HOMOTOPIES AND DEFORMATION

RETRACTS

THESIS

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This paper introduces the background concepts necessary to develop a detailed proof of a theorem by Ralph H. Fox which states that two topological spaces are the same homotopy type if and only if both are deformation retracts of a third space, the mapping cylinder. The concepts of homotopy and deformation are introduced in chapter 2, and retraction and deformation retract are defined in chapter 3. Chapter 4 develops the idea of the mapping cylinder, and the proof is completed. Three special cases are examined in chapter 5.
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CHAPTER I

INTRODUCTION

The purpose of this paper is to introduce some concepts which can be used to describe relationships between topological spaces, specifically homotopy, deformation, and retraction. These concepts can also be used to characterize an equivalence relation on the class of topological spaces. Another topological space called a mapping cylinder is the tool used. The idea of a mapping cylinder is described by R. H. Fox in his paper "On Homotopy Type and Deformation Retracts"¹, and this article provided the motivation for, and much of the material in this paper. The theorems and definitions quoted are from this paper unless otherwise noted.

The fundamental idea of this paper, that of homotopy, is introduced in chapter 2. The concept of homeomorphism can be thought of as a relationship between spaces. Then the question arises as to whether there is a more general relationship between two spaces which are not homeomorphic. One more general categorization of topological spaces, homotopy type, is described in chapter 3. The purpose of

Fox’s paper is to give a proof of his theorem which states that two spaces belong to the same homotopy type if and only if they both can be homeomorphically imbedded in a third space of which both are deformation retracts. The concepts of deformation and retract are defined in chapter 3. A generalized method of constructing this third space uses the concept of a mapping cylinder which is introduced in chapter 4. In chapter 5 some special cases are described. The concept of a space being inessential relative to a subset is defined, and several theorems are proved. Special homotopy and special homotopy inverse are defined, and a relationship between them is described. Then $\epsilon$-homotopies are defined and several results are shown.

The major result of this paper is to construct, or reconstruct in detail, the proofs of the theorems of Fox’s paper using a more precise notation. Additional theorems have been added to provide the background necessary to explicate his results.
Throughout this paper the terms "map" or "mapping" refer to a continuous function from one topological space to another. If there is a question as to whether a map is continuous, it will be referred to as a function until its continuity is established. The notation used generally follows that used in Willard’s General Topology.

Definition 2.1: Mappings \( f \) and \( g \) from a space \( A \) into a space \( D \) are homotopic, written \( f \sim g \), if and only if there is a mapping \( \xi \) from \( A \times [0,1] \) into \( D \) such that for all \( a \in A \)
\[ \xi(a,0) = f(a), \]
\[ \xi(a,1) = g(a). \]
The map \( \xi \) is a homotopy between \( f \) and \( g \).

From this definition one can see that a homotopy can be thought of as a collection of maps \( \xi_t : A \to D \) for every \( t \in [0,1] \). The notation "\( \xi_t \)" will refer to a map from \( A \) into \( D \) for a particular \( t \in [0,1] \), and the notation "\( \xi(a,t) \)" will refer to a map from \( A \times [0,1] \) into \( D \), but it should be clear that for every \( t \in [0,1] \) \( \xi_t(a) = \xi(a,t) \). It is often helpful to think of the parameter \( t \) as time. Thus for the homotopy \( \xi \), if \( \xi_0 = f \) and \( \xi_1 = g \) then the collection of

\[ ^2 \text{Stephen Willard, General Topology (Reading, Mass.: Addison-Wesley Publishing Co., 1970).} \]
The $\zeta_t$'s for $t \in [0,1]$ can be thought of as a kind of "continuous transformation" of $f$ into $g$ over time.

It is always important to maintain the distinction between a function and the set which is the image of the function; however, when a concept is illustrated geometrically the symbol for a map and the symbol for its image are sometimes interchanged for the sake of simplicity. For example, figure 1 is a way to illustrate the homotopy $\zeta : A \times [0,1] \to D$ between the maps $f : A \to D$ and $g : A \to D$.

![Figure 1](image)

**FIGURE 1**

On the one hand the rectangle can represent the space $A \times [0,1]$, the domain of $\zeta$. However, it can also be seen as the collection of images of $\zeta_t$ for $t \in [0,1]$. Thus the base of the rectangle when $t = 0$ is the image of $\zeta_0 = f$ in $D$, and the top of the rectangle is the image of $\zeta_1 = g$. Every point in the rectangle is the image of a point $\zeta_t(a)$ in $D$ for some $t \in [0,1]$. The rectangle then becomes an image of the map $\zeta$ in $D$ "stretched" over time to show the image of $\zeta$ at each time $t \in [0,1]$. If $D \subseteq A$ then the figure can be
seen both ways with the image of $\xi$ superimposed on its domain. For greatest clarity both the domain and the range of a homotopy can be shown. However, since the domain of a homotopy is usually clear from the written description, only its image is usually represented.

Just as $\xi$ can be seen as a collection of maps $\xi_t$ for $t \in [0,1]$, the homotopy can also be seen as a collection of maps $\xi_a : [0,1] \to D$, and this notation will also be useful.

The proof of a simple theorem about homotopies will illustrate some of the ways homotopies can be used to generate a new homotopy.

**Theorem 2.2:** Homotopy is an equivalence relation on the set of maps from one space to another.

**Proof:** Let $F$ be the set of maps from a space $A$ into a space $D$.

Suppose $f \in F$. $f \simeq f$ by the homotopy $\xi : A \times [0,1] \to D$ defined by $\xi(a,t) = f(a)$ for every $t \in [0,1]$. Thus homotopy is reflexive.

Suppose $f \simeq g$ by the homotopy $\xi : A \times [0,1] \to D$. Then $\xi(a,0) = f(a)$, $\xi(a,1) = g(a)$, and $\xi(a,t) \in D \forall a \in A$. Then by substituting $(1-t)$ for $t$, $\xi(a,1-0) = \xi(a,1) = g(a)$, and $\xi(a,1-1) = \xi(a,0) = f(a)$. Thus $g \simeq f$. Therefore homotopy is symmetric. The notation $\xi^{(r)}$ will be used to indicate that the homotopy, $\xi$ is reversed by substituting $(1-t)$ for $t$. 
Suppose \( f \simeq g \) by the homotopy \( \zeta \), and \( g \simeq h \) by the homotopy \( \lambda \). Define a map \( \zeta : A \times [0,1] \to D \) by:

\[
\zeta(a,t) = \begin{cases} 
\xi(a,2t) & \text{for } t \in [0,1/2] \\
\lambda(a,2t - 1) & \text{for } t \in [1/2,1]
\end{cases}
\]

Thus \( \zeta(a,0) = \xi(a,0) = f(a) \), \( \zeta(a,1) = \lambda(a,1) = h(a) \). Since \( \zeta \) is defined in terms of continuous functions, to show that \( \zeta \) is continuous it is sufficient to show that the definition is consistent where the different parts overlap. Thus \( \zeta(a,1/2) = \xi(a,1) = g(a) = \lambda(a,0) \). So \( \zeta \) is continuous and therefore a homotopy between \( f \) and \( h \). Therefore homotopy is transitive.

Therefore homotopy is an equivalence relation on the set of maps from one space to another.

**Definition 2.3**: A path in a space \( A \) between points \( a_0 \) and \( a_1 \) is a map \( p : [0,1] \to A \) such that \( p(0) = a_0 \) and \( p(1) = a_1 \).

**Lemma 2.4**: Suppose \( \xi : A \times [0,1] \to D \) is a homotopy between \( \xi_0 \) and \( \xi_1 \). Then for every \( a \in A \) there is a path in \( D \) between \( \xi(a,0) \) and \( \xi(a,1) \).

**Proof**: Let \( a \) be a member of \( A \). Define a function \( p_a : [0,1] \to D \) by \( p_a(t) = \xi(a,t) \). Thus \( p_a(0) = \xi(a,0) \) and \( p_a(1) = \xi(a,1) \). Notice that \( p_a = \xi_a \). Therefore \( p_a \) is continuous.

\( \square \)
Definition 2.5: If $A \subseteq D$, then a deformation $\delta$ is a homotopy between the identity on $A$, $\text{id}_A$, and a map $g : A \to D$. Thus $\delta : A \times [0,1] \to D$ where $\delta(a,0) = a$ and $\delta(a,1) = g(a)$ for all $a \in A$. The set $g(A)$ is called a deform of $A$, and $A$ is said to be deformable in $D$ into $g(A)$. If $A = D$, one simply says that $A$ can be deformed into $g(A)$. If $g(A)$ is a singleton set then $\delta$ is a contraction of $A$.

Theorem 2.6: If $A$ can be deformed into $B$ then a mapping $f$ of $A$ into $B$ is homotopic to the identity on $A$ if and only if $f$ restricted to $B$, written $f|_B$, is homotopic to the identity on $B$.

Proof: Suppose $A$ can be deformed into $B$. Let $f : A \to A$ be any mapping of $A$ into $A$. Since $A$ can be deformed into $B$, then $B \subseteq A$ and there is a homotopy $\delta$ between the identity on $A$ and some mapping $g : A \to A$ where $g(A) \subseteq B$. Thus $\delta : A \times [0,1] \to A$ such that $\delta(a,0) = a$ and $\delta(a,1) = g(a) \ \forall a \in A$.

Suppose $\text{id}_B \simeq f|_B$ by the homotopy $\eta$. Then $\eta : B \times [0,1] \to B$ such that $\eta(b,0) = \text{id}_B(b) = b$ and $\eta(b,1) = f|_B(b) \ \forall b \in B$. It needs to be shown that $\text{id}_A \simeq f$. Define $\zeta : A \times [0,1] \to A$ by:
Thus the map $\zeta(a,0) = \delta(a,0) = \text{id}_A(a) = a$, and the map $\zeta(a,1) = f(\delta(a,0)) = f(a)$. 

\[
\zeta(a, t) = \begin{cases} 
\delta(a, 3t) & \text{for } 0 \leq t \leq 1/3 \\
\eta(\delta(a,1), 3t - 1) & \text{for } 1/3 \leq t \leq 2/3 \\
f(\delta(a,3-3t)) & \text{for } 2/3 \leq t \leq 1 
\end{cases}
\]
In order to show that \( \zeta \) is continuous it is sufficient to show that \( \zeta \) is consistent at \( t = 1/3 \) and \( t = 2/3 \), since \( \zeta \) is defined in terms of continuous functions. The map

\[
\zeta(a,1/3) = \delta(a,3(1/3)) = \delta(a,1) = g(a) \in B, \text{ and } g(A) = B.
\]

Also, \( \zeta(a,1/3) = \eta(\delta(a,1),3(1/3) - 1) = \eta(\delta(a,1),0) = \eta(g(a),0) = \text{id}_B(g(a)) = g(a) \). The map \( \zeta(a,2/3) = \eta(\delta(a,1),3(2/3) - 1) = \eta(g(a),1) = f|_B(g(a)) \), and

\[
\eta(g(A),1) = \eta(B),1 = f|_B(B) = f(B).
\]

And \( \zeta(a,2/3) = f(\delta(a,3-3(2/3))) = f(\delta(a,1)) = f(g(a)) = f|_B(g(a)) \). Thus \( \zeta \) is consistent at \( t = 1/3 \) and \( t = 2/3 \) and is therefore continuous. Therefore \( \zeta \) is a homotopy between \( \text{id}_A \) and \( f(A) \).

Conversely, suppose \( f \simeq \text{id}_A \) by the homotopy \( \zeta \).

Thus \( \xi : A \times [0,1] \to A \) and \( \xi(a,0) = a \) and \( \xi(a,1) = f(a) \).

Since \( B \subseteq A \), \( \xi|_B : B \times [0,1] \to A \) such that \( \xi(b,0) = b \), and

\[
\xi(b,1) = f(b) = f|_B(b) \quad \forall b \in B.
\]

Therefore \( \xi|_B \) is a homotopy between \( \text{id}_B \) and \( f|_B \). \( \square \)
CHAPTER III

RETRACTION AND DEFORMATION RETRACT

The concept of a retraction is related to both set-theoretic topology and algebraic topology. In set-theoretic topology there are several generalizations of Tietze’s Extension Theorem which use this notion. In the following material two of these extension theorems will be used without proof, and in this paper the use of the concept will be confined to the set-theoretic applications.

**Definition 3.1:** A subset $\mathcal{B}$ of a space $\mathcal{A}$ is a **retract** of $\mathcal{A}$ if and only if there is a mapping $r : \mathcal{A} \rightarrow \mathcal{B}$ such that $r|_{\mathcal{B}} = \text{id}_\mathcal{B}$. The map $r$ is called a **retraction** of $\mathcal{A}$ into $\mathcal{B}$, and one says $\mathcal{A}$ is **retractable** into $\mathcal{B}$.

**Definition 3.2:** A retract $\mathcal{D}$ of a space $\mathcal{A}$ is a **deformation retract** if and only if a retraction $r : \mathcal{A} \rightarrow \mathcal{D}$ is homotopic to $\text{id}_\mathcal{A}$ by some homotopy, $\delta$. The retraction $r$ is then called a **deformation retraction** and the homotopy $\delta$ is called a **retracting deformation**.

**Theorem 3.3:** The subset $\mathcal{B}$ of a space $\mathcal{A}$ is a deformation retract of $\mathcal{A}$ if and only if $\mathcal{B}$ is a retract of $\mathcal{A}$, and $\mathcal{A}$ is deformable into $\mathcal{B}$.

---

Proof: Suppose B is a deformation retract of A. Then by the definition there is a retracting deformation 
\( \delta : A \times [0,1] \rightarrow A \) such that \( \delta(a,0) = a \) and \( \delta(a,1) = r(a) \) for some retraction \( r : A \rightarrow A \) such that \( r(A) = B \). Thus B is a retract of A, and A is deformable into B.

Conversely, suppose B is a retract of A and A is deformable into B. Since B is a retract of A, then there is a map \( r : A \rightarrow B \) such that \( r|_B = \text{id}_B \). Then clearly \( r|_B \simeq \text{id}_B \). Since A is deformable into B then by theorem 2.6, \( r \simeq \text{id}_A \). Thus B is a deformation retract of A.

As an example of a retract which is not a deform, consider the unit circle, \( S^1 \), in the plane along with the point \( (0,0) \). Let \( A = S^1 \cup \{(0,0)\} \) with the subspace topology. The constant map \( k : A \rightarrow A \) such that \( k(a) = (0,0) \) is evidently a retraction of A into \( \{(0,0)\} \), but A is not deformable into \( \{(0,0)\} \). Just suppose there was such a deformation \( \delta : A \times [0,1] \rightarrow A \). Then \( \delta_0 = \text{id}_A \) and \( \delta_1(A) = \{(0,0)\} \). By lemma 2.4, for a point \( x \) of \( S^1 \) there is a path from \( \delta_0(x) = x \) to \( \delta_1(x) = (0,0) \) in A. This is clearly a contradiction. Therefore A is not deformable into \( \{(0,0)\} \).

Definition 3.4: A closed subset \( X_0 \) of a Hausdorff space \( X \) is a **neighborhood retract** in \( X \) if and only if \( X_0 \) is a retract of some open subset of \( X \) containing \( X_0 \).

\(^4\text{ibid., 14.}\)
Notice that every retract of $X$ is a neighborhood retract in $X$, since the space $X$ is a neighborhood of every subset in $X$. But not every neighborhood retract is a retract. Borsuk gives the following example.\(^5\) Let $S^{n-1}$ be a sphere in $n$-dimensional Euclidean space, $\mathbb{R}^n$. The set, $\mathbb{R}^n$ minus the origin, $\mathbb{R}^n - \{0\}$, is a neighborhood of $S^{n-1}$. Define a map $r : \mathbb{R}^n - \{0\} \to S^{n-1}$ by $r(x) = x/\|x\|$, where $\|x\|$ is the distance of $x$ from the origin. Then $r$ is a retraction of $\mathbb{R}^n - \{0\}$ onto $S^{n-1}$. Then Borsuk cites a proof of the fact that $S^{n-1}$ is not a retract of $\mathbb{R}^n$ given by Hirsch.\(^6\) Hirsch's proof relies on theorems from algebraic topology and is beyond the scope of this paper.

**Definition 3.5:** An absolute neighborhood retract, abbreviated ANR, is a separable, metrizable space which is a neighborhood retract of every separable, metrizable space which contains it as a closed subset.

Thus an ANR is a separable, metrizable space $A$ such that for every separable, metrizable space $E$ which contains $A$ as a closed subset, there is an open neighborhood $U$ of $A$ in $E$ and a map $r : U \to A$ such that $r\mid_A = \text{id}_A$.

\(^5\)ibid., 12-14.


Theorem 3.6: If \( \xi \) is a deformation of an ANR, A, and B is the set of fixed points of \( \xi_1 \), then there is a homotopy \( \zeta \) between \( \text{id}_A \) and \( \xi_1 \) such that the points of B are fixed under each of the maps \( \xi_t \) for \( 0 \leq t \leq 1 \).

Proof: Let A be an ANR, \( \xi : A \times [0,1] \to A \) be a deformation, and B be the fixed points of \( \xi_1 \). Thus \( \xi_0 = \text{id}_A \) and \( B \subseteq \xi_1(A) \) such that \( \forall b \in B \: \xi_1(b) = b \). Define a function \( \eta : (A \times \{0\} \cup B \times [0,1] \cup A \times \{1\}) \times [0,1] \to A \) by:

\[
\eta((a,0),u) = \begin{cases} 
  a & \text{for } 0 \leq u \leq 1 \text{ and } a \in A \\
  \xi(a,t/u) & \text{for } 0 \leq t < u \leq 1 \text{ and } a \in B \\
  a & \text{for } 0 \leq t \leq 1 \text{ and } a \in B \\
\end{cases}
\]

Here the "time" variable for \( \xi \) is \( t \) and \( u \) for \( \eta \).

To show that \( \eta \) is continuous it is sufficient to show that the definition is consistent where the parts overlap. The first line and the second line of the definition of \( \eta \) overlap when \( t = 0 \) and \( a \in B \). Then \( \eta((a,0),u) = a \) for \( 0 \leq u \leq 1 \) and \( \xi(a,0/u) = \xi(a,0) = a \) for \( 0 = t < u \leq 1 \), and \( a \in B \). Thus when \( t = 0 \) the definition of \( \eta \) is consistent. Lines two and three overlap for members of B when \( 0 < t = u < 1 \). Then from line two \( \eta(a,t),u) = \xi(a,t/u) = \xi(a,1) = a \), since \( a \in B \) and B is the set B of fixed points of \( \xi_1 \). This
agrees with line three. Line three and line four overlap when \( t = 1 \) and \( a \in B \). By line four \( \eta((a,1),u) = \xi(a,1) \) which equals \( a \) if \( a \in B \). Thus lines three and four agree. Therefore \( \eta \) is a homotopy between \( \eta_0 \) and \( \eta_1 \).

Notice that for \( u = 1 \), when \( t = 0 \), \( \eta_1(a,0) = a = \xi(a,0) \) \( \forall a \in A \). And \( \eta_1(a,t) = \xi(a,t/1) = \xi(a,t) \) \( \forall a \in B \). When \( t = 1 \), \( \eta_1(a,1) = \xi(a,1) \) for all \( a \in A \). Thus \( \eta_1 \) and \( \xi \) agree for \( 0 \leq t \leq 1 \). But \( \xi \) is a map from \( A \times [0,1] \) into \( A \); so \( \xi \) is an extension of \( \eta_1 \) from the set 
\[{(A \times \{0\}) \cup (B \times [0,1]) \cup (A \times \{1\})} \] to the set \( A \times [0,1] \).

Claim: The set \( \{(A \times \{0\}) \cup (B \times [0,1]) \cup (A \times \{1\}) \} \) is closed in \( A \times [0,1] \). For brevity call this set \( S \).

Since \( A \times \{0\} \) and \( A \times \{1\} \) are each the product of \( A \) with a closed set, \( \{0\} \) and \( \{1\} \) respectively, they are both closed in \( A \times [0,1] \).

Sub-claim: The set of fixed points of \( \xi_1 B \) is closed in \( A \).

Suppose \( x \) is a limit point of \( B \) and \( x \notin B \). Since \( A \) is an ANR, a separable, metrizable space, \( A \) is 2nd countable and thus also 1st countable. Since \( x \) is a limit point of \( B \), there is a sequence of points of \( B \), \( \{b_k\} \), which converge to \( x \). Since \( \xi_1 \) is continuous, \( \xi_1(\{b_k\}) = \{\xi_1(b_k)\} \) is a sequence in \( \xi_1(A) \subseteq A \) which converges to \( \xi_1(x) \). But \( \xi_1(b_k) = b_k \) for every \( k \), since \( b_k \) is in \( B \). So the sequence \( \{b_k\} \) converges to \( \xi_1(x) \). Therefore \( \xi_1(x) = x \). This implies \( x \) is
a fixed point of $\xi_1$ and $x$ is in $B$. Therefore $B$ contains all of its limit points and is closed.

Thus $B \times [0,1]$ is closed in $A \times [0,1]$. Therefore the set $S$ is the finite union of closed sets and is closed in $A \times [0,1]$.

Borsuk's Theorem as generalized by Dowker states "Let $C$ be a closed subset of a space $X$ and $f$ and $g$ two homotopic mappings of $C$ in an arbitrary absolute neighborhood retract. Then if there is an extension $F$ of $f$ over $X$ there is also an extension $G$ of $g$ over $X$, with $F$ and $G$ homotopic." $^8$

In our case $C = S$ which is closed in $A \times [0,1]$, the space $X$ in our case. The maps $f$ and $g$ are $\eta_1$ and $\eta_0$, respectively which map $S$ into $A$, an ANR. Then since there is an extension of $\eta_1$ to $A \times [0,1]$, namely $\xi$, there is also an extension of $\eta_0$ to $A \times [0,1]$. Call this extension $\zeta$, and by the theorem $\xi \simeq \zeta$. Thus $\zeta : A \times [0,1] \to A$ such that $\zeta|_S = \eta_0$. Now when $t = 0$, $\zeta(a,0) = \eta_0(a,0) = a \ \forall a \in A$. So $\zeta_0 = \text{id}_A$. When $t = 1$ $\zeta(a,1) = \eta_0(a,1) = \xi(a,1)$. So $\zeta_1 = \xi_1$. Therefore $\zeta$ is a deformation of $A$ into $\xi_1(A)$. Furthermore, $\zeta(b,t) = \eta_0(b,t) = b \ \forall b \in B$ for $0 \leq t \leq 1$. So the points of $B$ are fixed under each of the maps, $\zeta_t$ for $0 \leq t \leq 1$. $\square$

Suppose $A$ and a subset $B$ are ANR sets.

---

Theorem 3.7a: The set $B$ is a deformation retract of $A$ if and only if there is a retracting deformation $\xi$ of $A$ onto $B$ such that the points of $B$ are fixed under each $\xi_t$ for $0 \leq t \leq 1$.

Theorem 3.7b: The set $B$ is a deformation retract of $A$ if and only if there is a deformation $\xi$ of $A$ onto $B$ such that $\xi_t(B) \subseteq B$ for $0 \leq t \leq 1$.

Proof of version a: Suppose there is a retracting deformation $\xi$ of $A$ onto $B$ such that the points of $B$ are fixed under each $\xi_t$ for $0 \leq t \leq 1$. It follows from the definition that $B$ is a deformation retract of $A$.

Conversely, suppose $B$ is a deformation retract of $A$ by the deformation $\delta : A \times [0,1] \rightarrow A$ such that $\delta_0 = \text{id}_A$ and $\delta_1(A) = B$ and $\delta_1|_B = \text{id}_B$. Then $B$ is the set of fixed points of $\delta_1$. By theorem 3.6 there is a homotopy $\zeta$ such that $\zeta_t(b) = \text{id}_B$ for $0 \leq t \leq 1$ and $\zeta$ is a retracting deformation of $A$ onto $B$.

Proof of version b: Suppose $B$ is a deformation retract of $A$. Then by the paragraph above there is a retracting deformation $\zeta$ from $A$ onto $B$ such that $\zeta_t(B) = \text{id}_B$, and therefore $\zeta_B(B) \subseteq B$ for $0 \leq t \leq 1$.

Conversely, suppose there is a deformation $\delta$ of $A$ onto $B$ such that $\delta_t(B) \subseteq B$ for $0 \leq t \leq 1$. Then $\delta_0 = \text{id}_A$ and $\delta_1(A) = B$. So $\delta_0|_B = \text{id}_B$ and $\delta_t|_B \subseteq B$ for every $t$. The map $\delta|_B$ is a homotopy between $\delta_0|_B$ and $\delta_1|_B$. Since $B$ is an ANR
and contained in $A$, by definition it is closed in $A$. Since $A$ is an ANR it is metrizable and therefore normal. By the Borsuk–Kuratowski Theorem which says, "If $W$ is a closed subset of a normal space, $Z$ and $X$ is an AR-set (ANR-set), then every continuous map of $W$ into $X$ can be extended to $Z$ (to a neighborhood of $W$ in $Z$)". Thus $\delta_0|_B$ can be extended to $A$. Call this extension $r$. So $r : A \rightarrow B$ and $r|_B = \delta_0 = \text{id}_B$. Therefore $r$ is a retraction of $A$ onto $B$. By theorem 3.3, $B$ is a deformation retract of $A$. \hfill \Box$

The necessity of the restriction that the space $A$ be an ANR is illustrated by the following example. Let $C$ be a subset of $\mathbb{R}^2$, and define $C$ as follows. Let the set $B = \{(x,0) \mid 0 \leq x \leq 1\}$; the set $S = \{(0,y) \mid 0 \leq y \leq 1\}$; and the set $X_n = \{(1/n,y) \mid 0 \leq y \leq 1, n \text{ is a natural number}\}$. Let $C = B \cup X_n \cup S$ with the subspace topology. $C$ can be called the "comb space". Notice that $C$ is closed and bounded and thus compact.

---

*Borsuk, 5.*
With a fact about ANR sets it can easily be shown that $C$ is not an ANR. "Each ANR(M)-space [ANR] is locally contractible."$^{10}$ A locally contractible space is a space which is locally contractible at every point. Locally contractible at a point $x$ is defined to mean every neighborhood $U$ of $x$ contains a neighborhood $V$ of $x$ which is contractible to a point.$^{11}$

Claim: The comb space is not an ANR.

Proof: Just suppose that $C$ is an ANR. The set $C$ is a closed subset in $\mathbb{R}^2$. Consider a point $(0,y)$ in $C$. Every neighborhood of $(0,y)$ contains a set $\{(1/n,y) | n > k\}$ for some natural number $k$. Let $U$ be a neighborhood of $(0,y)$ which does not contain $(0,0)$. Since $C$ is an ANR, by the theorem above, let $V$ be any other neighborhood of $(0,y)$

$^{10}$ibid., 87.
$^{11}$ibid., 28.
contained in U which is contractible to a point. By lemma 2.4 there is therefore a path in V from every point to the contraction point. But there cannot be a path to the contraction point from (0,y) and (1/n,y), since (0,0) \not\in V. Therefore C is not locally contractible and thus not an ANR.

Claim: The comb space, C, is deformable into S so that every point (x,y) is mapped to the point (0,y). Thus the points in S are mapped to themselves.

Define a map \( \delta : C \times [0,1] \to C \) by:

\[
\delta((x,y), t) = \begin{cases} 
(x,(1-4t)y) & \text{for } 0 \leq t \leq 1/4 \\
(x+(4t-1)(y-x),0) & \text{for } 1/4 \leq t \leq 1/2 \\
((3-4t)y,0) & \text{for } 1/2 \leq t \leq 3/4 \\
(0,(4t-3)y) & \text{for } 3/4 \leq t \leq 1
\end{cases}
\]

When \( t = 0 \), \( \delta((x,y),0) = (x,(1-0)y) = (x,y) \). From line one of the definition \( \delta((x,y),1/4) = (x,(1-1)y) = (x,0) \), and from line two \( \delta((x,y),1/4) = (x+(1-1)(y-x),0) = (x,0) \). From line two \( \delta((x,y),1/2) = (x+(2-1)(y-x),0) = (y,0) \), and from line three \( \delta((x,y),1/2) = ((3-2)y,0) = (y,0) \). From line three \( \delta((x,y),3/4) = ((3-3)y,0) = (0,0) \), and from line four \( \delta((x,y),3/4) = (0,(3-3)y) = (0,0) \). When \( t = 1 \) \( \delta((x,y),1) = (0,(4-3)y) = (0,y) \). Therefore \( \delta \) is a deformation of C into S. Furthermore, \( \delta_1|_S(0,y) = (0,y) \); so \( \delta_1|_S = \text{id}_S \).
Therefore $S$ is a deformation retract of $C$.

**Claim:** There is no retracting deformation of $C$ into $S$ that satisfies the condition that the points of $S$ are fixed under each $t \in [0,1]$.

**Proof:** Just suppose there is a retracting deformation $\xi : C \times [0,1] \to C$ such that $\xi_t|_S = \text{id}_S$ for all $t \in [0,1]$. Let $U$ be a basic open neighborhood of the point $(0,1) \in S$. Since by lemma 2.4 $\xi(1/n,1)^{(t)}$ is a path from $(1/n,1)$ to $\xi_1(1/n,1)$, for every natural number $n$, $\xi(1/n,1) = (0,0)$ for some $t \in [0,1]$. Let $t_0 = \inf\{t \mid \xi_{t_0}(1/n,1) = (0,0), \text{ and } n \text{ is a natural number}\}$. Then $\inf\{d(\xi_{t_0}(1/n,1),(0,0)) \mid n \text{ is a natural number}\} = 0$. Let $U$ be a neighborhood of $\xi_{t_0}(0,1) = (0,1)$ with radius $< 1/2$. Let $V$ be any neighborhood of $(0,1)$ such that $\xi_{t_0}(V) \subset U$. The set $V$ contains a set $\{(1/n,1) \mid n > k\}$ for some natural number $k$. Since $\inf\{d(\xi_{t_0}(1/n,1),(0,0))\} = 0$, there is a natural number $n$ such that $d(\xi_{t_0}(1/n,1),(0,0)) < 1/2$ then $d(\xi_{t_0}(1/n,1),(0,1)) > 1/2$ and $\xi_{t_0}(1/n,1) \notin \xi_{t_0}(V)$. Thus $\xi_{t_0}$ is not continuous. This contradicts the supposition that $\xi$ is a retracting deformation. Therefore there is no retracting deformation of $C$ into $S$ such that the points of $S$ remain fixed for all $t \in [0,1]$.

Therefore the restriction in theorem 3.7a that the space $A$ be an ANR is necessary and by the proof it is sufficient.
If $P$ is the set $\{(0,1)\}$, then $C$ can be deformed into $P$ by adding a retracting deformation of $S$ into $P$ to the retracting deformation $\delta$ in the previous example. Then $P$ is a deformation retract of $C$. But by a similar argument used in the first example there is no retracting deformation $\xi : C \to P$ satisfying the condition in theorem 3.7b that $\xi_t(P) \subseteq P$ for all $t \in [0,1]$. Thus the restriction that the space $A$ be an ANR is also necessary in theorem 3.7b.

Definition 3.8: Two spaces $A$ and $B$ are said to belong to the same homotopy type if and only if there are maps $f : A \to B$ and $g : B \to A$ such that $fg \simeq \text{id}_A$ and $gf \simeq \text{id}_B$.

Claim: Homotopy type is an equivalence relation on the class of topological spaces.

Proof: For any space $A$ define $f$ and $g$ to be $\text{id}_A$. Since $\text{id}_A$ composed with $\text{id}_A$ equals $\text{id}_A$, $\text{id}_A \text{id}_A \simeq \text{id}_A$. Thus $A$ is the same homotopy type as itself, and homotopy type is reflexive.

Suppose the space $A$ is the same homotopy type as the space $B$. By the definition $B$ is also the same homotopy type as $A$. So homotopy type is a symmetric relation.

Suppose that the space $A$ is the same homotopy type as the space $B$, and $B$ is the same homotopy type as the space $C$. Then there are maps $f : A \to B$ and $g : B \to A$ such that $fg \simeq \text{id}_B$ and $gf \simeq \text{id}_A$. Also there are maps $j : B \to C$ and $k : C \to B$ such that $jk \simeq \text{id}_C$ and $kj \simeq \text{id}_B$. 
Notice that the map $gkjf : A \to A$ and $jfgk : C \to C$.

Define the map $\lambda : A \times [0,1] \to A$ by:

$$
\lambda(a,t) = \begin{cases} 
\beta(a,2t) & \text{for } 0 \leq t \leq 1/2 \\
g(\delta(f(a),2t-1)) & \text{for } 1/2 \leq t \leq 1 
\end{cases}
$$

When $t = 1/2$, $\lambda(a,1/2) = \beta(a,1) = gf(a)$, and $\lambda(a,1/2) = g(\delta(f(a),0) = g(f(a)) = gf(a)$. So $\lambda$ is well defined.

Also, $\lambda(a,0) = \beta(a,0)$. Thus $\lambda_0 = \beta_0 = \text{id}_A$. When $t = 1$

$\lambda(a,1) = g(\delta(f(a),1)) = g(kj(f(a))) = gkjf(a)$. Therefore $\lambda$ is a homotopy between $\text{id}_A$ and $gkjf$.

Define the map $\rho : C \times [0,1] \to C$ by:

$$
\rho(c,s) = \begin{cases} 
g(c,2s) & \text{for } 0 \leq s \leq 1/2 \\
k(a(k(c),2t-1) & \text{for } 1/2 \leq s \leq 1 
\end{cases}
$$
When \( t = 1/2 \) \( \rho(c, 1/2) = \gamma(c, 1) = jk(c) \), and \( \rho(c, 1/2) = j(a(k(c), 0)) = j(k(c)) = jk(c) \). Thus \( \rho \) is well defined. Now when \( t = 0 \) \( \rho(c, 0) = \gamma(c, 0) \); so \( \rho_0 = \gamma_0 = \text{id}_C \). When \( t = 1 \) \( \rho(c, 1) = j(a(k(c), 1) = j(fg(k(c))) = jfgk(c) \).

Therefore \( \text{id}_C \sim jfgk \) by the homotopy \( \rho \). Since \( jf : A \to C \) and \( gk : C \to A \), \( C \) and \( A \) are the same homotopy type.

Therefore homotopy type is a transitive relation, and thus homotopy type is an equivalence relation on the class of topological spaces.

Notice, in the special case when \( g = f^{-1} \) that \( ff^{-1} = f^{-1}f = \text{id}_B \). Then \( ff^{-1}(B) = B \) which implies that \( f \) is an onto function. Since \( ff^{-1}(a) = a \) for all \( a \in A \), \( f \) is a one-to-one function. And since \( f \) and \( f^{-1} \) are both continuous, \( f \) is a homeomorphism between \( A \) and \( B \).

Thus the partition of the class of topological spaces resulting from homeomorphism is a refinement of the partition resulting from homotopy type.

**Definition 3.9** If maps \( f : A \to B \) and \( g : B \to A \) are such that \( gf \sim \text{id}_A \), then \( g \) is a left homotopy inverse of \( f \), and \( f \) is a right homotopy inverse of \( g \). A two-sided homotopy inverse of \( f \) is a map \( g \) which is both a left and right homotopy inverse of \( f \).

Therefore the spaces \( A \) and \( B \) are the same homotopy type if and only if there is a map \( f \) from \( A \) into \( B \) that has a two-sided homotopy inverse.
Theorem 3.10 Suppose $A$ and $B$ are topological spaces. If a map $f : A \to B$ has both a right and a left homotopy inverse, then $f$ has a two-sided homotopy inverse.\textsuperscript{12}

Proof: Let $L$ and $R$ be a left and a right homotopy inverse of $f$, respectively. Define $g = LfR$. Then $fg = fLfR$. Since $L$ is a left homotopy inverse of $f$, $Lf \simeq \text{id}_A$. So $fg \simeq f\text{id}_A R = fR$. Since $R$ is a right homotopy inverse of $f$, $fg \simeq fR \simeq \text{id}_B$. Thus $fg \simeq \text{id}_B$, and $g$ is a right homotopy inverse of $f$.

Similarly $gf = LfRf \simeq L\text{id}_B f = Lf \simeq \text{id}_A$. Thus $g$ is also a right homotopy inverse of $f$, and therefore $g$ is a two-sided homotopy inverse of $f$. \hfill \Box

\textsuperscript{12}Ralph H. Fox, "On Homotopy Type and Deformation Retracts," *Annals of Mathematics* 44 (January 1943):43. The author credits M. M. Day with the idea for this proof.
CHAPTER IV

THE MAPPING CYLINDER

Let $M$ and $N$ be topological spaces and $\alpha : M \to N$ be any map. Without loss of generality one can assume that $M \cap N$ is empty. Let $M \times [0,1]$, be a product space with the product topology.

**Definition 4.1:** The **cylinder**, $C_{\alpha}$ is the set:

$$\{(m,t) \mid m \in M, 0 \leq t < 1\} \cup \{(\alpha^{-1}(\alpha(m)) \times \{1\}) \cup \{\alpha(m)\} \mid m \in M\}.$$ 

Notice that $C_{\alpha}$ is a partition of $(M \times [0,1]) \cup \alpha(M)$ where for every $t < 1$ the equivalence classes are all singleton sets, $\{(m,t)\}$. When $t = 1$ for each $m \in M$ the equivalence class is the set $\alpha^{-1}(\alpha(m)) \times \{1\}$ with the singleton set $\{\alpha(m)\}$ attached. Let $N' = \{\{n\} \mid n \in N - \alpha(M)\}$. Give $M \times [0,1] \cup N$ the disjoint union topology.

Define a map, $i : (M \times [0,1]) \cup N) \to (C_{\alpha} \cup N')$ by:

$$i(m,t) = \begin{cases} 
\{(m,t)\} & \text{if } 0 \leq t < 1 \\
(\alpha^{-1}(\alpha(m)) \times \{1\}) \cup \{\alpha(m)\} & \text{if } t = 1
\end{cases}$$

and

$$i(n) = \begin{cases} 
\{n\} & \text{if } n \in N - \alpha(M) \\
(\alpha^{-1}(n) \times \{1\}) \cup \{n\} & \text{if } n \in \alpha(M)
\end{cases}$$

Notice that if $\alpha(m) = n$ then $i(m,1) = i(n)$. Thus $i$ is
well-defined and onto the set \( C_a \cup N' \); so \( i^{-1}(C_a \cup N') \) is a partition of \((M \times [0,1]) \cup N\). Give \( C_a \cup N' \) the quotient topology, \( Q_i \), induced by \( i \). Then \( i \) is a quotient map, and \( A \subseteq (C_a \cup N') \) is a member of \( Q_i \) if and only if \( i^{-1}(A) \cap (M \times [0,1]) \) and \( i^{-1}(A) \cap N \) are both open because \( A \times [0,1] \cup N \) has the disjoint union topology.

**Definition 4.2:** Let \( N + C_a \) symbolize the quotient space, \((C_a \cup N', Q_i)\) and call it the **mapping cylinder of \( a \)**.

![Diagram](https://via.placeholder.com/150)

**FIGURE 5**

The symbol "\(<m, t>\)" will denote the equivalence class \( i(m, t) \) for \( m \in M \) and \( 0 \leq t \leq 1 \). So the equivalence class \( i(n) = <n> \) for \( n \in N \). Notice that if \( a(m) = n \) then \( <m, 1> = \)
Thus \( N + C_\alpha = \{ <m, t> | m \in M, 0 \leq t \leq 1 \} \cup \{ <n> | n \in N \} \). Let \( \hat{N} \) denote the set \( \{ <n> | n \in N \} = \{ (a^{-1}(a(m))) \cup \{ a(m) \} | m \in M \} \cup N' \). Thus \( \hat{N} \subseteq N + C_\alpha \), and \( \hat{N} \) is the "top" of the mapping cylinder. Let \( M_0 \) denote the set \( \{ <m, 0> | m \in M \} \). So \( M_0 \subseteq N + C_\alpha \), and \( M_0 \) is the "base" of the mapping cylinder. For each \( t \in [0,1] \), the notation "\(<M \times \{ t \}>" \) will denote the set \( \{ <m, t> | m \in M \} \). Hereafter, the symbol "\( t \)" will be reserved for the elements in the interval \([0,1]\) used to define a mapping cylinder and some other letter, \( s \) or \( u \), will be used for the elements of other images of the unit interval. The set \( C_\alpha \), a subset of \( N + C_\alpha \), can be described several ways: \( C_\alpha = \{ \{(m, t)\} | m \in M, 0 \leq t < 1 \} \cup \{(a^{-1}(a(m)) \times \{1\}) \cup \{ a(m) \} | m \in M \} \) 
\[ = \{ <m, t> | m \in M, 0 \leq t < 1 \} \cup \{ <n> | n \in a(M) \} \]
\[ = \{ <m, t> | m \in M, 0 \leq t \leq 1 \}. \]

Notice that \( i|_{(M \times [0,1])} \cup a(M) \) is the projection function of the decomposition of \( (M \times [0,1]) \cup a(M) \) since \( i(a(m)) = <a(m)> \) and \( i(m, t) = <m, t> \) for all \( m \in M \) and \( 0 \leq t \leq 1 \). However, \( i|_{M \times [0,1]} \) is not the projection function of the decomposition of \( M \times [0,1] \) since \( <m, 1> \) contains the element \( \{ a(m) \} \in N \) as well as the set \( a^{-1}(a(m)) \subseteq M \times [0,1] \).

**Lemma 4.3**: Each map \( i|_{M \times \{ t \}} \) for \( 0 \leq t < 1 \) is a homeomorphism from \( M \times \{ t \} \) onto \( <M \times \{ t \}> \).

**Proof**: Let \( t \in [0,1] \). The image of \( i|_{M \times \{ t \}} \) is the
set \(<M \times \{t\}> = \{(m, t) \mid m \in M\} = \{(m, t)\mid m \in M\}. By the definition of \(i\), \(i\mid_{M \times \{t\}}(m, t) = (m, t)\). Thus \(i\mid_{M \times \{t\}}\) is one-to-one and onto.

Let \(U\) be open in \(<M \times \{t\}>\). Then \(U^c\), the complement of \(U\) in \(<M \times \{t\}>\), is closed. Thus there is a closed set \(V\) in \(C_\alpha\) such that \(U^c = V \cap <M \times \{t\}>\). Then \(V^c\) is open in \(C_\alpha\), and \(V^c \cap <M \times \{t\}> = U\). So \(i^{-1}(U) = i^{-1}(V^c \cap <M \times \{t\}> = i^{-1}(V^c) \cap i^{-1}(<M \times \{t\}>). But \(i^{-1}(<M \times \{t\}>) = M \times \{t\}. Since \(i\) is continuous \(i^{-1}(V^c)\) is open in \(M \times [0,1]\). Thus \(i^{-1}(U) = i^{-1}(V^c) \cap (M \times \{t\})\) is open in \(M \times \{t\}\). Since \(U \subseteq <M \times \{t\}>\), \(i^{-1}(U) = i^{-1}|_{M \times \{t\}}(U)\). Since \(i\) is one-to-one, \(i^{-1}|_{<M \times \{t\}> = i^{-1}|_{M \times \{t\}}^{-1}\); so \(i^{-1}|_{M \times \{t\}}^{-1}(U) = i^{-1}(U)\) which is open in \(M \times \{t\}\). Thus \(i^{-1}|_{M \times \{t\}}\) is continuous.

Now let \(U\) be open in \(M \times \{t\}\). Then \(U^c\) is closed. Then there is a closed set \(V\) in \(M \times [0,1]\) such that \(V \cap M \times \{t\} = U^c\), and \(U = V^c \cap M \times \{t\}\). Since \(i|_{M \times [0,1]}\) is the projection map for \(C_\alpha\), \(i|_{M \times [0,1]}(U) = a(M)^{-1}(V^c)\) is open in \(C_\alpha\). Since \(V^c \subseteq M \times [0,1]\), \(i|_{M \times [0,1]}(V^c) = i|_{M \times [0,1]}(V^c)\). So \(i|_{M \times [0,1]}(U) = i|_{M \times [0,1]}(V^c) \cap \{t\}\) which is open in \(<M \times \{t\}>\). Since \(U \subseteq M \times \{t\}\), \(i|_{M \times [0,1]}(U) = i|_{M \times \{t\}}(U)\). So \(i|_{M \times \{t\}}(U)\) is open. Thus \(i|_{M \times \{t\}}\) is an open map. Therefore \(i|_{M \times \{t\}}\) is a homeomorphism. Q.E.D.

The set \(C_{\alpha,N} = \{(m, t) \mid m \in M, 0 \leq t < 1\}\) is also a
subset of \( N + C_\alpha \). Observe that since \( i(m, t) = <m, t> \) for every \( m \in M \) and \( t \in [0,1) \), then \( i|_M \times [0,1) \) is continuous, one-to-one, and onto. If \( U \) is an open set in \( M \times [0,1) \), then \( i|_M \times [0,1)(U) = \{<m, t>| (m, t) \in U\} \), and

\[
i^{-1}|_M \times [0,1)(\{<m, t>| (m, t) \in U\}) = U.
\]

So \( i|_M \times [0,1) \) is an open function. Therefore \( i|_M \times [0,1) \) is a homeomorphism from \( M \times [0,1) \) onto \( C_\alpha - \hat{N} \).

Notice that \( C_\alpha - \hat{N} \) is the complement of \( \hat{N} \) in \( N + C_\alpha \). Now

\[
i^{-1}(C_\alpha - \hat{N}) = \{(m, t)| n \in M, 0 < t < 1\} = M \times [0,1) \text{ which is open in } M \times [0,1) \cup N.
\]

Therefore \( C_\alpha - \hat{N} \) is open and \( \hat{N} \) is closed in \( N + C_\alpha \).

Define a map, \( h: X \to X \times \{0\} \) for any space \( X \) by \( h(x) = (x,0) \forall x \in X \) where \( X \times \{0\} \) is contained in the product space, \( X \times [0,1] \) and has the subspace topology. Thus \( h \) is a homeomorphism between \( X \) and \( X \times \{0\} \). Then ,for example, \( i|_{M_0} h \) is a homeomorphism between \( M \) and \( M_0 \).

Lemma 4.4: The map \( i|_N \) is a homeomorphism from \( N \) onto \( \hat{N} \).

Proof: The map \( i|_N \) maps \( N \) into \( \hat{N} \) by:

\[
i(n) = \begin{cases} <n> & \text{if } n \in N - a(M) \\ <a^{-1}(n) \times \{1\} \cup \{n\}> & \text{if } n \in a(M) \end{cases}
\]

Since each \( n \) is mapped onto a singleton set containing itself unioned with \( a^{-1}(n) \) if \( n \in a(M) \), then \( i|_N \) is
one-to-one and onto its image.

Let $U$ be an open subset of $\mathbb{N}$. Since $i$ is continuous, $i^{-1}(U)$ is open in $\mathbb{N} \cup (M \times [0,1])$. Since $\mathbb{N}$ and $M$ are disjoint, $i^{-1}(U) \cap \mathbb{N}$ is open in $\mathbb{N}$. But $i^{-1}(U) \cap \mathbb{N} = i|_{\mathbb{N}}^{-1}(U)$; so $i|_{\mathbb{N}}$ is continuous.

Now suppose $U$ is an open set in $\mathbb{N}$. Then $i|_{\mathbb{N}}(U) = \{<n> | n \in U\}$. $i^{-1}(\{<n> | n \in U\}) = \{a^{-1}(a(m) \times \{1\} | a(m) \in U\} U \{n | n \in U\} = (a^{-1}(U) \times \{1\}) U U$. Since $a$ is continuous, $a^{-1}(U) \times [0,1]$ is open in $M \times [0,1]$; so $a^{-1}(U) \times \{1\}$ is open in $M \times \{1\}$. Thus $i|_{\mathbb{N}}(U)$ is open in the image of $i|_{\mathbb{N}}$.
Therefore $i|_{\mathbb{N}}$ is an open function.

Therefore $i|_{\mathbb{N}}$ is a homeomorphism. \(\Box\)

**Theorem 4.5:** Let $f : X \to Y$ and $\theta : X \to Z$ be maps defined for topological spaces $X, Y,$ and $Z$. Then there is a map $g : Y \to Z$ such that $\theta \simeq gf$ if and only if $\theta h^{-1} i^{-1}|_{X_0}$ can be extended to $Y + C_f$. 

**Proof:** Suppose \( \theta^* \) is an extension of \( \theta h^{-1} i^{-1} |_{X_0} \) to \( Y + C_f \). Let \( g = \theta^* i |_{Y} \). Define \( \xi : X \times [0,1] \rightarrow Z \) by \( \xi(x,t) = \theta^*(\iota(x,t)) \) for \( 0 \leq t \leq 1 \). Since \( \xi \) is the composition of continuous functions, it is continuous. \( \xi(x,0) = \theta^*(\iota(x,0)) = \theta^*(<x,0>) = \theta h^{-1} i^{-1} |_{X_0} (<x,0>) = \theta(x) \) and \( \xi(x,1) = \theta^*(i(x,1)) = \theta^*(<x,1>) = \theta^*(<f(x)>) = gf(x) \). Therefore \( \theta \simeq gf \) by the homotopy, \( \xi \).

Conversely, suppose there is a map \( g : Y \rightarrow Z \) such that \( \theta \simeq gf \) by the homotopy \( \xi \). Then \( \xi : X \times [0,1] \rightarrow Z \) such that \( \xi(x,0) = \theta(x) \) and \( \xi(x,1) = gf(x) \). Define \( \theta^* : Y + C_f \rightarrow Z \) by
\[ \theta^*(\langle x,t \rangle) = \xi_t(x) \text{ for } \langle x,t \rangle \in C_f, \text{ and } \theta^*(\langle y \rangle) = g(y) \text{ for } \langle y \rangle \in Y. \]

Notice that if \( y = f(x) \), then \( \theta^*(\langle y \rangle) = g(y) = g(f(x)) = \xi(x,1) = \theta^*(\langle x,1 \rangle) \). Therefore \( \theta^* \) is well defined.

Let \( V \) be an open subset of \( Z \). The set \( i^{-1}\theta^*^{-1}(V) = \{(x,t) \mid \xi(x,t) \in V\} \cup \{y \mid g(y) \in V\} = \xi^{-1}(V) \cup g^{-1}(V) \).

Since both \( \xi \) and \( g \) are continuous, \( i^{-1}\theta^*^{-1}(V) \) is open in \( Y \cup (X \times [0,1]) \). So by the definition of the topology on \( Y + C_f \), \( \theta^*^{-1}(V) \) is open in \( Y + C_f \). Therefore \( \theta^* \) is continuous.

Now \( \theta^*|_{X_0}(\langle x,0 \rangle) = \xi(x,0) = \theta(x) = \theta h^{-1} i^{-1}|_{X_0}(\langle x,0 \rangle) \).

Therefore \( \theta^* \) is an extension of \( \theta h^{-1} i^{-1}|_{X_0} \) from \( X_0 \) to \( Y + C_f \).

**Theorem 4.6**: Let \( X, Y, \) and \( Z \) be spaces and \( \theta : X \to Z \) as in the previous theorem. Suppose \( g \) is a map from \( Y \) into \( Z \).

Then there is a map \( f : X \to Y \) such that \( g f \simeq \theta \) if and only if \( i\theta \) is homotopic in \( Z + C_g \) to a map, \( (i\theta') \), from \( X \) into \( Y_0 \). Note that \( \theta' \) must be a map from \( X \) into \( Y \).

**Proof**: Suppose \( i\theta \simeq i\theta' \) in \( Z + C_g \) by \( \zeta : X \times [0,1] \to Z + C_g \) such that \( \zeta(x,0) = i\theta(x) \in Z \subseteq Z + C_g \) and \( \zeta(x,1) = i\theta'(x) \in Y_0 \subseteq Z + C_g \). Define a map \( \omega : Y_0 \times [0,1] \to Z + C_g \) by \( \omega(\langle y,0 \rangle,s) = \langle y,s \rangle \) for \( 0 \leq s \leq 1 \).

So \( \omega(\langle y,0 \rangle,0) = \langle y,0 \rangle \) or \( \omega_0 = \text{id}_{Y_0} \), and \( \omega(\langle y,0 \rangle,1) = \langle y,1 \rangle \).

Now each \( \langle y,1 \rangle \in \hat{Z} \) and for each \( y \in Y \), \( g(y) \in \langle y,1 \rangle \); so \( \langle y,1 \rangle = \langle g(y) \rangle \). Thus \( \omega_1 = ig \). Therefore \( \omega \) is a homotopy
between $\text{id}_{Y_0}$ and $ig$ in $Z + C_g$. Since $\omega_0 = \text{id}_{Y_0}$, then

$\omega_0 i \theta' = i \theta'$. Consider $\omega_1 i \theta' : X \to Z + C_g$. Notice that

$\omega_1 i \theta'(x) = \omega_1(<\theta'(x),0>)$ since $\theta'(x) \in Y$. And $\omega_1(<\theta'(x),0>) = ig\theta'(x)$. Thus $\omega_1 i \theta' = ig\theta'$. Also for $0 \leq s \leq 1$

$\omega_s(i \theta'(x)) = \omega_s(<\theta'(x),0>) = <\theta'(x),s>$. Define $F : X \times [0,1] \to Z + C_g$ by $F(x,s) = \omega_s i \theta'(x)$. Then $F(x,0) = \omega_0 i \theta'(x) = i \theta'(x) \in Y_0$, and $F(x,1) = \omega_1 i \theta'(x) = ig\theta'(x) \in \hat{Z}$. So $F$ is a homotopy between $i \theta'$ and $ig\theta'$ in $Z + C_g$.

Claim: The map $i \theta$ is homotopic to $ig\theta'$.

Define $\xi : X \times [0,1] \to Z + C_g$ by:

$$\xi(x,s) = \begin{cases} 
\zeta(x,2s) & \text{for } 0 \leq s \leq 1/2 \\
F(x,2s-1) & \text{for } 1/2 \leq s \leq 1
\end{cases}$$

Then $\xi(x,0) = \zeta(x,0) = i \theta(x)$, and $\xi(x,1) = F(x,1) = \omega_1 i \theta'(x) = ig\theta'(x)$. Since $\xi(x,1/2) = \zeta(x,1) = i \theta'(x)$ and also

$\zeta(x,1/2) = F(x,0) = \omega_0 i \theta'(x) = i \theta' x)$, $\xi$ is continuous and is a homotopy between $i \theta$ and $ig\theta'$ in $Z + C_g$.

Claim: The map $\theta$ is homotopic to $g\theta'$.

Define $\delta : Z + C_g \to Z$ by $\delta(<z>) = z$ and $\delta(<y,t>) = g(y)$

Let $U$ be open in $Z$. The set $i^{-1} \delta^{-1}(U)$

$= i^{-1}(<z> | z \in U) \cup <y,t> | y \in g^{-1}(U), 0 \leq t \leq 1)\)

$= \{z | z \in U\} \cup \{(y,1) | y \in g^{-1}(U)\} \cup \{(y,t) | y \in g^{-1}(U), 0 \leq t < 1\}$

$= U \cup \{(y,t) | y \in g^{-1}(U), 0 \leq t \leq 1\} = U \cup (g^{-1}(U) \times [0,1])$.\)
Since $g$ is continuous, $g^{-1}(U)$ is open, and thus $\delta^{-1}(U)$ is the union of open sets in $Z + C_g$. Therefore $\delta$ is continuous.

Then the composition $\delta \xi : X \times [0,1] \to Z$ is continuous. Also $\delta \xi(x,0) = \delta(i\theta(x)) = \delta<\theta(x)> = \theta(x)$, and $\delta \xi(x,1) = \delta(ig\theta'(x)) = \delta(<g\theta'(x)>)) = g\theta'(x)$. So $\delta \xi$ is a homotopy between $\theta$ and $g\theta'$. Let $f = \theta'$. Therefore $\theta \simeq gf$.

Conversely, suppose there is a map $f : X \to Y$ such that $gf \simeq \theta$ by some homotopy. Reverse that homotopy to define $\xi : X \times [0,1] \to Z$ such that $\xi(x,0) = \theta(x)$ and $\xi(x,1) = gf(x)$

Define $\omega : Y_0 \times [0,1] \to Z + C_g$ by $\omega(<y,0>,s) = <y,s>$. So $\omega_0 = \text{id}_{Y_0}$ and $\omega_1 = ig$. Now the map, $i\xi : X \times [0,1] \to Z + C_g$ is continuous. Also, $i\xi(x,0) = i(\theta(x)) = <\theta(x)> \in Z$, and $i\xi(x,1) = igf(x) = <gf(x)>$. Recall the map, $h : Y \to Y \times [0,1]$ such that $h(y) = (y,0)$ is a homeomorphism. Thus $igf = \omega_1ihf$. Since $\omega : Y_0 \times [0,1] \to Z + C_g$ and $ih(f(x)) \in Y_0$, then $\omega_0ihf \simeq \omega_1ihf$. But $\omega_0ihf = \text{id}_{Y_0}ihf = ihf : X \to Y_0$. Define $\lambda : X \times [0,1] \to Z + C_g$ by:

$$
\lambda(x,s) = \begin{cases} 
    i\xi(x,2s) & \text{for } 0 \leq s \leq 1/2 \\
    \omega(ihf(x),2-2s) & \text{for } 1/2 \leq s \leq 1 
\end{cases}
$$

Then $\lambda(x,0) = i\xi(x,0) = i\theta(x)$, and $\lambda(x,1) = \omega(ihf(x),0) = ihf(x)$. When $s = 1/2$, $\lambda(x,1/2) = i\xi(x,1) = igf(x)$, and $\lambda(x,1/2) = \omega(ihf(x),1) = igf(x)$. Therefore $i\theta \simeq ihf$ by the
homotopy $\lambda$, and $ihf : X \to Y_0$. 

If the space $Z$ in theorem 4.5 equals the space $X$, and $\theta$ is the identity on $X$ then the theorem becomes:

**Theorem 4.5a:** If $f : X \to Y$, then there is a map $g : Y \to X$ such that $gf \simeq id_X$ if and only if $id_X h^{-1} i^{-1} |_{X_0}$ can be extended to $Y + C_f$.

This gives a necessary and sufficient condition for $X$ to be the homeomorphic image of a retract of the mapping cylinder.

**Corollary 4.7:** Suppose $f$ is a map from a space $X$ into a space $Y$. The set $X_0$ is a retract of $Y + C_f$ if and only if the map $f$ has a left homotopy inverse

**Proof:** Suppose $X_0$ is a retract of $Y + C_f$. Then there is a map $r : Y + C_f \to X_0$ such that $r |_{X_0} = id_{X_0}$. Thus $r$ is an extension of $id_{X_0}$ from $X_0$ to $Y + C_f$, and $h^{-1} i^{-1} |_{X_0} r |_{X_0} = h^{-1} i^{-1} |_{X_0} id_{X_0} = h^{-1} i^{-1} |_{X_0} = id_X h^{-1} i^{-1} |_{X_0}$. So $h^{-1} i^{-1} |_{X_0} r$ is an extension of $id_X h^{-1} i^{-1} |_{X_0}$ to $Y + C_f$. Then by theorem 4.5a there is a map $g : Y \to X$ such that $gf \simeq id_X$, and $g$ is a left homotopy inverse of $f$.

Conversely, suppose $f$ has a left homotopy inverse, say $g$. So $g : Y \to X$ such that $gf \simeq id_X$. Then by theorem 4.5a $id_X h^{-1} i^{-1} |_{X_0}$ can be extended to $Y + C_f$; call the extension $r$. Then $r : Y + C_f \to X$ where $r |_{X_0} = id_X h^{-1} i^{-1} |_{X_0}$. Thus $r$ is a retraction and $X_0$ is a retract of $Y + C_f$. 

**Lemma 4.8:** Any mapping cylinder can be deformed into its top.
Proof: Suppose \( f \) maps \( X \) into \( Y \). Define a map
\[
\tau : Y + C_f \times [0,1] \to Y + C_f
\]
by \( \tau(x,t,s) = \)
\[
i(x,t + s(1-t)) \text{ for all } x,t \in C_f \text{ and } \tau(y,s) = y \text{ for all } y \in \hat{Y}.
\]
If \( f(x) = y \) then \( \tau(f(x)) = \tau(x,1,s) = i(x,1) = \langle x,1 \rangle = \langle f(x) \rangle = y = \tau(y,s) \). Thus \( \tau \) is well defined. Also \( \tau(x,t,0) = i(x,t + 0(1-t)) = i(x,t) = \langle x,t \rangle \), and \( \tau(y,0) = y \). So \( \tau_0 = id_{Y + C_f} \). Finally, \( \tau(x,t,1) = i(x,t + 1(1-t)) = i(x,1) \in \hat{Y} \), and \( \tau(y,1) = y \). Thus \( \tau_1(Y + C_f) = \hat{Y} \).

It remains to show that \( \tau \) is continuous.

Let \( U \) be open in \( Y + C_f \).

Claim: Every point in \( \tau^{-1}(U) \) is an interior point.

Let \( \langle x_0,t_0,s_0 \rangle \in \tau^{-1}(U) \). Then \( \tau(x_0,t_0,s_0) = \langle x_0,t_0 + s_0(1-t_0) = \langle x_0,t' \rangle, \text{ where } t' = t_0 + s_0(1-t_0) \).

Case 1: Suppose \( t_0 < 1 \) and \( s_0 < 1 \).

Claim: The point \( t' \in [0,1] \) is less than 1.

Just suppose \( t' = 1 = t_0 + s_0(1-t_0) \). If \( t_0 \neq 0 \), then \( s_0 = \frac{1-t_0}{1-t_0} = 1 \). But this is not the case. If \( t_0 = 0 \), then \( 1 = 0 + s_0(1-0) \text{ and } s_0 = 1 \). Again, this contradicts the hypothesis. Thus \( t' \neq 1 \); so \( t' < 1 \).
Let $V \subseteq U$ such that $V = i(G \times (p_1, p_2))$ where $G$ is a basic open neighborhood of $x_0$ in $X$, and $(p_1, p_2)$ is a basic open neighborhood of $t'$ such that $p_2 < 1$. Since $i|_{X \times [0,1]}$ is a homeomorphism and $C_f - Y$ is open in $Y + C_f$, then $V$ is an open neighborhood of $\langle x_0, t' \rangle$ in $Y + C_f$. 
Now \( \tau^{-1}(G \times \{p_2\}) = \{(x,t),s) \mid x \in G, t+s(1-t) = p_2 \} \). Choose \( t_2 = \frac{t_0 + p_2}{2} \). Thus \( t_0 < t_2 < p_2 < 1 \). Since \( V \) is open in \( Y + C_f \), there is a \( p' \) such that \( t' < p' < p_2 \) and \( \langle x_0, p' \rangle \in U \). So choose \( s_2 \) such that \( s_0 < s_2 \) and \( \tau(\langle x_0, t_2 \rangle, s_2) = \langle x_0, p' \rangle \in V \). Also \( \tau^{-1}(G \times \{p_1\}) = \{(x,t),s) \mid x \in G, t+s(1-t) = p_1 \} \) so choose \( s_1 < s_0 \) and \( t_1 < t_0 \) such that \( \tau(\langle x_0, t_1 \rangle, s_1) = \langle x, p' \rangle \) where \( p_1 < p' < t' \).

Let \( W = \{(x,t),s) \mid x \in G, s_1 < s < s_2, t_1 < t < t_2 \} \). Then \( W = \langle x,t \rangle \mid x \in G, t_1 < t < t_2 \} \times (s_1, s_2) \), or \( i(G \times (t_1, t_2)) \times (s_1, s_2) \). Thus \( W \) is open in \( Y + C_f \times [0,1] \), and contains \( \langle x_0, t_0 \rangle, s_0 \).

Let \( \langle x,t,s) \rangle \) be a member of \( W \). Then \( \tau(\langle x,t,s) \rangle) = \langle x, t+s(1-t) \rangle \). But \( x \in G \) and \( t_1+s_1(1-t_1) < t+s(1-t) < t_2+s_2(1-t_2) \); so \( \tau(\langle x,t,s) \rangle \in V \subseteq U \). Therefore \( W \subseteq \tau^{-1}(U) \).

**Case 2:** Suppose \( t_0 = 1 \). Then \( \tau(\langle x_0, 1 \rangle, s) = \langle x_0, 1 \rangle \) for every \( s \in [0,1] \). Then let \( V \subseteq Y + C_f \) such that \( V = i(G \times (p_1,1)) \) where \( G \) is a basic open neighborhood of \( x_0 \) in \( X \), and \( (p_1,1) \) is a basic open neighborhood of \( 1 \). Then \( V \) is an open neighborhood of \( \langle x_0, 1 \rangle \) in \( Y + C_f \). By the same argument as above every point in \( \tau^{-1}(U) \) is an interior point.

**Case 3:** Suppose \( s_0 = 1 \). Then \( t' = t + 1(1 - t) = 1 \) for every \( t \in [0,1] \). Then similarly every point in \( \tau^{-1}(U) \) is an interior point.
Thus $\tau^{-1}(U)$ is open, and $\tau$ is continuous.

Therefore $\tau$ is a deformation of $Y + C_f$ into $\hat{Y}$. \hfill \Box

Hereafter the symbol "$\tau$" will be used to denote this deformation in any mapping cylinder.

When the space $X$ is the space $Z$ and the map $\theta$ is the identity on $Y$, theorem 4.6 becomes:

**Theorem 4.6a:** If $g$ is a map from $Y$ into $Z$, then there is a map $f : Z \rightarrow Y$ such that $gf \simeq id_Z$ if and only if $\theta id_Z$ is homotopic in $Z + C_f$ to a map from $Z$ into $Y_0$.

Using this theorem the following corollary yields a necessary and sufficient condition for the mapping cylinder to be deformable into its base.

**Corollary 4.9:** Let $Y$ and $Z$ be spaces and $g : Y \rightarrow Z$ be any map. Then $Z + C_g$ can be deformed into $Y_0$ if and only if the map $g$ has a right homotopy inverse.

**Proof:** Suppose $Z + C_g$ can be deformed into $Y_0$. Then there is a deformation $\delta : Z + C_g \times [0,1] \rightarrow Z + C_g$ such that $\delta(<y,t>,0) = <y,t>$, and $\delta(<z>,0) = <z>$. Thus $\delta_0 = id_{Z + C_g}$.

Also $\delta(Z + C_g \times \{1\}) = Y_0$. For every $s \in [0,1]$, the map $\delta_s i|_Z$ is the composition of continuous maps, and $\delta_s$ maps $i|_Z(Z)$, a subset of $Z + C_g$, into $Z + C_g$. Thus

$\delta|_{\hat{Z}} : i|_Z(Z) \times [0,1] \rightarrow Z + C_g$ such that $\delta|_{\hat{Z}}(i|_Z(Z) \times \{0\}) = \delta|_{\hat{Z}}(Z \times \{0\}) = id_{\hat{Z}}$, and $\delta|_{\hat{Z}}(i|_Z(Z) \times \{1\}) = \delta|_{\hat{Z}}(\hat{Z} \times \{1\}) \subseteq Y_0$. Thus $\delta|_{\hat{Z}}$ is a homotopy between the map $id_{\hat{Z}} = i|_Z id_Z =$
\( \text{id}_Z \) and the map \( \delta | \hat{Z} \cdot i | Z \) which maps \( Z \) into \( Y_0 \). Therefore by theorem 4.6a, there is a map \( f : Z \rightarrow Y \) such that \( gf \simeq \text{id}_Z \); so \( g \) has a right homotopy inverse.

Conversely, suppose \( g \) has a right homotopy inverse, say \( f \). Then by theorem 4.6a, \( \text{id}_Z \) is homotopic to a map, say \( \iota \theta' \), which maps \( Z \) into \( Y_0 \). Call this homotopy \( \zeta \). Thus \( \zeta : \iota(Z) \times [0,1] \rightarrow Z + C_g \) such that \( \zeta(\iota(Z) \times \{0\}) = \text{id}_Z(Z) = \hat{Z} \), and \( \zeta(\iota(Z) \times \{1\}) = \iota \theta'(Z) = Y_0 \). By lemma 4.8, \( Z + C_g \) can be deformed into \( \hat{Z} \) by the deformation \( \tau \) where

\[
\tau_0 = \text{id}_Z + C_g \quad \text{and} \quad \tau_1(Z + C_g) = \hat{Z}.
\]

Define \( \delta : Z + C_g \times [0,1] \rightarrow Z + C_g \) by:

\[
\delta(y,t,s) = \begin{cases} 
\tau(y,t),2s & \text{for } 0 \leq s \leq 1/2 \\
\zeta(\tau_1(y,t),2s - 1) & \text{for } 1/2 \leq s \leq 1 
\end{cases}
\]

and

\[
\delta(z) = \begin{cases} 
\tau(z),2s & \text{for } 0 \leq s \leq 1/2 \\
\zeta(\tau_1(z),2s - 1) & \text{for } 1/2 \leq s \leq 1 
\end{cases}
\]

Notice that \( \delta(y,t),1/2) = \tau(y,t),1) = <y,1> \), and \( \delta(y,t),1/2) = \zeta(\tau_1(y,t)),2(1/2) - 1) = \zeta(\tau_1(y,t)),0) = <y,1> \). Also \( \delta(z),1/2) = \tau(z),1) = <z> \), and \( \delta(z),1/2) = \zeta(\tau_1(z),0) = \zeta(<z>,0) = <z> \). Clearly, when \( g(y) = z \) then \( \delta(z),s) = \delta(g(y),s) = \delta(y,1),0 \). So \( \delta \) is well defined. Then \( \delta(y,t),0) = \tau(y,t),0) = \tau_0(y,t) = <y,t> \), and \( \delta(z),0) = \tau(z),0) = <z> \). In other words, \( \delta_0 =
id_Z + C_g. Also \( \delta(<y,t>,1) = \zeta(\tau_1(<y,t>),2(1) - 1) = \zeta(<y,1>,1) \) by the definition of \( \tau \). So \( \delta(<y,t>,1) = \zeta(<g(y),1>1 = \iota(\theta'(g(y))) \in Y_0. \)

Finally, \( \delta(<z>,1) = \zeta(\tau_1(<z>),2(1) - 1) = \zeta(<z>,1) \) again, by the definition of \( \tau \). So \( \delta(<z>,1) = \zeta(<z>,1) = \zeta(i(z),1) = i\theta'(z) \in Y_0. \) Thus \( \delta_1(Z + C_g) \subseteq Y_0. \)

Therefore \( \delta \) is a deformation of \( Z + C_g \) into \( Y_0. \) \( \square \)

Theorem 4.10: There is a map \( f : X \to Y \) such that \( X_0 \) is a retract of \( Y + C_f \) if and only if there is a map \( g : Y \to X \) such that \( X + C_g \) can be deformed into \( Y_0. \)

Proof: Suppose there is a map \( f : X \to Y \) such that \( X_0 \) is a retract of \( Y + C_f \). By corollary 4.7, \( f \) has a left homotopy inverse, say \( g \). Then \( g : Y \to X \) such that \( gf \simeq \text{id}_X \).

Thus \( g \) has a right homotopy inverse. Then by corollary 4.9 \( X + C_g \) can be deformed into \( Y_0. \)

Conversely, suppose there is a map \( g : Y \to X \) such that \( X + C_g \) can be deformed into \( Y_0. \) By corollary 4.9, \( g \) has a right homotopy inverse, say \( f \). Then \( f : X \to Y \) such that \( gf \simeq \text{id}_X. \) Thus \( f \) has a left homotopy inverse. Then by corollary 4.7, \( X_0 \) is a retract of \( Y + C_f. \) \( \square \)

Theorem 4.11: Let \( X, Y, \) and \( Z \) be topological spaces and \( \theta \) be a map from \( X \) into \( Z \). There is a map \( f : X \to Y \) such that the map \( \theta h^{-1} i|^{-1}|_{X_0} \) which maps \( X_0 \) into \( Z \) can be extended to \( Y + C_f \), the mapping cylinder of \( f \), if and only if there

\[ \text{id}_Z + C_g. \]
is a map $g : Y \to Z$ such that $i\theta$ is homotopic in $Z + C_g$, the mapping cylinder of $g$, to a map from $X$ into $Y_0$.

Recall that $X_0$ is the base of the mapping cylinder of $f$ and $Y_0$ is the base of the mapping cylinder of $g$. The map $i$ is the projection map for a mapping cylinder. Which cylinder should be clear from the context. The map $h$ is the homeomorphism from any space $M$ to $M \times \{0\}$. Its particular domain and range should also be clear from the context.

Proof: Suppose there is a map $f : X \to Y$ such that $\theta h^{-1} i^{-1}|_{X_0}$ can be extended to $Y + C_f$. By theorem 4.5 there is a map $g : Y \to Z$ such that $gf \simeq \theta$. Then by theorem 4.6, $i\theta$ is homotopic in $Z + C_g$ to a map from $X$ to $Y_0$.

Conversely, suppose there is a map $g : Y \to Z$ such that $i\theta$ is homotopic to a map from $X$ to $Y_0$. By theorem 4.6 there is a map $f : X \to Y$ such that $fg \simeq \theta$. Then by theorem 4.5 $\theta h^{-1} i^{-1}|_{X_0}$ can be extended to $Y + C_f$. \qed

**Theorem 4.12**: Suppose $X$ and $Y$ are topological spaces, and $f$ is a map from $X$ into $Y$. Then $X_0$ is a deformation retract of $Y + C_f$ if and only if $f$ has a two-sided homotopy inverse.

Proof: Suppose $X_0$ is a deformation retract of $Y + C_f$. By theorem 3.3, $X_0$ is a retract of $Y + C_f$, and $Y + C_f$ can be deformed into $X_0$. By corollary 4.7, since $X_0$ is a retract of $Y + C_f$, $f$ has a left homotopy inverse. By corollary 4.9, since $Y + C_f$ can be deformed into $X_0$, $f$ has a right homotopy
inverse. By theorem 3.10, since \( f \) has both left and right homotopy inverses, \( f \) has a two-sided inverse.

Suppose, conversely, that \( f \) has a two-sided homotopy inverse. Since \( f \) a left homotopy inverse, by corollary 4.7 \( X_0 \) is a retract of \( Y + C_f \). And, since \( f \) has a right homotopy inverse, by corollary 4.9, \( Y + C_f \) can be deformed into \( X_0 \). Then by theorem 3.3, \( X_0 \) is a deformation retract of \( Y + C_f \). □

**Theorem 4.13:** Two topological spaces belong to the same homotopy type if and only if they both can be embedded in a third space, \( W \), in such a way that their images are both deformation retracts of \( W \).

**Proof:** Let \( X \) and \( Y \) be two topological spaces. Suppose \( X \) and \( Y \) belong to the same homotopy type. As a consequence of the definition of homotopy type, there is a map \( f : X \to Y \) such that \( f \) has a two-sided homotopy inverse. Then by corollaries 4.7 and 4.9, \( X_0 \) is a deformation retract of \( Y + C_f \). Notice that \( \hat{Y} \) is a retract of \( Y + C_f \) by the map \( r : Y + C_f \to \hat{Y} \) where \( r(<x,t>) = <x,1> \) and \( r(<y>) = <y> \). By lemma 4.8, \( Y + C_f \) can be deformed into \( \hat{Y} \). Thus by theorem 3.3, \( \hat{Y} \) is a deformation retract of \( Y + C_f \). Therefore \( X \) and \( Y \) are homeomorphically embedded in \( Y + C_f \) and both are deformation retracts of \( Y + C_f \).

Conversely, suppose \( X \) and \( Y \) are embedded in a space \( W \) such that the homeomorphic images of \( X \) and \( Y \) are both
deformation retracts of \( W \). Then there is a deformation 
\[
\delta : W \times [0,1] \to W \text{ such that } \delta(w,0) = w \text{ for all } w \in W, \text{ and }
\]
\[
\delta(W \times \{1\}) = X_0, \text{ and } \delta_1|_{X_0} = \text{id}_{X_0}. \text{ Notice also that }
\]
\[
\delta_1|_{Y_0} : Y_0 \to X_0. \text{ There is also a deformation }
\]
\[
\lambda : W \times [0,1] \to W \text{ such that } \lambda(w,0) = w \text{ for all } w \in W, \text{ and }
\]
\[
\lambda(W \times \{1\}) = Y_0, \text{ and } \lambda_1|_{Y_0} = \text{id}_{Y_0}. \text{ Similarly, }
\]
\[
\lambda_1|_{X_0} : X_0 \to Y_0.
\]
Define the homotopy, \( \xi : W \times [0,1] \to W \) by
\[
\xi(w,t) = \lambda(\delta(w,t),t). \text{ Then } \xi(w,0) = \lambda(\delta(w,0),0) =
\]
\[
\lambda(w,0) = w. \text{ That is, } \xi_0 = \text{id}_W. \text{ Also, } \xi(w,1) = \lambda(\delta(w,1),1) =
\]
\[
\lambda_1|_{X_0} \delta_1. \text{ Thus } \text{id}_W \simeq \lambda_1|_{X_0} \delta_1 \text{ by the homotopy } \xi. \text{ Therefore }
\]
\[
\text{id}_{Y_0} = \text{id}_W|_{Y_0} \simeq \lambda_1|_{X_0} \delta_1|_{Y_0}.
\]
Similarly, define the homotopy \( \zeta : W \times [0,1] \to W \) by
\[
\zeta(w,t) = \delta(\lambda(w,t),t). \text{ Then } \zeta(w,0) = \delta(\lambda(w,0),0) =
\]
\[
\delta(w,0) = w; \text{ so } \zeta_0 = \text{id}_W. \text{ Also } \zeta(w,1) = \delta(\lambda(w,1),1) =
\]
\[
\delta(\lambda_1(w),1) = \delta_1 \lambda_1(w); \text{ so } \zeta_1 = \delta_1|_{Y_0} \lambda_1. \text{ Thus } \text{id}_W \simeq \delta_1|_{Y_0} \lambda_1
\]
and \( \text{id}_{X_0} \simeq \delta_1|_{Y_0} \lambda_1|_{X_0} \). Therefore the map, \( \delta_1|_{Y_0} \) has a
two-sided homotopy inverse, namely, \( \lambda_1|_{X_0} \). By the
definition \( X_0 \) and \( Y_0 \) are the same homotopy type, and thus
t heir homeomorphic images, \( X \) and \( Y \), are the same homotopy
type. \( \square \)
CHAPTER V

APPLICATIONS

Definition 5.1: A space \( X \) is said to be \textit{inessential} relative to a subset \( B \) if and only if there is a deformation of \( X \) into a proper subset \( D \) such that the points of \( B \) remain fixed during the deformation.

Theorem 5.2: Let \( f : X \rightarrow Y \) such that \( Y + C_f \) is an ANR, and suppose \( f \) has a right homotopy inverse. If \( V \) is a proper subset of \( Y + C_f \) containing \( X_0 \) and \( k : Y \rightarrow V \) such that \( kf h^{-1} i^{-1} \mid_{X_0} \simeq id_V \mid_{X_0} \), then \( Y + C_f \) is inessential relative to \( X_0 \).

Proof: Let \( f : X \rightarrow Y \) be a map such that \( Y + C_f \) is an ANR and \( f \) has a right homotopy inverse.

Suppose \( V \) is a proper subset of \( Y + C_f \) which contains \( X_0 \) and \( k : Y \rightarrow V \) is a map such that \( kf h^{-1} i^{-1} \mid_{X_0} \simeq id_V \mid_{X_0} \).

Notice that by identifying \( V \) with \( Z \) in theorem 4.5 and \( \theta \) with \( \text{id}_{X_0} i h : X \rightarrow V \) then \( k \) can play the role of \( g \). Then theorem 4.5 says: Given spaces \( X, Y, \) and \( V \); the map \( f : X \rightarrow Y \); and the map \( \text{id}_{X_0} i h : X \rightarrow V \); then there is a map \( k : Y \rightarrow V \) such that \( \text{id}_{X_0} i h \simeq kf \) if and only if 
\[
(id_{X_0} i h) h^{-1} i^{-1} \mid_{X_0} = id \mid_{X_0} \text{ can be extended to } Y + C_f.
\]

Since \( i \mid_{X_0} \) and \( h \) are homeomorphisms, \( kf h^{-1} i^{-1} \mid_{X_0} \simeq id_V \mid_{X_0} \) implies that \( kf \simeq id_V \mid_{X_0} i h \). Thus by the above
version of theorem 4.5, \(\text{id}_V|_{X_0} = \text{id}_{X_0} : X_0 \to X_0 \subseteq V\) can be extended to \(Y + C_f\). Let \(\lambda : Y + C_f \to V\) be an extension; so \(\lambda|_{X_0} = \text{id}_{X_0}\). Since \(f\) has a right homotopy inverse, by corollary 4.9, \(Y + C_f\) can be deformed into \(X_0\). By theorem 2.4, \(\lambda\) is homotopic to \(\text{id}_Y + C_f\), and there is a deformation \(\eta\) such that \(\eta_0 = \text{id}_Y + C_f\) and \(\eta_1 = \lambda\). Since \(Y + C_f\) is an ANR, by theorem 3.6 there is a deformation \(\delta\) such that the fixed points of \(\lambda\), namely \(X_0\), remain fixed for every \(\delta_t\). Therefore \(Y + C_f\) is inessential relative to \(X_0\).

**Theorem 5.3:** If \(f\) maps the ANR \(X\) into \(Y = \{y\}\) and \(X_0\) can be contracted in a proper subset of \(Y + C_f\), then \(Y + C_f\) is inessential relative to \(X_0\).

**Proof:** Since \(X_0\) can be contracted in a proper subset of \(Y + C_f\), let \(V \subseteq Y + C_f\) be that set. Thus there is a deformation \(\rho : V \times [0,1] \to V\), and a point \(x_c\) in \(X_0\) such that \(\rho_0 = \text{id}_Y\) and \(\rho_1(X_0) = \{x_c\} \subseteq X_0\). Clearly \(\rho_0|_{X_0} = \text{id}_{X_0}\). Let \(k : \{y\} \to X\) be a map defined by \(k(y) = x_c\). Since \(fk : \{y\} \to \{y\}, fk = \text{id}_Y\), and \(fk \simeq \text{id}_Y\). Thus \(f\) has a right homotopy inverse. Now \(kh^{-1}i^{-1}|_{X_0}\) is a map from \(X_0\) into \(\{x_c\}\) so \(kh^{-1}i^{-1}|_{X_0} = \rho_1|_{X_0}\). Since \(\rho_1|_{X_0} \simeq \rho_0|_{X_0}\), then \(kh^{-1}i^{-1}|_{X_0} \simeq \text{id}_{X_0}\).

It only remains to show that \(Y + C_f\) is an ANR in order to conclude from theorem 5.2 that \(Y + C_f\) is inessential relative to \(X_0\).
Notice that since \( Y = \{y\} \), \( Y + C_f = C_f = \{<x,t>| x \in X, 0 \leq t \leq 1\} \)

**Claim:** The set \( Y + C_f \) is an ANR.

Let \( E \) be a separable, metrizable space which contains \( Y + C_f \) as a closed subset. Define a function \( P_I: Y + C_f \to [0,1] \) by \( P_I(<x,t>) = t \). Let \( V \) be an open subset of \([0,1]\). Then \( P^{-1}_I(V) = \{<x,t>| x \in X, t \in V\} \). Now \( i^{-1}(<x,t>| x \in X, t \in V) \) is open in \( X \times V \) possibly unioned with \( Y \) if \( 1 \in V \). But \( X \times V \) is a subbase element of \( X \times [0,1] \); so \( i^{-1}(<x,t>| x \in X, t \in V) \) is open in \( X \times [0,1] \cup Y \). Thus \( P^{-1}_I(V) \) is open in \( Y + C_f \). Therefore \( P_I \) is continuous.

Since \( E \) is metrizable and therefore normal and \( Y + C_f \) is closed in \( E \), then by Tietze's Extension Theorem\(^{13}\) \( P_I \) can be extended to all of \( E \). Call this extension \( P'_I \). Thus \( P'_I: E \to [0,1] \) such that \( P'_I|Y + C_f = P_I \).

Define a function \( P_X: Y + C_f \to X \) by \( P_X(<x,t>) = x \). Let \( U \) be an open subset of \( X \). The set \( P^{-1}_X(U) = \{<x,t>| x \in U, 0 \leq t \leq 1\} \). The set \( i^{-1}P^{-1}_X(U) = \{(x,t)| x \in U, 0 \leq t \leq 1\} \cup Y = (\{x| x \in U\} \times [0,1]) \cup Y \). Thus \( i^{-1}(P^{-1}_X(U)) \) is open in \( Y + C_f \). Therefore \( P_X \) is continuous.

Since \( X \) is an ANR and \( Y + C_f \) is closed in \( E \), then by

the Borsuk–Kuratowski Theorem\textsuperscript{14} $P_X$ can be extended to an open neighborhood, $\mathcal{W}$, of $Y + C_f$ in $E$. Call the extension $P'_X$.

Thus $P'_X : \mathcal{W} \to Y + C_f$ such that $P'_X|_{Y + C_f} = P_X$.

Define a function $r : \mathcal{W} \to Y + C_f$ by $r(e) = \langle P'_X(e), P'_I(e) \rangle$. Let $U$ be an open subset of $Y + C_f$. Then $r^{-1}(U) = \{ e | P'_X(e) = x, P'_I(e) = t \text{ for some } \langle x, t \rangle \in U \} = \{ e | P'_X(e) = x, 0 \leq t \leq 1 \text{ for some } \langle x, t \rangle \in U \} \cap \{ e | P'_I(e) = t, x \in X \text{ for some } \langle x, t \rangle \in U \} = P_X^{-1}(U) \cap P_I^{-1}(U)$. Since both $P_X$ and $P_I$ are continuous, $r^{-1}(U)$ is the intersection of open sets and $r$ is continuous.

Notice that if $e \in Y + C_f$, then $e$ is some $\langle x, t \rangle$, and $r(e) = r(\langle x, t \rangle) = \langle P'_X(x, t), P'_I(\langle x, t \rangle) \rangle = \langle x, t \rangle$. Thus $r|_{Y + C_f} = \text{id}_{Y + C_f}$. So $r$ is a retraction of $\mathcal{W}$ onto $Y + C_f$, and $Y + C_f$ is an ANR.

Therefore by theorem 5.2, $Y + C_f$ is inessential relative to $X_0$. 

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Fox describes an interesting subclass of retracting deformations, and shows a correspondence with a subclass of two-sided inverses.\textsuperscript{15}

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Definition 5.4: Let $Y + C_f$ be the mapping cylinder of $f : X \to Y$ and $\rho$ be the retraction of $Y + C_f$ into $\tilde{Y}$ defined by $\rho(<x,t>) = <f(x)> = <x,1> \text{ and } \rho(<y>) = <y>$. Then a retracting deformation $\delta$ of $Y + C_f$ into $X_0$ is special if and only if in addition to the usual conditions:

$\delta : Y + C_f \times [0,1] \to Y + C_f$ such that $\delta_0 = \text{id}_{Y + C_f}$,

$\delta_1(Y + C_f) = X_0$, and $\delta_1|_{X_0} = \text{id}_{X_0}$; $\delta$ also satisfies the following two conditions:

i) For every $s \in [0,1]$ $\delta_s|_{X_0} = \text{id}_{X_0}$.

ii) For every $x \in X$ whenever $t = s$

$\rho(\delta(<f(x),s))) = \rho(\delta(<x,t>),1)$

Here $s$ is the deformation parameter, and $t$ is the mapping cylinder parameter.

Definition 5.5: A two-sided inverse, $g$, of the map $f : X \to Y$ is special if and only if there are homotopies $F$ and $G$ such that in addition to the usual properties,

$F : X \times [0,1] \to X$ such that $F_0(x) = x$ and $F_1(x) = gf(x),

G : Y \times [0,1] \to Y$ such that $G_0(y) = y$ and $G_1(y) = fg(y)$, the condition $f(F(x,s)) = G(f(x),s)$ is satisfied.
Theorem 5.6: The map \( f: X \to Y \) has a special two-sided inverse if and only if there is a special retracting deformation of \( Y + C_f \) into \( X_0 \).

Proof: Suppose there is a special retracting deformation \( \delta \) of \( Y + C_f \) into \( X_0 \). Let \( g \) be a map such that \( g(y) = h^{-1} i^{-1} 1_{X_0} \delta_1 (<y>) \). Notice that \( i^{-1} 1_{X_0} = i 1_{X \times \{0\}} \), and \( i^{-1} 1_Y = i 1_Y \). Since \( i^{-1} 1_{X_0} \) and \( h \) are homeomorphisms, \( i 1_{X \times \{0\}} h g(y) = \delta_1 (<y>) \).

For each \( s \in [0,1] \) define the map \( F: X \times [0,1] \to X \) by \( F(x,s) = h^{-1} i^{-1} |_{X_0} \delta_1 (i(x,s)) \), and define the maps \( G: Y \times [0,1] \to Y \) by \( G(y,u) = i^{-1} |_Y \rho \delta_u (i(y)) \). Thus \( F(x,0) = h^{-1} i^{-1} |_{X_0} \delta_1 (<x,0>) = h^{-1} i^{-1} |_{X_0} (<x,0>) \), since \( \delta_s (<x,0>) = <x,0> \) for all \( s \in [0,1] \). So \( F(x,0) = h^{-1} i^{-1} |_{X_0} (<x,0>) = h^{-1}(x,0) = x \). In other words, \( F_0 = \text{id}_X \). And \( F(x,1) = \)
\[ h^{-1} i^{-1} |_{X_0 \delta_1(<x,1>)} = h^{-1} i^{-1} |_{X_0 \delta_1(<f(x)>)} = g(f(x)). \]  
So \( F_1 = gf. \) Also \( G_0(y) = i|_{Y}^{-1} \rho \delta(\langle y,0 \rangle) = i|_{Y}^{-1} \rho(\langle y \rangle) = i|_{X}^{-1}(\langle y \rangle) = y. \) So \( G_0 = \text{id}_Y. \) And \( G(y,1) = i|_{Y}^{-1} \rho(\langle x,0 \rangle) = i|_{X}^{-1} \rho(\langle x \rangle). \) So 
\[ G(y,1) = i|_{Y}^{-1} \rho(i|_{X}^{-1} \rho_1(<x,0>)) = i|_{X}^{-1} \rho(g(y),0) = i|_{Y}^{-1} \rho(\langle g(y) \rangle) = f(g(y)). \]  
So \( G_1 = fg. \) Then \( \text{id}_X \simeq gf \) by the homotopy \( F, \) and \( \text{id}_Y \simeq fg \) by the homotopy \( G. \) Therefore \( g \) is a two-sided homotopy inverse of \( f. \) In addition, however, 
\[ f(F(x,s)) = f(h^{-1} i^{-1} |_{X_0 \delta_1(<x,s>)}) = f(h^{-1} i^{-1} |_{X_0 \delta_1(<x,s'>)}) \]  
for some \( x' \in X. \) The map 
\[ i|_{Y}^{-1} \rho \delta(\langle x,s \rangle) = i|_{Y}^{-1} \rho(\langle x',0 \rangle), \]  
for the same \( x' \in X, \) so 
\[ i|_{Y}^{-1} \rho \delta(\langle x,s \rangle) = i|_{Y}^{-1} \rho(\langle f(x') \rangle) = f(x'). \]  
Thus 
\[ f(F(x,s)) = i|_{Y}^{-1} \rho \delta_1(\langle x,s \rangle) \]  
for every \( s \in [0,1]. \) Since \( \delta \) is a special retracting deformation, 
\[ \rho \delta_1(\langle x,s \rangle) = \rho \delta_1(\langle f(x),s \rangle). \]  
Thus 
\[ f(F(x,s)) = i|_{Y}^{-1} \rho \delta_1(\langle x,s \rangle) = i|_{Y}^{-1} \rho \delta_1(\langle f(x) \rangle) = G(f(x),s). \]  
Therefore \( g \) is a special two-sided homotopy inverse of \( f. \) 

Conversely, suppose \( g \) is a special two-sided homotopy inverse of \( f, \) and \( F \) and \( G \) are the homotopies satisfying the condition \( f(F(x,s)) = G(f(x),s). \) Recall that \( C_f \subseteq Y + C_f \) and \( C_f = \{<x,t> | x \in X, 0 \leq t \leq 1 \} \) where \( f(x) \in <x,1> \) for every \( x \in X. \) Define a function \( \xi : C_f \times [0,1] \to C_f \) by:

\[
\xi(<x,t>,u) =
\begin{cases}
  ihF(x,t(2u+1)); & 0 \leq t \leq 2u/2u+1, 0 \leq u \leq 1/2 \\
  <F(x,2u),t(2u+1)-2u>; & 2u/2u+1 \leq t \leq 1, 0 \leq u \leq 1/2 \\
  ihF(x,2t); & 0 \leq t \leq 1/2, 1/2 \leq u \leq 1 \\
  <g(f(x)), 2t-1)(2-2u)>; & 1/2 \leq t \leq 1, 1/2 \leq u \leq 1
\end{cases}
\]
Line one and line two of the definition of $\xi$ overlap when $t = 2u/2u+1$ and $0 \leq u \leq 1/2$. Then $\xi(\langle x, 2u/2u+1 \rangle, u) = \langle i\hbar F(x, (2u/2u+1)(2u+1)) = i\hbar F(x, 2u) = \langle F(x, 2u), 0 \rangle$. Also $\xi(\langle x, 2u/2u+1 \rangle, u) = \langle F(x, 2u), (2u/2u+1)(2u+1) - 2u \rangle = \langle F(x, 2u), 0 \rangle$.

Line three and line four of the definition of $\xi$ overlap when $t = 1/2$ and $1/2 \leq u \leq 1$. Then $\xi(\langle x, 1/2 \rangle, u) = \langle i\hbar F(x, 2(1/2)) = i\hbar F(x, 1) \rangle = \langle g(f(x)), 0 \rangle$. And $\xi(\langle x, 1/2 \rangle, u) = \langle g(f(x)), (2(1/2) - 1)(2 - 2(1/2)) \rangle = \langle g(f(x)), 0 \rangle$.

When line two and three overlap $u = 1/2$. The condition $2u/2u+1 \leq t \leq 1$ becomes $1/2 \leq t \leq 1$. Thus when line two and line three overlap $t = 1/2$ also. Then $\xi(\langle x, 1/2 \rangle, 1/2) = \langle i\hbar F(x, 2(1/2)) = \langle F(x, 1), 0 \rangle = \langle g(f(x)), 0 \rangle$. And $\xi(\langle x, 1/2 \rangle, 1/2) = \langle F(x, 1), (2(1/2) - 1)(2 - 2(1/2)) \rangle = \langle g(f(x)), 0 \rangle$. Since $\langle x, 1 \rangle = \langle f(x), \rangle$, when $0 \leq u \leq 1/2$

$\xi(\langle f(x), u \rangle) = \xi(\langle x, 1 \rangle, u) = \langle F(x, 2u), (1)(2u+1) - 2u \rangle = \langle F(x, 2u), 1 \rangle = \langle f(F(x, 2u)), = \langle G(f(x), 2u) \rangle \epsilon \hat{Y}$. When

$1/2 \leq u \leq 1$ $\xi(\langle f(x), u \rangle) = \xi(\langle x, 1 \rangle, u) = \langle g(f(x)), (1)(2 - 2(1/2)) \rangle$

Therefore $\xi$ is well defined on $C_f$ and is continuous. Thus $\xi$ is a deformation of $C_f$ onto $X_0$.

For every $\langle x, 0 \rangle \in X_0$, $\xi(\langle x, 0 \rangle, u) = i\hbar F(x, 2u) = \langle x, 0 \rangle$ for every $u \in [0,1]$. Thus $\xi$ is a retracting deformation that satisfies condition i.

Claim: Condition ii is also satisfied by $\xi'$.

Condition ii requires that $\rho(\xi(\langle f(x), u \rangle)) =$
\[ \rho(\xi(<x,u>,1)). \text{ If } 0 \leq u \leq 1/2, \rho(\xi(<f(x),u)> = \rho\xi(<x,1> = <F(x,2u),1> = <fF(x,2u)> , and \rho(\xi(<x,u>,1) = \rho h_F(x,2u) = \rho <F(x,2u),0> = <fF(x,2u)> . \text{ If } 1/2 \leq u \leq 1, \rho(\xi(<f(x),u) = \rho <g(f(x),2-2u)> = <fgf(x)>, and \rho(\xi(<x,u>,1) = \rho <g(f(x),0> = <fgf(x)> . \text{ Thus whenever } u = t \rho(\xi(<f(x),u) = \rho(\xi(<x,t>,1) . \text{ Therefore } \xi \text{ is a special retracting deformation of } C_f \text{ into } X_0. \]

Since G and g are both defined for all of Y, the definition for \( \xi \) can be extended to \( Y + C_f \) by:

\[
\xi'(y,u) = \begin{cases} 
\xi((x,1),u) & \text{for } y \in C_f \\
G(y,2u) & \text{for } y \in Y - C_f, 0 \leq u \leq 1/2 \\
g(y),2-2u) & \text{for } y \in Y - C_f, 1/2 < u \leq 1
\end{cases}
\]

Clearly \( \xi' \) agrees with \( \xi \) if \( y = f(x) \). Therefore \( \xi' \) is a retracting deformation \( Y + C_f \) onto \( X_0 \).

**Definition 5.7:** A homotopy \( \xi : X \times [0,1] \rightarrow M \) where M is a metric space with metric d is called a \( \epsilon \)-homotopy if and only if \( d(\xi(x,t_1),\xi(x,t_2)) < \epsilon \) for every \( t_1 \) and \( t_2 \) in \([0,1]\) and every \( x \) in \( X \). In particular, \( d(\xi(x,0),\xi(x,1)) < \epsilon \). If the homotopy is between maps \( f \) and \( g \), write \( f \simeq_{\epsilon} g \). If \( \xi \) is a deformation, call it a \( \epsilon \)-deformation.

**Definition 5.8:** Let \( X \) and \( Y \) be metric spaces and \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \) be maps such that \( gf \simeq_{\epsilon} id_X \). Then \( g \)
is a left $\epsilon$-homotopy inverse of $f$ and $f$ is a right
$\epsilon$-homotopy inverse of $g$.

**Theorem 5.9:** Suppose $X$ and $Y$ are metric spaces. Then
a map $f : X \to Y$ has a left $\epsilon$-homotopy inverse for every
$\epsilon > 0$ if and only if $f$ is a homeomorphism, and for every
$\epsilon > 0$, $id_f(X)$ is $\epsilon$-homotopic in $f(X)$ to a map extendable to
$Y$.

**Proof:** Let $\epsilon > 0$ be arbitrary. Suppose $g^\epsilon$ is a left
$\epsilon$-homotopy inverse of $f$. Then $id_X \simeq g^\epsilon f$ by a $\epsilon$-homotopy
$\xi^\epsilon : X \times [0,1] \to X$ such that $\xi^\epsilon(x,0) = x$, $\xi^\epsilon(x,1) = g^\epsilon f(x)$,
and for every $t_1, t_2 \in [0,1]$ and $x \in X$ $d(\xi(x,t_1), \xi(x,t_2)) < \epsilon$
In particular, for every $x \in X$, $d(\xi(x,0), \xi(x,1)) =
\epsilon$. $d(x, g^\epsilon f(x)) < \epsilon$.

**Claim:** The map $f$ is one-to-one.

Let $f(x_1) = f(x_2)$. By the hypothesis $d(x_1, g^\epsilon f(x_1)) < \epsilon$
and $d(x_2, g^\epsilon f(x_2)) < \epsilon$ for every $\epsilon > 0$. Since $f(x_1) = f(x_2)$,
then $d(x_2, g^\epsilon f(x_1)) < \epsilon$. By the triangle inequality
$d(x_1, x_2) \leq d(x_1, g^\epsilon f(x_1)) + d(x_2, g^\epsilon f(x_1)) < 2\epsilon$ for every
$\epsilon > 0$.

Just suppose $x_1 \neq x_2$. Let $d(x_1, x_2) = r > 0$. Choose
$\epsilon < r/2$. Then $2\epsilon < r = d(x_1, x_2) < \epsilon$. But this is a
contradiction. Therefore $x_1 = x_2$ and $f$ is one-to-one.

Clearly $f$ is onto its image.

**Claim:** The map $f$ is an open map.

Let $U$ be an open subset in $X$ and $y_0$ be a member of
f(U). Let \( \{y_k\} \to y_0 \) be a sequence in \( f(X) \) which converges to \( y_0 \). To prove that \( f \) is an open map it is sufficient to show that \( \{y_k\} \) is eventually in \( f(U) \). Since \( f \) is one-to-one, \( f^{-1}(y_k) \) is a single member of \( X \) for every \( k \), and \( f^{-1}(y_0) \in U \). Let \( \epsilon > 0 \) be chosen. Since \( f^{-1}(y_k) \in X \), and \( g^\epsilon \) is a left \( \epsilon \)-homotopy inverse of \( f \),

\[
d(\operatorname{id}_X(f^{-1}(y_k)), g^\epsilon f(f^{-1}(y_k))) = d(f^{-1}(y_k), g^\epsilon(y_k)) < \epsilon.
\]

Similarly, \( d(f^{-1}(y_0), g^\epsilon(y_0)) < \epsilon \). Since \( g^\epsilon \) is continuous, \( g^\epsilon(\{y_k\}) = \{g^\epsilon(y_k)\} \) is a sequence in \( X \) which converges to \( g^\epsilon(y_0) \). Thus there is a natural number \( N^\epsilon \) such that \( \forall n > N^\epsilon \)

\[
d(g^\epsilon(y_n), g^\epsilon(y_0)) < \epsilon.
\]

Therefore if \( n > N^\epsilon \)

\[
d(f^{-1}(y_0), f^{-1}(y_n)) \leq d(f^{-1}(y_0), g^\epsilon(y_0)) + d(g^\epsilon(y_0), g^\epsilon(y_n)) + d(g^\epsilon(y_n), f^{-1}(y_n)) < 3\epsilon.
\]

Therefore the sequence \( \{f^{-1}(y_k)\} \) converges to \( f^{-1}(y_0) \) in \( X \). Since \( f^{-1}(y_0) \) is in \( U \) which is open in \( X \), there is a natural number \( M \) such that \( \forall m > M \)

\[
f^{-1}(y_m) \in U. \text{ So } f f^{-1}(y_m) = y_m \text{ is in } f(U). \text{ Therefore } \{y_k\} \text{ is eventually in } f(U), \text{ and } f \text{ is an open map.}
\]

Therefore \( f \) is a homeomorphism.

Since \( g^\epsilon f \simeq \operatorname{id}_X \) and \( f \) is a homeomorphism, then

\[
fg^\epsilon f f^{-1} \simeq \operatorname{id}_X f^{-1}. \text{ But } \operatorname{id}_X f^{-1} = f f^{-1} = \operatorname{id}_f(X), \text{ and}
\]

\[
fg^\epsilon f f^{-1} = fg^\epsilon \operatorname{id}_f(X) = fg^\epsilon |_f(X). \text{ Thus } fg^\epsilon |_f(X) \simeq \operatorname{id}_f(X) \text{ for every } \epsilon > 0. \text{ Clearly } fg^\epsilon |_f(X) \text{ is a map that can be continuously be extended to } Y \text{ by } fg^\epsilon.
\]

Conversely, suppose the map \( f : X \to Y \) is a homeomorphism and for every \( \epsilon > 0 \), \( \operatorname{id}_f(X) \) is \( \epsilon \)-homotopic to
a map extendable to $Y$. Since $f$ is a homeomorphism, there is no loss of generality to assume that $X = f(X)$ or that $X \subseteq Y$ and that $f = \text{id}_X$. For each $\epsilon > 0$ let $g^\epsilon : Y \to X$ be the extension of the map $g^\epsilon|_{f(X)}$ which is $\epsilon$-homotopic to $\text{id}_X$. Thus $g^\epsilon|_{f(X)} : f(X) \to Y$. But since $f(X) = X$, $g^\epsilon|_{f(X)} : X \to X$, and $g^\epsilon|_{f(X)} \simeq _\epsilon \text{id}_X$. So for every $\epsilon > 0$ there is a map $\xi^\epsilon : X \times [0,1] \to X$ such that $\xi^\epsilon(x,0) = x$, $\xi^\epsilon(x,1) = g^\epsilon|_{f(X)}(x)$, and for every $x \in X$
$$d(\xi^\epsilon(x,s_1),\xi^\epsilon(x,s_2)) < \epsilon$$
for every $s_1, s_2$ in $[0,1]$. In particular, $d(x,g^\epsilon|_{f(X)}(x)) < \epsilon$. But since $f = \text{id}_X$, for all $x \in X$ $f(x) = x$. Thus $\xi^\epsilon(x,1) = g^\epsilon f(x)$, and
$$d((x,s_1),(x,s_2)) = d((f(x),s_1),(f(x),s_2)) < \epsilon$$
for all $x \in X$. Therefore $\text{id}_X \simeq _\epsilon g^\epsilon f$, and $g^\epsilon$ is a $\epsilon$-homotopy inverse of $f$ for every $\epsilon > 0$.

\textbf{Theorem 5.10:} Suppose $Y$ is a metric space and $X$ is an ANR. Then a map $f : X \to Y$ has a left $\epsilon$-homotopy inverse for every $\epsilon > 0$ if and only if $f$ is a homeomorphism from $X$ onto $f(X)$, and $f(X)$ is a retract of $Y$.

\textbf{Proof:} Let $X$ be an ANR and $Y$ be a metric space. Suppose a map $f : X \to Y$ has a $\epsilon$-homotopy inverse for every $\epsilon > 0$. Since an ANR is a separable metric space, by theorem 5.9, $f$ is a homeomorphism and $\text{id}_{f(X)}$ is homotopic to a map extendable to $Y$. Let $\theta : Y \to f(X)$ be an extension so that $\text{id}_{f(X)} \simeq _\epsilon \theta|_{f(X)}$. The set $f(X)$ is closed in $Y$ by the
same argument used in theorem 5.9 to show that \( f \) is an open map. Also since \( f \) is a homeomorphism and \( X \) is an ANR, then \( f(X) \) is an ANR. Then by the Borsuk–Dowker Theorem\(^{15}\) id\(_f(X)\) is also extendable to \( Y \). Call this extension \( r \); so \( r : Y \to f(X) \) such that \( r|_{f(X)} = \text{id}_f(X) \). Therefore \( r \) is a retraction and \( f(X) \) is a retract of \( Y \).

Conversely, suppose the map \( f : X \to Y \) is a homeomorphism onto \( f(X) \), and \( f(X) \) is a retract of \( Y \). Then there is a map \( r : Y \to f(X) \) such that \( r|_{f(X)} = \text{id}_f(X) \). Then clearly \( r|_{f(X)} \simeq \text{id}_f(X) \) and \( r \) is an extension of \( r|_{f(X)} \) to \( Y \). Therefore by theorem 5.9, \( f \) has a left \( \epsilon \)-homotopy inverse for every \( \epsilon > 0 \).

Theorem 5.11: Suppose that \( X \) and \( Y \) are metric spaces and \( f : X \to Y \) such that \( Y + C_f \) is metrizable. The map \( f \) has a right \( \epsilon \)-homotopy inverse for every \( \epsilon > 0 \) if and only if \( \hat{Y} \) is \( \eta \)-deformable into \( C_f - \hat{Y} \) for every \( \eta > 0 \).

The term "\( \eta \)-deformable" means that the deformation is a \( \eta \)-homotopy. Recall that \( C_f = \{<x,t>| x \in X, 0 \leq t \leq 1\} \), and \( <x,1> = <f(x)> \) for every \( x \in X \). Thus \( C_f - \hat{Y} = \{<x,t>| x \in X, 0 \leq t <1\} \).

Proof: Suppose \( f \) has a right \( \epsilon \)-homotopy inverse for every \( \epsilon > 0 \), say \( g^\epsilon \). So for every \( \epsilon > 0 \), id\(_Y \simeq \epsilon f g^\epsilon \) by a homotopy \( \xi^\epsilon : Y \times [0,1] \to Y \) where \( \xi^\epsilon(y,0) = y, \xi^\epsilon(y,1) = \)

$fg^{\epsilon}(y)$, and $d_Y(\xi^{\epsilon}(y,s_1), (\xi^{\epsilon}(y,s_2)) < \epsilon$ for every $s_1$, $s_2$ in $[0,1]$. For each $\epsilon > 0$ define $\delta^{\epsilon} : \hat{Y} \times [0,1] \to Y + C_f$ by:

$$\delta^{\epsilon}(<y>,u) = \begin{cases} 
    i\xi^{\epsilon}(y,2u) & 0 \leq u \leq 1/2 \\
    i(g^{\epsilon}(y), 1-\epsilon(2u-1)) & 1/2 \leq u \leq 1 
\end{cases}$$

Thus $\delta^{\epsilon}(<y>,0) = i\xi^{\epsilon}(y,0) = i(y) = <y>$; so $\delta^{\epsilon}_0 = id_{\hat{Y}}$. And $\delta^{\epsilon}(<y>,1) = i(g^{\epsilon}(y),1-\epsilon) = <g^{\epsilon}(y),1-\epsilon > \epsilon C_f - \hat{Y}$, since $g^{\epsilon}(y) \epsilon X$ and $\epsilon > 0$. When $u = 1/2$, $\delta^{\epsilon}(<y>,1/2) = i\xi^{\epsilon}(y,1) = i(fg^{\epsilon}(y)) = <fg^{\epsilon}(y)>$, and $\delta^{\epsilon}(<y>,1/2) = i(g^{\epsilon}(y),1-\epsilon(0)) = i(g^{\epsilon}(y),1) = <g^{\epsilon}(y),1> = <fg^{\epsilon}(y)>$. Thus $\delta^{\epsilon}$ is well defined and continuous for every $\epsilon > 0$.

Since $Y + C_f$ is metrizable, call its metric $d_c$. Let $d_p$ be the metric in $Y + C_f \times [0,1]$ defined by

$$d_p(<q_1>,s_1),(<q_2>,s_2)) = \sqrt{d_c(<c_1>,<c_2>)^2 + d_1(s_1,s_2)^2}$$

where $<q>$ represents any element of $Y + C_f$, and $d_1$ is the usual distance in the unit interval. Similarly, let $d_X$ be the metric in $X \times [0,1]$, and $d_Y$ be the metric in $Y$. Let $\eta > 0$ be chosen.

**Case 1:** Suppose $u_1$ and $u_2$ are in $[0,1/2]$. Now for every $\epsilon > 0$, $d_p(\delta^{\epsilon}(<y>,u_1), \delta^{\epsilon}(<y>,u_2)) = d_p(i\xi^{\epsilon}(y,2u_1), i(\xi^{\epsilon}(y,2u_2)).$ Since $i$ is continuous, if $d_Y(\xi^{\epsilon}(y,2u_1), \xi^{\epsilon}(y,2u_2)) \epsilon$, then $d_p(i\xi^{\epsilon}(y,2u_1), i(\xi^{\epsilon}(y,2u_2)) < \eta$ for every $\eta > 0$. So for the chosen $\eta$ choose an $\epsilon_1 > 0$ such that


Case 2: Suppose \( u_1 \) and \( u_2 \) are in \([1/2,1]\). Then
\[
d_p(\delta^\epsilon(\langle y, u_1 \rangle), \delta^\epsilon(\langle y, u_2 \rangle)) =
\]
\[
d_p(i(g^\epsilon(y), 1 - \epsilon(2u_1 - 1)), i(g^\epsilon(y), 1 - \epsilon(2u_2 - 1))) \text{ for every } \epsilon > 0.
\]
Notice that \( 1/2 \leq u \leq 1 \) implies \( 0 \leq 2u - 1 \leq 1 \) which implies
\( 1 - \epsilon \leq 2(2u - 1) \leq 1 \). Thus
\[
d^\epsilon((g^\epsilon(y), 1 - \epsilon(2u_1 - 1)), (g^\epsilon(y), 1 - \epsilon(2u_2 - 1))) =
\]
\[
0 + \left[ (d_1(1 - \epsilon(2u_1 - 1)), (1 - \epsilon(2u_2 - 1)))^2 \right] = \sqrt{(2\epsilon(u_2 - u_1))^2} = 2\epsilon|u_2 - u_1| \text{ for every } \epsilon > 0.
\]
Again since \( i \) is continuous, choose \( \epsilon_2 > 0 \) such that \( 2\epsilon_2|u_2 - u_1| < 2\epsilon_2|1 - 0| < \eta \). Then
\[
d_p(\delta^\epsilon(\langle y, u_1 \rangle), \delta^\epsilon(\langle y, u_2 \rangle)) < \eta.
\]

Case 3: Suppose \( u_1 \in [0,1/2] \) and \( u_2 \in [1/2,1] \). Since, when \( u = 1/2 \), \( i\epsilon(y,1) = i(g^\epsilon(y)) \), then
\[
d_p(\delta^\epsilon(\langle y, u_1 \rangle), \delta^\epsilon(\langle y, u_2 \rangle)) =
\]
\[
d_p(i\epsilon(y, 2u_1), i(g^\epsilon(y), 1 - \epsilon(2u_2 - 1))) \leq d_p(i\epsilon(y, 2u_1), i\epsilon(y,1)) + d_p(i(g^\epsilon(y), 1 - \epsilon(2u_2 - 1)), i(g^\epsilon(y),1)) \text{ for every } \epsilon > 0.
\]
Now from case 1 and case 2, \( d_p(i\epsilon^1(y, 2u_1), i\epsilon^1(y,1)) \) +
\[
d_p(i(g^\epsilon^2(y), 1 - \epsilon(2u_2 - 1)), i(g^\epsilon^2(y),1)) < 2\eta. \]
So choose \( \epsilon_3 \) so that \( d_p(\delta^\epsilon(\langle y, u_1 \rangle), \delta^\epsilon(\langle y, u_2 \rangle)) < 2\eta/2. \)

Choose \( \epsilon < \min \{\epsilon_1, \epsilon_2, \epsilon_3\} \). Then
\[
d_p(\delta^\epsilon(\langle y, u_1 \rangle), \delta^\epsilon(\langle y, u_2 \rangle)) < \eta. \]
Therefore for every \( \eta > 0 \), \( \hat{Y} \) is \( \eta \)-deformable into \( C_f - \hat{Y} \) by \( \delta^\epsilon \).

Conversely, suppose that \( \hat{Y} \) is \( \eta \)-deformable into \( C_f - \hat{Y} \)
for every \( \eta > 0 \). Then for every \( \epsilon > 0 \) there is a map 
\[
\delta^\eta : \hat{Y} \times [0,1] \to Y + C_f \text{ such that } \delta^\eta(<y>,0) = <y>, \\
\delta^\eta(<y>,1) \in C_f - \hat{Y}, \text{ and } d_p(\delta^\eta(<y>,u_1),\delta^\eta(<y>,u_2)) < \eta. \text{ In particular, } d_p(<y>,\delta^\eta(<y>,1)) < \eta. \text{ Let } \epsilon > 0 \text{ be chosen.}
\]

Define \( \nu : C_f - \hat{Y} \to X \) by \( \nu(<x,t>) = x. \) Suppose \( U \) is an open set in \( X. \) Then \( \nu^{-1}(U) = i\{(x,t) | x \in U, \ t \in [0,1]\} = U \times [0,1] \) which is open in \( X \times [0,1] \). So \( \nu^{-1}(U) \) is open in \( C_f - \hat{Y} \) and \( \nu \) is continuous.

Define \( g^\epsilon : Y \to X \) by \( g^\epsilon(y) = \nu(\delta^\epsilon(i(y),1)). \)

Notice that since \( \delta^\epsilon(<y>) \in C_f - \hat{Y}, \) then \( \delta^\epsilon(<y>) = <x,t> \) for some \( <x,t> \in C_f - \hat{Y}. \) Also by the definition of \( \nu, \) every \( <x,t> \in C_f - \hat{Y} \) is equal to \( \nu(<x,t>),t>. \)

Define \( \lambda : C_f - \hat{Y} \times [0,1] \to Y + C_f \) by \( \lambda(<x,t>,u) = <x,t+u(1-t)> \). Notice that \( \lambda \) is defined similarly to \( \tau \) in lemma 4.8, \( \eta = \tau|_{C_f - \hat{Y}}; \) so \( \lambda \) is continuous.

Define \( \xi^\epsilon : \hat{Y} \times [0,1] \to \hat{Y} \) by:

\[
\xi^\epsilon(<y>,v) = \begin{cases} 
\delta^\epsilon(<y>,2v) & 0 \leq v \leq 1/2 \\
\lambda(\delta^\epsilon_1(<y>),2v-1) & 1/2 \leq v \leq 1 
\end{cases}
\]
When \( v = 0 \), \( \xi^\epsilon(\langle y, 0 \rangle) = \delta^\epsilon(\langle y, 0 \rangle) = \langle y \rangle \); so \( \xi_0^\epsilon = \text{id}_{\hat{Y}} \).

When \( v = 1/2 \), \( \xi^\epsilon(\langle y, 1/2 \rangle) = \delta^\epsilon(\langle y, 1 \rangle) = \delta_1^\epsilon(\langle y \rangle) \), and

\[
\xi^\epsilon(\langle y, 1/2 \rangle) = \lambda(\delta_1^\epsilon(\langle y \rangle), 0) = \langle x, t+0 \rangle \text{ where } \langle x, t \rangle = \delta_1^\epsilon(\langle y \rangle).
\]

When \( v = 1 \), \( \xi^\epsilon(\langle y, 1 \rangle) = \lambda(\delta_1^\epsilon(\langle y \rangle), 2-1) \). Now \( \delta_1^\epsilon(\langle y \rangle) \) is some \( \langle x, t \rangle \in C_\hat{Y}; \) so \( \xi^\epsilon(\langle y, 1 \rangle) = \lambda(\langle x, t, 1 \rangle) = \langle x, t+1(1-t) \rangle = \langle x, 1 \rangle \). Notice that \( \nu(\delta_1^\epsilon(i(y))) = \nu(\delta_1^\epsilon(\langle y \rangle)) = \nu(\langle x, t \rangle) = x \).

Thus \( \xi^\epsilon(\langle y, 1 \rangle) = \langle x, 1 \rangle = \langle \nu(\delta_1^\epsilon(i(y))), 1 \rangle = \langle f\nu\delta_1^\epsilon(i(y)) \rangle = \langle fg^\epsilon(y) \rangle = i|_Y fg^\epsilon|^{-1} |_{\hat{Y}} \), since \( i|_Y \) is a homeomorphism. Thus \( \xi^\epsilon \) is a homotopy between \( \text{id}_{\hat{Y}} \) and \( \xi_1^\epsilon(\hat{Y}) = i|_Y fg^\epsilon|^{-1} |_{\hat{Y}} \). Then also \( \text{id}_{\hat{Y}} \simeq fg^\epsilon \). Since \( \epsilon \) was arbitrarily chosen, a homotopy \( \xi^\epsilon \) can be constructed for every \( \epsilon > 0 \). It remains to show that \( \xi^\epsilon \) is an \( \epsilon \)-homotopy.
Case 1: Suppose \( v_1 \) and \( v_2 \) are in \([0, 1/2]\). Then
\[
d_c(\xi\epsilon(\langle y\rangle, v_1), \xi\epsilon(\langle y\rangle, v_2)) = d_c(\delta\epsilon(\langle y\rangle, v_1), (\delta\epsilon(\langle y\rangle, v_2)) < \epsilon.
\]

Case 2: Suppose \( v_1 \) and \( v_2 \) are in \([1/2, 1]\). Then
\[
d_c(\xi\epsilon(\langle y\rangle, v_1), \xi\epsilon(\langle y\rangle, v_2)) = d_c(\lambda\delta_1\epsilon(\langle y\rangle), v_1), \lambda\delta_1\epsilon(\langle y\rangle), v_2)).
\]
Since \( \lambda \) is continuous, for every \( \epsilon > 0 \) there is an \( \eta > 0 \) such that if \( d_c(\lambda\delta_1\epsilon(\langle y\rangle), v_1), \lambda\delta_1\epsilon(\langle y\rangle), v_2)) < \epsilon \), then
\[
d_c(\delta\eta(\langle y\rangle, v_1), (\delta\eta(\langle y\rangle, v_2)) < \eta. \)
So choose \( \eta \) such that
\[
d_c(\lambda\delta_1\epsilon(\langle y\rangle), v_1), \lambda\delta_1\epsilon(\langle y\rangle), v_2)) < \epsilon.
\]

Case 3: Suppose \( v_1 \in [0, 1/2] \) and \( v_2 \in [1/2, 1] \). Recall that when \( v = 1/2 \), \( \xi\epsilon(\langle y\rangle, 1/2) = \delta\epsilon(\langle y\rangle, 1) = \lambda\delta_1\epsilon(\langle y\rangle), 0). \)
Then
\[
d_c(\xi\epsilon(\langle y\rangle, v_1), \xi\epsilon(\langle y\rangle, v_2)) \leq d_c(\xi\epsilon(\langle y\rangle, v_1), \xi\epsilon(\langle y\rangle, 1/2))
+ d_c(\xi\epsilon(\langle y\rangle, 1/2), \xi\epsilon(\langle y\rangle, v_2)) = d_c(\delta\epsilon(\langle y\rangle, v_1), (\delta\epsilon(\langle y\rangle, 1))
+ d_c(\lambda\delta_1\epsilon(\langle y\rangle), 0), \lambda\delta_1\epsilon(\langle y\rangle), v_1)) \). From case 2 there is an \( \eta > 0 \) such that
\[
d_c(\lambda\delta_1\epsilon(\langle y\rangle), v_1), \lambda\delta_1\epsilon(\langle y\rangle), v_2)) < \epsilon. \)
Choose \( \eta' \) such that
\[
d_c(\delta\eta'(\langle y\rangle, v_1), (\delta\eta'(\langle y\rangle, 1)) < \epsilon/2 \) and
\[
d_c(\lambda\delta_1\epsilon'(\langle y\rangle), 0), \lambda\delta_1\epsilon'(\langle y\rangle), v_1)) < \epsilon/2. \)
Then
\[
d_c(\lambda\delta_1\epsilon'(\langle y\rangle), v_1), \lambda\delta_1\epsilon'(\langle y\rangle), v_2)) < \epsilon. \)
Choose \( \gamma = \min \{\epsilon, \eta, \eta'\} \). Then \( \xi\gamma \) is an \( \epsilon \)-homotopy.

Since \( i^{-1}|_Y \) is a homeomorphism, by a similar argument there is a \( \gamma' > 0 \) such that \( i^{-1}|_Y \xi\gamma' \) is an \( \epsilon \)-homotopy between \( \text{id}_Y \) and \( f\epsilon \) for any \( \epsilon > 0 \).

\[ \square \]

**Theorem 5.12:** Suppose \( X \) is a compact metric space; \( Y \) is a metric space; and \( f : X \to Y \) is a map such that \( Y + C_f \) is metrizable. Then \( f \) has a two-sided \( \epsilon \)-homotopy inverse for every \( \epsilon > 0 \) if and only if \( f \) is a homeomorphism of \( X \) into \( Y \).
Proof: Suppose $f$ has a two-sided $\epsilon$-homotopy inverse for every $\epsilon > 0$.

Claim: The closure of $f(X)$ is equal to $Y$.

Clearly $\overline{f(X)} \subseteq Y$. Let $y \in Y$. Let $N$ be a basic open neighborhood of $y$ in $Y$. Then $iN = i|_Y(N)$ is an open neighborhood of $<y>$ in $\hat{Y}$ containing some basic open set $G$ of some radius, say $\rho$. By the previous theorem, $\hat{Y}$ is $\epsilon$-deformable into $C_f - \hat{Y}$; so for $\epsilon = \rho$ there is a $\delta^\rho : \hat{Y} \times [0,1] \to Y + C_f$ such that $\delta^\rho(<y>,0) = <y>$, $\delta^\rho(<y>,1) \in C_f - \hat{Y}$, and $d_c(\delta^\rho(<y>,s_1),\delta^\rho(<y>,s_2)) < \epsilon$. Since $\delta^\rho(<y>) \in C_f - \hat{Y}$, then $\delta^\rho(<y>)$ is some $<x,t>$ where $t < 1$. So $d_c(<y>,<x,t>) < \epsilon$. Since $\delta^\rho$ is a homotopy, $\delta^\rho_{<y>} : [0,1] \to Y + C_f$ is a path in $Y + C_f$ from $<y>$ to $<x,t>$. Furthermore, the distance between any two points in this path is less than $\rho$. Since $\delta^\rho_{<y>}$ is continuous, there is some $s \in [0,1]$ and some $x' \in X$ such that $\delta^\rho_{<y>}(s) = <x',1> = <f(x')>$, and $d_c(<y>,<x',1>) < \rho$. So $<f(x')> \in G$. Now $f(x') \in i|_Y^{-1}(<f(x')>) \subseteq N$; so $f(x') \in N$. Therefore $y$ is a closure point of $f(X)$, and $\overline{f(X)} = Y$.

Since $X$ is compact and $f$ is continuous, $f(X)$ is compact. Since $Y$ is a metric space, $f(X) = \overline{f(X)} = Y$. Then $\text{id}_f(X) = \text{id}_Y$; so $\text{id}_f(X) \simeq_\epsilon \text{id}_Y$ for every $\epsilon > 0$. Then by theorem 5.9, $f$ is a homeomorphism from $X$ onto $Y$.

Conversely, suppose $f$ is a homeomorphism from $X$ onto $Y$. Then every $y = f(x)$ for some $x \in X$. Thus every $<y> \in \hat{Y}$ can
be written as \(<f(x)>\) or \(<x,1>\).

Let \(\epsilon > 0\) be chosen. Define \(\delta^\epsilon : \hat{Y} \times [0,1] \to Y + C_f\) by \(\delta^\epsilon(\langle f(x)\rangle,s) = \langle x,1-s\epsilon\rangle\). Thus \(\delta^\epsilon(\langle f(x)\rangle,0) = \langle x,1-0\rangle = \langle x,1\rangle = \langle f(x)\rangle\); so \(\delta^\epsilon_0 = \text{id}_{\hat{Y}}\). When \(s = 1\), \(\delta^\epsilon(\langle f(x)\rangle,1) = \langle x,1-\epsilon\rangle \in C_f - \hat{Y}\). Also \(d_c(\delta^\epsilon(\langle f(x)\rangle,s_1),\delta^\epsilon(\langle f(x)\rangle,s_2)) = d_c(\delta^\epsilon(\langle x,1\rangle,s_1),\delta^\epsilon(\langle x,1\rangle,s_2)) \leq d_c(\langle x,1\rangle,\langle x,1-\epsilon\rangle) = \eta\). Thus for every \(\eta > 0\) \(\hat{Y}\) is \(\eta\)-deformable into \(C_f - \hat{Y}\). Then by theorem 5.11 \(f\) has a right \(\epsilon\)-homotopy inverse for every \(\epsilon > 0\).

Since \(f\) is a homeomorphism from \(X\) onto \(Y\), \(f(X) = Y\) and \(\text{id}_f(X) = \text{id}_Y\); so by theorem 5.9, \(f\) has a left \(\epsilon\)-homotopy inverse for every \(\epsilon > 0\).

Let \(\epsilon > 0\) be chosen, and let \(l^\epsilon\) be a left \(\epsilon\)-homotopy inverse and \(r^\epsilon\) be a right \(\epsilon\)-homotopy inverse of \(f\). Define a map \(g^\epsilon = l^\epsilon r^\epsilon\). Now \(fg^\epsilon = fl^\epsilon r^\epsilon\). Since \(l^\epsilon f \simeq f\text{id}_X\) then for some \(a > 0\), \(fl^\epsilon r^\epsilon\) is \(a\)-homotopic to \(f\text{id}_X r^\epsilon = fr^\epsilon\). Since \(fr^\epsilon \simeq f\text{id}_Y\), then \(fl^\epsilon r^\epsilon\) is \(a'\)-homotopic to \(f\text{id}_Y\) for some \(a' < 0\).

Similarly, \(g^\epsilon f = l^\epsilon fr^\epsilon f \simeq \gamma 1^\epsilon f\text{id}_X f = 1^\epsilon f \simeq \gamma' f\text{id}_Y\). So for \(\rho = \min \{\gamma', a'\}\) \(fg^\epsilon \simeq f\text{id}_X\) and \(g^\epsilon f \simeq f\text{id}_Y\). Therefore \(g^\epsilon\) is a two-sided \(\rho\)-homotopy inverse of \(f\).

Then for every \(\rho > 0\) there is some \(\epsilon > 0\) such that \(g^\epsilon\) is a two-sided \(\rho\)-homotopy inverse of \(f\). □
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