HAAR MEASURE ON THE CANTOR TERNARY SET

THESIS

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By

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The purpose of this thesis is to examine certain questions concerning the Cantor ternary set. The second chapter deals with proving that the Cantor ternary set is equivalent to the middle thirds set of [0,1], closed, compact, and has Lebesgue measure zero. Further a proof that the Cantor ternary set is a locally compact, Hausdorff topological group is given. The third chapter is concerned with establishing the existence of a Haar integral on certain topological groups. In particular if G is a locally compact and Hausdorff topological group, then there is a non-zero translation invariant positive linear form on G. The fourth chapter deals with proving that for any Haar integral I on G there exists a unique Haar measure on G that represents I.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The purpose of this thesis is to examine certain questions concerning the Cantor ternary set. A basic knowledge of analysis, topology, measure theory, and algebra is assumed.

The initial portion of Chapter II deals with defining the Cantor ternary set by the ternary expansion of elements of $[0,1]$ and proving its equivalence with the middle thirds set of $[0,1]$. This portion of Chapter II shows that the Cantor ternary set is closed, compact, and has Lebesgue measure zero. Also Chapter II deals with defining a continuous function on the Cantor ternary set that maps onto $[0,1]$; additionaly, this function is used to prove the existence of a measurable set which is not a Borel set. Finally Chapter II deals with proving that the Cantor ternary set is a locally compact, Hausdorff topological group.

Chapter III is concerned with establishing the existence of a Haar integral on certain topological groups. In particular if G is a locally compact and Hausdorff topological group, then there is a non-zero translation invariant positive linear form on G.

Chapter IV deals with proving that for any Haar integral I on G there exists a unique Haar measure on G that represents I. Further the Haar measure can be scaled so that the measure of G is one. Thus there exists a non-zero translation invariant regular Borel measure on the Cantor ternary set.

In this paper $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the reals, integers, and naturals respectively, the end of a proof is denoted by $\blacksquare$, the complement of a set A is denoted by $\tilde{A}$ or ~A, the closure of a set A is denoted by $\overline{A}$, and the empty set is denoted by ϕ. A <u>ring</u> R is a collection of sets such that if A and B are sets in R, then $A \cup B$ and $A \setminus B$ are both in R, and a <u>σ-ring</u> S is a ring so that if (A_n) is a sequence of sets in S, then $\bigcup_{i=1}^{\infty} A_i$ is in S. Also in this paper the <u>Borel</u> class of sets, denoted by $\mathfrak{B}$, is defined to be the σ-ring generated by the collection of all compact sets. By a <u>measure</u> μ we mean a non-negative and additive set function on a ring $\mathscr{R}$ so that $\mu(\phi) = 0$ and

$$\mu\left[\bigcup_{i=1}^{\infty} E_i\right] = \sup\{\ \mu(E_i)\ :\ i \in \mathbb{N}\ \}$$

where (E_n) is an increasing sequence of sets from the ring $\mathscr{R}$ so that $\bigcup_{i=1}^{\infty} E_i \in \mathscr{R}$. Observe that a measure is monotone and countably additive. The reader may consult Berberian [1], O'Neil [2], or Royden [3] for any terms in this paper that have been left undefined.

CHAPTER II

CANTOR TERNARY SET

In this chapter we establish some fundamental analytical and topological properties of the Cantor ternary set. In order to facilitate our definition of the Cantor ternary set, we begin by establishing the existence of ternary expansions for elements of the unit interval $[0,1]$. For $x \in [0,1]$ to have a <u>ternary expansion</u> means that there exists a sequence (a_n) so that $a_n \in \{0,1,2\}$ for each $n \in \mathbb{N}$ and $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$.

<u>Theorem 2.1</u>: If $x \in [0,1]$, then x has a ternary expansion. Conversely if (a_n) is a sequence so that $a_n \in \{0,1,2\}$ for every $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} \frac{a_n}{3^n} \in [0,1]$. Further if $x \in [0,1]$ so that $x = \frac{p}{3^m}$ for some $m \in \mathbb{N}$ and $0 < p < 3^m$, then x has two ternary expansions; otherwise, the ternary expansion for x is unique.

<u>Proof</u>: Let $x \in [0,1]$. Let

$$a_1 = \max\{\ p \in \{0,1,2\}\ |\ \tfrac{p}{3} \leq x\ \};$$

hence $\frac{a_1}{3} \leq x$. Let

$$a_2 = \max\{\ p \in \{0,1,2\}\ |\ \tfrac{p}{3^2} \leq (x - \tfrac{a_1}{3})\ \};$$

note that $\{\ p \in \{0,1,2\} \mid \frac{p}{3^2} \leq (x - \frac{a_1}{3})\ \}$ is not empty since $(x - \frac{a_1}{3}) \geq 0$. Now for $n \in \mathbb{N}$ and $n > 2$, assume that $a_1,\ldots,a_{n-1}$ have been chosen, $x - \sum_{i=1}^{n-1} \frac{a_i}{3^i} \geq 0$ and let

$$a_n = \max\{\ p \in \{0,1,2\} \mid \frac{p}{3^n} \leq (x - \sum_{i=1}^{n-1} \frac{a_i}{3^i})\ \}.$$

Hence (a_n) is a sequence so that $a_n \in \{0,1,2\}$ for every $n \in \mathbb{N}$. Note that for each $n \in \mathbb{N}$,

$$0 \leq (x - \sum_{i=1}^{n} \frac{a_i}{3^i}) \leq \frac{1}{3^n}.$$

Now $(x - \sum_{i=1}^{n} \frac{a_i}{3^i}) \xrightarrow[n]{} 0$ since $(\frac{1}{3^n}) \longrightarrow 0$. Therefore $\sum_{i=1}^{\infty} \frac{a_i}{3^i} = x$, and x has a ternary expansion.

Clearly if (a_n) is a sequence so that $a_n \in \{0,1,2\}$ for every $n \in \mathbb{N}$, then

$$0 = \sum_{i=1}^{\infty} \frac{0}{3^i} \leq \sum_{i=1}^{\infty} \frac{a_i}{3^i} \leq \sum_{i=1}^{\infty} \frac{2}{3^i} = 1$$

and $\sum_{i=1}^{\infty} \frac{a_i}{3^i} \in [0,1]$.

Now let $x \in [0,1]$ so that x has two ternary expansions. Let (a_n) be a ternary expansion of x; now let (b_n) be another ternary expansion of x so that $(a_n) \neq (b_n)$. Since $(a_n) \neq (b_n)$, let m be the least $i \in \mathbb{N}$ so that $a_i \neq b_i$. Assume that $a_m > b_m$. Now

$$\begin{aligned} x &= \sum_{i=1}^{\infty} \frac{b_i}{3^i} \leq \sum_{i=1}^{m-1} \frac{b_i}{3^i} + \frac{b_m}{3^i} + \sum_{i=m+1}^{\infty} \frac{2}{3^i} \\ &= \sum_{i=1}^{m-1} \frac{a_i}{3^i} + \frac{b_m+1}{3^m} \leq \sum_{i=1}^{m-1} \frac{a_i}{3^i} + \frac{a_m}{3^m} + \sum_{i=m+1}^{\infty} \frac{0}{3^i} \\ &\leq \sum_{i=1}^{\infty} \frac{a_i}{3^i} = x. \end{aligned}$$

Hence $b_m + 1 = a_m$, and, for $i > m$, $b_i = 2$, $a_i = 0$, and

$a_m \in \{1,2\}$. Clearly if (c_n) is also a ternary expansion of x, then $(c_n) = (a_n)$ or $(c_n) = (b_n)$. Thus x has at most two ternary expansions. Note that $x = \sum_{i=1}^{m} \frac{a_i}{3^i}$; let $p = 3^m \cdot \sum_{i=1}^{m} \frac{a_i}{3^i}$. Clearly $p \in \mathbb{N}$; also, $0 < p < 3^m$ since $0 \leq \sum_{i=1}^{m} \frac{a_i}{3^i} < 1$. Thus $x = \frac{p}{3^m}$ where $0 < p < 3^m$.

Now let $m,p \in \mathbb{N}$ so that $0 < p < 3^m$ and $GCD(p,3^m) = 1$, where GCD means the greatest common divisor. Let $x = \frac{p}{3^m}$. Note that by the division algorithm there exists integers b_{m-1} and p_{m-1} so that

$$p = b_{m-1}3^{m-1} + p_{m-1},$$

where $0 \leq b_{m-1} < 3$ and $0 \leq p_{m-1} < 3^{m-1}$. Hence for $i \in \mathbb{N}$ so that $2 \leq i < m$, there exists integers b_{m-i} and p_{m-i} so that

$$p_{m-i+1} = b_{m-i}3^{m-i} + p_{m-i},$$

where $0 \leq b_{m-i} < 3$ and $0 \leq p_{m-i} < 3^{m-i}$. Thus

$$p = p_1 + \sum_{i=1}^{m-1} b_{m-i}3^{m-i}$$

and

$$\frac{p}{3^m} = \frac{p_1}{3^m} + \sum_{i=1}^{m-1} \frac{b_{m-i}}{3^i}.$$

Note that $p_1 \in \{1,2\}$ since $GCD(p,3^m) = 1$. Now let $a_m = p_1$, and for $i \in \mathbb{N}$ so that $1 \leq i < m$ let $a_i = b_{m-i}$. Thus

$$x = \sum_{i=1}^{m} \frac{a_i}{3^i} = \sum_{i=1}^{m-1} \frac{a_i}{3^i} + \frac{a_m - 1}{3^m} + \sum_{i=m+1}^{\infty} \frac{2}{3^i}.$$

Therefore x has two ternary expansions. ■

We define the <u>Cantor</u> <u>ternary</u> <u>set</u>, denoted by $\mathfrak{C}$, to be all elements $x \in [0,1]$ so that x has a ternary expansion

(a_n) where $a_n \neq 1$ for each $n \in \mathbb{N}$. In the following theorem we will show that the Cantor ternary set may be obtained recursively by first removing the open middle third of [0,1] and then removing the open middle third of the two remaining closed intervals and then removing the middle third of the four remaining closed intervals, etc. More specifically, for $a < b$, define the "remove middle third" operator **RMT** on $[a,b]$ by

$$\mathrm{RMT}([a,b]) = [a, \tfrac{2}{3}\cdot a + \tfrac{1}{3}\cdot b] \cup [\tfrac{1}{3}\cdot a + \tfrac{2}{3}\cdot b\ ,b];$$

if $S = \bigcup_{i=1}^{n}[a_i,b_i]$ is a finite disjoint union of closed intervals, define

$$\mathrm{RMT}(S) = \bigcup_{i=1}^{n}\mathrm{RMT}([a_i,b_i]).$$

Denote n–fold composition of the operator RMT by $(\mathrm{RMT})^n$ where $n \in \mathbb{N}$. Define the middle third operator **MT** on $[a,b]$ by

$$\mathrm{MT}([a,b]) = \bigcap_{n=1}^{\infty}(\mathrm{RMT})^n([a,b]).$$

Now $\mathrm{MT}([a,b])$ is called the middle third set of $[a,b]$.

<u>Theorem</u> <u>2.2</u>: **The Cantor ternary set $\mathfrak{C}$ is closed, and $\mathfrak{C} = \mathrm{MT}([0,1])$.**

<u>Proof</u>: For each $n \in \mathbb{N}$, let

$$C_n = \{\ x \in [0,1] \mid \text{if } (a_i) \text{ is a ternary expansion of } x,\text{ then } a_n = 1\ \};$$

and let

$$D_n = \{\ x \in [0,1]\ \mid\ x \in (\frac{i\cdot 3+1}{3^n},\frac{i\cdot 3+2}{3^n}) \text{ where } i \in \mathbb{Z} \text{ so that } 0 \leq i < 3^{n-1}\ \},$$

the middle third open intervals. Note that the C_n's need not be disjoint and the D_n's need not be disjoint. Clearly $\mathfrak{C} = [0,1] \setminus \bigcup_{i=1}^{\infty} C_i$, and $MT([0,1]) = [0,1] \setminus \bigcup_{i=1}^{\infty} D_i$.

Next we assert that $D_n = C_n$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Let $x \in C_n$; and let (a_i) be a ternary expansion of x, hence $a_n = 1$. Clearly $\sum_{i=1}^{n-1} \frac{a_i}{3^i} = \frac{k}{3^{n-1}}$ for some $k \in \mathbb{Z}$ so that $0 \leq k < 3^{n-1}$. Since a_n must be 1, some $a_i = 0$ for $i > n$ and some $a_j = 2$ for $j > n$. Thus

$$\sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n} < \sum_{i=1}^{\infty} \frac{a_i}{3^i} < \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}.$$

Let k be such an integer. Now

$$\frac{k\cdot 3+1}{3^n} = \frac{k}{3^{n-1}} + \frac{1}{3^n} = \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n},$$

and

$$\frac{k\cdot 3+2}{3^n} = \frac{k}{3^{n-1}} + \frac{2}{3^n} = \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}$$

since $\sum_{i=n+1}^{\infty} \frac{2}{3^i} = \frac{1}{3^{n-1}}$. Hence $x \in (\frac{k\cdot 3+1}{3^n},\frac{k\cdot 3+2}{3^n})$, and $x \in D_n$. Thus $C_n \subseteq D_n$. Now if $x \in D_n$ with ternary expansion (a_i), hence $x \in (\frac{i\cdot 3+1}{3^n},\frac{i\cdot 3+2}{3^n})$ for some $i \in \mathbb{Z}$ so that $0 \leq i < 3^{n-1}$, and a_n must be 1. Thus $D_n \subseteq C_n$, and $C_n = D_n$ for every $n \in \mathbb{N}$. Hence the assertion follows.

Thus $\mathfrak{C} = [0,1] \setminus \bigcup_{i=1}^{\infty} D_n$. Therefore the Cantor ternary set is equal to the middle thirds set of $[0,1]$. Clearly the

Cantor ternary set is closed since $\bigcup_{i=1}^{\infty} D_n$ is open. ■

Note that the Cantor ternary set is compact since it is a closed subset of a compact set.

<u>Theorem</u> <u>2.3</u>: The Cantor ternary set can be put into a one-to-one correspondence with the closed interval [0,1].

<u>Proof</u>: Let $f:[0,1] \longrightarrow \mathfrak{C}$ be a function so that

$$f(x) = \begin{cases} x \text{ if } x \in \mathfrak{C} \\ 0 \text{ if } x \notin \mathfrak{C} \end{cases}$$

for $x \in [0,1]$. Clearly f is a function from $[0,1]$ onto $\mathfrak{C}$.

Now similar to the ternary expansion of [0,1], there is a binary expansion of [0,1]; i.e, for $x \in [0,1]$ there exists a sequence (b_n) so that $b_n \in \{0,1\}$ for all $n \in \mathbb{N}$ and $x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$. Let $g:\mathfrak{C} \longrightarrow [0,1]$ be a function defined by

$$g(x) = \sum_{n=1}^{\infty} \frac{\frac{1}{2}a_n}{2^n},$$

where (a_n) is the ternary expansion of $x \in \mathfrak{C}$ so that $a_n \neq 1$ for all $n \in \mathbb{N}$. Clearly g is well-defined since there is only one such expansion (a_n) for each $x \in \mathfrak{C}$, and g maps into $[0,1]$ since $\frac{1}{2}a_n \in \{0,1\}$ for each $n \in \mathbb{N}$. Let $y \in [0,1]$ and let (b_n) be a binary expansion of y. Now $(2b_n)$ is a ternary expansion of some number in [0,1] so that $2b_n \neq 1$ for every $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \frac{2b_n}{3^n} \in \mathfrak{C}$, and

$$g\left[\sum_{n=1}^{\infty} \frac{2b_n}{3^n}\right] = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = y.$$

Thus g is a function from $\mathfrak{C}$ onto $[0,1]$. Therefore there exists a one-to-one correspondence between $\mathfrak{C}$ and $[0,1]$. ■

A set P is said to be <u>perfect</u> if the set of accumulation points of P is the set P itself; hence P is also closed.

<u>**Theorem**</u> <u>**2.4**</u>: **The Cantor ternary set is perfect.**

<u>Proof</u>: Let $\mathfrak{C}'$ be the set of accumulation points of $\mathfrak{C}$. Clearly $\mathfrak{C}' \subseteq \mathfrak{C}$ since $\mathfrak{C}$ is closed. Let $\epsilon > 0$. Let $x \in \mathfrak{C}$; let (a_n) be the ternary expansion of x so that $a_n \neq 1$ for every $n \in \mathbb{N}$. Let m be the least $n \in \mathbb{N}$ so that $\frac{2}{3^n} < \epsilon$. Let (b_n) be a sequence so that $b_n = a_n$ for each $n \neq m$ and

$$b_m = \begin{cases} 0 \text{ if } a_m = 2 \\ 2 \text{ if } a_m = 0 \end{cases}.$$

Hence (b_n) is a ternary expansion so that $b_n \neq 1$ for all $n \in \mathbb{N}$. Thus $\sum_{n=1}^{\infty} \frac{b_n}{3^n} \in \mathfrak{C}$. Now

$$\left| x - \sum_{n=1}^{\infty} \frac{b_n}{3^n} \right| = \left| \sum_{n=1}^{\infty} \frac{a_n}{3^n} - \sum_{n=1}^{\infty} \frac{b_n}{3^n} \right|$$
$$= \left| \frac{a_m}{3^m} - \frac{b_m}{3^m} \right| = \frac{2}{3^m} < \epsilon.$$

Thus $x \in \mathfrak{C}$ and $\mathfrak{C} \subseteq \mathfrak{C}'$. Therefore $\mathfrak{C} = \mathfrak{C}'$, and the Cantor ternary set is perfect. ■

Now for $x \in [0,1]$, let (a_n) be a ternary expansion of x. Let $N = \infty$ if $a_n \neq 1$ for all $n \in \mathbb{N}$; otherwise let N be the least $n \in \mathbb{N}$ so that $a_n = 1$. Let (b_n) be a sequence so that

$$b_n = \begin{cases} \frac{1}{2}a_n \text{ if } n < N \\ 1 \text{ otherwise} \end{cases}.$$

Define $f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$ for $x \in [0,1]$. We shall call f the <u>Cantor ternary function</u>.

<u>Theorem 2.5</u>: **The Cantor ternary function f is well-defined, continuous, and monotone from $[0,1]$ to $[0,1]$. Additionaly, f is constant on each open interval of the complement of $\mathfrak{C}$, and $f(\mathfrak{C}) = [0,1]$.**

<u>Proof</u>: First we show that f is well-defined. Let $x \in [0,1]$ so that x has two different expansions. Hence let $m,p \in \mathbb{N}$ so that $x = \frac{p}{3^m}$ where $0 < p < 3^m$ and $GCD(p,3^m) = 1$. By the proof of Theorem 2.4, let (a_n) and (a_n') be ternary expansions for x so that $a_i = a_i'$ for $i < m$, $a_m' = a_m - 1$, and $a_i = 0$ and $a_i' = 2$ for $i > m$. Clearly $a_m > 0$ since $GCD(p,3^m) = 1$. Let N_a be the least $n \in \mathbb{N}$ so that $a_n = 1$, or else let $N_a = \infty$ if $a_n \neq 1$ for each $n \in \mathbb{N}$. Define N_a' with respect to (a_n'). Let (b_n) be a sequence so that

$$b_n = \begin{cases} \frac{1}{2}a_n \text{ if } n < N_a \\ 1 \text{ otherwise} \end{cases}.$$

Define (b_n') with respect to (a_n'). Clearly $N_a \leq m$, or $N_a = \infty$; likewise for N_a'. If $N_a < m$, then $N_a' = N_a < m$, and $(b_n) = (b_n')$. Thus

$$f\left[\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right] = \sum_{n=1}^{N_a} \frac{b_n}{2^n} = \sum_{n=1}^{N_a'} \frac{b_n'}{2^n} = f\left[\sum_{n=1}^{\infty} \frac{a_n'}{3^n}\right].$$

If $N_a = m$, then $a_m = 1$, $a_m' = 0$, $N_a' = \infty$, $b_i = b_i'$ for $i < m$, $b_m = 1$, $b_m' = 0$, and $b_i = b_i' = 1$ for $i > m$. Also,

$$f\left[\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right] = \sum_{n=1}^{m} \frac{b_n}{2^n} = \sum_{n=1}^{n-1} \frac{b_n}{2^n} + \frac{1}{2} = \sum_{n=1}^{n-1} \frac{b_n}{2^n} + 0 + \sum_{n=m-1}^{\infty} \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} \frac{b_n'}{2^n} = f\left[\sum_{n=1}^{\infty} \frac{a_n'}{3^n}\right].$$

If $N_a = \infty$, then $a_m = 2$, $a_m' = 1$, $N_a' = m$, $b_i = b_i'$ for $i < m$, $b_m = b_m' = 1$, and

$$f\left[\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right] = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = \sum_{n=1}^{m} \frac{b_n}{2^n} = \sum_{n=1}^{m} \frac{b_n'}{2^n} = f\left[\sum_{n=1}^{\infty} \frac{a_n'}{3^n}\right].$$

Therefore f is well defined. Clearly $\sum_{n=1}^{N_a} \frac{b_n}{2^n} \in [0,1]$.

Now to show that f is continuous on $[0,1]$, let $x \in [0,1]$ with ternary expansion (a_n). Let $\epsilon > 0$, and let m be the least $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \epsilon$. Let $0 < \delta < \frac{1}{3^m}$, and let $y \in [0,1]$ with ternary expansion (c_n) so that $| x - y | < \delta$. Assume that $y < x$. Let N_a be the least $n \in \mathbb{N}$ so that $a_n = 1$, or else let $N_a = \infty$ if $a_n \neq 1$ for each $n \in \mathbb{N}$; define N_c similarly. Let (b_n) be a sequence so that

$$b_n = \begin{cases} \frac{1}{2} a_n & \text{if } n < N_a; \\ 1 & \text{otherwise} \end{cases}$$

define (d_n) similarly with respect to (c_n).

Assume that $m < N_a \leq N_c$; then $a_n = c_n$ for $n \leq m$ since $0 < x - y < \frac{1}{3^m}$. Thus $b_n = d_n$ for $n \leq m$, and

$$| f(x) - f(y) | = \left| \sum_{n=1}^{N_a} \frac{b_n}{2^n} - \sum_{n=1}^{N_c} \frac{d_n}{2^n} \right|$$

$$= \left| \sum_{n=m+1}^{N_a} \frac{b_n}{2^n} - \sum_{n=m+1}^{N_c} \frac{d_n}{2^n} \right|$$

$$\leq \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m} < \epsilon.$$

The case where $m < N_c \leq N_a$ is similar.

Assume that $N_a \leq N_c \leq m$; then $a_n = c_n$ for $n < N_a$ and $N_a = N_c$ again since $0 < x - y < \frac{1}{3^m}$. Thus $b_n = d_n$ for $n \leq N_a = N_c$, and

$$| f(x) - f(y) | = | \sum_{n=1}^{N_a} \frac{b_n}{2^n} - \sum_{n=1}^{N_c} \frac{d_n}{2^n} | = 0 < \epsilon.$$

The argument is the same when $N_c \leq N_a \leq m$.

Now assume that $N_a \leq m < N_c$; then $a_n = c_n$ for $n < N_a$ and $c_{N_a} = 0$ and $c_n = 2$ for $N_a < n \leq m$ since $0 < x - y < \frac{1}{3^m}$. Thus $b_n = d_n$ for $n < N_a$, $b_{N_a} = 1$, $d_{N_a} = 0$, $d_n = 1$ for $N_a < n \leq m$, and

$$\begin{aligned} | f(x) - f(y) | &= | \sum_{n=1}^{N_a} \frac{b_n}{2^n} - \sum_{n=1}^{N_c} \frac{d_n}{2^n} | \\ &= \frac{1}{2^{N_a}} - \sum_{n=N_a+1}^{m} \frac{1}{2^n} - \sum_{n=m+1}^{N_c} \frac{d_n}{2^n} \\ &\leq \frac{1}{2^{N_a}} - \sum_{n=N_a+1}^{m} \frac{1}{2^n} = \frac{1}{2^m} < \epsilon. \end{aligned}$$

The case where $N_c \leq m < N_a$ is similar. Therefore f is continuous at x and on [0,1].

To show that f is monotone on [0,1], let $x,y \in [0,1]$ with ternary expansions (a_n) and (c_n) respectively so that $x > y$. Since $x > y$, let m be the least $n \in \mathbb{N}$ so that $a_n > c_n$. Define N_a, N_c, (b_n), and (d_n) as before.

Assume that $N_a < m$; then clearly $N_a = N_c$ by the definition of m. Hence $b_n = d_n$ for all $n \in \mathbb{N}$. Thus

$$f(y) = \sum_{n=1}^{N_c} \frac{d_n}{2^n} = \sum_{n=1}^{N_a} \frac{b_n}{2^n} = f(x).$$

The argument is similar when $N_c < m$.

Assume that $m = N_a$; then $m < N_c$, $a_m = 1$, and $c_m = 0$. Clearly $b_n = d_n$ for $n < m$. Thus

$$\begin{aligned} f(y) &= \sum_{n=1}^{N_c} \frac{d_n}{2^n} = \sum_{n=1}^{m-1} \frac{b_n}{2^n} + \frac{0}{2^m} + \sum_{n=m+1}^{N_c} \frac{d_n}{2^n} \\ &\le \sum_{n=1}^{N_a-1} \frac{b_n}{2^n} + \frac{0}{2^m} + \sum_{n=N_a+1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{N_a} \frac{b_n}{2^n} = f(x). \end{aligned}$$

The case where $m = N_c$ is identical.

Now assume that $m < N_a \le N_c$; then $a_m = 2$ and $c_m = 0$. Clearly $b_n = d_n$ for $n < m$, $b_m = 1$, and $d_m = 0$. Hence

$$\begin{aligned} f(y) &= \sum_{n=1}^{N_c} \frac{d_n}{2^n} = \sum_{n=1}^{m-1} \frac{b_n}{2^n} + \frac{0}{2^m} + \sum_{n=m+1}^{N_c} \frac{d_n}{2^n} \\ &\le \sum_{n=1}^{m-1} \frac{b_n}{2^n} + \frac{0}{2^m} + \sum_{n=m+1}^{\infty} \frac{1}{2^n} \\ &\le \sum_{n=1}^{m} \frac{b_n}{2^n} \le \sum_{n=1}^{N_a} \frac{b_n}{2^n} = f(x). \end{aligned}$$

The argument for $m < N_c \le N_a$ is identical. Therefore $f(y) \le f(x)$, and f is monotone on $[0,1]$.

Now to show that f is constant on each open interval of the complement of $\mathfrak{C}$, let $m \in \mathbb{N}$, $0 \le i \le 3^{m-1}$, and $x,y \in (\frac{i\cdot 3+1}{3^m}, \frac{i\cdot 3+2}{3^m})$ with ternary expansions (a_n) and (c_n) respectively. Recall from Theorem 2.2 that each open interval of the complement of $\mathfrak{C}$ is of the form $(\frac{i\cdot 3+1}{3^m}, \frac{i\cdot 3+2}{3^m})$ where $m \in \mathbb{N}$ and $0 \le i \le 3^{m-1}$. Define N_a, N_c, (b_n), and (d_n) as before. Note that $a_n = c_n$ for $n \le m$, and $a_m = c_m = 1$.

Hence $N_a = N_c \leq m$, and $b_n = d_n$ for $n \leq N_a = N_c$. Thus

$$f(x) = \sum_{n=1}^{N_a} \frac{b_n}{2^n} = \sum_{n=1}^{N_c} \frac{d_n}{2^n} = f(y).$$

Therefore f is constant on $(\frac{i\cdot 3+1}{3^m}, \frac{i\cdot 3+2}{3^m})$ and each open interval of the complement of $\mathfrak{C}$.

To show that $f(\mathfrak{C}) = [0,1]$, let $y \in [0,1]$ with binary expansion (b_n). Now $(2\cdot b_n)$ is a ternary expansion of some number in $[0,1]$; further $2\cdot b_n \neq 1$ for each $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty} \frac{2\cdot b_n}{3^n} \in \mathfrak{C}$. Thus

$$f\left[\sum_{n=1}^{\infty} \frac{2\cdot b_n}{3^n}\right] = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = y.$$

Therefore $f(\mathfrak{C}) = [0,1]$. ∎

Note that the Cantor ternary function is $\frac{1}{2}$ on the open middle third removed at the first stage, $\frac{1}{4}$ and $\frac{3}{4}$, respectively, on the open middle thirds removed at the second stage, and $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$, respectively, on the middle thirds removed at the third stage. Let ν be Lebesgue measure for the remainder of this paper.

<u>Theorem</u> <u>2.6</u>: **The Cantor ternary set has Lebesgue measure zero.**

<u>Proof</u>: Clearly $\mathfrak{C}$ is measurable since $\mathfrak{C}$ is closed. Let $D_1' = (\frac{1}{3}, \frac{2}{3})$; for $n \in \mathbb{N}$ so that $n > 1$, let

$D_n' = \{ x \in [0,1] \mid x \in \left[\frac{i\cdot 3+1}{3^n}, \frac{i\cdot 3+2}{3^n}\right] \setminus \bigcup_{j=1}^{n-1} D_j'$ for

$i \in \mathbb{Z}$ so that $0 \leq i < 3^{n-1}$ }.

Note that $\mathfrak{C} = [0,1] \setminus \bigcup_{n=1}^{\infty} D_n'$ since $\bigcup_{n=1}^{\infty} D_n' = \bigcup_{n=1}^{\infty} D_n$ where each D_n is as defined in Theorem 2.2. Hence $\nu(\mathfrak{C}) = 1 - \sum_{n=1}^{\infty} \nu(D_n')$ since all the D_n' are disjoint. Now define a sequence (d_n) so that d_n is the number of intervals in D_n'. Clearly $d_1 = 1$; and for $n \in \mathbb{N}$ so that $n > 1$, $d_n = 3^{n-1} - \sum_{j=1}^{n-1} d_j \cdot 3^{n-1-j}$ since

$$D_n' = \bigcup_{i=0}^{3^{n-1}-1} \left[\frac{i\cdot 3 + 1}{3^n}, \frac{i\cdot 3 + 2}{3^n}\right] \setminus \bigcup_{j=1}^{n-1} \bigcup_{i=0}^{3^{j-1}-1} \left[\frac{3^{n-j}(i\cdot 3 + 1)}{3^n}, \frac{3^{n-j}(i\cdot 3 + 2)}{3^n}\right].$$

Note that $d_n = 2^{n-1}$ for each $n \in \mathbb{N}$. For suppose not, and let m be the least $n \in \mathbb{N}$ so that $d_n \neq 2^{n-1}$. Clearly $m > 1$. Hence

$$\begin{aligned} d_m &= 3^{m-1} - \sum_{j=1}^{m-1} d_j \cdot 3^{m-1-j} \\ &= 3^{m-1} - \sum_{j=1}^{m-1} 3^{m-1-j}(2^{j-1}) \\ &= 3^{m-1} - \frac{3^{m-1}}{2} \cdot \sum_{j=1}^{m-1} \left[\frac{2}{3}\right]^j \\ &= 3^{m-1} - \frac{3^{m-1}}{2} \cdot \left[\frac{2\cdot 3^{m-1} - 2^m}{3^{m-1}}\right] = 2^{m-1}, \end{aligned}$$

which is a contradiction.

Clearly for each $n \in \mathbb{N}$, the measure of D_n' is the length of each interval in D_n' times the number of intervals in D_n'. Hence $\nu(D_n') = d_n \cdot \frac{1}{3^n} = \frac{2^{n-1}}{3^n}$. Thus

$$\sum_{n=1}^{\infty} \nu(D_n') = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1.$$

Therefore $\nu(\mathfrak{C}) = 1 - 1 = 0$. ■

Let f_1 be the function defined by

$$f_1(x) = f(x) + x$$

for $x \in [0,1]$, where f is the Cantor ternary function.

<u>Theorem</u> <u>2.7</u>: **The function f_1 is a homeomorphism between $[0,1]$ and $[0,2]$, $\nu(f_1(\mathfrak{C})) = 1$, there exists a Lebesgue measurable set A so that $f_1(A)$ is not Lebesgue measurable, and there exists a Lebesgue measurable function h so that $h \circ f_1^{-1}$ is not Lebesgue measurable. Further there exists a Lebesgue measurable set that is not a Borel set.**

<u>Proof</u>: To show that f_1 is a homeomorphism, let $x,y \in [0,1]$ so that $x < y$. Hence

$$f_1(x) = f(x) + x < f(y) + y = f_1(y),$$

and f_1 is strictly increasing. Clearly f_1 is continuous and one–to–one; since $f_1(0) = 0$ and $f_1(1) = 2$, f_1 is onto $[0,2]$ by the intermediate value theorem. Now since f_1 is strictly increasing and continuous, f_1 is open. Hence f_1^{-1} is continuous. Thus f_1 is a homeomorphism from $[0,1]$ onto $[0,2]$.

Now to show that $\nu(f_1(\mathfrak{C})) = 1$, note that

$$\nu(f_1(\mathfrak{C})) = 2 - \nu(f_1(\tilde{\mathfrak{C}})),$$

and

$$\nu(f_1(\tilde{\mathfrak{C}})) = \nu(f_1(\bigcup_{n=1}^{\infty} D_n')) = \sum_{n=1}^{\infty} \nu(f_1(D_n')),$$

where D_n' is as defined before. Since f is constant on the intervals of D_n' and Lebesgue measure is translation invariant, $\nu(f_1(D_n')) = \nu(D_n')$. Hence

$$\nu(f_1(\tilde{\mathfrak{C}})) = \sum_{n=1}^{\infty} \nu(D_n') = 1.$$

Thus $\nu(f_1(\mathfrak{C})) = 2 - 1 = 1$.

To show that there exists a Lebesgue measurable set A so that $f_1(A)$ is not Lebesgue measurable, let $P \subsetneq f_1(\mathfrak{C})$ so that P is not Lebesgue measurable. (Recall that $\nu(f_1(\mathfrak{C})) = 1$.) Now let $A = f_1^{-1}(P)$. Hence $A \subsetneq \mathfrak{C}$, and A is Lebesgue measurable since $\nu(\mathfrak{C}) = 0$. Thus

$$f_1(A) = f_1(f_1^{-1}(P)) = P,$$

and $f_1(A)$ is not Lebesgue measurable.

Now to show that there exists a Lebesgue measurable function h so that $h \circ f_1^{-1}$ is not Lebesgue measurable, let h be the characteristic function of A. Clearly h is Lebesgue measurable. Now

$$(h \circ f_1^{-1})^{-1}(1) = f_1(h^{-1}(1)) = f_1(A) = P,$$

which is not Lebesgue measurable. Thus $h \circ f_1^{-1}$ is not Lebesgue measurable.

To show the existence of a Lebesgue measurable set which is not a Borel set, we first have a lemma to prove.

<u>Lemma</u>: If g is a Lebesgue measurable function and B is a Borel set, then $g^{-1}(B)$ is Lebesgue measurable.

<u>Proof of Lemma</u>: Let g be a Lebesgue measurable

function (Recall that $\mathfrak{B}$ is the class of Borel sets). Let

$$\mathfrak{M} = \{ E \subseteq \mathbb{R} \mid g^{-1}(E) \text{ is Lebesgue measurable } \},$$

and let (E_n) be a sequence of sets from $\mathfrak{M}$. Now

$$g^{-1}(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} g^{-1}(E_n)$$

is Lebesgue measurable, and

$$g^{-1}(E_1 \setminus E_2) = g^{-1}(E_1) \setminus g^{-1}(E_2)$$

is Lebesgue measurable. Hence $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$, $(E_1 \setminus E_2) \in \mathfrak{M}$, and $\mathfrak{M}$ is a σ-ring. Also $g^{-1}[(a,b)]$ is Lebesgue measurable for $a,b \in \mathbb{R}$ so that $a < b$ since g is a Lebesgue measurable function. Now $\mathfrak{M}$ contains every open set; hence $\mathfrak{M}$ contains every closed set and every compact set. Thus $\mathfrak{B} \subseteq \mathfrak{M}$, and the lemma is true.

Now f_1 is a Lebesgue measurable function and A is Lebesgue measurable set, but $f_1^{-1}(A) = P$ is not Lebesgue measurable. Therefore A is not a Borel set. ■

A function g is said to be <u>absolutely</u> <u>continuous</u> on $[0,1]$ if for $\epsilon > 0$, there exits a $\delta > 0$ such that

$$\sum_{i=1}^{n} |g(x_i) - g(y_i)| < \epsilon$$

when $\{ (x_i,y_i) \mid 1 \leq i \leq n \}$ is a finite collection of non-overlapping open intervals from $[0,1]$ so that $\sum_{i=1}^{n} |x_i - y_i| < \delta$. Observe that an absolutely continuous function on $[0,1]$ is continuous on $[0,1]$.

<u>Proposition</u> <u>2.8</u>: **An absolutely continuous function on [0,1] maps a set of measure zero to a set of measure zero.**

<u>Proof</u>: Let g be an absolutely continuous function on [0,1]. Let $S \subseteq (0,1)$ so that the measure of S is zero. Now before continuing, we need the following lemma on absolutely continuous functions.

<u>Lemma</u>: If h is absolutely continuous on [0,1], then for $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\sum_{i=1}^{\infty} |h(x_i) - h(y_i)| < \epsilon$$

when $\{ (x_i,y_i) \mid i \in \mathbb{N} \}$ is a collection of non–overlapping open intervals so that $\sum_{i=1}^{\infty} |x_i - y_i| < \delta$.

<u>Proof</u> <u>of</u> <u>Lemma</u>: Let h be absolutely continuous on [0,1]. Let $\epsilon > 0$; and let $\delta > 0$ so that

$$\sum_{i=1}^{n} |h(x_i) - h(y_i)| < \frac{\epsilon}{2}$$

for $\{ (x_i,y_i) \mid 0 \leq i \leq n \}$ non–overlapping and $\sum_{i=1}^{n} |x_i - y_i| < \delta$. Now let $\{ (x_i,y_i) \mid i \in \mathbb{N} \}$ be non–overlapping so that $\sum_{i=1}^{\infty} |x_i - y_i| < \delta$. Hence $\left[\sum_{i=1}^{n} |h(x_i) - h(y_i)| \right]_n$ is an increasing sequence bounded by $\frac{\epsilon}{2}$. Thus

$$\sum_{i=1}^{\infty} |h(x_i) - h(y_i)| \leq \frac{\epsilon}{2} < \epsilon,$$

and the lemma is shown to be true.

Now let $\epsilon > 0$; and let $\delta > 0$ so that

$$\sum_{i=1}^{\infty} |g(x_i) - g(y_i)| < \epsilon$$

whenever $\{ (x_i,y_i) \mid i \in \mathbb{N} \}$ are non-overlapping and $\sum_{i=1}^{\infty} |x_i - y_i| < \delta$. Since the measure of S is zero, let $\{ (x_i,y_i) \mid i \in \mathbb{N} \}$ be non-overlapping so that $\sum_{i=1}^{\infty} |x_i - y_i| < \delta$ and $S \subseteq \bigcup_{i=1}^{\infty} (x_i,y_i)$. Now

$$g(S) \subseteq \bigcup_{i=1}^{\infty} g[(x_i,y_i)] \subseteq \bigcup_{i=1}^{\infty} g([x_i,y_i]).$$

Since g is absolutely continuous, g is also continuous. Hence g attains its maximum and minimum on each $[x_i,y_i]$. For each $i \in \mathbb{N}$ let $x_i', y_i' \in [x_i,y_i]$ so that $x_i' < y_i'$, $g[(x_i,y_i)] \subseteq g([x_i',y_i'])$, and $|g(x_i') - g(y_i')|$ is a maximum. Now $\{ (x_i',y_i') \mid i \in \mathbb{N} \}$ is non-overlapping and

$$g(S) \subseteq \bigcup_{i=1}^{\infty} g([x_i',y_i']).$$

Hence

$$\begin{aligned} \nu[g(S)] &\leq \sum_{i=1}^{\infty} \nu\{g[(x_i,y_i)]\} \\ &\leq \sum_{i=1}^{\infty} |g(x_i') - g(y_i')| < \epsilon \end{aligned}$$

since

$$\sum_{i=1}^{\infty} |x_i' - y_i'| \leq \sum_{i=1}^{\infty} |x_i - y_i| < \delta.$$

Therefore the measure of g(S) is zero. ■

Recall from Theorem 2.7 that $f_1 = f + \mathrm{Id}$, where f is the

Cantor ternary function and Id is the identity map on $[0,1]$, and that $\nu[f_1(\mathfrak{C})] = 1$. Thus f_1 is not absolutely continuous on $[0,1]$; hence the Cantor ternary function is also not absolutely continuous on $[0,1]$ since the identity function is absolutely continuous on $[0,1]$.

Define the set 2^ω as follows:

$$2^\omega = \{ (b_n) \mid (b_n) \text{ is a sequence where } b_n \in \{0,1\} \text{ for each } n \in \mathbb{N} \}.$$

In the next theorem, we demonstrate yet another realization of the Cantor ternary set. This realization will be particularly important in the subsequent construction of a non-trivial translation invariant measure on $\mathfrak{C}$.

Theorem 2.9: The Cantor ternary set is homeomorphic to 2^ω.

Proof: Let $h:\mathfrak{C} \to 2^\omega$ be a function so that $h(x) = (\frac{1}{2}a_n)$, where (a_n) is the ternary expansion of $x \in \mathfrak{C}$ such that $a_n \neq 1$ for each $n \in \mathbb{N}$. Clearly h is well-defined and bijective.

Now the topology on $\mathfrak{C}$ is the relative usual topology inherited from the real line, and the topology on 2^ω is the product of the discrete topology on $\{0,1\}$. Observe that 2^ω is compact since it is the product of compact sets. For $m \in \mathbb{N}$ let $\pi_m : 2^\omega \to \{0,1\}$ be a function so that for

$y = (b_n) \in 2^\omega$, $\pi_m(y) = b_m$. By definition, the product topology on 2^ω is the weakest topology on 2^ω so that each π_m is continuous for $m \in \mathbb{N}$. Hence for $y_0 = (b_n) \in 2^\omega$ and $k \in \mathbb{N}$, a neighborhood of y_0 may be described as

$$\begin{aligned} N(y_0,k) &= \bigcap_{i=1}^{k} \pi_i^{-1}(b_i) \\ &= \{\, y \in 2^\omega \mid \pi_i(y) = \pi_i(y_0) \text{ for } 1 \le i \le k \,\}. \end{aligned}$$

Clearly 2^ω with the product topology is Hausdorff.

To show that h is continuous, let $x_0 \in \mathfrak{C}$,and let (a_n) be the ternary expansion of x_0 such that $a_n \neq 1$ for each $n \in \mathbb{N}$. Let $N(h(x_0),k)$ be a neighborhood of $h(x_0)$, let $x \in (x_0 - \frac{1}{3^k}, x_0 + \frac{1}{3^k}) \cap \mathfrak{C}$, and let (c_n) be the ternary expansion of x such that $c_n \neq 1$ for each $n \in \mathbb{N}$. Note that $(x_0 - \frac{1}{3^k}, x_0 + \frac{1}{3^k}) \cap \mathfrak{C}$ is a neighborhood of x_0. Hence

$$\begin{aligned} |\, x - x_0 \,| &= \left|\, \sum_{n=1}^{\infty} \frac{c_n}{3^n} - \sum_{n=1}^{\infty} \frac{a_n}{3^n} \,\right| \\ &= \left|\, \sum_{n=1}^{\infty} \frac{c_n - a_n}{3^n} \,\right| < \frac{1}{3^k}. \end{aligned}$$

Thus $a_i = c_i$ and $\pi_i(h(x_0)) = \pi_i(h(x))$ for $1 \le i \le k$. Therefore $h(x) \in N(h(x_0),k)$, and h is continuous.

Since both 2^ω and $\mathfrak{C}$ are compact Hausdorff spaces and h is a continuous bijection, h^{-1} is continuous. Therefore h is a homeomorphism between 2^ω and $\mathfrak{C}$, and 2^ω is homeomorphic to the Cantor ternary set. ■

The statement that the ordered triple $(G,*,\mathcal{T})$ is a <u>topological group</u> means that G is a set and * is a binary operation on G so that $(G,*)$ is a group, $\mathcal{T}$ is a topology on G so that $*$ is continuous, and the function g on G defined by $g(x) = x^{-1}$ is continuous. Now define $* : 2^\omega \times 2^\omega \longrightarrow 2^\omega$ by

$$(a_n) * (b_n) = ((a_n + b_n) \bmod 2)$$

for $(a_n),(b_n) \in 2^\omega$. Clearly $(2^\omega,*)$ is a group with identity (0_n) and with $(a_n)^{-1} = (a_n)$ for all $(a_n) \in 2^\omega$. Let P denote the product topology on 2^ω.

<u>Theorem</u> <u>2.10</u>: The ordered triple $(2^\omega,*,P)$ is a topological group.

<u>Proof</u>: First to show that $*$ is continuous, let $(a_n),(b_n) \in 2^\omega$, and let $N((a_n) * (b_n),k)$ be a neighborhood of $(a_n) * (b_n)$. Now let

$$< (c_n),(d_n) > \in N((a_n),k) \times N((b_n),k).$$

Hence $(c_n) \in N((a_n),k)$, $(d_n) \in N((b_n),k)$, and $a_i = c_i$, $b_i = d_i$ for $1 \leq i \leq k$. Further

$$(c_i + d_i) \bmod 2 = (a_i + b_i) \bmod 2$$

for $1 \leq i \leq k$. Thus

$$(c_n) * (d_n) \in N((a_n) * (b_n),k),$$

and $*$ is continuous.

Now let g be a function defined on 2^ω such that $g[(a_n)] = (a_n)^{-1}$ for each $(a_n) \in 2^\omega$. Clearly g is continuous since $(a_n)^{-1} = (a_n)$ for all $(a_n) \in 2^\omega$. Therefore $(2^\omega,*,P)$ is a topological group. ■

Hence if h is the homeomorphism between $\mathfrak{C}$ and 2^{ω} from Theorem 2.9 and $*$: $\mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$ is defined by

$$x * y = h^{-1}(h(x) \ast h(y))$$

for $x,y \in \mathfrak{C}$, then clearly $*$ is continuous and $(\mathfrak{C},*)$ is a group since $\ast$ is continuous, h is a homeomorphism, and $(2^{\omega},\ast)$ is a group. Now to show that $(\mathfrak{C},*,\text{Usual})$ is a topological group, define $g : \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$ by $g(x) = x^{-1}$. Clearly g is continuous since $x^{-1} = x$ for all $x \in \mathfrak{C}$. Thus $(\mathfrak{C},*,\text{Usual})$ is a topological group.

CHAPTER III

HAAR INTEGRAL

In this chapter we establish the existence of a Haar integral on locally compact and Hausdorff topological groups. For the remainder of this paper, let G be a locally compact and Hausdorff topological group, and denote the binary operation on G by + and the identity element by θ. Observe that a locally compact and Hausdorff topological group is also completely regular; a proof of this fact may be found in O'Neal [2, p. 160]. Before proceeding, we establish some notation and terminology. A real valued function f on G is said to have <u>compact support</u> if there exists a compact $C \subseteq G$ so that $f(G \setminus C) = \{0\}$. Define the set $\mathscr{L}$ to be

$$\{\ f : f \text{ is a continuous real valued function on } G \text{ with compact support }\},$$

define the set $\mathscr{P}$ to be

$$\{\ g : g \in \mathscr{L} \text{ so that } g \geq 0 \text{ but } g \neq 0\ \},$$

and for $f \in \mathscr{L}$ and $s \in G$, define the real-valued function f_s on G by $f_s(x) = f(x - s)$. A <u>Haar integral</u> I on G is a linear real-valued function defined on $\mathscr{L}$ so that $I \neq 0$, $I(g) \geq 0$ for $g \in \mathscr{P}$, and $I(f_s) = I(f)$ for all $f \in \mathscr{L}$ and $s \in G$. A set $W \in G$ is said to be <u>symmetric</u> if W contains the inverse of each of its elements, and a real-valued function f on G is said to be symmetric if f maps each element in G and its inverse to the same real number.

<u>Theorem</u> <u>3.1</u>: **If $f \in \mathscr{L}$ and $\epsilon > 0$, then there exists a neighborhood V of θ so that $| f(x) - f(y) | < \epsilon$ when $x - y \in V$.**

<u>Proof</u>: Let $f \in \mathscr{L}$ and $\epsilon > 0$. Note that given $x,y \in G$, there exists $z \in G$ so that $x - y = z$ and $x = z + y$. Let $\epsilon > 0$, and let

$$V = \{ z \in G : | f(z + y) - f(y) | < \epsilon \text{ for all } y \in G \}.$$

Clearly $\theta \in V$ since

$$| f(\theta + y) - f(y) | = | f(y) - f(y) | = 0;$$

and $| f(x) - f(y) | < \epsilon$ whenever $(x - y) \in V$. All we need now is to show that V is a neighborhood of θ.

Let $h : G \times G \longrightarrow \mathbb{R}$ be a function so that

$$h(z,y) = | f(z + y) - f(y) |;$$

hence h is continuous since addition is jointly continuous in G and $|\cdot|$,f are continuous. Since f has compact support, let $C \subseteq G$ be compact so that $f(G \setminus C) = \{0\}$; and since G is locally compact, let W be a compact neighborhood of θ; and let $S = W \cap (-W)$. Clearly S is a symmetric compact neighborhood of θ.

Now let $D = S + C$. Note that D is compact since D is the continuous image of the compact set $S \times C$. Since $h(\theta,y) = 0$ for all $y \in G$, for each $x \in D$ let N_x be a neighborhood of (θ,x) so that $h(z,y) < \epsilon$ for all $(z,y) \in N_x$. Note that $N_x \supseteq U_x \times A_x$ for some neighborhood U_x of θ and some neighborhood A_x of x. For some $n \in \mathbb{N}$, let $\{x_1, \ldots, x_n\} \subseteq D$ so that $D \subseteq \bigcup_{i=1}^{n} A_{x_i}$ since D is compact. Now let $U = (\bigcap_{i=1}^{n} U_{x_i}) \cap S$. Clearly U is a neighborhood of θ.

Now we assert that $U \subseteq V$; for if $z \in U$ and $y \in G$, and if $y \in D$, then for some $1 \leq j \leq n$, $y \in A_{x_j}$. Also $z \in \bigcap_{i=1}^{n} U_{x_i} \subseteq U_{x_j}$, and thus $(z,y) \in N_{x_j}$. Hence

$$| f(z + y) - f(y) | = h(z,y) < \epsilon,$$

and $z \in V$. Now if $y \notin D$, then $y \notin C$ and $f(y) = 0$. Also $(z + y) \notin C$; for if $(z + y) \in C$, then

$$y \in (-z) + C \subseteq S + C \subseteq D,$$

which is a contradiction. Hence $f(z + y) = 0$ and

$$| f(z + y) - f(y) | = 0 < \epsilon.$$

Thus $z \in V$, and $U \subseteq V$. Therefore V is a neighborhood of θ.

■

Before establishing the existence of a Haar integral on G, we will establish the existence of a non-zero translation invariant positive linear real-valued function defined on $\mathscr{P}$. We will extend the function on $\mathscr{P}$ in order to obtain a

Haar integral on G.

Theorem 3.2: If I_0 is a real-valued function on $\mathscr{P}$ so that

i) $I_0(g) > 0$,

ii) $I_0(g + h) = I_0(g) + I_0(h)$,

iii) $I_0(c \cdot g) = c \cdot I_0(g)$, and

iv) $I_0(g_s) = I_0(g)$

for $g,h \in \mathscr{P}$, $c > 0$, and $s \in G$, then I_0 can be extended to a unique Haar integral on G.

Proof: Let $f \in \mathscr{L}$, and let $g,h \in \mathscr{P}$ so that $f = g - h$. Clearly there exists such g and h since

$$f = (f^+ + f_0) - (f^- + f_0)$$

for any $f_0 \in \mathscr{P}$. Now define $I : \mathscr{L} \longrightarrow \mathbb{R}$ by

$$I(f) = I_0(g) - I_0(h).$$

Suppose $g', h' \in \mathscr{P}$ so that $f = g' - h'$; hence $g + h' = g' + h$. Thus

$$I_0(g) + I_0(h') = I_0(g') + I_0(h),$$

and

$$I_0(g) - I_0(h) = I_0(g') - I_0(h').$$

Hence I is well-defined. Now

$$I(g) = I_0(2 \cdot g) - I_0(g) = 2 \cdot I_0(g) - I_0(g) = I_0(g),$$

and I extends I_0. Clearly I is translation invariant on $\mathscr{L}$ since I_0 is translation invariant on $\mathscr{P}$, and

$$I(0\cdot f) = I_0(g) - I_0(g) = 0\cdot I(f).$$

Now

$$\begin{aligned} I(-c\cdot f) &= I_0(c\cdot h) - I_0(c\cdot g) \\ &= c\cdot I_0(h) - c\cdot I_0(g) \\ &= -c\cdot[I_0(g) - I_0(h)] \\ &= -c\cdot I(f). \end{aligned}$$

Let $f_1, f_2 \in \mathscr{L}$, and let $g_1, h_1, g_2, h_2 \in \mathscr{P}$ so that $f_1 = g_1 - h_1$ and $f_2 = g_2 - h_2$. Note that

$$\begin{aligned} I(f_1 + f_2) &= I_0(g_1 + g_2) - I_0(h_1 + h_2) \\ &= I_0(g_1) + I_0(g_2) - I_0(h_1) - I_0(h_2) \\ &= I(f_1) + I(f_2). \end{aligned}$$

Thus I is linear. Therefore I is a Haar integral on G. Clearly I is unique since every $f \in \mathscr{L}$ can be decomposed as $f = g - h$ for some $g,h \in \mathscr{P}$. ■

For $f,g \in \mathscr{P}$, the statement that g <u>covers</u> f means that there exists elements $\{s_1,\ldots,s_n\} \subseteq G$ and positive numbers $c_1,\ldots,c_n$ so that $f \le \sum_{i=1}^{n} c_i\cdot g_{s_i}$ for some $n \in \mathbb{N}$. Now $\sum_{i=1}^{n} c_i\cdot g_{s_i}$ is said to be a covering of f by g.

<u>Theorem 3.3</u>: If $f,g \in \mathscr{P}$, then g covers f.

<u>Proof</u>: Let $f,g \in \mathscr{P}$, and let $C \subseteq G$ be compact so that $f(G \setminus C) = \{0\}$. Let $M > 0$ be a bound for f. Now since g is continuous and not zero, let U be open so that g is bounded

away from zero on U, and let $\eta > 0$ be such a bound. Let

$$\mathscr{U} = \{ U + \{s\} : s \in G\}.$$

Clearly $\mathscr{U}$ is an open covering of C; hence let

$\{s_1, \ldots, s_n\} \subseteq G$ so that $C \subseteq \bigcup_{i=1}^{n}(U + \{s_i\})$

for some $n \in \mathbb{N}$. Let $c_i = \frac{M}{\eta}$ for $1 \leq i \leq n$.

We assert that $f(x) \leq \sum_{i=1}^{n} c_i \cdot g_{s_i}(x)$ for all $x \in G$. The assertion is clearly so if $x \in G \setminus C$. Now if $x \in C$, then $x \in (U + \{s_i\})$ for some $1 \leq i \leq n$. Hence $(x - s_i) \in U$, $g(x - s_i) \geq \eta$, and $g_{s_i}(x) \geq \eta$. Now

$$c_i \cdot g_{s_i}(x) \geq \frac{M}{\eta} \cdot \eta = M \geq f(x).$$

Thus $f(x) \leq \sum_{i=1}^{n} c_i \cdot g_{s_i}(x)$ for all $x \in G$. Therefore g covers f, and $\sum_{i=1}^{n} c_i \cdot g_{s_i}$ is a covering of f by g. ■

Now for $f,g \in \mathscr{P}$, define $(f : g)$ to be

$$\inf\{ \sum_{i=1}^{n} c_i \mid \sum_{i=1}^{n} c_i \cdot g_{s_i} \text{ is a covering of } f \text{ by } g \text{ for some } s_1, \ldots, s_n \in G \}.$$

We call $(f : g)$ the <u>ratio</u> of f to g.

<u>Theorem 3.4</u>: If $f,g \in \mathscr{P}$, then $(f : g) \geq \frac{M}{N}$ where $M = \sup\{f(G)\}$, and $N = \sup\{g(G)\}$.

<u>Proof</u>: Let $f,g \in \mathscr{P}$. Let $M = \sup\{f(G)\}$, and let

$N = \sup\{g(G)\}$. Clearly $0 < M,N < \infty$ since f and g are non-zero and bounded. Now let $\sum_{i=1}^{n} c_i \cdot g_{s_i}$ be a covering of f by g. Hence

$$f \leq \sum_{i=1}^{n} c_i \cdot g_{s_i} \leq N \cdot \sum_{i=1}^{n} c_i;$$

thus $M \leq N \cdot \sum_{i=1}^{n} c_i$ and $\frac{M}{N} \leq \sum_{i=1}^{n} c_i$. Therefore $(f : g) \geq \frac{M}{N}$. ■

Observe that $(f : g) > 0$ for all $f,g \in \mathscr{P}$. Next we will show that $(\cdot : h)$ is translation invariant, linear, and monotone for any $h \in \mathscr{P}$.

<u>Theorem</u> <u>3.5</u>: If $f,g,h \in \mathscr{P}$, $c > 0$, and $s \in G$; then

i) $(f_s : h) = (f : h)$,

ii) $(c \cdot f : h) = c \cdot (f : h)$,

iii) $(f + g : h) = (f : h) + (g : h)$,

and

iv) if $f \leq g$, then $(f : h) \leq (g : h)$.

<u>Proof</u>: Let $f,g,h \in \mathscr{P}$, $c > 0$, and $s \in G$. Now let $\sum_{i=1}^{n} a_i \cdot h_{s_i}$ be a covering of f by h, and let $\sum_{i=1}^{m} b_i \cdot h_{t_i}$ be a covering of g by h. Note that

$$f_s \leq \sum_{i=1}^{n} a_i \cdot (h_{s_i})_s = \sum_{i=1}^{n} a_i \cdot h_{(s_i + s)};$$

hence $(f_s : h) \leq (f : h)$. Now $(f_{(s - s)} : h) \leq (f_s : h)$ since $f_{(s - s)} = (f_s)_{-s}$; hence

$$(f_\theta : h) = (f : h) \le (f_s : h).$$

Thus $(f_s : h) = (f : h)$.

Now note that $c\cdot f \le \sum_{i=1}^{n} c\cdot a_i\cdot h_{s_i}$; thus $\sum_{i=1}^{n}(c\cdot a_i)\cdot h_{s_i}$ is a covering of $c\cdot f$ by h. Hence $(c\cdot f : h) \le c\cdot\sum_{i=1}^{n} a_i$, and $(c\cdot f : h) \le c\cdot(f : h)$. Now

$$(f : h) = (\tfrac{1}{c}\cdot c\cdot f : h) \le \tfrac{1}{c}\cdot(c\cdot f : h),$$

and $c\cdot(f : h) \le (c\cdot f : h)$. Thus $(c\cdot f : h) = c\cdot(f : h)$.

Note that

$$f + g \le \sum_{i=1}^{n} a_i\cdot h_{s_i} + \sum_{i=1}^{m} b_i\cdot h_{t_i};$$

hence $(\sum_{i=1}^{n} a_i\cdot h_{s_i} + \sum_{i=1}^{m} b_i\cdot h_{t_i})$ is a covering of $(f + g)$ by h. Thus

$$(f + g : h) \le \sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i,$$

and

$$(f + g : h) \le (f : h) + (g : h).$$

Now if $f \le g$, then

$$\{ \sum_{i=1}^{n} c_i \mid \sum_{i=1}^{n} c_i\cdot h_{s_i} \text{ is a covering of } g \text{ by } h \}$$

is a subset of

$$\{ \sum_{i=1}^{n} c_i \mid \sum_{i=1}^{n} c_i\cdot h_{s_i} \text{ is a covering of } f \text{ by } h \}.$$

Thus $(f : h) \le (g : h)$. ∎

Lemma 3.6: If $f,g,h \in \mathscr{P}$, then

$$(f : h) \le (f : g)\cdot(g : h).$$

<u>Proof</u>: Let f,g,h $\in \mathscr{P}$, let $\sum_{i=1}^{n} c_i \cdot g_{s_i}$ be a covering of f by g, and let $\sum_{j=1}^{m} d_j \cdot h_{t_j}$ be a covering of g by h. Hence

$$f \leq \sum_{i=1}^{n} c_i \cdot g_{s_i} \leq \sum_{i=1}^{n} c_i \cdot \left(\sum_{j=1}^{m} d_j \cdot h_{t_j}\right)_{s_i}$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{m} c_i \cdot d_j \cdot h_{(s_i + t_j)}\right),$$

and $\sum_{i=1}^{n}\left(\sum_{j=1}^{m} c_i \cdot d_j \cdot h_{(s_i + t_j)}\right)$ is a covering of f by h. Thus

$$(f : h) \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{m} c_i \cdot d_j\right) = \left(\sum_{i=1}^{n} c_i\right) \cdot \left(\sum_{j=1}^{m} d_j\right).$$

Therefore $(f : h) \leq (f : g) \cdot (g : h)$. ■

<u>**Lemma 3.7**</u>: **If f,g,h $\in \mathscr{P}$, then**

$$\frac{1}{(h : f)} \leq \frac{(f : g)}{(h : g)} \leq (f : h).$$

<u>Proof</u>: Let f,g,h $\in \mathscr{P}$. Hence

$$(h : g) \leq (h : f) \cdot (f : g),$$

and

$$(f : g) \leq (f : h) \cdot (h : g).$$

Thus $\frac{1}{(h : f)} \leq \frac{(f : g)}{(h : g)}$, and $\frac{(f : g)}{(h : g)} \leq (f : h)$. Therefore

$$\frac{1}{(h : f)} \leq \frac{(f : g)}{(h : g)} \leq (f : h).$$

■

For the remainder of this chapter fix $f_0 \in \mathscr{P}$. Now for

$f, g \in \mathscr{P}$, define $A_f(g)$ to be $\frac{(f : g)}{(f_0 : g)}$; hence

$$\frac{1}{(f_0 : f)} \le A_f(g) \le (f : f_0).$$

For each $f \in \mathscr{P}$, let

$$X_f = \left[\frac{1}{(f_0 : f)}, (f : f_0)\right].$$

Clearly $A_f(g) \in X_f$ for each $g \in \mathscr{P}$. Define the set X to be $\prod_{f \in \mathscr{P}} X_f$; hence X is compact by the Tychonoff product theorem. Define $A(g) \in X$ by $(A(g))_f = A_f(g)$ for each $f \in \mathscr{P}$. Now for each neighborhood V of θ, define F_V to be

$$\{\, A(g) \mid g \in \mathscr{P} \text{ so that } g \text{ is symmetric and } g(G \setminus V) = \{0\} \,\}.$$

<u>Lemma</u> <u>3.8</u>: For each neighborhood V of θ, F_V is not empty.

<u>Proof</u>: Let V be a neighborhood of θ, and let W be a compact and symmetric neighborhood of θ so that $W \subseteq V$. Since G is completely regular, let $f \in \mathscr{P}$ so that $f(G \setminus W) = \{0\}$; and let $g(x) = f(x) + f(-x)$ for $x \in G$. Clearly $g \in \mathscr{P}$ and g is symmetric. Now $g(G \setminus W) = \{0\}$ since W is symmetric and $f(G \setminus W) = \{0\}$; hence $g(G \setminus V) = \{0\}$ since $W \subseteq V$. Therefore $A(g) \in F_V$. ■

<u>Theorem</u> <u>3.9</u>: There exists $I \in X$ so that if $n \in \mathbb{N}$,

$\{f_1,\ldots,f_n\} \subseteq \mathscr{P}$, V is a neighborhood of θ, and $\epsilon > 0$, then

$$| A_{f_i}(g) - I_{f_i} | < \epsilon$$

for $1 \leq i \leq n$ and some symmetric $g \in \mathscr{P}$ so that $g(G \setminus V) = \{0\}$.

Proof: Let

$$\mathscr{F} = \{ \overline{F}_V : V \text{ is a neighborhood of } \theta\}.$$

Now if U,V are neighborhoods of θ, then

$$F_{(U \cap V)} \subseteq F_U \cap F_V$$

since if $g \in \mathscr{P}$ is symmetric so that $g(G \setminus (U \cap V)) = \{0\}$, then

$$g(G \setminus U) = g(G \setminus V) = \{0\}.$$

Hence

$$F_{(U \cap V)} \subseteq \overline{F}_U \cap \overline{F}_V,$$

and $\overline{F}_U \cap \overline{F}_V$ is not empty. Thus $\mathscr{F}$ has the finite intersection property. Let $I \in \cap \mathscr{F}$ since X is compact.

Now let $n \in \mathbb{N}$ and $\{f_1,\ldots,f_n\} \subseteq \mathscr{P}$, let V be a neighborhood of θ, and let $\epsilon > 0$. Since $\cap \mathscr{F} \subseteq \overline{F}_V$, $I \in \overline{F}_V$. Let $N(I,f_1,\ldots,f_n,\epsilon)$ be a neighborhood of I; note that

$N(I,f_1,\ldots,f_n,\epsilon) = \{ x \in X : | x_{f_i} - I_{f_i}| < \epsilon \text{ for } 1 \leq i \leq n\}$.

Now let $A(g) \in F_V$ so that $A(g) \in N(I,f_1,\ldots,f_n,\epsilon)$; hence

$$| A_{f_i}(g) - I_{f_i}| < \epsilon$$

for $1 \leq i \leq n$. By the definition of $A(g)$, $g \in \mathscr{P}$, and

$g(G \setminus V) = \{0\}$. ■

For the remainder of this chapter, let I be an element of X that satisfies the conclusion of Theorem 3.9. Note that for $f \in \mathscr{P}$, I_f is the f–th coordinate of I.

Theorem 3.10: **If $f, f' \in \mathscr{P}$, $c > 0$, and $s \in G$, then**

i) $I_f > 0$,

ii) $I_{(f + f')} \leq I_f + I_{f'}$,

iii) $I_{c \cdot f} = c \cdot I_f$,

and

iv) $I_{f_s} = I_f$.

Proof: Let $f, f' \in \mathscr{P}$, $c > 0$, and $s \in G$. Clearly since $I_f \in X_f$, $I_f \geq \dfrac{1}{(f_0 : f)} > 0$. Let $\epsilon > 0$, and let symmetric $g \in \mathscr{P}$ so that

$$| A_f(g) - I_f | < \epsilon,$$

$$| A_{f'}(g) - I_{f'}| < \epsilon,$$

$$| A_{c \cdot f}(g) - I_{c \cdot f} | < \epsilon,$$

$$| A_{f_s}(g) - I_{f_s}| < \epsilon,$$

and

$$| A_{(f + f')}(g) - I_{(f + f')}| < \epsilon.$$

Note that $A_{c \cdot f}(g) = c \cdot A_f(g)$, $A_{f_s}(g) = A_f(g)$, and

$$A_{(f + f')}(g) \leq A_f(g) + A_{f'}(g)$$

by the ratio properties of Theorem 3.5.

Now

$$\begin{aligned} I_{(f + f')} &\leq A_{(f + f')}(g) + \epsilon \\ &\leq A_f(g) + A_{f'}(g) + \epsilon \\ &< I_f + I_{f'} + 3\cdot\epsilon, \end{aligned}$$

and

$$| c\cdot I_f - I_{c\cdot f} | < c\cdot\epsilon + \epsilon;$$

also

$$| I_f - I_{f_s} | < 2\cdot\epsilon$$

by the triangle inequality. Thus $I_{(f + f')} \leq I_f + I_{f'}$, $I_{c\cdot f} = c\cdot I_f$, and $I_{f_s} = I_f$. ■

<u>Lemma 3.11</u>: If $f,h,h' \in \mathscr{P}$ so that $h + h' \leq 1$, then

$$\mathbf{I_{fh} + I_{fh'} \leq I_f}.$$

<u>Proof</u>: Let $f,h,h' \in \mathscr{P}$ so that $h + h' \leq 1$, and let $\epsilon > 0$. Now by Theorem 3.1, let V be a neighborhood of θ so that

$$| h(x) - h(s) | < \epsilon \text{ and } | h'(x) - h'(s) | < \epsilon$$

whenever $(x - s) \in V$. Let $g \in \mathscr{P}$ so that $g(G \setminus V) = \{0\}$, and let $\sum_{i=1}^{n} c_i\cdot g_{s_i}$ be a covering of f by g. Hence for $x \in G$,

$$f(x)\cdot h(x) \leq \sum_{i=1}^{n} c_i\cdot g_{s_i}(x)\cdot h(x)$$

$$\leq \sum_{i=1}^{n} c_i \cdot g_{s_i}(x) \cdot (h(s_i) + \epsilon),$$

and

$$fh \leq \sum_{i=1}^{n} c_i \cdot (h(s_i) + \epsilon) \cdot g_{s_i}$$

since for $1 \leq i \leq n$, $g_{s_i}(x) = 0$ if $(x - s_i) \notin V$. Likewise

$$fh' \leq \sum_{i=1}^{n} c_i \cdot (h'(s_i) + \epsilon) \cdot g_{s_i}.$$

Hence

$$(fh : g) + (fh' : g) \leq \sum_{i=1}^{n} c_i \cdot (h(s_i) + h'(s_i) + 2 \cdot \epsilon)$$

$$\leq \sum_{i=1}^{n} c_i \cdot (1 + 2 \cdot \epsilon).$$

Thus

$$(fh : g) + (fh' : g) \leq (f : g) \cdot (1 + 2 \cdot \epsilon),$$

and

$$A_{fh}(g) + A_{fh'}(g) \leq A_f(g)$$

for all $g \in \mathscr{P}$ so that $g(G \setminus V) = \{0\}$.

Let symmetric $g' \in \mathscr{P}$ so that $g'(G \setminus V) = \{0\}$,

$$| A_f(g') - I_f | < \epsilon,$$

$$|A_{fh}(g') - I_{fh} | < \epsilon,$$

and

$$| A_{fh'}(g') - I_{fh'} | < \epsilon.$$

Thus

$$I_{fh} + I_{fh'} < A_{fh}(g') + A_{fh'}(g') + 2 \cdot \epsilon$$

$$\leq A_f(g') + 2 \cdot \epsilon < I_f + 3 \cdot \epsilon$$

again by the triangle inequality. Therefore

$I_{fh} + I_{fh'} \leq I_f$. ■

<u>Lemma 3.12</u>: **If $f, f' \in \mathscr{P}$, then**

$$I_{(f + f')} = I_f + I_{f'}.$$

<u>Proof</u>: Let $f, f' \in \mathscr{P}$, and let $C \subseteq G$ be compact so that $f(G \setminus C) = f'(G \setminus C) = \{0\}$. Before continuing, we need to establish the following sublemma.

<u>Sublemma</u>: If $D \subseteq U \subseteq G$ so that D is compact and U is open, then there exists $g \in \mathscr{P}$ so that $g(D) = \{1\}$ and $g(G \setminus U) = \{0\}$.

<u>Proof of sublemma</u>: Since G is completely regular, for each $x \in D$ there exists $g^x \in \mathscr{P}$ so that $(g^x)(x) = 2$ and $(g^x)(G \setminus D) = \{0\}$. Now for each $x \in D$, let $U_x = (g^x)^{-1}[(1,3)]$. Clearly each U_x is open and contains x. Hence $\{ U_x : x \in D \}$ is an open cover of D. For some $n \in \mathbb{N}$, let $\{x_1, \ldots, x_n\} \subseteq D$ so that $D \subseteq \bigcup_{i=1}^{n} U_{x_i}$. Now let $g' = \sum_{i=1}^{n} g^{x_i}$, and let $g = \min\{g', 1\}$. Clearly $g \in \mathscr{P}$, $g(D) = \{1\}$, and $g(G \setminus U) = \{0\}$. Thus the sublemma is true.

Now let $f'' \in \mathscr{P}$ so that $f''(C) = \{1\}$. Let $\epsilon > 0$ and $F = f + f' + \epsilon \cdot f''$. Clearly $F \in \mathscr{P}$, and $F(x) \geq \epsilon$ for $x \in C$. Let h be a function on G so that

$$h(x) = \begin{cases} \dfrac{f(x)}{F(x)} & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases};$$

clearly h is continuous on C. Let $A = f^{-1}(0)$; hence $G \setminus C \subseteq A$ and $G = A \cup C$. Now h is continuous on A, and A is closed since f is continuous. Note that C is closed since G is Hausdorff; hence h is continuous on G, and $h \in \mathscr{P}$. Define h' similarly; likewise $h' \in \mathscr{P}$.

Clearly $h(x) + h'(x) = 0 \leq 1$ for $x \in G \setminus C$. Now

$$h(x) + h'(x) = \frac{f(x) + f'(x)}{F(x)}$$

$$\leq \frac{f(x) + f'(x)}{f(x) + f'(x) + \epsilon \cdot f''(x)} \leq 1$$

for $x \in C$. Thus $h + h' \leq 1$. Note that $f = Fh$ and $f' = Fh'$. Hence

$$I_f + I_{f'} = I_{Fh} + I_{Fh'} \leq I_F,$$

and

$$I_F \leq I_{(f + f')} + \epsilon \cdot I_{f''}.$$

Thus $I_f + I_{f'} \leq I_{(f + f')}$. Therefore $I_{(f + f')} = I_f + I_{f'}$ since we already had that $I_{(f + f')} \leq I_f + I_{f'}$. ■

Now we are ready to show the existence of a Haar integral on G.

<u>Theorem</u> <u>3.13</u>: There exists a Haar integral on G.

<u>Proof</u>: Define the function I_0 on $\mathscr{P}$ so that $I_0(f) = I_f > 0$ for $f \in \mathscr{P}$. Now

$$I_0(f + g) = I_{(f + g)} = I_f + I_g = I_0(f) + I_0(g),$$

$$I_0(c\cdot f) = I_{c\cdot f} = c\cdot I_f = c\cdot I_0(f),$$

and

$$I_0(f_s) = I_{f_s} = I_f = I_0(f)$$

for all $f,g \in \mathscr{P}$, $c > 0$, and $s \in G$. Thus I_0 can be extended to a Haar integral on G. ■

Observe that given any $f \in \mathscr{P}$, there exists a Haar integral J on G such that if $n \in \mathbb{N}$ and $\{f_1,\ldots,f_n\} \subseteq \mathscr{P}$, V is a neighborhood of θ, and $\epsilon > 0$, then there exist a symmetric $g \in \mathscr{P}$ so that $g(G \setminus V) = \{0\}$ and

$$\left| \frac{(f_i : g)}{(f : g)} - J(f_i) \right| < \epsilon$$

for $1 \leq i \leq n$.

<u>Theorem</u> <u>3.14</u>: If $C \subseteq G$ is non-empty and compact, $n \in \mathbb{N}$, and $U_1,\ldots,U_n$ are open so that $C \subseteq \bigcup_{i=1}^{n} U_i$, then there exists continuous functions $f_1,\ldots,f_n \in \mathscr{L}$ such that $\left[\sum_{i=1}^{n} f_i\right](C) = \{1\}$, each $0 \leq f_i \leq 1$, and each U_i supports f_i.

<u>Proof</u>: Let $C \subseteq G$ be non-empty and compact. Let $n \in \mathbb{N}$, and let $U_1,\ldots,U_n$ be open and non-empty such that $C \subseteq \bigcup_{i=1}^{n} U_i$. Before continuing the proof, we establish the following

lemmas.

<u>Lemma</u> <u>1</u>: If D is compact and $V_1,\dots,V_n$ are open so that $D \subseteq \bigcup_{i=1}^{n} V_i$, then there exists continuous functions $g_1,\dots,g_n$ on D such that $\left[\sum_{i=1}^{n} g_i\right](D) = \{1\}$, and for $1 \leq i \leq n$, $0 \leq g_i \leq 1$ and $g_i(x) = 0$ for $x \in D \setminus V_i$.

<u>Proof</u> <u>of</u> <u>Lemma</u> <u>1</u>: Let D be compact, $n \in \mathbb{N}$, and $V_1,\dots,V_n$ be open so that $D \subseteq \bigcup_{i=1}^{n} V_i$. We assert that if U and V are open so that $D \subseteq U \cup V$, then there exists compact E and F so that $E \subseteq U$, $F \subseteq V$, and $D \subseteq E \cup F$. In fact, if U and V are open so that $D \subseteq U \cup V$, let A and B be open so that $D \setminus U \subseteq A$, $D \setminus V \subseteq B$, and $A \cap B$ is empty. Let $E = D \setminus A$ and $F = D \setminus B$; hence E and F are compact. Note that:

$$E = D \setminus A \subseteq D \setminus (D \setminus U) = U \cap D \subseteq U;$$

likewise, $F \subseteq V$. Also

$$E \cup F = D \setminus (A \cap B) = D$$

since $A \cap B$ is empty. Thus the assertion follows.

Hence, by using induction, we let $D_1,\dots,D_n$ be compact so that $D = \bigcup_{i=1}^{n} D_i$ and $D_i \subseteq V_i$ for $1 \leq i \leq n$. Further, for $1 \leq i \leq n$, (using the sublemma in the proof of Lemma 3.12) let h_i be continuous so that $0 \leq h_i \leq 1$, $h_i(D_i) = \{1\}$, and $h_i(x) = 0$ for $x \in X \setminus V_i$. Let $h = \sum_{i=1}^{n} h_i$; clearly h is continuous and $h(x) \geq 1$ for $x \in D$. Now let $g_i = \frac{h_i}{h}$ for $1 \leq i \leq n$. Hence each g_i is continuous, and $g_i(x) = 0$ when $x \in D \setminus V_i$ for $1 \leq i \leq n$. Note that

$$\sum_{i=1}^{n} g_i(x) = \sum_{i=1}^{n} \frac{h_i(x)}{h(x)} = \frac{h(x)}{h(x)} = 1$$

for $x \in D$. Thus lemma 1 is true.

<u>Lemma 2</u>: If D is compact, C is a closed subset of D, and $V_1, \ldots, V_n$ are open so that $C \subseteq \bigcup_{i=1}^{n} V_i$, then there exists continuous functions $g_1, \ldots, g_n$ on D such that $\left[\sum_{i=1}^{n} g_i\right](C) = \{1\}$, and for $1 \leq i \leq n$, $0 \leq g_i \leq 1$ and $g_i(x) = 0$ for $x \in D \setminus V_i$.

<u>Proof of Lemma 2</u>: Let D be compact, C be a closed subset of D, and $V_1, \ldots, V_n$ be open so that $C \subseteq \bigcup_{i=1}^{n} V_i$. Let V_{n+1} be open so that $V_{n+1} \cap D = D \setminus C$; hence $D \subseteq \bigcup_{i=1}^{n+1} V_i$. Now by lemma 1, let $g_1, \ldots, g_{n+1}$ be continuous on D such that $\left[\sum_{i=1}^{n+1} g_i\right](D) = \{1\}$, and for $1 \leq i \leq n+1$, $0 \leq g_i \leq 1$ and $g_i(x) = 0$ for $x \in D \setminus V_i$. Hence $\left[\sum_{i=1}^{n} g_i\right](C) = \{1\}$ since $g_{n+1}(x) = 0$ for $x \in D \setminus V_{n+1} = C$. Thus lemma 2 is true.

Now let $C_1, \ldots, C_n$ be compact so that $C \subseteq \bigcup_{i=1}^{n} C_i$ and each $C_i \subseteq U_i$. For each $1 \leq i \leq n$, let $h_i \in \mathscr{P}$ so that $h_i(C_i) = \{1\}$

and $h_i(X \setminus U_i) = \{0\}$, $V_i = h_i^{-1}[(\frac{1}{2},2)]$, and $D_i = h_i^{-1}([\frac{1}{2},2])$. Hence for $1 \le i \le n$, $C_i \subseteq V_i \subseteq D_i \subseteq U_i$, V_i is open, and D_i is compact since h_i has compact support. Let $D = \bigcup_{i=1}^{n} D_i$; clearly $C \subseteq \bigcup_{i=1}^{n} V_i \subseteq D$. Now by lemma 2, let functions $g_1, \dots, g_n$ be continuous on D such that $\left[\sum_{i=1}^{n} g_i\right](C) = \{1\}$, and for $1 \le i \le n$, $0 \le g_i \le 1$ and $g_i(x) = 0$ for $x \in D \setminus V_i$.

Now for each $1 \le i \le n$, define f_i by

$$f_i(x) = \begin{cases} g_i(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases};$$

thus $\left[\sum_{i=1}^{n} f_i\right](C) = \{1\}$, and for $1 \le i \le n$, $0 \le f_i \le 1$ and $f_i(x) = 0$ for $x \in G \setminus U_i$. Note that f_i is continuous on D since g_i is continuous on D, and f_i is continuous on $G \setminus V_i$ since $f_i(G \setminus V_i) = \{0\}$. Hence f_i is continuous on G since $G = (G \setminus V_i) \cup D$ and both D and $G \setminus V_i$ are closed. Clearly each f_i has D as compact support. ■

Lemma 3.15: If C is a non-empty compact subset of G and W is an open neighborhood of *0*, then for some $n \in \mathbb{N}$, there exists elements $s_1, \dots, s_n \in G$ and functions $f_1, \dots, f_n \in \mathcal{P}$ so that $C \subseteq \bigcup_{i=1}^{n} W + \{s_i\}$,

$$f_i[G \setminus (W + \{s_i\})] = \{0\}$$

for $1 \le i \le n$, and $\left[\sum_{i=1}^{n} f_i\right](C) = \{1\}$.

Proof: Let $C \subseteq G$ be compact and non-empty, and let W be an open neighborhood of θ. Clearly $\{ W + \{s\} \mid s \in C \}$ is an open covering of C. Let m be the least $n \in \mathbb{N}$ so that $C \subseteq \bigcup_{i=1}^{n} W + \{s_i\}$ where $s_1, \dots, s_n \in C$. Hence let $f_1, \dots, f_m$ be continuous functions with compact support so that $0 \leq f_i \leq 1$ and

$$f_i[G \setminus (W + \{s_i\})] = \{0\}$$

for $1 \leq i \leq m$ and $\left[\sum_{i=1}^{m} f_i\right](C) = \{1\}$.

Now suppose that for some $1 \leq j \leq m$, $f_j = 0$; assume that $j = 1$. Let $x \in C$; hence for some $1 \leq i \leq m$ such that $i > 1$, $f_i(x) > 0$ and $x \in W + \{s_i\}$. Thus $C \subseteq \bigcup_{i=2}^{n} W + \{s_i\}$, contradicting the definition of m. Therefore $f_i \neq 0$ and $f_i \in \mathscr{P}$ for $1 \leq i \leq m$. ■

Next we would like to show that a Haar integral on G is unique up to a positive scalar. We will first establish the following lemmas.

<u>Lemma</u> <u>3.16</u>: **If J is a Haar integral on G, then** $\dfrac{J(f)}{J(g)} \leq (f : g)$ **for all** $f,g \in \mathscr{P}$.

<u>Proof</u>: Let J be a Haar integral on $\mathscr{L}$, and let $f,g \in \mathscr{P}$. We claim that $J(g) > 0$. To see this let $f' \in \mathscr{P}$ so that $J(f') > 0$, and let $\sum_{i=1}^{n} c_i \cdot g_{s_i}$ be a covering of f' by g.

Clearly $f' \neq 0$, and $\sum_{i=1}^{n} c_i > 0$. Now

$$0 < J(f') \leq J\left[\sum_{i=1}^{n} c_i \cdot g_{s_i}\right] = \left[\sum_{i=1}^{n} c_i\right] \cdot J(g).$$

Thus $J(g) > 0$. Now let $\sum_{i=1}^{m} a_i \cdot g_{t_i}$ be a covering of f by g. Hence $J(f) \leq \left[\sum_{i=1}^{m} a_i\right] \cdot J(g)$, and $\dfrac{J(f)}{J(g)} \leq \sum_{i=1}^{m} a_i$. Therefore $\dfrac{J(f)}{J(g)} \leq (f : g)$. ∎

<u>Lemma 3.17</u>: If C is non-empty and compact and $f, f' \in \mathscr{P}$ so that C supports f and $f'(C) = \{1\}$, then for any $\epsilon > 0$, there exists a neighborhood U of θ so that

$$(f : g) \leq \epsilon \cdot (f' : g) + \frac{J(f)}{J(g)}$$

for any Haar integral J on G and any symmetric $g \in \mathscr{P}$ such that $g(G \setminus U) = \{0\}$.

<u>Proof</u>: Let $C \subseteq G$ be non-empty and compact, and let $f, f' \in \mathscr{P}$ so that C supports f and $f'(C) = \{1\}$. Let $\epsilon > 0$, and let U be a neighborhood of θ so that

$$| f(x) - f(y) | \leq \epsilon$$

whenever $(x - y) \in U$. Now let symmetric $g \in \mathscr{P}$ so that $g(G \setminus U) = \{0\}$, and let J be a Haar integral on G. We claim that $[f(x) - \epsilon] \cdot g_x \leq fg_x$ for all $x \in G$. For if $x \in G$ and if $y \in G$ so that $(y - x) \in G \setminus U$, then $g_x(y) = g(y - x) = 0$, and the claim holds. Now if $y \in G$ so that $(y - x) \in U$, then $f(x) - \epsilon \leq f(y)$, and

$$[f(x) - \epsilon]\cdot g_x(y) \leq f(y)\cdot g_x(y).$$

Thus this claim is true; hence

$$J([f(x) - \epsilon]\cdot g_x) \leq J(fg_x),$$

and

$$(*) \qquad [f(x) - \epsilon] \leq \frac{J(fg_x)}{J(g_x)}.$$

Now let $\delta > 0$, and let W be an open neighborhood of θ so that

$$| g(x) - g(y) | \leq \delta$$

whenever $(x - y) \in W$. Now as concluded in Lemma 3.15, for some $n \in \mathbb{N}$, let $s_1,\ldots,s_n \in C$ and $h_1,\ldots,h_n \in \mathcal{P}$ such that $C \subseteq \bigcup_{i=1}^{n} W + \{s_i\}$, $\left[\sum_{i=1}^{n} h_i\right](C) = \{1\}$ and $W + \{s_i\}$ supports h_i for $1 \leq i \leq n$. Since C supports f, $f = \sum_{i=1}^{n} h_i f$; hence $fg_x = \sum_{i=1}^{n} h_i f g_x$ and

$$(**) \qquad J(fg_x) = \sum_{i=1}^{n} J(h_i f g_x).$$

Now we assert that for $x \in G$ and $1 \leq i \leq n$,

$$h_i g_x \leq [g_x(s_i) + \delta]\cdot h_i.$$

For if $1 \leq i \leq n$ and $x \in G$ and if $y \in G$ so that $y \in G \setminus (W + \{s_i\})$, then $h_i(y) = 0$, and the assertion holds. Now if $y \in G$ so that $y \in W + \{s_i\}$, then $(y - s_i) \in W$. Hence

$$(y - x) - (s_i - x) = (y - s_i) \in W,$$

and

$$| g(y - x) - g(s_i - x) | = | g_x(y) - g_x(s_i) | \leq \delta.$$

Thus $g_x(y) \leq g_x(s_i) + \delta$, and the assertion is true. Hence

$$fh_i g_x \le [g_x(s_i) + \delta]\cdot fh_i,$$

and

$$J(fh_i g_x) \le [g_x(s_i) + \delta]\cdot J(fh_i).$$

Now note from (*) and (**) that

$$[f(x) - \epsilon] \le \frac{\sum_{i=1}^{n}[g_x(s_i) + \delta]\cdot J(fh_i)}{J(g_x)}$$

$$= \sum_{i=1}^{n}\frac{J(fh_i)}{J(g)}\cdot g_{s_i}(x) + \delta\cdot\frac{J(f)}{J(g)}$$

for all $x \in G$ since g is symmetric and $f = \sum_{i=1}^{n} h_i f$. Hence

$$f \le \left[\epsilon + \delta\cdot\frac{J(f)}{J(g)}\right]\cdot f' + \sum_{i=1}^{n}\frac{J(fh_i)}{J(g)}\cdot g_{s_i}$$

since C supports f and $f'(C) = \{1\}$. Now

$$(f : g) \le \left[\epsilon + \delta\cdot\frac{J(f)}{J(g)}\right]\cdot(f' : g) + \sum_{i=1}^{n}\frac{J(fh_i)}{J(g)}\cdot(g_{s_i} : g)$$

$$= \left[\epsilon + \delta\cdot\frac{J(f)}{J(g)}\right]\cdot(f' : g) + \frac{J(f)}{J(g)}$$

since $(g_{s_i} : g) = (g : g) = 1$. Therefore

$$(f : g) \le \epsilon\cdot(f' : g) + \frac{J(f)}{J(g)}$$

since δ is arbitrary. ■

<u>Lemma 3.18</u>: If J is a Haar integral on G and $f \in \mathscr{P}$, then there exists a neighborhood U of θ and a bound $0 < M < \infty$ so that

$$(f : g)\cdot J(g) \le M$$

for all symmetric $g \in \mathscr{P}$ such that $g(G \setminus U) = \{0\}$.

Proof: Let J be a Haar integral on G, and let $f \in \mathscr{P}$. Let $C \subseteq G$ so that C supports f and is compact and non-empty. Let $f' \in \mathscr{P}$ so that $f'(C) = \{1\}$, and let $0 < \epsilon < \dfrac{1}{(f' : f)}$. Now let U be a neighborhood of θ so that

$$(f : g) \leq \epsilon\cdot(f' : g) + \frac{J(f)}{J(g)}$$

for all symmetric $g \in \mathscr{P}$ such that $g(G \setminus U) = \{0\}$. Hence

$$(f : g)\cdot J(g) \leq \epsilon\cdot(f' : g)\cdot J(g) + J(f),$$

and

$$(f : g)\cdot J(g) \leq \epsilon\cdot(f' : f)\cdot(f : g)\cdot J(g) + J(f).$$

Now

$$[1 - \epsilon\cdot(f' : f)]\cdot(f : g)\cdot J(g) \leq J(f);$$

thus

$$(f : g)\cdot J(g) \leq \frac{J(f)}{[1 - \epsilon\cdot(f' : f)]}.$$

Clearly $0 < \dfrac{J(f)}{[1 - \epsilon\cdot(f' : f)]} < \infty$. ■

Now we are ready to establish the uniqueness, up to a positive scalar, of a Haar integral on G.

Theorem 3.19: If J and J' are a Haar integrals on G, then $J' = c\cdot J$ for some $c > 0$.

Proof: Let J and J' be a Haar integrals on G. Now as a consequence of Theorem 3.13, let I_0 be a Haar integral on G so that if $n \in \mathbb{N}$, $f_1,\ldots,f_n \in \mathscr{P}$, V is a neighborhood of θ, and $\epsilon > 0$, then there exists a symmetric $g \in \mathscr{P}$ such that $g(G \setminus V) = \{0\}$ and

$$\left| \frac{(f_i:\ g)}{(f_0:\ g)} - I_0(f_i) \right| < \epsilon$$

for $1 \leq i \leq n$. Let $C_0 \subsetneq G$ be compact so that C_0 supports f_0, and let $f_0' \in \mathscr{P}$ so that $f_0'(C_0) = \{1\}$. Let $f \in \mathscr{P}$ with compact support C, and let $f' \in \mathscr{P}$ so that $f'(C) = \{1\}$.

Now let $\epsilon > 0$, and let V be a neighborhood of θ and let $M > 0$ such that

$$(f_0':\ g)\cdot J(g) \leq M,$$
$$(f':\ g)\cdot J(g) \leq M,$$
$$(f_0:\ g) \leq \epsilon\cdot(f_0':\ g) + \frac{J(f_0)}{J(g)},$$

and

$$(f\ :\ g) \leq \epsilon\cdot(f':\ g) + \frac{J(f)}{J(g)}$$

for all symmetric $g \in \mathscr{P}$ such that g vanishes outside of V. Since $\frac{J(f_0)}{J(g)} \leq (f_0:\ g)$ and $\frac{J(f)}{J(g)} \leq (f\ :\ g)$,

$$(*) \quad J(f_0) \leq (f_0:\ g)\cdot J(g) \leq \epsilon\cdot(f_0':\ g)\cdot J(g) + J(f_0)$$

and

$$(**) \quad J(f) \leq (f\ :\ g)\cdot J(g) \leq \epsilon\cdot(f':\ g)\cdot J(g) + J(f)$$

for all symmetric $g \in \mathscr{P}$ such that g vanishes outside of V. Now fix symmetric $g \in \mathscr{P}$ so that

$$\left| \frac{(f\ :\ g)}{(f_0:\ g)} - I_0(f) \right| < \epsilon$$

and g vanishes outside of V.

Note that

$$\frac{J(f)}{\epsilon\cdot(f_0^{'}:\ g)\cdot J(g)\ +\ J(f_0)} \leq \frac{(f\ :\ g)}{(f_0:\ g)}$$

and

$$\frac{(f\ :\ g)}{(f_0:\ g)} \leq \frac{\epsilon\cdot(f^{'}:\ g)\cdot J(g)\ +\ J(f)}{J(f_0)}$$

by combining (*) and (**). Hence

$$\frac{J(f)}{\epsilon\cdot M\ +\ J(f_0)} \leq I_0(f)\ +\ \epsilon$$

and

$$I_0(f)\ -\ \epsilon \leq \frac{\epsilon\cdot M\ +\ J(f)}{J(f_0)}$$

by the triangle inequality. Thus

$$\frac{J(f)}{J(f_0)} \leq I_0(f) \leq \frac{J(f)}{J(f_0)},$$

and $J(f) = J(f_0)\cdot I_0(f)$ for all $f \in \mathscr{P}$ and all $f \in \mathscr{L}$.

Likewise $J^{'} = J^{'}(f_0)\cdot I_0$; hence $J = \frac{J(f_0)}{J^{'}(f_0)}\cdot J^{'}$. Clearly

$\frac{J(f_0)}{J^{'}(f_0)} > 0$. ■

Observe that since the Cantor ternary set is a Hausdorff and locally compact topological group, there exists a Haar integral on $\mathfrak{C}$.

CHAPTER IV

HAAR MEASURE

In this chapter, given a Haar integral I on G, we will establish the existence of a Haar measure on G which represents I. For the remainder of this paper, let I be a Haar integral on G. A <u>Borel measure</u> ρ on G is a measure defined on the class of Borel sets of G so that $\rho(C) < \infty$ for all compact C. A set $E \in \mathfrak{B}$ is said to be <u>regular</u> with respect to a Borel measure ρ if

$$\begin{aligned}\rho(E) &= \inf\{\ \rho(U)\ :\ E \subseteq U \text{ where U is open and Borel }\}\\ &= \sup\{\ \rho(C)\ :\ C \subseteq E \text{ where C is compact }\}.\end{aligned}$$

Further a <u>regular Borel measure</u> μ on G is a Borel measure on G so that E is regular with respect to μ for all $E \in \mathfrak{B}$. Now a <u>Haar measure</u> on G is a non-zero regular Borel measure μ on G so that $\mu(E + \{s\}) = \mu(E)$ for all $E \in \mathfrak{B}$ and $s \in G$. A set B is said to be <u>bounded</u> if there exists a compact set C so that $B \subseteq C$, and a set S is said to be <u>σ-bounded</u> if there exists a sequence (C_n) of compact sets such that $S \subseteq \bigcup_{i=1}^{\infty} C_i$. A <u>content</u> φ on G is a set function defined for all compact sets so that:

i) $0 \leq \varphi(C) < \infty$,

ii) if $C \subseteq D$, then $\varphi(C) \leq \varphi(D)$,

iii) $\varphi(C \cup D) \leq \varphi(C) + \varphi(D)$

and

iv) if $C \cap D$ is empty, then

$$\varphi(C \cup D) = \varphi(C) + \varphi(D)$$

for compact sets C and D. Observe that $\varphi(\phi) = 0$. Further a content φ on G is said to be regular if

$$\varphi(C) = \inf\{ \varphi(D) : C \subseteq U \subseteq D \text{ where } U \text{ is open and } D \text{ is compact} \}$$

for all compact C. In this paper, the characteristic function of a set A is denoted by $\aleph_A$. Now define the set function λ by

$$\lambda(C) = \inf\{ I(f) : f \in \mathscr{L} \text{ and } \aleph_C \leq f \}$$

for all compact $C \subseteq G$. This set function λ is important in constructing a Haar measure on G.

<u>Theorem</u> <u>4.1</u>: The set function λ is a regular content.

<u>Proof</u>: Clearly $\lambda(\phi) = 0$. To show that λ is non-negative, let C be compact; now let $f \in \mathscr{P}$ so that $\aleph_C \leq f$; clearly there is such a function (by the sublemma in Lemma 3.12). Note that $I(f) > 0$. Thus $0 \leq \lambda(C) < \infty$.

Now to show that λ is monotone, let A be compact so that $A \subseteq C$. Hence $\aleph_A \leq \aleph_C$ and $\aleph_A \leq f$. Thus $\lambda(A) \leq I(f)$ and $\lambda(A) \leq \lambda(C)$.

To show that λ is subadditive, let B be compact and

$g \in \mathscr{L}$ so that $\aleph_B \leq g$. Hence $\aleph_C + \aleph_B \leq f + g$, and $\aleph_{(C \cup B)} \leq f + g$. Thus $\lambda(C \cup B) \leq I(f) + I(g)$, and $\lambda(C \cup B) \leq \lambda(C) + \lambda(B)$.

Now to show that λ is additive, let D be compact so that $C \cap D$ is empty. Since G is Hausdorff and both C and D are compact, let U and V be open such that $C \subsetneq U$ and $D \subsetneq V$ and $U \cap V$ is empty. Let $f', g' \in \mathscr{L}$ so that $f'(C) = g'(D) = \{1\}$, $0 \leq f' \leq 1$, $0 \leq g' \leq 1$, and $f'(G \setminus U) = g'(G \setminus V) = \{0\}$. Clearly $\aleph_C \leq f'$ and $\aleph_D \leq g'$. Hence $f' + g' \leq 1$ since $U \cap V$ is empty. Now let $h \in \mathscr{L}$ so that $\aleph_{(C \cup D)} \leq h$; clearly $\aleph_C, \aleph_D \leq h$. Note that $h(f' + g') \leq h$, $\aleph_C \leq hf'$, and $\aleph_D \leq hg'$. Hence

$$\begin{aligned} \lambda(C) + \lambda(D) &\leq I(hf') + I(hg') \\ &= I[h(f' + g')] \leq I(h). \end{aligned}$$

Thus $\lambda(C) + \lambda(D) \leq \lambda(C \cup D)$ and $\lambda(C) + \lambda(D) = \lambda(C \cup D)$.

To show that λ is regular, let $\epsilon > 0$ and $f'' \in \mathscr{L}$ so that $\aleph_C \leq f''$ and $I(f'') \leq \lambda(C) + \epsilon$. Now for each $0 < t < 1$, let

$$U_t = \{ x \in G : f''(x) > t \}$$

and

$$E_t = \{ x \in G : f''(x) \geq t \}.$$

Clearly $C \subsetneq U_t \subsetneq E_t$, U_t is open, and E_t is compact. Note that $\aleph_{E_t} \leq \frac{1}{t}\cdot f''$, and

$$\lambda(E_t) \leq \tfrac{1}{t}\cdot I(f'') \leq \tfrac{1}{t}\cdot(\lambda(C) + \epsilon)$$

for all $0 < t < 1$. Now choose t so that

$$\tfrac{1}{t}\cdot(\lambda(C) + \epsilon) \leq \lambda(C) + 2\cdot\epsilon.$$

Thus $\lambda(E_t) \leq \lambda(C) + 2\cdot\epsilon$, and

$$\lambda(C) = \inf\{ \lambda(E) : C \subseteq U \subseteq E \text{ where } U \text{ is open and } E \text{ is compact} \}.$$

■

Now define the set function λ_* by

$$\lambda_*(U) = \sup\{ \lambda(C) : C \text{ is a compact subset of } U \}$$

for all open Borel sets U.

<u>Theorem</u> <u>4.2</u>: The set function λ_* has the following properties:

i) $\lambda_*(\phi) = 0$,

ii) $\lambda_*(U) < \infty$ for every bounded open set U,

iii) λ_* is monotone,

and

iv) λ_* is countably additive.

<u>Proof</u>: Clearly $\lambda_*(\phi) = 0$ since $\lambda(\phi) = 0$. Let U be open and bounded, and let D be compact so that $U \subseteq D$. Clearly U is Borel since $U = D \setminus (D \setminus U)$ and $D \setminus U$ is compact. Now for every compact $C \subseteq U$, $C \subseteq D$ and $\lambda(C) \leq \lambda(D)$. Thus $\lambda_*(U) \leq \lambda(D) < \infty$.

Now to show that λ_* is monotone, let U and V be open

Borel sets such that $U \subseteq V$. Let C be compact so that $C \subseteq U$; hence $C \subseteq V$. Thus $\lambda(C) \leq \lambda_*(V)$ and $\lambda_*(U) \leq \lambda_*(V)$.

To show that λ_* is countably additive, first let us show that λ_* is subadditive. Let U, V be open Borel sets, and let C be compact so that $C \subseteq U \cup V$. Now let D, E be compact such that $D \subseteq U$, $E \subseteq V$, and $C = D \cup E$. Hence

$$\lambda(C) \leq \lambda(D) + \lambda(E) \leq \lambda_*(U) + \lambda_*(V),$$

$$\lambda_*(U \cup V) \leq \lambda_*(U) + \lambda_*(V),$$

and λ_* is subadditive. Now let (U_n) be a sequence of open Borel sets; hence

$$\lambda_*\left[\bigcup_{i=1}^{m} U_i\right] \leq \sum_{i=1}^{m} \lambda_*(U_i) \leq \sum_{i=1}^{\infty} \lambda_*(U_i)$$

for each $m \in \mathbb{N}$. Thus $\lambda_*\left[\bigcup_{i=1}^{m} U_i\right] \leq \sum_{i=1}^{\infty} \lambda_*(U_i)$, and λ_* is countably subadditive.

Now to show that λ_* is countably additive, let us first show that λ_* is additive. Let U and V be Borel open and disjoint, and let C and D be compact such that $C \subseteq U$ and $D \subseteq V$. Clearly $C \cap D$ is empty. Hence

$$\lambda_*(U \cup V) \geq \lambda(C \cup D) = \lambda(C) + \lambda(D),$$

$$\lambda_*(U \cup V) \geq \lambda_*(U) + \lambda_*(V),$$

and λ_* is additive. Now let (U_n) be a sequence of disjoint open Borel sets. Clearly $\bigcup_{i=1}^{\infty} U_i$ is an open Borel set. Hence

$$\lambda_*\left[\bigcup_{i=1}^{\infty} U_i\right] \geq \lambda_*\left[\bigcup_{i=1}^{\infty} U_i\right] = \sum_{i=1}^{m} \lambda_*(U_i)$$

for each $m \in \mathbb{N}$ since λ_* is monotone. Thus $\lambda_*\left[\bigcup_{i=1}^{\infty} U_i\right] \geq \sum_{i=1}^{\infty} \lambda_*(U_i)$, and $\lambda_*\left[\bigcup_{i=1}^{\infty} U_i\right] = \sum_{i=1}^{\infty} \lambda_*(U_i)$. ■

A <u>hereditary</u> σ–ring $\mathscr{R}$ is a σ–ring so that if $A \in \mathscr{R}$ and $B \subseteq A$, then $B \in \mathscr{R}$. An <u>outer</u> <u>measure</u> ρ on a hereditary σ–ring $\mathscr{R}$ is a set function defined on $\mathscr{R}$ so that ρ is non–negative, monotone, countably subadditive, and $\rho(\phi) = 0$. For the remainder of this paper, define the set $\mathscr{H}$ to be

$$\{\ A \subseteq G : A \text{ is } \sigma\text{–bounded}\ \},$$

and define the set function λ^* by

$$\lambda^*(A) = \inf\{\ \lambda_*(U) : A \subseteq U \text{ where } U \text{ is Borel open}\ \}$$

for all $A \in \mathscr{H}$.

<u>Theorem</u> <u>4.3</u>: The collection $\mathscr{H}$ and the set function λ^* have the following properties:

- **i) $\mathscr{H}$ is a hereditary σ–ring,**
- **ii) λ^* is an outer measure on $\mathscr{H}$,**
- **iii) $\lambda^*(A) < \infty$ for all bounded $A \in \mathscr{H}$,**
- **iv) λ^* extends λ_*,**

and

- **v) if $U \subseteq D$ where U is open and D is compact, then**

$$\lambda^*(U) = \lambda_*(U) \leq \lambda(D) \leq \lambda^*(D).$$

<u>Proof</u>: Clearly $\mathscr{H}$ is a hereditary σ–ring since the countable union of σ–bounded sets and a subset of a

σ–bounded set is σ–bounded.

To show that λ^* is an outer measure on $\mathscr{H}$, first note that $\lambda^*(A) \geq 0$ for all $A \in \mathscr{H}$ since $\lambda_*(U) \geq 0$ for all open Borel set U, and $\lambda^*(\phi) = 0$ since $\lambda_*(\phi) = 0$. Let $A, B \in \mathscr{H}$ so that $A \subseteq B$, and let U be an open Borel set so that $B \subseteq U$. Hence $A \subseteq U$ and $\lambda^*(A) \leq \lambda_*(U)$. Thus $\lambda^*(A) \leq \lambda^*(B)$, and λ^* is monotone. Now let (A_n) be a sequence from $\mathscr{H}$, and let $\epsilon > 0$. For each $n \in \mathbb{N}$, let U_n be Borel open such that $A_n \subseteq U_n$ and $\lambda_*(U_n) \leq \lambda^*(A_n) + \frac{\epsilon}{2^n}$. Hence $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} U_i$, and

$$\lambda^*\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \lambda_*\left[\bigcup_{i=1}^{\infty} U_i\right] \leq \sum_{i=1}^{\infty} \lambda_*(U_i)$$

$$\leq \epsilon + \sum_{i=1}^{\infty} \lambda^*(A_i).$$

Thus $\lambda^*\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \lambda^*(A_i)$, and λ^* is an outer measure on $\mathscr{H}$.

Now let $A \in \mathscr{H}$ so that A is bounded, and let C be compact so that $A \subseteq C$. Since G is locally compact, let U be open and bounded so that $C \subseteq U$. Thus $A \subseteq U$, and

$$\lambda^*(A) \leq \lambda_*(U) < \infty.$$

To show that λ^* extends λ_*, let U be Borel open. Clearly $\lambda^*(U) \leq \lambda_*(U)$. Now $\lambda_*(U) \leq \lambda_*(V)$ for all open Borel V such that $U \subseteq V$. Hence $\lambda_*(U) \leq \lambda^*(U)$. Thus $\lambda^*(U) = \lambda_*(U)$ and λ^* extends λ_*.

Now let U be open and D be compact such that $U \subseteq D$; hence U is bounded and Borel, and $\lambda^*(U) = \lambda_*(U)$. Now $\lambda(D) \leq \lambda_*(V)$ for all open Borel V such that $D \subseteq V$; hence $\lambda(D) \leq \lambda^*(D)$. Also $\lambda(C) \leq \lambda_*(U)$ and $\lambda(C) \leq \lambda(D)$ for all

compact $C \subseteq U$; hence $\lambda_*(U) \leq \lambda(D)$. Thus
$$\lambda^*(U) = \lambda_*(U) \leq \lambda(D) \leq \lambda^*(D).$$
■

A set $E \subseteq \mathscr{H}$ is said to be <u>λ^*-measurable</u> if
$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap \tilde{E})$$
for all $A \in \mathscr{H}$. Let $\mathfrak{M}$ be the collection of all λ^*-measurable sets in $\mathscr{H}$.

<u>Theorem 4.4</u>: The collection $\mathfrak{M}$ is a σ-ring.

<u>Proof</u>: First to show that $\mathfrak{M}$ is a ring, let $E,F \in \mathfrak{M}$ and $A \in \mathscr{H}$. Now
$$\begin{aligned}\lambda^*[A \cap (E \cup F)] &= \lambda^*[A \cap (E \cup F) \cap E] + \lambda^*[A \cap (E \cup F) \cap \tilde{E}] \\ &= \lambda^*(A \cap E) + \lambda^*(A \cap \tilde{E} \cap F).\end{aligned}$$
Hence
$$\begin{aligned}&\lambda^*[A \cap (E \cup F)] + \lambda^*[A \cap (\tilde{E} \cap \tilde{F})] \\ &\qquad = \lambda^*(A \cap E) + \lambda^*(A \cap \tilde{E} \cap F) + \lambda^*(A \cap \tilde{E} \cap \tilde{F}) \\ &\qquad = \lambda^*(A \cap E) + \lambda^*(A \cap \tilde{E}) = \lambda^*(A),\end{aligned}$$
and $(E \cup F) \in \mathfrak{M}$. Note that $E \setminus F = E \cap \tilde{F}$, and $\tilde{}(E \setminus F) = \tilde{E} \cup F$. Now
$$\begin{aligned}\lambda^*[A \cap (\tilde{E} \cup F)] &= \lambda^*[A \cap (\tilde{E} \cup F) \cap F] + \lambda^*[A \cap (\tilde{E} \cup F) \cap \tilde{F}] \\ &= \lambda^*(A \cap F) + \lambda^*(A \cap \tilde{E} \cap \tilde{F}) \\ &= \lambda^*(A \cap F \cap E) + \lambda^*(A \cap F \cap \tilde{E}) \\ &\qquad + \lambda^*(A \cap \tilde{E} \cap \tilde{F}).\end{aligned}$$

Hence

$$\lambda^*[A \cap (E \cap \tilde{F})] + \lambda^*[A \cap (\tilde{E} \cup F)]$$
$$= \lambda^*(A \cap E \cap \tilde{F}) + \lambda^*(A \cap E \cap F)$$
$$+ \lambda^*(A \cap \tilde{E} \cap F) + \lambda^*(A \cap \tilde{E} \cap \tilde{F})$$
$$= \lambda^*(A \cap E) + \lambda^*(A \cap \tilde{E}) = \lambda^*(A).$$

Thus $(E \setminus F) \in \mathfrak{M}$, and $\mathfrak{M}$ is a ring.

Now to show that $\mathfrak{M}$ is a σ–ring, let (E_n) be a sequence of disjoint sets from $\mathfrak{M}$. Note that

$$\lambda^*[A \cap (E_1 \cup E_2)] = \lambda^*[A \cap (E_1 \cup E_2) \cap E_1]$$
$$+ \lambda^*[A \cap (E_1 \cup E_2) \cap \tilde{E}_1]$$
$$= \lambda^*[A \cap E_1] + \lambda^*[A \cap E_2]$$

since E_1 and E_2 are disjoint. Hence for each $n \in \mathbb{N}$,

$$\lambda^*[A \cap \left[\bigcup_{i=1}^{n} E_i\right]] = \sum_{i=1}^{n} \lambda^*(A \cap E_i).$$

Now

$$\lambda^*(A) = \lambda^*[A \cap \left[\bigcup_{i=1}^{n} E_i\right]] + \lambda^*[A \cap \sim\left[\bigcup_{i=1}^{n} E_i\right]]$$
$$\geq \sum_{i=1}^{n} \lambda^*(A \cap E_i) + \lambda^*[A \cap \sim\left[\bigcup_{i=1}^{\infty} E_i\right]]$$

for all $n \in \mathbb{N}$ since $\sim\left[\bigcup_{i=1}^{\infty} E_i\right] \subseteq \sim\left[\bigcup_{i=1}^{n} E_i\right]$. Hence

$$\lambda^*(A) \geq \sum_{i=1}^{\infty} \lambda^*(A \cap E_i) + \lambda^*[A \cap \sim\left[\bigcup_{i=1}^{\infty} E_i\right]]$$
$$\geq \lambda^*[A \cap \left[\bigcup_{i=1}^{\infty} E_i\right]] + \lambda^*[A \cap \sim\left[\bigcup_{i=1}^{\infty} E_i\right]].$$

Therfore $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}$, and $\mathfrak{M}$ is an σ–ring. ■

Observe that λ^* is a countably additive measure on $\mathfrak{M}$.

<u>Lemma</u> <u>4.5</u>: If $A \in \mathscr{H}$ so that

$$\lambda^*(U) = \lambda^*(U \cap A) + \lambda^*(U \cap \tilde{A})$$

for all open Borel sets U, then $A \in \mathfrak{M}$.

<u>Proof</u>: Let A be such a set in $\mathscr{H}$, and let $B \in \mathscr{H}$. Now

$$\begin{aligned}\lambda_*(U) = \lambda^*(U) &= \lambda^*(U \cap A) + \lambda^*(U \cap \tilde{A}) \\ &\geq \lambda^*(B \cap A) + \lambda^*(B \cap \tilde{A})\end{aligned}$$

for all open Borel sets U such that $B \subseteq U$. Hence

$$\lambda^*(B) \geq \lambda^*(B \cap A) + \lambda^*(B \cap \tilde{A}).$$

Thus $A \in \mathfrak{M}$. ■

<u>Theorem</u> <u>4.6</u>: Every Borel set is λ^*–measurable.

<u>Proof</u>: Clearly $\mathfrak{B} \subseteq \mathscr{H}$ since every Borel set is σ–bounded. Let C be compact, and let $U \in \mathfrak{B}$ so that U is open. Now $U \cap \tilde{C}$ is Borel open since C is closed. Let $D \subseteq U \cap \tilde{C}$ so that D is compact; hence $U \cap \tilde{D}$ is also Borel open. Let $E \subseteq U \cap \tilde{D}$ so that E is compact. Clearly $D \cap E$ is empty, $D \cup E \subseteq U$, and $U \cap C \subseteq U \cap \tilde{D}$. Hence

$$\lambda^*(U) = \lambda_*(U) \geq \lambda(D \cup E) = \lambda(D) + \lambda(E).$$

Now

$$\lambda_*(U) \geq \lambda(D) + \lambda_*(U \cap \tilde{D})$$

since E is an arbitrary compact subset of $U \cap \tilde{D}$, and

$$\lambda_*(U) \geq \lambda(D) + \lambda^*(U \cap C)$$

since λ^* extends λ_* and is monotone. Thus

$$\lambda^*(U) \geq \lambda^*(U \cap \tilde{C}) + \lambda^*(U \cap C)$$

since D is an arbitrary compact subset of $U \cap \tilde{C}$ and λ^* extends λ_*. Therefore $C \in \mathfrak{M}$, $\mathfrak{B} \subseteq \mathfrak{M}$, and every Borel set is λ^*-measurable. ∎

Note that λ^* is a Borel measure since $\lambda^*(C) < \infty$ for all compact sets C. For the rest of this paper, define μ to be $\lambda^*|_{\mathfrak{B}}$, and define the set $\mathfrak{R}$ to be

$$\{ A \in \mathfrak{B} : A \text{ is regular with respect to } \mu \}.$$

Clearly μ is a Borel measure.

Theorem 4.7: If (E_n) is a sequence from $\mathfrak{R}$ so that $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$ and (F_n) is a sequence from $\mathfrak{R}$, then $\bigcup_{i=1}^{\infty} F_i \in \mathfrak{R}$, $\bigcap_{i=1}^{\infty} E_i \in \mathfrak{R}$, and $(E_1 \setminus E_2) \in \mathfrak{R}$.

Proof: Let (E_n) be a sequence from $\mathfrak{R}$ so that $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$, and let (F_n) be a sequence from $\mathfrak{R}$. Let $\epsilon > 0$.

To show that $(E_1 \setminus E_2) \in \mathfrak{R}$, let U and V be Borel open such that $E_1 \subseteq U$, $E_2 \subseteq V$, $\mu(U) \leq \mu(E_1) + \frac{\epsilon}{2}$, and $\mu(V) \leq \mu(E_2) + \frac{\epsilon}{2}$, also, let C and D be compact such that $C \subseteq E_1$, $D \subseteq E_2$, $\mu(E_1) \leq \mu(C) + \frac{\epsilon}{2}$, and $\mu(E_2) \leq \mu(D) + \frac{\epsilon}{2}$. Hence

$$\mu(U) - \mu(D) - \frac{\epsilon}{2} \leq \mu(E_1) - \mu(E_2) + \frac{\epsilon}{2},$$

and

$$\mu(U \setminus D) \leq \mu(E_1 \setminus E_2) + \epsilon.$$

Also

$$\mu(E_1) - \mu(E_2) - \frac{\epsilon}{2} \le \mu(C) - \mu(V) + \frac{\epsilon}{2},$$

and

$$\mu(E_1 \setminus E_2) \le \mu(C \setminus V) + \epsilon.$$

Thus $(E_1 \setminus E_2) \in \mathfrak{R}$.

Now to show that the intersection of (E_n) is in $\mathfrak{R}$, first let us show that a finite intersection of (E_n) is in $\mathfrak{R}$. Let U and V be Borel open such that $E_1 \subseteq U$, $E_2 \subseteq V$, $\mu(U) \le \mu(E_1) + \frac{\epsilon}{2}$, and $\mu(V) \le \mu(E_2) + \frac{\epsilon}{2}$. Hence $\mu(U \setminus E_1) \le \frac{\epsilon}{2}$, and $\mu(V \setminus E_2) \le \frac{\epsilon}{2}$. Note that $(E_1 \cap E_2) \subseteq (U \cap V)$ and

$$(U \cap V) \setminus (E_1 \cap E_2) \subseteq (U \setminus E_1) \cup (V \setminus E_2).$$

Thus

$$\begin{aligned}\mu(U \cap V) - \mu(E_1 \cap E_2) &= \mu[(U \cap V) \setminus (E_1 \cap E_2)] \\ &\le \mu(U \setminus E_1) + \mu(V \setminus E_2) \le \epsilon.\end{aligned}$$

Now let C and D be compact such that $C \subseteq E_1$, $D \subseteq E_2$, $\mu(E_1) \le \mu(C) + \frac{\epsilon}{2}$, and $\mu(E_2) \le \mu(D) + \frac{\epsilon}{2}$. Note that $(C \cap D) \subseteq (E_1 \cap E_2)$ and

$$(E_1 \cap E_2) \setminus (C \cap D) \subseteq (E_1 \setminus C) \cup (E_2 \setminus D).$$

Hence

$$\begin{aligned}\mu(E_1 \cap E_2) - \mu(C \cap D) &= \mu[(E_1 \cap E_2) \setminus (C \cap D)] \\ &\le \mu(E_1 \setminus C) + \mu(E_2 \setminus D) \le \epsilon.\end{aligned}$$

Thus $(E_1 \cap E_2) \in \mathfrak{R}$, and $\left[\bigcap_{i=1}^{n} E_i\right] \in \mathfrak{R}$ for each $n \in \mathbb{N}$.

Now since $\left[\bigcap_{i=1}^{n} E_i\right] \in \mathfrak{R}$ for each $n \in \mathbb{N}$, we may assume that (E_n) is a decreasing sequence. Let $m \in \mathbb{N}$ so that

$\mu(E_m) \leq \mu\left[\bigcap_{i=1}^{\infty} E_i\right] + \frac{\epsilon}{2}$, and let U be Borel open so that $E_m \subseteq U$ and $\mu(U) \leq \mu(E_m) + \frac{\epsilon}{2}$. Hence $\mu(U) \leq \mu\left[\bigcap_{i=1}^{\infty} E_i\right] + \epsilon$, and $\bigcap_{i=1}^{\infty} E_i \subseteq U$. Now for each $n \in \mathbb{N}$, let C_n be compact such that $C_n \subseteq E_n$ and $\mu(E_n) \leq \mu(C_n) + \frac{\epsilon}{2^n}$. Clearly $\bigcap_{i=1}^{\infty} C_i$ is compact, $\bigcap_{i=1}^{\infty} C_i \subseteq \bigcap_{i=1}^{\infty} E_i$, and

$$\left[\bigcap_{i=1}^{\infty} E_i \setminus \bigcap_{i=1}^{\infty} C_i\right] \subseteq \bigcup_{i=1}^{\infty} (E_i \setminus C_i).$$

Hence

$$\mu\left[\bigcap_{i=1}^{\infty} E_i \setminus \bigcap_{i=1}^{\infty} C_i\right] \leq \sum_{i=1}^{\infty} \mu(E_i \setminus C_i) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \leq \epsilon.$$

Thus $\left[\bigcap_{i=1}^{\infty} E_i\right] \in \mathfrak{M}$.

To show that the union of (F_n) is in $\mathfrak{M}$, let us first show that a finite union of (F_n) is in $\mathfrak{M}$. Now if $\mu(F_1) = \infty$ or $\mu(F_2) = \infty$, then clearly

$$\begin{aligned} \mu(F_1 \cup F_2) &= \sup\{\ \mu(C) : C \subseteq (F_1 \cup F_2) \text{ where } C \text{ is compact }\} \\ &= \inf\{\ \mu(U) : (F_1 \cup F_2) \subseteq U \text{ for Borel open } U\ \} \\ &= \infty. \end{aligned}$$

Hence assume that $\mu(F_1) < \infty$ and $\mu(F_2) < \infty$. Let U and V be Borel open such that $F_1 \subseteq U$, $F_2 \subseteq V$, $\mu(U) \leq \mu(F_1) + \frac{\epsilon}{2}$, and $\mu(V) \leq \mu(F_2) + \frac{\epsilon}{2}$, and let C and D be compact such that $C \subseteq F_1$, $D \subseteq F_2$, $\mu(F_1) \leq \mu(C) + \frac{\epsilon}{2}$, and $\mu(F_2) \leq \mu(D) + \frac{\epsilon}{2}$. Note that $(F_1 \cup F_2) \subseteq (U \cup V)$ and

$$(U \cup V) \setminus (F_1 \cup F_2) \subseteq (U \setminus F_1) \cup (V \setminus F_2).$$

Hence

$$\mu[(U \cup V) \setminus (F_1 \cup F_2)] \leq \mu(U \setminus F_1) + \mu(V \setminus F_2) \leq \epsilon.$$

Also note that $(C \cup D) \subseteq (F_1 \cup F_2)$ and

$$(F_1 \cup F_2) \setminus (C \cup D) \subseteq (F_1 \setminus C) \cup (F_2 \setminus D).$$

Hence

$$\mu[(F_1 \cup F_2) \setminus (C \cap D)] \leq \mu(F_1 \setminus C) + \mu(F_2 \setminus D) \leq \epsilon.$$

Thus $(F_1 \cup F_2) \in \mathfrak{N}$, and $\left[\bigcup_{i=1}^{n} F_i\right] \in \mathfrak{N}$ for each $n \in \mathbb{N}$.

Now since $\left[\bigcup_{i=1}^{n} F_i\right] \in \mathfrak{N}$ for each $n \in \mathbb{N}$, assume that (F_n) is an increasing sequence. Note that

$$\mu\left[\bigcup_{i=1}^{\infty} F_i\right] = \lim_{i \to \infty} \mu(F_i).$$

If $\mu\left[\bigcup_{i=1}^{\infty} F_i\right] = \infty$, then clearly

$$\begin{aligned} \mu\left[\bigcup_{i=1}^{\infty} F_i\right] &= \sup\{\ \mu(C) : C \subseteq \left[\bigcup_{i=1}^{\infty} F_i\right] \text{ where } C \text{ is compact }\} \\ &= \inf\{\ \mu(U) : \left[\bigcup_{i=1}^{\infty} F_i\right] \subseteq U \text{ for Borel open } U\ \} \\ &= \infty. \end{aligned}$$

Thus assume that $\mu\left[\bigcup_{i=1}^{\infty} F_i\right] < \infty$. Let $m \in \mathbb{N}$ so that $\mu\left[\bigcup_{i=1}^{\infty} F_i\right] \leq \mu(F_m) + \frac{\epsilon}{2}$; and let C be compact so that $C \subseteq F_m$ and $\mu(F_m) \leq \mu(C) + \frac{\epsilon}{2}$. Hence $\mu\left[\bigcup_{i=1}^{\infty} F_i\right] \leq \mu(C) + \epsilon$, and $C \subseteq \bigcup_{i=1}^{\infty} F_i$. Now for each $n \in \mathbb{N}$, let U_n be Borel open such that $F_n \subseteq U_n$ and $\mu(U_n) \leq \mu(F_n) + \frac{\epsilon}{2^n}$. Clearly $\bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} U_i$, and

$$\bigcup_{i=1}^{\infty} U_i \setminus \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (U_i \setminus F_i).$$

Hence

$$\mu\left[\bigcup_{i=1}^{\infty} U_i \setminus \bigcup_{i=1}^{\infty} F_i\right] \leq \sum_{i=1}^{\infty} \mu(U_i \setminus F_i) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

Thus $\left[\bigcup_{i=1}^{\infty} F_i\right] \in \mathfrak{N}$. ■

Furthermore note that the class of bounded regular Borel sets form a ring. A <u>monotone class</u> $\mathscr{M}$ is a class of sets so that if (E_n) is an increasing sequence of sets in $\mathscr{M}$ and (F_n) is a decreasing sequence of sets in $\mathscr{M}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{M}$ and $\bigcap_{i=1}^{\infty} F_i \in \mathscr{M}$.

<u>Theorem 4.8</u>: If $\mathbf{R}$ is a ring, then $\mathscr{M}(\mathbf{R}) = \sigma(\mathbf{R})$; in other words, the monotone class generated by $\mathbf{R}$ is the σ–ring generated by $\mathbf{R}$.

<u>Proof</u>: Let $\mathbf{R}$ be a ring; clearly $\mathscr{M}(\mathbf{R}) \subseteq \sigma(\mathbf{R})$ since $\sigma(\mathbf{R})$ is a monotone class. Now for $E, F \subseteq G$, "E collaborates with F" means that $E \setminus F \in \mathscr{M}(\mathbf{R})$, $F \setminus E \in \mathscr{M}(\mathbf{R})$, and $E \cup F \in \mathscr{M}(\mathbf{R})$. For each $E \subseteq G$, let

$$K(E) = \{ F : E \text{ collaborates with } F \};$$

clearly $F \in K(E)$ if and only if $E \in K(F)$. Now we assert that if $F \subseteq G$ so that $K(F)$ is not empty, then $K(F)$ is a monotone class. Clearly if $A \in \mathbf{R}$, then $\mathbf{R} \subseteq K(A)$, and $K(A)$ is not empty; hence let $F \subseteq G$ so that $K(F)$ is not empty. Let (E_n) be an increasing sequence of sets from $K(F)$, and (D_n) be a decreasing sequence of sets from $K(F)$. Now for each $n \in \mathbb{N}$, $E_n \setminus F \in \mathscr{M}(\mathbf{R})$, $F \setminus E_n \in \mathscr{M}(\mathbf{R})$, and $E_n \cup F \in \mathscr{M}(\mathbf{R})$. Hence

$$\bigcup_{i=1}^{\infty} (E_i \setminus F) = \left[\bigcup_{i=1}^{\infty} E_i\right] \setminus F \in \mathscr{M}(\mathbf{R}),$$

$$\bigcup_{i=1}^{\infty}(E_i \cup F) = \left[\bigcup_{i=1}^{\infty} E_i\right] \cup F \in \mathscr{M}(\mathrm{R}),$$

and

$$\bigcap_{i=1}^{\infty}(F \setminus E_i) = F \setminus \left[\bigcup_{i=1}^{\infty} E_i\right] \in \mathscr{M}(\mathrm{R})$$

since $(E_n \setminus F)$ and $(E_n \cup F)$ are increasing sequences in $\mathscr{M}(\mathrm{R})$ and $(F \setminus E_n)$ is a decreasing sequence in $\mathscr{M}(\mathrm{R})$. Thus $\left[\bigcup_{i=1}^{\infty} E_i\right] \in K(F)$. Similarly $\left[\bigcap_{i=1}^{\infty} D_i\right] \in K(F)$. Hence $K(F)$ is a monotone class, and our assertion is true.

Thus $\mathscr{M}(\mathrm{R}) \subseteq K(A)$ for every $A \in \mathrm{R}$. Let $E,F \in \mathscr{M}(\mathrm{R})$; hence $E \in K(A)$ and $A \in K(E)$ for every $A \in \mathrm{R}$. Then $\mathrm{R} \subseteq K(E)$ and $\mathscr{M}(\mathrm{R}) \subseteq K(E)$. Thus $F \in K(E)$, and $E \setminus F \in \mathscr{M}(\mathrm{R})$, $F \setminus E \in \mathscr{M}(\mathrm{R})$, and $E \cup F \in \mathscr{M}(\mathrm{R})$. Therefore $\mathscr{M}(\mathrm{R})$ is a σ-ring and $\sigma(\mathrm{R}) = \mathscr{M}(\mathrm{R})$. ■

Further if M is any monotone class so that $\mathrm{R} \subseteq M$, then $\sigma(\mathrm{R}) = \mathscr{M}(\mathrm{R}) \subseteq M$.

__Theorem 4.9__: **The Borel measure μ is a regular Borel measure.**

__Proof__: Let C be compact. Clearly

$$\mu(C) = \sup\{\ \mu(D)\ :\ D \subseteq C \text{ where } D \text{ is compact }\}$$

since μ is monotone. Now

$$\mu(C) = \lambda^*(C) = \inf\{\ \lambda_*(U)\ :\ C \subseteq U \text{ for open Borel } U\ \}$$
$$= \inf\{\ \mu(U)\ :\ C \subseteq U \text{ for open Borel } U\ \}.$$

Thus C is regular with respect to μ, and $C \in \mathfrak{R}$.

Let $\mathscr{C}$ be the ring generated be the class of compact sets; hence $\mathscr{C} \subseteq \mathfrak{R}$ since the class of bounded regular Borel sets form a ring that is contained in $\mathfrak{R}$. Now let

$$\mathfrak{R}' = \{ E \in \mathfrak{B} : C \cap E \in \mathfrak{R} \text{ for all } C \in \mathscr{C} \}.$$

Clearly $\mathscr{C} \subseteq \mathfrak{R}'$ and $\mathfrak{R}'$ is not empty. Now we claim that $\mathfrak{R}'$ is a monotone class. For let (E_n) be a decreasing sequence from $\mathfrak{R}'$, and let (F_n) be an increasing sequence from $\mathfrak{R}'$. Clearly $\bigcap_{i=1}^{\infty} E_i \in \mathfrak{B}$ and $\bigcup_{i=1}^{\infty} F_i \in \mathfrak{B}$. Note that

$$C \cap \left[\bigcup_{i=1}^{\infty} F_i\right] = \bigcup_{i=1}^{\infty} (C \cap F_i) \in \mathfrak{R}$$

for all $C \in \mathscr{C}$ and that

$$C \cap \left[\bigcap_{i=1}^{\infty} E_i\right] = \bigcap_{i=1}^{\infty} (C \cap E_i) \in \mathfrak{R}$$

for all $C \in \mathscr{C}$ since each $\mu(C \cap E_n) < \infty$. Thus $\bigcap_{i=1}^{\infty} E_i \in \mathfrak{R}'$, $\bigcup_{i=1}^{\infty} F_i \in \mathfrak{R}'$, $\mathfrak{R}'$ is a monotone class, and the claim holds.

Hence $\mathfrak{R}' = \mathfrak{B}$ since a monotone class that contains a ring also contains the σ–ring generated by the ring. Now let $E \in \mathfrak{B}$; clearly E is σ–bounded. Hence let (C_n) be a sequence of compact sets such that $E \subseteq \bigcup_{i=1}^{\infty} C_i$. Note that $E = \bigcup_{i=1}^{\infty} (E \cap C_i)$. Thus $E \in \mathfrak{R}$ since each $(E \cap C_n) \in \mathfrak{R}$. Therefore $\mathfrak{B} = \mathfrak{R}$, and μ is a regular Borel measure. ■

Theorem 4.10: The set function μ is the unique regular Borel measure that extends λ.

<u>Proof</u>: Let C be compact and $\epsilon > 0$. Now let D be compact and U be open so that $C \subseteq U \subseteq D$ and $\lambda(D) \leq \lambda(C) + \epsilon$ since λ is a regular content. Hence

$$\lambda^*(C) \leq \lambda^*(U) \leq \lambda^*(D)$$

since λ^* is monotone. Note that $\lambda(C) \leq \lambda^*(C)$ and $\lambda(D) \leq \lambda^*(D)$ as shown before. Thus, by Theorem 4.3, $\lambda^*(C) \leq \lambda(C) + \epsilon$, and $\lambda^*(C) \leq \lambda(C)$. Therefore

$$\lambda(C) = \lambda^*(C) = \mu(C),$$

and μ extends λ.

Now to show that μ is unique, suppose that ρ is also a regular Borel measure that extends λ, but $\rho \neq \mu$. Let $E \in \mathfrak{B}$ so that $\rho(E) \neq \mu(E)$. Assume for now that $\rho(E) < \mu(E) < \infty$; let $\delta = \mu(E) - \rho(E)$. Now let $A \subseteq E$ be compact so that $\mu(E) < \mu(A) + \delta$. Hence

$$\mu(E) < \mu(A) + \mu(E) - \rho(E),$$

and

$$\rho(E) < \mu(A) = \rho(A);$$

which is a contradiction since $A \subseteq E$. We attain a similar contradiction if we assume that $\mu(E) < \rho(E) < \infty$.

Now assume, without loss of generality, that $\rho(E) < \mu(E) = \infty$. Let $B \subseteq E$ be compact so that $\mu(B) > \rho(E)$ since $\mu(E) = \infty$. Hence $\rho(E) < \mu(B) = \rho(B)$; which is a contradiction. Therefore μ is unique. ∎

The statement that a real-valued function f on G is

<u>measurable</u> means that

$$f^{-1}[(a,\infty)] \setminus f^{-1}(0) \in \mathfrak{B}$$

for all $a \in \mathbb{R}$. Observe that every function in $\mathscr{L}$ is measurable. For a measurable function g so that $g \geq 0$, define $\int g \, d\mu$ by

$$\int g \, d\mu = \sup\{ \sum_{i=1}^{n} \alpha_i \mu(F_i) \ : \ \sum_{i=1}^{n} \alpha_i \aleph_{F_i} \leq g \text{ where } n \in \mathbb{N}, \{\alpha_1,\ldots,\alpha_n\} \subseteq \mathbb{R}, \text{ and } \{F_1,\ldots,F_n\} \text{ is a collection of disjoint Borel sets } \}.$$

Clearly if $g = 0$, then $\int g \, d\mu = 0$, and if f is measurable, then f^+ and f^- are measurable, $f^+ \geq 0$, and $f^- \geq 0$. Further, if $f \in \mathscr{L}$, then $f^+ \in \mathscr{L}$ and $f^- \in \mathscr{L}$. Now for a measurable function f, define $\int f \, d\mu$ to be

$$\int f^+ d\mu - \int f^- d\mu.$$

A measurable function f is said to be <u>integrable</u> if $\int f \, d\mu < \infty$. Clearly if $f \in \mathscr{L}$, then f is integrable since f has compact support. Observe that $\int \cdot \, d\mu$ is linear on $\mathscr{L}$, e.g., see Royden [3, p. 267]. We now begin in the process of establishing that μ represents I.

<u>Lemma</u> <u>4.11</u>: If C is compact and $\epsilon > 0$, then there exists $f \in \mathscr{P}$ so that $\aleph_C \leq f$ and

$$I(f) \leq \int f \, d\mu + \epsilon.$$

<u>Proof</u>: Let C be compact and $\epsilon > 0$. Now let $f \in \mathscr{P}$ so

that $\aleph_C \leq f$ and $I(f) \leq \lambda(C) + \epsilon$. Hence

$$I(f) \leq \lambda(C) + \epsilon = \mu(C) + \epsilon \leq \int f \, d\mu + \epsilon$$

since μ extends λ and $\aleph_C \leq f$. ■

Lemma 4.12: **If $f \in \mathscr{P}$, then**

$$\int f \, d\mu \leq I(f).$$

Proof: Let $f \in \mathscr{P}$ and $\epsilon > 0$. Now let $n \in \mathbb{N}$, $\{\alpha_1,\ldots,\alpha_n\} \subseteq \mathbb{R}$ such that each $\alpha_i > 0$, and $\{F_1,\ldots,F_n\}$ be a finite collection of Borel sets so that $\sum_{i=1}^{n} \alpha_i \aleph_{F_i} \leq f$ and

$$\int f \, d\mu \leq \sum_{i=1}^{n} \alpha_i \mu(F_i) + \frac{\epsilon}{2}.$$

Note that each F_i is bounded since f has compact support. Before continuing, we need the following result.

Sublemma: There exists a finite collection $\{C_1,\ldots,C_n\}$ of compact sets so that each $C_i \subseteq F_i$ and

$$\left| \sum_{i=1}^{n} \alpha_i \mu(F_i) - \sum_{i=1}^{n} \alpha_i \mu(C_i) \right| \leq \frac{\epsilon}{2}.$$

Proof of sublemma: Let

$$d = 1 + \max\{ \alpha_i : 1 \leq i \leq m \}.$$

Since μ is regular, let each C_i be a compact such that $C_i \subseteq F_i$ and $\mu(F_i \setminus C_i) \leq \frac{\epsilon}{d \cdot n \cdot 2}$. Clearly the C_i's are disjoint. Thus

$$\left| \sum_{i=1}^{n} \alpha_i \mu(F_i) - \sum_{i=1}^{n} \alpha_i \mu(C_i) \right| = \sum_{i=1}^{n} \alpha_i \mu(F_i \setminus C_i)$$

$$\leq \sum_{i=1}^{n} d\cdot\frac{\epsilon}{d\cdot n\cdot 2} = \frac{\epsilon}{2},$$

and the sublemma is true.

Now let $\{C_1,\ldots,C_n\}$ be a collection of disjoint compact sets that satisfies the conclusion of the sublemma; clearly $f \geq \sum_{i=1}^{n} \alpha_i \aleph_{C_i}$ and

$$\int f\ d\mu \leq \sum_{i=1}^{n} \alpha_i \mu(C_i) + \epsilon.$$

Now let $\{U_1,\ldots,U_n\}$ be a collection of disjoint open sets so that each $C_i \subsetneq U_i$, and let $h_1,\ldots,h_n \in \mathscr{L}$ so that $0 \leq h_i \leq 1$, $h_i(C_i) = \{1\}$, and $h_i(G \setminus U_i) = \{0\}$ for $1 \leq i \leq n$. Hence $\sum_{i=1}^{n} h_i \leq 1$ since the U_i's are disjoint. Note that $\alpha_i \aleph_{C_i} \leq h_i f$ for $1 \leq i \leq n$. Thus $\alpha_i \lambda(C_i) \leq I(h_i f)$, and $\alpha_i \mu(C_i) \leq I(h_i f)$ for $1 \leq i \leq n$. Hence

$$\sum_{i=1}^{n} \alpha_i \mu(C_i) \leq \sum_{i=1}^{n} I(h_i f) = I\left[f \sum_{i=1}^{n} h_i\right] \leq I(f).$$

Therefore $\int f\ d\mu \leq I(f) + \epsilon$, and $\int f\ d\mu \leq I(f)$ since ϵ is arbitrary. ∎

Theorem 4.13: The regular Borel measure μ uniquely represents I; in other words, μ is unique so that

$$I(f) = \int f\ d\mu$$

for all $f \in \mathscr{L}$.

Proof: Let $f \in \mathscr{L}$. Since $\int \cdot\ d\mu$ and I are linear on $\mathscr{L}$, we may assume that $0 \leq f \leq 1$. Hence $\int f\, d\mu \leq I(f)$. Let $\epsilon > 0$, and let C be compact so that C supports f. Now apply Lemma 4.11 and let $g \in \mathscr{P}$ so that $\aleph_C \leq g$ and $I(g) \leq \int g\, d\mu + \epsilon$. Clearly $f \leq \aleph_C \leq g$; hence $g - f = 0$ or $g - f \in \mathscr{P}$. Thus $\int (g - f) d\mu \leq I(g - f)$,

$$\int g\, d\mu - \int f\, d\mu \leq I(g) - I(f),$$

and

$$\begin{aligned} I(f) &\leq I(g) - \int g\, d\mu + \int f\, d\mu \\ &\leq \int f\, d\mu + \epsilon. \end{aligned}$$

Hence $I(f) = \int f\, d\mu$.

To show that μ is unique, suppose that ρ is a regular Borel measure so that $I(f) = \int f\, d\rho$ for all $f \in \mathscr{L}$, but $\rho \neq \mu$. Since μ and ρ are regular, let A be compact so that $\rho(A) \neq \mu(A)$; and let $\epsilon = |\ \mu(A) - \rho(A)\ |$. Now let U be an open Borel set so that $A \subseteq U$, $\mu(U) \leq \mu(A) + \frac{1}{2}\epsilon$, and $\rho(U) \leq \rho(A) + \frac{1}{2}\epsilon$. Let $h \in \mathscr{P}$ so that $0 \leq h \leq 1$, $h(A) = \{1\}$, and $h(G \setminus U) = \{0\}$. Hence $\aleph_A \leq h \leq \aleph_U$. Now

$$\mu(A) \leq \int h\, d\mu = I(h) \leq \mu(U),$$

and

$$\rho(A) \leq \int h\, d\rho = I(h) \leq \rho(U).$$

Hence $|\ I(h) - \mu(A)\ | < \frac{1}{2}\epsilon$, and $|\ I(h) - \rho(A)\ | < \frac{1}{2}\epsilon$. Thus $|\ \mu(A) - \rho(A)\ | < \epsilon$, contradicting the definition of ϵ. Therefore μ is unique. ■

Now we are ready to show that μ is a Haar measure on G.

Theorem 4.14: The regular Borel measure μ is a Haar measure on G.

Proof: Since I is not zero, μ is not zero. Suppose that μ is not a Haar measure; thus μ would not be translation invariant. Since μ is regular, let C be compact and $s \in G$ so that $\mu(C + \{s\}) \neq \mu(C)$. Let

$$\epsilon = | \mu(C + \{s\}) - \mu(C) | > 0.$$

Note that $C + \{s\}$ is compact. Let $g,h \in \mathscr{P}$ so that $\aleph_C \leq g$, $\aleph_{C + \{s\}} \leq h$, $I(g) \leq \lambda(C) + \frac{1}{2}\epsilon$, and $I(h) \leq \lambda(C + \{s\}) + \frac{1}{2}\epsilon$. Now let $f = \min\{ g, h_{(-s)} \}$ Clearly $\aleph_C \leq f$, $\aleph_{C + \{s\}} \leq f_s$, $I(f) \leq \lambda(C) + \frac{1}{2}\epsilon$, and $I(f_s) \leq \lambda(C + \{s\}) + \frac{1}{2}\epsilon$. Since $I(f) = I(f_s)$,

$$| \lambda(C) - \lambda(C + \{s\}) | < \epsilon,$$

and

$$| \mu(C) - \mu(C + \{s\}) | < \epsilon;$$

contradicting the definition of ϵ. Thus μ is translation invariant. Therefore μ is a Haar measure on G. ■

Finally we are ready to show that μ is unique up to a positive scalar.

Theorem 4.15: If ρ is a Haar measure on G, then $\rho = c \cdot \mu$ for some $c > 0$.

<u>Proof</u>: Let ρ be a Haar measure on G. Let J be a positive linear form on $\mathscr{L}$ so that $J(f) = \int f\, d\rho$ for all $f \in \mathscr{L}$. Clearly J is linear, positive, and not zero since ρ is not zero. Let $f \in \mathscr{L}$, and $s \in G$. Now we assert that $\int f_s d\rho = \int f\, d\rho$. For if not, then assume without loss of generality that $\int f\, d\rho > \int f_s d\rho$, and let

$$\epsilon = \int f\, d\rho - \int f_s d\rho;$$

further we may assume that $f \geq 0$. Now let $n \in \mathbb{N}$, $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{R}$, and $\{F_1, \ldots, F_n\}$ be a collection of disjoint Borel sets so that $\sum_{i=1}^{n} \alpha_i \aleph_{F_i} \leq f$ and

$$\int f\, d\rho < \sum_{i=1}^{n} \alpha_i \rho(F_i) + \epsilon.$$

Note that $\sum_{i=1}^{n} \alpha_i \aleph_{(F_i + \{s\})} \leq f_s$, and

$$\sum_{i=1}^{n} \alpha_i \rho(F_i) = \sum_{i=1}^{n} \alpha_i \rho(F_i + \{s\})$$

since ρ is translation invariant. Thus

$$\int f\, d\rho < \sum_{i=1}^{n} \alpha_i \rho(F_i + \{s\}) + \int f\, d\rho - \int f_s d\rho,$$

and

$$\int f_s d\rho < \sum_{i=1}^{n} \alpha_i \rho(F_i + \{s\});$$

which is a contradiction. Hence $\int f_s d\rho = \int f\, d\rho$, and $J(f_s) = J(f)$. Thus J is a Haar integral.

Now let $c > 0$ so that $J = c \cdot I$. Hence $J(g) = c \cdot I(g)$,

$$\int g\, d\rho = c \cdot \int g\, d\mu,$$

and

$$\int g\, d\rho = \int g\, d(c \cdot \mu)$$

for all $g \in \mathscr{L}$. Therefore $\rho = c \cdot \mu$ since ρ is the unique regular Borel measure so that $J(g) = \int g \, d\rho$ for all $g \in \mathscr{L}$.
■

Observe that Lebesgue measure ν is certainly a Haar measure on $\mathbb{R}$, but $\nu|_{\mathfrak{C}} = 0$. The construction in chapters III and IV produces a translation invariant regular Borel measure ρ on $\mathfrak{C}$ so that $\rho(\mathfrak{C}) = 1$. Note that

$$\rho(\{\, x \in \mathfrak{C} : a_1 = 0 \text{ where } (a_n) \text{ is the ternary expansion of } x \text{ so that each } a_n \neq 1 \,\}) = \tfrac{1}{2},$$

$$\rho(\{\, x \in \mathfrak{C} : a_1 = 2 \text{ where } (a_n) \text{ is the ternary expansion of } x \text{ so that each } a_n \neq 1 \,\}) = \tfrac{1}{2},$$

and

$$\rho(\{\, x \in \mathfrak{C} : a_1 = 0 \text{ and } a_2 = 0 \text{ where } (a_n) \text{ is the ternary expansion of } x \text{ so that each } a_n \neq 1 \,\}) = \tfrac{1}{4}$$

since ρ is additive and translation invariant.

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