.40. **Theorem.** Suppose $F$ is an ordered field. Then the following statements are true:

1. If $x \in F$, then $|x| = |-x|$.
2. If $x, y \in F$, then $|xy| = |x| \cdot |y|$.
3. If $x, p \in F$ and $p > 0$, then $|x| \leq p$ iff $-p \leq x \leq p$.

**Proof.** (1) Suppose $F$ is an ordered field and $x \in F$.

If $x > 0$, then $-x \leq 0$. So $|x| = x = -(-x) = |-x|$. If $x \leq 0$, then $-x > 0$. Thus, $|x| = -x = |-x|$. If $x = 0$, then $x = -x$. So $|x| = |-x|$.

(2) If either $x = 0$ or $y = 0$, then $|x| \cdot |y| = 0 = |xy|$. If $x > 0$ and $y > 0$, then $|x| \cdot |y| = xy = |xy|$. If $x > 0$ and $y \leq 0$, then $|x| \cdot |y| = x(-y) = -(xy) = |xy|$. If $x \leq 0$ and $y \leq 0$, then $|x| \cdot |y| = (-x)(-y) = xy = |xy|$.

(3) Suppose $F$ is an ordered field, $x, p \in F$ and $p > 0$. Also assume $x \geq p$. If $x > 0$, then $-p \leq 0 < x < p$. If $x \leq 0$, then $-x = |x| \leq p$, so that $-p \leq x \leq 0 \leq p$. Now assume $-p \leq x \leq p$. If $x \geq 0$, then $|x| = x \leq p$. If $x \leq 0$, then $|x| = -x \leq p$. 


1.41. Corollary. Suppose $F$ is an ordered field and $a, b, x, y \in F$ such that $a \leq x \leq b$ and $a \leq y \leq b$. Then
\[ |x - y| \leq b - a. \]

Proof. Suppose $F$ is an ordered field and $a, b, x, y \in F$ such that $a \leq x \leq b$ and $a \leq y \leq b$. By Theorem 1.26(3)
\[-b \leq -y \leq -a \text{ so that } (a - b) \leq (x - y) \leq (b - a). \]
Thus,
\[-(b - a) \leq (x - y) \leq (b - a) \] and by Theorem 1.40(3)
\[ |x - y| \leq (b - a). \]

The following theorem is commonly referred to as the triangle inequality and two different proofs are presented.

1.42. Theorem. If $F$ is an ordered field and if $x, y \in F$, then
\[ |x + y| \leq |x| + |y|. \]

Proof. Suppose $F$ is an ordered field and $x, y \in F$. Since $x \leq |x|$, $-x \leq -|x| = |x|$, so that $-|x| \leq x$. Therefore,
\[-|x| \leq x \leq |x| \text{ and, similarly, } -|y| \leq y \leq |y|. \]
Hence,
\[-(|x| + |y|) = x + y \leq |x| + |y|. \]
Therefore, by Theorem 1.40(3)
\[ |x + y| \leq |x| + |y|. \]

Proof. Suppose $F$ is an ordered field and $x, y \in F$. Assume $x + y \geq 0$. Then by the definition of absolute value,
\[ |x + y| = x + y. \]
In addition, $x \leq |x|$ and $y \leq |y|$. Therefore,
\[ |x + y| = x + y \leq |x| + |y|. \]

Now assume $x + y \leq 0$. Then by Definition 1.39, $|x + y| = -(x + y) = (-x) + (-y) \leq |-x| + |-y|$. By Theorem 1.40(1)
\[ |-x| + |-y| = |x| + |y|. \] Thus, $|x + y| \leq |x| + |y|$. 
1.43. Definition. Suppose $F$ is a field. $E$ is an inductive set in $F$ iff:

1. $E \subseteq F$.
2. $1 \in E$.
3. If $x \in E$, then $(x + 1) \in E$.

1.44. Theorem. If $F$ is an ordered field, then $F$ is an inductive set.

Proof. Suppose $F$ is an ordered field, then $1$ is an element of $F$. Let $x \notin F$. Since "+" is a binary operation in $F$, $(x + 1) \notin F$. Therefore by definition, $F$ is an inductive set.

1.45. Theorem. Suppose $F$ is an ordered field and suppose $\{E_a \mid a \in A\}$ is the collection of all inductive sets in $F$. Then $E = \bigcap_{a \in A} E_a$ is the smallest inductive set in $F$.

Proof. Suppose $F$ is an ordered field and suppose $\{E_a \mid a \in A\}$ is the collection of all inductive sets in $F$. Since by Theorem 1.44 every ordered field is an inductive set, $F = E_a$ for some $a \in A$. Therefore, $\{E_a \mid a \in A\}$ is a nonempty set. Since $1 \in E_a$ for every $a \in A$, $E = \bigcap_{a \in A} E_a$ also contains $1$. Let $x \in E$. That is, $x \in E_a$ for every $a \in A$. Since each $E_a$ is an inductive set, $(x + 1) \in E_a$ for every $a \in A$. Hence $(x + 1) \in E$, and by Definition 1.43, $E$ is an inductive set in $F$.

Let $G$ be an inductive subset of $F$. Then $G = E_{a'}$ for some $a' \in A$, and $E = \bigcap_{a \in A} E_a \subseteq E_{a'} = G$. Thus, $E$ is a subset of every inductive set in $F$. Hence, $E$ is the smallest inductive set in $F$. 
1.46. Definition. If $F$ is an ordered field, the set of natural numbers in $F$, denoted by $N(F)$, is defined as the smallest inductive set in $F$.

1.47. Definition. The set of integers in an ordered field $F$, denoted by $I(F)$, is defined as follows: $I(F) = \{ x \in F \mid x \in N(F) \text{ or } -x \in N(F) \text{ or } x = 0 \}$.

1.48. Definition. The set of rational numbers in an ordered field $F$, denoted by $Q(F)$, is defined as follows: $Q(F) = \{ x \in F \mid \text{there exist } m, n \in I(F) \text{ such that } x = \frac{m}{n} \text{ and } n \neq 0 \}$.

The set of natural numbers in $R$ is often called the natural numbers and is denoted by $N$. The set of integers in $R$ is usually denoted by $I$ and referred to as the integers. Similarly, the set of rational numbers in $R$ is usually denoted by $Q$ and referred to as the rational numbers. A knowledge of $R$ has been assumed, but in the sequel it will be noted how $R$ can be defined in terms of the concepts in this thesis.

1.49. Definition. Suppose $G$ and $F$ are ordered fields and $A \subseteq G$ and $B \subseteq F$. If each of $A$ and $B$ is closed with respect to addition and multiplication, then $A$ is isomorphic to $B$ iff there exists a one-to-one function $f$ from $A$ onto $B$ such that:

1. $f(a + b) = f(a) + f(b)$.
2. $f(ab) = f(a)f(b)$.
3. $a \lhd b$ iff $f(a) \lhd f(b)$. 
Peano stated that there exists a set of elements M and a function S whose domain is M such that:
\[ \begin{align*}
P_1. & \quad \text{There exists an element } 1 \in M. \\
P_2. & \quad \text{If } x \in M, \text{ then } S(x) \in M. \\
P_3. & \quad \text{If } x \in M, \text{ then } S(x) \neq 1. \\
P_4. & \quad \text{If } x, y \in M \text{ and } S(x) = S(y), \text{ then } x = y. \\
P_5. & \quad \text{If } P \subseteq M \text{ and if (1) } 1 \in P \text{ and (2) for every } x \in M, \text{ if } x \in P \text{ then } S(x) \in P, \text{ then } P = M.
\end{align*} \]

Using the properties of the natural numbers and defining S as the successor function where \( S(x) = x + 1 \), it can be verified that Peano's set M is isomorphic to the set of natural numbers. Below is a proof of the fifth postulate which is also called the Principle of Mathematical Induction.

1.50. **Theorem.** Suppose \( F \) is an ordered field. If \( M \subseteq N(F) \) and if (1) \( 1 \in M \) and (2) for every \( x \in N(F) \), if \( x \in F \) then \( (x + 1) \in M \), then \( M = N(F) \).

**Proof.** Suppose \( F \) is an ordered field and \( M \subseteq N(F) \). In addition assume \( 1 \in M \) and if \( x \in M \) then \( (x + 1) \in M \). Thus, \( M \) is an inductive set in \( F \). Since \( N(F) \) is the intersection of all inductive sets in \( F \), \( N(F) \subseteq M \). By hypothesis, \( M \subseteq N(F) \). Therefore, \( M = N(F) \).

1.51. **Theorem.** If \( A \) and \( B \) are ordered fields, then \( N(A) \) is isomorphic to \( N(B) \).

**Proof.** Let \( l' \) represent the multiplicative identity in \( B \). Define the relation \( f \) from \( N(A) \) into \( N(B) \) as follows: \( f(1) = l' \) and for \( n \in N(A) \) and \( f(n) \in N(B) \), \( f(n + 1) = f(n) + l' \).
Let \( M = \{ m \in A \mid m \text{ is an element of the domain of } f \} \). Clearly, \( 1 \in M \). If \( k \in M \), then \( f(k) \) and \( f(k) + 1' \) are defined. Hence, by the definition of \( f \), \( f(k + 1) \) is defined and \( (k + 1) \in M \). Therefore, by mathematical induction, \( M = N(A) \).

Now let \( P = \{ p \in N(B) \mid \text{there exists some } a \in N(A) \text{ such that } f(a) = p \} \). Clearly \( 1' \in P \). If \( k \in P \), then there exists some \( a \in N(A) \) such that \( f(a) = k \). Since \( k + 1' = f(a) + 1' = f(a + 1) \), \( (k + 1') \in P \). Thus, by mathematical induction, \( P = N(B) \).

In order to demonstrate that \( f \) is a function, let \( Q = \{ n \in N(A) \mid \text{if } m \in N(A) \text{ and } m = n, \text{ then } f(m) = f(n) \} \). Since \( f(l) \) has a unique value of \( 1' \), \( l \in Q \). Suppose \( k \in Q \); that is, suppose \( f(k) \) has a unique value. Then from the definition, \( f(k + 1) = f(k) + 1' \), so that \( f(k + 1) \) also has a unique value. Hence, \( (k + 1) \in Q \). Therefore, by mathematical induction \( Q = N(A) \) which means that \( f \) is a function.

Now let \( S = \{ n \in N(A) \mid \text{if } m \in N(A) \text{ and } f(m) = f(n), \text{ then } m = n \} \). Suppose \( 1 \notin S \); that is, suppose there exists an \( m \in N(A) \) such that \( f(m) = f(1) \) and \( m \neq 1 \). Since \( m \in N(A) \) and \( m \neq 1 \), then \( (m - 1) \in N(A) \). Now, \( f((m - 1) + 1') = f(1) = 1' \), so that \( f(m - 1) + 1' = 1' \). Thus, \( f(m - 1) = 0 \). Since \( 0 \notin N(B) \), this conclusion contradicts the fact that the range of \( f \) is \( N(B) \). Therefore the assumption that \( 1 \notin S \) must be false. Thus, \( 1 \in S \). Now suppose \( k \in S \); that is, if \( j \in N(A) \) and \( f(j) = f(k) \),
then $j = k$. Choose $m \in N(A)$ such that $f(m) = f(k + 1)$.

Note that $m > 1$ since from above if $m = 1$, $k + 1 = 1$ which
contradicts the assumption that $k \in N(A)$. Thus, $f(m) =
f(m - 1 + 1) = f(k + 1)$ so that $f(m - 1) + 1' = f(k) + 1'$.

Since $(m - 1) \in N(A)$ and since $k \in S$, $(m - 1) = k$. Thus,
$m = k + 1$ which implies that $(k + 1) \in S$. Therefore by
mathematical induction, $S = N(A)$, and $f$ is a one-to-one
function.

Let $T = \{ n \in N(A) \mid \text{if } m \in N(A) \text{ then } f(m + n) = f(m) + f(n) \}$. For each $m \in N(A)$,
$f(m + 1) = f(m) + 1' = f(m) + f(1)$ which
implies that $1 \in T$. Suppose $k \in T$. Then for $m \in N(A),
\begin{align*}
f[m + (k + 1)] &= f[(m + 1) + k] = f(m + 1) + f(k) = 
\end{align*}
\begin{align*}
f(m) + f(1) + f(k) &= f(m) + f(k) + f(1) = f(m) + f(k + 1).
\end{align*}
Thus, if $k \in T$, then $(k + 1) \in T$. Therefore by mathematical
induction $T = N(A)$. Hence for all $m, n \in N(A)$, $f(m + n) =
f(m) + f(n)$.

Let $U = \{ n \in N(A) \mid \text{if } m \in N(A), \text{ then } f(mn) = f(m)f(n) \}$. Note that $f(m \cdot 1) = f(m) = f(m) \cdot 1' = f(m)f(1)$. Thus $1 \in U.$

Suppose $k \in U$. Then for $m \in N(A), f[m(k + 1)] =
\begin{align*}
f[(m \cdot k) + (m \cdot 1)] &\quad \text{and since } f \text{ is isomorphic with respect to addition, } f[(m \cdot k) + (m \cdot 1)] = f(m \cdot k) + f(m \cdot 1) = 
\end{align*}
\begin{align*}
f(m)f(k) + f(m)f(1). \quad \text{Applying the distributive property results in the following: } f(m)f(k) + f(m)f(1) &= f(m)f(k + 1). \text{ Therefore } (k + 1) \in U. \text{ Thus by mathematical induction, if } n, m \in N(A), \text{ then } f(mn) = f(m)f(n).
In order to determine the order preserving property of \( f \), suppose \( m, n \in \mathbb{N}(A) \) and \( m > n \). That is, \( m = n + (m - n) \) where \( (m - n) \in \mathbb{N}(A) \). Since \( f \) is isomorphic with respect to addition, \( f(m) = f(n) + f(m - n) \) where \( f(m - n) \in \mathbb{N}(B) \). Thus \( f(m) > f(n) \). Conversely, it can be verified that if \( f(m) > f(n) \) then \( m > n \).

Thus \( f \) is a one-to-one function from \( \mathbb{N}(A) \) onto \( \mathbb{N}(B) \) with the three properties of 1.49. Therefore by definition, \( \mathbb{N}(A) \) is isomorphic to \( \mathbb{N}(B) \). This seems to be adequate justification to call \( \mathbb{N}(A) \) the set of natural numbers.

As the following theorem indicates, it can be verified that an isomorphism exists between the sets of integers in any two ordered fields. In addition, the sets of rational numbers of any two ordered fields are isomorphic.

1.52. Theorem. If \( A \) and \( B \) are ordered fields, then \( \mathbb{I}(A) \) is isomorphic to \( \mathbb{I}(B) \).

1.53. Theorem. If \( A \) and \( B \) are ordered fields, then \( \mathbb{Q}(A) \) is isomorphic to \( \mathbb{Q}(B) \).

The proofs of Theorems 1.52 and 1.53 are omitted. In Theorem 1.51 it was verified that \( \mathbb{N}(A) \) is isomorphic to \( \mathbb{N}(B) \). Using the fact that the integers consist of the natural numbers, their additive inverses, and zero it can be verified that \( \mathbb{I}(A) \) is isomorphic to \( \mathbb{I}(B) \). Then, since every rational number can be expressed as the quotient of two integers, it can be proven that \( \mathbb{Q}(A) \) is isomorphic to \( \mathbb{Q}(B) \).
1.54. **Theorem.** If $G$ is a subfield of an ordered field $F$, then $N(G) = N(F)$.

**Proof.** Suppose $G$ is a subfield of an ordered field $F$. By Theorem 1.45, $N(G)$ is a subset of every inductive set in $G$ and $N(F)$ is a subset of every inductive set in $F$. Since $N(G)$ is an inductive set in $F$, $N(F) \subseteq N(G)$.

In addition, the fact that $N(F) \subseteq N(G)$ implies that $N(F) \subseteq G$. Thus, $N(F)$ is an inductive set in $G$. Hence, $N(G) \subseteq N(F)$.

Therefore, $N(G) = N(F)$.

1.55. **Theorem.** Suppose $F$ is an ordered field. If $m, n \in N(F)$ such that $m < n$, then $m + 1 \leq n$.

The proof is omitted.

1.56. **Theorem.** If $F$ is an ordered field, then $Q(F)$ is the smallest ordered subfield in $F$.

**Proof.** Suppose $G$ is an ordered subfield in an ordered field $F$. Then $G \subseteq F$ and $1 \in G$. In addition, if $x \in G$, then $(x + 1) \in G$. Thus, $G$ is an inductive set in $F$ and $N(F) \subseteq G$.

Suppose $x \in I(F)$. Then either $x \in N(F)$, $(-x) \in N(F)$, or $x = 0$. If $x \in N(F)$, then $x \in G$. If $(-x) \in N(F)$, then $(-x) \in G$. Since $G$ is an ordered field, $-(-x) = x \in G$. Also, $0 \in G$.

Therefore, $I(F) \subseteq G$.

Now suppose $x \in Q(F)$. Then there exist $p, q \in I(F)$ such that $x = (p)(q^{-1})$ and $q \neq 0$. From above, $p \in G$ and since $q \in I(F) \subseteq G$, $(q^{-1}) \in G$. Thus, $(p)(q^{-1}) \in G$. Therefore $Q(F) \subseteq G$. 
Since $\mathbb{Q}(F)$ is a subset of every ordered subfield in $F$, $\mathbb{Q}(F)$ is the smallest ordered subfield in $F$.

1.57. **Theorem.** If $F$ is an ordered field and $n \in N(F)$, then $n \geq 1$.

**Proof.** Suppose $F$ is an ordered field. Define the following set $S = \{x \in F \mid x \geq 1\}$. Clearly, $S \subseteq F$ and $1 \in S$. If $y \in S$, then $y \geq 1$. Hence, $1 \leq y \leq y + 1$ and $(y + 1) \in S$. Thus, $S$ is an inductive set in $F$. Since $N(F)$ is defined to be the smallest inductive set in $F$, $N(F) \subseteq S$. Therefore, if $n \in N(F)$, then $n \in S$. That is, if $N \subseteq N(F)$, then $n \geq 1$.

The following theorem is often referred to as the Well Ordering Principle.

1.59. **Theorem.** If $S$ is a nonempty subset of the natural numbers, then $S$ contains a least element.

**Proof.** Let $M = \{x \in N \mid x \leq y \text{ for all } y \in S\}$. By Theorem 1.57, $1 \leq y$ for all $y \in N$. Therefore, $1 \in M$.

Clearly $M \subseteq N$. In addition $M \neq N$, since if $s \in S \subseteq N$, then $(s + 1) \notin M$. By mathematical induction since $M \neq N$, there exists $m \in M$ such that $(m + 1) \notin M$.

Since $m \in M$, $m \leq y$ for all $y \in S$. Suppose $m \notin S$. Then $m \leq y$ for all $y \in S$. By Theorem 1.55, $m + 1 \leq y$ for all $y \in S$. Thus, $(m + 1) \in M$. But this contradicts the fact that $(m + 1) \notin M$. Thus, the assumption that $m \notin S$ must be false. Hence, $m \in S$ and $m$ is less than every element of $S$. Therefore $m$ is the least element in $S$. 
CHAPTER II

ARCHIMEDEAN FIELDS AND THE REAL NUMBERS

2.1. Definition. An Archimedean field \( F \) is an ordered field satisfying the property that for each \( x \in F \) there exists some \( n \in N(F) \) such that \( x \leq n \).

2.2. Theorem. An ordered field \( F \) is an Archimedean field iff for \( x, y \in F \) such that \( 0 \leq x \leq y \), there exists some \( n \in N(F) \) such that \( y \leq nx \).

Proof. Suppose \( F \) is an Archimedean field and assume \( x, y \in F \) where \( 0 \leq x \leq y \). Then \( (x^{-1}) \in F \) and \( (y)(x^{-1}) \in F \). Since \( F \) is an Archimedean field, there exists some \( n \in N(F) \) such that \( (y)(x^{-1}) \leq n \). Since \( x > 0 \), \( (y)(x^{-1})x \leq nx \). Thus \( y \leq nx \).

Now suppose \( F \) is an ordered field satisfying the property that if \( x, y \in F \) and \( 0 \leq x \leq y \), then there exists some \( n \in N(F) \) such that \( y \leq nx \). Choose \( x \in F \). If \( x \leq 1 \), then \( x \leq x + 1 \leq 1 + 1 = 2 \). Since \( 2 \in N(F) \), let \( n = 2 \) so that \( x \leq n \). If \( x > 1 \), then \( 0 \leq 1 \leq x \). By assumption there exists \( n \in N(F) \) such that \( x \leq n(1) = n \). Therefore, \( F \) is an Archimedean field.

2.3. Theorem. An ordered field \( F \) is Archimedean iff for each \( x, y \in F \) such that \( x > 0 \) and \( y > 0 \) there exists a natural number \( n \) such that \( nx > y \).
Proof. Suppose $F$ is an Archimedean field and $x$ and $y$ are elements of $F$ such that $x > 0$ and $y > 0$. If $x \geq y$, then $2x \in N(F)$ and $2x > y$. If $x \not\geq y$, then by Theorem 2.2 there exists some $n \in N(F)$ such that $y < nx$.

Conversely, assume $F$ is an ordered field having the property that if $x$ and $y$ are elements of $F$ such that $x > 0$ and $y > 0$, then there exists a natural number $n$ such that $nx > y$. By Theorem 2.2, $F$ is an Archimedean field.

2.4. Theorem. An ordered field $F$ is Archimedean iff for $p \in F$ such that $p > 0$ there exists some $n \in N(F)$ such that $\frac{1}{n} < p$.

Proof. Suppose $F$ is an Archimedean field. Let $p \in F$ such that $p > 0$. By Theorem 1.26(6), $\frac{1}{p} > 0$. By the definition of an Archimedean field, there exists some $n \in N(F)$ such that $\frac{1}{p} < n$. Since $p > 0$ and $\frac{1}{n} > 0$, $\frac{p}{n} > 0$. Thus, $\left(\frac{1}{p}\right)\left(\frac{p}{n}\right) < n\left(\frac{p}{n}\right)$ which implies that $\frac{1}{n} < p$.

Now suppose $F$ is an ordered field with the property that if $p \in F$ and $p > 0$ then there exists some $n \in N(F)$ such that $\frac{1}{n} < p$. Let $x \in F$. If $x \leq 0$, then $1 \in N(F)$ and $x \leq 0 < 1$. If $x > 0$, then $\frac{1}{x} \in F$ and $\frac{1}{x} > 0$. Thus, by hypothesis there exists some $n \in N(F)$ such that $0 < \frac{1}{n} < \frac{1}{x}$. By Theorem 1.26(6), $0 < x < n$. Therefore $F$ is an Archimedean field.

2.5. Lemma. If $F$ is an ordered field and $n \in N(F)$, then $n < 2^n$. 
Proof. Suppose \( F \) is an ordered field. Let \( S = \{ n \in N(F) \mid n < 2^n \} \). Clearly, \( 1 \in S \). Assume \( k \in S \). That is, assume \( k < 2^k \). By Theorem 1.55, \( k + 1 \leq 2^k \). Since \( 1 < 2 \), \( 2^k(1) < 2^k(2) \). So that \( 2^k \leq 2^{k+1} \). Thus, \( k + 1 \leq 2^k \leq 2^{k+1} \). Therefore, \((k + 1) \in S \). By the principle of mathematical induction, \( S = N(F) \). Thus, for each \( n \in N(F) \), \( 0 \leq n \leq 2^n \).

2.6. Theorem. An ordered field \( F \) is Archimedean iff for each element \( z > 0 \) there exists \( n \in N(F) \) such that \( 0 < \frac{1}{2^n} < z \).

Proof. Suppose \( F \) is an Archimedean field and \( z > 0 \). By Theorem 2.4 there exists some \( n \in N(F) \) such that \( 0 < \frac{1}{n} < z \). In addition, by Lemma 2.5, \( 0 < n < 2^n \) so that by Theorem 1.26(6) \( 0 < \frac{1}{2^n} < \frac{1}{n} \). Thus, by the transitive property \( 0 < \frac{1}{2^n} < z \).

Now suppose \( F \) is an ordered field satisfying the property that for each element \( z > 0 \) there exists some \( n \in N(F) \) such that \( 0 < \frac{1}{2^n} < z \). Let \( x \in F \). If \( x \leq 0 \), then \( x \leq 0 < 1 \) where \( 1 \in N(F) \). If \( x > 0 \), then \( 0 < \frac{1}{x} \). By assumption there exists an \( n \in N(F) \) such that \( 0 < \frac{1}{2^n} < \frac{1}{x} \). From Theorem 1.26(6), \( 0 < x < 2^n \) where \( 2^n \in N(F) \). Thus, \( F \) is an Archimedean field.

2.7. Corollary. An ordered field \( F \) is Archimedean iff for each element \( z > 0 \) there exists a natural number \( n \) such that \( 0 < \frac{1}{2^{n-1}} < z \).

2.8. Theorem. If \( F \) is an Archimedean field and \( x > 0 \) then there exists some \( n \in N(F) \) such that \( n - 1 \leq x \leq n \).
Proof. Suppose $F$ is an Archimedean field and $x > 0$.

Let $S = \{ n \in N(F) \mid x \leq n \}$. Since $F$ is an Archimedean field, $S$ is a nonempty subset of $N(F)$. By the well ordering principle, $S$ has a least element. Let $m$ be the least element in $S$.

Since $m$ is the least element in $S$, $(m - 1) \not\in S$. Therefore, $(m - 1) \leq x < m$.

2.9. Example. In Example 1.25 it was shown that the field of rational functions $Q(t)$ is an ordered field. Suppose $n$ is an element of the set of natural numbers in $Q(t)$. Note that the rational function defined by $\frac{p(t)}{q(t)} = t$ is an element of $Q(t)$. Suppose $n$ is a natural number in $Q(t)$. Then $\frac{p(t)}{q(t)} - n = t - n = \frac{t - n}{1}$. By definition, $\frac{t - n}{1}$ is positive which implies that $\frac{p(t)}{q(t)} > n$. Therefore, $Q(t)$ is not an Archimedean field.

2.10. Theorem. If $F$ is an ordered field, then $Q(F)$ is an Archimedean field.

Proof. Suppose $F$ is an ordered field. From Theorem 1.56 $Q(F)$ is an ordered field. Suppose $x \in Q(F)$. If $x \leq 0$, then $1 \in N(F)$ and $x \leq 1$. Now suppose $x > 0$; then there exist $m, n \in I(F)$ such that $m(n^{-1}) = x$. Without loss of generality assume $m > 0$ and $n > 0$. Then $m, n \in N(F)$ and $x = m(n^{-1})$.

Thus, $x \leq xn = m(n^{-1})n = m < m + 1$. Hence, there exists a natural number $(m + 1)$ such that $x < (m + 1)$. Therefore, $Q(F)$ is an Archimedean field.
2.11. Theorem. An ordered field $F$ is Archimedean iff $Q(F)$ is dense in $F$.

Proof. Suppose $F$ is an Archimedean field, $a, b \in F$, and without loss of generality assume $0 \leq a \leq b$.

If $0 = a \leq b$, by Theorem 2.4 there exists some $n \in N(F)$ such that $\frac{1}{n} \leq b$. Thus, $\frac{1}{n} \in Q(F)$ and $a = 0 \leq \frac{1}{n} \leq b$.

Now assume $0 \leq a < b$ which implies $0 \leq (b - a)$. Let $p$ represent the minimum of $a$ and $(b - a)$. Since $p > 0$, by Theorem 2.4 there exists a natural number $m$ such that $\frac{1}{m} < p$. Therefore, $0 \leq \frac{1}{m} \leq a$ and $0 < \frac{1}{m} \leq (b - a)$. By Theorem 2.3 since $a > 0$ and $\frac{1}{m} > 0$, there exists a natural number $k$ in $F$ such that $\frac{k}{m} = k(\frac{1}{m}) > a$. Let $S = \{ k \in N(F) \mid \frac{k}{m} > a \}$.

By the well ordering principle, $S$ contains a least element $n$. Thus, $\frac{n-1}{m} \leq a < \frac{n}{m}$. By way of contradiction, suppose $b \leq \frac{n}{m}$.

Then $\frac{n-1}{m} \leq a < b \leq \frac{n}{m}$. By Corollary 1.41, $(b - a) = \lfloor b - a \rfloor \leq \frac{n}{m} - \frac{n-1}{m} = \frac{1}{m}$. But this conclusion contradicts the fact that $(b - a) > \frac{1}{m}$. Therefore, $b > \frac{n}{m}$ so that $a \leq \frac{n}{m} < b$. Since $\frac{n}{m} \in Q(F)$, by definition $Q(F)$ is dense in $F$.

Conversely, assume $F$ is an ordered field such that $Q(F)$ is dense in $F$ and $x \in F$. If $x \leq 0$, then $1 \in N(F)$ and $x \leq 0 < 1$.

If $x > 0$, then $0 \leq x \leq x + 1$. Since $Q(F)$ is dense in $F$, there exists a rational number $\frac{p}{q}$ such that $0 \leq x \leq \frac{p}{q} \leq x + 1$. Since $\frac{p}{q} > 0$, it can be assumed without loss of generality that $p, q \in N(F)$. Thus, $0 \leq x \leq \frac{p}{q} \leq p \leq p + 1$ where $(p + 1)$ is an element of $N(F)$. Hence, $F$ is an Archimedean field.
2.12. **Theorem.** Every subfield of an Archimedean field is Archimedean.

**Proof.** Suppose $G$ is a subfield of an Archimedean field $F$. Let $x \in G$. Then since $x$ is also in $F$, there exists some $n \in N(F)$ such that $x \not\leq n$. By Theorem 1.54, $N(F) = N(G)$ which implies that $n \in N(G)$ and $x \not\leq n$. Thus, $G$ is Archimedean.

2.13. **Definition.** Suppose $S$ is a subset of an ordered field $F$. An element $x$ of $F$ is said to be an *upper bound* of $S$ iff $x \geq s$ for every $s$ in $S$. If $S$ has an upper bound, $S$ is said to be bounded above.

2.14. **Definition.** Suppose $S$ is a subset of an ordered field $F$. An element $x$ of $F$ is said to be a *least upper bound* of $S$ iff:

1. $x$ is an upper bound of $S$.
2. If $y$ is an upper bound of $S$, then $x \leq y$.

2.15. **Theorem.** If $S$ is a subset of an ordered field $F$, then $S$ has at most one least upper bound.

**Proof.** Suppose $S$ is a subset of an ordered field such that each of $x$ and $y$ is a least upper bound of $S$. Since $x$ is a least upper bound and $y$ is an upper bound of $S$, by Definition 2.14, $x \leq y$. Similarly, since $y$ is a least upper bound and $x$ is an upper bound, $y \leq x$. Thus, $x = y$ which means that $S$ has at most one least upper bound.

2.16. **Theorem.** Suppose $S$ is a subset of an ordered field $F$. If $x$ is an upper bound of $S$ and $x \in S$, then $x$ is the least upper bound of $S$. 
Proof. Suppose $S$ is a subset of an ordered field $F$ and suppose $x$ is an upper bound of $S$ such that $x \in S$. Assume $y$ is also an upper bound of $S$. Then, since $x \in S$, $x \leq y$.

Therefore, by Definition 2.14, $x$ is the least upper bound of $S$.

2.17. Theorem. Suppose $S$ is a subset of an ordered field $F$. An element $x$ of $F$ is the least upper bound of $S$ iff:

(1) If $s \in S$, then $s \leq x$.

(2) If $y \not< x$ and $y \in F$, then there exists an element $t$ of $S$ such that $y \leq t \leq x$.

Proof. First assume $S$ is a subset of an ordered field $F$ and that $x$ is the least upper bound of $S$. By definition, $s \leq x$ for every $s$ in $S$. By way of contradiction, suppose there exists an element $y$ in $F$ such that $y \not< x$ and if $t \in S$, then $t \not< y$. Hence, $y$ is an upper bound of $S$ and $y \not< x$. This contradicts the assumption that $x$ is the least upper bound of $S$. Therefore, there exists some $t \in S$ such that $y \leq t \leq x$.

Conversely, assume $S$ is a subset of an ordered field $F$. In addition, suppose (1) if $s \in S$, then $s \leq x$ and (2) if $y \not< x$ and $y \in F$, then there exists an element $t$ of $S$ such that $y \leq t \leq x$. By assumption $x$ is an upper bound of $S$. Suppose $y$ is an upper bound of $S$, and by way of contradiction, assume $y \not< x$. By hypothesis, there exists some $t \in S$ such that $y \leq t \leq x$. This contradicts the hypothesis that $y$ is an upper bound. Therefore the assumption that $y \not< x$ is false. Thus, $x \leq y$. So, by definition, $x$ is the least upper bound of $S$.  
Note that every element in an ordered field is an upper bound of the empty set and thus the empty set has no least upper bound.

2.18. Definition. A complete ordered field is an ordered field in which every nonempty set that is bounded above has a least upper bound.

2.19. Remark. It should be noted that the following is equivalent to Definition 2.18: A complete ordered field is an ordered field in which every nonempty set that is bounded below has a greatest lower bound.

The question of whether a complete ordered field exists is a difficult one. The Italian mathematician Peano assumed the existence of a mathematical system which satisfies the so-called Peano Axioms, and this system is usually called the natural numbers. Upon this assumption, one may extend, by definition and proof, the system of natural numbers to the set of integers, the set of rational numbers, and finally to a complete ordered field. As Theorem 3.9 indicates any two complete ordered fields are isomorphic. Thus, in a valid sense, one may conclude that there is at most one complete ordered field which is often referred to as the set of real numbers (denoted by \( \mathbb{R} \)).

2.20. Theorem. The set of real numbers is an Archimedean field; that is, a complete ordered field is Archimedean.
Proof. By way of contradiction assume $R$ is not Archimedean. Then there exists some $x \in R$ such that $x \geq n$ for every $n \in \mathbb{N}$. Thus, $x$ is an upper bound of $\mathbb{N}$ so that by the completeness property $\mathbb{N}$ has a least upper bound $c$. That is, if $n \in \mathbb{N}$, then $n \leq c$. Since $(n + 1) \in \mathbb{N}$, $(n + 1) \leq c$ which implies $n \leq (c - 1) \leq c$. Thus $(c - 1)$ is an upper bound of $\mathbb{N}$ less than the least upper bound. This contradiction implies that the assumption that $R$ is not Archimedean is false. Thus, $R$ is an Archimedean field.

2.21. Theorem. The set of real numbers is not a proper subfield of any Archimedean field.

Proof. By way of contradiction, suppose $R$ is a proper subfield of an Archimedean field $F$. Then there exists an element $z$ of $F$ such that $z$ is not an element of $R$. Since $F$ is an ordered field exactly one of the following is true: $z = 0$, $z > 0$ or $z < 0$.

The assumption that $z = 0$ leads to a contradiction since $0 \not\in R$.

Suppose $z > 0$ and let $S = \left\{ x \in R \mid x \leq z \right\}$. Since $0 \in S$, $S$ is a nonempty set. Since $F$ is Archimedean, there exists a natural number $n$ such that $z < n$. By Theorem 1.54 $N = N(R) = N(F)$ which implies that $n \not\in R$. Therefore, $n$ is an upper bound for $S$. Thus, by the completeness property $S$ has a least upper bound $k$ in $R$. If $k < z$, then by Theorem 2.11 there exists a rational number $q$ in $Q(F) = Q(R) = Q \subseteq R$ such that $k < q < z$. Thus, $q \not\in S$ which contradicts the fact that $k$ is the least upper.
bound of $S$. If $z \preceq k$, by the same argument there exists a rational number $p$ such that $z \preceq p \preceq k$. Hence, $p$ is an upper bound for $s$ and $p \preceq k$. This conclusion contradicts the fact that $k$ is the least upper bound of $S$. The assumption that $k = z$ also leads to a contradiction since $k \in \mathbb{R}$ but $z \notin \mathbb{R}$. Thus, each case of the conclusion results in a contradiction which means that the assumption that $z > 0$ must be false.

Suppose $z < 0$. Then $(-z) > 0$ and $(-z) \notin \mathbb{R}$. Let $S = \{x \in \mathbb{R} \mid x \preceq (-z)\}$. Using the same argument presented above, a contradiction can be reached. Thus, the assumption that $z \preceq 0$ must be false.

Therefore, the original assumption that $\mathbb{R}$ is a proper subfield of an Archimedean field $F$ implies that $z \in F$ such that $z \neq 0$, $z \notin \mathbb{R}$ and $z \neq 0$. This conclusion contradicts the fact that $F$ is an ordered field. Thus, $\mathbb{R}$ is not a proper subfield of any Archimedean field.

2.22. **Example.** There exists an ordered field which contains the set of real numbers as a proper subfield.

Consider the ordered field of rational functions $\mathbb{Q}(t)$ described in Example 1.25. Since $R = \left\{ \frac{p(t)}{q(t)} \mid q(t) = 1 \text{ and } p(t) = c \text{ where } c \in \mathbb{R} \right\}$, the set of all constant real-valued functions, $R$ is a subset of $\mathbb{Q}(t)$. Also, the addition and multiplication operations for rational functions are equivalent to the usual operations for the field $\mathbb{R}$. Thus, by Definition 1.8 $R$ is a subfield of $\mathbb{Q}(t)$. Consider the function $\frac{p(t)}{q(t)} = t^2$. Since $t^2 \in \mathbb{Q}(t)$ and $t^2 \notin \mathbb{R}$, $R$ is a proper subfield of the ordered field $\mathbb{Q}(t)$. 
2.23. **Lemma.** If $s, y \in \mathbb{R}$, $0 \leq s \leq 1$ and $0 \leq y$, then 

$$(y + s)^2 \leq y^2 + s(y + 1)^2.$$ 

**Proof.** Suppose $s, y \in \mathbb{R}$, $0 \leq s \leq 1$ and $0 \leq y$. Since 

$1 \leq (1 + s), \ y^2 \leq y^2(1 + s).$ 

Since $0 \leq s \leq 1, \ 0 \leq s^2 \leq s \leq 1.$ Therefore, 

$y^2 + 2sy + s^2 \leq y^2(1 + s) + 2sy + s$ which implies that 

$(y + s)^2 \leq y^2 + s(y^2 + 2y + 1) = y^2 + s(y + 1)^2.$ 

2.24. **Lemma.** If $s, y \in \mathbb{R}$, $0 \leq s \leq 1$, and $0 \leq s \leq y$, then 

$$(y - s)^2 > y^2 - s(y + 1)^2.$$ 

**Proof.** Suppose $s, y \in \mathbb{R}$, $0 \leq s \leq 1$ and $0 \leq s \leq y$. Since $s > 0$, $s(s + 1 + y^2) > 0$ which implies that $s^2 > -sy^2 - s$. Thus, 

$y^2 - 2sy + s^2 > y^2 - 2sy - sy^2 - s.$ 

Factoring results in the following inequality: 

$(y - s)^2 > y^2 - s(y^2 + 2y + 1).$ 

Therefore, 

$(y - s)^2 > y^2 - s(y + 1)^2.$ 

2.25. **Theorem.** If $x \in \mathbb{R}$ and $x > 0$, then there exists a unique $y \in \mathbb{R}$ such that $y > 0$ and $y^2 = x$. 

**Proof.** Suppose $x \in \mathbb{R}$ and $x > 0$. Let $S = \{ s \in \mathbb{R} \mid s > 0 \text{ and } s^2 < x \}$. If $x < 1$, then $x^2 < x$ and $x \in S$. 

If $x \geq 1$, then $(\frac{1}{x})^2 = \frac{1}{x} < 1$ and $\frac{1}{x} \in S$. Therefore, $S$ is a nonempty subset of $\mathbb{R}$. 

Suppose $x \leq 1$. If $s \in S$, then $s > 0$ and $s^2 < 1$. By Theorem 1.27 $s < 1$ so that 1 is an upper bound of $S$. Suppose $x > 1$. If $s \in S$, then $s^2 < x^2$. By Theorem 1.27, $s < x$ so that $x$ is an upper bound of $S$. 

Therefore, $S$ is a nonempty subset of $\mathbb{R}$ and $S$ is bounded above. By the least upper bound principle, there exists a unique real number $y$ such that $y$ is the least upper bound of $S$. In addition, $y > 0$. 


Assume $y^2 \leq x$. Let $p = \frac{1}{2} \min \left\{ 1, \frac{x - y^2}{(y + 1)^2} \right\}$. Then $0 \leq p \leq 1$ and by Lemma 2.23 and the definition of $p$, $\left( y + p \right)^2 \leq \left( y + 1 \right)^2 = y^2 + x - y^2 = x$. Therefore $\left( y + p \right)^2 \leq x$ which implies that $(y + p) \notin S$. The conclusion that $(y + p) \notin S$ contradicts the fact that $y$ is an upper bound of $S$. Therefore, the assumption that $y^2 \leq x$ must be false.

Now assume $x < y^2$. Let $p = \frac{1}{2} \min \left\{ 1, y, \frac{y^2 - x}{(y + 1)^2} \right\}$. Then $0 \leq p \leq 1$ and $0 \leq p \leq y$. By Lemma 2.24 and the definition of $p$, $(y - p)^2 > y^2 - p(y + 1)^2 = y^2 - y^2 + x = x$. Therefore $(y - p)^2 > x$ which implies that $(y - p)$ is an upper bound for $S$. The conclusion that $(y - p)$ is an upper bound for $S$ contradicts the fact that $y$ is the least upper bound of $S$. Thus, the assumption that $x < y^2$ must be false.

Therefore, by Definition 1.18(1), $y^2 = x$.

The element $y$ is the positive square root of $x$ and is denoted by $\sqrt{x}$.

2.26. **Definition.** An integer $n$ of an ordered field is said to be **even** iff there exists a unique integer $k$ such that $n = 2k$.

2.27. **Definition.** An integer $n$ of an ordered field is said to be **odd** iff there exists a unique integer $k$ such that $n = 2k + 1$. 
2.28. **Theorem.** If $a$ and $b$ are integers in an ordered field such that $b \neq 0$, then there exist unique integers $q$ and $r$ such that $a = bq + r$, where $0 \leq r < |b|$. The proof is omitted.

2.29. **Corollary.** If $a$ is an integer in an ordered field, then there exists a unique integer $q$ such that either $a = 2q$ or $a = 2q + 1$.

**Proof.** Suppose $a$ is an integer. Since $2$ is also an integer and $2 \neq 0$, by Theorem 2.28 there exist unique integers $q$ and $r$ such that $a = 2q + r$, where $0 \leq r < |2|$. Thus, $a = 2q$ or $a = 2q + 1$.

Note that Corollary 2.29 is equivalent to the statement that every integer is either even or odd.

2.30. **Theorem.** If $m^2$ is an even integer, then $m$ is an even integer.

**Proof.** The contrapositive form of the theorem is: If $m$ is an odd integer, then $m^2$ is an odd integer.

Suppose $m$ is an odd integer. By Definition 2.27, there exists an integer $k$ such that $m = 2k + 1$. Thus, $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. Let $p = (2k^2 + 2k)$. Then, $m^2 = 2p + 1$, where $p$ is an integer. By Definition 2.27, $m^2$ is odd.

2.31. **Theorem.** $\sqrt{2}$ is not an element of the set of rational numbers.

**Proof.** By way of contradiction, suppose $\sqrt{2}$ is an element of the set of rational numbers. Then, there exist integers
m and n such that $\frac{m}{n} = \sqrt{2}$. In addition, assume m and n are relatively prime. Since $\frac{m}{n} = \sqrt{2}$, $(\frac{m}{n})^2 = 2$. Hence, $m^2 = 2n^2$ which means that $m^2$ is even, and by Theorem 2.30, m is also even. Thus, there exists an integer k such that $m = 2k$. Therefore, $4k^2 = (2k)^2 = m^2 = 2n^2$. Hence, $2k^2 = n^2$ and by Theorem 2.30, n is an even integer. The conclusion that m and n are even integers contradicts the fact that m and n are relatively prime. Therefore, the assumption that $\sqrt{2} \in \mathbb{Q}$ must be false.

2.32. Theorem. The set of rational numbers is an ordered field that is not complete.

Proof. From Example 1.4 the set of rational numbers $\mathbb{Q}$ is a field. It can be verified that the set $P = \{ x \in \mathbb{Q} \mid x > 0 \}$ is a positive class in $\mathbb{Q}$, so that $\mathbb{Q}$ is an ordered field.

By way of contradiction, suppose $\mathbb{Q}$ is complete. Since $2 \in \mathbb{Q}$ and $2 > 0$, by Theorem 2.25, there exists a unique element $\sqrt{2}$ in $\mathbb{Q}$ such that $0 < \sqrt{2}$ and $(\sqrt{2})^2 = 2$. However, this conclusion contradicts Theorem 2.31 which states that $\sqrt{2} \notin \mathbb{Q}$. Therefore, the assumption that $\mathbb{Q}$ is complete must be false.
CHAPTER III

COMPLETENESS

3.1. Definition. Suppose $F$ is an ordered field and $a, b \in F$ such that $a \not\leq b$. Then the closed interval from $a$ to $b$, denoted by $[a, b]$, is the set of all elements $x$ in $F$ such that $a \leq x \leq b$.

3.2. Definition. Suppose $F$ is an ordered field and $a, b \in F$ such that $a \not\leq b$. Then the open interval from $a$ to $b$, denoted by $(a, b)$, is the set of all elements $x$ in $F$ such that $a < x < b$.

3.3. Remark. Suppose $[a, b]$ and $[c, d]$ are intervals in an ordered field $F$. Then, $[a, b]$ contains $[c, d]$ iff $a \leq c \leq d \leq b$.

3.4. Definition. A sequence in a field $F$ is a function whose domain is $\mathbb{N}(F)$ and whose range is a subset of $F$. A sequence may be denoted by $\langle a_n \rangle$ or by $\langle a_1, a_2, a_3, \ldots \rangle$.

3.5. Definition. In an ordered field a sequence $\langle b_n \rangle$ is a subsequence of a sequence $\langle a_n \rangle$ iff there exists a strictly increasing sequence of natural numbers $\langle m_n \rangle$ such that $b_n = a_{m_n}$ for every natural number $n$.

3.6. Definition. A sequence of closed intervals $[a_n, b_n]$ in $F$ is said to be nested iff $[a_n, b_n]$ contains $[a_{n+1}, b_{n+1}]$ for every $n \in \mathbb{N}(F)$. 

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3.7. **Theorem.** F is a complete ordered field iff F is an Archimedean field such that if \( \langle I_n \rangle \) is a nested sequence of closed intervals in F then there exists an element of F common to all the intervals in the sequence.

**Proof.** Suppose F is a complete ordered field. By Theorem 2.20, F is an Archimedean field. Suppose \( \langle I_n \rangle = \langle [a_n, b_n] \rangle \) is a nested sequence of closed intervals in F. Hence, if \( m, n \in N(F) \), then either \( I_m \subset I_n \) or \( I_n \subset I_m \). Assume \( k \in N(F) \). Then \( I_k \subset I_1 \), since the intervals are nested. Hence, \( [a_k, b_k] \subset [a_1, b_1] \) which implies \( a_1 \leq a_k \leq b_k \leq b_1 \).

Since this statement is true for each \( k \in N(F) \), \( a_1 \) is a lower bound for the set \( B = \{ b_n \mid [a_n, b_n] = I_n \text{ for some } n \in N(F) \} \) and \( b_1 \) is an upper bound for the set \( A = \{ a_n \mid [a_n, b_n] = I_n \text{ for some } n \in N(F) \} \). By the definition of a complete ordered field there exists a unique element \( b \) in F such that \( b \) is the greatest lower bound of \( B \) and there exists a unique element \( a \) in F such that \( a \) is the least upper bound of \( A \).

By way of contradiction, suppose \( b < a \). By Theorem 2.17 there exists an element \( a_i \) in A such that \( b < a_i \leq a \), and, similarly, there exists an element \( b_j \) in B such that \( b \leq b_j < a_i \). Thus, \( a_j \leq b_j < a_i \leq b_i \) which implies that \( I_j \not\supset I_i \) and \( I_i \not\supset I_j \).

This conclusion contradicts the fact that the intervals are nested. Therefore, the assumption that \( b < a \) must be false which means \( a \leq b \).
If \( n \in \mathbb{N}(F) \), then \( a_n \leq a \leq b \leq b_n \), and therefore \([a, b] \subseteq [a_n, b_n]\). Thus, each element \( m \) in \([a, b]\) is common to all the intervals in the sequence.

Conversely, assume \( F \) is an Archimedean field such that every nested sequence of closed intervals in \( F \) contains an element of \( F \) which belongs to all the intervals in the sequence. Suppose \( S \) is a nonempty set in \( F \) such that \( S \) is bounded above. If \( S \) contains only one element, then that element is the least upper bound of \( S \). Therefore, assume \( S \) contains at least two elements and let \( b \) denote an upper bound of \( S \). If \( b \in S \), then by Theorem 2.16, \( b \) is the least upper bound of \( S \). Thus, suppose \( b \notin S \) and let \( a \) represent an element of \( S \) that is not an upper bound of \( S \). Define a sequence of closed intervals in \( F \) as follows:

\[
I_1 = [a_1, b_1] = [a, b];
\]

if \( \frac{1}{2}(a_1 + b_1) \) is an upper bound of \( S \), define \( I_2 = [a_2, b_2] = [a_1, \frac{1}{2}(a_1 + b_1)] \);

if \( \frac{1}{2}(a_1 + b_1) \) is not an upper bound of \( S \), define \( I_2 = [a_2, b_2] = [\frac{1}{2}(a_1 + b_1), b_1] \);

and, in general, \( I_{n+1} = [a_{n+1}, b_{n+1}] = [a_n, \frac{1}{2}(a_n + b_n)] \) if \( \frac{1}{2}(a_n + b_n) \) is an upper bound of \( S \) and \( I_{n+1} = [a_{n+1}, b_{n+1}] = [\frac{1}{2}(a_n + b_n), b_n] \) if \( \frac{1}{2}(a_n + b_n) \) is not an upper bound of \( S \).

Suppose \( n \in \mathbb{N}(F) \) and consider \( I_n \) and \( I_{n+1} \). Since either \( I_{n+1} = [a_n, \frac{1}{2}(a_n + b_n)] \) or \( I_{n+1} = [\frac{1}{2}(a_n + b_n), b_n] \), \( I_{n+1} \subseteq I_n \). Therefore, \( \langle I_n \rangle \) is a nested sequence of closed intervals in \( F \)
such that for each \( n \in N(F) \) \( I_n = [a_n, b_n] \) where \( a_n \) is not an upper bound for \( S \) and \( b_n \) is an upper bound for \( S \).

It can be verified that for each \( n \in N(F) \), \( b_n - a_n = \frac{b-a}{2^{n-1}} \). Let \( L(I_n) = b_n - a_n = \frac{b-a}{2^{n-1}} \). By hypothesis, there exists an element \( x \) in \( F \) such that \( x \in I_n \) for every \( n \in N(F) \).

By way of contradiction, assume \( x \) is not an upper bound for \( S \). Then there exists an element \( y \) in \( S \) such that \( x < y \). Thus, \( 0 < (y - x) \) and \( 0 < (b - a) \) which implies that \( 0 < \frac{y - x}{2(b - a)} \).

By Theorem 2.6, there exists a natural number \( n \) such that \( 0 < \frac{1}{2^n} < \frac{y - x}{2(b - a)} \). Since \( \frac{1}{2^n} > 0 \), \( 0 < \frac{1}{2^n} \frac{y - x}{2(b - a)} \) which implies that \( 0 < \frac{b-a}{2^{n-1}} \).

Thus, \( 0 < \frac{1}{2^{n-1}} \frac{y - x}{b - a} \), and since \( (b - a) > 0 \), \( 0 < \frac{b-a}{2^{n-1}} \).

Substituting \( (b_n - a_n) \) for \( \frac{b-a}{2^{n-1}} \) gives \( 0 < b_n - a_n < y - x \).

Since \( x \in I_n \), \( a_n \leq x \leq b_n \). By Theorem 1.26(3), \( -x \leq -a_n \) which implies that \( b_n - x \leq b_n - a_n \). From above, \( b_n - a_n \leq y - x \).

so that by the transitive property \( b_n - x \leq y - x \) and, hence, \( b_n \leq y \). Therefore, \( a_n \leq x \leq b_n \leq y \) which means \( S \) contains an element \( y \) that is greater than \( b_n \). This conclusion contradicts the fact that \( b_n \) is an upper bound of \( S \), so that the assumption that \( x \) is not an upper bound of \( S \) must be false. Therefore, \( x \) is an upper bound of \( S \).

In order to verify that \( x \) is the least upper bound of \( S \), suppose \( z \in S \) such that \( z \leq x \). Then \( 0 \leq x - z \) and \( 0 \leq \frac{x - z}{b - a} \).

Since \( F \) is Archimedean, by Corollary 2.7 there exists a natural number \( n \) such that \( 0 < \frac{1}{2^n-1} < \frac{x - z}{b - a} \). Since \( (b - a) > 0 \), \( 0 < \frac{b-a}{2^{n-1}} \).

so that \( 0 < b_n - a_n \leq x - z \). Since \( x \in I_n \), \( a_n \leq x \leq b_n \) which implies that \( x - a_n \leq b_n - a_n \). By the transitive...
property, \( x - a_n \not\leq x - z \). Therefore, \(-a_n \not\leq -z\) which implies \( z \not\leq a_n \). By the definition of \( I_n\), \( a_n \) is not an upper bound of \( S\). Therefore, there exists an element \( t \) in \( S\) such that \( a_n \not\leq t \). So, \( z \not\leq a_n \not\leq t \). Since \( x \) is an upper bound of \( S\), \( z \not\leq a_n \not\leq t \leq x \). Thus, for every \( z \in S\) where \( z \not\leq x \), there exists an element \( t \) in \( S\) such that \( z \not\leq t \leq x \). By definition \( x \) is the least upper bound of \( S\).

Therefore, every nonempty subset of \( F\) having an upper bound has a least upper bound. By definition, \( F\) is a complete ordered field.

3.8. **Definition.** A **Dedekind cut** in an ordered field \( F\), denoted by \((A,B)\), is an ordered pair of nonempty subsets \( A\), \( B\) of \( F\) such that:

1. \( A \cap B = \emptyset \).
2. \( A \cup B = F \).
3. If \( a \in A\) and \( b \in B\), then \( a \not\leq b \).
4. \( A\) contains no largest element.

3.9. **Theorem.** Any two complete ordered fields are isomorphic.

**Proof.** Suppose \( R\) is a complete ordered field and let \( Q\) denote the set of rational numbers in \( R\). For each \( a \in R\) let \( M_a = \{ r \in Q \mid r \not\leq a \} \). Then for each \( a \in R\), \((M_a, Q - M_a)\) is a Dedekind cut in \( Q\). Let \( f \) be a function from \( R\) into the set of all cuts of \( Q\), denoted by \( Q'\), where \( f(a) = (M_a, Q - M_a)\) or, equivalently, \( f(a) = M_a \).
In order to show that $f$ is onto $Q'$, suppose $(M, Q - M)$ is a Dedekind cut in $Q$ and let $a$ denote the least upper bound of $M$. If $x \in M$, then since $M \subset Q$ and $x \not \subset a$, $M \subset M_a$. Also if $x \notin M_a$, then $x \in Q$ and $x \not \subset a$, so that $x \in M$. Thus, $M = M_a$ which implies that $f$ is onto $Q'$.

In order to verify that $f$ is a one-to-one function suppose $(K, Q - K)$ and $(L, Q - L)$ are cuts in $Q$ such that $K = L$. Let $a$ denote the least upper bound of $K$ and let $b$ denote the least upper bound of $L$. As was shown above, $K = M_a$ and $L = M_b$, so that $M_a = M_b$. Since each of $a$ and $b$ is a least upper bound of $M_a = M_b$, by Theorem 2.15 $a = b$. Thus, $f$ is a one-to-one function.

In order to prove the order preserving property of $f$, let proper set containment represent the ordering in $Q'$. Suppose $a, b \in R$ and $a \subset b$. Then by definition, $M_a \subset M_b$ and $M_a \neq M_b$. Conversely, suppose $(K, Q - K), (L, Q - L) \in Q'$, $K \subset L$ and $K \neq L$. Then $K = M_a$ and $L = M_b$, where $a$ denotes the least upper bound of $K$ and $b$ denotes the least upper bound of $L$. Since $M_a \subset M_b$ and $M_a \neq M_b$, by definition $a \subset b$.

Define addition in $Q'$ as follows: $M_a + M_b = \left\{ r + s \mid r \in M_a \text{ and } s \in M_b \right\}$. Note that $M_{a+b} = \left\{ r \in Q \mid r \not \subset a + b \right\}$. Suppose $x \in M_a + M_b$. Then there exist elements $r$ and $s$ such that $r \in M_a$, $s \in M_b$, and $x = r + s$. Since $r \in M_a$ and $s \in M_b$, $r \not \subset a$ and $s \not \subset b$. Hence, $x = r + s \not \subset a + b$. Therefore, $x \in M_{a+b}$ and $M_a + M_b \subset M_{a+b}$. Now suppose $x \in M_{a+b}'$, which means
that \( x \in \mathbb{Q} \) and \( x \leq a + b \). Let \( a + b - x = p > 0 \). Since the rationals are dense in \( \mathbb{R} \), there exists some \( r \in \mathbb{Q} \) such that \( a - p < r < a \). Also, let \( s = x - r \). Hence \( r + s = x = b + a - p < b + r \) which implies that \( s < b \). Therefore, \( r \in M_a \), \( s \in M_b \) and \( x = r + s \). Thus, \( M_{a+b} \subseteq M_a + M_b \), so that \( M_{a+b} = M_a + M_b \). Therefore, \( \mathbb{R} \) is isomorphic to \( \mathbb{Q}' \) with respect to addition.

Before defining multiplication in \( \mathbb{Q}' \), for each \( a \in \mathbb{R} \) such that \( a > 0 \), let \( M_a^+ = \{ x \in M_a \mid 0 < x \} \) and note that \(-M_a = M_{-a}\). Now define multiplication in \( \mathbb{Q}' \) as follows:

1. If \( a > 0 \) and \( b > 0 \), \( M_a \cdot M_b = \{ r \cdot s \mid r \in M_a^+ \text{ and } x \in M_b \} \).
2. If either \( a = 0 \) or \( b = 0 \), \( M_a \cdot M_b = M_0 \).
3. If \( a < 0 \) and \( b < 0 \), \( M_a \cdot M_b = M_{-a} \cdot M_{-b} \).
4. If \( a \leq 0 \leq b \), \( M_a \cdot M_b = -(M_{-a} \cdot M_b) \).

Although the proof is omitted, it can be verified that in all four cases \( M_a \cdot M_b = M_{a \cdot b} \). Thus, \( \mathbb{R} \) is isomorphic to \( \mathbb{Q}' \) with respect to multiplication.

Therefore, every complete ordered field is isomorphic to the set of all Dedekind cuts in its set of rational numbers. By Theorem 1.53 the sets of rational numbers of any two ordered fields are isomorphic. Thus, any two complete ordered fields are isomorphic.

3.10. **Theorem.** An ordered field \( F \) is complete iff for each Dedekind cut \((A,B)\) in \( F \) there exists an element \( z \) in \( F \) such that \( a \leq z \) for every element \( a \) in \( A \) and \( z \leq b \) for every element \( b \) in \( B \).
Proof. Suppose $F$ is a complete ordered field and suppose $(A,B)$ is a Dedekind cut in $F$. Then by Definition 3.8, $A$ and $B$ are nonempty subsets of $F$. Let $b \in B$. Then by the same definition $a \leq b$ for every $a$ in $A$. That is, $b$ is an upper bound for $A$. Since, by hypothesis, $F$ is a complete ordered field, $A$ has a least upper bound $z$. Let $a$ be an element of $A$. Then $a \leq z$ because $z$ is an upper bound of $A$. Further, $a \leq z$; for otherwise, $z$ would be the largest element in $A$. Let $b$ be an element of $B$. Then $b$ is also an upper bound of $A$. Since $z$ is the least upper bound of $A$, by Definition 2.14 $z \leq b$. Therefore $z$ is an element of $F$ such that $a \leq z$ for every $a \in A$ and $z \leq b$ for every $b \in B$.

Conversely, assume $F$ is an ordered field such that for each Dedekind cut $(C,D)$ in $F$ there exists an element $z$ in $F$ such that $c \leq z$ for every element $c$ in $C$ and $z \leq d$ for every element $d$ in $D$. Suppose $S$ is a nonempty subset of $F$ and suppose there exists an element $t$ of $F$ such that $t$ is an upper bound of $S$. Let $A = \{ a \in F \mid a \leq s \text{ for some } s \in S \}$, and let $B = \{ b \in F \mid b \notin A \}$. Since if $s \in S$, $a = s - 1 \in A$, $A$ is a nonempty subset of $F$. In addition, $(t + 1) \in F$ and $(t + 1) \notin A$; Thus, $(t + 1) \in B$ and $B$ is also a nonempty subset of $F$. Since an element of $F$ is an element of $B$ iff it is not an element of $A$, it can be concluded that $A \cap B = \emptyset$. Moreover, the definition of $B$ also implies that $A \cup B = F$. Suppose $a \in A$ and $b \in B$. Then by definition of $A$, there
exists an element \( r \) in \( S \) such that \( a \not< r \). Since \( b \not\in A \), \( s \leq b \) for every \( s \in S \). Thus, \( a \not< r \leq b \) and thus \( a \not< b \). \( A \) contains no largest element \( a \); for this would imply \( a \not< a \), a contradiction. Therefore, by Definition 3.8, \((A,B)\) is a Dedekind cut in \( F \). Hence, by assumption there exists an element \( z \) in \( F \) such that \( a \not< z \) for every \( a \in A \) and \( z \not< b \) for every \( b \in B \).

By way of contradiction, suppose there exists an element \( s \) in \( S \) such that \( z \not< s \). Since \( F \) is dense in \( F \), there exists an element \( x \) in \( F \) such that \( z \not< x \not< s \). By definition of \( A \), \( x \in A \). Therefore, \( x \in A \) such that \( z \not< x \). This conclusion contradicts the fact that \( a \not< z \) for every \( a \) in \( A \). Thus, the assumption is false which implies that \( z \) is an upper bound of \( S \).

Now suppose \( y \) is also an upper bound of \( S \). Then \( s \leq y \) for each \( s \in S \). That is, there is no element \( s \) in \( S \) such that \( y \not< s \). Thus, \( y \notin A \), so that by definition \( y \notin B \). Since \((A,B)\) is a Dedekind cut, by hypothesis \( z \leq y \). Therefore, by definition \( z \) is the least upper bound of \( S \).

Hence, every nonempty subset of \( F \) which is bounded above has a least upper bound. Therefore, by definition \( F \) is a complete ordered field.

3.11. **Definition.** An element \( x \) in an ordered field \( F \) is called a cluster point of \( H \), where \( H \) is a subset of \( F \), iff for each open interval \((a,b)\) containing \( x \), there exists an element \( y \) in \( H \) such that \( y \leq (a,b) \) and \( y \neq x \).
3.12. **Theorem.** If \( H \) is a subset of an ordered field \( F \), then \( x \) is a cluster point of \( H \) if and only if for each open interval \((a, b)\) containing \( x \), \( H \cap (a, b) \) is an infinite set.

A proof is omitted.

3.13. **Theorem.** If \( F \) is a complete ordered field, then every infinite, bounded subset of \( F \) has a cluster point.

**Proof.** Suppose \( F \) is a complete ordered field, and by way of contradiction assume there exists an infinite, bounded subset \( H \) of \( F \) such that \( H \) has no cluster points. Since \( H \) is a bounded set, there exists a closed interval \([a, b]\) such that \( H \subseteq [a, b] \). Let \( K \) be the set of all \( x \in [a, b] \) such that \( \{ y \mid y \in H \text{ and } x \leq y \leq b \} \) is a finite set. Note that \( b \in K \) and that \( a \) is a lower bound for \( K \). Since by hypothesis \( F \) is a complete ordered field, \( K \) has a greatest lower bound \( p \). By assumption, \( H \) has no cluster points. Thus, \( p \) is not a cluster point of \( H \). That is, there exists an open interval \((c, d)\) containing \( p \) such that for each \( y \in H \), either \( y = p \) or \( y \notin (c, d) \).

If \( a \not\leq p \not\leq b \), then since \( p \in (c, d) \), the following open interval can be defined. Let \((u, v) = (\max\{a, c\}, \min\{b, d\})\). Since \( p \not\leq v \), there is only a finite number of elements of \( H \) contained in the interval \([v, b]\). Since \((u, v) \subseteq (c, d)\), \((u, v)\) contains at most one element of \( H \). Thus, \([u, b]\) contains a finite number of elements of \( H \), which implies that \( u \in K \). But \( u \not\leq p \) which contradicts the fact that \( p \) is the greatest lower bound of \( K \). Thus, the assumption that \( a \not\leq p \not\leq b \) must be false.
If \( p = b \), again let \( u = \max\{a, c\} \). Then, since \((u, b)\) is a subset of \((c, d)\), \((u, b)\) contains at most one element of \( H \). Hence, \( u \in K \). But either \( u = a \leq b = p \) or \( u = c \leq p \). Thus, \( u \in K \) and \( u \leq p \). This conclusion contradicts the fact that \( p \) is the greatest lower bound of \( K \). Therefore, the assumption that \( p = b \) must be false.

Finally, if \( p = a \), let \( v = \min\{b, d\} \). Then since \( c \leq p \leq d \), \((a, v) = (p, v) \subseteq (c, d)\) and \((a, v)\) contains at most one element of \( H \). In addition, since \( p \leq v \), the interval \([v, b]\) contains only finitely many elements of \( H \). Thus, \([a, b]\) contains only finitely many elements of \( H \). But this conclusion contradicts the fact that \( H \) is an infinite subset of \([a, b]\). Thus, \( p \neq a \).

Since each case results in a contradiction, the original assumption must be false. Therefore, every infinite, bounded subset of \( F \) has a cluster point.

3.14. **Lemma.** If \( F \) is an ordered field, \( x \in F \) and \( x \neq 0 \), then the set \( X = \{nx \mid n \in \mathbb{N}(F)\} \) is an infinite set which has no cluster points.

**Proof.** Suppose \( F \) is an ordered field, \( x \in F \), and \( x \neq 0 \). Without loss of generality, suppose \( x > 0 \). Let \( X = \{nx \mid n \in \mathbb{N}(F)\} \) and suppose \( p \in F \).

If \( p = mx \) where \( m \in \mathbb{N}(F) \), consider the open interval \( I = ((m - \frac{1}{2})x, (m + \frac{1}{2})x) \) and note that \( p \in I \). By Theorem 1.55 there is no natural number \( z \) such that \( (m - 1) \leq z \leq m \) nor is there a natural number \( z \) such that \( m \leq z \leq (m + 1) \).
Now \((m - 1)x \leq (m - \frac{1}{2})x \leq mx = p \leq (m + \frac{1}{2})x \leq (m + 1)x\) which implies that there are no elements of \(X\) other than \(p\) contained in \(I\). Hence, \(p\) is not a cluster point of \(X\).

If for each \(m \in \mathbb{N}(F)\), \(p \neq mx\), let \(I = (p - \frac{x}{2}, p + \frac{x}{2})\). If \(I\) contains no element of \(X\), then \(p\) is not a cluster point of \(X\). So suppose there exists an element \(q\) of \(X\) such that \(q \in I\) and \(q = hx\) for some \(h \in \mathbb{N}(F)\). Then, \(p - \frac{x}{2} < q < p + \frac{x}{2}\). Without loss of generality assume \(q < p\), and let \(\varepsilon = p - q\). Consider the interval \((p - \varepsilon, p + \varepsilon)\) and note that 
\[(p - \varepsilon, p + \varepsilon) \subset (p - \frac{x}{2}, p + \frac{x}{2})\] which implies that \(2\varepsilon < x\). Now \(hx = q = p - \varepsilon < p \leq p + \varepsilon = q + 2\varepsilon < q + x = (h + 1)x\). Thus, \((p - \varepsilon, p + \varepsilon) \subset (hx, (h + 1)x)\). Since there is no natural number between \(h\) and \(h + 1\), there is no element of \(X\) between \(hx\) and \((h + 1)x\). Hence, \((hx, (h + 1)x)\) is an open interval containing \(p\) but no element of \(X\). By definition, \(p\) is not a cluster point of \(X\).

Thus, \(X\) has no cluster points.

3.15. **Lemma.** If \(F\) is an ordered field and if \(\langle [a_n, b_n] \rangle\) is a nested sequence of closed intervals in \(F\), then \(a_i \leq b_j\) for every \(i, j \in \mathbb{N}(F)\).

**Proof.** Suppose \(F\) is an ordered field and let \(\langle I_n \rangle = \langle [a_n, b_n] \rangle\) be a nested sequence of closed intervals in \(F\). Let \(i\) and \(j\) be distinct natural numbers and first suppose \(i < j\). Then since the intervals are nested \(I_j \subset I_i\) which implies that \(a_i \leq a_j \leq b_j \leq b_i\). Now suppose \(j < i\) which
implies that $I_i \subseteq I_j$. Hence, $a_j \leq a_i \leq b_i \leq b_j$. Thus, in both cases $a_i \leq b_j$.

3.16. Theorem. If $F$ is an ordered field such that every bounded, infinite subset of $F$ has a cluster point, then (1) $F$ is Archimedean, and (2) if $\langle I_n \rangle$ is a nested sequence of closed intervals in $F$, then there exists an element of $F$ common to all the intervals in the sequence.

Proof. Suppose $F$ is an ordered field such that every bounded, infinite subset of $F$ has a cluster point. By way of contradiction, assume $F$ is not Archimedean. Then by Theorem 2.3 there exist elements $x$ and $y$ in $F$ such that $0 \leq x \leq y$ and $nx \leq y$ for each $n \in N(F)$. Hence, $x \leq nx \leq y$ which means that the set $X = \{nx \mid n \in N(F)\}$ is bounded and infinite. By Lemma 3.14, $X$ has no cluster points. Since this conclusion contradicts the hypothesis, $F$ must be Archimedean.

Now assume $\langle I_n \rangle$ is a nested sequence of closed intervals in $F$. For $n \in N(F)$, let $I_n = [a_n, b_n]$. Also let $A = \{a_n \mid n \in N(F)\}$. Since the intervals are nested $a_1 \leq a_n \leq b_n \leq b_1$ for all $n \in N(F)$. Thus, $A$ is a bounded subset of $F$. If there exists some $i \in N(F)$ such that $a_j = a_i$ for all $j \geq i$, then $a_n \leq a_i \leq b_n$ for all $n \in N(F)$ and, hence, $a_i \in \bigcap_{n \in N(F)} I_n$. On the other hand, if for each $i \in N(F)$ there exists some $j \in N(F)$ such that $a_i \leq a_j$, then by Theorem 1.30 $A$ is an infinite subset of $F$. Thus, by hypothesis $A$ has a cluster point $z$. 

Suppose there exists some \( n \in N(F) \) such that \( z < a_n \).

Then \( z < a_n \leq a_m \) for all \( m \geq n \). By Theorem 1.38 there exists an \( \varepsilon \) such that \( 0 < \varepsilon < a_n - z \). Consider the open interval \((z - \varepsilon, z + \varepsilon)\) and without loss of generality, assume \( a_1 \leq (z - \varepsilon) \). Since \((z - \varepsilon, z + \varepsilon) \subseteq (a_1, a_n), (z - \varepsilon, z + \varepsilon)\)
contains finitely many elements of \( A \). Since \( z \in (z - \varepsilon, z + \varepsilon)\)
this conclusion contradicts the fact that \( z \) is a cluster point
of \( A \). Thus, \( z \geq a_n \) for all \( n \in N(F) \).

Now suppose there exists some \( n \in N(F) \) such that \( b_n \not< z \).
Then by Lemma 3.15, \( a_n \leq b_n < z \) for every \( m \in N(F) \). By
Theorem 1.38 there exists some \( \varepsilon \in F \) such that \( 0 < \varepsilon < (z - b_n) \).
Since \((z - \varepsilon) > b_n \) and \( b_n \geq a_m \) for every \( m \in N(F) \), the interval
\((z - \varepsilon, z + \varepsilon)\) contains no element of \( A \). But this conclusion
contradicts the fact that \( z \) is a cluster point of \( A \). Thus,
\( b_n \geq z \) for every \( n \in N(F) \).

Therefore, \( a_n \leq z \leq b_n \) for all \( n \in N(F) \) which means that
\( z \) is an element of \( F \) common to all the intervals in the sequence.

3.17. **Theorem.** An ordered field is complete iff
every bounded infinite subset has a cluster point.

A proof follows from Theorems 3.7, 3.13, and 3.16.

3.18. **Definition.** A subset \( A \) of an ordered field \( F \) is
said to be open iff for each \( a \in A \) there exists some \( p > 0 \)
such that the open interval \((a - p, a + p)\) is a subset of \( A \).

3.19. **Definition.** A subset \( A \) of an ordered field \( F \) is
said to be closed iff every cluster point of \( A \) is an
element of \( A \).
3.20. **Theorem.** In an ordered field $F$ the empty set $\emptyset$ is both open and closed.

**Proof.** Suppose $F$ is an ordered field. Then, in a vacuous sense the following statement is true: for each $a \in \emptyset$ there exists some $p > 0$ such that the open interval $(a - p, a + p)$ is a subset of $\emptyset$. Thus, $\emptyset$ is an open set.

In addition, since the empty set has no cluster points, it contains all of its cluster points. Thus, by definition the empty set is a closed set.

3.21. **Theorem.** In an ordered field $F$ a set is open iff its complement in $F$ is closed.

**Proof.** Suppose $F$ is an ordered field and $A$ is an open set in $F$. Let $A'$ represent the complement of $A$ in $F$ and by way of contradiction assume $A'$ is not closed. Thus, there exists an element $x$ of $A$ such that $x$ is a cluster point of $A'$. By Definition 3.18 there exists an open interval $I$ such that $x \in I \subset A$. Since $I \subset A$, $x$ is not a cluster point of $A'$. Hence, the assumption that $A'$ is not a closed set leads to a contradiction. Therefore, $A'$ is a closed set.

Now suppose $A$ is a set in $F$ such that $A'$ is closed. By way of contradiction, suppose $A$ is not open. Then there exists a point $x$ in $A$ such that for every $p > 0$, the set $\{y \in F \mid x - p < y < x + p\}$ is not a subset of $A$. If $(c, d)$ is an open interval containing $x$, then the interval $(x - q, x + q)$ where $q = \min \{(x - c), (d - x)\}$ is a subset of $(c, d)$ containing
x. Since \((x - q, x + q)\) is not a subset of \(A\), \((c,d)\) is not a subset of \(A\). Thus, no open interval containing \(x\) is a subset of \(A\). That is, every open interval containing \(x\) contains at least one point in \(A'\). Since \(x \in A\), the point in \(A'\) must be distinct from \(x\). Thus, \(x\) is a cluster point of \(A'\) and since \(x \in A\), \(A'\) is not closed. Since this conclusion contradicts the fact that \(A'\) is closed, \(A\) must be open.

3.22. Definition. A collection \(T\) of sets in an ordered field \(F\) is called a \textit{covering} of a set \(E\) iff \(E \subseteq \bigcup_{t \in T} t\).

3.23. Definition. Suppose \(E\) is a subset of an ordered field \(F\) and suppose \(T\) is a covering of \(E\). Then \(T\) is said to be an \textit{open covering} of \(E\) iff for each \(t \in T\), \(t\) is an open set in \(F\).

3.24. Definition. Suppose \(T\) is a covering of a set \(E\). Then \(S\) is a \textit{subcovering} of \(T\) iff \(S\) covers \(E\) and \(S \subseteq T\).

3.25. Theorem. Suppose \(F\) is a complete ordered field and \([a,b]\) is a closed interval in \(F\). If \(T\) is an open covering of \([a,b]\), then there exists a finite subcovering of \(T\) which also covers \([a,b]\).

Proof. Suppose \(F\) is a complete ordered field and \([a,b]\) is a closed interval in \(F\). Let \(T\) be an open covering of \([a,b]\). Define the set \(L\) as follows: \(L = \{x \in F\mid x \in [a,b] \text{ and } [a,x] \text{ is covered by a finite subset of } T\}\). Since \(T\) covers \([a,b]\) there exists an open set \(t_a \in T\) such that \(a = [a,a] \in T\). Thus, \(a \in L\). If \(x \in L\), then \(a \leq x \leq b\). Hence, \(b\) is an upper
bound of the nonempty set \( L \). Since \( F \) is a complete ordered field, there exists an element \( k \) in \( F \) such that \( k \) is the least upper bound of \( L \).

Since \( a \in L \), \( a \leq k \); and, since \( b \) is an upper bound of \( L \), \( k \leq b \). Thus, \( a \leq k \leq b \) which means that \( k \in [a, b] \). In addition, there exists an open set \( t_k \in T \) containing an open interval \((p, q)\) such that \( p \leq k \leq q \). Without loss of generality, assume \( a \leq p \). Since \( k \) is the least upper bound of \( L \) and \( p \leq k \), there exists an element \( z \) in \( L \) such that \( p \leq z \leq k \). Then by the definition of \( L \), there exists a finite subset \( S \) of \( T \) such that \( S \) covers \([a, z] \). Since \( a \leq p \leq z \leq k \), \( \{t_k\} \cup S \) is a finite subset of \( T \) that covers \([a, k] \). Thus, \( k \in L \).

By way of contradiction, suppose \( k \neq b \). Then \( k \leq b \).

Since \( k \leq q \), \( k \leq \min\{b, q\} \). Therefore since \( F \) is an ordered field, by Theorem 1.38, there exists an element \( y \) in \( F \) such that \( k \leq y \leq \min\{b, q\} \). Thus, \( a \leq p \leq k \leq y \leq \min\{b, q\} \leq q \), so that \( \{t_k\} \cup S \) also covers \([a, y] \). Hence, \( y \in L \). However this conclusion contradicts the fact that \( k \) is an upper bound of \( L \). Thus, the assumption that \( k \neq b \) must be false. Therefore, \( k = b \), and there exists a finite subcovering of \( T \) that covers \([a, b] \).

3.26. Definition. A sequence \( \langle a_n \rangle \) in an ordered field \( F \) is said to be increasing iff \( a_n \leq a_{n+1} \) for every \( n \in N(F) \). If for every \( n \in N(F) \), \( a_n \not\leq a_{n+1} \), then \( \langle a_n \rangle \) is strictly increasing.
3.27. **Definition.** A sequence \( \langle a_n \rangle \) in an ordered field \( F \) is said to be **decreasing** iff \( a_n \geq a_{n+1} \) for every \( n \in N(F) \). If for every \( n \in N(F) \), \( a_n > a_{n+1} \), then \( \langle a_n \rangle \) is strictly decreasing.

3.28. **Definition.** A sequence in an ordered field \( F \) is **monotonic** iff it is an increasing sequence or it is a decreasing sequence.

3.29. **Definition.** In an ordered field \( F \) a sequence \( \langle a_n \rangle \) **converges to** \( a \) iff for each \( \varepsilon > 0 \), \( \varepsilon \in F \), there exists some \( n \in N(F) \) such that if \( m \in N(F) \), \( m \geq n \), \( |a_m - a| < \varepsilon \).

In addition, \( a \) is called the **limit of the sequence**.

3.30. **Lemma.** If \( F \) is an ordered field, then every nonconvergent increasing sequence contains a strictly increasing subsequence.

**Proof.** Suppose \( F \) is an ordered field and \( \langle a_n \rangle \) is a nonconvergent increasing sequence in \( F \). Let \( c_1 = a_1 \). Define the set \( K_2 = \{ a_n \mid a_1 < a_n \} \). If \( K_2 \) is the empty set, then \( a_n = a_1 \) for every \( n \) which implies that \( \langle a_n \rangle \) converges. Thus, \( K_2 \) is not empty. Let \( H_2 = \{ m \mid a_m \in K_2 \} \). Since \( H_2 \) is a nonempty set of natural numbers, \( H_2 \) contains a least element \( i \).

Let \( c_2 = a_i \). Note that for each \( n \), \( K_n \) is a nonempty set because \( \langle a_n \rangle \) does not converge. Thus, the above process can be continued so that by finite mathematical induction the sequence \( \langle c_n \rangle \) is defined and is a strictly increasing subsequence of \( \langle a_n \rangle \).
3.31. **Lemma.** In an ordered field an increasing sequence converges iff the set of its values has a least upper bound.

**Proof.** Suppose $F$ is an ordered field and $\langle a_n \rangle$ is an increasing sequence in $F$ that converges to $a$. If there exists some $a_i$ such that $a < a_i$, then for $\varepsilon = a_i - a$, $|a_j - a| = (a_j - a) \geq (a_i - a) = \varepsilon$, for every $j \geq i$. Since this conclusion contradicts the fact that $\langle a_n \rangle$ converges to $a$, $a_i \leq a$ for every natural number $i$. Thus, $a$ is an upper bound for $\{a_n\}$.

By way of contradiction, suppose $a$ is not the least upper bound of $\{a_n\}$. Then, there exists an element $q$ in $F$ such that $a_n \leq q < a$ for every $n$. If $\varepsilon = \frac{1}{2}(a - q)$, then $|a - a_n| = (a - a_n) \geq (a - q) > \varepsilon$ for every $n$. This conclusion contradicts the fact that $\langle a_n \rangle$ converges to $a$. Thus, $a$ is the least upper bound of $\{a_n\}$.

Conversely, assume $F$ is an ordered field and that $\langle a_n \rangle$ is an increasing sequence such that $\{a_n\}$ has a least upper bound $a$. Let $\varepsilon > 0$ so that $(a - \varepsilon) < a$. Then there exists some $a_m$ such that $(a - \varepsilon) \leq a_m \leq a$. Since $\langle a_n \rangle$ is an increasing sequence bounded above by $a$, $(a - \varepsilon) \leq a_i \leq a$ for every $i \geq m$. Thus, for each $i \geq m$, $(a - \varepsilon) \leq a_i \leq a \leq (a + \varepsilon)$ which implies that $|a_i - a| \leq \varepsilon$. Therefore, $\langle a_n \rangle$ converges to $a$.

3.32. **Lemma.** In an ordered field any monotonic sequence that contains a convergent subsequence is convergent.
Proof. Suppose \( \langle a_n \rangle \) is a monotonic sequence in an ordered field \( F \). In addition suppose \( \langle a_n \rangle \) contains a convergent subsequence \( \langle b_n \rangle \). Without loss of generality suppose \( \langle a_n \rangle \) is increasing. By Lemma 3.31, there exists some \( k \) in \( F \) such that \( k \) is the least upper bound of \( \{ b_n \} \).

By way of contradiction, suppose \( k \) is not an upper bound of \( \{ a_n \} \). Then there must exist a natural number \( i \) such that \( k < a_i \). Since \( \langle b_n \rangle \) is a subsequence of \( \langle a_n \rangle \), it follows that \( b_i = a_j \) where \( j \geq i \) and \( b_i \geq a_i > k \). This conclusion contradicts the fact that \( k \) is the least upper bound of \( \{ b_n \} \). Thus, \( k \) must be an upper bound of \( \{ a_n \} \).

Now suppose there exists an element \( h \) in \( F \) such that \( h \) is an upper bound of \( \{ a_n \} \) and \( h \leq k \). Since \( \{ b_n \} \subset \{ a_n \} \), \( h \) is also an upper bound of \( \{ b_n \} \) which contradicts the fact that \( k \) is the least upper bound of \( \{ b_n \} \). Hence, there is no upper bound of \( \{ a_n \} \) less than \( k \). Therefore, \( k \) is the least upper bound of \( \{ a_n \} \).

By Lemma 3.31, the sequence \( \langle a_n \rangle \) is convergent.

3.33. Lemma. In an ordered field, a strictly increasing sequence converges iff the set of its values has a cluster point.

Proof. Suppose \( F \) is an ordered field and \( \langle a_n \rangle \) is a strictly increasing sequence in \( F \) that converges to \( k \). As the proof of Lemma 3.31 indicates, \( k \) is the least upper bound of \( \{ a_n \} \). Since \( \langle a_n \rangle \) is a strictly increasing sequence,
\( a_n \not\leq k \) for every natural number \( n \). Suppose \((p,q)\) is an open interval containing \( k \). Then since \( p \not\leq k \), there must exist some \( a_m \) such that \( p \not\leq a_m \not\leq k \). Thus, every open interval that contains \( k \) also contains an element of the sequence distinct from \( k \). Therefore, by definition \( k \) is a cluster point of \( \langle a_n \rangle \).

Conversely, assume \( \langle a_n \rangle \) is a strictly increasing sequence in an ordered field \( F \) and that there exists an element \( k \) that is a cluster point of \( \{a_n\} \). By way of contradiction, suppose \( k \) is not an upper bound for \( \{a_n\} \). Then there exists some \( m \) such that \( k \not\leq a_m \). Hence, the open interval \( (k-1, a_m) \) contains \( k \) and, at most, finitely many elements of \( \{a_n\} \). By Theorem 3.12, \( k \) is not a cluster point of \( \{a_n\} \) which contradicts the hypothesis. Thus, \( k \) is an upper bound for \( \{a_n\} \).

If \( k \) is not the least upper bound of \( \{a_n\} \) then there exists an upper bound \( h \) such that \( h \not\leq k \). Since for every \( n \), \( a_n \not\leq h \not\leq k \), the open interval \((h, k+1)\) contains no points of \( \{a_n\} \). Hence by Theorem 3.12, \( k \) is not a cluster point of \( \{a_n\} \) which contradicts the hypothesis. Thus, \( k \) is the least upper bound of \( \{a_n\} \). By Lemma 3.31, \( \langle a_n \rangle \) converges.

3.34. Theorem. If \( F \) is an ordered field such that every open covering of a closed interval contains a finite subcovering, then every bounded monotonic sequence in \( F \) converges.

Proof. Suppose \( F \) is an ordered field such that every open covering of a closed interval contains a finite subcovering.
By way of contradiction, assume $F$ contains a bounded monotonic sequence $\langle a_n \rangle$ that fails to converge. Without loss of generality assume $\langle a_n \rangle$ is an increasing sequence. Since $\langle a_n \rangle$ is bounded there exist elements $a$ and $b$ in $F$ such that $a \leq a_n \leq b$ for every $n$. By Lemma 3.30 $\langle a_n \rangle$ contains a strictly increasing subsequence $\langle c_n \rangle$. Since $\langle a_n \rangle$ is not a convergent sequence, by Lemma 3.32 $\langle c_n \rangle$ also fails to converge. Thus, by Lemma 3.33, $\{c_n\}$ does not have a cluster point.

Let $C_1 = \{c_1, c_2, c_3, \ldots, c_n, \ldots\}$. Since $C_1 = \{c_n\}$, $C_1$ has no cluster points. Thus, by Definition 3.19 $C_1$ is a closed set. Similarly, let $C_2 = \{c_2, c_3, c_4, \ldots, c_n, \ldots\}$ and note that $C_2$ also has no cluster points. Thus, $C_2$ is a closed set. Therefore, in general, the set $C_n = \{c_n, c_{n+1}, c_{n+2}, \ldots\}$ has no cluster points and is a closed set.

Now for each $n$, let $G_n$ represent the complement of $C_n$ in $F$. By Theorem 3.21, $G_n$ is an open set. Suppose $x \in [a, b]$. If $x \in \{c_n\}$, then $x = c_i$ for some $i$ and, hence, $x \in G_{i+1}$. If $x \notin \{c_n\}$, then $x \in G_i$ for every $i$. Thus, if $x \in [a, b]$ there exists at least one $G_i$ such that $x \notin G_i$. Therefore, the collection $G = \{G_n\}$ is an open cover of $[a, b]$. By hypothesis, there exists a finite subcollection $G'$ that also covers $[a, b]$. Note that for each $n$, $G_n \subseteq G_{n+1}$. Since $G'$ is a finite subcollection, there exists some natural number $k$ such that $G_i \subseteq G_k$ for every $G_i \in G'$. Thus, since $G'$ covers
and since \( G_i \subseteq G_k \) for every \( G_i \in G' \), \([a, b] \subseteq G_k\).

However, \( c_k \notin [a, b] \) which implies that \( c_k \notin G_k \). This conclusion contradicts the definition that \( G_k \) is the complement of \( C_k \). Thus, the assumption that \( F \) contains a bounded, monotonic sequence that fails to converge is untenable.

Therefore, every bounded monotonic sequence in \( F \) converges.

3.35. **Lemma.** Every sequence in an ordered field contains a monotonic subsequence.

**Proof.** Suppose \( F \) is an ordered field and \( \langle a_n \rangle \) is a sequence in \( F \). Denote the following subsequences of \( \langle a_n \rangle \) as follows:

- \( a_0 = \langle a_n \rangle \), \( a_1 = \langle a_2, a_3, a_4, \ldots \rangle \), \( a_2 = \langle a_3, a_4, a_5, \ldots \rangle \) so that \( a_n = \langle a_{n+1}, a_{n+2}, a_{n+3}, \ldots \rangle \).

Then, either for every \( n \in \{0, 1, 2, \ldots\} \) \( a_n \) contains a largest term or there exists an \( n \) such that \( a_n \) contains no largest term.

If for every \( n \), \( a_n \) contains a largest term, let \( n_1 \), denote the largest term of \( a_0 \), let \( n_2 \), denote the largest term of \( a_1 \), and in general, let \( n_m \), denote the largest term of \( a_{n_{m-1}} \).

Since \( A_n \supseteq A_{n+1} \) for every \( n \in \{0, 1, 2, \ldots\} \), \( a_{n_m} \geq a_{n_{m+1}} \) for every natural number \( m \). In addition, suppose \( m \) is a natural number. By definition of the \( A_n \)'s, \( a_{n_m} \notin A_n \). In fact, if \( a_{n_m} \in A_n \), then \( n_m \notin n_j \). Since \( a_{n_{(m+1)}} \in A_{n_m} \), \( n_m \notin n_{(m+1)} \). Thus \( \langle a_{n_1}, a_{n_2}, a_{n_3}, \ldots \rangle \) is a decreasing subsequence of \( \langle a_n \rangle \).

Now suppose there exists some \( N \) such that \( A_N \) contains no largest term. Then if \( a_i \notin A_N \), there exists some \( j > i \) such
that $a_i \preceq a_j$. For each $n > N$ consider the set $S_n = \{ j \mid a_j > a_n \}$ and $j > n \}$. By the well ordering principle $S_n$ has a least element $n'$. Let $a_n = a_{N+1}$ and for each $m \geq 1$, let $a_{n(m+1)} = a_{n_m}$. That is, for $m \geq 1$, $a_{n(m+1)}$ is the first term following $a_{n_m}$ such that $a_{n_{m+1}} > a_{n_m}$. Thus, $\langle a_{n_m} \rangle$ is an increasing subsequence of $\langle a_n \rangle$.

3.36. Theorem. If $F$ is an ordered field such that every bounded monotonic sequence converges, then every bounded sequence contains a convergent subsequence.

Proof. Suppose $F$ is an ordered field such that every bounded monotonic sequence converges and let $\langle a_n \rangle$ be a bounded sequence in $F$. By Lemma 3.35, $\langle a_n \rangle$ contains a monotonic subsequence $\langle b_n \rangle$. Since $\langle a_n \rangle$ is bounded, $\langle b_n \rangle$ is also bounded. Thus, by hypothesis $\langle b_n \rangle$ converges.

3.37. Theorem. Suppose $F$ is an ordered field such that every bounded sequence contains a convergent subsequence; then if $\langle I_n \rangle$ is a nested sequence of closed intervals in $F$, there exists an element of $F$ common to all the intervals in the sequence.

Proof. Suppose $F$ is an ordered field such that every bounded sequence contains a convergent subsequence. Let $\langle I_n \rangle = \langle [a_n, b_n] \rangle$ be a nested sequence of closed intervals in $F$. Define the sequence $\langle c_n \rangle = \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots \}$ so that $b_i = c_{2i}$ and $a_i = c_{2i-1}$. Note that $\langle c_n \rangle$ is bounded by $[a_1, b_1]$ and that for each natural number $i$, $\{c_{2i-1}, c_{2i}\}$. 
\[c_{2i+1}, c_{2i+2}, \ldots\] = \{c_{2i-1}, c_{2i}, c_{2(i+1)-1}, c_{2(i+1)}-1, \ldots\} \subseteq [a_i, b_i] = [c_{2i-1}, c_{2i}].\] Since \(\langle c_n \rangle\) is a bounded sequence, by hypothesis it contains a subsequence \(\langle d_n \rangle\) that converges to an element \(k\). Let \(d_N\) be an element of \(\{d_n\}\). Then there exists a natural number \(m_N\) such that \(d_N = c_{m_N}\). As was shown above there exists a natural number \(j\) such that \(\{d_N, d_{N+1}, d_{N+2}, \ldots\} \subseteq \{c_{m_N}, c_{m_N+1}, c_{m_N+2}, \ldots\} \subseteq [a_j, b_j] \).

By way of contradiction, suppose there exists an interval \([a_i, b_i]\) of the sequence such that \(k \notin [a_i, b_i]\). Then, either \(k \leq a_i\) or \(k > b_i\). Suppose \(k \leq a_i\) and let \(\epsilon = a_i - k\). Let \(S = \{p \mid d_p = c_{m_p}, m_p \leq 2i - 1\}\). \(S\) is not empty since 1 is in \(S\). Hence, \(S\) is a finite collection of natural numbers and, therefore, \(S\) has a greatest element \(q\). Let \(j = q + 1\). Thus,

\[d_q = c_{m_q} \leq c_{2i-1}\]

which implies that \(c_{2i-1} \leq d_{q+1} = d_j\). As was shown above, \(\{d_j, d_{j+1}, d_{j+2}, \ldots\} \subseteq \{c_{2i-1}, c_{2i}, c_{2i+1}, \ldots\} \subseteq [a_i, b_i]\) which implies that \(k \leq a_i \leq d_n\) for every \(n \geq j\). Thus if \(n \geq j\), then \(|d_j - k| = d_j - k > a_i - k = \epsilon\). Therefore, there is no natural number \(n\) such that if \(m \geq n\), \(|d_m - k| \leq \epsilon\). This conclusion contradicts the fact that \(\langle d_n \rangle\) converges to \(k\). Hence \(a_i \leq k\). An analogous argument can be used to demonstrate that \(k \leq b_i\).

Thus, \(k \in [a_i, b_i]\) for every natural number \(i\). That is, there exists an element of \(F\) common to all the intervals in the sequence.
3.38. **Theorem.** If $F$ is an ordered field such that every bounded sequence contains a convergent subsequence, then $F$ is an Archimedean field.

**Proof.** Suppose $F$ is an ordered field such that every bounded sequence contains a convergent subsequence. By way of contradiction suppose $F$ is not Archimedean. That is, there exists an element $x$ in $F$ such that $x \geq n$ for every natural number $n$. Define the following strictly increasing sequence $\langle a_n \rangle = \{1, 2, 3, \ldots\}$, and note that $\langle a_n \rangle$ is bounded by the closed interval $[1, x]$. Thus, by hypothesis $\langle a_n \rangle$ contains a subsequence $\langle b_n \rangle$ that converges to some element $k$. In addition, $\langle b_n \rangle$ is also a strictly increasing sequence. By Theorem 1.55, if $i$ and $j$ are natural numbers such that $i < j$ then $b_j - b_i \geq 1$. Let $\varepsilon = \frac{1}{2}$. Then since $\langle b_n \rangle$ converges to $k$, there exists a natural number $N$ such that if $m > N$, $|b_m - k| < \frac{1}{2}$. Suppose $n > N$. Then $|k - b_n| = |b_n - k| < \frac{1}{2}$. In addition, $|b_{n+1} - k| < \frac{1}{2}$. Thus, $k - b_n < \frac{1}{2}$ and $b_{n+1} - k < \frac{1}{2}$, which implies that $(k - b_n) + (b_{n+1} - k) = b_{n+1} - b_n < 1$. Since this conclusion contradicts Theorem 1.55 the assumption that $F$ is not Archimedean must be false. Thus, $F$ is an Archimedean field.

3.39. **Theorem.** If $F$ is an ordered field then each of the following is a necessary and sufficient condition for $F$ to be a complete ordered field:
(1) Every open covering of a closed interval in \( F \) contains a finite subcovering of the interval.

(2) Every bounded monotonic sequence in \( F \) converges.

(3) Every bounded sequence in \( F \) contains a convergent subsequence.

**Proof.** The result follows by Theorems 3.7, 3.25, 3.34, 3.36, 3.37, and 3.38.

3.40. **Definition.** A sequence \( \langle a_n \rangle \) in an ordered field \( F \) is a Cauchy sequence iff for each \( \varepsilon > 0 \), there exists some \( N \in \mathbb{N}(F) \) such that if \( m, n \in \mathbb{N}(F) \) and \( m, n > N \), then \( |a_m - a_n| < \varepsilon \).

3.41. **Theorem.** Every Cauchy sequence in an ordered field is bounded.

**Proof.** Suppose \( \langle a_n \rangle \) is a Cauchy sequence in an ordered field \( F \). Let \( \varepsilon = 1 \). Then there exists some \( N \in \mathbb{N}(F) \) such that if \( n > N \), \( |a_n - a_{n+1}| < 1 \). By Theorem 1.40, 

\[-1 < a_n - a_{n+1} < 1 \]

which implies that \( a_{n+1} - 1 < a_n < a_{n+1} + 1 \), where \( n > N \). Therefore for every \( m \in \mathbb{N}(F) \), 

\[
\min\{a_1, a_2, a_3, \ldots, a_N, a_{N+1}\} - 1 < a_m < \max\{a_1, a_2, a_3, \ldots, a_N, a_{N+1}\} + 1.
\]

That is, the sequence \( \langle a_n \rangle \) is bounded.

3.42. **Theorem.** If \( \langle a_n \rangle \) is a Cauchy sequence in an ordered field \( F \) such that \( \langle a_n \rangle \) contains a convergent subsequence, then \( \langle a_n \rangle \) is convergent.

**Proof.** Suppose \( F \) is an ordered field and suppose \( \langle a_n \rangle \) is a Cauchy sequence in \( F \) such that \( \langle a_n \rangle \) contains a convergent
subsequence \( \langle b_n \rangle \) where for each \( n \in \mathbb{N}(F) \) \( b_n = a_{m_n} \). Let \( b \) represent the limit of \( \langle b_n \rangle \). Suppose \( \varepsilon > 0 \). Then there exists some \( N_1 \in \mathbb{N}(F) \) such that if \( m, n > N_1, |a_m - a_n| < \frac{\varepsilon}{2} \). In addition, there exists an \( N_2 \) in \( \mathbb{N}(F) \) such that if \( i > N_2 \), then \( |b_i - b| < \frac{\varepsilon}{2} \). Let \( N = \max\{N_1, N_2\} \). Thus if \( n > N \), \( m_n \geq n \), so that \( |a_n - b| = |a_n - b_n + b_n - b| \leq |a_n - b_n| + |b_n - b| = |a_n - a_m| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \). Hence, the sequence \( \langle a_n \rangle \) converges to \( b \).

3.43. Theorem. If \( F \) is a complete ordered field, then \( F \) is Archimedean and every Cauchy sequence in \( F \) converges.

Proof. Suppose \( F \) is a complete ordered field. By Theorem 2.20, \( F \) is Archimedean. Assume \( \langle a_n \rangle \) is a Cauchy sequence in \( F \). By Lemma 3.35 \( \langle a_n \rangle \) contains a monotonic subsequence \( \langle b_n \rangle \). Since by Theorem 3.41 \( \langle a_n \rangle \) is bounded, \( \langle b_n \rangle \) is also bounded. Thus, \( \langle b_n \rangle \) is a monotonic bounded sequence in a complete ordered field. By Corollary 3.39 \( \langle b_n \rangle \) converges, so that by Theorem 3.42 \( \langle a_n \rangle \) converges.

3.44. Theorem. If \( F \) is an Archimedean field such that every Cauchy sequence converges, then every bounded monotonic sequence in \( F \) converges.

Proof. Suppose \( F \) is an Archimedean field such that every Cauchy sequence converges. By way of contradiction, assume there exists a bounded monotonic sequence \( \langle a_n \rangle \) in \( F \) that does not converge. Without loss of generality assume \( \langle a_n \rangle \) is an increasing sequence. By hypothesis, \( \langle a_n \rangle \) is
not a Cauchy sequence. That is, there exists some \( \varepsilon > 0 \) such that if \( N \in N(F) \) there exist natural numbers \( n, m > N \) having the property that \( \left| a_m - a_n \right| \geq \varepsilon \). Without loss of generality, assume \( n > m \) so that \( a_n - a_m = \left| a_m - a_n \right| \geq \varepsilon \). Thus, \( a_n \geq a_m + \varepsilon \geq a_N + \varepsilon \). Let \( N = 1 \). Then there exists a natural number \( i \) such that \( a_i \geq a_1 + \varepsilon \). Let \( S_1 = \{ i \in N(F) \mid a_i \geq a_1 + \varepsilon \} \). Since \( S_1 \) is a nonempty set of natural numbers, \( S_1 \) has a least element \( m_1 \). Let \( S_2 = \{ i \in N(F) \mid a_i \geq a_{m_1} + \varepsilon \} \) and let \( m_2 \) denote the least element in \( S_2 \). Thus, for each natural number \( n \), \( m_{n+1} \) denotes the smallest natural number greater than \( m_n \) where \( a_{m_{n+1}} \geq a_{m_n} + \varepsilon \). Hence, the sequence \( \langle b_n \rangle = \{ a_{m_1}, a_{m_2}, \ldots, a_{m_n}, \ldots \} \) is a strictly increasing subsequence of \( \langle a_n \rangle \). In addition, since \( \langle a_n \rangle \) is bounded, \( \langle b_n \rangle \) is also bounded.

Let \( T = \{ t \in N(F) \mid b_t \geq a_1 + t\varepsilon \} \). By definition, \( b_1 = a_{m_1} \) so that \( b_1 = a_{m_1} \geq a_1 + \varepsilon \). Hence, \( 1 \in T \). Suppose \( k \in T \). That is, \( b_k \geq a_1 + k\varepsilon \). Note that \( b_{k+1} \geq a_{m_k} + \varepsilon = b_k + \varepsilon \geq (a_1 + k\varepsilon) + \varepsilon = a_1 + (k + 1)\varepsilon \). Thus, \( (k + 1) \in T \). Therefore, by mathematical induction, \( T = N(F) \) which means that \( b_n \geq a_n + n\varepsilon \) for every \( n \in N(F) \).

Let \( b \) denote an upper bound of \( \{ b_n \} \). Then \( b \geq b_n \geq a_1 + n\varepsilon \) for every \( n \). Thus, \( \frac{b - a_1}{\varepsilon} \geq n \) for every \( n \) which contradicts the assumption that \( F \) is an Archimedean field. Therefore, the assumption that \( F \) contains a nonconvergent bounded monotonic sequence must be false.
3.45. **Theorem.** An ordered field $F$ is complete iff $F$ is Archimedean and every Cauchy sequence in $F$ converges.

**Proof.** A proof follows from Theorems 3.39, 3.43, and 3.44.

3.46. **Definition.** Suppose $F$ is an ordered field, $f$ is a function from $F$ into $F$ and $x$ is an element of $F$. Then $f$ is continuous at $x$ iff for each positive $\varepsilon$ in $F$ there exists a positive $\delta$ in $F$ such that if $y \in F$ and $|y - x| \leq \delta$ then $|f(y) - f(x)| \leq \varepsilon$. A function is said to be continuous iff it is continuous at every point of its domain.

3.47. **Lemma.** Suppose $F$ is an ordered field and let $\langle a_n \rangle$ denote a sequence in $F$ that converges to $a$. If $\langle a_n \rangle \subseteq [c, d]$ then $a \in [c, d]$.

**Proof.** Suppose $F$ is an ordered field and let $\langle a_n \rangle$ denote a sequence in $F$ that converges to $a$. Assume $\langle a_n \rangle \subseteq [c, d]$. If $a \not< c$, let $\varepsilon = c - a$. Then for every natural number $n$, $|a_n - a| = a_n - a \geq c - a = \varepsilon$ which contradicts the fact that $\langle a_n \rangle$ converges to $a$. Thus, $c \leq a$. In a similar manner it can be shown that $a \leq d$.

3.48. **Lemma.** Suppose $F$ is an ordered field and $\langle a_n \rangle$ is a sequence in $F$ that converges to $a$. If $f$ is a function continuous at $a$, then $\langle f(a_n) \rangle$ converges to $f(a)$.

**Proof.** Suppose $F$ is an ordered field and $\langle a_n \rangle$ is a sequence in $F$ that converges to $a$. Let $f$ be a function continuous at $a$. Choose $\varepsilon > 0$. There exists a $\delta > 0$ such
that if $x \in F$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Now, since $\delta > 0$ there exists a natural number $N$ such that
if $n \geq N$, $|a_n - a| < \delta$ and thus $|f(a_n) - f(a)| < \varepsilon$.
Therefore, $\langle f(a_n) \rangle$ converges to $f(a)$.

3.49. **Lemma.** If $F$ is an ordered field and $\langle a_n \rangle$ is a
convergent sequence in $F$, then $\{a_n\}$ is bounded.

**Proof.** Suppose $F$ is an ordered field and let $\langle a_n \rangle$ be
a sequence in $F$ that converges to an element $a$. Since $1 > 0$
there exists a natural number $N$ such that for $n > N$
$|a_n - a| < 1$; that is, $-1 < a_n - a < 1$ and $a - 1 < a_n < a + 1$.
Let $c = \min\{a-1, a_1, a_2, a_3, \ldots, a_N\}$ and let $d = \max\{a+1, a_1, a_2, \ldots, a_N\}$.
Then $\{a_n\} \subseteq [c, d]$.

3.50. **Theorem.** Suppose $F$ is an ordered field such
that every bounded sequence contains a convergent subsequence.
Then if $f$ is a function from $F$ into $F$ and $f$ is continuous on
a closed interval $I$, there exists some $n \in N(F)$ such that for
each $x \in I$, $|f(x)| \leq n$.

**Proof.** Suppose $F$ is an ordered field such that every
bounded sequence contains a convergent subsequence. By way
of contradiction, assume there exists a function $f$ from $F$ into
$F$ such that $f$ is continuous on a closed interval $[a, b]$ and
for each $n \in N(F)$ there exists some $x_n \in [a, b]$ such that
$|f(x_n)| > n$. Since $\{x_n\} \subseteq [a, b]$, by hypothesis the sequence
$\langle x_n \rangle$ contains a convergent subsequence $\langle x_{n_k} \rangle$. Let $z$
denote the limit of $\langle x_{n_k} \rangle$. Since $\{x_{n_k}\} \subseteq [a, b]$, by Lemma
3.47 $z \in [a, b]$ which implies that $f$ is continuous at $z$. By Lemma 3.48, since $\langle x_{n_k} \rangle$ converges to $z$, $\langle f(x_{n_k}) \rangle$ converges to $f(z)$. Thus, by Lemma 3.49 $\{f(x_{n_k})\}$ is bounded. That is, there exists a closed interval $[c, d]$ such that $\{f(x_{n_k})\} \subseteq [c, d]$. Since by Theorem 3.39 $F$ is complete, there exists a natural number $N$ such that $|c| < N$ and $|d| < N$. Hence, $\{f(x_{n_k})\} \subseteq [0, N)$ which means that $|f(x_{n_k})| < N$. However, $|f(x_{n_k})| = |f(x_{n_k})| > N$. Since the original assumption leads to a contradiction, it must be false. Thus, there exists a natural number $n$ such that for each $x \in [a, b]$, $|f(x)| \leq n$.

3.51. Theorem. Suppose $F$ is an ordered field such that if $f$ is a function from $F$ into $F$ and $f$ is continuous on a closed interval $I$, then there exists some $n \in N(F)$ such that for each $x \in I$, $|f(x)| \leq n$. Then every bounded monotonic sequence in $F$ converges.

Proof. Suppose $F$ is an ordered field such that if $f$ is a function from $F$ into $F$ and $f$ is continuous on a closed interval $I$, then there exists some natural number $n$ such that for each $x \in I$, $|f(x)| \leq n$. By way of contradiction, suppose $F$ contains a bounded monotonic sequence $\langle a_n \rangle$ that fails to converge. Without loss of generality assume $\langle a_n \rangle$ is increasing. Then, by Lemma 3.30 $\langle a_n \rangle$ contains a strictly increasing subsequence $\langle b_n \rangle$. Let $a = b_1$ and let $b$ represent an upper bound of $\{b_n\}$. Define the function $f$ from $F$ into $F$ as follows:
\[
    f(x) = \begin{cases} 
    n + \frac{x - b_n}{b_{n+1} - b_n}, & \text{if } b_n \leq x \leq b_{n+1}, \ n \in N(F); \\
    0, & \text{otherwise}
    \end{cases}
\]

Note that if \( x = b_n, \ n \in N(F), \) then \( f(x) = f(b_n) = n. \)

In order to verify that \( f \) is continuous on \([a,b]\), consider three cases. Firstly assume \( x \notin [a,b] \) and that there exists some \( n \in N(F) \) such that \( b_n \leq x \leq b_{n+1}. \) Let \( \varepsilon > 0 \) and let \( \delta = \min\{b_{n+1} - x, x - b_n, \varepsilon (b_{n+1} - b_n)\} \). Suppose \( y \in [a,b] \) such that \( |y - x| < \delta. \) By the definition of \( \delta, \ y \in [b_n, b_{n+1}]. \)

Thus, \( |f(y) - f(x)| = \left| n + \frac{y - b_n}{b_{n+1} - b_n} - (n + \frac{x - b_n}{b_{n+1} - b_n}) \right| = \left| \frac{y - x}{b_{n+1} - b_n} \right| \leq \frac{\varepsilon (b_{n+1} - b_n)}{b_{n+1} - b_n} = \varepsilon. \)

Similarly, it can be verified that if \( x \in [a,b] \) and \( x = b_n \) for some \( n \in N(F), \) \( f \) is continuous at \( x. \) Finally, if \( x \in [a,b] \) and if for every \( n \in N(F) \) \( x \notin [b_n, b_{n+1}] \) then \( x \) is an upper bound for \( [b_n]. \) By Lemma 3.32, \( \langle b_n \rangle \) fails to converge, so that by Lemma 3.31 the set \( \{b_n\} \) has no least upper bound.

Hence, there exists an upper bound \( z \) for \( \{b_n\} \) such that \( z \leq x. \) Suppose \( \varepsilon > 0 \) and let \( \delta = x - z. \) If \( y \in [a,b] \) and \( |y - x| < \delta = x - z, \) then \( z - s \leq y - x \leq x - z \) which implies that \( z \leq y. \) Therefore, \( |f(y) - f(x)| = |0 - 0| = 0 < \varepsilon. \)

Thus, \( f \) is continuous on \([a,b]. \)

However, if \( n \) is a natural number and \( q \in (b_n, b_{n+1}], \) then \( |f(q)| \geq f(q) > n. \) This conclusion contradicts the hypothesis that if \( f \) is continuous on a closed interval then
f is bounded above by some natural number. Thus, the assumption that \( \{a_n\} \) fails to converge is false. Therefore, every bounded monotonic sequence in \( F \) converges.

3.52. **Theorem.** If \( F \) is an ordered field then the following condition is necessary and sufficient for \( F \) to be a complete ordered field: If \( f \) is a function from \( F \) into \( F \) and \( f \) is continuous on a closed interval \( I \), then there exists some \( n \in \mathbb{N}(F) \) such that for each \( x \in I \), \( |f(x)| \leq n \).

**Proof.** A proof follows from Theorems 3.39, 3.50, and 3.51.
BIBLIOGRAPHY


