SUFFICIENT CRITERIA FOR TOTAL DIFFERENTIABILITY OF A REAL VALUED FUNCTION OF A COMPLEX VARIABLE IN $\mathbb{R}^n$: AN EXTENSION OF H. RADEMACHER'S RESULT FOR $\mathbb{R}^2$

THESIS

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This thesis provides sufficient conditions for total differentiability almost everywhere of a real-valued function of a complex variable defined on a bounded region in $\mathbb{R}^n$. This thesis extends H. Rademacher's 1918 results in $\mathbb{R}^2$, which culminated in total differentiability, to $\mathbb{R}^n$.

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CHAPTER I

INTRODUCTION

In 1918 the German mathematician Hans Rademacher published a significant extension of the then-existing criteria needed to insure the property of total differentiability of a real-valued function of a complex variable. In the same paper, using two such functions concurrently, a uniform mapping of their bounded region of domain onto an area in $\mathbb{R}^2$ was produced in which the ratio of magnification of the mapping was identical to the absolute value of the functional determinant almost everywhere. Making the additional assumption that this mapping be 1-1 and continuous, the previously known transformation formula for the double integral of a finite, measurable function was valid.

This thesis is devoted to Part I of that 1918 paper, translated from the original German. In it, we will extend Rademacher's results in $\mathbb{R}^2$ which culminated in total differentiability to $\mathbb{R}^n$. Though alluded to by Rademacher himself as being possible, this extension of the vast body of lemmas and propositions to $\mathbb{R}^n$ comes only after a vigorous substantiation of these claims in the setting of $\mathbb{R}^n$. We must even modify Rademacher's plan of attack to accommodate an appeal to induction on the dimension of the space.

Although set in $\mathbb{R}^n$, all terms used in the thesis derive their meanings from analogous counterparts in real analysis, in particular, Lebesgue measure and integration theory. One term, however, that calls for illumination is the concept of total differentiability. Its formal definition is as follows:

<u>Definition</u>: Let the function $f: G \to \mathbb{R}$, G a subset of $\mathbb{R}^n$, possess all 1st order partial derivatives with respect to any variable throughout G. f is said to be totally differentiable at the point $\chi = (x_1, x_2, \ldots, x_n) \in G$ if there exists a function $R: H \to \mathbb{R}$ where $H = \{H \in \mathbb{R}^n | \chi + H \in G\}$ such that for any $H = (h_1, h_2, \ldots, h_n) \in H$,

$$(1)\quad f(\chi + H) = f(\chi) + \frac{\partial f(\chi)}{\partial x_1} h_1 + \frac{\partial f(\chi)}{\partial x_2} h_2 + \ldots + \frac{\partial f(\chi)}{\partial x_n} h_n$$

$$+ |H| \; R(H)$$

$$(2)\quad R(H) \to 0 \text{ as } H \to 0 \quad .$$

Equivalently, f is totally differentiable at the point χ if there exists the linear function $L: H \to \mathbb{R}$ where, again, $H = \{H \in \mathbb{R}^n | \chi + H \in G\}$ such that for any $H = (h_1, h_2, \ldots, h_n) \in H$,

$$(1)\quad L(H) = \frac{\partial f(\chi)}{\partial x_1} h_1 + \frac{\partial f(\chi)}{\partial x_2} h_2 + \ldots + \frac{\partial f(\chi)}{\partial x_n} h_n$$

(2) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|H| < \delta$ then

$$|[f(\chi + H) - f(\chi)] - L(H)| < \varepsilon |H| \quad .$$

Observe that for dimension $n = 1$, differentiability of f is synonymous with total differentiability of the function.

CHAPTER II

UNDERLYING HYPOTHESIS; EXISTENCE OF ALL 1ST ORDER PARTIAL DERIVATIVES ON A MUTUAL SET

Let f be a real-valued function whose domain is G, a bounded region in $\mathbb{R}^n$. We take the norm on $\mathbb{R}^n$ to be the familiar: $|\chi| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ where $\chi = (x_1, x_2, \ldots, x_n)$ and assume the measure μ on $\mathbb{R}^n$ to be n-dimensional Lebesgue measure. The directed system $(\mathbb{R}^+, \succ)$ is established in which $\beta \succ \alpha$ is interpreted $\beta < \alpha$ in the usual ordering of the reals and we summarily form the net $\langle \omega_f(\rho) \rangle_{\rho \varepsilon \mathbb{R}^+}$ where each $\omega_f(\rho)\colon G \to \mathbb{R}_{+\infty}$ is defined by

$$\omega_f(\rho)(\chi) = \sup_{\substack{0<|H|<\rho, \\ \chi + H \varepsilon G}} \left| \frac{f(\chi + H) - f(\chi)}{H} \right| .$$

We pose the question of convergence, the answer of which is affirmative.

Proposition 1: $\langle \omega_f(\rho) \rangle_{\rho \varepsilon \mathbb{R}^+}$ converges pointwise on G.

Proof

For each $\chi \varepsilon G$ let $\Gamma_\chi = \{\omega_f(\rho)(\chi) : \rho \varepsilon \mathbb{R}^+\}$. As Γ_χ is bounded below by 0, define the mapping $L_f\colon G \to \mathbb{R}_{+\infty}$ by

$$L_f(\chi) = \inf \Gamma_\chi .$$

We now demonstrate pointwise convergence of the net to L_f. $L_f(\chi) = +\infty$ implies $\omega_f(\rho)(\chi) = +\infty$ for all $\rho \in \mathbb{R}^+$, so, surely, we have convergence for such χ. For χ such that $L_f(\chi) < +\infty$, letting $\varepsilon > 0$ there exists a $\rho' > 0$ such that

$$L_f(\chi) \leq \omega_f(\rho')(\chi) < L_f(\chi) + \varepsilon \quad .$$

But as ρ increases in our system, the corresponding functional values $\omega_f(\rho)(\chi)$ are nonincreasing, so we have

$$L_f(\chi) \leq \omega_f(\rho)(\chi) \leq \omega_f(\rho')(\chi) < L_f(\chi) + \varepsilon$$

for all $\rho \succ \rho'$. Hence, $\langle\omega_f(\rho)\rangle \to L_f$ pointwise on $\{\chi{:}L_f(\chi) < +\infty\}$. As all possibilities for $\chi \in G$ have been accounted for, the proof is complete.

Q.E.D.

Let us make the added restriction that the values of L_f be finite. The finiteness of L_f along with the definition of $\omega_f(\rho)$ and the "squeeze theorem" for limits easily give us f's contin- uity on G; however, we may go further and even assert measur- ability of L_f. But first we need

<u>Lemma 1</u>: Each $\omega_f(\rho)$ is measurable.

<u>Proof</u>: Let $\{H_i\}$ be a dense sequence of $B(0;\rho) \subset \mathbb{R}^n$; define $F_i{:}G \to \mathbb{R}$ by

$$F_i(X) = \begin{cases} \left| \dfrac{f(\chi + H_i) - f(\chi)}{H_i} \right| & \text{if } \chi + H_i \in G \\ 0 & \text{if } \chi + H_i \notin G \end{cases}$$

<u>Claim 1</u>: $\omega_f(\rho)(\chi) = \sup_i F_i(\chi)$ for all $\chi \in G$.

<u>Proof</u>

Obviously, if $\sup_i F_i(\chi) = +\infty$ then $\omega_f(\rho)(\chi) = +\infty$, so let us restrict our attention to the case where $\sup_i F_i(\chi)$, $\omega_f(\rho)(\chi) < +\infty$, the proof of the other case being similar in nature. Let us also suppose to the contrary.

<u>Subcase 1</u>: $\omega_f(\rho)(\chi') > \sup_i F_i(\chi')$ for some $\chi' \in G$.

Let $\varepsilon = \omega_f(\rho)(\chi') - \sup_i F_i(\chi')$; by definition of $\omega_f(\rho)(\chi')$, there exists an $H' \in B(0;\rho)$ such that $\chi' + H' \in G$ and

$$\omega_f(\rho)(\chi') - \frac{\varepsilon}{2} < \left| \frac{f(\chi' + H') - f(\chi')}{H'} \right| .$$

Note that H' cannot be a member of the range of our dense sequence, so H' must be a limit point of $R_{\{H_i\}}$.

<u>Lemma 1.1</u>: Given $a \geq 0$ and $b, d, > 0$, there exists a $\delta \in \mathbb{R}^+$ such that

$$c \in \mathbb{R} \text{ and } a < b < c/d \Longrightarrow a < \frac{c}{d+\delta}$$

<u>Proof</u>

If $a = 0$, choose any positive real for δ; if $a > 0$, the necessary δ is $\frac{db}{a} - d$.

Q.E.D.

Let $a = \sup_i F_i(\chi')$, $b = \omega_f(\rho)(\chi')$, $d = |H'|$, and δ be the one guaranteed to exist in the lemma. As preliminaries, we record:

(i) f continuous on G implies there exists a $\delta' > 0$ such that if $|H' - H| < \delta'$,

$$|f(\chi' + H') - f(\chi' + H)| < |H'| \cdot \varepsilon/4 \quad .$$

(ii) Letting $\delta'' = \min(\delta, \delta')$ and H' a limit point of $R_{\{H_i\}}$ implies there exists an $H_{i'}$ such that

$$| H' - H_{i'}| < \delta'' \quad .$$

(iii) By the left-hand side of the triangle inequality,

$$\frac{1}{|H'| + \delta''} < \frac{1}{|H_{i'}|}$$

Now we are ready to rewrite inequality (*):

$$\omega_f(\rho)(\chi') - \varepsilon/2 < \left| \frac{[f(\chi' + H') - f(\chi' + H_{i'})] + [f(\chi' + H_{i'}) - f(\chi')]}{H'} \right|$$

$$\leq \left| \frac{f(\chi' + H') - f(\chi' + H_{i'})}{H'} \right| + \left| \frac{f(\chi' + H_{i'}) - f(\chi')}{H'} \right|$$

$$\leq \frac{|H'|}{|H'|} \cdot \varepsilon/4 + \frac{|f(\chi' + H_{i'}) - f(\chi')|}{|H'|} \text{ by (i) and (ii),}$$

which implies that

$$\sup F_i(\chi') < \omega_f(\rho)(\chi') - \frac{3}{4}\varepsilon < \frac{|f(\chi' + H_{i'}) - f(\chi')|}{|H'|} \quad .$$

Notice that we have the antecedent of the conditional in Lemma 1.1 with a, b, d as previously designated and $c = |f(\chi' + H_{i'}) - f(\chi')|$. Hence,

$$\sup_i F_i(\chi') < \frac{|f(\chi' + H_{i'}) - f(\chi')|}{|H'| + \delta}$$

$$\leq \frac{|f(\chi' + H_{i'}) - f(\chi')|}{|H'| + \delta''}$$

$$< \left|\frac{f(\chi' + H_{i'}) - f(\chi')}{H_{i'}}\right|$$

$$= F_{i'}(\chi')$$

by (iii); but this is absurd. Therefore, Subcase 1 leads to a contradiction.

<u>Subcase 2</u>: $\sup_i F_i(\chi_o) > \omega_f(\rho)(\chi_o)$ for some $\chi_o \in G$.

Again, let $\varepsilon = \sup_i F_i(\chi_o) - \omega_f(\rho)(\chi_o)$; by definition of $\sup_i F_i(\chi_o)$, there exists on i_o such that

$$0 < \sup F_i(\chi_o) - \varepsilon/2 < F_{i_o}(\chi_o) \quad .$$

Observe that $\chi_o + H_{i_o}$ must lie in G; recall, also, that $H_{i_o} \varepsilon B(0;\rho)$. Hence, $F_{i_o}(\chi_o)$ is a member of the set of difference quotients over which we took the supremum to form $\omega_f(\rho)(\chi_o)$; yet $F_{i_o}(\chi_o)$ clearly dominates that supremum. Thus, Subcase 2 leads to a contradiction.

We are, therefore, forced to conclude equality between $\omega_f(\rho)$ and $\sup_i F_i$ on the subset of G for which their functional values are finite. This completes the proof.

Q.E.D.

Claim 2: For all i, F_i is a measurable function of G.

Proof

We establish this claim by considering two restrictions of F_i, namely, $F_i|G_i$ and $F_i|G \sim G_i$ where $G_i = \{\chi \varepsilon G | \chi + H_i \varepsilon G\}$, an open, hence, measurable set. By the formulation of F_i, the latter restriction is simply the constant function 0, so surely $F_i|G \sim G_i$ is measurable. In regard to the former restriction, we will show a little more, namely, that $F_i|G_i$ is, in fact, continuous, therfore, measurable. As was pointed out prior to Lemma 1, f is continuous throughout G, so it sufficies to demonstrate continuity of the composite function $f_i(\chi) = f(\chi + H_i)$. But this becomes a trivial matter due to the fact that G is open, for then any sequence of points converging to χ will have its corresponding sequence of translates by H_i eventually contained in G. Therefore, the problem reverts back to the continuity of f at $\chi + H_i$, so we are done.

As both restrictions are measurable, F_i takes on measurability on its entire domain G.

G.E.D.

Measurability of $\omega_f(\rho)$ now follows from Claims 1 and 2.

Q.E.D. (Lemma 1)

Proposition 2: L_f is a measurable function on G.

Proof

This follows from Lemma 1 and the observation that the subnet $<\omega_f(\frac{1}{m})>_{m \in \mathbb{N}}$ also converges pointwise to L_f on G.

Q.E.D.

We now make the final assumption that L_f is integrable on all G. It is interesting to note in passing that the class of functions f for which L_f is finite and integrable encompasses those functions satisfying a Lipschitz condition on G. In fact, due to L_f's formulation, the Lipschitz constant for such f serves as a bound for L_f. Thus, L_f becomes a bounded measurable function defined on a set of finite measure, hence, integrable.

We would like to make some statements concerning the 4 partial derivatives of f with respect to a given variable x_i of χ and the value $L_f(\chi)$. In the ensuing discussion, for any $h \in \mathbb{R} \sim \{0\}$, by $\chi \oplus h_i$ we mean the element $(x_1, x_2, \ldots x_i + h, x_{i+1}, \ldots, x_n)$. Let us first consider: $D^+_{x_i} f(\chi)$.

As

$$\left\{ \left| \frac{f(\chi \oplus h_i) - f(\chi)}{h} \right| : 0 < h < \rho \text{ and } \chi \oplus h_i \in G \right\}$$

$$\subset \left\{ \left| \frac{f(\chi + H) - f(\chi)}{H} \right| : 0 < |H| < \rho \text{ and } \chi + H \in G \right\} ,$$

$$\sup_{\substack{0<h<\rho,\\ \chi\oplus h_i\varepsilon G}} \left|\frac{f(\chi\oplus h_i) - f(\chi)}{h}\right| \leq \sup_{\substack{0<|H|<\rho,\\ \chi+H\varepsilon G}} \left|\frac{f(\chi + H) - f(\chi)}{H}\right| \equiv \omega_f(\rho)(\chi)$$

$$\Rightarrow \inf_{\rho>0} \sup_{\substack{0<h<\rho,\\ \chi\oplus h_i\varepsilon G}} \left|\frac{f(\chi\oplus h_i) - f(\chi)}{h}\right| \leq \inf_{\rho>0} \omega_f(\rho)(\chi) = L_f(\chi) \quad ,$$

so,

$$\overline{\lim_{h\to 0^+}} \left|\frac{f(\chi\oplus h_i) - f(\chi)}{h}\right| \leq L_f(\chi) \quad .$$

<u>Lemma 2</u>: Let $g: \mathbb{R} \to \mathbb{R}$; then

$$\left|\overline{\lim_{h\to 0}} \; g(h)\right| \leq \overline{\lim_{h\to 0}} \; |g(h)| \quad .$$

By Lemma 2, we have

$$\left|D^+_{x_i} \; f(\chi)\right| = \left|\overline{\lim_{h\to 0^+}} \frac{f(\chi\oplus h_i) - f(\chi)}{h}\right| \leq \overline{\lim_{h\to 0^+}} \left|\frac{f(\chi\oplus h_i) - f(\chi)}{h}\right| ,$$

so, combining with our last inequality we finally arrive at

$$|D^+_{x_i} \; f(\chi)| \leq L_f(\chi) \quad .$$

We have analogous statements for $D^-_{x_i} \; f(\chi)$, $D_{+,x_i} \; f(\chi)$, and $D_{-,x_i} \; f(\chi)$. If $D_{x_i} \; f(\chi)$ signifies any one of these 4 partial derivates, then we may express succinctly what we have obtained by

$$| D_{x_i} \; f(\chi)| \leq L_f(\chi) \qquad \text{for all } 1 \leq i \leq n \quad .$$

Note the similarity between $D^+_{x_i} f$ and L_f; the only differences lie in the set of H values used to generate the difference quotients and an application of the norm. As a result, $D^+_{x_i} f$ as well as the other partial derivates acquire measurability from L_f through proofs analogous to the one establishing that very same property for L_f. In light of this new information, the last inequality yields the fact that $D_{x_i} f$ is also finite and integrable on G.

This second occurrence of finiteness and integrability has some far-reaching effects. We will presently define a real-valued function of a real variable, f_X, in terms of a restriction of f; integrability of $D_{x_i} f$ will imply integrability of all derivates of f_X, for nearly all X. After a long descent we will work our way back up to arrive at the prized property of differentiability of f_X a.e. in its domain. From here we reduce matters back to our original function f and obtain the sought-after property which closes Chapter II: existence of all 1st order partial derivatives of f on a single set whose measure matches that of G.

But first for preliminaries. As G is bounded, there exist reals a_j, b_j, $1 \leq j \leq n$, such that $G \subset \prod_{j=1}^{n} X_j$ where $X_j = (a_j, b_j)$. For $X = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \prod_{j \neq i}^{n} X_j$ and $x \in X_i$ we define the symbol,

$$X[x] = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in \prod_{j=1}^{n} X_j \quad .$$

With this in mind, define the set

$$G_i = \left\{ X \in \prod_{j\neq i}^{n} X_j \middle| X[x'] \in G \text{ for some } x' \in X_i \right\}$$

and for fixed $X \in G_i$ let

$$E_X = \left\{ x \in X_i \middle| X[x] \in G \right\} .$$

Define the mapping $f_X : E_X \to \mathbb{R}$ by

$$f_X(x) = f(X[x]) .$$

Observe that we may write

$$D^+ f_X(x) = D^+_{x_i} f(X[x]) ,$$

with analogous identities holding for the remaining three pairs of derivates. If Df_X denotes any one of the 4 derivates of f_X, then we may condense these into

$$Df_X(x) = D_{x_i} f(X[x])$$

where corresponding derivates must match. We are now ready for the following lemma.

<u>Lemma 3</u>: If $D_{x_i} f$ is integrable on G, then Df_X is integrable on E_X for almost all $X \in G_i$.

Proof

By assumption, $D_{x_i} f$ is integrable on G, so we may surely extend the function to all of $\prod_1^n X_j$, still maintaining its integrability, and, following suit, the derivate Df_X extends from E_X to all of X_i. Define $D_{x_i}\hat{f}: X_i \times \prod_{j\neq i}^n X_j \to \mathbb{R}_{+\infty}$ by

$$D_{x_i}\hat{f}\left((x,X')\right) = D_{x_i} f(X'[x]) \quad .$$

Integrability of $D_{x_i} f$ on $\prod_1^n X_j$ allows us to claim integrability of $D_{x_i}\hat{f}$ on $X_i \times \prod_{j\neq i}^n X_j$, for, in fact, we have equality of the integrals for nonnegative simple functions related as $D_{x_i} f$ and $D_{x_i}\hat{f}$ are, and, consequently, we may extend this to nonnegative functions so related. After recalling that Lebesgue measure as well as (n - 1) - dimensional Lebesgue measure are complete measures relative to the spaces X_i and $\prod_{j\neq i}^n X_j$, respectively, we are in a position to apply the Fubini Theorem, giving us the result: $(D_{x_i}\hat{f})_X : X_i \to \mathbb{R}_{+\infty}$, defined by $(D_{x_i}\hat{f})_X(x) = D_{x_i}\hat{f}\left((x,X)\right)$, is integrable on X_i for almost all X. However, a simple calculation shows that

$$(D_{x_i}\hat{f})_X(x) = Df_X(x) \qquad \text{for all } x \in X_i \quad ,$$

so that Df_X becomes integrable on X_i for almost all $X \in \prod_{j\neq i}^n X_j$. Restricting X to G_i and observing that $E_X \subset X_i$ is open, hence, measurable, the conclusion of the lemma falls out--Df_X is integrable on E_X for almost all $X \in G_i$.

Q.E.D.

Let X_o be such a point in G_i for which we have integrability of Df_{X_o} on E_{X_o}. Recalling that E_{X_o} is open, we may decompose it into a countable union of disjoint open intervals of the form $(\alpha_\ell, \beta_\ell)$. Noting that $(\alpha_\ell, \beta_\ell) = \bigcup_{m=1}^{+\infty} [\alpha_\ell + \frac{1}{m}, \beta_\ell - \frac{1}{m}]$ and the fact that we have integrability of Df_{X_o} on $(\alpha_\ell, \beta_\ell)$, surely,

$$\int_{\alpha_\ell + 1/m}^{\beta_\ell - 1/m} Df_{x_o} < +\infty \quad .$$

Let $P_{\ell,m} = \{\alpha_\ell + 1/m = c_o < c_1 < \ldots < c_p = \beta_\ell - 1/m\}$ be a partition of $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$. Because of the affinity between f_{X_o} and f created in the former's definition, the continuity of f at points $X_o[x]$, where $x \in E_{X_o}$, translates into continuity of f_{X_o} at all $x \in E_{X_o}$, in particular, continuity on $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$. Consider now an arbitrary subinterval generated by two consecutive points of $P_{\ell,m}$: $[c_{q-1}, c_q]$. Our two critical properties--continuity of f_{X_o} and integrability of Df_{X_o} on $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$ -- carry over to $[c_{q-1}, c_q]$. We now introduce the following general theorem concerning continuous, real-valued functions defined on a closed interval.

<u>Theorem</u>: Let g be a continuous, real-valued function defined on [a,b] and let Dg denote any one of the 4 derivates of g, which we assume to be measurable, real-valued also. If, in addition, Dg is integrable on [a,b], then

$$|g(b) - g(a)| \leq \int_a^b |Dg| \quad .$$

Proof

We carry out the proof for the case where $Dg \equiv D^+g$; analogous proofs hold for the remaining three derivates. We must credit Lebesgue for the ensueing construction process leading up to Lemma 4.[1] In the following preliminary lemma as well as Lemmas 4 and 5, we allow the term interval to encompass a singleton, in which case we call the interval "degenerate" and denote it by $[z,z]$, where z is the single point in question.

Lemma: Suppose that to every $x \in [a,b]$ there corresponds an interval $[x, x+h] \subset [a,b]$, $h > 0$. Then there exists a countable sequence of nonoverlapping intervals $[x_i, x_i + h_i]$ whose union is $[a,b]$, $h_i \geq 0$. If, moreover, the singleton $\{b\}$ occurs in the sequence, it appears only once. The intervals of the sequence are said to form a chain from a to b.

Proof

Let W be the set of all countable ordinals; so, 1, ω, ω^2, etc. are elements of W. Under its natural ordering, W forms an uncountable, well-ordered set with initial element 1. To 1, let us assign an interval $[x_1, x_1 + h_1] \subset [a,b]$ where $x_1 = a$, whose existence is guaranteed by hypothesis. Relabel $x_1 + h_1 \equiv x_2$ and to 2 assign an intervale $[x_2, x_2 + h_2]$ corresponding by

[1]Thomas Hawkins, "Section 1, Chapter 5," Lebesgue's Theory of Integration (New York, 1975), pp. 134-135.

hypothesis to x_2. Now, for an arbitrary $\beta \in W$, $\beta \neq 1$, suppose that we have assigned to each $\alpha < \beta$ an interval $[x_\alpha, x_\alpha + h_\alpha] \subset [a,b]$, degenerate if ever $x_\alpha = b$; suppose further that, collectively, the intervals have the property

$$\bigcup_{\alpha<\beta} [x_\alpha, x_\alpha + h_\alpha] = [a,c) \text{ or } [a,c] \quad .$$

Let $x_\beta = \sup \{ x_\alpha + h_\alpha | \alpha < \beta \} \in [a,b]$ and assign to β:

(1) some non-degenerate $[x_\beta, x_\beta + h_\beta]$ — if $x_\beta \neq b$ (guaranteed by hypothesis)

(2) $\{x_\beta\}$ — if $x_\beta = b$.

It is clear that, in either case, we have maintained the property

$$\bigcup_{\alpha<\beta+1} [x_\alpha, x_\alpha + h_\alpha] = (a,d) \text{ or } [a,d] \quad .$$

So, the set of all ordinals to which we have assigned sets displaying the collective property above forms on inductive subset of W containing initial element 1. Since W satisfies the transfinite induction principle, we may, therefore, assert that to every $\tau \in W$, there exists an interval $[x_\tau, x_\tau + h_\tau] \subset [a,b]$, $h_\tau \geq 0$, such that

$$\bigcup_{\sigma<\tau} [x_\sigma, x_\sigma + h_\sigma] = [a,c) \text{ or } [a,c] \quad .$$

Now, either there exists some $\gamma' \varepsilon W$ to which we have assigned $\{b\}$ or not; if so, there exists a first such ordinal, γ_o, and, by definition, it must have been the case that

$$x_{\gamma_o} = b \quad .$$

Then, surely,

$$\bigcup_{\alpha<\gamma_o} [x_\alpha, x_\alpha + h_\alpha] = [a,b] \text{ or } [a,b) \quad .$$

If the former occurs, we have established what we wished to prove since γ_o is a countable ordinal and all of our intervals are pairwise disjoint except possibly for endpoints. For the latter occurrence, our countable sequence will simply be

$$\left\{[x_\alpha, x_\alpha + h_\alpha]\right\}_{\alpha\leq\gamma_o} \quad .$$

However, if no such γ' exists, we have generated an uncountable number of nonoverlapping subintervals of [a,b], so we have produced an uncountable number of rationals--absurd. Thus, our first supposition must have been the case, thus completing the proof.

Q.E.D.

Let $\eta > 0$; define $\rho > 0$ such that $\rho(b - a) = \eta/3$ and let P denote any partition of $(-\infty, +\infty)$: $P = \left\{\ldots < L_{-2} < L_{-1} < L_0 < L_1 < L_2 < \ldots\right\}$ where we insist that $\| P \| < \rho$. Let us arrange the partition points into a sequence for brevity. Hence,

$$P = \left\{L_{n_i}\right\}_1^{+\infty} \text{ where } \|P\| < \rho \quad .$$

Define

$$e_{n_i} = \left\{x \in [a,b] \mid L_{n_i} < D^+g(x) \leq L_{n_i+1}\right\}$$

and positive sequence $\left\{a_{n_i}\right\}_1^{+\infty}$ such that $|L_{n_i}|\ a_{n_i} \leq \eta/4^i$. Observe that the terms of our sequence are chosen so that

$$\sum_{i=1}^{+\infty} |L_{n_i}|\ a_{n_i} \leq \eta/3 \quad .$$

Finally, as e_{n_i} is measurable, for any i, there exists open set $A_{n_i} \supset e_{n_i}$ such that $mA_{n_i} - m(e_{n_i}) < a_{n_i}$.

We summarily obtain a collection of closed intervals covering [a,b]. For each $x \in [a,b)$, if $x \in e_{n_i}$ then there exists an $h > 0$ such that the following three conditions are met:

(i) $h \leq \rho$

(ii) $(x, x + h) \subset A_{n_i} \cap [a,b]$

(iii) $L_{n_i} < \dfrac{g(x + h) - g(x)}{h} < L_{n_i+1} + \rho \quad .$

Notice that with statement (ii) we satisfy the hypothesis of the preceding lemma. So, there exists a chain of subintervals

$[x_i, x_i + h_i]$ from a to b, and w.l.g. we may take the chain to be countably infinite. Let $\left\{[x_i, x_i + h_i]\right\}_1^{+\infty}$ denote such a chain and B_{n_i} represent $\bigcup_{x_k \varepsilon e_{n_i}} [x_k, x_k + h_k]$.

For $x_k \varepsilon e_{n_i}$, $x_k \neq b$, (iii) implies

$$L_{n_i} h_k < g(x_k + h_k) - g(x_k) < (L_{n_i} + \rho) h_k$$

$$(|L_{n_i}| - \rho) h_k < |g(x_k + h_k) - g(x_k)| < (|L_{n_i}| + \rho) h_k .$$

Therefore,

$$\sum_{x_k \varepsilon e_{n_i}} |g(x_k + h_k) - g(x_k)| \leq (|L_{n_k}| + \rho) m B_{n_i}$$

$$\sum_{i=1}^{+\infty} \sum_{x_k \varepsilon e_{n_i}} |g(x_k + h_k) - g(x_k)| \leq \sum_{i=1}^{+\infty} |L_{n_i}| m B_{n_i} + \rho (b - a) .$$

<u>Lemma 4</u>: $|g(b) - g(a)| \leq \sum_i \sum_k |g(x_k + h_k) - g(x_k)|$.

<u>Proof</u>

Let $\varepsilon > 0$ be given and consider the left and right endpoints of an element in the chain: $[x_i, x_i + h_i]$. g continuous at x_i implies there exists $\delta_{i_1} > 0$ such that

$$|g(x_i) - g(x)| < \frac{\varepsilon}{2^i} \text{ if } |x_i - x| < \delta_{i_1}, \ x \ \varepsilon [a,b] ;$$

g continuous at $x_i + h_i$ implies there exists $\delta_{i_2} > 0$ such that

$$|g(x_i + h_i) - g(x)| < \frac{\varepsilon}{2^i} \quad \text{if} \quad |(x_i + h_i) - x| < \delta_{i_2},\ x \in [a,b] \quad .$$

Let $\delta_i = \min(\delta_{i_1}, \delta_{i_2})$ and $I_j = (x_i - \frac{\delta_i}{2}, (x_i + h_i) + \frac{\delta_i}{2})$.

$\{I_j\}_1^{+\infty}$ forms an open cover of [a,b], so there exists finite subcover $\{I_{j_i}\}_1^p$ of [a,b], ordered increasingly w.r.t. left and right endpoints. Considering only the non-overlapping elements of the chain associated with the I_{j_i}'s, we have:

$$a \le x_{j_1} < x_{j_1} + h_{j_1} \le x_{j_2} < x_{j_z} + h_{j_z} \le \ldots \le x_{j_p} < x_{j_p} + h_{j_p} \le b \quad .$$

<u>Note</u>: $x_{j_i} - (x_{j_{i-1}} + h_{j_{i-1}}) < \max(\delta_{j_i}, \delta_{j_{i-1}})$ for $1 \le i \le p$.

Observe that

$$[a, x_{j_1}] \cup \bigcup_2^p [(x_{j_{i-1}} + h_{j_{i-1}}), x_{j_i}] \cup \bigcup_1^p [x_{j_i}, (x_{j_i} + h_{j_i})] \cup [(x_{j_p} + h_{j_p}), b]$$

$$= [a,b] \quad .$$

Therefore,

$$|g(b) - g(a)| \le |g(x_{j_1}) - g(a)| + \sum_1^p |g(x_{j_i} + h_{j_i}) - g(x_{j_i})|$$

$$+ \sum_2^p |g(x_{j_i}) - g(x_{j_{i-1}} + h_{j_{i-1}})|$$

$$+ |g(b) - g(x_{j_p} - h_{j_p})|$$

$$\leq \varepsilon/2^{j_1} + \sum_{1}^{p} |g(x_{j_i} + h_{j_i}) - g(x_{j_i})| + \sum_{2}^{p} \max\left(\frac{\varepsilon}{2^{j_i}}, \frac{\varepsilon}{2^{j_{i-1}}}\right) + \varepsilon/2^{j_p}$$

$$< \sum_{1}^{p} |g(x_{j_i} + h_{j_i}) - g(x_{j_i})| + 3\varepsilon$$

$$< \sum_{i=1}^{+\infty} \sum_{x_k \varepsilon e_{n_i}} |g(x_k + h_k) - g(x_k)| + 3\varepsilon$$

since we are working with two series, both of which are absolutely convergent. As $\varepsilon > 0$ was arbitrary, the lemma follows.

Q.E.D.

Lemma 4 and the inequality preceding it furnishes us with

$$|g(b) - g(a)| \leq \sum_{1}^{+\infty} |L_{n_i}| \, m\, B_{n_i} + \rho(b - a)$$

$$= \sum_{1}^{+\infty} |L_{n_i}| \, m\, B_{n_i} + \eta/3$$

by choice of ρ.

<u>Lemma 5</u>: $\sum_{1}^{+\infty} |L_{n_i}| \, m\, B_{n_i} \leq \sum_{1}^{+\infty} |L_{n_i}| \, m(e_{n_i}) + \eta/3$

<u>Proof</u>

It suffices to show

$$\sum_{1}^{q} |L_{n_i}| \, m\, B_{n_i} < \sum_{1}^{q} |L_{n_i}| \, m(e_{n_i}) + \eta/3$$

for any $q \in \mathbb{N}$. Recall that $B_{n_i} = \bigcup_{x_k \varepsilon e_{n_i}} [x_k, x_k + h_k]$ and observe that $m\, B_{n_i} \leq m\, A_{n_i}$ for all i; let q ε and define

$$D = \left\{ i \le q \mid m(e_{n_i}) \le m\, B_{n_i} \right\} ,$$

$$E = \left\{ j \le q \mid m\, B_{n_j} < m\,(e_{n_j}) \right\} .$$

Note: For all $i \varepsilon D$, $m\, B_{n_i} - m(e_{n_i}) < a_{n_i}$.

Therefore,

$$\sum_{1}^{q} |L_{n_i}|\, m\, B_{n_i} = \sum_{i \,\varepsilon\, D} |L_{n_i}|\, m\, B_{n_i} + \sum_{j \,\varepsilon\, E} |L_{n_j}|\, m\, B_{n_j}$$

$$< \sum_{i \,\varepsilon\, D} |L_{n_i}|\, [m(e_{n_i}) + a_{n_i}] + \sum_{j \,\varepsilon\, E} |L_{n_j}|\, m(e_{n_j})$$

$$\le \sum_{i=1}^{q} |L_{n_i}|\, [m(e_{n_i}) + a_{n_i}]$$

$$< \sum_{1}^{q} |L_{n_i}|\, m\,(e_{n_i}) + \eta/3$$

by our original formulation of $\left\{ a_{n_i} \right\}_1^{+\infty}$.

Q.E.D.

With Lemma 5, our bound for $|g(b) - g(a)|$ may now be stated in terms of the e_{n_i}'s. Our prior inequality becomes

$$|g(b) - g(a)| \le \sum_{1}^{+\infty} |L_{n_i}|\, m(e_{n_i}) + \frac{2}{3}\eta .$$

Lemma 6: $\left| \int_a^b |D\overset{+}{g}| - \sum_{1}^{+\infty} |L_{n_i}|\, m(e_{n_i}) \right| \le \eta/3$.

Proof

By definition,

$$L_{n_i} < D_g^+(x) \leq L_{n_{i+1}} < L_{n_i} + \rho \quad \text{for all} \quad x \in e_{n_i}$$

which implies

$$|L_{n_i}| - \rho < |D_g^+(x)| < |L_{n_i}| + \rho$$

$$[|L_{n_i}| - \rho]\ m(e_{n_i}) \leq \int_{e_{n_i}} |D_g^+| \leq [|L_{n_i}| + \rho]\ m(e_{n_i})$$

$$\sum_1^{+\infty} [|L_{n_i}| - \rho]\ m(e_{n_i}) \leq \sum_1^{+\infty} \int_{e_{n_i}} |D_g^+| \leq \sum_1^{+\infty} [|L_{n_i}| + \rho]\ m(e_{n_i})$$

$$\sum_1^{+\infty} |L_{n_i}|\ m(e_{n_i}) - \rho\ (b-a) \leq \int_a^b |D_g^+| \leq \sum_1^{+\infty} |L_{n_i}|\, m(e_{n_i}) + \rho\,(b-a)$$

$$-\eta/3 \leq \int_a^b |D_g^+| - \sum_1^{+\infty} |L_{n_i}|\, m(e_{n_i}) \leq \eta/3$$

$$\left| \int_a^b |D_g^+| - \sum_1^{+\infty} |L_{n_i}|\ m(e_{n_i}) \right| \leq \eta/3$$

as $\int_a^b |D_g^+| < +\infty$.

Q.E.D.

With Lemma 6 and our prior inequality, we may safely say that

$$|g(b) - g(a)| \leq \int_a^b |D^+g| + \eta$$

which, thus, completes the bounding process. For $\eta > 0$ was arbitrary! Therefore, the long-awaited and essential result of our theorem,

$$|g(b) - g(a)| \leq \int_a^b |D^+g| \quad ,$$

is valid.

Q.E.D.

Thus, for our particular function f_{X_o} and Df_{X_o} restricted to $[c_{q-1}, c_q]$, the theorem yields the result

$$|f_{X_o}(c_q) - f_{X_o}(c_{q-1})| \leq \int_{c_{q-1}}^{c_q} |Df_{X_o}| \quad .$$

As q was arbitrary,

$$\sum_{q=1}^{p} |f_{X_o}(c_q) - f_{X_o}(c_{q-1})| \leq \int_{\alpha_\ell+1/m}^{\beta_\ell-1/m} |Df_{X_o}| < +\infty \ .$$

Therefore, the set of all such sums for finite partitions of $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$ is bounded, implying that $f_{X_o} \in BV$ on $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$. Consequently, by a standard theorem from

Lebesgue measure theory, f'_{X_o} exists a.e. in $[\alpha_\ell + 1/m, \beta_\ell - 1/m]$. But, as the union of countably many sets of measure 0 is again a set of measure 0, f'_{X_o} exists a.e. in $(\alpha_\ell, \beta_\ell)$ and, likewise, f'_{X_o} exists a.e. in E_{X_o}. We record this crucial result as

Proposition 3: f'_{X_o} exists a.e. in E_{X_o}.

In terms of our original function f, this translates into $\frac{\partial f(X_o[\![x]\!])}{\partial x_i}$ exists a.e. in E_{X_o}. However, we can go further and claim a much stronger result, namely,

<u>Theorem I</u>: For each i, $\frac{\partial f}{\partial x_i}$ exists a.e. in G.

<u>Proof</u>

We analyze this from the position of considering the set of points for which $\frac{\partial f(\chi)}{\partial x_i}$ does not exist. Since this is the complement of the measurable set on which all 4 partial derivates do agree,

$$L = \left\{\chi \in G \,\middle|\, \frac{\partial f(\chi)}{\partial x_i} \text{ does not exist}\right\}$$

is measurable. It suffices to show that L has 0-measure.

Let us transfer discussion to the product space $\mathbb{R}_i \times \prod_{j\neq i}^{n} \mathbb{R}_j$ with its product measure $m \times \lambda$ which we will denote by ν; we assume, of course, each $\mathbb{R}_j \equiv \mathbb{R}$ for $1 \leq j \leq n$ and that m, λ denote real and $(n - 1)$ - dimensional Lebesgue measures, respectively. Define the isomorphism $\Phi: \mathbb{R}^n \to \mathbb{R}_i \times \prod_{j\neq i}^{n} \mathbb{R}_j$ by

$$\Phi(X) = (x,X)$$

when X is expressed as $X[x]$; accordingly, let $\chi_{\Phi(L)}$ be the characteristic function on $\Phi(L)$. We wish to extract measurability of $\Phi(L)$, hence, measurability of $\chi_{\Phi(L)}$, from our knowledge of L. This will follow by straightforward means if we look at an equivalent formulation of product measure ν, that of the restriction of outer measure ν^* to the σ-algebra of measurable sets in $\mathbb{R}_i \times \prod_{j\neq i}^{n} \mathbb{R}_j$; ν^* is simply $\mu^* \circ \Phi^{-1}$. In light of this, measurability of $\Phi(L)$ stems from measurability of L. Hence, $\chi_{\Phi(L)}$ is measurable and

$$\int\int_{\mathbb{R}_i \times \prod_{j\neq i}^{n} \mathbb{R}_j} \chi_{\Phi(L)}\, d\nu = \nu[\Phi(L)] = \mu^* L = \mu L \quad .$$

As $\Phi(\prod_1^n X_j) = X_i \times \prod_{j\neq i}^{n} X_j$,

$$\mu L = \int\int_{X_i \times \prod_{j\neq i}^{n} X_j} \chi_{\Phi(L)}\, d\nu \quad .$$

Recall that $G_i = \left\{ X \in \prod_{j\neq i}^{n} X_j \mid X[x'] \in G \text{ for some } x' \in X_i \right\}$ and define $I_i = \left\{ X \in G_i \mid Df_X \text{ is integrable on } E_X \right\}$; observe that both G_i and I_i are measurable. So, by the Fubini Theorem,

$$\mu L = \int_{\substack{n \\ \pi X_j \\ j\neq i}} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX$$

$$= \int_{G_i} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX + \int_{\substack{n \\ \pi X_j \sim G_i \\ j\neq i}} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX$$

$$= \int_{I_i} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX + \int_{G_i \sim I_i} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX .$$

Considering the 2nd iterated integral, for any given X,

$$0 \leq \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \leq mX_i .$$

Therefore,

$$0 \leq \int_{G_i \sim I_i} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX \leq mX_i \cdot \nu(G_i \sim I_i) = 0$$

by Lemma 3. Consequently, the 2nd summand vanishes and we are left with

$$\mu L = \int_{I_i} \left\{ \int_{X_i} \chi_{\Phi(L)}(x,X)\ dx \right\} dX .$$

Finally, for $X \in I_i$ define $\mathcal{E}_X = \left\{ x \in E_X \mid \frac{\partial f(X[x])}{\partial x_i} \text{ exists} \right\}$;

then

$$\mu L = \int_{I_i} \left\{ \int_{\mathcal{E}_X} \chi_{\Phi(L)}(x,X)\ dx + \int_{X_i \sim \mathcal{E}_X} \chi_{\Phi(L)}(x,X)\ dx \right\} dX$$

$$= \int_{I_i} \left\{ \int_{\mathcal{E}_x} \chi_{\Phi(L)}(x,X)\ dx + \int_{E_X \sim \mathcal{E}_X} \chi_{\Phi(L)}(x,X)\ dx \right\} dX$$

$$= \int_{I_i} \left\{ \int_{\mathcal{E}_X} \chi_{\Phi(L)}(x,X)\ dx \right\} dX$$

from the discussion immediately preceding the theorem. But, now we have reduced matters to the point where (x,X) is no longer in $\mu(L)$. Hence, the final iterated integral vanishes and we are left with the sufficient result for the theorem.

Q.E.D.

Finally, if K_i denotes the measure equivalent subset of G on which $\frac{\partial f}{\partial x_i}$ exists, then Theorem I enables us to obtain quite easily a measurable set on which $\frac{\partial f}{\partial x_i}$ exists for any i and whose measure still equals that of G--namely, $K = \bigcap_{i=1}^{n} K_i$.

CHAPTER III

THE INDUCTION PROCESS PROPER; EXISTENCE OF THE APPROPRIATE SUBSET OF WHICH TO ESTABLISH TOTAL DIFFERENTIABILITY

We formally state the major theorem which this thesis proposes to prove.

<u>Theorem</u>: Let f be a real-valued function defined on a bounded region G of $\mathbb{R}^n$, $<\omega_f(\rho)>_{\rho\in\mathbb{R}^+}$ be a net of extended real-valued functions where each $\omega_f(\rho)\colon G \to \mathbb{R}_{+\infty}$ is defined by

$$\omega_f(\rho)(\chi) = \sup_{\substack{0<|H|<\rho,\\ \chi + H \in G}} \left|\frac{f(\chi + H) - f(\chi)}{H}\right| ,$$

and L_f be the pointwise limit of $<\omega_f(\rho)>_{\rho\in\mathbb{R}^+}$ on G. If L_f is finite and integrable on G then there exists a measure-equivalent subset of G on which f is totally differentiable.

Proof

We proceed with the proof of total differentiability of f a.e. in G by induction on the dimension of the space.

Part I: For n = 1, i = 1 in Chapter II and $\frac{\partial f(x)}{\partial x_1}$ is simply f'. Recalling the observation made earlier that total differentiability is equivalent to differentiability in dimension n = 1, the assertion follows from Theorem I, Chapter II.

Part II: So, we shall assume for all functions $g: G \to \mathbb{R}$, G a bounded region in $\mathbb{R}^{k-1}$, whose corresponding L_g function, defined, as before, to be the pointwise limit of the net $\langle \omega_g(\rho) \rangle$, is finite and integrable on G, that the contention of total differentiability a.e. in G is valid, $(k - 1)$ - dimensional Lebesgue measure assumed. Note that we have a little bit more going for us, namely, that the bounded region G may be a subset of a hyperplane of dimension $k - 1$ in $\mathbb{R}^k$ since any such hyperplane is isomorphic to $\mathbb{R}^{k-1}$. Thus, the isomorphic copy of g would inherit all of the necessary requirements to insure total differentiability a.e. in its bounded region of domain in $\mathbb{R}^{k-1}$, this latter property being passed back up to g defined on G.

Consequently, let $n = k$ and $f: G \to \mathbb{R}$, G a bounded region in $\mathbb{R}^k$, be a mapping whose corresponding L_f function is both finite and integrable on G. We retain all of the results of Chapter II (primarily, Theorem I) since $n \in \mathbb{N}$ was arbitrary and we now proceed to strengthen our properties of convergence. In order to do so, however, we must further decompose G and, for the function $\frac{\partial f}{\partial x_i}$, introduce the net concept in its formulation. In order to accomplish the latter, we must reexamine our natural ordering of the reals. Under the directed system $(\mathbb{R} \{0\}, ⧁)$, $\beta ⧁ \alpha$ if and only if $|\beta| < |\alpha|$. Note that the restriction of ⧁ to $\mathbb{R}^+$ is our previous ordering $\succ$ for that set. Therefore, by convention, we will write $\succ$ for ⧁ and simply specify whenever we are working with a restricted subset of $\mathbb{R} \sim \{0\}$. Our directed system now becomes $(\mathbb{R} \sim \{0\}, \succ)$.

To begin with, let $\eta > 0$ be given such that $\eta < \frac{\mu G}{k+3}$. We introduce a latent result which is used, at present, only for the set it generates.

Property 1: There exists a measurable subset $H \subset K$ possessing the two properties:

(i) $\mu H > \mu K - \eta = \mu G - \eta$

(ii) L_f is bounded on H.

Proof

This follows simply from the integrability and nonnegativeness of L_f. Let M_η be a positive real such that

$$M_\eta > 1/_\eta \int_G L_f \, d\mu \quad .$$

Consider $A_\eta = \left\{ \chi \in K \,|\, L_f(X) \geq M_\eta \right\}$; we have

$$\mu A_\eta \cdot M_\eta \leq \int_{A_\eta} L_f \, d\mu \leq \int_G L_f d\mu \quad ,$$

so, $\mu A_\eta < \eta$ and L_f is bounded on $H = K \sim A_\eta$.

Q.E.D.

Let us return now to the convergence of the net $\langle \omega_f(\rho) \rangle_{\rho \in \mathbb{R}^+}$ to L_f on G. We have the following theorem.

Theorem II: There exists a measurable subset $S_o \subset H$ possessing the two properties:

(i) $\mu S_o > \mu H - \eta$

(ii) $\langle \omega_f(\rho) \rangle_{\rho \in \mathbb{R}^+}$ converges uniformly to L_f on S_o.

Proof

Consider the subnet $\langle \omega_f(\frac{1}{m}) \rangle$. By Egoroff's Theorem for $\mathbb{R}^k$ there exists subset $A_o \subset H$ with $\mu A_o < \eta$ such that $\langle \omega_f(\frac{1}{m}) \rangle \to L_f$ uniformly on $H \sim A_o$. Set $S_o = H \sim A_o$ and let $\varepsilon > 0$. There exists $m_o \in \mathbb{N}$ such that for all $\chi \in S_o$

$$|L_f(\chi) - \omega_f(\tfrac{1}{m})(\chi)| < \varepsilon$$

if $m > m_o$. But, we have the relation

$$L_f(\chi) \leq \omega_f(\rho)(\chi) \leq \omega_f(\tfrac{1}{m_o})(\chi)$$

for any $\rho \leq m_o$, holding for all $\chi \in G$; so, choose $\rho_o = \frac{1}{m_o}$, and for any $\chi \in S_o$

$$|L_f(\chi) - \omega_f(\rho)(\chi)| < \varepsilon$$

if $\rho \succ \rho_o$, which is (ii) above. (i) follows readily from the definition of S_o.

Q.E.D.

Recall that for any $h \in \mathbb{R} \sim \{0\}$, by $\chi \oplus h_i$ we mean the element $(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_k)$. With this in mind, for such non-zero h define $f_h : G \to \mathbb{R}$ by

$$f_h(\chi) = \begin{cases} \dfrac{f(\chi \oplus h_i) - f(\chi)}{h} & \text{if } \chi \oplus h_i \in G \\ 0 & \text{if } \chi \oplus h_i \notin G . \end{cases}$$

We form the net $\langle f_h \rangle_{h \in \mathbb{R} \sim 0}$ and in the process observe that pointwise convergence of $\langle f_h \rangle$ to the function $\frac{\partial f}{\partial x_i}$ on a subset of G is equivalent to the more standard interpretation of the existence of $\frac{\partial f}{\partial x_i}$ on that subset. An analogous statement to that of Theorem II concerning the convergence of $\langle f_h \rangle$ to $\frac{\partial f}{\partial x_i}$ may now be put forth. We assert

<u>Theorem III</u>: For each i, there exists a measurable subset $S_i \subset H$ having the two properties:

(i) $\mu S_i > \mu H - \eta$

(ii) $\langle f_h \rangle$ converges uniformly to $\frac{\partial f}{\partial x_i}$ on S_i .

<u>Proof</u>

Define the set

$$S(m,t) = \bigcap_{h \succ \frac{1}{m}} \left\{ \chi \in H: B(\chi;\ 2|h|) \subset G \text{ and } \left|f_h(\chi) - \frac{\partial f(\chi)}{\partial x_i}\right| \leq 1/t \right\}$$

where $m \in \mathbb{N}$, $t > 0$. Our first goal is to establish its measurability.

<u>Proposition</u>: For each $m \in \mathbb{N}$, $t > 0$, the set $S(m,t)$ is measurable.

Proof

We establish this claim by considering two restrictions of f_h, namely, $f_h|G_h$ and $f_h|G \sim G_h$ where $G_h = \left\{\chi \in G | \chi \oplus h_i \in G\right\}$, an open, hence, measurable set. By the formulation of f_h, the latter restriction is simply the constant function 0, so, surely, $f_h|G \sim G_h$ is measurable. In regard to the former restriction, we will show a little bit more, namely, that $f_h|G_h$ is in fact continuous, therefore, measurable. As was pointed out early in Chapter II, f is continuous throughout G, so it sufficies to demonstrate continuity of the composite function $g(\chi) = f(\chi \oplus h_i)$. But this becomes a trivial matter due to the fact that G is open, for then any sequence of points converging to χ will have its corresponding sequence of translates by $\oplus h_i$ eventually contained in G. Therefore, the problem reverts back to the continuity of f at $\chi \oplus h_i$, so that we are done.

As both restrictions are measurable, f_h takes on measurability on its entire domain, G.

Q.E.D.

With Claim 1 at our disposal, we now know that the individual members of the intersection are, themselves, measurable since $f_h - \frac{\partial f}{\partial x_i}$ now becomes a measurable function on G. To see that the uncountable intersection is also measurable, define

$$S^*(m,t) = \bigcap_{\substack{h \succ \frac{1}{m} \\ h \in \mathbf{Q}}} \left\{ \chi \in H \colon B(\chi;\ 2|h|) \subset G \text{ and } \left|f_h(\chi) - \frac{\partial f(\chi)}{\partial x_i}\right| \leq 1/t \right\}$$

where, of course, $\mathbf{Q}$ is the set of rationals. Obviously,

$$S(m,t) \subset S^*(m,t)$$

but, what is more, we actually have the reverse inclusion.

Claim 2: $S^*(m,t) \subset S(m,t)$ for all $m \in \mathbb{N}$, $t > 0$.

Proof

Let $\chi_o \in S^*(m,t)$. We wish to show that for any irrational $h' \succ \frac{1}{m}$:

(i) $B(\chi_o; 2|h'|) \subset G$

(ii) $|f_{h'}(\chi_o) - \frac{\partial f(\chi_o)}{\partial x_i}| \leq 1/t$.

(i) is obvious if one observe that we may always choose a sequence of rationals convering to $|h'|$ from below. In regard to (ii), let $\{\eta_q\}$ be any sequence of rationals which converge to h' such that each term satisfies $\eta_q \succ \frac{1}{m}$. Since $\chi_o \in S^*(m,t)$, for any $q \in \mathbb{N}$, $|f_{\eta_q}(\chi_o) - \frac{\partial f(\chi_o)}{\partial x_i}| \leq 1/t$, which implies

$$\left|f_{h'}(\chi_o) - \frac{\partial f(\chi_o)}{\partial x_i}\right| = \left|\lim_{q\to+\infty} f_{\eta_q}(\chi_o) - \frac{\partial f(\chi_o)}{\partial x_i}\right| \leq 1/t$$

after observing the f is continuous at $\chi_o \oplus h'_i$ since we have taken the trouble to insure that the point lies in G. Therefore, Claim 2 is valid.

Q.E.D.

With $S(m,t) = S^*(m,t)$, our proof of measurability of $S(m,t)$ comes to a close.

Q.E.D.

In light of the preceding proposition, for a fixed $t > 0$, $\left\{S(m,t)\right\}_1^{+\infty}$, now becomes a nested, monotonically increasing sequence of measurable sets; $H = \bigcup_{m=1}^{+\infty} S(m,t)$ due to pointwise convergence of the net $\langle f_n \rangle$ to $\partial f/\partial x_i$ and the fact that G is open. Consequently, for all $t > 0$, $\mu(S(m,t)) \to \mu H$ as $m \to +\infty$. So, for $t_p = 2^p/\eta$, $p \in \mathbb{N}$, there exists an $N_p \in \mathbb{N}$ such that if $m \geq N_p$, we have

$$\mu[H \sim S(m,t_p)] < 1/t_p = \eta/2^p \quad .$$

In particular, $\mu[H \sim S(N_p,t_p)] < \eta/2^p$. With this in mind, let

$$S_i = \bigcap_{p=2}^{+\infty} S(N_p,t_p) \quad .$$

By construction, $\sum_{p=2}^{+\infty} \mu[H \sim S(N_p,t_p)] \leq \eta/2$ so that

$$\mu[H \sim \bigcap_{2}^{+\infty} S(N_p,t_p)] < \eta \quad ,$$

which gives us property (i). For property (ii), if $\varepsilon > 0$ then, surely, there exists a p_o such $1/t_{p_o} = \eta/2^{p_o} < \varepsilon$; therefore, for the constant $1/N_{p_o}$, if $\chi \in S(N_{p_o},t_{p_o})$ then

$$\left|f_h(\chi) - \frac{\partial f(\chi)}{\partial x_i}\right| < \varepsilon$$

for all $h \succ 1/N_{p_o}$. But $S(N_{p_o},t_{p_o}) \supset S_i$, and, so, we have uniform convergence of the net $\langle f_h \rangle$ to $\partial f/\partial x_i$ on S_i.

Q.E.D.

We now obtain just one set for which the uniform convergence stipulation of the last two theorems, property (ii), holds:

$$S = \bigcap_{i=0}^{k} S_i \quad .$$

Observe that

$$\mu S > \mu H - (k + 1)\eta > \mu G - (k + 2)\eta > 0 \quad .$$

Due to the measurability of S, there exists a closed set $T \subset S$ such that

$$\mu T > \mu S - \eta > \mu G - (k + 3)\eta > 0 \ ;$$

in addition, T is compact since G is bounded. It is this piece of information that enables us to assert something about the function $\frac{\partial f}{\partial x_i}$ itself, namely,

<u>Theorem IV</u>: For each i, $\frac{\partial f}{\partial x_i}$ is uniformly continuous on T.

<u>Proof</u>

As T is compact and G is a bounded region in $\mathbb{R}^k$, $d(T,\partial G) > 0$. Letting $\delta = d(T, \partial G)$, we know that for any $\chi \in T$, $B(\chi;\delta) \subset G$. Let $\{h_j\}$ denote a monotonic sequence of positive reals converging to 0 such that $h_1 < \delta$. Then $\langle f_{h_j} \rangle$ is a subnet of continuous functions on T whose uniform limit is $\frac{\partial f}{\partial x_i}$ restricted to T. So, $\frac{\partial f}{\partial x_i}$ inherits this continuity on T. In fact, we have a bit more-- $\frac{\partial f}{\partial x_i}$ is uniformly continuous on T due to this set's compactness.

Q.E.D.

Recall, now, what it means for the point $\chi \in \mathbb{R}^k$ to be a "point of density" of the measurable subset U of $\mathbb{R}^k$ -- for any $\xi > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\mu(U \cap S_r)}{\mu S_r} - 1 \right| < \xi$$

whenever S_r is a non-degenerate, k-dimensional, open sphere of radius r with $r < \delta$; less formally,

$$\lim_{\substack{r \to 0, \\ \chi \in S_r}} \frac{\mu(U \cap S_r)}{\mu S_r} = 1 \quad .$$

The following proposition is the Lebesgue Density Theorem for $\mathbb{R}^k$, the proof of which may be easily adapted from the corresponding proof using non-degenerate, k-dimensional, closed cubes in the definition.[1]

Theorem: Let U be a Lebesgue measurable subset of $\mathbb{R}^k$. Then almost every point of U is a point of density of U.

For our purposes, however, we need only consider open spheres centered at the point in question. Therefore, using T and this

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[1]Donald Cohn, "Section 2, Chapter 6," Measure Theory (Boston, 1980), pp. 177-184.

weaker version of the Density Theorem for dimension k produces a subset V of T, $\mu V = \mu T$, on which

$$\lim_{r \to 0} \frac{\mu[T \cap B(X;r)]}{\mu B(X;r)} = 1$$

for any $X \in V$.

As will become more clear in Chapter IV, it behooves us to delimit V further to arrive at a measure - equivalent subset Z having the property that on any hyperplanar section of Z orthogonal to the last coordinate axis of $\mathbb{R}^k$, the restriction of f to that hyperplanar section is totally differentiable throughout that section. Conceptually, we modify V by retaining only those points for which our earlier-made induction hypothesis is valid for each restriction of f to the hyperplanar section of V determinted by the last coordinate of each of those points. We then delete from each of those hyperplanar sections, and, thus, from V, the "bad" points of the induction hypothesis; that is, the exceptional points for which total differentiability of the restricted function fail. We accomplish this through still more applications of the Fubini Theorem.

Consider the product space $\prod_{j=1}^{k-1} \mathbb{R}_j \times \mathbb{R}_k$ with its product measure $\lambda \times m$; we assume, of course, each $\mathbb{R}_j \equiv \mathbb{R}$ for $1 \leq j \leq k$ and that λ, m denote (k - 1) - dimensional and real Lebesgue measures, respectively. Define the isomorphism $\Phi_k : \mathbb{R}^k \to \prod_1^{k-1} \mathbb{R}_j \times \mathbb{R}_k$ by

$$\Phi_k(\chi) = (X, x_k)$$

where $X = (x_1, x_2, \ldots, x_{k-1})$ for $\chi = (x_1, x_2, \ldots, x_k)$. With this in mind, define the set

$$\mathbf{G}_k = \{x_k \in \mathbb{R}_k \mid \Phi_k^{-1}((X', x_k)) \in G \text{ for some } X' \in \prod_1^{k-1} \mathbb{R}_j\}$$

and for fixed $x_k \in \mathbf{G}_k$ let

$$E_{x_k} = \left\{ X \in \prod_1^{k-1} \mathbb{R}_j \mid \Phi_k^{-1}((X, x_k)) \in G \right\} .$$

Finally, for arbitrary $\chi \in \mathbb{R}^k$, let P_{x_k} denote the (k - 1) - dimensional hyperplane of $\mathbb{R}^k$ determined by the last coordinate, x_k, of χ; G_{x_k} denotes the hyperplanar section $P_{x_k} \cap G$.

By assumption, L_f is integrable on G, so, extend the function to all of $\mathbb{R}^k$ by defining it to be 0 on $\mathbb{R}^k \sim G$, thus, maintaining its integrability. Integrability of L_f on $\mathbb{R}^k$ allows us to claim integrability of the function $\Gamma_f = L_f \circ \Phi_k^{-1}$ on $\prod_1^{k-1} \mathbb{R}_j \times \mathbb{R}_k$ after observing that we have, in fact, equality of the integrals for nonnegative simple functions related at L_f and Γ_f are, allowing us to make an identical claim for nonnegative functions so related. After recalling that m as well as λ are complete measures, we are in a position to apply the Fubini Theorem again, giving us the result: $(\Gamma_f)_{x_k} : \prod_1^{k-1} \mathbb{R}_j \to \mathbb{R}$, defined by $(\Gamma_f)_{x_k}(X) = \Gamma_f((X, x_k))$, is integrable on $\prod_1^{k-1} \mathbb{R}_j$ for almost all $x_k \in \mathbb{R}_k$. But,

$$(\Gamma_f)_{x_k}(X) = (\Gamma_f \circ \Phi_k)(\chi)$$

where χ must come from P_{x_k}; yet, for such χ, $(\Gamma_f \circ \Phi_k) \equiv L_f | P_{x_k}$.

So,

$$(\Gamma_f)_{x_k}(X) = (L_f|P_{x_k})(X)$$

for all $X \in \prod_1^{k-1} \mathbb{R}_j$, $X \in P_{x_k}$, telling us that $L_f|P_{x_k}$ is integrable on P_{x_k} for almost all $x_k \in \mathbb{R}_k$. Restricting x_k to $\mathbf{G}_k$ and observing that $G_{x_k} \subset P_{x_k}$ is open, hence, measurable in dimension k - 1 yields the preliminary result we seek -- $L_f| G_{x_k}$ is integrable on G_{x_k} for almost all $x_k \in \mathbf{G}_k$. Let

$$I_k = \left\{x_k \in \mathbf{G}_k : L_f|G_{x_k} \text{ is integrable on } G_{x_k}\right\}$$

and we have finally arrived at the closing theorem of this chapter which gives rise to the set we wish to work with in Chapter IV.

Theorem IV: $f|G_{x_k}$ is totally differentiable at the point X, for almost all $X \in V$.

Proof

The method of attack will be quite analogous to that employed in Theorem I, Chapter II. We will, however, encounter a little bit more difficulty in demonstrating the measurability of our primary set under consideration.

Again, we analyze this problem from the position of considering the set

$$W = \left\{X \in V: f|G_{x_k} \text{ is not totally differentiable at } X\right\}.$$

Lemma 1: The set W is measurable.

Proof

Observe, first, that $V \subseteq K$ so that $\frac{\partial f}{\partial x_i}$ exists throughout V for all $1 \leq i \leq k$. In particular, this holds for $1 \leq i \leq k-1$, telling us that for any $\chi \in V$, all 1st order partial derivatives of $f|G_{x_k}$ w.r.t. any variable other than x_k exist at χ. Consequently,

$$W = \Big\{ \chi \in V\text{: there exists } \varepsilon > 0 \text{ such that for any } \delta > 0 \text{ there exists a } \chi' = (x_1', x_2', \ldots, x_{k-1}', x_k) \in G_{x_k} \text{ with the property that } 0 < |\chi' - \chi| < \delta \text{ but}$$

$$\left| [f|G_{x_k}(\chi') - f|G_{x_k}(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f|G_{x_k}(\chi)}{\partial x_i} (x_i' - x_i) \right| \geq \varepsilon\, |\chi' - \chi| \Big\}$$

$$= \Big\{ \chi \in V\text{: there exists } \varepsilon > 0 \text{ such that for any } \delta > 0 \text{ there exists a } \chi' \in G_{x_k} \text{ with the property that } 0 < |\chi' - \chi| < \delta \text{ but}$$

$$\left| [f(\chi') - f(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f(\chi)}{\partial x_i} (x_i' - x_i) \right| > \varepsilon\, |\chi' - \chi| \Big\}$$

$$= \bigcup_{p=1}^{+\infty} \Big\{ \chi \in V\text{: for any } \delta > 0 \text{ there exists a } \chi' \in G_{x_k} \text{ with the property that } 0 < |\chi' - \chi| < \delta \text{ but}$$

$$\left| [f(\chi') - f(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f(\chi)}{\partial x_i} (x_i' - x_i) \right| > \frac{1}{p}\, |\chi' - \chi| \Big\} .$$

$$= \bigcup_{p=1}^{+\infty} \bigcap_{q=1}^{+\infty} \Bigg\{ \chi \in V\text{: there exists a } \chi' \in G_{x_k} \text{ with the property that } 0 < |\chi' - \chi| < \frac{1}{q} \text{ but } \left|[f(\chi') - f(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f(\chi)}{\partial x_i} (x_i' - x_i)\right| > \frac{1}{p} |\chi' - \chi| \Bigg\} .$$

For each p, q $\in \mathbb{N}$, let

$$W(p,q) = \Bigg\{ \chi \in V\text{: there exists a } \chi' \in G_{x_k} \text{ with the property that } 0 < |\chi' - \chi| < \frac{1}{q} \text{ but } \left|[f(\chi') - f(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f(\chi)}{\partial x_i} (x_i' - x_i)\right| > \frac{1}{p} |\chi' - \chi| \Bigg\}$$

and we make the following assertion:

<u>Claim</u>: W(p,q) is open relative to V.

<u>Proof</u>

For brevity, we sketch the outline of the proof. Select $\chi \in W(p,q)$ with corresponding $\chi' \in G_{x_k}$. Let

$$\varepsilon_1 = 1/q - |\chi' - \chi| \quad ,$$

$$\varepsilon_2 = \left|[f(\chi') - f(\chi)] - \sum_{i=1}^{k-1} \frac{\partial f(\chi)}{\partial x_i} (x_i' - x_i)\right| - 1/p\, |\chi' - \chi| \, ,$$

and ε_3 be such that $B(\chi';\varepsilon_3) \subset G$. We wish to find a ball about χ relative to V of radius ζ completely contained in W(p,q). First of all, we insist that $\zeta \le \min\left(\varepsilon_3, \frac{\varepsilon_1}{4}, \frac{\varepsilon_2}{4}, \frac{|\chi' - \chi|}{2}\right)$; we then appeal to f's continuity at χ, $\chi' \in G$ and $\frac{\partial f}{\partial x_i}$'s continuity at $\chi \in V$ together with judiciously chosen ε's ($\varepsilon_2/4k$, $1/4k$) to ascertain δ's which further restrict ζ. Finally, insist that

$$\zeta \leq \frac{\varepsilon_2}{4\,k \cdot \max\left(\left|\frac{\partial f(\chi)}{\partial x_1}\right|, \left|\frac{\partial f(\chi)}{\partial x_2}\right|, \ldots, \left|\frac{\partial f(\chi)}{\partial x_k}\right|\right) + 1} .$$

Then, for $\overline{\chi} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k) \varepsilon B(\chi;\zeta)$ relative to V, simply choose the $\overline{\chi}' \varepsilon G_{x_k}$ to be the orthogonal projection of χ' onto $G_{\overline{x}_k}$.

Q.E.D.

But, with V measurable, the above claim yields what we need, namely, that W(p,q) is also measurable for all p,q $\varepsilon \mathbb{N}$. So, W is nothing more than a countable union of countable intersections of measurable sets and the proof is complete.

Q.E.D.

With the measurability of W disposed of it now sufficies to demonstrate that W has 0 - measure to prove the theorem. Accordingly, let $\chi_{\Phi_k(W)}$ be the characteristic function on $\Phi_k(W)$. We wish to extract measurability of $\Phi_k(W)$, hence, measurability of $\chi_{\Phi_k(W)}$, from our knowledge of W. This will follow by straightforward means if we look at an equivalent formulation of the product measure $\lambda \times m$, which we will denote by μ_k -- that of the restriction of outer measure μ_k^* to the σ-algebra of measurable sets in $\mathbb{R}_i \times \prod_{j\neq i}^{n} \mathbb{R}_j$; μ_k^* is simply $\mu^* \circ \Phi_k^{-1}$. In light of this, measurability of $\Phi_k(W)$ stems from measurability of W. Hence, $\chi_{\Phi_k(W)}$ is measurable and

$$\int_{\overset{k-1}{\underset{1}{\pi}}} \int_{\mathbb{R}_j \times \mathbb{R}_k} \chi_{\Phi_k(W)} d\mu_k = \mu_k(\Phi_k(W)] = \mu^* W = \mu W \quad .$$

As a result of the Fubini Theorem and $\mathbf{G}_k$, I_k's measurability,

$$\mu W = \int_{\mathbb{R}_k} \left\{ \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$= \int_{\mathbf{G}_k} \left\{ \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$+ \int_{\mathbb{R}_k \sim \mathbf{G}_k} \left\{ \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$= \int_{I_k} \left\{ \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$+ \int_{\mathbf{G}_k \sim I_k} \left\{ \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k \quad .$$

Considering the second iterated integral, as our region G is bounded, there exist reals a_j, b_j, $1 \leq j \leq k$, such that $G \subset \overset{k}{\underset{j=1}{\pi}} X_j$ where $X_j = (a_j, b_j)$. For any given x_k,

$$0 \leq \int_{\overset{k-1}{\underset{1}{\pi}} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX = \int_{\overset{k-1}{\underset{1}{\pi}} X_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \leq \lambda \left(\overset{k-1}{\underset{1}{\pi}} X_j \right) ;$$

therefore,

$$0 \leq \int_{\mathbb{G}_k \sim I_k} \left\{ \int_{\prod_1^{k-1} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k \leq \prod_1^{k-1} (b_j - a_j) \cdot m(\mathbb{G}_k \sim I_k) = 0.$$

Thus, the second iterated integral vanishes and we are left with

$$\mu W = \int_{I_k} \left\{ \int_{\prod_1^{k-1} \mathbb{R}_j} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k \quad .$$

Finally, for $x_k \in I_k$ define $\mathcal{E}_{x_k} = \{X \in E_{x_k} : f|G_{x_k}$ is totally differentiable at $\Phi_k^{-1}((x, x_k))\}$. But, I_k has the property that our induction hypothesis is valid for $f|G_{x_k}$, as G_{x_k} is always a bounded region in dimension k - 1. Therefore, for such x_k, $f|G_{x_k}$ is totally differentiable a.e. in G_{x_k}, implying that

$$\lambda \mathcal{E}_{x_k} = \lambda E_{x_k} \quad .$$

So, we may proceed with the iteration:

$$\mu W = \int_{I_k} \left\{ \int_{E_{x_k}} \chi_{\Phi_k(W)}(X, x_k)\, dX + \int_{\prod_1^{k-1} \mathbb{R}_j \sim E_{x_k}} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$= \int_{I_k} \left\{ \int_{\mathcal{E}_{x_k}} \chi_{\Phi_k(W)}(X, x_k)\, dX + \int_{E_{x_k} \sim \mathcal{E}_{x_k}} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k$$

$$= \int_{I_k} \left\{ \int_{\mathcal{E}_{x_k}} \chi_{\Phi_k(W)}(X, x_k)\, dX \right\} dx_k \quad .$$

But, now we have reduced matters to the point where (X, x_k) no longer lies in $\Phi_k(W)$. Hence, the final iterated integral vanishes and we are left with the sufficient result for the theorem.

Q.E.D.

Let

$$Z = \left\{ \chi \in V: f|G_{x_k} \text{ is totally differentiable at } \chi \right\}.$$

We have finally finished with our decomposition of G and arrived at the appropriate and penultimate set on which to establish the property of total differentiability of our function f. Toward this end, let ε be an arbitrary positive real number and N be a positive integer, $N \neq 1$. In closing this chapter, let us amass our results for the set Z and the real numbers η, ε, and N.

Results

(i) By Property 1 of this chapter, L_f is bounded by M_η on Z; so, by the uniform convergence property of Theorem II, there exists a $\rho_o(\eta) > 0$ such that for each $\chi \in Z$,

$$\omega_f(\rho)(\chi) < L_f(\chi) + M_\eta$$

$$< 2M_\eta$$

for all $\rho \succ \rho_o(\eta)$.

(ii) By repeated applications of Theorem III (k times), for ε there exists on $h_o(\eta,\varepsilon)$, which we may assume to be positive, such that for each i, $1 \leq i \leq k$, and for every $\chi \in Z$,

$$\left| f_h(\chi) - \frac{\partial f(\chi)}{\partial x_i} \right| < \varepsilon$$

for all $h \succ h_o(\eta,\varepsilon)$.

(iii) Likewise, by repeated applications of the uniform continuity property of Theorem IV (k times), for ε there exists a $\delta(\eta,\varepsilon) > 0$ such that for each i, $1 \leq i \leq k$,

$$\left| \frac{\partial f(\chi)}{\partial x_i} - \frac{\partial f(\chi')}{\partial x_i} \right| < \varepsilon$$

if $|\chi - \chi'| < \delta(\eta,\varepsilon)$.

(iv) The pointwise property:

$$\lim_{r\to 0} \frac{\mu[T \cap B(\chi,r)]}{\mu B(\chi;r)} = 1$$

on all of Z, implies that for the given $N (N\neq 1)$, if $\chi \in Z$ there exists an $r_N(\chi) > 0$ such that

$$1 - 1/N^k < \frac{\mu T(\chi;r)}{\mu B(\chi;r)}$$

for all $0 < r \leq r_N(\chi)$, $T(\chi;r)$ denoting $T \cap B(\chi;r)$. But, since $\mu Z = \mu V = \mu T$, then, surely,

$$1 - 1/N^k < \frac{\mu Z(\chi;r)}{\mu B(\chi;r)}$$

for all $0 < r \leq r_N(\chi)$, $Z(\chi,r)$ denoting $Z \cap B(\chi;r)$.

As a concluding remark, note also that $\mu Z > \mu G - (k+3)\eta > 0$.

CHAPTER IV

THE CONSTRUCTION PROCESS; THE APPROXIMATION PROCESS CULMINATING IN TOTAL DIFFERENTIABILITY

Let $\chi_o \in Z$ have coordinate representation $(x_1, x_2, \ldots x_k)$ and $\xi > 0$ be such that $B(\chi_o;\xi) \subset G$. Designate

$$r_o = \min \left\{ h_o(\eta,\varepsilon),\ \delta(\eta,\varepsilon),\ r_N(\chi_o),\ \xi \right\} \quad ,$$

guaranteed to exist by Results (ii), (iii), and (iv). Form concentric spheres $B(\chi_o;r_o)$ and $B(\chi_o;\nu_o)$ where ν_o is defined by

$$\nu_o = \min \left\{ r_o \frac{N-1}{N} ,\ \rho_o(\eta)(N-1) \right\} \quad ,$$

$\rho_o(\eta)$ originating from (i) of Results. Let $\chi_1 \in B(\chi_o;\nu_o) \sim \{\chi_o\}$ with coordinate representation $(y_1, y_2, \ldots, y_k)$ and the real number r be defined by

$$r = |\chi_1 - \chi_o| \cdot \frac{N}{N-1} \quad .$$

We summarily form $B(\chi_1;r/N)$ and in the process observe that, by construction, we have $B(\chi_1;r/N) \subsetneq B(\chi_o;r)$. Also in the construction process, we have nicely arranged it so that we may make the following claim concerning volumes.

<u>Lemma 1</u>: $\mu[B(\chi_o;r) \sim Z] < \mu B(\chi_1;r/N)$.

<u>Proof</u>

First, recall that, by convention, $Z(\chi_o;r) \equiv Z \cap B(\chi_o;r)$ and that the volume of a k-dimensional sphere of radius ρ is $\kappa_k \rho^k$ where κ_k denotes the volume of the unit ball expressed in terms of the Γ function.[1] Therefore, as $r_o \leq r_N(\chi_o)$, we have

$$\mu[B(\chi_o;r) \sim Z] = \mu B(\chi_o;r) - \mu Z(\chi_o;r)$$

$$< \frac{\mu B(\chi_o;r)}{N^k}$$

$$= \frac{\kappa_k r^k}{N^k}$$

$$= \kappa_k (r/N)^k$$

which is precisely $\mu B(\chi_1;r/N)$.

Q.E.D.

The importance of Lemma 1 is demonstrated by the following existence claim. By ℓ_{x_k} we mean the line in $\mathbb{R}^k$ determined by the last coordinate, x_k, of χ_o.

[1] Luis Santaló, "Section 5, Chapter 1," <u>Integral Geometry and Geometric Probability</u>, Vol. I of <u>Encyclopedia of Mathematics and its Applications</u>, 2 vols. (Reading, 1976), p. 9.

<u>Property 2</u>: $[B(\chi_1;r/N) \sim \ell_{x_k}] \cap Z$ is nonempty.

<u>Proof</u>

As $\mu[B(\chi_1;r/N) \cap \ell_{x_k}] = 0$, it sufficies to show

$$\mu[B(\chi_1;r/N) \cap Z] > 0 \quad .$$

But,

$$\mu[B(\chi_1;r/N) \sim Z] \leq \mu[B(\chi_o;r) \sim Z]$$

$$< \mu B(\chi_1;r/N)$$

by Lemma 1, so the result follows from the additivity of the measure.

Q.E.D.

Let $\chi_2 \in [B(\chi_1;r/N) \sim \ell_{x_k}] \cap Z$ and χ_3 denote the orthogonal projection of χ_2 onto the hyperplane P_{x_k}, $(z_1, z_2, \ldots, z_{k-1}, x_k)$. Note that by choice of r_o we have insured that both χ_1 and χ_3 belong to G. We are now in a position to state the central equality about which all of Chapter IV hinges:

$$(*) \quad \frac{f(\chi_1) - f(\chi_o)}{|\chi_1 - \chi_o|} = \frac{f(\chi_1) - f(\chi_2)}{|\chi_1 - \chi_o|} + \frac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_o|} + \frac{f(\chi_3) - f(\chi_o)}{|\chi_1 - \chi_o|} \quad .$$

Part A: Approximation of the 1st Summand of Equation (✱)

Case 1: $\chi_2 \equiv \chi_1$.

In such an event, the quotient vanishes.

Case 2: $\chi_2 \not\equiv \chi_1$.

By construction,

$$\left| \frac{f(\chi_1) - f(\chi_2)}{\chi_1 - \chi_2} \right| \leq \omega_f(r/N)(\chi_2)$$

in which $r/N = \frac{|\chi_1 - \chi_o|}{N-1} < \rho_o(\eta)$. As $\chi_2 \varepsilon Z$, (i) of Results yields

$$\left| \frac{f(\chi_1) - f(\chi_2)}{\chi_1 - \chi_2} \right| < 2M_\eta \quad ;$$

so, returning to our first summand, we have

$$\left| \frac{f(\chi_1) - f(\chi_2)}{\chi_1 - \chi_o} \right| < 2M_\eta \frac{|\chi_1 - \chi_2|}{|\chi_1 - \chi_o|} < \frac{2M_\eta}{N - 1} \quad .$$

Letting $\zeta_o = \frac{f(\chi_1) - f(\chi_2)}{|\chi_1 - \chi_o|}$, we may express the above succinctly as

$$|\zeta_o| < \frac{2M_\eta}{N - 1} \quad ,$$

valid for both Cases 1 and 2.

<u>Part B</u>: Approximation of the 2nd Summand of Equation (*)

Note: $z_k \neq x_k$.

<u>Case 1</u>: $y_k = x_k$.

Since

$$|\chi_3 - \chi_2| = |x_k - z_k| = |z_k - y_k| \leq |\chi_2 - \chi_1| < r/N < \rho_o(\eta)$$

where $\chi_2 \in Z$, (i) of Results again yields

$$\left|\frac{f(\chi_2) - f(\chi_3)}{\chi_2 - \chi_3}\right| < 2M_\eta \quad ;$$

thus, returning to our second summand we have

$$\left|\frac{f(\chi_2) - f(\chi_3)}{\chi_1 - \chi_o}\right| < 2M_\eta \frac{|\chi_2 - \chi_3|}{|\chi_1 - \chi_o|} < \frac{2M_\eta}{N - 1} \quad .$$

Letting $\eta_o = \dfrac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_o|} - \dfrac{\partial f(\chi_o)}{\partial x_k} \dfrac{(y_k - x_k)}{|\chi_1 - \chi_o|}$, we may write

$$\frac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_o|} = \frac{\partial f(\chi_o)}{\partial x_k} \frac{(y_k - x_k)}{|\chi_1 - \chi_o|} + \eta_o$$

where $|\eta_o| < \dfrac{2M_\eta}{N - 1}$.

<u>Case 2</u>: $y_k \neq x_k$.

We rewrite the second summand as follows:

$$(*) \quad \frac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_2|} = \frac{f(\chi_2) - f(\chi_3)}{z_k - x_k} \cdot \frac{z_k - x_k}{y_k - x_k} \cdot \frac{y_k - x_k}{|\chi_1 - \chi_o|} \quad .$$

Lemma 2: $\dfrac{f(\chi_2) - f(\chi_3)}{z_k - x_k} = \dfrac{\partial f(\chi_o)}{\partial x_k} + \eta_1$

where $|\eta_1| < 2\varepsilon$.

Proof

By construction, $|z_k - x_k| \leq |\chi_2 - \chi_o| < r < r_o$ where both χ_o, $\chi_2 \in Z$; observe, also, that

$$\frac{f(\chi_2) - f(\chi_3)}{z_k - x_k} = \frac{f(\chi_3) - f(\chi_2)}{x_k - z_k} \equiv f_h(\chi_2)$$

in which $h = x_k - z_k$. Hence, $h \succ r_o$, so that by (ii) of Results,

$$\left| \frac{f(\chi_2) - f(\chi_3)}{z_k - x_k} - \frac{\partial f(\chi_2)}{\partial x_k} \right| < \varepsilon \quad ;$$

(iii) of Results yields

$$\left| \frac{\partial f(\chi_2)}{x_k} - \frac{\partial f(\chi_o)}{x_k} \right| < \varepsilon \; .$$

Setting $\eta_1 = \dfrac{f(\chi_2) - f(\chi_3)}{z_k - x_k} - \dfrac{\partial f(\chi_o)}{\partial x_k}$ and combining inequalities furnishes the desired conclusion. Q.E.D.

Let us define the quantities:

$$\eta_2 = \frac{z_k - x_k}{y_k - x_k} - 1$$

so that

$$\frac{z_k - x_k}{y_k - x_k} = 1 + \eta_2 \quad ,$$

and

$$\eta_3 = \frac{y_k - x_k}{|\chi_1 - \chi_o|} \quad .$$

Returning now to equation (*) and inserting the values η_1, η_2, and η_3, we have

$$(**) \quad \frac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_o|} = \frac{\partial f(\chi_o)}{\partial x_k} \eta_3 + \Theta((\eta_1, \eta_2, \eta_3))$$

where $\Theta((\eta_1, \eta_2, \eta_3)) = \eta_1(\eta_3 + \eta_2\eta_3) + \dfrac{\partial f(\chi_o)}{\partial x_k} \eta_2\eta_3 \quad .$

Note: $|\eta_2\eta_3| < \frac{1}{N - 1} \quad .$

We now deviate from our usual bounding process and attempt something more, the thrust of which is embodied in Property 3. First, let us realize that our choice of χ_o at the outset of Chapter IV in no way depended upon our choice of ε and N. Consequently, we may view χ_o as being fixed throughout all of the preceding analysis. As a result, the range of values for η_o, η_1, η_2, and η_3 vary according to the range of values assumed by the points χ_1 and χ_2; χ_1 and χ_2 are, of course, restricted by the initial choice of ε and N. If we now consider ε and N as variables, free to range throughout $\mathbb{R}^+$ and $\mathbb{N} \sim \{1\}$, respectively, we may define the function Ψ of the complex variable (ε, N) by

$$\Psi((\varepsilon,N)) = 2\varepsilon \frac{N}{N-1} + 2 \max\left(\left|\frac{\partial f(\chi_o)}{\partial x_k}\right| ,M_\eta\right) \cdot \frac{1}{N-1}$$

and note that from the preceding analysis of Part B, Ψ possesses the two properties:

(i) $|R_1(\chi_1 - \chi_o, \chi_2)| < \Psi((\varepsilon,N))$

where $R_1(\chi_1 - \chi_o, \chi_2)$ is defined by

$$R_1(\chi_1 - \chi_o, \chi_2) = \begin{cases} \eta_o & \text{if } y_k = x_k \\ \theta((\eta_1, \eta_2, \eta_3)) & \text{if } y_k \neq x_k \end{cases} .$$

(ii) $\Psi((\varepsilon,N)) \to 0$ as $\begin{cases} \varepsilon \to 0 \\ N \to +\infty \end{cases}$.

In light of the above discussion, we propose the following proposition:

<u>Property 3</u>: Suppose that for every $(\varepsilon,N) \in \mathbb{R}^{+'} \times \mathbb{N} \sim \{1\}$ there exists a constant $\nu_o(\varepsilon,N)$ such that if $|\chi_1 - \chi_o| < \nu_o(\varepsilon,N)$ then there exists a point $\chi_2 \in G$ such that

$$|R_1(\chi_1 - \chi_o, \chi_2)| < \Psi((\varepsilon,N)) .$$

Assume, in addition, Ψ has the property that

$$\Psi((\varepsilon,N)) \to 0 \quad \text{as} \quad \begin{cases} \varepsilon \to 0 \\ N \to \infty \end{cases} .$$

Then $R_1(\chi_1 - \chi_o, \chi_2) \to 0$ as $(\chi_1 - \chi_o) \to 0$. (The latter statement of convergence is to be interpreted as: given $\alpha > 0$, there exists

$\beta > 0$ such that if $|\chi_1 - \chi_o| < \beta$ then there exists a point $\chi_2 \varepsilon G$ such that $|R_1 (\chi_1 - \chi_o, \chi_2)| < \alpha$.)

Proof

Let $\alpha > 0$; there exists order pair (ε_1, N_1) such that $\Psi((\varepsilon_1, N_1)) < \alpha$. By assumption, for this (ε_1, N_1) there exists the constant $\nu_o(\varepsilon_1, N_1)$ such that if $|\chi_1 - \chi_o| < \nu_o(\varepsilon_1, N_1)$ then there exists a point $\chi_2 \varepsilon G$ such that

$$|R_1(\chi_1 - \chi_o, \chi_2)| < \Psi((\varepsilon_1, N_1)) < \alpha.$$

Letting $\beta \equiv \nu_o(\varepsilon_1, N_1)$, the proof is complete.

Q.E.D.

In order for us to claim the conclusion of Property 3 at this point, we only need for the given (ε, N) the existence of such a constant $\nu_o(\varepsilon, N)$. But such a constant does, in fact, exist in Part B -- namely, ν_o, which was introduced at the very outset of Chapter IV and for which property (i) of Ψ is valid whenever $|\chi_1 - \chi_o| < \nu_o$ and $\chi_2 \varepsilon Z$ is obtained via Property 2. Consequently, we have all of the hypotheses of Property 3 holding, allowing us to claim

$$R_1(\chi_1 - \chi_o, \chi_2) \to 0 \text{ as } (\chi_1 - \chi_o) \to 0 .$$

Therefore, we may restate equation (**) as:

$$\frac{f(\chi_2) - f(\chi_3)}{|\chi_1 - \chi_o|} = \frac{\partial f(\chi_o)}{\partial x_k} \frac{(y_k - x_k)}{|\chi_1 - \chi_o|} + R_1(\chi_1 - \chi_o, \chi_2)$$

where $R_1(\chi_1 - \chi_o, \chi_2) \to 0$ as $(\chi_1 - \chi_o) \to 0$, valid for both Cases 1 and 2 of Part B.

In retrospect, we could have stated a proposition analogous to Property 3 earlier, in Part A, and obtained a similar result on convergence since both ν_o and χ_2 were chosen prior to equation (*) and are, therefore, independent of any part following. Letting $R_o(\chi_1 - \chi_o, \chi_2) \equiv \zeta_o$, we have

$$R_o(\chi_1 - \chi_o, \chi_2) \to 0 \text{ as } (\chi_1 - \chi_o) \to 0$$

true for Part A.

<u>Part C</u>: Approximation of the 3rd Summand of Equation (*).

We rewrite the final summand as

$$\frac{f(\chi_3) - f(\chi_o)}{|\chi_1 - \chi_o|} \equiv \frac{f|G_{x_k}(\chi_3) - f|G_{x_k}(\chi_o)}{|\chi_1 - \chi_o|} \quad .$$

By definition of our set Z, $f|G_{x_k}$ is totally differentiable at χ_o. Consequently, the above quotient becomes

$$\sum_{i=1}^{k-1} \left[\frac{\partial f|G_{x_k}(\chi_o)}{\partial x_i} \frac{(z_i - x_i)}{|\chi_1 - \chi_o|} \right] + \frac{|\chi_3 - \chi_o|}{|\chi_1 - \chi_o|} R_{x_k}(\chi_3 - \chi_o) \equiv$$

$$\sum_{i=1}^{k-1} \left[\frac{\partial f(\chi_o)}{\partial x_i} \; \frac{(z_i - x_i)}{|\chi_1 - \chi_o|} \right] + \frac{|\chi_3 - \chi_o|}{|\chi_1 - \chi_o|} \; R_{x_k}(\chi_3 - \chi_o)$$

with R_{x_k} having the essential property:

$$R_{x_k}(\chi' - \chi_o) \to 0 \text{ as } (\chi' - \chi_o) \to 0, \; \chi' \in G_{x_k} \; .$$

Claim 1: $R_{x_k}(\chi_3 - \chi_o) \to 0$ as $(\chi_1 - \chi_o) \to 0$.

Proof

If suffices to demonstrate that $(\chi_3 - \chi_o) \to 0$ as $(\chi_1 - \chi_o) \to 0$; but this is a simple matter as, given $\alpha > 0$, let $\beta = \alpha/2$. We have the following string of implications:

$$|\chi_1 - \chi_o| < \alpha/2 \leq \alpha \frac{N-1}{N} \qquad \text{for all } N \in \mathbb{N},$$

then

$$r \equiv |\chi_1 - \chi_o| \cdot \frac{N}{N-1} < \alpha \qquad \text{for all } N \in \mathbb{N},$$

then

$$|\chi_2 - \chi_o| < \alpha \quad ,$$

then

$$|\chi_3 - \chi_o| < \alpha \quad .$$

Q.E.D.

Claim 2: For any i, $1 \leq i \leq k - 1$,

$$\frac{\partial f(\chi_o)}{x_i} \frac{(z_i - x_i)}{|\chi_1 - \chi_o|} = \frac{\partial f(\chi_o)}{\partial x_i} \frac{(y_i - x_i)}{|\chi_1 - \chi_o|} + \zeta_i$$

where $|\zeta_i| < \left|\frac{\partial f(\chi_o)}{\partial x_i}\right| \cdot \frac{1}{N - 1}$.

Claim 3: $\frac{|\chi_3 - \chi_o|}{|\chi_1 - \chi_o|} = 1 + \zeta_k$ where $|\zeta_k| \leq 1$.

Hence, by Claims 2 and 3 we may write

$$\frac{f(\chi_3) - f(\chi_o)}{|\chi_1 - \chi_o|} = \sum_{i=1}^{k-1} \frac{\partial f(\chi_o)}{\partial x_i} \frac{(y_i - x_i)}{|\chi_1 - \chi_o|} + \sum_{i=1}^{k-1} \zeta_i + (1 + \zeta_k) R_{x_k}(\chi_3 - \chi_o) .$$

The latter half of Claim 2 allows us to conclude that $\sum_{i=1}^{k-1} \zeta_i \to 0$ as $(\chi_1 - \chi_o) \to 0$; the latter half of Claim 3 along with Claim 1 allow us to assert that $(1 + \zeta_k) R_{x_k}(\chi_3 - \chi_o) \to 0$ as $(\chi_1 - \chi_o) \to 0$. Letting $R_2(\chi_1 - \chi_o, \chi_2) \equiv \sum_{i=1}^{k-1} \zeta_i + (1 + \zeta_k) R_{x_k}(\chi_3 - \chi_o)$, we may finally state the third summand succinctly as

$$\frac{f(\chi_3) - f(\chi_o)}{|\chi_1 - \chi_o|} = \sum_{i=1}^{k-1} \frac{\partial f(\chi_o)}{\partial x_i} \frac{(y_i - x_i)}{|\chi_1 - \chi_o|} + R_2(\chi_1 - \chi_o, \chi_2)$$

where $R_2(\chi_1 - \chi_o, \chi_2) \to 0$ as $(\chi_1 - \chi_o) \to 0$.

We have finally reached the culmination of our approximation process. Pooling the results of Parts A, B, and C, we may reformulate equation (✯) as

$$\frac{f(\chi_1) - f(\chi_o)}{|\chi_1 - \chi_o|} = \frac{\partial f(\chi_o)}{\partial x_k} \frac{(y_k - x_k)}{|\chi_1 - \chi_o|} + \sum_{i=1}^{k-1} \frac{\partial f(\chi_o)}{\partial x_i} \frac{(y_i - x_i)}{|\chi_1 - \chi_o|}$$

$$+ [R_o(\chi_1 - \chi_o, \chi_2) + R_1(\chi_1 - \chi_o, \chi_2) + R_2(\chi_1 - \chi_o, \chi_2)]$$

where each $R_\ell(\chi_1 - \chi_o, \chi_2) \to 0$ as $(\chi_1 - \chi_o) \to 0$, $\ell = 0, 1$, and 2.

Let $H = \{\chi_1' - \chi_o \in \mathbb{R}^k \mid \chi_1' = (y_1', y_2', \ldots, y_k') \in G\}$ and define the function $R: H \to \mathbb{R}$ by

$$R(\chi' - \chi_o) = \begin{cases} 0 & \text{if } \chi_1' = \chi_o \\ \dfrac{f(\chi_1') - f(\chi_o)}{|\chi_1' - \chi_o|} - \displaystyle\sum_{i=1}^{k} \frac{\partial f(\chi_o)}{\partial x_i} \frac{(y_i' - x_i)}{|\chi_1' - \chi_o|} & \text{if } \chi_1' \neq \chi_o \end{cases} .$$

We claim that our preceding work shows that $R(\chi_1' - \chi_o) \to 0$ as $(\chi_1' - \chi_o) \to 0$. The reason: for χ' sufficiently close to χ_o, $\chi_1' \equiv \chi_1$ and

$$R(\chi_1 - \chi_o) = R_o(\chi_1 - \chi_o, \chi_2) + R_1(\chi_1 - \chi_o, \chi_2) + R_2(\chi_1 - \chi_o, \chi_2) \quad ,$$

if $\chi_1 \neq \chi_o$, in which each summand converges to 0 as $(\chi_1 - \chi_o) \to 0$.

Hence, we have met the two conditions required to insure total differentiability of f at $\chi_o \in Z$. As χ_o was arbitrary, f achieves total differentiability throughout our set Z, $\mu Z > \mu G - (k + 3)\eta > 0$.

This latter result combined with the relatively arbitrary nature of η now permits a swift coup de grace to the proof of total differentiability of our function a.e. in G. For, let $N' \in \mathbb{N}$ be such that $1/N' < \frac{\mu G}{(k+3)}$. If we now consider the sequence $\{\frac{1}{n}\}_{n=N'}^{+\infty}$ of values of η, then Chapter IV furnishes us with a corresponding sequence of sets, $\{Z_n\}_{n=N'}^{+\infty}$, sets on which f is totally differentiable and for which

$$\mu Z_n > \mu G - \frac{(k+3)}{n}$$

for all $n \geq N'$. So, let $\Omega = \bigcup_{n=N'}^{+\infty} Z_n$ and we claim that f is totally differentiable on this measure-equivalent subset of G. The former property of f is obvious; the latter equivalence of measure follows readily from the observation that $G \sim \Omega \subset G \sim Z_n$ for all $n \geq N'$.

Hence, Part II of the induction process begun at the outset of Chapter III is now, at long last, complete. By the principle of mathematical induction it follows that the theorem which this thesis proposes to prove is true.

Q.E.D.

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