HYPERSPACES

THESIS

Presented to the Graduate Council of the North Texas State University in Partial Fulfillment of the Requirements

For the Degree of

MASTER OF SCIENCE

By

Charles H. Voas, B. A.
Denton, Texas
December, 1976

This paper is an exposition of the theory of the hyperspaces $2^X$ and $C(X)$ of a topological space $X$. These spaces are obtained from $X$ by collecting the nonempty closed and nonempty closed connected subsets respectively, and are topologized by the Vietoris topology.

The paper is organized in terms of increasing specialization of spaces, beginning with $T_1$ spaces and proceeding through compact spaces, compact metric spaces and metric continua. Several basic techniques in hyperspace theory are discussed, and these techniques are applied to elucidate the topological structure of hyperspaces.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. HYPERSPACES OF GENERAL SPACES</td>
<td>1</td>
</tr>
<tr>
<td>II. HYPERSPACES OF A COMPACT SPACE</td>
<td>17</td>
</tr>
<tr>
<td>III. PROPERTIES OF CONTINUA</td>
<td>29</td>
</tr>
<tr>
<td>IV. HYPERSPACES OF A COMPACT METRIC SPACE</td>
<td>45</td>
</tr>
<tr>
<td>V. HYPERSPACES OF A METRIC CONTINUUM</td>
<td>54</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>79</td>
</tr>
</tbody>
</table>
CHAPTER I

HYPERSPACES OF GENERAL SPACES

For a topological space $X$, the hyperspace of $X$, denoted by $2^X$, is the collection of all nonempty closed subsets of $X$. Before defining a topology on $2^X$, we shall establish the following notation: for a finite collection \( \{A_i: i=1,2,\ldots,n\} \) of subsets of $X$, \( \langle A_i \rangle_{i=1}^n = \{K \in 2^X: K \cap A_i \neq \emptyset \text{ for each } i = 1,2,\ldots,n \text{ and } K \subset \bigcup_{i=1}^n A_i \} \). Note that $\langle A \rangle$ is the subcollection of $2^X$ consisting of subsets of $A$, and $\langle X,A \rangle$ consists of those elements of $2^X$ which meet $A$. Now the totality of sets of the form $\langle U \rangle$ or $\langle X,V \rangle$, where $U$ and $V$ are open in $X$, forms a subbase for the Vietoris, or finite, topology on $2^X$.

For notational clarity, we shall consistently employ small letters to denote points of $X$, capital letter to denote subsets of $X$ and points of $2^X$, and script letters to denote subsets of $2^X$. We note further that one may form higher order hyperspaces by collecting the closed subsets of $2^X$, and we denote this space by $2^{2^X}$. For a set $A$, $\overline{A}$ will denote the closure of $A$, $A^0$ the interior of $A$, and $B(A)$ the boundary of $A$. Finally, we wish to emphasize that ALL SPACES CONSIDERED IN THIS PAPER ARE ASSUMED TO SATISFY THE $T_1$ SEPARATION AXIOM. Hence finite
subsets of X are closed and are therefore elements of 
2X.

Theorem 1.1. The collection of all sets of the 
form \( \langle U_i \rangle_{i=1}^n \), where \( U_i \) is open in X, forms a base for 
the Vietoris topology on \( 2^X \).

Proof. Let \( \bigcap_{i=1}^n \langle U_i \rangle \bigcap \bigcap_{j=1}^m \langle X, V_j \rangle = \emptyset \) be a basic open 
set of \( 2^X \), and let \( A \) be an element of \( \emptyset \). Let \( U = \bigcap_{i=1}^n U_i \)
and \( W_j = V_j \cap U \). Then we have \( A \in \langle U, W_1, \ldots, W_m \rangle \subset \emptyset \).

Theorem 1.2. Let \( A_i \subset X \), for \( i = 1, 2, \ldots, n \). Then
\( \langle A_i \rangle_{i=1}^n = \langle \overline{A_i} \rangle_{i=1}^n \).

Proof. We first show that \( \langle \overline{A_i} \rangle_{i=1}^n \) is closed in \( 2^X \).
Hence let \( K \in 2^X - \langle \overline{A_i} \rangle_{i=1}^n \). Then either \( K \notin \bigcup_{i=1}^n \overline{A_i} \), or
\( K \cap \overline{A_j} = \emptyset \) for some \( j \). In the former case, there is an 
open set \( U \) which meets \( K \) but is disjoint from \( \bigcup_{i=1}^n \overline{A_i} \).
Then \( \langle X, U \rangle \) is a neighborhood of \( K \) which does not meet
\( \langle \overline{A_i} \rangle_{i=1}^n \). In the latter case, \( \langle X - \overline{A_j} \rangle \) is a neighborhood 
of \( K \) which does not meet \( \langle \overline{A_i} \rangle_{i=1}^n \). Hence \( \langle \overline{A_i} \rangle_{i=1}^n \) is closed 
in \( 2^X \), and since \( \langle A_i \rangle_{i=1}^n \subset \langle \overline{A_i} \rangle_{i=1}^n \), we have \( \langle A_i \rangle_{i=1}^n \subset \langle \overline{A_i} \rangle_{i=1}^n \).

Now let \( K \in \langle \overline{A_i} \rangle_{i=1}^n \), and let \( \langle U_j \rangle_{j=1}^m \) be a neighborhood 
of \( K \). Let \( U = \bigcup_{j=1}^m U_j \) and \( A = \bigcup_{i=1}^n \overline{A_i} \). For each \( i = 1, 2, \ldots, n \),
there is a point \( x_i \in A_i \cap K \subseteq A_i \cap U \), so there is a point \( y_i \in A_i \cap U \). For each \( j = 1, 2, \ldots, m \), there is a point \( z_j \in K \cap U_j \subseteq U_j \cap A \), so \( z_j \in U_j \cap A_k \) for some \( k \). Hence there is a point \( w_j \in U_j \cap A_k \). Now if \( F = \{y_1, \ldots, y_m, w_1, \ldots, w_m\} \), we have \( F \in \langle A_i \rangle_{i=1}^n \cap \langle U_j \rangle_{j=1}^m \). It follows that \( K \in \langle A_i \rangle_{i=1}^n \).

**Corollary 1.3.** Let \( A \subseteq 2^X \). Then \( \langle A \rangle \), \( \langle X, A \rangle \), and the set \( K = \{F \in 2^X : F \supseteq A \} \) are each closed in \( 2^X \).

**Proof.** The first two assertions are immediate consequences of the preceding theorem. Let \( H \in 2^X - K \). Then \( A \not\subseteq H \), so there exists a point \( x \in A - H \). Now \( \langle X - x \rangle \) is a neighborhood of \( H \) which does not meet \( K \).

**Theorem 1.4.** The map \( f : X \to 2^X \) by \( f(x) = \{x\} \) is an embedding.

**Proof.** If \( U \) is open in \( X \), \( f(U) = \langle U \rangle \cap f(X) \) which is open in \( f(X) \), so \( f \) is an open map. Also \( f^{-1}(\langle U \rangle) = f^{-1}(\langle X, U \rangle) = U \), so since inverse images of subbasic open sets are open, \( f \) is continuous. Since \( f \) is clearly one to one, \( f \) is an embedding.

We shall henceforward denote the image of \( X \) under the above embedding by \( \hat{X} \) and call this set the base of \( 2^X \).

**Theorem 1.5.** \( X \) is Hausdorff if and only if \( \hat{X} \) is closed in \( 2^X \).

**Proof.** If \( \hat{X} \) is closed in \( 2^X \), and \( x_1 \) and \( x_2 \) are distinct points of \( X \), then there is a basic open set
\( \{U_i\}_{i=1}^n \) which contains \( \{x_1, x_2\} \) but does not meet \( \hat{X} \). If \( V_1 \) is the intersection of the \( U_i \)'s which contain \( x_1 \), and \( V_2 \) the intersection of the \( U_i \)'s which contain \( x_2 \), then \( V_1 \) and \( V_2 \) are disjoint open sets, since any point in their intersection would be in \( \{U_i\}_{i=1}^n \). Hence \( X \) is Hausdorff.

Conversely, assume \( X \) is Hausdorff and \( A \in 2^X - \hat{X} \). Then there are disjoint sets \( U_1 \) and \( U_2 \) so \( U_i \cap A \neq \emptyset \) for \( i = 1, 2 \). Then \( \langle X, U_1, U_2 \rangle \) is an open set containing \( A \) which does not meet \( \hat{X} \).

**Theorem 1.6.** The collection \( \mathcal{F} \) of finite subsets of \( X \) is dense in \( 2^X \).

**Proof.** Let \( \{U_i\}_{i=1}^n \) be a basic open set and for each \( i \), choose \( x_i \in U_i \). Then \( \{x_1, \ldots, x_n\} \in \mathcal{F} \cap \{U_i\}_{i=1}^n \).

**Theorem 1.7.** \( 2^X \) is separable if and only if \( X \) is separable.

**Proof.** If \( X \) is separable, and \( D \) is a countable dense subset of \( X \), the collection of all finite subsets of \( D \) is a countable dense subset of \( 2^X \).

Conversely, let \( \mathcal{D} \) be a countable dense subset of \( 2^X \) and \( x_i \in A_i \) for each \( A_i \in \mathcal{D} \). Then if \( U \) is open in \( X \), there is \( A_i \in \{U\} \), so \( x_i \in U \). Hence \( \{x_i : i = 1, 2, \ldots\} \) is dense in \( X \).

**Theorem 1.8.** \( 2^X \) is \( T_1 \). If \( X \) is \( T_3 \), then \( 2^X \) is \( T_2 \).

**Proof.** Let \( A, B \in 2^X \) so \( A \neq B \). Then this is \( x \in B - A \),
say. Then \( (X - x) \) is a neighborhood of \( A \) which does not contain \( B \), while \( (X, X - A) \) is a neighborhood of \( B \) but not \( A \). Hence \( 2^X \) is \( T_1 \).

If in addition \( X \) is regular, let \( U \) and \( V \) be disjoint open sets so \( x \in U \) and \( A \subset V \). Then \( (X, U) \) and \( (V) \) are the required disjoint open sets to make \( 2^X \) Hausdorff.

Among the many ways in which one may view the space \( 2^X \) is as a topological semigroup (= a semigroup with a jointly continuous multiplication). This is a consequence of the following theorem, which incidentally implies that connectedness is an invariant of the hyperspace construction.

**Theorem 1.9.** Let \( n \) be a positive integer, and denote by \( 2^X_n \) the Cartesian product \( \bigoplus_{i=1}^n 2^X \). Define \( f: 2^X_n \to 2^X \) by union; that is, \( f((A_1, \ldots, A_n)) = \bigcup_{i=1}^n A_i \). Then \( f \) is continuous.

**Proof.** It is evident that \( f \) maps into \( 2^X \). We shall show that \( f \) is continuous by showing that inverse images of subbasic open sets are open. Hence let \( U \) be open in \( X \), then:

\[
\begin{align*}
 f^{-1}(\langle U \rangle) &= \{(A_1, \ldots, A_n) \in 2^X_n : \bigcup_{i=1}^n A_i \subset U \} \\
 &= \{(A_1, \ldots, A_n) \in 2^X_n : A_i \subset U \text{ for each } i \} \\
 &= \bigoplus_{i=1}^n \langle U_i \rangle \text{ where } U_i = U \text{ for each } i.
\end{align*}
\]
Similarly:

\[ f^{-1}(\langle X, U \rangle) = \{(A_1, \ldots, A_n) \in 2^X: \bigcup_{i=1}^n A_i \text{ meets } U\} \]

\[ = \{(A_1, \ldots, A_n) \in 2^X: A_j \text{ meets } U \text{ for some } j\} \]

\[ = \bigcup_{i=1}^n \pi_i^{-1}(\langle X, U \rangle), \] where \( \pi_i \) denotes the \( i \)-th projection map onto \( \mathbb{2}^X \). Since this set is open in \( \mathbb{2}^n \), the theorem is proved.

**Corollary 1.9a.** Under union, \( \mathbb{2}^X \) is a commutative topological semigroup.

**Corollary 1.9b.** For each position integer \( n \), the map \( \tau_n : X^n \rightarrow \mathbb{2}^X \) by \( \tau_n((x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\} \) is continuous.

**Proof.** The map \( \tau_n \) may be regarded as the restriction of the union map to the subspace \( \prod_{i=1}^n \mathbb{2}X(X) \) of \( \mathbb{2}^X \), and is therefore continuous.

**Theorem 1.10.** The space \( \mathbb{2}^X \) is connected if and only if \( X \) is connected.

**Proof.** If \( X \) is not connected, let \( U \) and \( V \) be non-empty disjoint open sets whose union is \( X \). Then \( \langle U \rangle \) and \( \langle X, V \rangle \) separate \( \mathbb{2}^X \). Hence assume that \( X \) is connected. Then for each \( n \), \( X^n \) is connected, and the function \( \tau_n \) of Corollary 1.9b maps onto the subset \( \mathcal{F}_n \) of \( \mathbb{2}^X \) consisting of all elements of \( \mathbb{2}^X \) of cardinality \( \leq n \). Since \( \tau_n \) is continuous, we conclude that \( \mathcal{F}_n \) is connected.
Furthermore, \( \bigcap_{n=1}^{\infty} F_n = \hat{X} \neq \emptyset \), and so \( \bigcup_{n=1}^{\infty} F_n = \mathcal{F} \) is connected. But by Theorem 1.6, \( \mathcal{F} \) is dense in \( 2^X \), so \( 2^X \) is connected.

The subset relation induces a natural partial order on \( 2^X \) which is particularly useful in elucidating its structure. All order theoretic terminology refers to the order \( A \leq B \) iff \( A \subseteq B \).

**Theorem 1.11.** If \( \mathcal{C} \) is a chain in \( 2^X \), then \( S = \bigcup \mathcal{C} \) is in \( \mathcal{C} \), and \( S \) is the least upper bound of \( \mathcal{C} \).

**Proof.** It is clear that \( S \) is the smallest closed set containing each element of \( \mathcal{C} \), and so we show that \( S \in \mathcal{C} \). Hence let \( \langle U_i \rangle_{i=1}^{n} \) be a basic open set containing \( S \). Then for each \( i = 1,2,\ldots,n \), there exists a point \( x_i \in S \cap U_i \), and since \( S \) is the closure of \( \bigcup \mathcal{C} \), and \( U_i \) is open, there exists \( C_i \in \mathcal{C} \), so \( C_i \cap U_i \neq \emptyset \). But since \( \mathcal{C} \) is a chain, one can choose a \( C_j \in \mathcal{C} \) which is the largest of the sets \( C_i, i = 1,2,\ldots,n \). Then \( C_j \in \langle U_i \rangle_{i=1}^{n} \cap \mathcal{C} \), so \( S \in \mathcal{C} \).

The corresponding assertion for greatest lower bounds requires the added assumption of compactness, since without this property, the intersection of a chain of closed sets may be empty. We defer the statement and proof of this theorem until Chapter II.

**Theorem 1.12.** Let \( \mathcal{F} \) be a closed subset of \( 2^X \), and \( \mathcal{C} \) be a maximal chain in \( \mathcal{F} \). Then \( \mathcal{C} \) is closed in \( 2^X \).
Proof. Let \( A \in 2^X - \mathcal{C} \). Then if \( A \notin \mathcal{F} \), \( 2^X - \mathcal{F} \) is an open set containing \( A \) which does not meet \( \mathcal{C} \). Hence assume \( A \in \mathcal{F} \). Then there exists a set \( K \in \mathcal{C} \) so \( A \) and \( K \) are not comparable, since otherwise one could extend the chain \( \mathcal{C} \) by adjoining \( A \). Hence we have \( A - K \neq \emptyset \) and \( K - A \neq \emptyset \). Let \( p \in K - A \) and consider the open set \( (X - p, X - K) \). This set contains \( A \), but contains no set which is comparable to \( K \), hence no set in \( \mathcal{C} \). It follows that \( 2^X - \mathcal{C} \) is open, and so \( \mathcal{C} \) is closed in \( 2^X \).

Recall that if \((X, \leq)\) is a totally ordered set, the order topology on \( X \) is the topology generated by the intervals \((-\infty, a) = \{x \in X : x \leq a \text{ and } x \neq a\} \), and \((a, \infty) = \{x \in X : a \leq x \text{ and } a \neq x\} \), for \( a \in X \).

Theorem 1.13. If \( X \) is regular, and \( \mathcal{C} \) is a chain in \( 2^X \), then the order topology on \( \mathcal{C} \) is weaker than the relative Vietoris topology on \( \mathcal{C} \).

Proof. Let \( A \in \mathcal{C} \), and assume that \((-\infty, A) \) and \((A, \infty) \) are nonempty. Then the regularity of \( X \) implies that if \( K \in (-\infty, A) \), there is an open set \( U \) so that \( K \subset U \) and \( A \notin U \). Then \( K \in (U) \cap \mathcal{C} \subset (-\infty, A) \). Also, if \( K \in (A, \infty) \), we have \( K \in (X, X - A) \cap \mathcal{C} \subset (A, \infty) \). Hence each of \((A, \infty) \) and \((-\infty, A) \) are open in the relative Vietoris topology on \( \mathcal{C} \), and so the theorem is proved.

We shall give an example below to illustrate that the order topology may be strictly weaker than the Vietoris topology, even for maximal chairs in \( 2^X \). The equality of
these two topologies is explored in Chapter II in the presence of compactness.

Even for very simple spaces, $2^X$ is a large and unwieldy space. We shall see later, for example, that if $X = [0,1]$, then $2^X$ contains topologically the Hilbert Cube, and hence each separable metric space. A more tractable hyperspace is the subspace $C(X)$ of $2^X$ consisting of connected sets.

Theorem 1.14. If $X$ is normal, then $C(X)$ is closed in $2^X$.

Proof. Let $A \in 2^X - C(X)$. Then $A$ is not connected, and since $A$ is closed, there are disjoint closed sets $M$ and $N$ so $M \cup N = A$. Since $X$ is normal, there are disjoint open sets $U$ and $V$ so $M \subseteq U$ and $N \subseteq V$. Then we have $A \in \langle U, V \rangle$, while $\langle U, V \rangle \cap C(X) = \emptyset$. Hence $2^X - C(X)$ is open, so $C(X)$ is closed.

We should note that the converse of this theorem is false, as may be seen by exhibiting a non-normal totally disconnected Hausdorff space, such as $X = S \times S$, where $S$ is the Sorgenfrey line. In this case, $C(X) = \dot{X}$ is closed by Theorem 1.5.

Unlike the corresponding result for $2^X$, the connectedness of $X$ is not sufficient to insure the connectedness of $C(X)$. There are at least two reasons for this phenomenon. In the first place, there exist connected Hausdorff spaces which are locally totally
disconnected [1] and in this situation, the base \( \hat{X} \) is a proper clopen subset of \( C(X) \). A property of \( X \) which is calculated to remedy this problem is called property A. A space \( X \) has property A provided there exists a point \( p \in X \) so that each neighborhood of \( p \) contains a non-degenerate closed connected set.

**Theorem 1.15.** A space \( X \) fails to have property A if and only if \( \hat{X} \) is open in \( C(X) \).

**Proof.** Note that \( X \) does not have property A if and only if each point \( x \) of \( X \) has a neighborhood \( U_x \) which contains no nondegenerate element of \( C(X) \). Then

\[
\hat{X} = \bigcup \{ \langle U_x \rangle : x \in X \} \cap C(X)
\]

which is open in \( C(X) \).

Hence if \( X \) is a connected Hausdorff space, property A is necessary for \( C(X) \) to be connected. That it is not sufficient will be seen from an example given below.

We first state and prove two facts about the "explosion point" space of Knaster and Kuratowski. This space is defined as follows. In the plane, let \( C \) be the Cantor set situated in the interval \([0,1]\) of the x-axis. Let \( p \) be the point \( \left( \frac{1}{2}, \frac{1}{2} \right) \), and let \( L(x) \) be the line segment joining \( x \) to \( p \), for each \( x \in C \). Let \( X \) be the union of the sets \( L(x) \), for \( x \in C \). Let \( C = P \cup Q \), where \( Q \) is the set of endpoints of removed intervals, and \( P \) is its complement in \( C \). The explosion point space is the set \( K \) of all
(a,b) ∈ X = ∪L(x) where b is rational iff x ∈ Q and b is irrational iff x ∈ P. Note that $\overline{A}$ and $B(A)$ mean closure and boundary in X, not in K. The following lemma readily implies that K is connected.

**Lemma 1.16a.** If $H ∈ 2^K$, $p ∈ H$, and U is open in X so $H ⊆ U$, then the set $F = \{x ∈ C:L(x) ∩ \overline{H} ∩ B(U) ≠ ∅\}$ is of the first category in $C$.

**Proof.** Let $\{r_i:i = 1,2,\ldots\}$ be an enumeration of the rationals in $[0,\frac{1}{2})$, and define $G_i = \{(a,r_i) ∈ \overline{H} ∩ B(U): (a,r_i) ∈ L(x) for some x ∈ P\}$. Now if $x ∈ P ∩ F$, there is a point $g ∈ L(x) ∩ \overline{H} ∩ (U)$. Since U is open in X, $g \notin U$, so $g \notin H$. But $H = \overline{H} ∩ K$, so evidently $g \notin K$. It follows that the ordinate of $g$ is rational, and since $g ≠ p$, we have $g ∈ G_i$ for some i. Now if $F_i$ denotes the image of the radial projection from $p$ of $G_i$ into the Cantor set C, we have shown that $P ∩ F ⊆ \bigcup_{i=1}^{∞} F_i$.

We now show that each set $F_i$ is nowhere dense in C. To see this, note that $\overline{G_i} ⊆ B(U) ∩ \overline{H}$, and so $\overline{G_i}$ does not meet K. But if $q ∈ \overline{G_i}$, q has a rational ordinate, so $q ∈ L(x)$ for some $x ∈ P$. Hence the projection of $\overline{G_i}$ into C is a closed set containing $F_i$ which does not meet Q. Since Q is dense in C, $F_i$ is nowhere dense in C, so

$\bigcup_{i=1}^{∞} F_i$ is first category in C. Since C is perfect, any countable set is first category, and since Q is countable,
is first category in $C$.

**Theorem 1.16.** $K$ is connected.

**Proof.** Let $H$ be a clopen subset of $K$ containing $p$. Then there is an open set $U$ of $X$ so $H = K \cap U$. Note that since $U \cap K$ is dense in $U$, $U = U \cap K = H$.

Now since $H \in 2^K$ and $p \in H \subset U$, we may apply the lemma to conclude that

$$F = \{x \in C : L(x) \cap H \cap B(U) \neq \emptyset\}$$

$$= \{x \in C : L(x) \cap B(U) \neq \emptyset\}$$

is first category in $C$.

Now since $U$ is open in $X$, and $L(x)$ is connected, $L(x) \cap B(U) \neq \emptyset$ whenever $L(x) \cap U \neq L(x)$. Hence if $x \in C - F$,

$$L(x) \cap B(U) = \emptyset,$$

and so

$$L(x) \cap U = L(x),$$

or equivalently,

$$L(x) \subset U.$$ 

Hence there is a dense subset $M = \bigcup \{L(x) : x \in C - F\}$ of $X$ so that $M \subset U$. Then $X = \overline{M} \subset \overline{U} = \overline{H}$, and so $K = K \cap \overline{H} = H$, since $H$ is closed in $K$. Hence $K$ is connected.

It is readily apparent that $K - \{p\}$ is totally disconnected, hence the appellation of $p$ as an explosion point. Now, for each $\alpha \in [0, \frac{1}{2}]$, denote by $K_\alpha$ the subset of $K$ lying above and upon the line $y = \alpha$. For rational $\alpha$ less than $\frac{1}{2}$, $K_\alpha$ is homeomorphic with $K$, and for
irrational α, $K_\alpha = \text{Cl}_K(\bigcup\{K_\beta : \beta > \alpha \text{ and } \beta \text{ rational}\})$. Hence $K_\alpha$ is in $C(K)$ for each α. Note that $\mathcal{C} = \{K_\alpha : \alpha \in [0, \frac{1}{2}]\}$ is a maximal chain in $C(K)$ which is not connected, since if $U$ is an open circular disc about $p$ whose boundary meets $L(0)$ at an irrational point, $(U) \cap \mathcal{C} = \{\bar{U} \cap \mathcal{C}\}$ is clopen in $\mathcal{C}$. It follows that the order topology on $\mathcal{C}$, which is homeomorphic with $[0, \frac{1}{2}]$, is strictly weaker than the Vietoris topology on $\mathcal{C}$. Also, note that $p$ is the unique point at which $K$ has property A.

The following modification of $K$ is an example of a connected metric space with property A for which $C(X)$ is not connected. Let $K^1$ be the set symmetric to $K$ with respect to the line $x = 1$, and let $p^1 = (\frac{3}{2}, \frac{1}{2})$ be the point corresponding to $p$ under this symmetry. Then $M = K \cup K^1$ is connected, since $K \cap K^1 = \{(1,0)\}$. Now any nondegenerate element of $C(M)$ must contain at least one of the points $p$ or $p^1$, and if it contains exactly one, it is a subset of either $K$ or $K^1$. Now define the sets $\mathcal{U} = \{A \in C(M) : \{p, p^1\} \subset A\}$ and $\mathcal{V} = \{A \in C(M) : A \text{ contains exactly one of the points } p, p^1\} \cup \hat{M}$. Then $\mathcal{U} = C(M) \cap \langle M, p, p^1 \rangle$ is closed by Theorem 1.2, while $\mathcal{V} = C(M) \cap \langle \hat{M} \cup \{K_0 \cup \langle K^1, p^1 \rangle \} \rangle$ is also closed. Since $M \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$, $\mathcal{U}$ and $\mathcal{V}$ disconnect $C(M)$.

The problem with the space $M$ is that it does not possess the following property. A space $X$ is said to be C-connected provided that whenever $K$ is a proper,
nondegenerate element of $C(X)$, and $U$ is an open set containing $K$, then there exists $H \in C(X)$ so $K \subset H \subset U$ and $K \neq H$.

Note that $C$-connectedness does not imply connectedness, since a totally disconnected space is vacuously $C$-connected. However:

**Theorem 1.17.** If $X$ is $C$-connected and is not locally totally disconnected, then $X$ is connected.

**Proof.** Since $X$ is not locally totally disconnected, there exists a point $p \in X$ so that every neighborhood of $U$ has a nondegenerate component. Then if $U$ is a clopen subset of $X$ containing $p$, $U$ has a closed, nondegenerate component $K$. If $K \neq X$, $C$-connectedness implies that there exists $H \in C(X)$ so $K \subset H \subset U$ and $K \neq H$, which is impossible. Hence $K = X$, or $X$ is connected.

**Corollary 1.17a.** If $X$ has property $A$ and is $C$-connected, then $X$ is connected.

**Proof.** Since $X$ has property $A$, it is not locally totally disconnected.

**Theorem 1.18.** If $X$ is $C$-connected, connected and Hausdorff, then $C(X) - \hat{X}$ is connected.

**Proof.** If $C(X) - \hat{X}$ is not connected, there is a clopen subset $\mathcal{U}$ of $C(X) - \hat{X}$ which does not contain $X$. Let $A \in \mathcal{U}$ and let $\mathcal{X} = \{K \in \mathcal{U}: K \supseteq A\}$. Choose a maximal chain $\mathcal{C}$ in $\mathcal{X}$, and let $C = \bigcup \mathcal{C}$. By Theorem 1.11, $C \in \text{Cl}_{2X}(\mathcal{C})$
and since $C$ is connected, $C \in \text{Cl}_{C(X)}(\mathcal{U})$. Since $C$ is nondegenerate, $C \in \mathcal{U}$. Now $C(X) - \hat{X}$ is open in $C(X)$, by Theorem 1.5, and so is $\mathcal{U}$. Hence there is a basic open set $\langle U_i \rangle_{i=1}^n$ so $C \in \langle U_i \rangle_{i=1}^n \cap C(X) \subset \mathcal{U}$. If $C \neq X$, we have $C \subset \bigcup_{i=1}^n U_i$, so $C$-connectedness implies there is $H \in C(X)$ so $C \subset H \subset U$ and $C \neq H$. But clearly $H \in \langle U_i \rangle_{i=1}^n \cap C(X) \subset \mathcal{U}$, so $C \cup \{H\}$ is a chain in $X$ properly containing $C$. This contradiction implies that $X = C$, so $X \in \mathcal{U}$. This is a contradiction, so $C(X) - \hat{X}$ is connected.

**Theorem 1.19.** If the Hausdorff space $X$ has property A and is $C$-connected, then $C(X)$ is connected.

**Proof.** By Theorem 1.18 and Corollary 1.17a, each of $\hat{X}$ and $C(X) - \hat{X}$ are connected. Hence this is the only possible separation of $C(X)$, but this would make $X$ open, in contradiction to property A.

Since the explosion point space is in a sense minimal with respect to having property A, it is interesting to note that it is also $C$-connected, and hence has connected $C(X)$. The proof of this fact is omitted.
CHAPTER BIBLIOGRAPHY

CHAPTER II

HYPERSPACES OF A COMPACT SPACE

As in so many other areas of topology, the property of compactness plays a central role in concluding a satisfying theory of hyperspaces. This is particularly true since the Vietoris topology is very difficult to visualize, and so geometric intuition plays almost no role in discovering its properties. It is fortunate, therefore, that for the class of compact Hausdorff spaces a characterization of net convergence in $2^X$ exists which is quite intuitive, and provides the tool for a broad extension of the theory. We shall begin with a definition.

**Definition 2.1.** Let $X$ be a topological space and \{${A_\alpha : \alpha \in D}$\} a net of subsets of $X$. If $U$ is a subset of $X$, we say that \{${A_\alpha}$\} eventually (frequently) meets $U$ if the set \{${\alpha \in D : A_\alpha \cap U \neq \emptyset}$\} is residual (cofinal) in $D$. We define

$$\lim_\alpha A_\alpha = \{x \in X : \text{${A_\alpha}$ eventually meets each neighborhood of } x\},$$

and

$$\limsup_\alpha A_\alpha = \{x \in X : \text{${A_\alpha}$ frequently meets each neighborhood of } x\}.$$  

If $\lim_\alpha A_\alpha = A = \limsup_\alpha A_\alpha$, then we say that \{${A_\alpha}$\} converges
topologically to \( A \), and write
\[
\lim A_\alpha = A.
\]

It is easily verified that if \( \{A_\alpha\} \) is a net of subsets of \( X \), then each of \( \overline{\lim} A_\alpha \) and \( \overline{\lim} A_\alpha \) are closed, and that \( \overline{\lim} A_\alpha \subseteq \overline{\lim} A_\alpha \).

The following theorem is an illustration of this type of convergence.

**Theorem 2.2.** Let \( X \) be a compact Hausdorff space and \( \{A_\alpha : \alpha \in D\} = \mathcal{F} \) a filterbase of closed subsets of \( X \). For \( \alpha, \beta \in D \), define \( \alpha \preceq \beta \) iff \( A_\alpha \supseteq A_\beta \). Then if \( A = \bigcap \{A_\alpha : \alpha \in D\} \), \( \lim A_\alpha = A \).

**Proof.** It is evident that \( \preceq \) defines a direction on \( D \), and since \( \mathcal{F} \) has the finite intersection property, \( A \) is closed and nonempty. We shall demonstrate the desired equality by showing that \( \overline{\lim} A_\alpha \subseteq A \subseteq \lim A_\alpha \). Hence let \( x \in \overline{\lim} A_\alpha \) and let \( U \) be a neighborhood of \( x \). By regularity, we may choose an open set \( V \) so that \( x \in V \subseteq \overline{V} \subseteq U \). Then there is a cofinal subset \( E \) of \( D \) so that \( A_\alpha \cap V \neq \emptyset \) for each \( \alpha \in E \). Choose \( x_\alpha \in A_\alpha \cap V \) for each \( \alpha \in E \). Then \( \{x_\alpha : \alpha \in E\} \) is a net which clusters at some point \( y \in \overline{V} \subseteq U \).

Suppose \( y \notin A \). Then \( y \notin A_\beta \) for some \( \beta \in D \), and since \( X - A_\beta \) is a neighborhood of \( y \), one may choose \( \delta \in E \) so \( \delta \geq \beta \) and \( x_\delta \in X - A_\beta \). But \( x_\delta \in A_\delta \subseteq A_\beta \), so we have a contradiction. Hence \( y \in A \cap U \), and so \( x \in A = A \). Since it is clear that \( A \subseteq \lim A_\alpha \), the theorem is proved.
The following sequence of theorems characterize convergence in $2^X$ when $X$ is compact Hausdorff, and show that $2^X$ is compact whenever $X$ is compact. The following lemma is an obvious consequence of the definitions.

**Lemma 2.3.** If $\{A_\alpha : \alpha \in D\}$ is a net of subsets of a space $X$; if $\{A_\beta : \beta \in E\}$ is a subnet of $\{A_\alpha\}$; if $x_\beta \in A_\beta$ for each $\beta \in E$; and if $\{x_\beta : \beta \in E\}$ clusters at a point $x \in X$, then $x \in \overline{\lim} A_\alpha$.

**Lemma 2.4.** Let $X$ be compact and $\{A_\alpha : \alpha \in D\}$ an ultranet in $2^X$. Then $\overline{\lim} A_\alpha = \overline{\lim} A_\alpha$ is in $2^X$.

**Proof.** Since $X$ is compact and each $A_\alpha$ is nonempty, the previous lemma implies that $\overline{\lim} A$ is nonempty. By the remark following Definition 2.1, $\overline{\lim} A_\alpha \in 2^X$ and $\overline{\lim} A_\alpha \subset \overline{\lim} A_\alpha$, so it remains to show that $\overline{\lim} A_\alpha \subset \lim A_\alpha$.

Let $x \in \overline{\lim} A_\alpha$, and let $U$ be a neighborhood of $x$. Then since $\{A_\alpha\}$ is an ultranet, $\{A_\alpha\}$ is eventually in either $(X,U)$ or $2^X - (X,U) = (X-U)$. But since $\{A_\alpha\}$ frequently meets $U$, the latter possibility is impossible. Hence $\{A_\alpha\}$ eventually meets $U$, so $x \in \lim A_\alpha$.

Hence, if $X$ is compact, any ultranet in $2^X$ converges topologically to a point of $2^X$. The obvious question is to determine the relationship between topological convergence and Vietoris convergence.

**Theorem 2.5.** If $X$ is compact and $\{A_\alpha : \alpha \in D\}$ is a net in $2^X$ so that $\{A_\alpha\}$ converges topologically to $A$, then
\( \{A_\alpha\} \) converges to \( A \) in the Vietoris topology. If \( X \) is compact Hausdorff, the converse is true.

**Proof.** Assume that \( \varprojlim A_\alpha = A = \varprojlim A_\alpha \). By the proof of Lemma 2.4, \( A \) is closed and nonempty, and so is in \( 2^X \). Let \( \langle V_i \rangle_{i=1}^n \) be a neighborhood of \( A \) in \( 2^X \), and let \( V = \bigcup_{i=1}^n V_i \). Then \( A \subset V \). Let \( E = \{ \alpha \in D : A_\alpha \cap (X - V) \neq \emptyset \} \), and suppose that \( E \) is cofinal in \( D \). Then if we choose \( x_\alpha \in A_\alpha \cap (X - V) \) for each \( \alpha \in E \), \( \{x_\alpha : \alpha \in E\} \) is a net in the compact set \( X - V \) which clusters at some point \( x \in X - V \). But by Lemma 2.3, \( x \in \varprojlim A_\alpha = A \subset V \), which is a contradiction. Hence \( E \) is not cofinal in \( D \), so there exists \( \delta \in D \) so \( \alpha \geq \delta \) implies that \( \alpha \notin E \), or \( A_\alpha \cap (X - V) = \emptyset \). Hence \( A_\alpha \subset V \) for all \( \alpha \geq \delta \).

Now choose \( x_i \in V_i \cap A \) for \( i = 1, 2, \ldots, n \). Then \( V_i \) is a neighborhood of \( x_i \) for each \( i \), and since \( x_i \in \varprojlim A_\alpha \), there exists \( \delta_i \in D \) so \( \alpha \geq \delta_i \) implies that \( A_\alpha \cap V_i \neq \emptyset \). Now choose \( \beta \in D \) larger than each of \( \delta_1, \delta_2, \ldots, \delta_n \), and \( \delta \). Then if \( \alpha \geq \beta \), \( A_\alpha \in \langle V_i \rangle_{i=1}^n \), so \( A_\alpha \) converges to \( A \) in the Vietoris topology.

Assume now that \( X \) is compact Hausdorff and \( A_\alpha \) converges to \( A \) in \( 2^X \). We shall show that \( \varprojlim A_\alpha \subset A \subset \varprojlim A_\alpha \).

Let \( x \in \varprojlim A_\alpha \). If \( x \notin A \), the regularity of \( X \) allows us to choose disjoint open sets \( U \) and \( V \) so that \( A \subset U \) and \( x \in V \). Now since \( \langle U \rangle \) is a neighborhood of \( A \), \( \{A_\alpha\} \) is eventually a subset of \( U \), so could not frequently meet \( V \).
But $x \in V \cap \overline{\lim A_\alpha}$ shows that $\{A_\alpha\}$ frequently meets $V$. This contradiction shows that $x \in A$, and so $\overline{\lim A_\alpha} \subset A$.

Now let $x \in A$ and let $U$ be a neighborhood of $x$. Then $(X,U)$ is a neighborhood of $A$, and so $\{A_\alpha\}$ is eventually in $(X,U)$, or eventually meets $U$. Hence $x \in \overline{\lim A_\alpha}$, and $A \subset \overline{\lim A_\alpha}$.

In view of this theorem, we shall hence forward make no distinction between Vietoris and topological convergence when dealing with compact $T_2$ spaces.

Since any chain in $2^X$ is a filterbase of closed subsets of $X$, the following theorem is a corollary to Theorem 2.2. and 2.5.

Theorem 2.6. If $X$ is compact Hausdorff, and $\mathcal{C}$ is a chain in $2^X$, then $A = \cap \mathcal{C}$ is in $\overline{\mathcal{C}}$, and $A$ is the greatest lower bound of $\mathcal{C}$.

Proof. $A$ is the limit of a net in $\mathcal{C}$.

Theorem 2.7. $2^X$ is compact if and only if $X$ is compact.

Proof. Assume $X$ is compact and $\{A_\alpha : \alpha \in D\}$ is an ultranet in $2^X$. By Lemma 2.4, $\{A_\alpha\}$ converges topologically, and so $\{A_\alpha\}$ converges in $2^X$. Hence $2^X$ is compact.

Conversely, assume $2^X$ is compact and $\mathcal{C}$ is an open cover of $X$. Then $\{\langle X,U \rangle : U \in \mathcal{C}\}$ is an open cover of $2^X$, so there is a finite subcover $\{\langle X,U_i \rangle : i = 1,2,\ldots,n\}$. Let $x \in X$. Then $\{x\} \in \langle X,U_i \rangle$ for some $i$. Hence $x \in U_i$, so $\{U_i : i = 1,2,\ldots,n\}$ is a finite subcover of $X$ and $X$ is compact.
Corollary 2.8. If any one of the spaces $X$, $2^X$ or $C(X)$ is compact Hausdorff, then all of them are.

Proof. If $X$ is compact $T_2$, it is regular, and so $2^X$ is Hausdorff by Theorem 1.8. $2^X$ is compact by the previous theorem.

If $2^X$ is compact $T_2$, $X$ is normal, and so $C(X)$ is closed in $2^X$ by 1.14 and is therefore compact Hausdorff.

If $C(X)$ is compact $T_2$, then $X$ is closed in $C(X)$ by Theorem 1.5, and so is compact $T_2$.

The fact that $C(X)$ is closed in $2^X$ whenever $X$ is compact Hausdorff produces a hyperspace proof of the following well known theorem. Recall that a continuum is a compact connected Hausdorff space.

Theorem 2.9. The intersection of a nest of continua is a continuum.

Proof. Let $\mathcal{C}$ be a nest of continua. One may assume that $\mathcal{C} \subseteq C(X)$ for some compact Hausdorff space $X$. Then by Theorem 2.6, $\bigcap \mathcal{C} \subseteq \text{Cl}_{2^X}(\mathcal{C}) \subseteq \text{Cl}_{2^X}(C(X)) = C(X)$, and so $\bigcap \mathcal{C}$ is a continuum.

In Chapter I, we showed that the union operation was continuous in $2^X$. In our present setting, there is a different type of union map whose continuity is perhaps somewhat more surprising. For a subset $\mathcal{A}$ of $2^X$, we define $\sigma(\mathcal{A}) = \bigcup \mathcal{A}$.

Theorem 2.10. If $X$ is a compact Hausdorff space, then the map $\sigma$ defined above is a continuous map from $2^{2^X}$ into $2^X$. 
Proof. Our first task is to show that $\sigma$ is well defined; that is, that $\sigma$ maps into $2^X$ as claimed. Hence let $\mathcal{F} \in 2^{2^X}$. Then $\mathcal{F}$ is closed in $2^X$, so $\mathcal{F}$ is compact. Let $x \in X - \sigma(\mathcal{F}) = X - U[F : F \in \mathcal{F}]$. For each $F \in \mathcal{F}$, $x \notin F$, so we may choose disjoint open sets $U_F$ and $V_F$ such that $x \in U_F$ and $F \subseteq V_F$. Then the collection $\{V_F : F \in \mathcal{F}\}$ forms an open cover of $\mathcal{F}$, and so let $\{V_{F_1} : i = 1, 2, \ldots, n\}$ be a finite subcover. Let $U = \bigcap_{i=1}^n U_F$. Then $U$ is a neighborhood of $x$ which is disjoint from $\sigma(\mathcal{F})$. Hence $\sigma(\mathcal{F})$ is closed in $X$, and $\sigma$ maps into $2^X$.

To show that $\sigma$ is continuous, we shall show that inverse images of subbasic open sets are open in $2^{2^X}$. Hence let $U$ be open in $X$, and note that each of $\langle \langle U \rangle \rangle$ and $\langle 2^X, \langle X, U \rangle \rangle$ are open in $2^{2^X}$. Further,

$$\langle \langle U \rangle \rangle = \{ \mathcal{A} \in 2^{2^X} : \mathcal{A} \subseteq \langle U \rangle \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : A \subseteq U \text{ for each } A \in \mathcal{A} \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : \bigcup \mathcal{A} \subseteq U \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : \sigma(\mathcal{A}) \subseteq \langle U \rangle \}$$

$$= \sigma^{-1}(\langle U \rangle).$$

Also,

$$\langle 2^X, \langle X, U \rangle \rangle = \{ \mathcal{A} \in 2^{2^X} : \mathcal{A} \cap \langle X, V \rangle \neq \emptyset \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : \text{there exists } A \in \mathcal{A} \text{ so } A \cap V \neq \emptyset \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : \sigma(\mathcal{A}) \cap V \neq \emptyset \}$$

$$= \{ \mathcal{A} \in 2^{2^X} : \sigma(\mathcal{A}) \in \langle X, V \rangle \} = \sigma^{-1}(\langle X, V \rangle).$$

Hence $\sigma$ is continuous.
**Theorem 2.11.** Let $X$ be a $T_1$ space, and $A$ a connected subset of $2^X$. If $A \cap C(X) \neq \emptyset$, then $\sigma(A)$ is connected.

**Proof.** Assume that $H$ and $K$ are nonempty mutually separated sets whose union is $\sigma(A)$. Let $A \in A \cap C(X)$, and assume that $A \subset H$, so that $A \in A \cap (H)$. Also, since $K \neq \emptyset$, there exists $B \in A$ so $B$ meets $K$, or $B \in A \cap (X,K)$. But $(H)$ and $(X,K)$ are mutually separated sets whose union contains $A$, and since $A$ is connected, either $A \subset (H)$ or $A \subset (X,K)$. Since neither of these cases may hold, we conclude that $\sigma(A)$ is connected.

The following sequence of theorems illustrate a method for producing arcs in $2^X$. We define an arc as a continuum with exactly two non-cut points, and so separability is not assumed. Equivalently, an arc is a continuum whose topology is generated by a linear order. If an arc in $2^X$ is also a chain with respect to the partial order of set containment, it is called an arc chain.

**Theorem 2.12.** Let $X$ be a compact $T_2$ space and let $\mathcal{C}$ be a chain which is closed in $2^X$. Then the relative Vietoris topology on $\mathcal{C}$ is the order topology.

**Proof.** Since $X$ is regular, the order topology on $\mathcal{C}$ is weaker than the Vietoris topology by Theorem 1.13. Hence the identity map from $(\mathcal{C}, \text{Vietoris})$ to $(\mathcal{C}, \text{order})$ is continuous. Since $\mathcal{C}$ is closed in $2^X$, it is compact, and since the order topology on $\mathcal{C}$ is Hausdorff, the
identity map is closed and is therefore a homeomorphism.

A subset \( \mathscr{F} \) of \( 2^X \) is said to be order dense provided that whenever \( A \) and \( B \) are distinct elements of \( \mathscr{F} \) so \( A \subset B \), there exists \( C \in \mathscr{F} \) so \( A \subset C \subset B \) and \( A \neq C \neq B \).

Theorem 2.13. If \( X \) is compact Hausdorff, \( \mathscr{F} \) is a closed and order dense subset of \( 2^X \), and \( \mathcal{C} \) is a maximal chain in \( \mathscr{F} \) then \( \mathcal{C} \) is an arc chain.

Proof. By Theorem 1.12, \( \mathcal{C} \) is closed in \( 2^X \), and so is compact. The previous theorem shows that \( \mathcal{C} \) has the order topology, and so \( \mathcal{C} \) is an arc if and only if \( \mathcal{C} \) is connected. But a compact ordered space is connected if and only if it is order dense, and since \( \mathscr{F} \) is order dense, the maximality of \( \mathcal{C} \) implies that \( \mathcal{C} \) is order dense. Hence \( \mathcal{C} \) is an arc chain.

We shall exploit this theorem when we have identified some order dense subsets of \( 2^X \), which we shall do in Chapter III.

The concluding theorems of this chapter concern a natural hyperspace map induced by a continuous map between spaces. Specifically, let \( X \) and \( Y \) be spaces, and let \( f \) be a closed, continuous map from \( X \) onto \( Y \). Then the induced maps are the maps

\[
f^* : 2^X \to 2^Y \text{ defined by } f^*(A) = f(A), \text{ and } \]

\[
f_* : 2^Y \to 2^X \text{ defined by } f_*(A) = f^{-1}(A). \]
Our theorems give conditions on $f$ under which the induced maps are continuous.

**Theorem 2.14.** If $f: X \rightarrow Y$ is closed and continuous, then $f^*: 2^X \rightarrow 2^Y$ is continuous.

**Proof.** Let $U$ be open in $Y$. Then
\[
f^{-1}(\langle U \rangle) = \{ A \in 2^X : f(A) \in \langle U \rangle \}
\]
\[
= \{ A \in 2^X : f(A) \subset U \}
\]
\[
= \{ A \in 2^X : A \subset f^{-1}(U) \}
\]
\[
= \langle f^{-1}(U) \rangle \text{ which is open in } 2^X.
\]

Also,
\[
f^{-1}(\langle Y, U \rangle) = \{ A \in 2^X : f(A) \in \langle Y, U \rangle \}
\]
\[
= \{ A \in 2^X : f(A) \text{ meets } U \}
\]
\[
= \{ A \in 2^X : A \text{ meets } f^{-1}(U) \}
\]
\[
= \langle X, f^{-1}(U) \rangle \text{ which is open in } 2^X.
\]

Since inverse images of subbasic open sets are open, $f^*$ is continuous.

**Remark.** Since connected sets are preserved by continuous functions, $f^*$ maps $C(X)$ into $C(Y)$.

**Theorem 2.16.** Let $X$ and $Y$ be compact Hausdorff spaces, and let $f: X \rightarrow Y$ be continuous and onto. Then $f^*: 2^Y \rightarrow 2^X$ is continuous if and only if $f$ is open.

**Proof.** Assume that $f^*$ is continuous, and let $U$ be open in $X$ and $y \in f(U)$. Then $\langle X, U \rangle$ is open in $2^X$, and $f^*(\{y\}) = f^{-1}(y) \in \langle X, U \rangle$. Hence there exists a basic open set $\langle V_i \rangle_{i=1}^n$ of $2^Y$ so $\{y\} \in \langle V_i \rangle_{i=1}^n$ and $f^*(\langle V_i \rangle_{i=1}^n) \subset \langle X, U \rangle$. 

Set $V = \bigcap_{i=1}^{n} V_i$, and note that $V$ is a neighborhood of $y$. Also, if $z \in V$, then $[z] \in \langle V_i \rangle_{i=1}^{n}$, so $f_{\times}(\{z\}) \in \langle X, U \rangle$. That is, $f^{-1}(z)$ meets $U$, or $z \in f(U)$. Hence we have $y \in V \subset f(U)$, so $f(U)$ is open in $Y$. Therefore $f$ is open.

Assume conversely that $f$ is open. We shall use our characterization of net convergence to show that $f_{\times}$ is continuous. Hence let $[B_\alpha : \alpha \in D]$ be a net in $2^Y$ so that $B_\alpha$ converges to $B$. Set $A_\alpha = f_{\times}(B_\alpha)$, and $A = f_{\times}(B)$. Now let $x \in \overline{\text{lim}} A_\alpha$. If $f(x) \notin B$, then there exist disjoint open subsets $U$ and $V$ of $Y$ so that $f(x) \in U$ and $B \subset V$. Then since $B_\alpha \rightarrow B$, there exists $\alpha \in D$ so that $\beta \geq \alpha$ implies that $B_\beta \in \langle V \rangle$, or $B_\beta \subset V$. But $x \in f^{-1}(U)$ which is open in $X$, so since $x \in \overline{\text{lim}} A_\alpha$, there exists $\beta \geq \alpha$ so that $A_\beta$ meets $f^{-1}(U)$. That is, there exists a point $p \in f^{-1}(B_\beta) \cap f^{-1}(U) = f^{-1}(B_\beta \cap U)$, which is impossible. Hence $f(x) \in B$, or $x \in f^{-1}(B) = A$. Therefore, $\overline{\text{lim}} A_\alpha \subset A$.

Now, let $x \in A = f^{-1}(B)$, and let $U$ be a neighborhood of $x$. Then $f(x) \in f(U) \cap B$. Since $f(U)$ is open in $Y$ and $B = \overline{\text{lim}} B_\alpha$, there exists $\alpha \in D$ so that $\beta \geq \alpha$ implies that $B_\beta$ meets $f(U)$. Let $\beta \geq \alpha$ and let $p \in B_\beta \cap f(U)$. Then there exists $g \in U$ so $p = f(g)$. But $g \in f^{-1}(B) = A_\beta$. Hence $A_\beta \cap U \neq \emptyset$, and so $x \in \overline{\text{lim}} A_\alpha$.

Therefore, $\lim A_\alpha = A$, and so $f_{\times}$ is continuous.

Recall that a map is said to be monotone if point-inverses are connected.
Corollary 2.17. If \( f: X \to Y \) is a continuous open onto map of compact Hausdorff spaces, then \( Y \) is embeddable in \( 2^X \). If in addition \( f \) is monotone, then \( Y \) is embeddable in \( C(X) \).

Proof. \( f_\# | \hat{Y} \) is certainly such an embedding.

Remark. An equivalent formulation of the preceding theorem is the following.

If \( \mathcal{I} \) is a continuous (i.e. upper and lower semi-continuous) decomposition of the compact Hausdorff space \( X \) into compact sets, then the quotient topology on \( \mathcal{I} \) is the topology it inherits as a subset of \( 2^X \).
CHAPTER III

PROPERTIES OF CONTINUA

Since the succeeding sections of this paper are connected primarily with hyperspaces of a continuum, this chapter is a digression into the relevant terminology and theorems about continua which are needed in the sequel. Throughout this section, a continuum refers to a compact connected Hausdorff space. If a theorem has been proved only for the more restricted class of metric continua, this fact will be noted in the hypothesis.

Our first goal is to show that both $2^X$ and $C(X)$ are arc-wise connected when $X$ is a continuum. To this end, we need several preliminary theorems. Recall that in a topological space $X$, a component of $X$ is a maximal connected subset, while a quasi-component of $X$ is the intersection of all open-closed subsets containing a fixed point. It is evident that components and quasi-components are always closed, and also that if $p \in X$, the component containing $p$ is contained in the quasi-component to which $p$ belongs. It is an important fact that for the class of compact Hausdorff spaces, the two concepts coincide.

**Theorem 3.1.** If $X$ is compact $T_2$, then the components are the quasi-components.
Proof. Let \( p \in X \), and let \( R(p) \) and \( Q(p) \) be respectively the component and the quasi-component of \( X \) containing \( p \). We have \( R(p) \subseteq Q(p) \), and since \( R(p) \) is a maximal connected subset, it suffices to show that \( Q(p) \) is connected. Hence assume that \( Q(p) \) is the union of nonempty mutually separated sets \( H \) and \( K \), and that \( p \in H \). Since \( Q(p) \) is closed, each of \( H \) and \( K \) are closed in \( X \). Since \( X \) is normal, there is an open set \( U \) so that \( H \subseteq U \) and \( \overline{U} \) does not meet \( K \). If \( B(U) \) denotes the boundary of \( U \), then

\[
B(U) \cap Q(p) = \phi.
\]

Hence for each point \( x \in B(U) \), there is an open-closed set \( V_x \) containing \( x \) so \( p \notin V_x \). The collection \( \{V_x : x \in B(U)\} \) is an open cover of the compact set \( B(U) \), and so there exists a finite subcover

\[
\{V_{x_i} : i = 1, 2, \ldots, n\}.
\]

Set

\[
M = U - \bigcup_{i=1}^{n} V_{x_i} = \overline{U} - \bigcup_{i=1}^{n} V_{x_i}.
\]

Then \( M \) is a clopen set containing \( p \) which is disjoint from \( K \). This is a contradiction, and so \( Q(p) \) is connected.

Corollary 3.2. If \( X \) is compact \( T_2 \), \( C \) is a component of \( X \), and \( U \) is open in \( X \) so \( C \subseteq U \), then there exists a clopen set so

\[
C \subseteq V \subseteq U.
\]

Proof. Denote by \( \mathcal{F} \) the collection of clopen subsets of \( X \) containing \( C \). Direct \( \mathcal{F} \) by \( F_1 \leq F_2 \) iff \( F_2 \subseteq F_1 \).
Since $C = \bigcap F$ by the previous theorem, and $F$ may be considered as a net in $2^X$, we have

$$C = \lim\{F : F \in F\}$$

by Theorem 2.2. Since $C \in \langle U \rangle$ which is open in $2^X$, there certainly exists $V \in F$ so $V \in \langle U \rangle$, or

$$C \subseteq V \subseteq U.$$

**Corollary 3.3.** If $X$ is compact $T_2$, $C$ is a component of $X$, and $A$ is a closed subset of $X$ which is disjoint from $C$, then there exist disjoint closed sets $M$ and $N$ so $C \subseteq M$, $A \subseteq N$, and $X = M \cup N$.

**Proof.** Put $U = X - A$. Then $U$ is open and $C \subseteq U$, so by Corollary 3.2, there is a clopen set $M$ so

$$C \subseteq M \subseteq U.$$

Set $N = X - M$ to obtain the desired separation of $X$.

**Theorem 3.4.** If $X$ is a continuum, $U$ is a proper open subset of $X$, and $C$ is a component of $X$, then $\overline{C}$ meets the boundary of $U$.

**Proof.** Assume that $\overline{C} \cap \partial(U) = \emptyset$. Since $C \subseteq \overline{U} = U \cup \partial(U)$, our assumption forces $\overline{C} \subseteq U$. Since $C$ is a component of $U$, and $\overline{C}$ is connected, we have $\overline{C} \subseteq C$, and so $C$ is closed in $X$. Then by the normality of $X$, there is an open set $V$ so that

$$C \subseteq V \subseteq \overline{V} \subseteq U,$$

and $C$ is a component of the compact $T_2$ space $V$. By the Corollary 3.2, there exists a set $M$ which is clopen in $\overline{V}$ so that

$$C \subseteq M \subseteq V \subseteq \overline{V}.$$
Since $V$ is closed, $M$ is clearly closed in $X$. Also, $M = O \cap \overline{V}$ for some open subset $O$ of $X$. Then
\[ O \cap V \subseteq O \cap \overline{V} = M \subseteq O \cap V, \]
and so
\[ M = O \cap V \]
is open in $X$. Hence $M$ is a nonempty proper clopen subset of $X$. Since $X$ is a continuum, we have a contradiction.

**Theorem 3.5.** If $X$ is a nondegenerate continuum, and $K$ is a proper subcontinuum of $X$, then there exists a continuum $H$ so
\[ K \subset H \subset X \]
and
\[ K \neq H \neq X. \]

**Proof.** Let $X \in X - K$. By the regularity of $X$, there is an open set $V$ so that $K \subset V$ and $x \notin \overline{V}$. Let $C$ be the component of $V$ which contains $K$. Then $\overline{C}$ is the desired continuum.

An arc chain $\mathcal{A} \subset 2^X$ is called an arc chain from $A$ to $B$ provided that $A$ and $B$ are the end points of $\mathcal{A}$ and $A \subset B$.

**Theorem 3.6.** If $X$ is a compact Hausdorff space and $A, B \in 2^X$, then there exists an arc chain from $A$ to $B$ if and only if $A \subset B$, $A \neq B$ and every component of $B$ intersects $A$.

**Proof.** Assume that $\mathcal{A}$ is an arc chain from $A$ to $B$. From the definition of this statement, we have $A \subset B$ and
A \neq B. Suppose then that there is a component \( C \) of \( B \) which is disjoint from \( A \). We may apply Corollary 3.3 to the compact \( T_2 \) space \( B \) to obtain disjoint sets \( M \) and \( N \) which are closed in \( B \) so that \( C \subseteq M, A \subseteq N, \) and \( M \cup N = B \). \( M \) and \( N \) are each closed in \( X \), and so the sets \( \langle B, M \rangle \) and \( \langle N \rangle \) are each closed in \( 2^X \). Then we have

\[
\mathcal{A} \subseteq \langle B \rangle = \langle B, M \rangle \cup \langle N \rangle,
\]

while \( A \in \langle N \rangle \) and \( B \in \langle B, M \rangle \). Since \( \langle B, M \rangle \) and \( \langle N \rangle \) are disjoint, they form a separation of \( \mathcal{A} \), which is contrary to the hypothesis. Therefore, each component of \( B \) must intersect \( A \).

Conversely, assume that \( A \subseteq B, A \neq B \) and each component of \( B \) intersects \( A \). Consider the set \( \mathcal{F} = \{ K \in 2^X : A \subseteq K \subseteq B \) and each component of \( K \) meets \( A \} \). Note that \( A \) and \( B \) are both in \( \mathcal{F} \). We shall show that \( \mathcal{F} \) is closed in \( 2^X \), and that \( \mathcal{F} \) is order dense.

To show that \( \mathcal{F} \) is closed, define the set \( \mathcal{F}_A = \{ K \in 2^X : A \subseteq K \) and each component of \( K \) meets \( A \} \), and note that \( \mathcal{F} = \mathcal{F}_A \cap \langle B \rangle \). Since \( \langle B \rangle \) is closed in \( 2^X \), it suffices to show that \( \mathcal{F}_A \) is closed. Hence let \( K \in 2^X - \mathcal{F}_A \). If \( A \nsubseteq K \), then there is an open set containing \( K \) which is disjoint from \( \mathcal{F}_A \) by Theorem 1.3. Therefore, assume that \( A \subseteq K \) and some component \( C \) of \( K \) is disjoint from \( A \). Use Corollary 3.3 to obtain disjoint closed sets \( M \) and \( N \) so \( K = M \cup N, C \subseteq M \) and \( A \subseteq N \). Use the normality of \( X \) to separate \( M \) and \( N \) with disjoint open sets \( U \) and \( V \). Now
note that $K \in \langle U, V \rangle$, while if $H \in \langle U, V \rangle$, there is a component of $H$ which is contained in $U$ and hence does not meet $A$. Hence $H \notin \mathcal{F}_A$, and so $K \in \langle U, V \rangle \subseteq 2^X - \mathcal{F}_A$. It follows that $\mathcal{F}_A$, and hence $\mathcal{F}$ is closed in $2^X$.

To show that $\mathcal{F}$ is order dense, let $H$ and $K$ be distinct sets in $\mathcal{F}$ so $H \subset K$. Let $x \in K - H$, and let $C$ be the component of $K$ containing $x$. Since $C$ intersects $A$ and $A \subset H$, it follows that $C$ intersects $H$. Let $D$ be a component of $C \cap H$. Then $D$ is also a component of $H$, since if $M$ is a connected subset of $H$ which intersects $D$, then $M \subset K$ and $M$ intersects $C$ and so $M \subset C$. Hence $M \subset C \cap H$ and so $M \subset D$. Now we have $D \subset C$ and $x \in C - D$. Since $C$ and $D$ are continua, there exists a continuum $T$ so $D \subset T \subset C$ and $C \neq T \neq D$ by Theorem 3.5. Set $S = H \cup T$. Then $S \in 2^X$ $A \subset H \subset S \subset K \subset B$ and $H \neq S \neq K$. Also any component of $S$ contains a component of $H$, and hence meets $A$. It follows that $S \in \mathcal{F}$, and so $\mathcal{F}$ is order dense.

Now, by Theorem 2.13, any maximal chain in $\mathcal{F}$ is clearly an arc chain from $A$ to $B$.

**Corollary 3.7.** Let $X$ be a continuum, and $A \in 2^X$ so $A \neq X$. Then there is an arc chain in $2^X$ from $A$ to $X$.

**Proof.** $X$ has only one component, which certainly meets $A$.

**Corollary 3.8.** If $X$ is a compact Hausdorff space and $A, B \in C(X)$ so $A \subset B$ and $A \neq B$, then there is an arc chain from $A$ to $B$. Also, if $\mathcal{A}$ is any such arc chain,
then

\[ \mathcal{A} \subset C(X). \]

**Proof.** The first statement is an immediate application of the preceding corollary. To prove the second statement, let \( \mathcal{A} \) be an arc chain from \( A \) to \( B \) and \( K \in \mathcal{A} \). One may assume that \( K \neq A \), and so \( A \subset K \). Then \( \mathcal{B} = \{ H \in \mathcal{A} : H \subset K \} \) is an arc chain from \( A \) to \( K \), and so by Theorem 3.6, each component of \( K \) meets \( A \). Since \( A \) is connected, \( K \) is connected. Hence \( K \in C(X) \) and so \( \mathcal{A} \subset C(X) \).

A topological space is arc-wise connected if every two distinct points are the endpoints of an arc. Of course, an arc-wise connected space is connected.

**Theorem 3.9.** If \( X \) is a continuum, then each of \( 2^X \) and \( C(X) \) are arc-wise connected.

**Proof.** If \( A \) and \( B \) are in \( 2^X(C(X)) \), then each of \( A \) and \( B \) may be connected to \( X \) by an arc which lies entirely in \( 2^X(C(X)) \). Since the union of two arcs whose intersection is nonempty is arc-wise connected, \( A \) and \( B \) lie in an arc-wise connected subspace of \( 2^X(C(X)) \), which proves the theorem.

The preceding theorems are useful in proving that local connectedness is a hyperspace invariant for the class of compact Hausdorff spaces.

**Theorem 3.10.** Let \( X \) be a compact Hausdorff space. Then \( X \) is locally connected (l.c.) iff \( 2^X \) is l.c. iff \( C(X) \) is l.c.
Proof. We shall use repeatedly the fact that a space is locally connected iff it is connected im kleinen (c.i.k.) at each point.

Assume that $X$ is l.c. Let $K \subseteq 2^X$ and let $\mathcal{U} = \langle U_i \rangle_{i=1}^n$ be an open set containing $K$. For each $x \in K$, let $U_x = \cap \{U_i : x \in U_i \}$. Then $U_x$ is a neighborhood of $x$, so there exists for each $x \in K$ a connected open set $V_x$ so that $x \in V_x \subseteq \overline{V}_x \subseteq U_x$. The collection $\{V_x : x \in K\}$ is an open cover of the compact set $K$, and so let $\{V_i : i = 1, 2, \ldots, m\}$ be a finite subcover. Let $\mathcal{V} = \langle V_i \rangle_{i=1}^m$. Then $\mathcal{V}$ is open in $2^X$, and

$$K \in \mathcal{V} \subseteq \overline{\mathcal{V}} = \langle \overline{V}_i \rangle_{i=1}^m \subseteq \mathcal{U}.$$ 

Put $V = \bigcup_{i=1}^m V_i$, and note that

$$\overline{V} = \bigcup_{i=1}^m \overline{V}_i \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U}.$$ 

Now let $A$ and $B$ be points of $\mathcal{V}$. Since $\overline{V}_j$ is connected for each $j$, $\overline{V}$ has at most $m$ components, and each components of $\overline{V}$ meets both $A$ and $B$. Since $A \subseteq \overline{V}$ and $B \subseteq \overline{V}$, there are arc chains $\mathcal{A}$ and $\mathcal{B}$ from $A$ to $\overline{V}$ and from $B$ to $\overline{V}$ respectively. Note that each of $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{U}$, and so $\mathcal{A} \cup \mathcal{B}$ is a subcontinuum of $\mathcal{U}$ which contains both $A$ and $B$. Therefore $2^X$ is c.i.k. at $K$. It follows that $2^X$ is l.c.

Suppose that $2^X$ is locally connected and $K \in C(X)$. Let $\mathcal{U} = \langle U_i \rangle_{i=1}^n$ be a basic open set containing $K$. Then
there is a connected open subset $\mathcal{V}$ of $2^X$ so that $K \in \mathcal{V} \subset \overline{\mathcal{V}} \subset U$. Let $H = \bigcup \overline{\mathcal{V}}$. By Theorems 2.10 and 2.11, $H \in C(X)$, and clearly $H \in U$. Let $A$ and $B$ be points of $C(X) \cap \mathcal{V}$. Then $A \subset H$ and $B \subset H$, and so there exist arc chains $\mathcal{A}$ and $\mathcal{B}$ from $A$ to $H$ and from $B$ to $H$ respectively. Then $\mathcal{A} \cup \mathcal{B}$ is a connected subset of $C(X) \cap U$ containing $A$ and $B$, and so $C(X)$ is c.i.k. at $K$. Hence $C(X)$ is l.c.

Assume now that $C(X)$ is l.c. and let $x \in X$. Let $U$ be open in $X$ so $x \in U$. Then $\{x\} \in C(X) \cap \langle U \rangle$, and so there is a connected set $\mathcal{V}$ which is open in $C(X)$ so that $\{x\} \in \mathcal{V} \subset \langle U \rangle \cap C(X)$. There exists an open set $V$ so that $\{x\} \in \langle V \rangle \cap C(X) \subset \mathcal{V}$. Then we have $x \in V \subset U$, and if $a$ and $b$ are points of $V$, they lie in the subset $\cup \mathcal{V}$ of $U$. Note that $\cup \mathcal{V}$ is connected by Theorem 2.11. Hence $X$ is c.i.k. at $x$, and so $X$ is l.c.

One of the most important concepts in the classification of continua has proven to be that of decomposability. A connected space $X$ is decomposable if there exist proper closed connected sets $M$ and $N$ so $X = M \cup N$. A connected space is indecomposable if it is not decomposable.

**Lemma 3.11.** If $X$ is connected, $K$ is a proper connected subset of $X$ so that $X - K$ is the union of mutually separated sets $M$ and $N$, then $K \cup M$ is connected.

**Theorem 3.12.** Let $X$ be an indecomposable connected space. Then every proper connected subset of $X$ has empty interior.
Proof. Let $K \subseteq X$ be proper, closed, and connected. Assume that $K^0 \neq \varnothing$. Then $X - K^0$ is a proper closed subset of $X$ so that $X - K \subseteq X - K^0$. Hence $\overline{X - K}$ is a proper closed subset of $X$. Now $X - K$ is not connected, since if it were, $K$ and $\overline{X - K}$ would be proper closed connected subsets whose union is $X$. Hence there are subsets $M$ and $N$ of $X$ which disconnect $X - K$. Now $X - K$ is open in $X$, and so are $M$ and $N$. Hence $K \cup M = X - N$ and $K \cup N = X - M$ are proper closed subsets of $X$ which are connected by Lemma 3.11. Also, $X = (K \cup M) \cup (K \cup N)$, which contradicts $X$ being indecomposable. Hence $K^0 = \varnothing$.

Corollary 3.13. A connected space $X$ is decomposable if and only if there exists a proper closed connected set $K \subseteq X$ so that $K^0 \neq \varnothing$.

Proof. If $X$ is decomposable, let $X = A \cup B$, where $A$ and $B$ are proper, closed and connected. Then $X - A$ is nonempty and open, and $X - A \subseteq B$, so $B^0 \neq \varnothing$.

The converse is Theorem 3.12.

A concept closely related to decomposability is that of a composant. If $X$ is a topological space, and $x \in X$, then the composant of $x$, denoted by $C(x)$, is the union of all proper closed connected sets which contain $x$.

Theorem 3.14. If $X$ is a continuum and $p \in X$, then $C(p)$ is dense in $X$.

Proof. Suppose that $C(p) \neq X$. Note that $\overline{C(p)}$ is a continuum, and so by Theorem 3.5, there is a continuum
K so $\overline{C(p)} \subseteq K \subseteq X$ and $\overline{C(p)} \neq K \neq X$. But since $C(p)$ is a composant and $p \in K$, we have $K \subseteq C(p)$, which is a contradiction.

**Theorem 3.15.** If $X$ is an indecomposable metric continuum, then $X$ has uncountably many distinct composants.

**Proof.** Since $X$ is a complete metric space, it is not of the first category in itself, and so the theorem will be proved if we show that each composant of $X$ is of the first category in $X$; or, that each composant is the countable union of closed, nowhere dense sets.

Hence, let $C(x)$ be a composant of $X$, and let

$$\{B_i : i = 1, 2, \ldots\}$$

be a countable base for the topology on $X$. Let $\{B_{n_i} : i = 1, 2, \ldots\}$ be the subsequence of $\{B_i\}$ containing precisely the $B_i$'s for which $x \notin B_{n_i}$. Let $K_i$ be the component of $X - B_{n_i}$ containing $x$. Then, for each $i$, $K_i$ is a proper subcontinuum of $X$ containing $x$, and is therefore nowhere dense by Theorem 3.12. Hence, if we set $K = \bigcup_{i=1}^{\infty} K_i$, we have $K \subseteq C(x)$, and $K$ is first category in $X$. Let $p \in C(x)$. Then there is a proper subcontinuum $H$ of $X$ so $\{p, x\} \subseteq H$. Since $X - H$ is open and $x \notin X - H$, choose $j \in \mathbb{N}$ so $B_{n_j} \subseteq X - H$. Then clearly $p \in H \subseteq K_j \subseteq K$. It follows that $C(x) \subseteq K$. Therefore $C(x) = K$, and so $C(x)$ is of the first category in $X$. 
Theorem 3.16. The composants of an indecomposable continuum are either equal or disjoint.

Proof. Let \( C(p) \) and \( C(q) \) be composants of the indecomposable continuum \( X \). Assume that there is a point \( x \in C(p) \cap C(q) \). We shall show that \( C(p) = C(q) \).

Since \( x \in C(p) \cap C(q) \), there exist proper subcontinua \( K \) and \( L \) of \( X \) so that \( \{x,p\} \subseteq K \) and \( \{x,q\} \subseteq L \). Then \( K \cup L \) is a continuum, which is proper since \( X \) is indecomposable. Now let \( y \in C(p) \). There is a proper subcontinuum \( M \) of \( X \) so \( \{y,p\} \subseteq M \). Now \( K \cup L \cup M \) is a proper subcontinuum of \( X \) containing \( q \) and \( y \), and so \( y \in C(q) \). Therefore \( C(p) \subseteq C(q) \). An analogous argument shows that \( C(q) \subseteq C(p) \).

An equivalent method of considering the composants of a continuum is through the concept of irreducibility. If \( p \) and \( q \) are points of the continuum \( X \), then \( X \) is irreducible between \( p \) and \( q \) if no proper subcontinuum of \( X \) contains both \( p \) and \( q \). It is clear that \( X \) is irreducible between \( p \) and \( q \) if and only if \( q \notin C(p) \). A continuum is said to be irreducible if it is irreducible between some two of its points. Note that Theorems 3.14 and 3.15 imply that in an indecomposable metric continuum \( X \) there is an uncountable subset \( D \) so that \( X \) is irreducible between any two points of \( D \).

A more general type of irreducibility may be defined as follows. Let \( P \) be any property capable of being
possessed by continua. If property P is shared by the intersection of any chain of continua, each of which has property P, then property P is said to be inductive. If continuum X has property P, while no proper subcontinuum of X has property P, then X is said to be irreducible with respect to property P.

**Theorem 3.17.** If X is a continuum, and P is an inductive property of X, then there exists a subcontinuum K of X so that K is irreducible with respect to property P.

**Proof.** Set \( \mathcal{X} = \{ K \in C(X) : K \) has property P \}. Then \( X \in \mathcal{X} \). Choose a maximal chain \( \mathcal{C} \) of \( K \) and set \( K = \bigcap \mathcal{C} \). Then \( K \) is the desired continuum.

Examples of inductive properties are those of containing a certain set, or interesting each of a fixed collection of sets. The following theorem is yet another application of this principle.

**Theorem 3.18.** Let X and Y be continua and let \( f: X \to Y \) be a continuous surjection. If Y is indecomposable, then X contains an indecomposable continuum.

**Proof.** For \( K \in C(X) \), let us say that K maps onto Y iff \( f(K) = Y \). We shall show that the property of mapping onto Y is inductive. Hence let \( \mathcal{C} = \{ K_\alpha : \alpha \in D \} \) be a chain into \( C(X) \) so that each \( K_\alpha \) maps into Y. Direct D by \( \alpha \leq \beta \) iff \( K_\alpha \supset K_\beta \). Put \( K = \bigcap \mathcal{C} \). Then \( K = \lim K_\alpha \) by Theorem 2.2.
Since the induced map $f^*: 2^X \to 2^Y$ is continuous, we have $f(K) = f^*(K) = \lim f^*(K_\alpha) = Y$. Hence $K$ maps onto $Y$.

Now let $C$ be a subcontinuum of $X$ which is irreducible with respect to mapping onto $Y$. Suppose that $C$ is decomposable. Then $C$ is the union of proper subcontinua $A$ and $B$. Since $C$ is irreducible with respect to mapping onto $Y$, each of $f(A)$ and $f(B)$ are proper subcontinua of $Y$. Hence $Y = f(C) = f(A) \cup f(B)$ is decomposable, which is a contradiction.

A property of a continuum which is shared by each of its nondegenerate subcontinua is said to be hereditary. Hence a continuum is hereditarily decomposable, for example, if and only if each of its nondegenerate subcontinua are decomposable. Hereditarily indecomposable continua are also called Knaster continua, in honor of the man who first demonstrated their existence [2].

A continuous mapping $f: X \to Y$ is called monotone if $f^{-1}(K)$ is connected for each $K \in C(Y)$; it is called confluent if, for each $K \in C(Y)$, $f(C) = K$ for each component $C$ of $f^{-1}(K)$; and it is called weakly confluent if, for each $K \in C(Y)$, there is a component $C$ of $f^{-1}(K)$ so that $f(C) = K$.

The concept of dimension of a separable metric space is an important unifying idea in topology. Since
an exposition of this theory is out of place here, we refer the reader to the book by Hurewicz and Wallman [1] for the needed definitions and theorems.

The following two theorems are illustrations of the preceding definitions.

**Theorem 3.19.** If $X$ is a hereditarily indecomposable continuum, and $f:X \rightarrow Y$ is a continuous, monotone surjection, then $Y$ is hereditarily indecomposable.

**Proof.** Suppose that $K \subseteq C(Y)$ is decomposable. Then $K = A \cup B$ for proper subcontinua $A$ and $B$ of $K$. Since $f^{-1}(K)$ is a subcontinuum of $X$ which is the union of proper subcontinua $f^{-1}(A)$ and $f^{-1}(B)$, we contradict the fact that $X$ is hereditarily indecomposable.

**Theorem 3.20.** A hereditarily decomposable metric continuum is one dimensional.

**Proof.** If the continuum $X$ is not one dimensional, then the dimension of $X$ is greater than or equal to two. In [3], Mazurkiewicz proved that there exists a continuous weakly confluent surjection $f:X \rightarrow I^2$, where $I$ is the unit interval. There exists an indecomposable continuum $P \subseteq I^2$. Since $f$ is weakly confluent, there is a subcontinuum $K$ of $X$ so that $f(K) = P$. By Theorem 3.18, $K$, and hence $X$, contains an indecomposable continuum. Hence $X$ is not hereditarily decomposable. The theorem follows by contraposition.
CHAPTER BIBLIOGRAPHY


CHAPTER IV

HYPERSPACES OF A COMPACT METRIC SPACE

In any construction of a new topological space from given spaces, an important criterion in defining a topology on the new space is how well it inherits topological properties possessed by the old spaces. The Tychonoff topology on a product is an outstanding example of the pre-eminence of this requirement. We have seen in previous chapters that the Vietoris topology on $2^X$ is the "right" topology for preserving compactness and connectedness. In this chapter we shall find that $2^X$ is metrizable whenever $X$ is a compact metric space. We shall demonstrate this fact by explicitly defining a metric, called the Hausdorff metric, on $2^X$ and proving that it generates the Vietoris topology.

**Definition 4.1.** Let $(X,d)$ be a bounded metric space. For subsets $A$ and $B$ of $X$, a point $x \in X$, and positive number $\epsilon$, we make the definitions

\[
d(x,B) = \inf\{d(x,y) : y \in B\}
\]

\[
dist(A,B) = \inf\{d(a,b) : a \in A, b \in B\}
\]

\[
S(x,\epsilon) = \{p \in X : d(x,p) < \epsilon\}
\]

\[
V_\epsilon(A) = \{p \in X : d(p,A) < \epsilon\}
\]
\[ \rho(A,B) = \inf \{ \varepsilon : A \subset V_{\varepsilon}(B) \text{ and } B \subset V_{\varepsilon}(A) \} \]
\[ \text{diam}(A) = \sup \{ d(a,b) : a,b \in A \} . \]

The function \( \rho \) is called the Hausdorff metric when it is restricted to \( 2^X \times 2^X \).

**Lemma 4.2.** If \((X,d)\) is a metric space, \( \varepsilon > 0 \), and \( A \subset X \), then \( V_{\varepsilon}(A) = \bigcup \{ S(a,\varepsilon) : a \in A \} \).

**Proof.** The proof is easy, and so is omitted.

**Theorem 4.3.** If \((X,d)\) is a bounded metric space, then the Hausdorff metric \( \rho \) is a metric on \( 2^X \).

**Proof.** Let \( A, B \in 2^X \). If \( d \) is bounded by \( M \), we have \( \rho(A,B) \leq M < \infty \). Also \( \rho \geq 0 \) since it is the infimum of positive numbers. Now let \( \varepsilon > 0 \). Then \( A \subset V_{\varepsilon}(A) \), and so \( \rho(A,A) < \varepsilon \). Hence \( \rho(A,A) = 0 \). Conversely, assume that \( A \neq B \) and that there exists \( x \in B - A \). Since \( A \) is closed, there exists \( \varepsilon > 0 \) so that \( S(x,\varepsilon) \cap A = \emptyset \). Then \( B \notin V_{\varepsilon}(A) \).

Since the same result holds for all positive \( \delta < \varepsilon \), we have \( \rho(A,B) \geq \varepsilon > 0 \). Since the definition of \( \rho \) is symmetric in \( A \) and \( B \), we have \( \rho(A,B) = \rho(B,A) \). Hence we have only to prove the triangle inequality to show that \( \rho \) is a metric.

Let \( A_1, A_2 \) and \( A_3 \) be in \( 2^X \) and define the sets
\[ E_{12}, E_{23} \text{ and } E_{13} \] by \( E_{ij} = \{ \varepsilon > 0 : A_i \subset V_{\varepsilon}(A_j) \text{ and } A_j \subset V_{\varepsilon}(A_i) \} \).
Define \( E_{12} + E_{23} = \{ \delta_1 + \delta_2 : \delta_1 \in E_{12} \text{ and } \delta_2 \in E_{23} \} \). The triangle inequality for \( d \) implies that \( E_{12} + E_{23} \subset E_{13} \).

It follows that
\[ \rho(A_1, A_3) = \inf E_{13} \leq \inf (E_{12} + E_{23}) \]
\[ = \inf E_{12} + \inf E_{23} \]
\[ = \rho(A_1, A_2) + \rho(A_2, A_3). \]

Hence \( \rho \) is a metric on \( 2^X \).

Observe that the above proof uses the fact that the sets involved are closed only in proving that \( \rho(A, B) = 0 \) only if \( A = B \). It follows that \( \rho \) is a pseudo-metric on the space of all nonempty subsets of a bounded metric space \( X \).

**Theorem 4.4.** If \((X, d)\) is a bounded metric space, and \( \rho \) is the Hausdorff metric on \( 2^X \), then \((\hat{X}, \rho)\) is isometric to \((X, d)\).

**Proof.** Let \( x \) and \( y \) be two points of \( X \), and \( \varepsilon = d(x, y) \). Then \( y \notin S(x, \varepsilon) = V_\varepsilon(\{x\}) \), and so \( \rho(\{x\}, \{y\}) \geq \varepsilon \). On the other hand, for any \( \delta > 0 \), we have \( x \in S(y, \varepsilon + \delta) \) and \( y \in S(x, \varepsilon + \delta) \), and so \( \rho(\{x\}, \{y\}) \leq \varepsilon + \delta \). Hence \( \rho(\{x\}, \{y\}) \leq \varepsilon \).

It is easy to find examples of metric spaces for which the Hausdorff metric topology on \( 2^X \) differs from the Vietoris topology. For example, let \( X \) be an infinite discrete space with metric \( d(x, y) = 1 \) iff \( x \neq y \) and \( d(x, y) = 0 \) iff \( x = y \). Then the metric topology on \( 2^X \) is discrete, while the Vietoris topology cannot be discrete, since the collection of finite sets is a proper dense subset. However, if \( X \) is compact, the two topologies coincide.
Theorem 4.5. If \((X,d)\) is a compact metric space, then the Vietoris topology on \(2^X\) is the Hausdorff metric topology.

Proof. Since \(X\) is compact, \(d\) is bounded and so the Hausdorff metric is defined. Let \(A \in 2^X\), \(\varepsilon > 0\), and \(\mathcal{S}(A,\varepsilon) = \{K \in 2^X : \rho(A,K) < \varepsilon\}\) be a basic open set in the metric topology on \(2^X\). Choose \(\delta\) so that \(0 < \delta < \varepsilon/2\). Since \(A\) is compact, there is a finite subset \(\{a_i : i = 1, 2, \ldots, n\}\) of \(A\) so that \(A \subset \bigcup_{i=1}^{n} S(a_i,\delta)\). Let \(U_i = S(a_i,\delta)\) for \(i = 1, 2, \ldots, n\) and let \(\mathcal{B} = \langle U_i \rangle_{i=1}^{n}\). Clearly \(A \in \mathcal{B}\) which is open in the Vietoris topology, and so we must show that \(\mathcal{B} \subset \mathcal{S}(A,\varepsilon)\).

Let \(B \in \mathcal{B}\). Then \(B \subset \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} S(a_i,\delta) \subset V_{2\delta}(A)\). To show that \(A \subset V_{2\delta}(B)\), let \(x \in A\). Then \(x \in U_i\) for some \(i\), and since \(B \in \mathcal{B}\), \(B \cap U_i \neq \emptyset\). Let \(y \in B \cap U_i\). Then \(d(x,y) \leq d(x,a_i) + d(a_i,y) < \delta + \delta = 2\delta\), and so

\[x \in S(y,2\delta) \subset V_{2\delta}(B).\]

Hence \(A \subset V_{2\delta}(B)\). It follows that \(\rho(A,B) \leq 2\delta < \varepsilon\). Therefore \(B \in \mathcal{S}(A,\varepsilon)\), and so \(\mathcal{B} \subset \mathcal{S}(A,\varepsilon)\).

Since \(\mathcal{S}(A,\varepsilon)\) is open in the Vietoris topology, we have shown that the metric topology is weaker than the Vietoris topology. Since the stronger topology is compact and the weaker is Hausdorff, they must be identical.
For a bounded metric space $X$, there are numerous interesting real-valued functions defined on $2^X$. We shall find it useful to know that they are continuous in the Hausdorff metric topology.

**Theorem 4.6.** If $(X,d)$ is a bounded metric space, the functions $\text{dist}:2^X \times 2^X \to \mathbb{R}$ and $\text{diam}:2^X \to \mathbb{R}$ are each uniformly continuous in the Hausdorff metric topology.

**Proof.** We shall first show that $\text{dist}$ is continuous. Let $A, B \in 2^X$ and $\varepsilon > 0$. Take $\delta = \varepsilon/3$ and suppose that $H, K \in 2^X$ so that $\rho(A, H) < \delta$ and $\rho(B, K) < \delta$. Then $H \subseteq V_\delta(A)$ and $K \subseteq V_\delta(B)$. There exists $(h, k) \in H \times K$ so that $d(h, k) < \text{dist}(H, K) + \delta$. There exists $(a, b) \in A \times B$ so that $d(h, a) < \delta$ and $d(k, b) < \delta$. Then we have

$$\text{dist}(A, B) \leq d(a, b) \leq d(a, h) + d(h, k) + d(k, b)$$

$$< \delta + \text{dist}(H, K) + \delta + \delta = \text{dist}(H, K) + \varepsilon.$$ 

A similar argument shows that $\text{dist}(H, K) < \text{dist}(A, B) + \varepsilon$, and so $\text{dist}$ is uniformly continuous.

To show that $\text{diam}$ is continuous, let $A \in 2^X$ and $\varepsilon > 0$. Take $\delta = \varepsilon/2$ and suppose that $B \in 2^X$ so that $\rho(A, B) < \delta$. Then $A \subseteq V_\delta(B)$. Hence if $x, y \in A$, there are $s, t \in B$ so that $d(x, s) < \delta$ and $d(y, t) < \delta$. Hence

$$d(x, y) \leq d(x, s) + d(s, t) + d(t, y) < \delta + \text{diam}(B) + \delta$$

$$= \text{diam}(B) + 2\delta.$$ 

Therefore

$$\text{diam}(A) \leq \text{diam}(B) + 2\delta < \text{diam}(B) + \varepsilon.$$
A similar argument shows that diam(B) < diam(A) + ε, and so diam is uniformly continuous.

In 1933, and in an entirely different context, Whitney [3] defined a realvalued function on 2^X which has come to occupy a prominent place in the theory of hyperspaces. Kelley [1] used Whitney's function to greatly extend our knowledge of 2^X when X is a metric continuum.

Definition 4.7. Let X be a topological space. Then μ: 2^X → [0,1] is called a Whitney map provided that
(1) μ is continuous
(2) μ(\{x\}) = 0 for each x ∈ X
(3) If A, B ∈ 2^X, A ⊆ B and A ≠ B, then μ(A) < μ(B)
(4) μ(X) = 1.

Note that condition (3) above implies that if C is any compact chain in 2^X, then μ|C is a homeomorphism.

Theorem 4.8. If (X,d) is a compact metric space, then a Whitney map from 2^X into [0,1] exists.

Proof. Since the details of the proof are routine but tedious, we shall merely outline the construction, and refer the reader to [3] for the details.

Let \{a_i : i = 1,2,...\} be a countable dense subset of X. For each i, and for each p ∈ X, define

f_i(p) = \frac{1}{1 + d(a_i, p)}. 
Then we have

(1) \[ |f_i(p) - f_i(q)| \leq |d(a_i, p) - d(a_i, q)| \leq d(p, q). \]

For each \( A \in 2^X \), and for each \( i \), define

(2) \[ \mu_i(A) = \sup_{p \in A} f_i(p) - \inf_{p \in A} f_i(p) \]

\[ = \sup_{p,q \in A} |f_i(p) - f_i(q)|. \]

Note that \( 0 \leq \mu_i(A) \leq 1 \). Define

(3) \[ \mu(A) = \sum_{i=1}^{\infty} \frac{\mu_i(A)}{2^i} \text{ for each } A \in 2^X. \]

Then \( 0 \leq \mu(A) \leq 1 \). It may be shown that \( \mu \) has properties 1, 2 and 3 of a Whitney map, and the map \( \nu = \frac{\mu}{\mu(X)} \) has all the desired properties.

It should be noted that the Whitney map constructed above is quite dependent upon the metric properties of \( X \). Nadler has produced an example of a metric on an arc \( X \) for which \( \mu \) is not monotone as a map from \( 2^X \to [0,1] \) [2]. Since we shall show in Chapter V that a Whitney map is always monotone and open when it is restricted to the subspace \( C(X) \) of \( 2^X \), Whitney maps are most useful in studying the structure of \( C(X) \).

The following theorem is a useful result relating Whitney maps and the Hausdorff metric.

**Theorem 4.9.** If \( X \) is a compact metric space, and \( \mu \) is a Whitney map on \( 2^X \), then for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that if \( A \) and \( B \) are in \( 2^X \) so that \( A \subset B \) and \( \mu(B) - \mu(A) < \delta \), then \( \rho(A, B) < \varepsilon \).
Proof. Let $\varepsilon > 0$, and suppose, to the contrary, that no such $\delta$ exists. Then, for each positive integer $n$, there exist sets $A_n$ and $B_n$ in $2^X$ so that $A_n \subseteq B_n$, $\mu(B_n) - \mu(A_n) < \frac{1}{n}$, and $\rho(A_n, B_n) \geq \varepsilon$. Since $2^X$ is compact metric, we may assume, by taking subsequences if necessary, that each of $\{A_n\}$ and $\{B_n\}$ converge. Let $A = \lim A_n$ and $B = \lim B_n$. Then clearly $A \subseteq B$, and the continuity of $\mu$ gives $\mu(A) = \mu(B)$. Hence $A = B$ by a property of $\mu$. But our assumption forces $\rho(A, B) \geq \varepsilon$, which is a contradiction.
CHAPTER BIBLIOGRAPHY


CHAPTER V

HYPERSONCES OF A METRIC CONTINUUM

We now have several useful techniques at our disposal for elucidating the structure of $2^X$ and $C(X)$, where $X$ is a metric continuum. We begin with a theorem which gives us good reason for restricting our attention to $C(X)$. Recall that the Hilbert Cube is the space $I^\omega$, when $I$ is the interval $[0,1]$ and $\omega$ is the set of natural numbers.

**Lemma 5.1.** Let $X$ be a nondegenerate metric continuum. There exists a sequence $\{K_i : i = 1, 2, \ldots\}$ in $C(X)$, a sequence $\{p_i\}$ in $X$ so that $p_i \in K_i$ for each $i$, and a point $q \in X - \bigcup_{i=1}^\infty K_i$, so that

1. $K_i \cap K_j = \emptyset$ if $i \neq j$
2. $K_i$ is nondegenerate for all $i$
3. diam $K_i \rightarrow 0$
4. $K_i \rightarrow \{q\}$ in $C(X)$.

**Proof.** The construction is a repeated application of the "boundary bumper" Theorem 3.4. Let $U_1$ be a proper open subset of $X$ so that diam $U_1 \leq 1$. Let $p_1 \in U_1$ and choose an open set $V_1$ so that $p_1 \in V \subset \overline{V_1} \subset U_1$. Note that $\overline{V_1} \neq U_1$ since $X$ is connected. Let $C_1$ be the component
of \( V_1 \) containing \( p_1 \), and put \( K_1 = \overline{C_1} \). Then \( K_1 \) is a non-
degenerate element of \( C(X) \) which is a proper subset of
\( U_1 \), and diam \( K_1 \leq 1 \).

Let \( p_2 \in U_1 - K_1 \). Since \( U_1 - K_1 \) is open in \( X \), there
is an open set \( U_2 \) so that \( p_2 \in U_2 \subseteq \overline{U_2} \subseteq U_1 - K_1 \) and diam \( U_2 \leq \frac{1}{2} \).
Choose an open set \( V_2 \) so that \( p_2 \in V_2 \subseteq \overline{V_2} \subseteq U_2 \), and let \( C_2 \)
be the component of \( V_2 \) containing \( p_2 \). Then \( K_2 = \overline{C_2} \) is a
nondegenerate element of \( C(X) \) so that \( p_2 \in K_2 \), diam \( K_2 \leq \frac{1}{2} \),
and \( K_1 \cap K_2 = \emptyset \).

This process may be continued inductively by letting
\( p_{n+1} \in U_n - K_n \) and repeating the above construction. We
therefore obtain sequences \( \{K_n\} \), \( \{p_n\} \) and \( \{U_n\} \) so that
for each \( n \in \mathbb{N} \), \( K_n \in C(X) = \check{X} \), \( U_n \) is open in \( X \),
\[
\begin{align*}
p_n &\in K_n \subseteq \overline{U_n} \subseteq U_{n-1} - K_{n-1},
\end{align*}
\]
and
\[
\text{diam } \overline{U_n} = \text{diam } U_n \leq \frac{1}{n}.
\]
It is clear that the sequence \( \{K_n\} \) satisfies conditions
(1), (2) and (3) of the theorem.

To find the point \( q \), note that \( \{\overline{U_n}\} \) is a chain in \( 2^X \),
and so
\[
M = \bigcap_{n=1}^{\infty} \overline{U_n} = \lim_{n \to \infty} \overline{U_n}.
\]
Since diam is continuous, \( \text{diam } M = \lim_{n \to \infty} \text{diam } U_n = 0 \), and so
\( M \) is degenerate. Let \( M = \{q\} \). Note that if \( q \in K_j \) for some
\( j \), then the relation \( \overline{U_{j+1}} \subseteq U_{j} - K_j \) implies that \( q \notin \overline{U_{j+1}} \)
which is a contradiction. Hence \( q \in X - \bigcup_{i=1}^{\infty} K_i \).
It is true that \( \{K_i\} \) converges to \( \{q\} \). We shall not prove this, however, but rather observe that since \( C(X) \) is compact metric, there is a convergent sequence \( \{K_{n_i}\} \) of \( \{K_n\} \). Let \( K_{n_i} \to K \) in \( C(X) \). Since \( K_{n_i} \subseteq \overline{U_{n_i}} \), we have

\[
K = \lim_{n_i} K_{n_i} \subseteq \lim_{n_i} \overline{U_{n_i}} = \{q\}. 
\]

Since \( K \neq \emptyset \), we have \( K = \{q\} \).

Therefore the sequences \( \{K_{n_i}\} \) and \( \{p_{n_i}\} \) satisfy the conclusion of the theorem.

**Theorem 5.2.** If \( X \) is a nondegenerate metric continuum, then \( 2^X \) contains a subspace homeomorphic to the Hilbert Cube. Hence any separable metric space is embeddable in \( 2^X \).

**Proof.** Let \( \{K_i\} \) and \( \{p_i\} \) be the sequences constructed in the preceding lemma, and let \( \{q\} = \lim_i K_i \). For each \( i \), there exists an arc chain \( A_i \) in \( C(X) \) from \( \{p_i\} \) to \( K_i \).

Note that since \( C(X) \) is metrizable, \( A_i \) is homeomorphic to \( I \). Let \( H = \prod_{i=1}^{\infty} A_i \), and let \( (A_i)_{i=1}^{\infty} \) be a point of \( H \).

Since \( A_i \subseteq K_i \) for each \( i \), and \( \lim_i K_i = \{q\} \) and \( \text{diam } K_i \to 0 \), we obtain \( \lim_i A_i = \{q\} \). It follows that the set

\[
\{A_i : i = 1, 2, \ldots\} \cup \{\{q\}\}
\]

is closed in \( 2^X \), and so by Theorem 2.10, \( \bigcup_{i=1}^{\infty} A_i \cup \{\{q\}\} \) is closed in \( X \).

Hence we may define a function \( G : H \to 2^X \) by

\[
G(A_i) = \bigcup_{i=1}^{\infty} A_i \cup \{\{q\}\}.
\]
Then $G$ is one to one, since if $(A_i)$ and $(B_i)$ are distinct elements of $H$, there is an index $k$ so that $A_k \neq B_k$.

Since $A_k$ and $B_k$ are in the chain $\mathcal{A}_k$, we may take $A_k \subseteq B_k$.

There exists $x \in B_k - A_k$, and so $x \in G((B_i))$. Now $x \notin K_j$ for all $j \neq k$, and clearly $x \neq q$, and so

$$x \notin \bigcup_{i=1}^{\infty} A_i \cup \{q\} = G((A_1)).$$

Hence $G((A_1)) \neq G((B_1))$, and so $G$ is one to one.

To show that $G$ is continuous, let $(A_i) \in H$ and let $\mathcal{B} = \langle U_i \rangle_{i=1}^{n}$ be a neighborhood of $G((A_1)) = \bigcup A_i \cup \{q\}$.

Then $q \in U_j$ for some $j$, and since $K_i \subseteq \{q\}$, $K_i \subseteq U_j$ for all sufficiently large $i$, say $i \geq N$. Now, for all $i < N$, define $\mathcal{M}_i = \{U_k : U_k \cap A_i \neq \emptyset\}$. Then $\mathcal{M}_i \neq \emptyset$ since $A_i \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{k=1}^{n} U_i$. Further, $\mathcal{M}_i$ is finite, so let

$$\mathcal{M}_i = \{V_{i_1}^1, V_{i_2}^1, \ldots, V_{i_t}^i \} \text{ for } i = 1, 2, \ldots, N-1.$$ Put

$$\mathcal{B}_i = \langle V_{j}^i \rangle_{j=1}^{t_i} \text{ for } i = 1, 2, \ldots, N-1,$$ and define

$$\mathcal{U} = \mathcal{B}_1 \cap \mathcal{A}_1 \times \mathcal{B}_2 \cap \mathcal{A}_2 \times \ldots \times \mathcal{B}_{N-1} \cap \mathcal{A}_{N-1} \times \prod_{i \geq N} \mathcal{A}_i.$$ Then $\mathcal{U}$ is open in $H$. Also, if $i < N$, we have $A_i \in \mathcal{B}_i$ by the choice of the sets $V_{i_j}^i$, $j = 1, 2, \ldots, t_i$. Hence $\mathcal{U}$ is a neighborhood of $(A_1)$ in $H$. Finally, let $(B_i) \in \mathcal{U}$. Then for $i \geq N$, $B_i \subseteq K_i \subseteq U_j$, and for $i < N$, $B_i \in \mathcal{B}_i$, and so

$$B_i \subseteq \bigcup_{j=1}^{t_i} V_{i_j}^i \subseteq \bigcup_{k=1}^{n} U_k.$$
Therefore $G((B_i)) = \bigcup_{i=1}^{\infty} B_i \cup \{q\} \subset \bigcup_{k=1}^{n} U_k$. Also, if $k$ is an index from 1 to $n$, then some $A_i$ meets $U_k$, and so $U_k \in \mathcal{K}_i$. Hence $U_k = V_{i_j}$ for some $j$. But $B_i \in \mathcal{B}_i$ implies that $B_i$ meets $V_{i_j} = U_k$. Hence $G((B_i))$ meets $U_k$. It follows that $G((B_i))$ is in $\langle U_k \rangle_{k=1}^{n}$, and so $G$ is continuous.

Since $H$ is compact, $G$ is a homeomorphism from the Hilbert Cube $H$ into $2^X$.

Since $2^X$ is always so large when $X$ is a metric continuum, we shall henceforward restrict our attention to its subspace $C(X)$. The following theorem was proved by Rogers in [6], and is a useful tool for investigating the dimension of $C(X)$. An $n$-cell is a space homeomorphic to the product $I^n$.

**Theorem 5.3.** Let $X$ be a metric continuum. If there exist proper subcontinua $M, K_1, K_2, \ldots, K_n$ of $X$ satisfying

1. $M \cap K_i \neq \emptyset$ for each $i$
2. $K_i \neq M$ for each $i$
3. $(K_i - M) \cap (K_j - M) = \emptyset$ if $i \neq j$,

then $C(X)$ contains an $n$-cell.

**Proof.** For each $i = 1, 2, \ldots, n$ $M \cup K_i \in C(X)$ by (1), and we have $M \neq M \cup K_i$ by (2). Since $M \subset M \cup K_i$, there exists an arc chain $\mathcal{A}_i$ in $C(X)$ from $M$ to $M \cup K_i$ for $i = 1, 2, \ldots, n$. Note that $\mathcal{A}_i$ is homeomorphic to $I$, and so $\bigcup_{i=1}^{n} \mathcal{A}_i$ is an $n$-cell.
Define the function $f: \bigotimes_{i=1}^{n} A_i \to 2^X$ by $f((A_i)) = \bigcup_{i=1}^{n} A_i$. Then $f$ is continuous by Theorem 1.9. Note that $f$ actually maps into $C(X)$, since $\bigcap_{i=1}^{n} A_i \neq \emptyset$ for any $(A_i) \in \bigotimes_{i=1}^{n} A_i$.

Condition (3) above readily implies that $f$ is 1-1, and so $f$ is the required embedding.

**Corollary 5.4.** $C(I^2)$ contains an $n$-cell for any $n \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$ and take $M = \{(0,y): 0 \leq y \leq 1\}$ and $K_i = \{(x,\frac{i}{n}): 0 \leq x \leq 1\}$ for $i = 1, 2, \ldots, n$. Then $M$ and the $K_i$'s satisfy the conditions of Theorem 5.3.

A continuum $X$ is a triod if it contains subcontinuum $K$ so that $X - K$ is the union of three mutually separated sets. $X$ is atriodic if it contains no triod.

**Theorem 5.5.** Let $X$ be a metric continuum. If $\dim C(X) \leq 2$, then $X$ is atriodic.

**Proof.** If $X$ is not atriodic, it contains a triod $T$. There is a subcontinuum $M$ of $T$ so that $T - M$ is the union of three mutually separated sets $K_1, K_2$, and $K_3$. Then $M, K_1 \cup M, K_2 \cup M$ and $K_3 \cup M$ are each continua (Lemma 3.11) which satisfy the hypotheses of Theorem 5.3. Hence $C(T)$ contains a 3-cell. Since the dimension of a subspace is less than or equal to the dimensions of a space, we have

$$2 \geq \dim C(X) \geq \dim C(T) \geq 3,$$

which is a contradiction. Therefore $X$ is atriodic.

We shall proceed to identify $C(X)$ for some simple spaces $X$. Consider first the closed interval $I = [0,1]$.
If \( A \) is a subcontinuum of \( I \), then \( A = [a, b] \) for some \( 0 \leq a \leq b \leq 1 \). Hence we may define a bijection
\[
f : C(I) \rightarrow \{(x, y) \in I^2 : x \leq y \} = M,
\]
where \( f([a, b]) = (a, b) \). Consideration of the endpoints of a convergent sequence of subcontinua of \( I \) show that \( f \) is continuous, and hence is a homeomorphism.

Let \( S = \{z \in \mathbb{C} : |z| = 1\} \) be the unit circle in the complex plane \( \mathbb{C} \). Let us agree that the "midpoint" of a degenerate subcontinuum \( \{e^{i\theta}\} \) is \( e^{i\theta} \), \( 0 \leq \theta < 2\pi \). If \( A \) is a proper subcontinuum of \( S \), let \( e^{i\alpha} \), \( 0 \leq \alpha < 2\pi \), be the midpoint of \( A \), and let \( \beta, 0 \leq \beta < 2\pi \), be the length of \( A \). Define a bijection \( f : C(X) \rightarrow \{z \in \mathbb{C} : |z| \leq 2\pi\} \) by
\[
f(A) = \begin{cases} (2\pi - \beta)e^{i\alpha} \text{ if } A \neq S, \\ 0 \quad \text{if } A = S, \end{cases}
\]
when \( \alpha \) and \( \beta \) depend on \( A \) as described above. Since the functions which assign to \( A \in C(X) \) its midpoint and length are each continuous, \( f \) is a homeomorphism.

Denote by \( T \) a continuum which is homeomorphic to its name. That is, \( T \) is the union of three arcs \( A_1, A_2 \) and \( A_3 \) so that \( A_i \cap A_j = \{p\} \) iff \( i \neq j \), where \( p \) is an end point of each \( A_i \). Let \( [a_i, p] = A_i \) for \( i = 1, 2, 3 \), and so \( \{a_i : i = 1, 2, 3\} \) are the non-cut points of \( T \). By Theorem 5.3, \( C(T) \) contains a 3-cell. In fact, if we denote by \( C(T, p) \) the set of subcontinua which contains \( p \), the argument of Theorem 5.3 may be used to show that
$C(T, p)$ is a three cell. Then $C(T) = \bigcup_{i=1}^{3} C(A_i) \cup C(T, p)$. Since we have identified $C(A_i)$ above, and $C(T, p) \cap C(A_i)$ is an arc chain for each $i$, the continuum $C(T)$ is a "3-cell with 3 fins" pictured below.

The subspace $\bar{A}$ of $C(T)$ is denoted by the heavy line, and other easily recognized points are identified.

Let $X$ be a metric continuum and let $\mu : C(X) \to I$ be a Whitney map on $C(X)$. For $p \in X$ and $t \in [0,1]$, define the sets $C(X, p) = \{K \in C(X) : p \in K\}$ and $p^t = \mu^{-1}(t) \cap C(X, p)$.
Our immediate goal is to show that \( p \) is a monotone open map. A metric continuum is called Peano iff it is locally connected.

**Theorem 5.6.** If \( X \) is a metric continuum and \( \mu:C(X) \to I \) is a Whitney map, then \( \mu \) is open.

**Proof.** Let \( K \in C(X) \) and \( \mathcal{U} \) be open in \( C(X) \) so \( K \in \mathcal{U} \). Let \( p \in K \). Let \( \mathcal{A} \) be an arc chain from \( \{p\} \) to \( X \) which contains \( K \). Then \( \mu|_{\mathcal{A}} \) is a homeomorphism, and hence is open. Since \( \mathcal{U} \cap \mathcal{A} \) is open in \( \mathcal{A} \), \( \mu|_{\mathcal{A}(\mathcal{U} \cap \mathcal{A})} \) is an open set in \( I \) which contains \( \mu(K) \) and is a subset of \( \mu(\mathcal{U}) \). Hence \( \mu \) is an open map.

**Theorem 5.7.** If \( X \) is a metric continuum, \( \mu:C(X) \to I \) is a Whitney map, and \( p \in X, t \in [0,1] \). Then \( p^t \) is an arc-wise connected continuum.

**Proof.** Note that since there is an arc chain from \( \{p\} \) to \( X \), \( p^t \) is nonempty. Also, \( p^t = \langle X, p \rangle \cap C(X) \cap \mu^{-1}(t) \), each of which is closed in \( 2^X \), and so \( p^t \) is a compact subset of \( C(X) \). We assume with no loss of generality that \( p^t \) is nondegenerate.

Let \( A, B \) be distinct points of \( p^t \). Then \( p \in A \cap B \), and since \( \mu(A) = t = \mu(B) \), neither \( A \) nor \( B \) is a subset of the other. Let \( C \) be the component of \( A \cap B \) containing \( p \). Then \( C \subset A, C \neq A, C \subset B, C \neq B \), and so there are arc chains \( \mathcal{A} \) and \( \mathcal{B} \) in \( C(X) \) from \( C \) to \( A \) and from \( C \) to \( B \), respectively. Define function \( f: \mathcal{A} \times \mathcal{B} \to C(X,p) \) by the
formula \( f(H,K) = H \cup K \). This function is continuous, and maps into \( C(X,p) \) since if \( H \in A \) and \( K \in B \), \( p \in H \cap K \). Denote the image of \( f \) by \( \mathcal{K} \), and let \( \mathcal{B} = \mathcal{K} \cap \mu^{-1}(t) \subset p^t \). Note that \( A = A \cup C \) and \( B = C \cup B \) are each in \( \mathcal{B} \). Let \( T = f^{-1}(\mathcal{B}) \). We wish to show that \( \mathcal{B} \) is a locally connected continuum. The following diagram illustrates the situation.

\[
\begin{array}{ccc}
A \times B & \xrightarrow{f} & \mathcal{K} \\
\downarrow{\text{inc}} & & \downarrow{\text{inc}} \\
A & \xrightarrow{g} & B \\
\downarrow{\text{inc}} & & \downarrow{f|T} \\
T & \xrightarrow{\mu} & \mathcal{B}
\end{array}
\]

Note that \( \mathcal{K} \) is compact, and \( \mathcal{B} \) is a closed subset of \( \mathcal{K} \). It follows that \( T \) is a compact subset of \( A \times B \).

Define a function \( g:T \to A \) by \( g(H,K) = H \). Then \( g \) is the restriction of the projection function \( \mu \) to \( T \), and so \( g \) is continuous. We wish to show that \( g \) is onto \( A \).

Hence let \( H \in A \). Then \( H \subset A \), so \( \mu(H) \leq t \). If \( \mu(H) = t \), then \( H = A \), and since \( (A,C) \in T \), we have \( g(A,C) = A \). Hence assume that \( \mu(H) < t \). Since \( H \subset H \cup B \), and \( \mu(H \cup B) \geq \mu(B) = t \), we have that \( H \neq H \cup B \). Let \( \mathcal{R} = \{H \cup K: K \in B\} \). \( \mathcal{R} \) is clearly the continuous image of \( \mathcal{B} \), and so \( \mathcal{R} \) is a continuum. Hence \( \mu(\mathcal{R}) = [\mu(H), \mu(H \cup B)] \) which contains \( t \). It follows that there exists \( K \in \mathcal{B} \) so that \( \mu(H \cup K) = t \). Hence \( (H,K) \in T \), and so \( H \in g(T) \). Therefore \( g \) is onto \( A \).
Finally, let us define a map \( k: \mathbb{A} \to \mathbb{B} \) by \( K(H) = H \cup K \) for any \((H, K) \in T\). We must show that this map is well defined, so let \((H, K_1)\) and \((H, K_2)\) be in \(T\). We may assume that \(K_1 \subset K_2\). Then \(H \cup K_1 \subset H \cup K_2\), and so 
\[
\mu(H \cup K_1) \leq \mu(H \cup K_2) = t.
\]
It follows that \(H \cup K_1 = H \cup K_2\), and so \(k\) is well defined.

Clearly \(k \cdot g = f\vert T\), and since \(g\) is a quotient map and \(f\vert T\) is continuous and onto, we obtain that \(\mathbb{B}\) is the continuous image of \(\mathbb{A}\). Since \(\mathbb{A}\) is homeomorphic to \(I\), \(\mathbb{B}\) is a Peano continuum, and is therefore arc-wise connected.

Hence \(A\) and \(B\) lie in an arc-wise connected subspace of \(pt\). It follows that \(pt\) is an arc-wise connected continuum.

Theorem 5.8. If \(X\) is a metric continuum and \(\mu: C(X) \to I\) is a Whitney map, then \(\mu\) is a monotone map. That is, \(\mu^{-1}(t)\) is a continuum for each \(t \in I\).

Proof. Since \(\mu^{-1}(0) = \emptyset\) and \(\mu^{-1}(1) = \{X\}\), we may assume that \(0 < t < 1\). Suppose that \(\mu^{-1}(t)\) is the union of nonempty mutually separated sets \(\mathcal{M}\) and \(\mathcal{N}\). Since \(pt\) is a connected subset of \(\mu^{-1}(t)\) for each \(p \in X\), we know that \(pt\) is a subset of either \(\mathcal{M}\) or \(\mathcal{N}\). Let \(M = \{p \in X: pt \subset \mathcal{M}\}\) and \(N = \{p \in X: pt \subset \mathcal{N}\}\). Note that \(M\) and \(N\) are nonempty disjoint sets whose union is \(X\).

To show that \(M\) is closed in \(X\), let \(\{p_n\}\) be a sequence in \(M\) so that \(p_n \to p\). Let \(K_n \in p_n^t\) for each \(n\). Note that \(p_n^t \subset \mathcal{M}\), and so \(K_n \in \mathcal{M}\) for each \(n\). Since \(\mathcal{M}\) is
closed in the compact metric space $C(X)$, we may assume
with no loss of generality that $K_n \to K$, and $K \in \mathcal{K}$. Then
$\mu(K) = t$, and $p \in \lim K_n = K$, and so $K \in p^t \cap \mathcal{K}$. It follows
that $p^t \in \mathcal{K}$, and so $p \in M$. Therefore $M$ is closed in $X$.

Since a similar argument shows that $N$ is closed in
$X$, $M$ and $N$ disconnect $X$, which is a contradiction.
Hence $\mu^{-1}(t)$ is connected, and so $\mu$ is monotone.

If $X$ and $Y$ are spaces, a multivalued function $F: X \to Y$ is a correspondence which assigns to each $x \in X$
a subset of $Y$. If $F(x)$ is compact (a continuum) for each
$x \in X$, then $F$ is said to be compact (continuum) valued.
If $X$ and $Y$ are compact metric spaces and $F$ is a compact
valued function from $X$ to $Y$, then $F$ is said to be upper
(lower) semi-continuous iff $p_n \to p$ in $X$ implies that
$\lim F(p_n) \subseteq F(p)$ ($F(p) \subseteq \lim F(p_n)$). $F$ is said to be con-
tinuous if it is both upper and lower semi-continuous.
It is evident that a compact valued multifunction $F: X \to Y$
is continuous iff the induced single valued function
$F: X \to 2^Y$ is continuous.

If $X$ is a metric continuum, $\mu$ is a Whitney map on
$C(X)$ and $t \in [0,1]$, define $F_t(x) = x^t$ for $x \in X$. Then $F_t$
is a continuum valued function from $X$ onto $\mu^{-1}(t)$. We
shall show that $F_t$ is upper semi-continuous. Let \{p_n\}
be a sequence in $X$ so that $p_n \to p$, and let
$K \in \lim F_t(p_n) = \lim p_n^t$. 

Now certainly $\mu(K) = t$, and so $K \in p^t$ iff $p \in K$. Suppose, to the contrary that $p \notin K$. Then there exist disjoint open sets $U$ and $V$ so $p \in U$ and $K \subset V$. Since $\lim p_n = p$. There is an index $n$ so $i \geq n$ implies that $p_i \in U$. Since $K \subset \lim p_n$ and $(V)$ is a neighborhood of $K$, there is an index $i > n$ so $p_i \cap (V) \neq \emptyset$. Let $L \in p_i \cap (V)$. Then $p_i \in L \subset V$, which contradicts $p_i \in U$. Therefore $p \in K$, and so $K \in p^t$. Hence $\lim F_t(p_n) \subset F_t(p)$, and so $F_t$ is upper semi-continuous.

The following example shows that $F_t$ need not be lower semi-continuous.

Let $X$ be the closure of the graph of the real valued function $f(x) = \begin{cases} \sin \left( \frac{1}{x} \right) + 2, & -\frac{2}{\pi} \leq x < 0 \\ \sin \left( \frac{1}{x} \right), & 0 < x \leq \frac{2}{\pi}. \end{cases}$

Then $X$ is a metric continuum. Let $\mu$ be a Whitney map on $C(X)$. Let $p = (0,1) \in X$. Let

$$K_1 = \{(0,y): 1 \leq y \leq 3\}$$

and

$$K_2 = \{(0,y): -1 \leq y \leq 1\}.$$

Then $K_1$ and $K_2$ are points of $C(X)$. Let $t$ be the smaller of the numbers $\mu(K_1)$ and $\mu(K_2)$. Assume that $t = \mu(K_1)$. Consider the sequence defined by

$$p_n = \left( \frac{2}{(4n+1)\pi}, 1 \right).$$

Then $\{p_n\}$ is a sequence in $X$ and $p_n \to p$. If $F_t$ were lower semi-continuous, then $F_t(p) \subset \lim F_t(p_n)$, and, in particular,
Let \( U \) be a small open disc in \( X \) so that \( U \) meets \( K_1 \) but not \( K_2 \). Then \( K_1 \in \langle X,U \rangle \) and so there exists an index \( N \) and element \( H \in F_t(p_N) \) so that \( H \in \langle X,U \rangle \). Then \( p_n \in H \) and \( H \cap U \neq \emptyset \) implies that \( K_2 \subset H \) and \( K_2 \neq H \). Hence \( \mu(H) > t \). This is a contradiction, since \( H \in p_n^t \). Therefore \( F_t \) is not lower semi-continuous.

It is a consequence of Theorems 5.6 and 5.8 that any Whitney map on \( C(X) \) induces a continuous decomposition of \( C(X) \) into continua so that the resulting decomposition space is an arc. It is useful, therefore, to visualize \( C(X) \) as a cone, with the base \( \hat{X} \) at one end and the vertex, \( X \), at the other. The continua \( \mu^{-1}(t) \) may be considered as sections of the cone. A study of the continua \( \mu^{-1}(t) \), which are called Whitney continua in the literature, has been a fruitful source of information about \( C(X) \).

We shall compute the Whitney continua for some simple spaces. First, let \( \mu \) be a Whitney map on \( C(I) \), where \( I \) is the interval \([0,1]\). Since \( \mu^{-1}(0) = \hat{I} \), we may restrict our attention to the case \( 0 < t < 1 \).

For each \( A \in \mu^{-1}(t) \), define \( f(A) = m \), where \( m \) is the midpoint of \( A \). It is easy to see that \( f \) is continuous. To see that \( f \) is one to one, note that if two subcontinua of \( I \) have the same midpoint, then one contains the other, and so they have different \( \mu \) values. It follows that \( f \) is a homeomorphism of \( \mu^{-1}(t) \) onto a subarc of \( I \).
Consider now a Whitney map $\mu$ on $C(S)$, where $S$ is the unit circle. Let $0 < t < 1$. Geometric considerations show the function which assigns to each $A \in \mu^{-1}(t)$ its midpoint is a homeomorphism of $\mu^{-1}(t)$ onto $S$.

**Theorem 5.9.** If $X$ is an arc-wise connected metric continuum and $\mu$ is a Whitney map on $C(X)$, then $\mu^{-1}(t)$ is arc-wise connected for each $t \in [0,1]$.

**Proof.** We may assume with no loss of generality that $0 < t < 1$. Let $A$ and $B$ be distinct points of $\mu^{-1}(t)$. If $A \cap B \neq \emptyset$, let $p \in A \cap B$. Then $A$ and $B$ lie in the arc-wise connected subset $p^t$ of $\mu^{-1}(t)$. Hence assume that $A \cap B = \emptyset$.

Let $Q$ be an arc in $X$ which intersects each of $A$ and $B$. Let $p \in A \cap Q$ and let $q \in B \cap Q$. We shall distinguish two cases: either $\mu(Q) \leq t$ or $\mu(Q) > t$.

If $\mu(Q) \leq t$, let $\mathcal{A}$ be an arc chain from $Q$ to $Q \cup B$. Then $\mu(\mathcal{A}) = [\mu(Q), \mu(Q \cup B)]$ which contains $t$, since $t = \mu(B) < \mu(Q \cup B)$. Let $K \in \mathcal{A} \cap \mu^{-1}(t)$. Then $Q \subseteq K$ and so $\{p, q\} \subseteq K$. Hence $K \subseteq p^t \cap q^t$. It follows that $p^t \cup q^t$ is an arc-wise connected subset of $\mu^{-1}(t)$ containing both $A$ and $B$.

Assume then that $\mu(Q) < t$. Note that $\mu|C(Q)$ may not be a Whitney map on $C(Q)$, but only because $\mu(Q) < 1$. Since $Q$ is an arc, however, the same argument used in showing that $\mu^{-1}(t)$ is an arc when $\mu$ is a Whitney map on $I$ may be used to conclude that $\mu^{-1}(t) \cap C(Q)$ is an arc in $\mu^{-1}(t)$.
Since \( C(Q) \cap p^t \neq \emptyset \neq C(Q) \cap q^t \), we conclude that 
\( p^t \cup (\mu^{-1}(t) \cap C(Q)) \cup q^t \) is an arc-wise connected subset of \( \mu^{-1}(t) \) containing \( A \) and \( B \).

Hence \( \mu^{-1}(t) \) is arc-wise connected.

We remark that there exist simple examples of continua \( X \) which are not arc-wise connected, but for which \( \mu^{-1}(t) \) is an arc for some \( t \). For example, if \( X \) is the closure of the graph of \( \sin \frac{1}{x} \), \( 0 < x \leq \frac{2}{\pi} \), and \( \mathbf{K} \) is the limit line 
\[ \{(0,y): -1 \leq y \leq 1\}, \]
then \( \mu^{-1}(t) \) is an arc for any \( \mu(\mathbf{K}) < t < 1 \).

We shall now turn our attention to hyperspaces of hereditarily indecomposable continua. It is ironic that these most "pathological" of continua have hyperspaces which are particularly easy to visualize due to the fact that their subcontinua are chained.

**Lemma 5.10.** A continuum \( X \) is hereditarily indecomposable iff whenever \( A \) and \( B \) are in \( C(X) \), then either \( A \cap B = \emptyset \), \( A \subset B \), or \( B \subset A \).

**Lemma 5.11.** Let \( X \) be a hereditarily indecomposable metric continuum, and \( \mu \) be a Whitney map on \( C(X) \). If \( A, B \in C(X) \) so \( A \cap B \neq \emptyset \) and \( \mu(A) = \mu(B) \), then \( A = B \).

**Proof.** The proof is immediate from Lemma 5.10 and a property of \( \mu \).

Now let \( X \) be a nondegenerate hereditarily indecomposable metric continuum, and let \( \mu \) be a Whitney map on \( C(X) \). For each \( t \in [0,1] \), it is evident from Lemma 5.11
that the sets $p^t$ are degenerate for each $p \in X$. Hence we may define a function $f_t: X \to \mu^{-1}(t)$ by the requirement $f_t(p) = A$ iff $\{A\} = p^t$. We have shown above that $f_t$ is always upper semi-continuous when viewed as a multi-valued map. Since in this case it is single valued, it follows easily that $f_t$ is continuous. Note that the subcontinua in $\mu^{-1}(t)$ are all disjoint, and so we may view $\mu^{-1}(t)$ as a decomposition of $X$ into continua. The map $f_t$ is then the natural map induced by this decomposition, and so $f_t$ is a monotone map. We observe that $f_t$ is an open map, since if $U$ is open in $X$, then $f_t(U) = \{A \in \mu^{-1}(t): A \cap U \neq \emptyset\} = \mu^{-1}(t) \cap (X, U)$.

We have therefore proved the following theorem.

**Theorem 5.12.** If $X$ is a hereditarily indecomposable metric continuum and $\mu$ is a Whitney map on $C(X)$, then there exists for each $t \in [0,1]$ a continuous, monotone, and open map $f_t$ from $X$ onto $\mu^{-1}(t)$.

**Corollary 5.13.** If $X$ is a hereditarily indecomposable metric continuum, and $\mu$ is a Whitney map on $C(X)$, then $\mu^{-1}(t)$ is hereditarily indecomposable for each $t \in [0,1]$.

**Proof.** $\mu^{-1}(t)$ is the monotone image of $X$, and hence is hereditarily indecomposable by Theorem 3.19.

The best known example of a hereditarily indecomposable continuum is undoubtedly the pseudo-arc. This continuum was originally defined by Knaster, and has been
extensively studied by Moise and Bing among others. The reader is referred to [2] for a description of this continuum.

Bing [1] has shown that the pseudo-arc is the only hereditarily indecomposable metric continuum $X$ having the property that for each $\varepsilon > 0$, there is a finite open cover $\{V_i : i = 1, 2, \ldots, n\}$ of $X$ so that $V_i \cap V_j \neq \emptyset$ iff $|i-j| \leq 1$, and $\text{diam } V_i < \varepsilon$ for each $i$. Such a cover of $X$ is called on $\varepsilon$-chain, and continua which may be covered by an $\varepsilon$-chain for each positive number $\varepsilon$ are called chainable (also arc-like, or snake-like). Bing [1] also showed that chainable continua are preserved by continuous monotone maps. These observations imply the following corollary.

**Corollary 5.14.** If $P$ is the pseudo arc, and $\mu$ is a Whitney map on $C(P)$, then $\mu^{-1}(t)$ is homeomorphic to $P$ for each $0 < t < 1$.

The maps $\{f_t : t \in I\}$ may be used to define a natural correspondence between $X \times I$ and $C(X)$, when $X$ is a hereditarily indecomposable continuum. For such a continuum, define $\hat{Q} : X \times I \rightarrow C(X)$ by the formula $\hat{Q}(x,t) = f_t(x)$. It is clear that $\hat{Q}$ is onto. $\hat{Q}$ is open, since if $U$ and $V$ are open in $X$ and $I$ respectively, then $\hat{Q}(U \times V) = \mu^{-1}(V) \cap \langle X, U \rangle$, which is open in $C(X)$. $\hat{Q}$ is monotone, since if $A \in C(X)$, then $\hat{Q}^{-1}(A) = A \times \{\mu(A)\}$. The following lemma is useful in proving that $\hat{Q}$ is continuous.
Lemma 5.15. If $X$ is a hereditarily indecomposable metric continuum, and $\mu$ is a Whitney map on $C(X)$, then for each $\varepsilon > 0$ there exists $\delta > 0$ so that if $A, B \in C(X)$, 
\[ \text{dist}(A, B) < \delta \text{ and } |\mu(A) - \mu(B)| < \delta, \] 
then $\rho(A, B) < \varepsilon$.

Proof. The proof proceeds exactly as in Theorem 4.9, with the obvious modifications.

Theorem 5.16. If $X$ is a hereditarily indecomposable metric continuum, $\mu$ is a Whitney map on $C(X)$, and $\tilde{Q}: X \times I \to C(X)$ is the map defined above, then $\tilde{Q}$ is continuous, monotone and open.

Proof. We have already seen that $\tilde{Q}$ is monotone and open. To show that $\tilde{Q}$ is continuous, let $\varepsilon > 0$. Choose $\delta (= \delta(\varepsilon))$ to satisfy the requirements of the preceding lemma. Let $a, b \in X$ so $d(a, b) < \delta$ and $s, t \in I$ so $|s - t| < \delta$. Let $A = \tilde{Q}(a, s)$ and $B = \tilde{Q}(b, t)$. Note that $a \in A$ and $b \in B$, so $\text{dist}(A, B) \leq d(a, b) < \delta$, and $|\mu(A) - \mu(B)| = |s - t| < \delta$. Hence $\rho(A, B) < \varepsilon$, and so $\tilde{Q}$ is continuous.

For any topological space $X$, the cone over $X$ is the space $K(X)$ obtained from $X \times I$ by identifying the subset $X \times \{1\}$ of $X \times I$. Rogers [4], [5] has made a detailed investigation of continua which have cones homeomorphic to hyperspaces.

Theorem 5.17. If $X$ is a hereditarily indecomposable metric continuum, there exists a continuous monotone surjection $h: K(X) \to C(X)$.

Proof. Let $g: X \times I \to K(X)$ be the quotient map. Define $h$ by $h(g(x, t)) = \tilde{Q}(x, t)$. Then $h$ is continuous by a
property of quotient maps. It is obvious that h is monotone.

In spite of the above theorem, it is not possible to embed $C(X)$ in $K(X)$ when $X$ is hereditarily indecomposable. This is a consequence of the following two theorems. We denote the point $X \times \{1\}$ of $K(X)$ by $v$.

**Lemma 5.18.** If $X$ is a hereditarily indecomposable metric continuum, and $A$ is an arc in $K(X) - \{v\}$, then there exists a point $p \in X$ so $A \subseteq \{p\} \times I$.

**Proof.** Note that $K(X) - \{v\}$ is homeomorphic to $X \times [0,1)$. If the theorem is false there are two points $p$ and $q$ of $X$ and numbers $s$ and $t \in I$ so that $(p,s)$ and $(q,t)$ are in $A$. If $\pi : X \times [0,1) \to X$ is the projection, $\pi(A)$ is a nondegenerate Peano subcontinuum of $X$. Such a continuum has a proper subcontinuum with nonempty interior, and hence is decomposable by Corollary 3.13. This is a contradiction, and so the theorem is true.

**Theorem 5.19.** If $X$ is a hereditarily indecomposable metric continuum, then $K(X)$ contains no subset homeomorphic to a capital H.

**Proof.** Such a subset is the union of three arcs so that the intersection of any two is a point. The only possible intersection two such arcs can have the set $\{v\}$.

**Theorem 5.20.** If $X$ is an indecomposable metric continuum, then $C(X)$ contains a set homeomorphic to a capital H.
Proof. Let $K$ be a proper subcontinuum of $X$, and $p_1$ and $p_2$ be distinct points of $K$. Let $p_3$ and $p_4$ be points of $X$ so that no two of the points $p_2, p_3, p_4$ lie in the same composant. Let $\mathcal{A}_i$, $i = 1, 2$, be arc chains from $\{p_i\}$ to $X$ so that $K \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$, and let $\mathcal{A}_i$, $i = 3, 4$, be arc chains from $\{p_i\}$ to $X$. Then the union of the sets $\mathcal{A}_i$ contains the required set.

The following sequence of theorems were originally proved by Kelley in [3]. They give an elegant characterization of hereditarily indecomposable continua in terms of their hyperspace $C(X)$. Let us recall that if $X$ is a compact Hausdorff space, the map $\sigma : 2^X \to 2^X$ defined by $\sigma(\mathcal{A}) = \bigcup \mathcal{A}$ is continuous, and $\sigma(C(C(X))) \subseteq C(X)$. For $M \in C(X)$, we write $C(M)$ to denote the subspace $(M) \cap C(X)$.

Theorem 5.21. If $X$ is an indecomposable metric continuum, and $\mathcal{A}$ is an arc in $C(X)$ so $\sigma(\mathcal{A}) = X$, then $X \in \mathcal{A}$.

Proof. Let $\mu$ be a Whitney map on the subcontinuum $C(\mathcal{A})$ of $C(C(X))$. Let $\Lambda = \{B \in C(\mathcal{A}) : \sigma(B) = X\}$. Then $\mathcal{A} \in \Lambda$. Since $\Lambda = \sigma^{-1}(X) \cap C(\mathcal{A})$, and $\sigma$ is continuous, $\Lambda$ is a compact subset of $C(\mathcal{A})$. Therefore $\mu$ attains its minimum value on $\Lambda$, and so there exists $\mathcal{M} \in \Lambda$ so that $\mu(\mathcal{M}) = \inf \mu(\Lambda)$.

If $\mu(\mathcal{M}) > 0$, then $\mathcal{M}$ is a sub arc of $\mathcal{A}$. It is therefore the union of proper subarcs $\mathcal{M}_1$ and $\mathcal{M}_2$. Since $\mu(\mathcal{M}_i) < \mu(\mathcal{M})$ for $i = 1, 2$, it follows that neither $\mathcal{M}_1$ nor $\mathcal{M}_2$ are in $\Lambda$. Hence $\sigma(\mathcal{M}_1)$ and $\sigma(\mathcal{M}_2)$ are proper subcontinua.
of \( X \) whose union is \( \sigma(\mathcal{M}) = X \). This is a contradiction, and so \( \nu(\mathcal{M}) = 0 \). Therefore \( \mathcal{M} \) is degenerate, and since \( \sigma(\mathcal{M}) = \{X\} \), we have \( \mathcal{M} = \{X\} \). Hence \( X \in \mathcal{A} \).

Note that proof of the preceding theorem is valid if we assume the weaker hypothesis that \( \mathcal{A} \) is a hereditarily decomposable subcontinuum of \( C(X) \).

**Theorem 5.22.** The metric continuum \( X \) is indecomposable if and only if \( C(X) - \{X\} \) fails to be arc-wise connected.

**Proof.** Assume that \( X \) is indecomposable. There exist \( A \) and \( B \) in \( C(X) - \{X\} \) so that \( A \) and \( B \) are in different composants of \( X \). Suppose, to the contrary, that \( C(X) - \{X\} \) is arc-wise connected. Then there exists an arc \( \mathcal{A} \) in \( C(X) - \{X\} \) containing \( A \) and \( B \). Then \( \sigma(\mathcal{A}) \) is a subcontinuum of \( X \) which meets two distinct composants. Hence \( \sigma(\mathcal{A}) = X \), and so by the previous theorem, \( X \in \mathcal{A} \). This is a contradiction, and so \( C(X) - \{X\} \) is not arc-wise connected.

Assume conversely that \( X \) is decomposable. Then \( X = A \cup B \) for proper subcontinua \( A \) and \( B \) of \( X \). Let \( p \in A \cap B \). Then \( \{p\} \in C(A) \cap C(B) \), and so \( C(A) \cup C(B) \) is an arc-wise connected subset of \( C(X) - \{X\} \). Let \( M, N \in C(X) - \{X\} \). Then each of \( C(M) \) and \( C(N) \) meets either \( C(A) \) or \( C(B) \), and so \( C(M) \cup C(N) \cup C(A) \cup C(B) \) is an arc-wise connected subset of \( C(X) - \{X\} \) which contains both \( M \) and \( N \). Hence \( C(X) - \{X\} \) is arc-wise connected.
A space $X$ is said to be uniquely arc-wise connected if there is a unique arc joining each pair of points in $X$.

**Theorem 5.23.** The metric continuum $X$ is hereditarily indecomposable if and only if $C(X)$ is uniquely arc-wise connected.

**Proof.** Assume that $X$ is not hereditarily indecomposable. Then there is a subcontinuum $M$ of $X$ which is the union of its proper subcontinua $A$ and $B$. Let $p \in A \cap B$. There are arc chains $\mathcal{A}_i$, $i = 1,2$, connecting $\{p\}$ to $A$ and $A$ to $M$ respectively, and there are arc chains $\mathcal{B}_i$ connecting $\{p\}$ to $B$ and $B$ to $M$ respectively. Then $\mathcal{A}_2 \cup \mathcal{B}_2$ is an arc from $A$ to $B$ and $\mathcal{A}_1 \cup \mathcal{B}_1$ contains an arc from $A$ to $B$. These two arcs are clearly distinct, and so $C(X)$ is not uniquely arc-wise connected.

Assume conversely that $X$ is hereditarily indecomposable. Let $A$ and $B$ be points of $C(X)$, and let $\mathcal{A} = [A,B]$ be an arc with end points $A$ and $B$. Since $\sigma(\mathcal{A})$ is indecomposable and $\mathcal{A}$ is an arc in $C(\sigma(\mathcal{A}))$, we infer from 5.21 that $\sigma(\mathcal{A}) \in \mathcal{A}$. Let $M = \sigma(\mathcal{A})$. Now let $\phi$ be an arc chain in $C([A,M])$ from $[A]$ to $[A,M]$. We may use 5.21 to conclude that for each $[A,K] \in \phi$, $\sigma([A,K]) \in [A,K] \subset [A,M]$. Since $\phi$ is connected, $\sigma([A,M]) = M$, and $\sigma([A]) = A$, the continuity of $\sigma$ implies that $\sigma(\phi) = [A,M]$. Since $\sigma(\phi)$ is a chain, we infer that $[A,M]$ is a chain. Similar considerations may be used to conclude that $[M,B]$ is a chain.
Since there is only one continuum containing $A$ with a specified $p$ value, $[A,M]$ is the unique arc chain in $C(X)$ from $A$ to $M$. Similarly, $[B,M]$ is the unique arc chain from $B$ to $M$. Hence, $A$ is the only arc in $C(X)$ from $A$ to $B$. Thus, $C(X)$ is uniquely arc-wise connected.

We conclude this chapter with a theorem which may be considered an application of the preceding result. It is an example of hyperspace techniques being used to solve a seemingly "non-hyperspace" problem.

**Theorem 5.24.** Let $X$ and $Y$ be metric continua, with $Y$ hereditarily indecomposable. If $F:X \rightarrow Y$ is a continuous surjection, the $f$ is confluent.

**Proof.** Let $K \in C(Y)$ and let $C$ be a component of $f^{-1}(K)$. Let $q \in C$. Then $f(q) = p$ for some point $p \in K$. There exists a unique arc chain $\mathcal{A}$ in $C(X)$ from $\{p\}$ to $Y$. Note that $K \in \mathcal{A}$. Now let $\mathcal{B}$ be an arc chain in $C(X)$ from $\{q\}$ to $X$ which contains $C$. Since the induced map $f^*:C(X) \rightarrow C(Y)$ is continuous, $f^*(\mathcal{B})$ is a Peano subcontinuum of $C(Y)$ which contains $\{p\}$ and $Y$. Hence there is an arc in $f^*(\mathcal{B})$ from $\{p\}$ to $Y$, and this arc must be $\mathcal{A}$. There exists, therefore, a continuum $H \in \mathcal{B}$ so $f(H) = f(K) = K$. Since $H \subseteq f^{-1}(K)$ and $H$ meets $C$, we have $H \subseteq C$. Hence

$$K = f(H) \subseteq f(C) \subseteq K,$$

and so $f(C) = K$. Therefore $f$ is confluent.
CHAPTER BIBLIOGRAPHY


BIBLIOGRAPHY

Books


Articles


