MANIFOLDS, VECTOR BUNDLES, AND STIEFEL–WHITNEY CLASSES

THESIS

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Michael D. Green, Honors B.A.
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The problem of embedding a manifold in Euclidean space is considered. Manifolds are introduced in Chapter I along with other basic definitions and examples. Chapter II contains a proof of the Regular Value Theorem along with the "Easy" Whitney Embedding Theorem. In Chapter III, vector bundles are introduced and some of their properties are discussed. Chapter IV introduces the Stiefel-Whitney classes and the four properties that characterize them. Finally, in Chapter V, the Stiefel-Whitney classes are used to produce a lower bound on the dimension of Euclidean space that is needed to embed real projective space.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION AND BASIC DEFINITIONS</td>
<td>1</td>
</tr>
<tr>
<td>II. THE WHITNEY EMBEDDING THEOREM</td>
<td>9</td>
</tr>
<tr>
<td>III. VECTOR BUNDLES</td>
<td>26</td>
</tr>
<tr>
<td>IV. STIEFEL–WHITNEY CLASSES</td>
<td>48</td>
</tr>
<tr>
<td>V. ADDITIONAL PROPERTIES OF STIEFEL–WHITNEY CLASSES</td>
<td>75</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>90</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION AND BASIC DEFINITIONS

The first part of this thesis is concerned with manifolds. Basically speaking, a manifold is an object that looks like Euclidean space in some neighborhood around every point. For example, a balloon is a two dimensional manifold because a small bug with limited eyesight walking on the balloon’s surface would think that the balloon is flat, i.e. around every point of the balloon there is a neighborhood that looks like $\mathbb{R}^2$.

We start off by giving some basic definitions, and see what facts we can derive from these. Probably the most interesting result we encounter is the Whitney Embedding Theorem. The discovery of this theorem leads us to ask a very natural question which then leads us to the topic of vector bundles. As a tool for studying vector bundles we develop the Stiefel–Whitney classes.

The development of vector bundles is based largely on the treatment given in [4] and the notation in the entire thesis is consistent with that in [4]. The discussion of manifolds and the proof of the Whitney Embedding Theorem follow, for the most part, the presentation in [2] although the Local Submersion Theorem and the Mini–Sard Theorem come
from [1]. The development of Stiefel–Whitney classes given here is suggested by a problem in [4, p.171].

We begin, then, by giving a concrete definition of a manifold.

**DEFINITION 1.1.** Let \( U \subseteq \mathbb{R}^n \) be open. A function \( f: U \to \mathbb{R}^k \) (\( k > 0 \)) is smooth or differentiable of class \( C^0 \) if its partial derivatives of all orders exist and are continuous.

**DEFINITION 1.2.** A subset \( M \subseteq \mathbb{R}^k \) is a smooth manifold of dimension \( n \geq 0 \) if, for each \( x \in M \) there exists a smooth function

\[
h: U \to \mathbb{R}^k
\]

defined on an open set \( U \subseteq \mathbb{R}^n \) such that

1) \( h \) maps \( U \) homeomorphically onto an open neighborhood \( V \) of \( x \) in \( M \).

2) For each \( u \in U \) the vectors \( \frac{\partial h}{\partial u_1}, \ldots, \frac{\partial h}{\partial u_n} \) evaluated at \( u \), are linearly independent.

The image \( h(U) = V \) of such a mapping is called a coordinate neighborhood in \( M \), and the triple \( (U,V,h) \) is called a local parametrization of \( M \).

If \( M \) is a smooth manifold and \( A \subseteq M \) is also a smooth manifold then we call \( A \) a smooth submanifold of \( M \).
Sometimes we denote an \( n \)-dimensional smooth manifold \( M \) by \( M^n \) when we wish to emphasize its dimension.

Let \( M^n \subset \mathbb{R}^k \) be an \( n \)-dimensional manifold. Let \((U,V,h)\) be a local parameterization of \( M \) with \( h(u) = x \).

**DEFINITION 1.3.** A vector \( v \in \mathbb{R}^k \) is tangent to \( M^n \) at the point \( x \in M^n \) if \( v \) can be expressed as a linear combination of the vectors

\[
\frac{\partial h(u)}{\partial u_1}, \ldots, \frac{\partial h(u)}{\partial u_n}
\]

**DEFINITION 1.4.** The set of all vectors tangent to \( M \) at the point \( x \) is called the tangent space of \( M \) at \( x \) and is denoted by \( DM_x \).

Thus \( DM_x \) is an \( n \)-dimensional vector space over the real numbers.

It would appear that the tangent space depends on the local parameterization. This, however, is not the case as we now show:

First recall the following Theorem from Analysis:

**THEOREM 1.5 (Inverse Function Theorem)** Suppose that \( f: \mathbb{R}^n \to \mathbb{R}^n \) is a smooth function in an open set containing
a, and det(Df_a) \neq 0. Then there is an open set V containing \( a \) and an open set W containing \( f(a) \) such that \( f:V \to W \) has a smooth inverse \( f^{-1}:W \to V \). i.e. \( f \) is a smooth local diffeomorphism at \( a \).

Let \((U,V,h)\) and \((W,Y,k)\) be two local parameterizations for \( M^n \). Let \( x \in V \cap Y \). Then \( x = h(u) = k(w) \) for some \( u \in U, w \in W \). Define

\[
\begin{align*}
DM_h^x &= \{ t \in M | t = \sum_{i=1}^n a_i \frac{\partial h}{\partial w_i}, \ a_i \in \mathbb{R} \} \\
DM_k^x &= \{ t \in M | t = \sum_{i=1}^n b_i \frac{\partial k}{\partial w_i}, \ b_i \in \mathbb{R} \}
\end{align*}
\]

Let \( t \in DM_h^x \). We want to show that \( t \in DM_k^x \).

Since the matrix \( \left[ \frac{\partial k_i}{\partial w_j}(w) \right] \) is non-singular we have, from the inverse function theorem, that \( k \) has a smooth inverse \( k^{-1} \) in some neighborhood of \( x \).

Then the function

\[
f = k^{-1} \circ h : h^{-1}(V \cap Y) \to W
\]

is a smooth homeomorphism in some neighborhood of \( u \). Thus \( h = k \circ f \) and \( \frac{\partial h}{\partial u_i} = \frac{\partial (k \circ f)}{\partial w_i} \). By the chain rule this is just a linear combination of the \( \frac{\partial k}{\partial w_i} \). Explicitly we can write

\[
\frac{\partial h}{\partial u_i} = \sum_{j=1}^n \frac{\partial k}{\partial w_j} \frac{\partial f_j}{\partial u_i}.
\]

Thus \( t \in DM_k^x \).

So \( DM_h^x \subseteq DM_k^x \).

A similar argument shows that \( DM_k^x \subseteq DM_h^x \). Therefore \( DM_h^x = DM_k^x \) and the tangent space is independent of the local parameterization.
DEFINITION 1.6. The *tangent manifold* of $M$ is defined to be the subspace

$$DM \subset M \times \mathbb{R}^k$$

consisting of all pairs $(x,v)$ with $x \in M$, $v \in DM_x$. So $DM$ is a smooth manifold of dimension $2n$.

Let $M \subset \mathbb{R}^n$, $N \subset \mathbb{R}^m$ be smooth manifolds. Let $x \in M$ and $(U,V,h)$ be a local parameterization of $M$ with $h(u) = x$.

DEFINITION 1.7. A function $f : M \to N$ is said to be *smooth* at $x$ if the composition

$$f \circ h : U \to N \in \mathbb{R}^m$$

is smooth throughout some neighborhood of $u$. We say $f$ is *smooth* if it is smooth at every point $x \in M$.

DEFINITION 1.8. $f : M \to N$ is called a *diffeomorphism* if $f$ is one-to-one, onto, and both $f$ and $f^{-1}$ are smooth.

Let $f : M \to N$ be smooth at $x$ and let $v \in DM_x$. Then we can write

$$v = \sum_{i=1}^{n} a_i \frac{\partial h}{\partial u_i}.$$
We now define a map \( Df_x : \mathcal{D}M \to \mathcal{D}N_{f(x)} \) by

\[
Df_x \left( \sum_{i=1}^{n} a_i \frac{\partial h}{\partial u_i} \right) = \sum_{i=1}^{n} a_i \frac{\partial (f \circ h)}{\partial u_i}.
\]

**DEFINITION 1.9.** The linear transformation \( Df_x \) is called the derivative, or the Jacobian of \( f \) at \( x \).

Just as with the tangent space of a manifold we can show that the derivative of a smooth map is independent of the local parameterization used.

If \( f : M \to N \) is smooth we can define \( Df : \mathcal{D}M \to \mathcal{D}N \) where \( Df(x,v) = (f(x), Df_x(v)) \).

We conclude this chapter with some examples of manifolds which will prove useful later on.

**DEFINITION 1.10.** An \( n \)-frame in \( \mathbb{R}^{n+k} \) is an \( n \)-tuple of linearly independent vectors of \( \mathbb{R}^{n+k} \). The collection of all \( n \)-frames in \( \mathbb{R}^{n+k} \) forms an open subset of the \( n \)-fold cartesian product \( \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k} \) called the Stiefel manifold \( V_n(\mathbb{R}^{n+k}) \).
DEFINITION 1.11. The Grassmann manifold $\mathcal{G}_n(\mathbb{R}^{n,k})$ is the set of all $n$-dimensional planes through the origin of the coordinate space $\mathbb{R}^{n,k}$. This is topologized as follows: Define the function
\[ q : V_n(\mathbb{R}^{n,k}) \to \mathcal{G}_n(\mathbb{R}^{n,k}) \]
which maps each $n$-frame to the $n$-plane which it spans. Now give $\mathcal{G}_n(\mathbb{R}^{n,k})$ the quotient topology induced by $q$. That is, a subset $U \subseteq \mathcal{G}_n(\mathbb{R}^{n,k})$ is open if and only if its inverse image $q^{-1}(U) \subseteq V_n(\mathbb{R}^{n,k})$ is open.

DEFINITION 1.12. Let $\mathbb{R}^\infty$ denote the vector space consisting of those infinite sequences
\[ x = (x_1, x_2, x_3, \ldots) \]
of real numbers where $x_i = 0$ for all but finitely many values of $i$.

For a fixed $k$ the subspace consisting of
\[ \{(x_1, x_2, \ldots, x_k, 0, 0, \ldots)\} \subseteq \mathbb{R}^\infty \]
will be identified with $\mathbb{R}^k$. Thus
\[ \mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \]
with $\mathbb{R}^\infty = \bigcup_{k=1}^{\infty} \mathbb{R}^k$.

DEFINITION 1.13. The infinite Grassmann manifold
\[ G_n = \mathcal{G}_n(\mathbb{R}^\infty) \]
is the set of all $n$-planes through the origin of $\mathbb{R}^\infty$. 
A subset \( U \subseteq G_n \) is open if and only if its intersection with \( G_n(\mathbb{R}^{n+k}) \) is open as a subset of \( G_n(\mathbb{R}^{n+k}) \) for each \( k \). This makes sense since

\[
G_n(\mathbb{R}^n) = \bigcup_{k=0}^{\infty} G_n(\mathbb{R}^{n+k}).
\]
CHAPTER II

THE WHITNEY EMBEDDING THEOREM

We now take a closer look at manifolds and eventually we come to the embedding theorems. First we need more definitions.

DEFINITION 2.1. Let \( f: M \to N \) be smooth. Then \( f \) is immersive at \( x \in M \) if \( Df_x: DM_x \to DN_{f(x)} \) is injective. If \( f \) is immersive at every point of \( M \) we call \( f \) an immersion. If \( Df_x \) is surjective then we call \( f \) submersive at \( x \in M \). If \( f \) is submersive at every point of \( M \) we call \( f \) a submersion.

DEFINITION 2.2. \( f: M \to N \) is an embedding if \( f \) is an immersion in which \( f \) maps \( M \) homeomorphically onto its image. We indicate this by writing \( f: M \hookrightarrow N \).

DEFINITION 2.3. Let \( f: M \to N \) be a smooth map. Then \( y \in N \) is called a regular value for \( f \) if \( Df_x: DM_x \to DN_{f(x)} \) is surjective at every point \( x \in M \) such that \( f(x) = y \).
THEOREM 2.4. (Local Submersion Theorem) If \( f: \mathbb{M}^m \rightarrow \mathbb{N}^n \) is a submersion at \( x \), and \( y = f(x) \) then there exist local parameterizations \((U, V, h)\) and \((W, Y, k)\) around \( x \) and \( y \) for \( \mathbb{M}^m \) and \( \mathbb{N}^n \) respectively such that

\[
f \circ h(x_1, \ldots, x_m) = k(x_1, \ldots, x_n)
\]

i.e. such that we get the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{M}^m & \xrightarrow{f} & \mathbb{N}^n \\
\uparrow h & & \uparrow k \\
U \subset \mathbb{R}^m & \xrightarrow{\pi} & V \subset \mathbb{R}^n
\end{array}
\]

where \( \pi: \mathbb{R}^m \rightarrow \mathbb{R}^n \) is the projection map given by

\[
\pi(x_1, \ldots, x_m) = (x_1, \ldots, x_n)
\]

PROOF: Choose local parameterizations \((U', V, h')\) and \((W, Y, k)\) such that \( h'(0) = x \), \( k(0) = f(x) = y \) and choose a function \( g \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{M}^m & \xrightarrow{f} & \mathbb{N}^n \\
\uparrow h' & & \uparrow k \\
U' \subset \mathbb{R}^m & \xrightarrow{g} & V \subset \mathbb{R}^n
\end{array}
\]
We can always do this since we could just let
\( g = k^{-1} \circ f \circ h' \). This definition actually defines a
function \( g \) for some open set containing \( 0 \) which we call
\( U' \).

Then since \( Df_x \) is surjective we must have that \( Dg_0 \)
is surjective. Since \( Dg_0 \) is linear we can think of it as
being a matrix with the usual base. Thus, by a linear
change of coordinates, we can write \( Dg_0 = [I_n \mid 0] \) where
\( I_n \) is the \( n \times n \) identity matrix.

Now define \( \mathcal{C}:U \to \mathbb{R}^m \) by \( \mathcal{C}(a) = (g(a), a_{n+1}, \ldots, a_m) \)
where \( a = (a_1, \ldots, a_m) \).

Then \( D\mathcal{C}_0 = I_m \)

Thus, by the inverse function theorem, \( \mathcal{C}^{-1} \) exists and
maps some open neighborhood \( U \) of \( 0 \) diffeomorphically onto
\( U' \).

We now have the following commutative diagram:

\[
\begin{array}{ccc}
M^n & \xrightarrow{f} & N^n \\
\downarrow{h'} & & \uparrow{k} \\
U' & \xrightarrow{g} & V \\
\downarrow{\mathcal{C}} & \,
\nearrow{\iota} & \\
U & \simeq & U
\end{array}
\]

Then by setting \( h = h' \circ \mathcal{C}^{-1} \) we get the commutative
diagram:
THEOREM 2.5. (Regular Value Theorem) If \( y \) is a regular value of \( f: M^m \to N^n \) then \( f^{-1}(y) \) is a submanifold of \( M \) with \( \dim f^{-1}(y) = \dim M^m - \dim N^n \).

PROOF: From the local submersion theorem we can find two local parameterizations \((U, V, h)\) and \((W, Y, k)\) for \( M^m \) and \( N^n \) respectively, with \( h(0) = x \) and \( k(0) = y \) such that

\[
f \circ h(x_1, \ldots, x_m) = k(x_1, \ldots, x_n)
\]

Then the set \( f^{-1}(y) \cap V \) is an open neighborhood of \( x \). Also, in \( V \) we have

\[
f^{-1}(y) = \{(0, \ldots, 0, x_{n+1}, \ldots, x_m) \mid x_i \in \mathbb{R}\}.
\]

So the set \( f^{-1}(y) \cap V \) is an open neighborhood of \( x \) in \( f^{-1}(y) \) that is homeomorphic to \( \mathbb{R}^{m-n} \).

Thus \( f^{-1}(y) \) is a submanifold of \( M^m \) with

\[
\dim f^{-1}(y) = \dim M^m - \dim N^n.
\]

DEFINITION 2.6. Let \( f: M \to N \) be smooth. Let \( A \subset N \) be a smooth submanifold of \( N \). Then we say \( f \) is
transverse to $A$ if

$$\text{Image}(Df_x) + DA_y = DN_y$$

whenever $f(x) = y \in A$. That is, the vectors in $\text{Image}(Df_x)$ taken with the vectors in $DA_y$ span $DN_y$.

Notice that if the submanifold $A$ consists of a single point then the notion of transverse is equivalent to the notion of regularity. We can use the idea of transversality to extend the regular value theorem.

**DEFINITION 2.7.** The codimension of a submanifold $A \subset N$ is $\text{codim } A = \dim N - \dim A$.

**THEOREM 2.8.** Let $f: M \to N$ be smooth and let $A \subset N$ be a submanifold. If $f$ is transverse to $A$, then $f^{-1}(A)$ is a submanifold of $M$. Also the codimension of $f^{-1}(A)$ in $M$ is the same as the codimension of $A$ in $N$.

**PROOF:** Let $p = \dim A$ and let $q = \text{codim } A$. Then locally we can write a neighborhood in $N$ around the point $(0,0)$ as $U \times V \subset \mathbb{R}^p \times \mathbb{R}^q$ where $U \times 0$ is a neighborhood around 0 in $A$. We need to show that $f^{-1}(U \times 0)$ is a smooth submanifold of $M$.

Since $f: M \to U \times V$ is transverse to $U \times 0$ so we have

$$\text{Image } Df_x + D(U \times 0)_0 = D(U \times V)_0$$
Where \( f(x) = 0 \). Now consider the map \( g: M \xrightarrow{f} U \times V \xrightarrow{r} V \).

The transversality of \( f \) implies that the map \( Dg_x \) is surjective. So \( 0 \) is a regular value for \( g \). Thus, by the regular value theorem, \( g^{-1}(0) \) is a smooth submanifold of \( M \). But \( g^{-1}(x) = f^{-1}(U \times 0) \). So \( f^{-1}(U \times 0) \) is a smooth submanifold of \( M \). Also \( \dim f^{-1}(U \times 0) = \dim g^{-1}(0) \)

\[
\dim M - \dim V = \dim M - \text{codim } A \quad \text{so} \\
\dim M - \dim f^{-1}(U \times 0) = \text{codim } A
\]

and we have

\[
\text{codim } f^{-1}(U \times 0) = \text{codim } A. \quad \square
\]

We have defined a manifold in such a way that it is already embedded in some Euclidean space, however this is not necessary. Sometimes a manifold is defined as follows:

**DEFINITION 2.9.** A 2nd countable Hausdorff space \( M^n \) is called an \( n \)-dimensional manifold if it is locally homeomorphic to \( \mathbb{R}^n \). That is, around every point \( x \) of \( M^n \) there is a continuous function

\[
h: U \to M^n
\]

that takes an open subset \( U \subset \mathbb{R}^n \) homeomorphically onto an open neighborhood \( V \) of \( x \) in \( M^n \).

Just as before we call the triple \( (U, V, h) \) a local parameterization of \( M^n \).
Now suppose that for $M^n$ we have a local parameterizations $\{(U_\alpha, V_\alpha, h_\alpha)\}$ so that the collection $\{V_\alpha\}$ converges $M^n$. We call $M^n$ smooth if whenever two of the parameterizations, say $(U_1, V_1, h_1)$ and $(U_2, V_2, h_2)$, are such that $V_1 \cap V_2 \neq \emptyset$ the composition
$$h_2^{-1} \circ h_1 : h_1^{-1}(V_1 \cap V_2) \to U_2$$
is smooth.

The fact that the two definitions we have for a manifold are equivalent, follows from the following:

**THEOREM 2.10.** Let $M^n$ be a compact smooth manifold. Then there exists a smooth embedding of $M$ into $\mathbb{R}^q$ for some $q$.

**PROOF:** Let $m = \dim M^n$. Let $D^n(\rho) \subset \mathbb{R}^n$ be the closed disk of radius $\rho$ and center $0$.

Now for each point $x$ of $M^n$ there exists a set $U_i \in \mathbb{R}^n$ and a homeomorphism $h_i : U_i \to V_i$ where $V_i$ is an open neighborhood of $x$. By changing the $h_i$'s if necessary we can require that $0 \in U_i$ and $D^n(2) \subset U_i$. Now we cover $M^n$ by open sets of the form $\text{Int} \ h_i(D^n(1))$. Thus, since $M^n$ is compact, we can find a finite number of local parameterizations $\{(U_i, V_i, h_i)\}_{i=1}^n$ with the following two properties:
1) \( D^n(2) \subseteq U_i \) and
2) \( \mathbb{M}^n = \bigcup \{ \text{Int} \ h_i(D^n(1)) \} \)

We will now define a smooth function \( \lambda: \mathbb{R}^n \to [0,1] \) that is equal to one on the ball \( B^n(1) \) and equal to zero outside the ball \( B^n(2) \). We construct the function \( \lambda \) by first defining the smooth function \( \alpha: \mathbb{R} \to \mathbb{R} \) by

\[
\alpha(x) = \begin{cases} 
    e^{-\frac{x}{2}} & \text{for } x > 0 \\
    0 & \text{for } x \leq 0
\end{cases}
\]

The graph of \( \alpha \) is shown in Figure 2.1.

Next define \( \beta: \mathbb{R} \to \mathbb{R} \) by

\[
\beta(x) = \alpha(x - 1)\alpha(2 - x).
\]

The graph of \( \beta \) is shown in Figure 2.2.
This function is close to the shape of the function we want but we still need to work with it. We define another function $\gamma: \mathbb{R} \rightarrow [0,1]$ by

$$\gamma(x) = \frac{\int_x^2 \beta}{\int_1^2 \beta}$$

see Figure 2.3 for a graph of $\gamma$.

Finally we can define $\lambda: \mathbb{R}^n \rightarrow [0,1]$ by

$$\lambda(x) = \gamma(|x|).$$

A graph of $\lambda$ defined on $\mathbb{R}^1$ is shown in Figure 2.4.
So we have a smooth map $\lambda: \mathbb{R}^n \to [0,1]$ that is equal
to 1 only on $D^n(1)$ and equal to 0 on $\mathbb{R}^n - D^n(2)$.
Such a function is called a bump function.

Define smooth maps:

$$\lambda_i = \begin{cases} 
\lambda \circ h_i^1 & \text{on } h(U_i) \\
0 & \text{on } \mathbb{M} - h(U_i)
\end{cases}$$

Since $\mathbb{M}^m = \bigcup \{\text{Int } h_i(D^n(1))\}$ we have the sets
$B_i = \lambda_i^1(1) \cap h(U_i)$ cover $\mathbb{M}^m$. Define maps $f_i: \mathbb{M} \to \mathbb{R}^m$ by

$$f_i(x) = \begin{cases} 
\lambda_i(x)h_i(x) & \text{if } x \in h_i(U_i) \\
0 & \text{if } x \in \mathbb{M} - h_i(U_i)
\end{cases}$$

and set

$$g_i = (f_i, \lambda_i): \mathbb{M} \to \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{m+1}$$

and

$$g = (g_1, g_2, \ldots, g_n): \mathbb{M} \to \mathbb{R}^{m+1} \times \ldots \times \mathbb{R}^{m+1} = \mathbb{R}^{n(m+1)}$$

Then $g$ is smooth since it is composed of smooth functions.

If $x \in B_i$ then $g_i$ is immersive at $x$ since

$$f_i(x) = \lambda_i(x)h_i^1(x) = h_i^1(x)$$

which is an immersion.

Therefore $g$ is an immersion.
Now we must show that \( g \) is injective. Suppose \( x \neq y \) with \( y \in B_i \) for some \( i \). If \( x \in B_i \) then \( g(x) \neq g(y) \) since \( f_i|B_i = h_i|B_i \) which is a homeomorphism. If \( x \notin B_i \) then \( \lambda_i(y) = 1 \neq \lambda_i(x) \). Therefore \( g(x) \neq g(y) \). So \( g \) is injective.

Now since \( g \) is a continuous one-to-one map from \( M \) (a compact space) onto \( g(M) \) (a Hausdorff space) we have that \( g \) is a homeomorphism from \( M \) onto \( g(M) \).

Therefore \( g \) is an embedding. \( \Box \)

The preceding theorem says that every compact manifold embeds in \( \mathbb{R}^q \) for some \( q \). However, we have no idea of how large \( q \) must be. Fortunately we will soon prove a theorem that does give us a bound for \( q \).

First we need some more definitions.

**DEFINITION 2.11.** Let \( M \) be a metric space and let \( A \) be a subset of \( M \). We say that \( A \) is **nowhere dense** in \( M \) if \((M - \overline{A}) = M\).

Note that a closed set \( A \) is nowhere dense in \( M \) if its complement is dense in \( M \).

**DEFINITION 2.12** A subset \( A \) of \( \mathbb{R}^n \) has \((n - \text{dimensional}) \text{ measure zero}\) if for all \( \epsilon > 0 \) there
exists a cover \( \{U_1, U_2, \ldots \} \) of \( A \) by closed rectangles such that \( \sum_{i=1}^{n} v(U_i) < \epsilon \), where \( v(U_i) \) is the volume of the rectangle \( U_i \).

Note that if a closed set \( A \) has measure zero in \( \mathbb{R}^n \) then \( A \) must be nowhere dense in \( \mathbb{R}^n \), for suppose not, then there exists a point \( x \in \mathbb{R}^n \) such that \( x \notin (\mathbb{R}^n - \bar{A}) \). That is, there is an open ball \( B \) containing \( x \) such that \( B \cap (\mathbb{R}^n - \bar{A}) = \emptyset \). So \( B \subseteq \bar{A} = A \). Then \( A \) does not have measure zero, which is a contradiction.

More generally suppose that \( A \subseteq \mathbb{R}^n \) has measure zero and is \( \sigma \)-compact, that is, \( A \) is the union of a countable collection of compact sets. Each of these is nowhere dense and so \( A \) is nowhere dense by the Baire category theorem (since \( \mathbb{R}^n \) is a complete metric space).

**THEOREM 2.13.** (Mini-Sard Theorem) Let \( M, N \) be manifolds with \( \dim(M) < \dim(N) \). If \( f : M \to N \) is smooth, then \( f(M) \) is nowhere dense in \( N \).

In order to prove this we first need some lemmas.

**LEMMA 2.14** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^n \) be smooth. If \( A \subseteq U \) has measure zero, then so does \( f(A) \).
**PROOF:** We may assume $\bar{A}$ is compact and contained in $U$ since $A$ may be written as a countable union of subsets of $U$ with this property. Let $\epsilon > 0$. Let $W$ be an open neighborhood of $A$ such that $W$ is compact and contained in $U$. Since $W$ is compact, there exists a constant $M$ such that $|f(x) - f(y)| < M|x - y|$ for all $x, y \in W$.

Cover $A$ with a sequence $\{U_1, U_2, \ldots\}$ of closed rectangles, each of which is contained in $W$, such that

$$\sum_{i=1}^{\infty} v(U_i) < \frac{\epsilon}{M}.$$  

If $v(U_i) = k_i$ then there is a rectangle $U'_i$ containing $f(U_i)$ with $v(U'_i) < Mk_i$. So

$$\sum_{i=1}^{n} v(U'_i) < M \sum_{i=1}^{n} k_i < M \frac{\epsilon}{M} = \epsilon.$$

So $f(A)$ has measure zero. \(\square\)

**LEMMA 2.15** Let $U \subset \mathbb{R}^m$ and let $m < n$. Then $U \times 0$ has measure zero in $U \times \mathbb{R}^{n-m}$.

**PROOF:** Let $\epsilon > 0$. We can cover $U \times 0$ by a countable number of closed rectangles $\{U_1, U_2, \ldots\}$ such that every side of $U_i$ has length 1 except the side that extends in the direction of the $n^{th}$ coordinate axis which we let have length $\frac{\epsilon}{\Sigma_{i=1}^{\infty} 1}$. Then

$$\sum_{i=1}^{\infty} v(U_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{\Sigma_{i=1}^{\infty} 1} < \epsilon.$$  

So $U \times 0$ has measure zero in $U \times \mathbb{R}^{n-m}$. \(\square\)
LEMMA 2.16. If $U \subset \mathbb{R}^n$ is open and $g: U \rightarrow \mathbb{R}^n$ is smooth with $m < n$, then $g(U)$ has measure zero.

PROOF: Consider the composition of maps:

$$U \rightarrow U \times 0 \subset U \times \mathbb{R}^{n-m} \xrightarrow{\pi} U \xrightarrow{g} \mathbb{R}^n$$

Since $\pi$ and $g$ are smooth and since $U \times 0$ has $n$-measure zero in $U \times \mathbb{R}^{n-m}$ we have $(g \circ \pi)(U \times 0) = g(U)$ has measure zero in $\mathbb{R}^n$. □

We can define a subset $A$ of a smooth $k$-dimensional manifold $M$ to be of measure zero provided that, for every local parameterization $f: U \rightarrow M$, the preimage $f^{-1}(A)$ has measure zero in $U \subset \mathbb{R}^k$.

PROOF: (of Mini-Sard Theorem) We can cover $M$ with a countable number of local parameterizations $(U_i, V_i, h_i)$. Then $f \circ h_i(U_i)$ has measure zero. Since a countable collection of sets of measure zero has measure zero we see that $f(M)$ has measure zero. Therefore $f(M)$ is nowhere dense in $N$ since $f(M)$ is $\sigma$-compact. □

We are now finally ready to give a stronger embedding theorem:
THEOREM 2.17. (Easy Whitney embedding theorem) Let $M^n$ be a compact Hausdorff smooth $n$-dimensional manifold. Then there is an embedding of $M$ in $\mathbb{R}^{2n+1}$.

PROOF: We have already shown that $M$ embeds in some $\mathbb{R}^q$. So we can replace $M^n$ by its image under an embedding. Thus we assume $M^n$ is a smooth submanifold of $\mathbb{R}^q$. If $q \leq 2n + 1$ there is nothing to prove. Assume that $q > 2n + 1$. It suffices to show that $M^n$ embeds in $\mathbb{R}^{q-1}$ since we could then repeat this process until we have finally embedded $M^n$ in $\mathbb{R}^{2n+1}$. We identify $\mathbb{R}^{q-1}$ with

$$\{(x_1, x_2, \ldots, x_q) \in \mathbb{R}^q | x_q = 0\}.$$

For $v \in \mathbb{R}^q - \mathbb{R}^{q-1}$ let $\pi_v: \mathbb{R}^q \to \mathbb{R}^{q-1}$ be projection parallel to $v$. We want to find a vector $v$ such that $\pi_v|_M$ is an embedding of $M$ into $\mathbb{R}^{q-1}$. We limit our search to unit vectors.

For $\pi_v$ to be injective we must require that $v$ is not parallel to any secant of $M$. That is, if $x, y \in M$ then

$$v \neq \frac{x - y}{|x - y|}$$

(1)

The kernel of $\pi_v$ is just the line through $v$. Therefore a tangent vector $z \in DM_x$ is in the kernel of $D(\pi_v)_x$ only if $z$ is parallel to $v$. So $\pi_v|_M$ is an immersion if, for all $(x, z) \in DM$ with $z \neq 0$,
(2) $v \neq \frac{z}{|z|}$.

To find a vector that satisfies (1) consider the map

$$\sigma: (M \times M) - \Delta \rightarrow S^{q-1}$$

by

$$\sigma(x, y) = \frac{x - y}{|x - y|}$$

where $\Delta = \{(x, y) \in M \times M | x = y\}$ and $S^{q-1} = \{v \in \mathbb{R}^q : |v| = 1\}$. Then $v$ satisfies (1) if and only if $v$ is not in the image of $\sigma$. Since $\sigma$ is a smooth map, and

$$\dim((M \times M) - \Delta) = 2n < \dim(S^{q-1})$$

we have from the Mini-Sard theorem that the image of $\sigma$ is nowhere dense in $S^{q-1}$. So every nonempty open subset of $S^{q-1}$ contains a point $v$ which is not in the image of $\sigma$.

To find a vector that satisfies (2) note that we need only consider points $(x, z) \in DM$ where $|z| = 1$. Define the Unit Tangent Manifold of $M$ to be

$$D_tM = \{(x, z) \in DM : |z| = 1\}$$

Since $DM \subset M \times \mathbb{R}^q$, $D_tM \subset M \times S^{q-1}$. Also $D_tM$ is a compact submanifold of $DM$ (since $M$ is compact). Define a smooth map $\tau: D_tM \rightarrow S^{q-1}$ by projection onto $S^{q-1}$. That is $\tau(x, z) = z$. Geometrically this is just a parallel translation of unit vectors based at a point of $M$ to unit vectors based at $0$. Then a vector $v$ satisfies (2) only if $v$ is not in the image of $\tau$. Again note that

$$\dim(D_tM) = 2n - 1 < \dim(S^{q-1})$$
So the Mini–Sard theorem says that the image of \( \tau \) is nowhere dense in \( S^{q-1} \).

We see that we have many vectors that satisfy (1) and many vectors that satisfy (2). We want a vector that satisfies both (1) and (2). Since \( D_1M \) is compact \( \tau(D_1M) \) is compact. Since every compact subset of a Hausdorff space is closed, \( \tau(D_1M) \) is closed in \( S^{q-1} \). So \( S^{q-1} - \tau(D_1M) \) is an open dense subset of \( S^{q-1} \). Therefore the set 
\[ W = (S^{q-1} - \tau(D_1M)) \cap (\mathbb{R}^q - \mathbb{R}^{q-1}) \]
is a nonempty open subset of \( S^{q-1} \). Every \( v \in W \) satisfies (2) and from our previous work we know that there is a vector \( v \in W \) that satisfies (1). So we have found a vector \( v \in \mathbb{R}^q - \mathbb{R}^{q-1} \) that satisfies (1) and (2). Therefore \( \pi_v|_{\mathcal{M}}:\mathcal{M} \to \mathbb{R}^{q-1} \) is an injective immersion. Furthermore \( \pi_v|_{\mathcal{M}} \) is a one-to-one continuous mapping from a compact space onto its image (which is Hausdorff), so we have that \( \pi_v|_{\mathcal{M}} \) is an injective immersion that maps \( \mathcal{M} \) homeomorphically onto its image. Therefore \( \pi_v|_{\mathcal{M}} \) is an embedding of \( \mathcal{M}^n \) into \( \mathbb{R}^{q-1} \).

In fact, we can even improve upon this result. Whitney has proven the following [8]:

**Theorem 2.18.** (Whitney embedding theorem) Every paracompact Hausdorff manifold \( \mathcal{M}^n \) of dimension \( n \) embeds in \( \mathbb{R}^{2n} \).
Whitney's embedding theorem says any \( n \)-dimensional manifold \( M^n \) can be embedded in \( \mathbb{R}^{2n} \). This theorem is the best possible in the sense that there are \( n \)-dimensional manifolds that cannot be embedded in \( \mathbb{R}^{2n-1} \). One example is the Klein Bottle which is a 2-dimensional manifold that cannot be embedded in \( \mathbb{R}^3 \).

Of course there are 2-dimensional manifolds that embed in \( \mathbb{R}^3 \), such as the torus or the 2-sphere. So Whitney's theorem does not give the smallest dimension of Euclidean space that we need to embed these manifolds. How then, given a specific manifold \( M^n \), can we know if we can embed it in Euclidean space of dimension less than \( 2n \)? To answer this question we must know more than just the dimension of the manifold. We now develop tools that tell us more about the manifold in question.

The first tool we need is a generalization of the tangent manifold:

**Definition 3.1.** Let \( B \) be a fixed topological space called the base space. A real vector bundle \( \xi \) over \( B \) consists of the following:
1) a topological space $E = E(\xi)$ called the total space,

2) a continuous map $\pi: E \to B$ called the projection map, and

3) for each $b \in B$, the set $\pi^{-1}(b)$ has the structure of a vector space over the real numbers.

In addition a vector bundle must satisfy the following:

**Local triviality condition:** For each $b \in B$ there exists a neighborhood $U \subset B$, an integer $n \geq 0$, and a homeomorphism

$$h: U \times \mathbb{R}^n \to \pi^{-1}(U)$$

so that, for each $b \in U$, the correspondence $x \mapsto h(b, x)$ defines an isomorphism between the vector space $\mathbb{R}^n$ and the vector space $\pi^{-1}(b)$.

The pair $(U, h)$ is called a **local coordinate system** for $\xi$ about $b$. If the set $U$ can be chosen to be the entire base space then we call $\xi$ a **trivial bundle**.

The vector space $\pi^{-1}(b)$ is called the **fiber** over $b$ and is denoted by $F_b$ or $F_b(\xi)$. If the dimension of $F_b$ is a constant for all $b$, say $n$, then we say that $\xi$ is an $n$-plane bundle. If $n = 1$ then we call $\xi$ a **line bundle**.

**DEFINITION 3.2.** Let $\xi$ and $\eta$ be two vector bundles over the same base space. We say $\xi$ is **isomorphic**
to \( \eta \) if there exists a homeomorphism \( f: E(\xi) \rightarrow E(\eta) \) between the total spaces which maps each vector space \( F_b(\xi) \) isomorphically onto the corresponding vector space \( F_b(\eta) \). If \( \xi \) is isomorphic to \( \eta \) we write \( \xi \cong \eta \).

Actually we only need the function \( f \) to be continuous and map fibers isomorphically onto fibers because of the following:

**Theorem 3.3.** Let \( \xi \) and \( \eta \) be vector bundles over \( B \) and let \( f: E(\xi) \rightarrow E(\eta) \) be a continuous function which maps each vector space \( F_b(\xi) \) isomorphically onto the corresponding vector space \( F_b(\eta) \). Then \( f \) is necessarily a homeomorphism. Hence \( \xi \) is isomorphic to \( \eta \).

**Proof:** Given any point \( b_0 \in B \), choose local coordinate systems \( (U, g) \) for \( \xi \), and \( (V, h) \) for \( \eta \), with \( b_0 \in U \cap V \). Then we must show that the composition

\[
(U \cap V) \times \mathbb{R}^n \xrightarrow{h^{-1} \circ f \circ g} (U \cap V) \times \mathbb{R}^n
\]

is a homeomorphism.

We know that \( h^{-1} \circ f \circ g \) is continuous, one-to-one, and onto, we must show that it has a continuous inverse. Set \( h^{-1} \circ f \circ g(b, x) = (b, y) \). Then since \( f \) maps \( F_b(\xi) \) isomorphically onto \( F_b(\eta) \) there is a non-singular \( n \times n \) matrix \( [f_{ij}(b)] \) of real numbers that allows us to express
\[ y = (y_1, \ldots, y_n) \text{ in the form} \]
\[ y_i = \sum_{j=1}^{n} f_{ij}(b) x_j. \]

Let \([F_{ji}(b)]\) denote the inverse matrix of \([f_{ij}(b)]\).
Then \(g^{-1} \circ f^{-1} \circ h(b,y) = (b,x)\) where
\[ x_j = \sum_{i=1}^{n} F_{ji}(b)y_i. \]

Since the \(F_{ji}(b)\) depend continuously on the matrix \([f_{ij}(b)]\), they depend continuously on \(b\).
Thus \(g^{-1} \circ f^{-1} \circ h\) is continuous. \(\square\)

EXAMPLE 3.4. Let \(B\) be some topological space. The trivial bundle over \(B\) is the bundle with total space \(B \times \mathbb{R}^n\) for some \(n > 0\). The projection map is \(\pi(b,x) = b\) and the vector space structure given to the fibers is defined by
\[ t_1(b,x_1) + t_2(b,x_2) = (b, t_1x_1 + t_2x_2) \]
for every \(t_1, t_2 \in \mathbb{R}\). We denote the trivial \(n\)-plane bundle over \(B\) by \(\xi^n_B\).

THEOREM 3.5. An \(n\)-plane bundle over \(B\) is trivial if and only if it is isomorphic to \(\xi^n_B\).

PROOF: Let \(\xi\) be a trivial \(n\)-plane bundle. Then there is a homeomorphism \(h: B \times \mathbb{R}^n \to \pi^{-1}(B)\) so that the correspondence \(x \mapsto h(b,x)\) defines an isomorphism between \(\mathbb{R}^n\) and \(\pi^{-1}(b)\). But \(\pi^{-1}(B) = E(\xi)\) and \(B \times \mathbb{R}^n = E(\xi^n_B)\). So
we have a homeomorphism \( h : E(\varepsilon^n_B) \to E(\xi) \) that maps each fiber of \( \varepsilon^n_B \) isomorphically onto the corresponding fiber of \( \xi \). Thus \( \xi \cong \eta \).

Now suppose \( \xi \cong \eta \). Then there is a homeomorphism \( h : E(\varepsilon^n_B) \to E(\xi) \) which maps each fiber of \( \varepsilon^n_B \) isomorphically onto the corresponding fiber of \( \xi \). But then we have \( h : B \times \mathbb{R} \to \pi^{-1}(B) \) so that for each \( b \in B \) the correspondence \( x \mapsto h(b, x) \) defines an isomorphism between \( \mathbb{R}^n \) and \( \pi^{-1}(b) \). Thus \( \xi \) is trivial. \( \square \)

**EXAMPLE 3.6.** The tangent bundle \( \tau_M \) of a smooth manifold \( M \) is defined to be the vector bundle with base space \( M \), total space \( DM \) and projection map \( \pi : DM \to M \) defined by \( \pi(x, v) = x \). Again the vector space structure in the fibers \( \pi^{-1}(x) \) is defined by

\[
t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2).
\]

**DEFINITION 3.7.** Real Projective Space \( \mathbb{RP}^n \) is the set of all unordered pairs \( \{x, -x\} \) where \( x \) ranges over the unit sphere \( S^n \subset \mathbb{R}^{n+1} \). We topologize \( \mathbb{RP}^n \) as a quotient space of \( S^n \).

**DEFINITION 3.8.** The canonical line bundle \( \gamma^1_n \) over \( \mathbb{RP}^n \) is obtained as follows: The total space \( E(\gamma^1_n) \subset \mathbb{RP}^n \times \mathbb{R}^{n+1} \) consists of all pairs \( (\{\pm x\}, v) \) such
that the vector \( v \) is a multiple of \( x \). The projection map 
\( \pi: E(\gamma_d^n) \rightarrow \mathbb{R}P^n \) is defined be \( \pi(\{\pm x\}, v) = \{\pm x\} \). Thus each fiber \( \pi^{-1}(\{\pm x\}) \) can be identified with the line through \( x \) and \( -x \) in \( \mathbb{R}^{n+1} \). We give each such line its usual vector space structure.

To show that \( \gamma_d^n \) is, in fact, a vector bundle we must show that the local triviality condition is satisfied: Let \( U \subseteq S^n \) be any open set which contains no antipodal points, and let \( U_1 \) denote the image of \( U \) in \( \mathbb{R}P^n \) under the map \( x \mapsto \{\pm x\} \). Then we define a homeomorphism 
\[
h: U_1 \times \mathbb{R} \rightarrow \pi^{-1}(U_1)
\]
by
\[
h(\{\pm x\}, t) = (\{\pm x\}, tx)
\]
for each \((x, t) \in U \times \mathbb{R}\). Then \((U_1, h)\) is a local coordinate system for \( \gamma_d^n \). Hence \( \gamma_d^n \) is locally trivial. □

Since a line through the origin in \( \mathbb{R}^{n+1} \) intersects the unit sphere \( S^n \) at two antipodal points we can think of a point in \( \mathbb{R}P^n \) as being a line through the origin instead of the pair \( \{\pm x\} \). But a line through the origin is just a 1-plane, so we see that \( \mathbb{R}P^n = G_1(\mathbb{R}^{n+1}) \) the Grassmann manifold.

**DEFINITION 3.9.** The canonical vector bundle \( \gamma^n(\mathbb{R}^{n\times k}) \) over \( G_n(\mathbb{R}^{n\times k}) \) is constructed as follows: Let 
\( E = E(\gamma^n(\mathbb{R}^{n\times k})) \) be the set of all pairs
(n–plane through 0 in $\mathbb{R}^{n+k}$, vector in that n–plane).

This is to be topologized as a subspace of $G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}$.

The projection map $\pi: E \rightarrow G_n(\mathbb{R}^{n+k})$ is defined by $\pi(X,x) = X$, and the vector space structure in the fiber over $X$ is defined by

$$t_1(X,x_1) + t_2(X,x_2) = (X, t_1x_1 + t_2x_2).$$

Note that $\gamma^1(\mathbb{R}^{n+1})$ is the same as the line bundle $\gamma_n^1$ over $\mathbb{R}P^n$.

Similarly we can construct the canonical bundle $\gamma^n$ over $G_n$ by letting $E(\gamma^n) \subset G_n \times \mathbb{R}^n$ be the set of all pairs

(n–plane through 0 in $\mathbb{R}^n$, vector in that n–plane).

The projection map $\pi: E(\gamma^n) \rightarrow G_n$ is given by $\pi(X,x) = X$. The vector space structure in the fibers is defined as before.

The infinite Grassmann manifold is useful to us because of the following property.

THEOREM 3.10. Any n–plane bundle $\xi$ over a paracompact base space admits a bundle map $\xi \rightarrow \gamma^n$.

In other words, every n–plane bundle over the paracompact base space $B$ is isomorphic to the induced
bundle \( f^*(\gamma^n) \) for some function \( f:B \to G_n \). For this reason \( \gamma^n \) is called the \textit{universal} \( n \)-\textit{plane bundle}.

**PROOF:** Let \( \xi \) be an \( n \)-plane bundle over the paracompact base space \( B \). We first show that there exists a locally finite covering of \( B \) by countably many open sets \( U_1, U_2, U_3, \ldots \) so that \( \xi|U_i \) is trivial for each \( i \). To do this we use the fact that every paracompact space is normal \cite[p.147]{7}.

Choose a locally finite open covering \( \{V_\alpha\} \) so that each \( \xi|V_\alpha \) is trivial, and choose an open covering \( \{W_\alpha\} \) with \( W_\alpha \subseteq V_\alpha \) for each \( \alpha \). Let

\[
\lambda_\alpha : B \to \mathbb{R}
\]

be a continuous function which takes the value 1 on \( W_\alpha \) and the value 0 outside of \( V_\alpha \).

For each non-empty finite subset \( S \) of the index set \( \{\alpha\} \), let \( U(S) \subseteq B \) denote the set of all \( b \in B \) for which

\[
\min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \notin S} \lambda_\alpha(b).
\]

Let \( U_k \) be the union of those sets \( U(S) \) for which \( S \) has precisely \( k \) elements. Then each \( U_k \) is open and

\[ B = U_1 \cup U_2 \cup U_3 \cup \cdots \]

since, given \( b \in B \), if precisely \( k \) of the numbers \( \lambda_\alpha(b) \) are positive, then \( b \in U_k \). If \( \alpha \) is any element of the set \( S \) note that

\[ U(S) \subseteq V_\alpha. \]
Since the covering \( \{ V_\alpha \} \) is locally finite, it follows that \( \{ U_k \} \) is locally finite. Furthermore, since each \( \xi|V_\alpha \) is trivial, each \( \xi|U(S) \) is trivial.

But the set \( U_k \) is equal to the disjoint union of its open subsets \( U(S) \). Therefore \( \xi|U_k \) is also trivial.

So we have found a countable number of sets \( U_1, U_2, U_3, \ldots \) that cover \( B \) such that \( \xi|U_i \) is trivial for each \( i \).

Now to construct a bundle map \( f: \xi \to \gamma^n \) we first construct a map

\[
\hat{f}: E(\xi) \to \mathbb{R}^n
\]

which is linear and injective on each fiber of \( \xi \).

Let \( U_1, U_2, U_3, \ldots \) be as above. Again using the fact that \( B \) is normal, there exist open sets \( Q_1, Q_2, Q_3, \ldots \) covering \( B \) with \( Q_i \subset U_i \). Similarly construct \( D_1, D_2, D_3, \ldots \) with \( D_i \subset Q_i \). Let

\[
\lambda_i: B \to \mathbb{R}
\]

denote a continuous function which takes the value 1 on \( D_i \) and the value 0 outside of \( Q_i \).

Since \( \xi|U_i \) is trivial there exists a map

\[
h_i: \pi^{-1}(U_i) \to \mathbb{R}^n
\]

which maps each fiber of \( \xi|U_i \) linearly onto \( \mathbb{R}^n \). Define \( h_i': E(\xi) \to \mathbb{R}^n \) by

\[
h_i' = \begin{cases} 
\lambda_i(\pi(e)) h_i(e) & \text{for } \pi(e) \in U_i \\
0 & \text{for } \pi(e) \notin Q_i
\end{cases}
\]
Then \( h'_1 \) is continuous, and is linear on each fiber. Now define

\[
\tilde{f}: E(\xi) \to \mathbb{R}^n \oplus \mathbb{R}^n \oplus \cdots \cong \mathbb{R}^n
\]

by \( \tilde{f}(e) = (h'_1(e), h'_2(e), \ldots) \). Then \( \tilde{f} \) is also continuous and maps each fiber injectively. Now define

\[
f: E(\xi) \to E(\gamma^n)
\]

by

\[
f(e) = (\tilde{f}(\text{fiber through } e), \tilde{f}(e))
\]

Then \( f \) is continuous and maps each fiber of \( E(\xi) \) isomorphically onto a fiber of \( E(\gamma^n) \). Therefore \( f \) is a bundle map from \( \xi \) to \( \gamma^n \). \( \square \)

**Definition 3.11.** A cross-section of a vector bundle \( \xi \) over \( B \) is a continuous function

\[
s: B \to E(\xi)
\]

which maps each \( b \in B \) into the corresponding fiber \( F_b(\xi) \). A cross-section is nowhere zero if \( s(b) \) is a non-zero vector of \( F_b(\xi) \) for each \( b \).

Now consider a collection \( \{s_1, \ldots, s_n\} \) of cross-sections of a vector bundle \( \xi \).

**Definition 3.12.** The cross-sections \( s_1, \ldots, s_n \) are nowhere dependent if, for each \( b \in B \), the vectors \( s_1(b), \ldots, s_n(b) \) are linearly independent.
THEOREM 3.13. An $\mathbb{R}^n$-bundle $\xi$ is trivial if and only if $\xi$ admits $n$ cross-sections $s_1, \ldots, s_n$ which are nowhere dependent.

PROOF: Let $s_1, \ldots, s_n$ be cross-sections which are nowhere dependent. Define

$$f : B \times \mathbb{R}^n \to E$$

by

$$f(b, x) = x_1 s_1(b) + \cdots + x_n s_n(b).$$

Then $f$ is continuous and since the $s_1, \ldots, s_n$ are nowhere dependent, $f$ maps each fiber of the trivial bundle $e^n_B$ isomorphically onto the corresponding fiber of $\xi$. From Theorem 3.3, $f$ is a bundle isomorphism. Thus $\xi \cong e^n_B$. So $\xi$ is trivial.

Now suppose that $\xi$ is trivial with coordinate system $(B, h)$. Define

$$s_i(b) = h(b, (0, \ldots, 0, 1, 0, \ldots, 0)) \in F_b(\xi)$$

(with the 1 in the $i^{th}$ place). Since we know the correspondence $x \mapsto h(b, x)$ is an isomorphism, $s_1(b), \ldots, s_n(b)$ are $n$ nowhere dependent cross sections. $\square$

We need to consider vector bundles in which each fiber has the structure of a Euclidean Vector Space.

DEFINITION 3.14. A real valued function $\mu$ on a finite dimensional vector space $V$ is quadratic if $\mu$ can
be expressed in the form

\[ \mu(v) = \sum_{i=1}^{n} l_i(v) h_i(v) \]

where \( l_i \) and \( h_i \) are linear. \( \mu \) is called positive definite if \( \mu(v) > 0 \) for \( v \neq 0 \).

**DEFINITION 3.15.** If \( \mu \) is quadratic, then we define an inner product on the vector space \( V \) to be the map from \( V \times V \) to \( \mathbb{R} \) given by

\[ (v,w) \mapsto v \cdot w \]

where

\[ v \cdot w = \frac{1}{2}(\mu(v + w) - \mu(v) - \mu(w)) \]

Notice that

\[ v \cdot v = \frac{1}{2}(\mu(v + v) - \mu(v) - \mu(v)) \]
\[ = \frac{1}{2} \mu(v + v) - \mu(v) \]
\[ = \frac{1}{2} \sum_{i=1}^{n} l_i(v+v) h_i(v+v) - \mu(v) \]
\[ = 2 \sum_{i=1}^{n} l_i(v) h_i(v) - \mu(v) \]
\[ = 2\mu(v) - \mu(v) \]
\[ = \mu(v). \]

**DEFINITION 3.16.** A Euclidean vector space is a real vector space \( V \) together with a positive definite quadratic function

\[ \mu: V \to \mathbb{R} \]

For each \( v \in V \) we define the norm of \( v \), denoted \( |v| \), by \( |v| = \sqrt{v \cdot v} = \sqrt{\mu(v)} \).
DEFINITION 3.17. A Euclidean vector bundle is a real vector bundle $\xi$ together with a continuous function $\mu: E(\xi) \rightarrow \mathbb{R}$ such that the restriction of $\mu$ to each fiber of $\xi$ is positive definite and quadratic. The function $\mu$ is called a Euclidean metric on $\xi$.

In the case of the tangent bundle $\tau_M$ of a smooth manifold a Euclidean metric $\mu: DM \rightarrow \mathbb{R}$ is called a Riemannian metric, and $M$ together with $\mu$ is called a Riemannian manifold.

Probably the most natural Euclidean metric we can give the trivial bundle $\varepsilon^n_B$ is $\mu(b, x) = x_1^2 + \cdots + x_n^2$ so that we have the usual notion of distance in the fibers.

Since the tangent bundle of $\mathbb{R}^n$ is trivial it follows that the smooth manifold $\mathbb{R}^n$ possesses a standard Riemannian metric. Now for any smooth manifold $M \subset \mathbb{R}^n$ the composition

$$DM \subset D\mathbb{R}^n \xrightarrow{\mu} \mathbb{R}$$

makes $M$ into a Riemannian manifold. Thus, since every paracompact manifold $M^n$ of dimension $n$ embeds in $\mathbb{R}^{2n}$, $M^n$ can be given a Riemannian structure. Note, however, that this structure depends upon the embedding.
THEOREM 3.18. Let $\xi$ be a trivial vector bundle of dimension $n$ over $B$, and let $\mu$ be any Euclidean metric on $\xi$. Then there exist $n$ cross-sections $s_1, \ldots, s_n$ of $\xi$ which are normal and orthogonal in the sense that
\[
s_i(b) \cdot s_j(b) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]
for each $b \in B$.

PROOF: Let $s'_1, \ldots, s'_n$ be any $n$ cross-sections which are nowhere dependent. We can apply the Gram-Schmidt process to $s'_1(b), \ldots, s'_n(b)$ and obtain a normal orthogonal basis $s_1(b), \ldots, s_n(b)$ for $F_b(\xi)$. Since the resulting functions $s_1, \ldots, s_n$ are continuous we have our required cross-sections. □

Given a vector bundle $\xi$ with projection $\pi: E \to B$ we can construct other vector bundles from this, as we now show.

DEFINITION 3.19. Let $\xi$ be as above. Let $B'$ be a subset of $B$. Setting $E' = \pi^{-1}(B')$ and letting
\[
\pi: E' \to B'
\]
be the restriction of $\pi$ to $E'$ we obtain a vector bundle which will be denoted by $\xi|B'$ and called the restriction of $\xi$ to $B'$. Each fiber $F_b(\xi|B')$ is equal to the
corresponding fiber \( F_b(\xi) \), and is to be given the same vector space structure.

Note that using the above terminology we can restate the local triviality condition by saying that for each \( b \in B \) there is a neighborhood \( U \) of \( b \) such that the bundle \( \xi|U \) is trivial.

DEFINITION 3.20. Let \( \xi \) be as defined above and let \( B_1 \) be a topological space. Given any map \( f:B_1 \to B \) we can construct the induced bundle \( f^*\xi \) over \( B \) as follows: The total space \( E_1 \) of \( f^*\xi \) is the subset \( E_1 \subseteq B_1 \times E \) consisting of all pairs \( (b,e) \) with \( f(b) = \pi(e) \). The projection map \( \pi_1:E_1 \to B_1 \) is defined by \( \pi_1(b,e) = b \). Thus we have the commutative diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\tilde{f}} & E \\
\pi_1 \downarrow & & \downarrow \pi \\
B_1 & \xrightarrow{f} & B
\end{array}
\]

where \( \tilde{f}(b,e) = e \).

The vector space structure in \( \pi_1^{-1}(b) \) is defined by \( t_1(b,e_1) + t_2(b,e_2) = (b, t_1e_1 + t_2e_2) \).

Thus \( \tilde{f} \) carries each vector space \( F_b(f^*\xi) \) isomorphically onto the vector space \( F_{f(b)}(\xi) \).
To see that $f^*\xi$ is locally trivial let $(U, h)$ be a local coordinate system for $\xi$, set $U_1 = f^{-1}(U)$ and define

$$h_1: U_1 \times \mathbb{R}^n \rightarrow \pi^{-1}(U_1)$$

by $h_1(b, x) = (b, h(f(b), x))$.

Then $(U_1, h_1)$ is a local coordinate system for $f^*\xi$. So $f^*\xi$ is locally trivial.

More generally we have:

**DEFINITION 3.21.** A **Bundle map** from $\eta$ to $\xi$ is a continuous function

$$g: E(\eta) \rightarrow E(\xi)$$

which carries each vector space $F_b(\eta)$ isomorphically onto one of the vector spaces $F_d(\xi)$. If we set $\bar{g}(b) = d$ then the function $\bar{g}: B(\eta) \rightarrow B(\xi)$ is continuous and we have the following commutative diagram

$$
\begin{array}{ccc}
E(\eta) & \xrightarrow{g} & E(\xi) \\
\pi(\xi) \downarrow & & \downarrow \pi(\eta) \\
B(\eta) & \xrightarrow{\bar{g}} & B(\xi)
\end{array}
$$

**THEOREM 3.22.** If $g: E(\eta) \rightarrow E(\xi)$ is a bundle map, and if $\bar{g}: B(\eta) \rightarrow B(\xi)$ is the corresponding map of base spaces, then $\eta$ is isomorphic to the induced bundle $\bar{g}^*\xi$. 
PROOF: Define $h:E(\eta) \to E(\tilde{g}^*\xi)$ by

$$h(e) = (\pi(e), g(e))$$

where $\pi$ denotes the projection map of $\eta$. Since $h$ is continuous and maps each fiber $F_b(\eta)$ isomorphically onto the corresponding fiber $F_b(\tilde{g}^*\xi)$, $h$ must be a homeomorphism by Theorem 3.3. Thus $\eta \cong \tilde{g}^*\xi$. $\square$

Natural operations on vector spaces can also be extended to vector bundles.

DEFINITION 3.23. Let $\xi_1$ and $\xi_2$ be vector bundles with projection maps $\pi_i:E_i \to B_i$ $i = 1, 2$. The Cartesian Product $\xi_1 \times \xi_2$ is defined to be the bundle with projection map

$$\pi_1 \times \pi_2:E_1 \times E_2 \to B_1 \times B_2$$

where the vector space structure on each fiber

$$(\pi_1 \times \pi_2)^{-1}(c, d) = F_c(\xi_1) \times F_d(\xi_2)$$

is given by

$$t_1(c \times d, e_1 \times e_2) + t_1(c \times d, e_2 \times e_4)$$
$$= (c, t_1e_1 + t_2e_2) \times (d, t_1e_3 + t_2e_4).$$

DEFINITION 3.24. Let $\xi_1$ and $\xi_2$ be two vector bundles over the same base space $B$. Let $d:B \to B \times B$ be given by $d(b) = (b, b)$. The bundle $d^*(\xi_1 \times \xi_2)$ over $B$ is called the Whitney sum of $\xi_1$ and $\xi_2$, and will be denoted by $\xi_1 \oplus \xi_2$. 


Now consider the fiber $F_b(d^*(\xi_1 \times \xi_2)) = F_b(\xi_1 \oplus \xi_2)$. We have the following commutative diagram

$$
\begin{array}{ccc}
F_b(\xi_1 \oplus \xi_2) & \xrightarrow{d} & F_b(\xi_1) \oplus F_b(\xi_2) \\
\downarrow \pi_2 & & \downarrow \pi_1 \\
B & \xrightarrow{d} & B \times B
\end{array}
$$

and we see that $F_b(\xi_1 \oplus \xi_2)$ is isomorphic to $F_b(\xi_1) \oplus F_b(\xi_2)$ where the isomorphism is given by $\dot{d}$.

**Definition 3.25.** Let $\xi$ and $\eta$ be vector bundles over the same base space with $E(\xi) \subset E(\eta)$. Then $\xi$ is a sub-bundle of $\eta$ (written $\xi \subset \eta$) if each fiber $F_b(\xi)$ is a sub-vector-space of the corresponding fiber $F_b(\eta)$.

**Definition 3.26.** Let $\eta$ be a vector bundle on which a Euclidean metric has been defined and let $\xi \subset \eta$ be a sub-bundle. Then the orthogonal complement $\xi^\perp$ of $\xi$ in $\eta$ is defined by letting the fibers $F_b(\xi^\perp)$ be the subspace of $F_b(\eta)$ consisting of all vectors $v$ such that $v \cdot w = 0$ for all $w \in F_b(\xi)$ and letting the total space be

$$
E(\xi^\perp) = \bigcup_{b \in B} F_b(\xi^\perp).
$$
THEOREM 3.27. \( E(\xi^+) \) is the total space of a sub-bundle \( \xi^+ \subset \eta \). Furthermore \( \eta \cong \xi \oplus \xi^+ \).

Before we prove this we need the following

LEMMA 3.28. Let \( \xi_1 \) and \( \xi_2 \) be sub-bundles of \( \eta \) such that each vector space is equal to the direct sum of the sub-spaces \( F_b(\xi_1) \) and \( F_b(\xi_2) \). Then \( \eta \cong \xi_1 \oplus \xi_2 \).

PROOF (of Lemma): Define \( f : E(\xi_1 \oplus \xi_2) \to E(\eta) \) by \( f(b,e_1,e_2) = e_1 + e_2 \). Then \( f \) is continuous and since \( F_b(\xi_1 \oplus \xi_2) \cong F_b(\xi_1) \oplus F_b(\xi_2) \) which equals \( F_b(\eta) \), we have from Theorem 3.3 that \( f \) is a homeomorphism and thus \( \eta \cong \xi_1 \oplus \xi_2 \). □

PROOF (of Theorem): Each vector space \( F_b(\eta) = F_b(\xi) \oplus F_b(\xi^+) \) so from Lemma 3.26 \( \eta \cong \xi \oplus \xi^+ \) provided that \( \xi^+ \) is, in fact, a sub-bundle of \( \eta \). To show this we must show that \( \xi^+ \) is locally trivial. Let \( b_0 \in B \) and let \( U \) be a neighborhood of \( b_0 \) which is sufficiently small that both \( \xi|U \) and \( \eta|U \) are trivial. From Theorem 3.16 there exist orthonormal cross-sections \( s_1, \ldots, s_m \) of \( \xi|U \) and \( r_1, \ldots, r_n \) of \( \eta|U \) where \( m \) and \( n \) are the respective fiber dimensions. Thus the \( m \times n \) matrix \( [s_i(b_0) \cdot r_j(b_0)] \) has rank \( m \). Renumbering the \( r_j \) if
necessary we may assume that the first $m$ columns are linearly independent.

Let $V \subset U$ be the set consisting of all points $b$ for which the first $m$ columns of the matrix $[s_i(b) \cdot r_j(b)]$ are linearly independent. Consider the square matrix that consists of the first $m$ rows and $m$ columns. Since the determinant function is continuous $\det^{-1}([0])$ is closed. Also the complement of $V$, $V^c = \det^{-1}([0])$. Hence $V$ is open.

We now show that the sections $s_1, s_2, \ldots, s_m, r_{m+1}, \ldots, r_n$ of $\eta|U$ are not linearly dependent at any point of $V$. Suppose that the sections were linearly dependent at some point $b \in V$. Without loss of generality we can assume that

$$s_1(b) = \sum_{i=2}^{m} a_i s_i(b) + \sum_{j=m+1}^{n} c_j r_j(b)$$

where at least one $c_j$ is not zero. Hence

$$\sum_{i=1}^{m} a_i s_i(b) = \sum_{j=m+1}^{n} c_j r_j(b).$$

So at least one of the $s_1(b), \ldots, s_m(b)$ is in the subspace spanned by $r_{m+1}(b), \ldots, r_n(b)$ and thus is orthogonal to $r_1(b), \ldots, r_m(b)$. That is, for at least one value of $i$, $s_i(b) \cdot r_j(b) = 0$ if $j \leq m$. Hence at least one of the first $m$ columns of the matrix $[s_i(b) \cdot r_j(b)]$ is zero. This contradicts the fact that the first $m$ columns of the matrix are linearly independent.
Therefore $s_1, s_2, \ldots, s_m, \tau_{m+1}, \ldots, \tau_n$ are not linearly dependent at any point of $V$.

Applying the Gram-Schmidt process to this sequence of cross-sections, we obtain normal orthogonal cross-sections $s_1, \ldots, s_m, s_{m+1}, \ldots, s_n$ of $\eta|V$.

Now a local coordinate system
\[ h: V \times \mathbb{R}^{n-m} \to E(\xi^1) \]
for $\xi^1$ is given by the formula
\[ h(b, x) = x_1 s_{m+1}(b) + \cdots + x_{n-m} s_n(b). \]
Since the inverse function
\[ h^{-1}(e) = \left( \pi(e), (e \cdot s_{m+1}(\pi(e)), \ldots, e \cdot s_n(\pi(e))) \right) \]
is continuous, $h$ is a homeomorphism.

Thus $\xi^1$ is locally trivial and we are done. $\square$

DEFINITION 3.29. Let $M \subset N$ be smooth manifolds, and suppose that $N$ is provided with a Riemannian metric. Then the tangent bundle $\tau_M$ is a sub-bundle of the restriction $\tau_N|M$. We define the normal bundle $\nu$ of $M$ in $N$ to be the orthogonal complement $\tau_M^\perp \subset \tau_N|M$.

From the preceding theorem we have
COROLLARY 3.30. For any smooth submanifold $M$ of a smooth Riemannian manifold $N$ the normal bundle $\nu$ is defined, and

$$\tau_M \Theta \nu \cong \tau_N | M.$$
CHAPTER IV

STIEFEL–WHITNEY CLASSES

In this chapter we use the following facts from Algebraic Topology without proof.

To each topological space $B$ there corresponds a sequence of groups $H^k(B; \mathbb{Z}_2)$ called the $k^{th}$ singular cohomology groups with coefficients in $\mathbb{Z}_2$ defined for all integers $k \geq 0$. If we let

$$H^*(B; \mathbb{Z}_2) = H^0(B; \mathbb{Z}_2) \times H^1(B; \mathbb{Z}_2) \times H^2(B; \mathbb{Z}_2) \times \cdots$$

then $H^*(B; \mathbb{Z}_2)$ has the structure of a ring with the multiplication operation being given by the cup product $\cup$. Since, in what follows, we always use cohomology and homology with coefficients in $\mathbb{Z}_2$ we just write $H^n(B)$ instead of $H^n(B; \mathbb{Z}_2)$.

Let $\xi$ be a real $n$–plane bundle with total space $E = E(\xi)$, base space $B$, and projection $\pi:E \to B$. Furthermore assume that a Euclidean metric has been defined on $\xi$. Let $E_0$ denote the set of all non–zero vectors in $E$. Then we define an $(n-1)$–plane bundle $\xi_0$ over $E$ as follows: A point in $E_0$ is specified by a fiber $F$ of $\xi$ together with a non–zero vector $v$ in that fiber. The fiber of $\xi_0$ over $v$ is by definition, the orthogonal
complement of $v$ in the vector space $F$. Then this is a real vector space of dimension $(n - 1)$. In terms of local coordinates, for each point $(b, v) \in E_0$ ($b \in B, v \in \pi^{-1}(b), v \neq 0$) define the fiber of $\xi_0$ over $(b, v)$ to be the orthogonal complement of $v$ in $\pi^{-1}(b)$. Then a point in the total space of $\xi_0$ looks like $(b, v, z)$ where $(b, v) \in E_0$ and $z \in \pi^{-1}(b)$ with $z \cdot v = 0$. The projection map $\pi_0$ of $\xi_0$ is then given by $$\pi_0(b, v, z) = (b, v).$$ Thus we have constructed an $(n - 1)$-plane bundle $\xi_0$ over $E_0$.

Note that $E_0$ (and hence $\xi_0$) is independent of the Euclidean metric used since any Euclidean metric $\mu$ is positive definite which means that $\mu(v) > 0$ for $v \neq 0$.

We now have the following important Theorem from Algebraic Topology [4, p.106] or [6, p.144]:

**THEOREM 4.1.** (Thom Isomorphism Theorem) The group $H^n(E, E_0)$ contains a unique class $u$ such that for each fiber $F = \pi^{-1}(b)$ the restriction $$u|(F, F_0) \in H^n(F, F_0)$$ is non-zero. Furthermore the correspondence $x \mapsto x \circ u$ defines an isomorphism $H^k(E) \rightarrow H^{k+m}(E, E_0)$ for every $k$. (Here $\circ$ denotes the cup product operation.)
DEFINITION 4.2. Let \( u \) be as in Theorem 4.1. Then \( u \) is called the fundamental cohomology class or the Thom class.

If \( E \) is the total space of a vector bundle over \( B \) with projection map \( \pi \), then the zero section embeds \( B \) as a deformation retract of \( E \) with retraction mapping \( \pi \). Thus the map \( \pi:E \to B \) induces an isomorphism:

\[
\pi^*: H^k(B) \to H^k(E).
\]

DEFINITION 4.3. The Thom isomorphism \( \phi:H^k(B) \to H^k(E,E_0) \) is defined to be the composition of the two isomorphisms:

\[
H^k(B) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\nu} H^{k+n}(E,E_0).
\]

Let \( \xi \) be an \( n \)-plane bundle with projection map \( \pi:E \to B \). Restricting \( \pi \) to the non-zero vectors in \( E \) we obtain an associated map \( \pi_0:E_0 \to B \). We now have the following.

THEOREM 4.4. To any \( n \)-plane bundle \( \xi \) there is associated an exact sequence of the form

\[
\cdots \to H^i(B) \xrightarrow{\nu_n} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \to H^{i+1}(B) \xrightarrow{\nu} \cdots.
\]

The above sequence is called the Gysin sequence of \( \xi \).
PROOF: We start with the cohomology exact sequence
\[ \cdots \to H^i(E, E_0) \to H^i(E) \to H^i(E_0) \to \delta \to H^{i+1}(E, E_0) \to \cdots \]
of the pair \((E, E_0)\).

The isomorphism
\[ \varphi_u : H^i_n(E) \to H^i(E, E_0) \]
allows us to substitute \(H^i_n(E)\) for \(H^i(E, E_0)\) to get
\[ \cdots \to H^i_n(E) \xrightarrow{\varphi_u} H^i(E) \to H^i(E_0) \to \delta \to H^{i+1}(E) \to \cdots \]
where \(g(x) = (x \circ u)|E = x \circ (u|E)\).

Now substitute the isomorphic cohomology ring \(H^*(B)\) in place of \(H^*(E)\). Since the cohomology class \(u|E\) in \(H^n(E)\) corresponds to the top Stiefel–Whitney class \(w_n \in H^n(B)\) we have the required exact sequence
\[ \cdots \to H^i_n(B) \xrightarrow{\varphi_{u_n}} H^i(B) \to H^i(E_0) \to H^{i+1}(B) \to \cdots. \]

Notice that for \(j < n - 1\) in the proof of the above theorem that the groups \(H^i_n(B)\) and \(H^{i+1}(B)\) are zero. So we get
\[ 0 \xrightarrow{\varphi_{u_n}} H^i(B) \to H^i(E_0) \to 0. \]

Thus, by exactness, \(\pi_0^*\) is both injective and surjective. Therefore \(\pi_0^*\) is an isomorphism from \(H^i(B)\) to \(H^i(E_0)\) for \(j < n - 1\).

We are now ready to define the Stiefel–Whitney classes of a vector bundle.
Let $\xi$ be an $n$-plane bundle over the base space $B(\xi)$ with total space $E(\xi)$. Let $u \in H^n(E,E_0)$ be the Thom class. The inclusion $i:(E,\phi) \to (E,E_0)$ gives rise to the homomorphism:

$$i^*: H^*(E,E_0) \to H^*(E).$$

Also the projection $\pi:E \to B$ gives rise to an isomorphism:

$$\pi^*: H^*(B) \to H^*(E).$$

We now define the $n^{th}$ **Stiefel–Whitney class** to be:

$$w_n(\xi) = (\pi^*)^{-1}(i^*(u)) \in H^n(B(\xi)).$$

For $i < n$ we define the $i^{th}$ **Stiefel–Whitney class** inductively by the formula:

$$w_i(\xi) = (\pi_0^*)^{-1}(w_i(\xi_0))$$

For $i > n$ we define $w_i(\xi) = 0$.

The **total Stiefel–Whitney class** of an $n$-plane bundle $\xi$ is the element:

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots + w_n(\xi)$$

in the ring $H^*(B)$.

To see that the above definition really does define all the Stiefel–Whitney classes, consider the following example: Suppose $\xi$ is a 3-plane bundle and we wish to find $w_2(\xi)$. By definition $w_2(\xi) = (\pi_0^*)^{-1}(w_2(\xi_0))$. Now $\xi_0$ is a 2-plane bundle, so $w_2(\xi_0) = (\pi^*)^{-1}(i^*(u))$, thus we have defined $w_2(\xi)$. If, instead, we wanted to find $w_1(\xi)$ we would first need to know $w_1(\xi_0)$. To find this we would construct a 1-plane bundle $(\xi_0)_0$ from $\xi_0$ in the same way
we constructed $\xi_0$ from $\xi$. Now since $((\xi_0)_0)$ is a 1-plane bundle, $w_1((\xi_0)_0)$ is defined in terms of the Thom class and thus, by backward substitution, we know $w_1(\xi_0)$ and finally $w_1(\xi)$.

Note that at this moment we do not know that this definition makes sense. This is because we do not yet know that $w_{n-1}(\xi_0) \in \text{Im}(\pi_0)$. We will prove that this is true later (after we have proven Property 1 below).

We are interested in proving four main properties of the Stiefel-Whitney classes. These are

PROPERTY 1. To each $n$-plane vector bundle $\xi$ there corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i(B(\xi)) \quad i = 0,1,2,\ldots$$

with

$$w_0(\xi) = 1 \in H^0(B(\xi))$$

and $w_i(\xi) = 0$ for $i > n$.

PROPERTY 2. (Naturality) If $f:B(\xi) \to B(\eta)$ is covered by a bundle map from $\xi$ to $\eta$ then

$$w_i(\xi) = f^*w_i(\eta)$$
PROPERTY 3. (The Whitney Product Theorem) If $\xi$ and $\eta$ are vector bundles over the same base space then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cdot w_{k-i}(\eta)$$

Note: The symbol $\cdot$ will often be omitted. For example:

$$w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)$$
$$w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)w_1(\eta) + w_2(\eta)$$

In terms of the total Stiefel–Whitney class we can write

$$w(\xi \oplus \eta) = w(\xi)w(\eta)$$

PROPERTY 4. For the line bundle $\gamma_1^i$ over the circle $\mathbb{R}P^1$, the Stiefel–Whitney class $w_1(\gamma_1^i)$ is non–zero.

We first prove Property 2.

PROOF (of Property 2): Let $\xi$ and $\eta$ be $n$–plane bundles, and let $f:B(\xi) \to B(\eta)$ be covered by a bundle map from $\xi$ to $\eta$. That is, we have the following commutative diagram

$$\begin{array}{ccc}
E(\xi) & \to & E(\eta) \\
\downarrow \pi(\xi) & & \downarrow \pi(\eta) \\
B(\xi) & \to & B(\eta)
\end{array}$$

where $\tilde{f}$ carries each vector space $F_b(\xi)$ isomorphically onto some vector space $F_d(\eta)$. 
We proceed by induction by assuming that we know that Property 2 holds for \((n - 1)\)-plane bundles. We are then able to show directly that Property 2 holds for line bundles (Case 2 below), and this is what the induction hinges on.

Case 1: \((i < n)\)

We can assume that the map \(\hat{f}\) preserves inner products since, if not, we can redefine the metric in \(E(\xi)\) as follows: for \(v, w \in E(\xi)\) define \(v \cdot w \equiv \hat{f}(v) \cdot \hat{f}(w)\).

Forming the spaces \(E_o(\xi)\) and \(E_o(\eta)\) as described above, the map \(\hat{f}\) gives rise to a map

\[
E_o(\xi) \rightarrow E_o(\eta)
\]

which is covered by a bundle map \(\xi_o \rightarrow \eta_o\) of \((n - 1)\)-plane bundles.

Since \(\xi_o\) and \(\eta_o\) are \((n - 1)\)-plane bundles we know \(w_i(\xi_o) = f_o^* w_i(\eta_o)\) by the induction hypothesis.

The following diagram:

\[
\begin{array}{ccc}
E_o(\xi) & \xrightarrow{f_o} & E_o(\eta) \\
\pi_o(\xi) & & \pi_o(\eta) \\
B(\xi) & \xrightarrow{f} & B(\eta)
\end{array}
\]

commutes. Also, by definition

\[
w_i(\xi_o) = \pi_o^*(\xi)(w_i(\xi)) \\
w_i(\eta_o) = \pi_o^*(\eta)(w_i(\eta)).
\]
So

\[ w_i(\xi) = \pi_0^*(\xi)^{-1}(w_i(\xi_0)) = \pi_0^*(\xi)^{-1}(f_0^*w_i(\eta_0)) = \pi_0^*(\xi)^{-1}f_0^*\pi_0^*(\eta)(w_i(\eta)) = \pi_0^*(\xi)^{-1}(\pi_0(\eta)f_0)^*(w_i(\eta)) = \pi_0^*(\xi)^{-1}(f\pi_0(\xi))^*(w_i(\eta)) = \pi_0^*(\xi)^{-1}\pi_0^*(\xi)f^*(w_i(\eta)) = f^*(w_i(\eta)). \]

Therefore: \( w_i(\xi) = f^*(w_i(\eta)). \)

Case 2: \( (i = n) \)

We have the following commutative diagram

\[
\begin{array}{ccc}
E(\xi) & \xrightarrow{\tilde{f}} & E(\xi) \\
\pi(\xi) & & \pi(\xi) \\
B(\xi) & \xrightarrow{f} & B(\xi)
\end{array}
\]

where \( \tilde{f} \) maps each fiber \( F_b(\xi) \) isomorphically onto some \( F_d(\eta) \). Now consider the following diagram for any fiber \( F \)

\[
\begin{array}{ccc}
(F(\xi),F_0(\xi)) & \xrightarrow{\tilde{f}} & (F(\eta),F_0(\eta)) \\
k & & l \\
(E(\xi),E_0(\xi)) & \xrightarrow{\tilde{f}} & (E(\eta),E_0(\eta))
\end{array}
\]

where \( k \) and \( l \) are injections. Here we are abusing notation slightly, by \( \tilde{f} \) in the bottom half of the diagram we really mean \( \tilde{f} \circ \tilde{f}|E_0 \) and in the top half of the diagram we really mean \( \tilde{f}|F \circ \tilde{f}|F_0 \). In cohomology we get
Let \( u' \in H^n(E(\eta), E_0(\eta)) \) be the Thom class and let \( u \in H^n(E(\xi), E_0(\xi)) \) be the Thom class. We want to show that \( u = \hat{f}^*(u') \).

By the definition of the Thom class, \( l^*(u') \) is nonzero in \( H^0(F(\eta), F_0(\eta)) \) and \( k^*(u) \) is nonzero in \( H^0(F(\xi), F_0(\xi)) \). But there are only two elements in \( H^0(F(\eta), F_0(\eta)) \) and \( H^0(F(\xi), F_0(\xi)) \), and since \( \hat{f}|_{F^*} \) is an isomorphism we must have \( \hat{f}^*(l^*(u')) = k^*(u) \). That is \( u = \hat{f}^*(u') \).

From the following commutative diagram

\[
\begin{array}{c}
H^n(E(\eta), E_0(\eta)) \xrightarrow{j^*} H^n(E(\eta)) \xrightarrow{(\pi^*(\eta))^{-1}} H^n(B(\eta)) \\
\downarrow \hat{f}^* \downarrow \hat{f}^* \downarrow f^* \\
H^n(E(\xi), E_0(\xi)) \xrightarrow{i^*} H^n(E(\xi)) \xrightarrow{(\pi^*(\xi))^{-1}} H^n(B(\xi))
\end{array}
\]

we get: \( w_n(\xi) = (\pi^*(\xi))^{-1}(i^*(u)) = (\pi^*(\xi))^{-1}(i^*(\hat{f}^*(u'))) \)
\( = (\pi^*(\xi))^{-1}(\hat{f}^*(j^*(u'))) = f^*(\pi^*(\eta))^{-1}(j^*(u')) \)
\( = \hat{f}^*w_n(\eta). \)

Thus \( w_n(\xi) = f^*w_n(\eta) \). \( \Box \)
We are now ready to show that \( w_{n-1}(\xi_0) \in \text{Im}(\pi_0^*) \). It suffices to show that \( w_{n-1}(\xi_0) \) restricts to zero in each fiber \( F_0 \). But \( S^{n-1} \) is a retract of the fiber \( F_0 \) so we can think of the vectors in the total space of \( \xi_0 \) as being tangent vectors of \( S^{n-1} \) (they are orthogonal to vectors in \( S^{n-1} \)). Therefore it suffices to show that \( w_{n-1}(\tau) = 0 \) where \( \tau \) is the tangent bundle of the sphere \( S^{n-1} \). Let \( w_n(M) \) denote the Stiefel–Whitney class of the tangent bundle of the smooth manifold \( M \).

The natural map \( f:S^{n-1} \to \mathbb{R}P^{n-1} \) given by \( f(x) = \{\pm x\} \) is locally a diffeomorphism so the induced map
\[
Df:DS^{n-1} \to D(\mathbb{R}P^{n-1})
\]
is a bundle map. Using naturality of the top Stiefel–Whitney class only, we know that
\[
w_{n-1}(S^{n-1}) = f^*w_{n-1}(\mathbb{R}P^{n-1}).
\]
We now use the following theorem from Algebraic Topology [5, p.403].

**THEOREM 4.5.** The mod 2 cohomology of projective space \( \mathbb{R}P^n \) is given by
\[
H^i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq i \leq n \\ 0 & \text{for } i > n \end{cases}.
\]

We now compute the homomorphism \( f^*:H^n(\mathbb{R}P^n) \to H^n(S^n) \) provided that \( n > 1 \).
Let \( a \in H^1(\mathbb{RP}^n) \) be the generator for the cohomology of \( \mathbb{RP}^n \). Then
\[
f^*a^n = (f^a)^n.
\]
But \( f^*a \in H^1(S^n) = 0 \) for \( n > 1 \).
Thus \( w_{n-1}(S^{n-1}) = f^*(w_{n-1}(\mathbb{RP}^n)) = 0 \).
Therefore \( w_{n-1}(\xi_0) \in \text{Im}(\pi_*^0) \) and we have finally proven that our definition of Stiefel–Whitney classes makes sense.

PROOF (of Property 1): We only need to show that \( w_0(\xi) = 1 \in H^0(B(\xi)) \) since the rest follows immediately by definition.

Since the Thom class \( u \) is a non-zero element of \( H^0(E,E_0) \), and since \( i^* \) and \( (\pi^*)^{-1} \) are isomorphisms it follows that \( w_0(\xi) = (\pi^*)^{-1}(i^*(u)) \) is non-zero in \( H^0(B(\xi)) \). Now if \( B(\xi) \) is connected \( H^0(B(\xi);\mathbb{Z}_2) \) has only two elements, and we have \( w_0(\xi) = 1 \in H^0(B(\xi)) \). If \( B(\xi) \) is not connected consider a point \( x \in B(\xi) \). We want to show that \( w_0(\xi) \) evaluated at \([x]\), the homology class of \( x \), is equal to 1.

We have the following commutative diagram:

\[
\begin{array}{ccc}
F_x(\xi) & \overset{i}{\longrightarrow} & E(\xi) \\
\downarrow{\pi|_{F_x}} & & \downarrow{\pi} \\
\{x\} & \overset{i}{\longrightarrow} & B(\xi)
\end{array}
\]

where \( i \) and \( \overset{i}{\ } \) are the inclusion maps.
Let $w' \in H^0(\{x\})$ be the zero-th Stiefel-Whitney class of the vector bundle $\xi|\{x\}$. Since the single point $\{x\}$ is certainly connected we know that $w' = 1$. Also the inclusion $i: \{x\} \hookrightarrow B(\xi)$ induces an inclusion map in homology $i_*: [x] \hookrightarrow H_0(B(\xi))$. Thus $[x] = i_*[x]$.

If we let $<w_0,[x]>$ denote $w_0$ evaluated at $[x]$ we have:

$$<w_0,[x]> = <w_0,i_*[x]> = <i^*w_0,[x]>$$

$$= <w',[x]> = 1$$

where $i^*w_0 = w'$ by the naturality property of Stiefel-Whitney classes.

So $<w_0,[x]> = 1$ for all $x \in B(\xi)$. Therefore $w_0(\xi) = 1$. □

We save the proof of Property 3 until last.

**PROOF** (of Property 4): We want to show that $w_1(\gamma_1^i)$ is nonzero. First recall the definition of $\gamma_1^i$.

The total space $E = E(\gamma_1^i)$ consists of all pairs $([\pm x],v)$ where $\{\pm x\} \in \mathbb{R}P^1$ and $v$ is a multiple of $x$. We can represent every point of $E(\gamma_1^i)$ as

$$(\pm(\cos(\theta),\sin(\theta)), t(\cos(\theta),\sin(\theta)))$$

for $\theta \in [0,\pi]$ and $t \in \mathbb{R}$.

This representation is unique except that

$$(\pm(\cos(0),\sin(0), t(\cos(0),\sin(0))) = (\pm(1,0), t(1,0)) =$$
\((\pm(-1,0),-t(-1,0)) = (\pm(\cos(\pi),\sin(\pi),-t(\cos(\pi),\sin(\pi)))\).

That is, \(E\) can be obtained from the strip \([0,\pi] \times \mathbb{R}\) in the \((\theta,t)\)-plane by identifying the points \((0,t)\) with \((\pi,-t)\). Thus \(E\) can be thought of as an open Mobius strip (see Figure 4.1).

\(\text{(0, } t)\) \([0,\pi] \times \mathbb{R}\)

\((\pi, -t)\)

Figure 4.1

Now the space of vectors of length \(\leq 1\) in \(E\) is a Mobius strip \(M\) bounded by a circle \(\partial M\). \(M\) is a deformation retract of \(E\) and \(\partial M\) is a deformation retract of \(E_0\). So

\[H^*(M, \partial M) \cong H^*(E, E_0).\]

Also, if we embed the disk \(D^2\) in \(\mathbb{RP}^2\) the closure of \(\mathbb{RP}^2 - D^2\) is homeomorphic to \(M\). To see this consider Figure 4.2 where we have shown the closure of \(\mathbb{RP}^2 - D^2\).
Now if we cut along the sides labeled $b$ and $c$ in Figure 4.2 we get the two strips that are shown in Figure 4.3.

Now we identify the sides labeled $a$ as shown in the left side of Figure 4.4. Then identifying sides $b$ and $c$ again we get the Mobius strip shown in the right side of Figure 4.4.
So, by the Excision Theorem:

\[ H^*(M, \partial M) \cong H^*(\mathbb{RP}^2, D^2). \]

Hence we have the isomorphisms:

\[ H^i(E, E_0) \rightarrow H^i(M, \partial M) \rightarrow H^i(\mathbb{RP}^2, D^2) \leftarrow H^i(\mathbb{RP}^2) \]

for \( i \neq 0 \). So

\[ H^i(E, E_0) = H^i(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 0, 1, 2 \\ 0 & \text{if } i > 2 \end{cases} \]

From the sequence of injective maps

\[(E_0, \phi) \hookrightarrow (E, \phi) \hookrightarrow (E, E_0)\]

we get the exact sequence of the pair \((E, E_0)\):

\[ \cdots \rightarrow H^1(E, E_0) \rightarrow H^1(E) \rightarrow H^1(E_0) \rightarrow \cdots \]

where \( H^i(E, E_0) \cong \mathbb{Z}_2 \), \( H^i(E) \cong H^i(\mathbb{RP}^1) = H^1(S^1) \cong \mathbb{Z}_2 \), and \( H^1(E_0) \cong H^1(\partial M) = H^1(S^1) \cong \mathbb{Z}_2 \). So we have

\[ \cdots \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \cdots. \]

Since \( \partial M \) is a deformation retract of \( E_0 \) and \( M \) is a deformation retract of \( E \) we see that \( j^* \) sends the generator \( a \in H^1(E) \) to twice a generator \( 2a \in H^1(E_0) \). (We
see this by noticing that if we trace a path once around $\partial M$ this corresponds to tracing a path twice around the center circle of $M$). Since we are working with coefficients in $\mathbb{Z}_2$ we have $j^* \equiv 0$. so $\text{kernel}(j^*) = \mathbb{Z}_2$ and, by exactness, the map $i^*$ is onto.

Now since the Thom class $u \in H^1(E,E_0)$ is nonzero, $i^*(u) \neq 0$. And since the map $\pi^* : H^1(B) \to H^1(E)$ is an isomorphism $w_1 = (\pi^*)^{-1}(i^*(u)) \neq 0$.

Thus $w_1(\gamma^1)$ is nonzero. $\Box$

We now wish to prove Property 3, but before we can do this we need several lemmas:

**Lemma 4.6.** Let $\xi$ and $\eta$ be $m$ and $n$ dimensional vector bundles respectively over the base space $B$. Then $w_{m+n}(\xi \oplus \eta) = w_m(\xi)w_n(\eta)$.

**Proof:** This proof makes use of the cohomology cross product operation [5, p.355].

We first wish to compute the Stiefel-Whitney class of the product bundle $\xi \times \eta$. Let $E = E(\xi)$, $E' = E(\eta)$, and let $\pi \times \pi' : E \times E' \to B \times B$ be the projection map.

Consider the Thom classes
\[ u \in H^m(E,E_0) \quad u' \in H^n(E',E'_0). \]
From Algebraic Topology the cross product is defined and
\(u \times u' \in H^{m+n}(E \times E', (E \times E_0) \cup (E_0 \times E')).\) Then

\[
\begin{align*}
\nu_{n+m}(\xi \times \eta) &= (\pi^*)^{-1}(i^*(u \times u')) \\
&= (\pi^*)^{-1}(i^*(u) \times i^*(u')) \\
&= (\pi^*)^{-1}i^*(u) \times (\pi^*)^{-1}i^*(u') \\
&= \nu_n(\xi) \times \nu_n(\eta).
\end{align*}
\]

So \(\nu_{n+m}(\xi \times \eta) = \nu_n(\xi) \times \nu_n(\eta).\) Lifting both sides of this equation back to \(B\) by means of the diagonal map \(d:B \to B \times B\) we get \(\nu_{m+n}(\xi \oplus \eta) = \nu_m(\xi) \cdot \nu_n(\eta).\) \(\square\)

So we have proven the Whitney Product Theorem for the top Stiefel-Whitney class.

**Lemma 4.7.** If \(\xi\) is an \(n\)-plane bundle that possesses a nowhere zero cross-section, then \(\nu_n(\xi) = 0.\)

**Proof:** Let \(s:B \to E_0\) be a cross-section so that the composition

\[
B \xrightarrow{s} E_0 \xrightarrow{} E \xrightarrow{\pi} B
\]

is the identity map of \(B\). Then the corresponding composition

\[
H^n(B) \xrightarrow{\pi^*} H^n(E) \to H^n(E_0) \xrightarrow{s^*} H^n(B)
\]

is the identity map of \(H^n(B)\). By definition \(\pi^*\) maps \(\nu_n(\xi)\) to the restriction of the Thom class \(u|E\). Hence the first two homomorphisms in the above composition map \(\nu_n(\xi)\) to the restriction \((u|E)|E_0\) which is zero since the
composition \(H^n(E,E_0) \to H^n(E) \to H^n(E_0)\) is zero. Then
\[w_n(\xi) = s^*((u|E)|E_0) = s^*(0) = 0.\]
Therefore \(w_n(\xi) = 0.\) □

**Lemma 4.8.** Let \(\xi\) be an \(n\)-plane bundle, and let \(\varepsilon^k\) be the trivial \(k\)-plane bundle. Then \(w(\xi \oplus \varepsilon^k) = w(\xi).\)

**Proof:** It suffices to prove the Lemma for the case \(k = 1\) since the Lemma then follows from induction.

Let \(s:B \to E_0(\xi \oplus \eta)\) be defined by \(s(b) = (v,1)\) where \(v \in F_b(\xi)\). Then \(s\) is a nowhere zero cross-section.

By Lemma 4.6 \(w_{n+1}(\xi \oplus \varepsilon^1) = 0.\) Hence
\[w_{n+1}(\xi \oplus \varepsilon^1) = w_{n+1}(\xi) = 0.\]

Now since \(s\) is nonzero it is covered by a bundle map \(\xi \to (\xi \oplus \varepsilon^1)_0\). Hence \(s^*w_i((\xi \oplus \varepsilon^1)_0) = w_i(\xi)\) by naturality.

If the projection is \(\pi_0:(\xi \oplus \varepsilon^1)_0 \to (\xi \oplus \varepsilon^1),\) then
\[w_i((\xi \oplus \varepsilon^1)_0) = \pi_0^*w_i(\xi \oplus \varepsilon^1)\] again by naturality. So
\[w_i(\xi) = s^*w_i((\xi \oplus \varepsilon^1)_0) = s^*\pi_0^*w_i(\xi \oplus \varepsilon^1).\] But \(s^*\pi_0^*\) is the identity. So
\[w_i(\xi) = w_i(\xi \oplus \varepsilon^1).\] □

In order to prove the Whitney product theorem we first show that the total Stiefel–Whitney class of a Whitney sum \(w(\xi \oplus \eta)\) can be expressed as one, and only one, polynomial. We are then able to explicitly compute the polynomial
and see that it is what we want, i.e. \( w(\xi \oplus \eta) = w(\xi)w(\eta) \).

To make things easier we make use of the "universal" \( n \)-plane bundle \( \gamma^n \) over the infinite Grassmann manifold \( G_n \). The reason for this is that we can prove the Whitney Product Theorem for the specific case of the Whitney sum of two universal bundles, then since every \( n \)-plane vector bundle over the base space \( B \) is isomorphic to the induced bundle \( f^*(\gamma^n) \) for some map \( f:B \to G_n \), we can get the Whitney Product Theorem for any \( n \)-plane bundle.

**LEMMA 4.9.** \( H^*(G_n(\mathbb{R}^\infty)) \) is a polynomial ring over \( \mathbb{Z}_2 \) generated by the Stiefel-Whitney classes \( w_1(\gamma^n), \ldots, w_n(\gamma^n) \).

**PROOF:** We proceed by induction on \( n \). From Theorem 4.5 we know the Lemma is true for \( n = 1 \).

Assume the Lemma is true for \( n - 1 \).

Consider the Gysin sequence

\[
\cdots \to H^i(G_n) \xrightarrow{\cup w_i} H^{i+n}(G_n) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \to H^{i+n}(G_n) \to \cdots.
\]

We first show that \( H^*(E_0) \) can be identified with \( H^*(G_{n-1}) \).

Construct a map \( f:E_0 \to G_{n-1} \) as follows: A point \( (X,v) \) in \( E_0 \) consists of an \( n \)-plane through \( 0 \) in \( \mathbb{R}^\infty \) and a non-zero vector \( v \) in \( X \). Let

\[
f(X,v) = X \cap v^\perp
\]

be the orthogonal complement of \( v \) in \( X \) using the
standard metric

\[(v_1, v_2, \ldots, w_1, w_2, \ldots) = \sum_{j=1}^{\infty} v_j w_j\]
on \(\mathbb{R}^\infty\).

Then \(f(X, v)\) is a well defined \((n-1)-\text{plane in } \mathbb{R}^\infty\).

Now we wish to show that \(f\) induces cohomology isomorphisms. We first consider the subbundle \(\gamma^n(\mathbb{R}^N) \subseteq \gamma^n\) consisting of \(n\)-planes through 0 in \(N\)-space where \(N\) is large but finite. Let

\[f_N: E_0(\gamma^n(\mathbb{R}^N)) \to G_{n-1}(\mathbb{R}^N)\]

be the corresponding restriction of \(f\). For any \((n-1)-\text{plane } Y\) in \(G_{n-1}(\mathbb{R}^N)\) the inverse image

\[f^{-1}(Y) \subseteq E_0(\gamma^n(\mathbb{R}^N))\]

consists of all pairs \((X, v)\) where \(v \in \mathbb{R}^N\) is a non-zero vector perpendicular to \(Y\) and where \(X\) is determined by \(Y\) and \(v\) (since every vector in \(X\) is a linear combination of a vector in \(Y\) and \(v\)).

Thus \(f_N\) can be identified with the projection map

\[E_0(\omega^{N-n+1}) \to G_{n-1}(\mathbb{R}^N)\]

where \(\omega = \omega^{N-n+1}\) is the \((N-n+1)\)-dimensional vector bundle whose fiber over \(Y \in G_{n-1}(\mathbb{R}^N)\) is the orthogonal complement of \(Y\) in \(\mathbb{R}^N\).

The Gysin sequence of \(\omega\) is

\[
\cdots \to H^i(G_{n-1}(\mathbb{R}^N)) \xrightarrow{\omega^*} H^{i+N-n+1}(G_{n-1}(\mathbb{R}^N)) \xrightarrow{\pi_0^*(\omega)} H^{i+N-n+1}(E_0(\omega)) \to H^{i+1}(G_{n-1}(\mathbb{R}^N)) \to \cdots
\]

where \(\pi_0^*(\omega)\) is an isomorphism when \((i+N-n+1) < (N-n+1)\), i.e. when \(i < 2(N-n+1)\).
But since $f_N$ can be identified with $\pi_0(\omega)$ we have that $f_N$ induces cohomology isomorphisms in dimensions less than $2(N - n + 1)$. Letting $N$ go to infinity we see that $f$ induces cohomology isomorphisms in all dimensions.

Therefore we can insert $G_{n-1}$ in place of $E_0$ in the original Gysin sequence to get

\[
\cdots \to H^i(G_n) \to H^{i+n}(G_n) \xrightarrow{\lambda} H^{i+n}(G_{n-1}) \to H^{i+1}(G_n) \to \cdots
\]

where $\lambda = (f^*)^{-1}\pi_0^*$. 

We must show that $\lambda$ maps $w_i(\gamma^n)$ to $w_i(\gamma^{n-1})$. For $i = n$ we have $\lambda w_n(\gamma^n) = (f^*)^{-1}(\pi_0^*)(\pi_*^{-1}(i^* (u))) = 0 = w_i(\gamma^{n-1})$. For $i < n$ the Stiefel-Whitney classes are defined by the formula $w_i(\gamma^n) = (\pi_0^*)^{-1}(w_i(\gamma_0^n))$ so $\pi_0^* w_i(\gamma^n) = w_i(\gamma_0^n)$. But the map $f : E_0 \to G_{n-1}$ is covered by a bundle map $\gamma_0^n \to \gamma^{n-1}$. Therefore $f^* w_i(\gamma^{n-1}) = w_i(\gamma_0^n)$ by naturality.

So $\lambda w_i(\gamma^n) = (f^*)^{-1}(\pi_0^*) w_i(\gamma^n) = (f^*)^{-1}(\pi_0^*)(\pi_*^{-1} w_i(\gamma_0^n)) = (f^*)^{-1} w_i(\gamma_0^n) = w_i(\gamma^{n-1})$.

We have shown that $\lambda w_i(\gamma^n) = w_i(\gamma^{n-1})$.

Now, we have assumed that $H^*(G_{n-1})$ is a polynomial ring over $\mathbb{Z}_2$ generated by $w_1(\gamma^{n-1}), \ldots, w_n(\gamma^{n-1})$. Therefore $\lambda$ is surjective since it maps to each generator of $H^*(G_{n-1})$. Our Gysin sequence reduces to

\[
\cdots \to H^i(G_n) \to H^{i+n}(G_n) \xrightarrow{\lambda} H^{i+n}(G_{n-1}) \to 0
\]

We now show, using induction on $i$, that every element
$x \in H^{i,n}(G_n)$ can be expressed uniquely as a polynomial in the Stiefel–Whitney classes $w_1(\gamma^n), \ldots, w_n(\gamma^n)$.

Certainly the image of $\lambda(x)$ can be expressed uniquely as a polynomial $P(w_1(\gamma^{n-1}), \ldots, w_{n-1}(\gamma^{n-1}))$ by our main induction hypothesis that $H^*(G_{n-1})$ is generated by $w_1(\gamma^{n-1}), \ldots, w_{n-1}(\gamma^{n-1})$. Therefore the element $x - P(w_1(\gamma^n), \ldots, w_{n-1}(\gamma^n))$ belongs to the kernel of $\lambda$, and hence can be expressed as a product $y w_n(\gamma^n)$ for some uniquely determined $y \in H^i(G_n)$. Now $y$ can be expressed uniquely as a polynomial $Q(w_1(\gamma^n), \ldots, w_n(\gamma^n))$ by our induction hypothesis for $i$. Hence

$$x = P(w_1(\gamma^n), \ldots, w_{n-1}(\gamma^n)) + w_n(\gamma^n) Q(w_1(\gamma^n), \ldots, w_n(\gamma^n)).$$

The polynomials on the right are unique since if we also had $x = P'(w_1(\gamma^n), \ldots, w_{n-1}(\gamma^n)) + w_n(\gamma^n) Q'(w_1(\gamma^n), \ldots, w_n(\gamma^n))$ we could apply $\lambda$ to get

$$P(w_1(\gamma^{n-1}), \ldots, w_{n-1}(\gamma^{n-1})) = P'(w_1(\gamma^{n-1}), \ldots, w_{n-1}(\gamma^{n-1})).$$

So $P = P'$. Then we could subtract both polynomials to get

$$w_n(\gamma^n) Q'(w_1(\gamma^n), \ldots, w_n(\gamma^n)) = w_n(\gamma^n) Q(w_1(\gamma^n), \ldots, w_n(\gamma^n))$$

which implies that $Q = Q'$.

**Lemma 4.10.** There exists one and only one polynomial

$$P_{mn}(w_1, \ldots, w_n; w'_1, \ldots, w'_n)$$

with $\mathbb{Z}_2$ coefficients in $m + n$ indeterminants, so that the identity
\[ w(\xi \oplus \eta) = \text{P}_{mn}(w_1(\xi), \ldots, w_n(\xi); w_1(\eta), \ldots, w_n(\eta)) \]

is valid for every real \( m \)-plane bundle \( \xi \) and \( n \)-plane bundle \( \eta \) over a paracompact base space \( B \).

**PROOF:** Let \( \pi_1: G_m \times G_n \to G_m \) be projection onto the first factor, and let \( \pi_2: G_m \times G_n \to G_n \) be projection onto the second factor. Let

\[ \gamma_1^m = \pi_1(\gamma^m) \quad \text{and} \quad \gamma_2^n = \pi_2(\gamma^n). \]

Then the Whitney sum \( \gamma_1^m \oplus \gamma_2^n \) can be identified with \( \gamma^m \times \gamma^n \). From Algebraic Topology we know that the external cohomology cross product operation

\[ a \times b \mapsto a \times b = \pi_1^*a \circ \pi_2^*b \]

induces an isomorphism

\[ H^*(G_m) \otimes H^*(G_n) \to H^*(G_m \times G_n) \]

since \( \mathbb{Z}_2 \) is a field. Therefore, by Lemma 4.9, \( H^*(G_m \times G_n) \) is a polynomial ring over \( \mathbb{Z}_2 \) with the algebraically independent generators

\[ \pi_1^*w_i(\gamma^m) = w_i(\gamma_1^m) \quad 1 \leq i \leq m \]

and

\[ \pi_2^*w_j(\gamma^n) = w_j(\gamma_2^n) \quad 1 \leq j \leq n \]

hence the total Stiefel–Whitney class of \( \gamma_1^m \oplus \gamma_2^n \) can be expressed uniquely as a polynomial

\[ w(\gamma_1^m \oplus \gamma_2^n) = \text{P}_{mn}(w_1(\gamma_1^m), \ldots, w_n(\gamma_1^m); w_1(\gamma_2^n), \ldots, w_n(\gamma_2^n)). \]

Now if \( \xi \) is an \( m \)-plane bundle over \( B \) and \( \eta \) is an \( n \)-plane bundle over \( B \) then we can find maps \( f: B \to G_m \)
and $g:B \rightarrow G_n$ such that

$$\xi \cong f^*(\gamma^m) \quad \text{and} \quad \eta \cong g^*(\gamma^n).$$

Defining the map $h:B \rightarrow G_m \times G_n$ by

$$h(b) = (f(b), g(b))$$

we get the following commutative diagram:

$$\begin{array}{ccc}
B & \xrightarrow{f} & G_m \\
\downarrow{h} & & \downarrow{g} \\
G_m \times G_n & \xrightarrow{x_1} & G_m \\
\end{array}$$

it follows that

$$h^*(\gamma^m) \cong \xi \quad \text{and} \quad h^*(\gamma^n) \cong \eta.$$ 

Hence

$$w(\xi \oplus \eta) = h^*w(\gamma^m_1 \oplus \gamma^n_2) = P_{mn}(w_1(\xi), \ldots, w_m(\xi); w_1(\eta), \ldots, w_n(\eta)).$$

**PROOF (of Property 3):** We now wish to compute the polynomials $P_{mn}$. We proceed by induction.

Suppose that

$$w(\gamma^{-1}_1 \oplus \gamma^n_2) = (1 + w_1(\gamma^{-1}_1) + \cdots + w_m(\gamma^{-1}_1))(1 + w_1(\gamma^n_2) + \cdots + w_n(\gamma^n_2)).$$

Consider the two vector bundles $\gamma^{-1}_1 \oplus \varepsilon^1$ and $\gamma^n_2$ over $G_{m-1} \times G_n$, where $\varepsilon^1$ is a trivial line bundle. From Lemma 4.7 we have

$$w(\gamma^{-1}_1 \oplus \varepsilon^1 \oplus \gamma^n_2) = P_{mn}(w_1(\gamma^{-1}_1 \oplus \varepsilon^1), \ldots, w_m(\gamma^{-1}_1 \oplus \varepsilon^1); w_1(\gamma^n_2), \ldots, w_2(\gamma^n_2)).$$

From Lemma 4.6 we can ignore the $\varepsilon^1$ summand. Thus

$$w(\gamma^{-1}_1 \oplus \gamma^n_2) = P_{mn}(w_1(\gamma^{-1}_1), \ldots, w_{m-1}(\gamma^{-1}_1), 0; w_1(\gamma^n_2), \ldots, w_2(\gamma^n_2)).$$
Substituting $w_i$ for $w_i(\gamma^{m-1})$ and $w'_i$ for $w_i(\gamma^n)$ we get from the induction hypothesis that

$$P_{mn}(w_1, \ldots, w_{m-1}, 0, w'_1, \ldots, w'_n) = (1 + w_1 + \cdots + w_{m-1})(1 + w'_1 + \cdots + w'_n).$$

Introducing a new indeterminant $w_m$ we have the congruence

$$P_{mn}(w_1, \ldots, w_{m-1}, w_m, w'_1, \ldots, w'_n) \equiv (1 + w_1 + \cdots + w_m)(1 + w'_1 + \cdots + w'_n) \pmod{w_m}.$$ But by a similar induction argument we also have

$$P_{mn}(w_1, \ldots, w_{m-1}, w_m, w'_1, \ldots, w'_n) \equiv (1 + w_1 + \cdots + w_m)(1 + w'_1 + \cdots + w'_n) \pmod{w'_n}.$$ Since $\mathbb{Z}_2[w_1, \ldots, w_m, w'_1, \ldots, w'_n]$ is a unique factorization domain, it follows that the above polynomials are congruent modulo the product $w_m w'_n$. That is

$$P_{mn}(w_1, \ldots, w_{m-1}, w_m, w'_1, \ldots, w'_n) = (1 + w_1 + \cdots + w_m)(1 + w'_1 + \cdots + w'_n) + tw_m w'_n$$

where $t$ must be a constant for otherwise the Whitney sum $\gamma^m_1 \oplus \gamma^n_2$ would have non-zero Stiefel-Whitney classes in dimensions greater than $m + n$.

But we know that $w_{mn}(\xi \oplus \eta) = w_m(\xi) w_n(\eta)$ from Lemma 4.4. Therefore $t$ must be zero.

So we have

$$w(\xi \oplus \eta) = h^* w(\gamma^m_1 \oplus \gamma^n_2) = P_{mn}(w_1(\xi), \ldots, w_m(\xi); w_1(\eta), \ldots, w_n(\eta)) = (1 + w_1(\xi) + \cdots + w_m(\xi))(1 + w_1(\eta) + \cdots + w_n(\eta)) = w(\xi) w(\eta). \quad \Box$$
We have now proven that the Stiefel–Whitney classes have Properties 1–4. It turns out that these four properties completely characterize Stiefel–Whitney classes [4].
CHAPTER V
ADDITIONAL PROPERTIES OF STIEFEL–WHITNEY CLASSES

Now we wish to find out more about the Stiefel–Whitney classes. As noted at the end of the last chapter the Stiefel–Whitney classes are completely characterized by Properties 1–4. Hence, in what follows, we shall not need to refer back to the original definition of $w(\xi)$.

For the Stiefel–Whitney classes to be of any use we must have the following

THEOREM 5.1. If $\xi$ is isomorphic to $\eta$ then $w_i(\xi) = w_i(\eta)$.

PROOF: We have the following diagram

$$
\begin{array}{ccc}
E(\xi) & \xrightarrow{i} & E(\eta) \\
\downarrow & & \downarrow \\
B(\xi) & \xrightarrow{i} & B(\eta)
\end{array}
$$

where the base spaces are equal and $i$ is the identity map. From Property 2, $w_i(\xi) = i^* w_i(\eta)$. But since $i$ is the
identity $i^*$ must be an isomorphism. Therefore

\[ w_i(\xi) = w_i(\eta). \]

THEOREM 5.2. If $\epsilon$ is a trivial vector bundle, then

\[ w_i(\epsilon) = 0 \text{ for } i > 0. \]

PROOF: Let $\epsilon$ be a trivial $n$-plane bundle. Then there is a bundle map

\[ g: E(\epsilon) \rightarrow \{x\} \times \mathbb{R}^n \]

for some point $x$.

If we let $\overline{g}$ be the corresponding map of base spaces we get the commutative diagram

\[
\begin{array}{ccc}
E(\epsilon) & \xrightarrow{g} & \{x\} \times \mathbb{R}^n \\
\downarrow & & \downarrow \\
B(\epsilon) & \xrightarrow{\overline{g}} & \{x\} \\
\end{array}
\]

By Theorem 3.20 $\epsilon$ is isomorphic to the bundle over the point $\{x\}$. But $H^i(\{x\}) = 0$ for $i > 0$. Since $w_i($bundle over $\{x\}) \in H^i(\{x\}) = 0$ we have

\[ w_i(\epsilon) = \overline{g}^*w_i($bundle over $\{x\}) = \overline{g}^*(0) = 0. \]

Thus $w_i(\epsilon) = 0$. □

THEOREM 5.3. If $\epsilon$ is trivial then

\[ w_i(\epsilon \oplus \eta) = w_i(\eta). \]
PROOF: From Property 1 we know \( w_0(\epsilon) = w_0(\eta) = 1 \).
From Theorem 5.2 we have \( w_i(\epsilon) = 0 \) for \( i > 0 \). Finally from Property 3 we get

\[
w_i(\epsilon \oplus \eta) = \sum_{j=0}^{i} w_j(\epsilon) \cup w_{i-j}(\eta) = (1) \cup w_i(\eta) + \sum_{j=1}^{i} 0 \cup w_{i-j}(\eta) = w_i(\eta).
\]

THEOREM 5.4. If \( \xi \) is an \( n \)-plane bundle with a Euclidean metric which possesses a nowhere zero cross-section, then \( w_n(\xi) = 0 \). If \( \xi \) possesses \( k \) cross-sections which are nowhere linearly dependent, then

\[
w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \cdots = w_n(\xi) = 0.
\]

That is, the last \( k \) Stiefel-Whitney classes are zero.

PROOF: From Theorem 3.27 \( \xi \cong \epsilon \oplus \epsilon^\perp \) where \( \epsilon \) is a trivial \( k \)-plane bundle and \( \epsilon^\perp \) is an \((n-k)\)-plane bundle. From Theorem 5.3 \( w_i(\xi) = w_i(\epsilon \oplus \epsilon^\perp) = w_i(\epsilon^\perp) \) and since \( \epsilon^\perp \) is an \((n-k)\)-plane bundle, \( w_i(\epsilon^\perp) = 0 \) for \( i > n-k \).

THEOREM 5.5. The total Stiefel-Whitney class

\[
w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \cdots \in H^*(B(\xi))
\]

has a multiplicative inverse in \( H^*(B(\xi)) \).
PROOF: Let $w = 1 + w_1 + \cdots \in \text{H}^*(B)$. An element

$$\overline{w} = 1 + \overline{w}_1 + \overline{w}_2 + \cdots \in \text{H}^*(B)$$

is an inverse of $w$ if $w\overline{w} = 1$. Computing this product we see that

$$w\overline{w} = 1 + (w_1 + \overline{w}_1) + (w_2 + w_1\overline{w}_1 + \overline{w}_2) + \cdots$$

$$+ \ (w_n + w_{n-1}\overline{w}_1 + \cdots + w_1\overline{w}_{n-1} + \overline{w}_n) + \cdots$$

So $w$ has an inverse if there are Stiefel-Whitney classes $\overline{w}_1, \ldots, \overline{w}_n$ that solve the following equations:

$$w_1 + \overline{w}_1 = 0$$
$$w_2 + w_1\overline{w}_1 + \overline{w}_2 = 0$$
$$\vdots$$
$$w_n + w_{n-1}\overline{w}_1 + \cdots + w_1\overline{w}_{n-1} + \overline{w}_n = 0$$
$$\vdots$$

We can solve the first equation for $\overline{w}_1$ to get

$$\overline{w}_1 = w_1 \in \text{H}^1(B)$$

(Remember we are working with coefficients in $\mathbb{Z}_2$ so that $w_1 = -\overline{w}_1$). Plugging this into the second equation we are able to solve for $\overline{w}_2$:

$$\overline{w}_2 = w_1^2 + w_2 \in \text{H}^2(B).$$

We may continue in this manner so that we are able to compute $\overline{w}_n$ for any $n$ by first computing $\overline{w}_i$ for $i < n$. The formula for $\overline{w}_n$ is
\[ \overline{w}_n = w_1\overline{w}_{n-1} + w_2\overline{w}_{n-2} + \cdots + w_{n-1}\overline{w}_1 \in H^n(B). \]

Thus \( \overline{w} = 1 + \overline{w}_1 + \overline{w}_2 + \cdots \) is an inverse for \( w \) in \( H^*(B) \).

**COROLLARY 5.6.** Let \( \xi \) and \( \eta \) be vector bundles over the same base space. The equation

\[ w(\xi \oplus \eta) = w(\xi)w(\eta) \]

can be solved as

\[ w(\eta) = \overline{w}(\xi)w(\xi \oplus \eta). \]

In particular, when \( \xi \oplus \eta \) is trivial

\[ w(\eta) = \overline{w}(\xi). \]

**PROOF:** When \( \xi \oplus \eta \) is trivial \( w(\xi \oplus \eta) = 1 \) from Theorem 5.2. The rest follows immediately from Theorem 5.5.

We will make use of a special case of the above corollary.

**COROLLARY 5.7.** (Whitney Duality Theorem) If \( \tau_M \) is the tangent space of a manifold in Euclidean space and \( \nu \) is the normal bundle, then

\[ w_i(\nu) = \overline{w}_i(\tau_M). \]
PROOF: From Corollary 3.28 \( \nu \oplus \tau_{\mathbb{M}} \) is trivial. Now by Corollary 5.6 \( w_i(\nu) = \overline{w}_i(\tau_{\mathbb{M}}) \). □

The following theorem illustrates a weakness of the Stiefel-Whitney classes. It states that the tangent bundle of the unit sphere \( S^n \) has the same Stiefel-Whitney class as the trivial bundle over \( S^n \).

NOTATION: We write \( w(M) \) for the total Stiefel-Whitney class of a tangent bundle \( \tau_M \).

THEOREM 5.8. For the tangent bundle \( \tau \) of the unit sphere \( S^n \), the class \( w(S^n) = 1 \).

PROOF: For the standard embedding \( S^n \to \mathbb{R}^{n+1} \), the normal (line) bundle \( \nu \) is trivial since the function \( h:E(S^n) \to S^n \times \mathbb{R} \) given by \( h(x, tx) = (x, t) \) is a homeomorphism.

Now from the Whitney Product Theorem

\[ 1 = w(\tau \oplus \nu) = w(\tau)w(\nu). \]

But \( w(\nu) = 1 \) since \( \nu \) is trivial. Thus \( w(\tau) = 1 \). □

In order to compute the Stiefel-Whitney classes for \( \gamma_n^1 \) over \( \mathbb{RP}^n \) we need the following fact from Algebraic Topology [5, p.403].
THEOREM 5.9. The mod 2 cohomology of projective space $\mathbb{RP}^n$ is given by

$$H^i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}_2 & \text{for } 0 \leq i \leq n \\ 0 & \text{for } i > n \end{cases}.$$ 

Furthermore, if $a$ denotes the nonzero element of $H^1(\mathbb{RP}^n)$ then each $H^i(\mathbb{RP}^n)$ is generated by the $i$–fold cup product $a^i$.

THEOREM 5.10. The total Stiefel–Whitney class of the canonical line bundle $\gamma_i^1$ over $\mathbb{RP}^n$ is given by

$$w(\gamma_i^1) = 1 + a.$$ 

PROOF: From Property 1, $w_0(\gamma_i^1) = 1$ and $w_i(\gamma_i^1) = 0$ for $i > 1$. The inclusion $j: \mathbb{RP}^1 \to \mathbb{RP}^n$ is covered by a bundle map $\gamma_i^1 \to \gamma_i^1$. By Property 2

$$j^*w_i(\gamma_i^1) = w_i(\gamma_i^1) \neq 0.$$ 

Therefore $w_i(\gamma_i^1)$ cannot be zero and hence $w_i(\gamma_i^1) = a$. So $w(\gamma_i^1) = 1 + a$. □

The following Theorem shows that all of the $n$ Stiefel–Whitney classes of an $n$–plane bundle may be non–zero.
THEOREM 5.11. By definition, the line bundle $\gamma^n_1$ over $\mathbb{R}P^n$ is contained, as a subbundle, in the trivial bundle $\varepsilon^{n+1}$. Let $\gamma^\perp$ denote the orthogonal complement of $\gamma^n_1$ in $\varepsilon^{n+1}$. Then

$$w(\gamma^\perp) = 1 + a + a^2 + \cdots + a^n.$$  

PROOF: Since $\gamma^n_1 \oplus \gamma^\perp \cong \varepsilon^{n+1}$ which is trivial, we have $w(\gamma^\perp) = \overline{w}(\gamma^n_1)w(\gamma^n_1 \oplus \gamma^\perp) = \overline{w}(\gamma^n_1)$. Thus

$$w(\gamma^\perp) = (1 + a)^{-1}. \text{ To compute } (1 + a)^{-1} \text{ consider the product}$$

$$(1 + b_1 + b_2 + \cdots) \in H^*(\mathbb{R}P^n). \text{ We wish to find } b_1, b_2, \ldots \text{ so}$$

$$\text{that the product is } 1. \text{ From the formula introduced in the}$$

proof of Theorem 5.5 we have $b_1 = a$, $b_2 = a^2$, $\ldots$, $b_n = a^n$, $b_{n+1} = a^{n+1} = 0$, $b_{n+2} = a^{n+2} = 0, \ldots$. Thus

$$(1 + a)^{-1} = 1 + a + a^2 + \cdots + a^n + 0 + 0 + \cdots. \text{ So}$$

$$w(\gamma^\perp) = 1 + a + a^2 + \cdots + a^n. \ \Box$$

We now wish to compute the total Stiefel–Whitney class of the tangent bundle of $\mathbb{R}P^n$. To do this we first have the following:

DEFINITION 5.12. Given two vector bundles $\xi$ and $\eta$ over the same base space $B$ we can construct the vector bundle $\text{Hom}(\xi, \eta)$ in the following way: A fiber $F_b$ of $\text{Hom}(\xi, \eta)$ is just
\[ F_b = \text{Hom}(F_b(\xi), F_b(\eta)) \]

the space consisting of all linear transformations from \( F_b(\xi) \) to \( F_b(\eta) \). The total space \( E \) is defined to be the disjoint union

\[ E(\text{Hom}(\xi, \eta)) = \bigcup_{b \in B} F_b. \]

We will be interested in the vector bundle \( \text{Hom}(\gamma^1_n, \gamma^1) \). In this case the fibers are \( F_b = \text{Hom}(F_b(\gamma^1_n), F_b(\gamma^1)) = \text{Hom}(L, L'^\perp) \) where \( L \) is the line through the origin in \( \mathbb{R}^{n+1} \) that passes through \( b \).

**Lemma 5.13.** Let \( \tau \) be the tangent bundle of \( \mathbb{R}P^n \). Then \( \tau \cong \text{Hom}(\gamma^1_n, \gamma^1) \).

**Proof:** Let \( L \) be a line through the origin in \( \mathbb{R}^{n+1} \), intersecting \( S^n \) in the points \( \pm x \), and let \( L'^\perp \) be the complementary \( n \)-plane. Let \( f: S^n \to \mathbb{R}P^n \) be the map \( f(x) = \{ \pm x \} \).

The two tangent vectors \( (x, v) \) and \( (-x, -v) \) both have the same image under the map \( Df: DS^n \to D(\mathbb{R}P^n) \).

Thus the tangent manifold \( D(\mathbb{R}P^n) \) can be identified with the set of all pairs \( \{(x, v), (-x, -v)\} \) satisfying \( x \cdot x = 1 \) and \( x \cdot v = 0 \).

(Remember that \( x \in S^n \) which is the unit \( n \)-sphere, so
\( x \cdot x = 1, \) and if \( v \) is tangent to \( S^n \) at \( x \) it is orthogonal to \( x \), hence \( x \cdot v = 0 \).

Then we can define a linear map \( l : L \rightarrow L^\perp \) by \( l(x) = v \). Also note that any map \( l : L \rightarrow L^\perp \) determines a unique pair \( \{(x,v),(-x,-v)\} \) satisfying the above conditions.

Thus the tangent space to \( \mathbb{RP}^n \) at \( \{\pm x\} \) is isomorphic to the vector space \( \text{Hom}(L,L^\perp) \). So we have a continuous function that maps each fiber of \( \tau \) isomorphically onto the corresponding fiber in \( \text{Hom}(\gamma_n^1,\gamma^1) \). By Theorem 3.3, \( \tau \cong \text{Hom}(\gamma_n^1,\gamma^1) \).

**Theorem 5.14.** Let \( \varepsilon^1 \) be a trivial line bundle over \( \mathbb{RP}^n \). The Whitney sum \( \tau \oplus \varepsilon^1 \) is isomorphic to the \((n + 1)\)-fold Whitney sum \( \gamma_n^1 \oplus \gamma_n^1 \oplus \cdots \oplus \gamma_n^1 \). Hence the total Stiefel-Whitney class is given by

\[
\begin{align*}
\text{w}(\mathbb{RP}^n) &= (1 + a)^{n+1} \\
&= 1 + \left(\begin{array}{c} n+1 \\ 1 \end{array}\right)a + \left(\begin{array}{c} n+1 \\ 2 \end{array}\right)a^2 + \cdots + \left(\begin{array}{c} n+1 \\ n \end{array}\right)a^n.
\end{align*}
\]

**Proof:** A fiber \( F \) over the bundle \( \text{Hom}(\gamma_n^1,\gamma_n^1) \) is the collection of all linear functions that map the line \( L \), that passes through the point \( \{\pm x\} \) in \( \mathbb{RP}^n \), to itself. So each linear function \( f \in F \) is a stretching by a factor of \( t \) for some \( t \in \mathbb{R} \). That is, each function in \( F \) can be
written as \( ft \) where \( ft(v) = tv \). Then the correspondence \( ft \mapsto t \) shows that \( \text{Hom}(\gamma^1_n, \gamma^1_n) \) is a line bundle.

Furthermore \( \text{Hom}(\gamma^1_n, \gamma^1_n) \) has a nowhere zero cross-section defined by \( \{\pm x\} \mapsto f_1 \in \text{Hom}(L, L) \) where \( f_1 \) is the identity function that maps the line \( L \) to itself by \( f_1(v) = v \).

Therefore \( \text{Hom}(\gamma^1_n, \gamma^1_n) \) is a trivial line bundle.

Now \( \tau \oplus \epsilon^1 \cong \text{Hom}(\gamma^1_n, \gamma^1_n) \oplus \text{Hom}(\gamma^1_n, \gamma^1_n) \)

\[ \cong \text{Hom}(\gamma^1_n, \gamma^1_n \oplus \gamma^1_n) \]

\[ \cong \text{Hom}(\gamma^1_n, \epsilon^{n+1}) \]

\[ \cong \text{Hom}(\gamma^1_n, \epsilon^1 \oplus \epsilon^1 \oplus \cdots \oplus \epsilon^1) \]

\[ \cong \text{Hom}(\gamma^1_n, \epsilon^1) \oplus \text{Hom}(\gamma^1_n, \epsilon^1) \oplus \cdots \oplus \text{Hom}(\gamma^1_n, \epsilon^1). \]

Each fiber \( F = \pi^1(\{\pm x\}) \) of \( \text{Hom}(\gamma^1_n, \epsilon^1) \) has the form \( F = \text{Hom}(L, R) \) where \( L \) is the line through \( \{\pm x\} \).

Since \( \gamma^1_n \) has a Euclidean metric, the correspondence \( w \mapsto \varphi_w \in \text{Hom}(L, R) \) where \( \varphi_w \) is defined by \( \varphi_w(v) = w \cdot v \) gives an isomorphism \( L \to \text{Hom}(L, R) \). Thus, by Theorem 3.3

\[ \gamma^1_n \cong \text{Hom}(\gamma^1_n, \epsilon^1). \]

So we have

\[ \tau \oplus \epsilon^1 \cong \gamma^1_n \oplus \gamma^1_n \oplus \cdots \oplus \gamma^1_n \]

\[ \text{times} \]

Now from the Whitney Product Theorem

\[ w(\tau) = w(\tau \oplus \epsilon^1) = w(\gamma^1_n)w(\gamma^1_n) \cdots w(\gamma^1_n) = (1 + a)^{n+1}. \]

Using the binomial theorem (and the fact that \( a^{n+1} = 0 \)) we have
$$w(r) = (1 + a)^{n+1}$$
$$= 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \cdots + \binom{n+1}{n}a^n. \quad \square$$

Remember that we are working with coefficients in $\mathbb{Z}_2$, so we are only interested in the binomial coefficients modulo 2. Table 5.1., reproduced from [4, p.46], makes it easy to find the total Stiefel–Whitney class $w(\mathbb{R}P^n)$ for $n = 1, \ldots, 10$.

<table>
<thead>
<tr>
<th>$\mathbb{R}P^n$</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}P^1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
</tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbb{R}P^{10}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.1

The table lists the binomial coefficients modulo 2. Since
\[ a^{n+1} = 0 \] we ignore the ones on the right hand side of the triangle. As an example, to find \( w(\mathbb{RP}^4) \) we look at the line on the table:

\[
\mathbb{RP}^4 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1
\]

This means that
\[
w(\mathbb{RP}^4) = (1) + (1)a + (0)a^2 + (0)a^3 + (1)a^4
\]
\[
= 1 + a + a^4
\]

Remember that we ignore the last 1 in the row since \( a^5 = 0 \).

Now suppose that a manifold \( M^n \) of dimension \( n \) can be immersed in Euclidean space \( \mathbb{R}^{n+k} \). Recall that a function \( f: M^n \to \mathbb{R}^{n+k} \) is an immersion if at every point \( x \in M^n \)

\[
Df_x: DM_x \to \mathbb{R}^{n+k}
\]
is injective. This means that \( \tau_M \) is an \( n \)-dimensional vector bundle, and the normal bundle \( \nu \) is a \( k \)-dimensional vector bundle.

From the Whitney Duality Theorem

\[
w_i(\nu) = \overline{w}_i(M^n).
\]

So the dual Stiefel-Whitney classes \( \overline{w}_i(M^n) = 0 \) for \( i > k \).

We have proven the following

**THEOREM 5.15.** If an \( n \)-dimensional manifold \( M^n \) can be immersed in \( \mathbb{R}^{n+k} \) then \( \overline{w}_i(M^n) = 0 \) for \( i > k \).
EXAMPLE: For $\mathbb{RP}^5$ we have
\[ w(\mathbb{RP}^5) = 1 + a^2 + a^4 \]
then
\[ \overline{w}(\mathbb{RP}^5) = 1 + a^2 \]
So if we want to immerse $\mathbb{RP}^5$ in $\mathbb{R}^{5^*k}$ then $k$ must be at least 2 (since $\overline{w}_2(\mathbb{RP}^5) \neq 0$).

EXAMPLE: For $\mathbb{RP}^9$
\[ w(\mathbb{RP}^9) = 1 + a^2 + a^8 \]
\[ \overline{w}(\mathbb{RP}^9) = 1 + a^2 + a^4 + a^6 \]
So if $\mathbb{RP}^9$ can be immersed in $\mathbb{R}^{9^*k}$ then $k$ must be at least 6.

**COROLLARY 5.16.** Let $n = 2^r$ for some integer $r$. If $\mathbb{RP}^n$ can be immersed in $\mathbb{R}^{n^*k}$, then $k$ must be at least $n - 1$.

**PROOF:** $w(\mathbb{RP}^n) = (1 + a)^{n+1} = (1 + a)(1 + a)^n$. Since we are working modulo 2 and $n = 2^r$ we have

\[ (1 + a)^n = 1 + a^n. \]
So $w(\mathbb{RP}^n) = (1 + a)(1 + a)^n = 1 + a + a^n + a^{n+1} = 1 + a + a^n$. Thus
\[ w(\mathbb{RP}^n) = 1 + a + a^n. \]

Then
\[ \overline{w}(\mathbb{RP}^n) = 1 + a + a^2 + \cdots + a^{n-1}. \]
Therefore $k$ must be at least $n - 1$. \(\square\)
Whitney has proven [8] that any smooth compact manifold of dimension \( n > 1 \) can be immersed in \( \mathbb{R}^{2n-1} \). We have shown that for \( \mathbb{RP}^n \), where \( n \) is a power of 2, that this is the best we can do.

So we have accomplished, at least in part, what we set out to do in Chapter 3. That is, we now have a way to find a lower bound on the dimension of Euclidean space that we need to embed \( \mathbb{RP}^n \).
BIBLIOGRAPHY


