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STUDIES OF REACTOR CONTAINMENT STRUCTURES

ARF Report No. 8262-2
(Bi-Monthly Progress Report)

by

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TABLE OF CONTENTS

ILLUSTRATIONS.		iv
<u>Section</u>		<u>Page</u>
Task I: <u>ANALYSES OF THE RESPONSE OF</u> <u>CONTAINMENTS VESSELS</u>		1
A. Objectives		1
B. Progress During Review Period		1
C. Plans for Future Work		2
D. Contributors		2
Task II: <u>EFFECT OF OPENINGS ON THE STRENGTH</u> <u>VESSELS</u>		2
A. Objectives		2
B. Progress During Review Period		3
C. Plans for Future Work		6
D. Contributors		6
Task III: <u>BULGING EXPERIMENTS ON PRESSURIZED</u> <u>CYLINDERS OF FINITE LENGTH</u>		6
A. Objectives		6
B. Progress During Review Period		7
C. Plans for Future Work		8
D. Contributors		8
Task IV: <u>DEVELOPMENT OF CONTAINMENT</u> <u>HANDBOOK</u>		8
A. Objectives		8
B. Progress During Review Period		8
C. Plans for Future Work		8
D. Contributors		9

TABLE OF CONTENTS

<u>Section</u>	<u>Page</u>
APPENDIX A APPROXIMATE BURST STRENGTH OF FINITE CYLINDERS WITH VARIOUS END CONDITIONS.	A-1
APPENDIX B BEHAVIOR OF THE INFINITE CYLINDRICAL SHELL UNDER DYNAMIC LOADS.	B-1
APPENDIX C THE EFFECTS OF A CENTRAL RIGID INSERT ON THE STRENGTH OF A FLAT CIRCULAR MEMBRANE.	C-1
I. Introduction	C-1
II. Nomenclature	C-3
III. Solution.	C-4

LIST OF ILLUSTRATIONS

<u>Figure</u>	<u>Page</u>
A-1 Pressure Radius Relation for Cylinder With One End Hemispherically Capped, One End Rigid, $H=0$	A-3
A-2 Pressure Radius Relation for Cylinder With One End Hemispherically Capped, One End Rigid, $H=2$	A-4
A-3 Pressure-Radius Relation for Cylinder With One End Hemispherically Capped One End Rigid, $H=4$	A-5
A-4 Pressure Radius Relation for Cylinder With One End Hemispherically Capped, One End Rigid, $H=6$	A-6
A-5 Burst Pressure as a Function of Hardness for Finite Cylinder With Both Ends Rigid	A-7
A-6 Burst Pressure as a Function of Hardness for Finite Cylinder With One End Rigid, One End Hemispherically Capped	A-8
A-7 Burst Pressure as a Function of Hardness for Finite Cylinder With Both Ends Hemispherically Capped	A-9
A-8 Excess Strength of Finite Cylinder Over Infinite Cylinder, Linear Hardening.	A-10
B-1 Dynamic Load vs Duration for Constant Final Displacement, $n = 0.1$	B-3
B-2 Dynamic Load vs Duration for Constant Final Displacement, $n = 0.5$	B-4
B-3 Dynamic Load vs Duration for Constant Final Displacement, $n = 1.0$	B-5
C-1 Annular Plate With Rigid Insert	C-2
C-2 Deformed Configuration of Plate	C-5
C-3 Linear Isotropic Strain Hardening Characteristics	C-7
C-4 Maximum Displacement vs Insert Size	C-14
C-5 Effect of Strain Hardening on Burst Pressure	C-17

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LIST OF ILLUSTRATIONS (Cont'd)

<u>Figure</u>		<u>Page</u>
C-6	Effect of Insert Size on Burst Pressure.	C-18
C-7	Approximate Relations for Burst Pressure for $0 \leq \bar{c} \leq 0.6$	C-19
C-8	Load-Displacement Curves for $H=2$	C-20
C-9	Load-Displacement Curves for $H = 4$	C-21
C-10	Load-Displacement Curves for $H = 6$	C-22
C-11	Load-Displacement Curves for $H = 8$	C-23
C-12	Load-Displacement Curves for $c = 0$	C-24
C-13	Load-Displacement Curves for $c = 0.3$	C-25
C-14	Load-Displacement Curves for $c = 0.6$	C-26
C-15	Load-Displacement Curves for $c = 0.8$	C-27
C-16	Reduction in Plate Thickness at Burst	C-28

STUDIES OF REACTOR CONTAINMENT STRUCTURES

This is the second bi-monthly progress report (ARF Report No. 8262-2) on Contract No. AT (11-1)-578, Project Agreement 22, of the U. S. Atomic Energy Commission. The following is an account of the work done on the specific Tasks during the months of July-August, 1962.

TASK 1 ANALYSIS OF THE RESPONSE OF CONTAINMENT VESSELS

A. Objectives

The three problem areas in which work on this Task is being carried on under this year's program are:

1. The extension of the incremental method to apply to the solution of cylinders with a rigid base and a hemispherical head.
2. The investigation of shield-structure interaction for spherical and cylindrical shells with an ideal locking shield material.
3. The determination of the dynamic response of vessels with various end conditions.

B. Progress During Review Period

In the first of these problem areas numerical results have been obtained for the static burst pressure of cylinders with one end rigid and one end hemispherically capped using the approximate method previously developed. These results are given in Appendix A along with a comparison of the results for other types of end closures. Work on the development of an exact method for the solution of cylinders with other than rigid end closures continued. These results will be useful in determining the applicability of the approximate solutions.

Work continued on the development of the machine program for the

analysis of shield-structure interaction. There are as yet no results to report.

In the area of dynamic response, solutions have been obtained for the infinite cylindrical shell for a range of material parameters. The analysis and some preliminary results are given in Appendix B. Additional solutions have also been obtained for the complete sphere and will be reported as soon as a convenient method for graphical presentation has been developed.

C. Plans for Future Work

It is planned to continue work in the directions outlined above during the next report period. Specifically, it is expected that working computer programs for the shield-structure interaction problem and for the analysis of the dynamic response of finite cylinders will be completed in this period. It is also expected that the computation of the dynamic response of complete spheres and cylinders for the full range of values of material parameters will be completed.

D. Contributors

Contributions to this Task for the period under review were made by C. Costantino and M. A. Salmon.

TASK 2 EFFECT OF OPENINGS ON THE STRENGTH OF CONTAINMENT VESSELS.

A. Objectives

The purpose of this Task is to investigate the effect of attachments and cutouts on the strength of containment vessels. The program for the current year's funding on this Task consists of the following three phases:

Phase I Review of the literature

Phase II A theoretical study of the effect of circular openings (unreinforced and reinforced) on the plastic behavior of plates. The principal purpose of this phase is to extend the findings of previous

Phase II (cont'd) theoretical studies on this subject to consideration of response under conditions of strain hardening and/or large strains.

Phase III The effect of circular openings (unreinforced and reinforced) on the plastic behavior of thin shallow shells with one direction of curvature (infinite cylinders).

B. Progress During Review Period

During the review period work was continued on Phase I and initial analytical studies were conducted on Phase II.

The literature search of Phase I was essentially completed. No pertinent reports or papers were discovered during the review period. A list of papers previously collected for the literature review was presented in the first bi-monthly progress report.

The first problem which was investigated for Phase II represents a modification of the one which was originally chosen and which was discussed in the first bi-monthly report. The original problem was that of an annular plate with a uniformly distributed transverse load and a uniform tensile load applied at the outer edge. The outer edge was simply supported and free to move in the plane of the plate. The inner edge boundary condition was left undefined at that time. It was intended that the solution of this problem include the effect of bending moments on the behavior of the plate. However, after a careful reconsideration, it was decided to neglect bending moments and to change the outer edge boundary condition to a fixed edge condition. The bending moments were neglected in order to be consistent with investigations which have been conducted in preceding years on this program for momentless vessels. With this assumption the solutions to cutout problems would then provide a fair comparison with the past investigations in order to determine the effect of cutouts on the behavior of the vessels. The outer edge boundary condition was changed to a fixed-

edge condition so as to be able to make use of an existing solution to a similar problem¹ with a fixed outer edge. This solution gives the tensile instability ("burst") pressure for a solid circular plate rather than an annular plate and was based on linear isotropic hardening, large deformations, the Tresca failure criterion and the flow theory of plasticity.

During the period under review, the solution was obtained to the corresponding annular plate problem with a rigid circular insert or "plug" at the center of the plate, fixed outer edge and a uniform pressure load. This work is given in Appendix C. The results revealed that the burst pressure depends on the problem parameters as follows:

- (1) The burst pressure is relatively unaffected by the presence of the rigid insert for "small" inserts, i. e., for ratios, $\bar{r}$, of insert radius to plate radius between 0 and 0.5. The burst pressure is slightly modified for "medium"-size inserts ($0.5 < \bar{r} < 0.8$) and is appreciably lowered only as the insert size approaches the limiting value ($\bar{r} = 1$) where the burst pressure is reduced on the order of 5%-20% depending on the rate of strain hardening. The absolute maximum burst pressure occurs for the case of zero insert radius at low rates of strain hardening and occurs for some non-zero insert size at high rates of hardening.
- (2) The burst pressure is approximately a linearly increasing function of the rate of strain hardening and of the yield stress.
- (3) The burst pressure is proportional to the initial plate thickness and is approximately inversely proportional to the plate radius.

¹ E. W. Ross, Jr. and W. Prager, "On the Theory of the Bulge Test", Quarterly of Applied Mathematics, Vol. 12 pp. 86-91 (1954).

Because of the results stated in (1) above, it was possible to develop a simple approximate analytical expression for the burst pressure which compares well with computed values for small and medium size inserts.

The solution predicts a spherical deformation pattern for the plate throughout the entire loading range (from zero pressure to burst pressure). At low rates of hardening and large insert sizes the plate "bursts" before it reaches hemispherical proportions but at high rates of hardening and small insert sizes the plate will bulge out to a spherical cap corresponding to something greater than a hemisphere at the time of burst.

The burst pressures were obtained under the assumption that the material has an unlimited amount of ductility and therefore could withstand large reductions in thickness without fracturing. The percentage reductions in thickness at burst ranged from 26% for low rates of hardening and small inserts to 66% for high rates of hardening and large inserts.

The development of the solution and the results are given in Appendix C.

In addition to determining the solution to the plate problem an analysis was begun to extend the plate solution to the solution of the spherical cap problem. So far the solution has been approached by two methods. The first method, which might be called the direct approach, consists of determining the burst pressure directly from the governing relations for the spherical cap as was done for the flat plate. This approach has progressed to the point where the solution has been obtained but no numerical computations have been made. The presentation of the results for a wide range of parameters would require numerous sets of graphs. In an attempt to circumvent this apparent necessity, the problem was approached by a second or indirect method. This method employs the concept of an "equivalent" flat plate. By this method it is theoretically possible that the burst pressure of a spherical cap can be determined from knowledge of the burst pressure for an equivalent flat plate thereby

greatly simplifying the graphical presentation of results. However, it appears that this method may work only for a limited range of parameters. Further investigation is required to answer this question.

C. Plans for Future Work

The work on determining burst pressures for spherical caps with rigid inserts by means of an equivalent plate concept will be continued. The range of applicability of the method will be fully established and an attempt will be made to develop a rapid procedure for determination of burst pressure by use of a few simple graphs.

It is also expected that work will begin on determining the effect of flexibility of the insert on the burst pressure.

D. Contributors

Contributions to Task 2 during the review period were made by R. L. Chiapetta, P. G. Hodge, Jr., M. A. Salmon and R. M. Verette.

TASK 3 BULGING EXPERIMENTS ON PRESSURIZED CYLINDERS OF FINITE LENGTH

A. Objectives

This task consists of an experimental investigation of the behavior of thin cylindrical shells with rigid caps and complete pressure vessels when subjected to a series of incrementally increasing internal pressures to failure.

The first test series will utilize thin cylindrical shells suitably closed at each end providing complete radial constraint in the form of rigid end caps; axial displacements of the cylinder, however, will not be restrained. The active shell lengths between closures will vary over the values of approximately 1, 2, 3 and 4 diameters.

The second series of experiments will consist of plastic deformation and bursting tests on vessels of three different forms for each material investigated. These will be represented by cylindrical shells provided with the following end closures, (1) hemispherical heads on both ends, (2) hemispherical head at one end, rigid cap at the other end and (3) elliptical or

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torispherical heads at both ends.

Each form of specimen will be fabricated from three different materials as follows: (1) type 304 austenitic stainless steel, (2) a low carbon ferritic steel and (3) a low-alloy steel represented by ASTM A-302 steel. The specific choice of materials will need to be limited to those available in the form of drawn seamless tubing.

B. Progress During Review Period

During the second bi-monthly report period, the materials for all specimens have been purchased and fabrication has been commenced on the first of the specimens which are to be tested with the ends closed by rigid, self-sealing end plates.

The materials selected were as follows, all being seamless pipe of nominal 8-5/8 in. O.D.:

1. A335P22 (high Mo)
2. Type 304 stainless steel
3. Carbon Steel black seamless pipes, A53, Grade B.

Each lot of material was specified to be from a single heat and adequate evidence was furnished by the suppliers that this specification was followed. Upon receipt, drillings from each length of material were chemically analyzed for alloying metals and significant impurities. Within each lot, the variations in analysis were small. Sufficient excess material if each type was obtained to insure that under any reasonably conceivable circumstances of spoilage or bad luck there will be plenty to perform the planned program.

After cutting material for all the specimens roughly to length, the maximum inside diameter of each individual piece has been carefully determined so that one I.D. could be specified to which all pieces can be accurately bored so that all will be truly circular, have a satisfactory, very smooth finish and accurately fit one set of end closures. This boring operation is currently in progress and no further machining will be authorized until it is completed and a report is received from the fabricator.

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C. Plans for Future Work

It is anticipated that the fabrication of the specimens will be completed during the coming report period and that the fixtures required for the conduct of the tests will be built.

D. Contributors

Contributions to this Task during the report period have been made by S. A. Morse and N. A. Weil.

TASK 4 DEVELOPMENT OF CONTAINMENT HANDBOOK

A. Objectives

The objectives of this Task is the preparation of a handbook which will present the designer the results of the investigations undertaken on this project in a useable form. The scope of this year's work includes the completion of those parts of the handbook dealing with the determination of the static strength of containment vessels as well as those on the dynamic response of spheres and cylinders. In addition a report is to be prepared presenting the data on crushable materials collected in the explosive-driven impact experiments.

B. Progress During Review Period

Work continued on the development of convenient formulas for the static pressure-radius relationship for finite cylinders for both linear and power law types of hardening in order to facilitate the use of these results by the designer. The handbook sections dealing with failure theories and material properties including the effects of strain rate are being prepared and the work on the compilation of the data obtained on the explosive driven plate experiments is continuing.

C. Plans for Future Work

Work will continue along present lines during the next report period. It is expected that the simplified representation of the static response of finite shells will be completed during this period.

D. Contributors

Contributions to this Task during the report period were made by H. Napadenski and M. A. Salmon.

Respectfully submitted,

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APPENDIX A

APPROXIMATE BURST STRENGTH OF FINITE CYLINDERS

WITH VARIOUS END CONDITIONS

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APPENDIX A
APPROXIMATE BURST STRENGTH OF FINITE CYLINDERS
WITH VARIOUS END CONDITIONS

Approximate solutions have been presented for the bursting strength of thin-walled cylinders of finite length with various end conditions.^{1, 2, 3/} The end conditions considered have been (a) both ends rigid, (b) one end rigid, one end hemispherically capped, and (c) both ends hemispherically capped. Numerical results have been presented previously for cases (a) and (c) above.^{1, 2/} Similar results for case (b) are presented herein together with the comparison of the burst strength for all three end conditions. Solutions are presented for hardening materials obeying the linear strain hardening law. Only numerical results are shown herein, and the reader is referred to the references listed below for details of the solution.

In Figs. (A-1), (A-2), (A-3), and (A-4), the pressure-center radius relations are shown for the cylinder with one end rigid and one end hemispherically capped. As for the case of both ends rigid⁽¹⁾, instability conditions cannot be obtained for very short cylinders. This is due to the fact that these cylinders become spherical prior to achieving the instability condition, invalidating the analysis. The curves for these cases are shown dotted in the figures.

-
- 1/ Weil, N. A., "An Approximate Solution for the Bursting of Thin-Walled Cylinders", Report No. ARF 4132-17, U. S. Atomic Energy Commission, Contract No. AT (11-1)-528, Armour Research Foundation, January, 1961.
 - 2/ Costantino, C. J., and Salmon, M. A., "Approximate Bursting Strength of Thin-Walled Cylinders with Hemispherical Caps", Report No. ARF 4132-20, U. S. Atomic Energy Commission, Contract No. AT (11-1)-528, Armour Research Foundation, May 1962.
 - 3/ Weil, N. A., et al, "Studies of Reactor Containment Structures", Appendix A, Progress Report ARF 8262-1, for U. S. Atomic Energy Commission, Contract No. AT (11-1)-578, Armour Research Foundation, July, 1962.

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Figures (A-5), (A-6), and (A-7) contain the burst strength, as a function of the hardening constant, H , for various values of the length-diameter ratio, ℓ , for the three different end conditions. Figure (A-8) shows a comparison of these results. The curves indicate the excess strength of the finite length cylinder over the infinite length cylinder. From these curves, it may be noted that for large length-diameter ratios ($\ell > 3$), end effects are insignificant, in that the excess strength of the finite cylinder over the infinite cylinder is small (less than 10%) even for high values of the hardening constants. For smaller length-diameter ratios, however, the finite cylinder is significantly stronger than the infinite cylinder. In addition, the type of end restraint plays a more important role in determining the bursting strength. Naturally, the cylinder with two rigid ends is significantly stronger than the cylinder with two hemispherically capped ends, with the cylinder with one cap and one rigid end falling in between these two cases.

Fig. A-1 PRESSURE RADIUS RELATION FOR CYLINDER WITH ONE
END HEMISPHERICALLY CAPPED, ONE END RIGID, $H=0$

Fig. A-2 PRESSURE RADIUS RELATION FOR CYLINDER WITH ONE END
HEMISPHERICALLY CAPPED, ONE END RIGID, $H = 2$

Fig. A-3 PRESSURE-RADIUS RELATION FOR CYLINDER WITH ONE END HEMISPHERICALLY
CAPPED ONE END RIGID, $H = 4$

Fig. A-4 PRESSURE RADIUS RELATION FOR CYLINDER WITH ONE
END HEMISPHERICALLY CAPPED, ONE END
RIGID, $H = 6$

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Fig. A-5 BURST PRESSURE AS A FUNCTION OF HARDNESS FOR
FINITE CYLINDER WITH BOTH ENDS RIGID

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Fig. A-6 BURST PRESSURE AS A FUNCTION OF HARDNESS FOR
FINITE CYLINDER WITH ONE END RIGID, ONE END
HEMISPHERICALLY CAPPED

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Fig. A-7 BURST PRESSURE AS A FUNCTION OF HARDNESS FOR
FINITE CYLINDER WITH BOTH ENDS HEMISPHERICALLY
CAPPED

Fig. A-8 EXCESS STRENGTH OF FINITE CYLINDER OVER INFINITE CYLINDER, LINEAR HARDENING

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APPENDIX B

BEHAVIOR OF THE INFINITE CYLINDRICAL SHELL UNDER
DYNAMIC LOADS

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APPENDIX B

BEHAVIOR OF THE INFINITE CYLINDRICAL SHELL UNDER DYNAMIC LOADS

The dynamic response of an infinite cylindrical shell can be found from the analysis similar to that already presented for the spherical shell.^{1/} Therefore, only the final equations will be shown here together with the numerical results for a material obeying Ludwik power strain hardening law.

The static strength of the infinite shell is given by^{2/}

$$\alpha = \left(\frac{2}{\sqrt{3}} \right)^{n+1} \frac{\log \frac{p}{p^*}}{p^*} \quad (\text{B-1})$$

while the burst conditions are given by

$$\alpha^* = \frac{2}{\sqrt{3}} \left(\frac{n}{\sqrt{3}} e \right)^n \quad (\text{B-2})$$
$$p^* = e^{n/2}$$

Considering a dynamic internal pressure pulse applied to the shell, the radius-time relation can be found from the differential equation

$$\frac{d^2 r}{dt^2} = \frac{p}{2} (\alpha_d - \alpha_s) \quad (\text{B-3})$$

1. Weil, N. A., et al., "Studies of Reactor Containment Structures", Appendix B, Progress Report, ARF 8262-1, for U. S. Atomic Energy Commission, Contract No. AT (11-1)-528, Armour Research Foundation, July, 1962.
2. Costantino, C. J., and Salmon, M. A., "Approximate Burst Strength of Thin-Walled Cylinders with Hemispherical Caps", ARF Report No. 4132-20 for U. S. Atomic Energy Commission, Contract No. AT (11-1)-528, Armour Research Foundation, May, 1962.

where τ = dimensionless time
 α_d = dynamic pressure
 α_s = static strength given by Eq. (B-1)

Considering the problem of a linear decaying pressure pulse, of load duration τ_o , and initial peak pressure, α_o , the pressure-time relation is given by

$$\alpha_d = \alpha_o \left(1 - \frac{\tau}{\tau_o}\right) \quad (B-4)$$

The aim is now to determine the final radius that the shell will reach, r_f , when a pressure of given magnitude and duration is applied. This condition is found by determining when the velocity, $dr/d\tau$, first goes to zero.

Numerical methods were used to integrate Eq. (B-3), and the results are shown in Figs. (B-1), (B-2), and (B-3) for values of the hardening exponent of 0.1, 0.5, and 1.0, respectively. These figures relate the parameters α_o/α_s with τ_o for constant values of $\bar{r}$, where $\bar{r} = (r_f^{n+1}/r_s^{n+1})^{1/(n+1)}$ = ratio of final radius to the static burst radius, and α_s = static strength corresponding to r_s , from Eq. (B-1). It should be noted from these figures that for short duration loads, the relation between the dynamic load, α_d/α_s , and load duration is linear on the log-log plot with a slope of 1/2. Hence,

$$\frac{\alpha_o}{\alpha_s} = \frac{f(\bar{r}, n)}{\sqrt{\tau_o}} \quad (B-5)$$

where $f(\bar{r}, n)$ is some unknown function. For longer duration loads, this relationship is no longer valid. The aim now is to determine the approximate analytic form of the relation between α_o/α_s and $\bar{r}$, τ_o , and n , valid over the entire range of load duration, from the numerical results presented herein.

Fig. B-1 DYNAMIC LOAD VS. DURATION FOR CONSTANT FINAL DISPLACEMENT, $n=0.1$

B-5

Fig. B-3 DYNAMIC LOAD VS. DURATION FOR CONSTANT FINAL DISPLACEMENT, $n = 1.0$

APPENDIX C

THE EFFECTS OF A CENTRAL RIGID INSERT ON THE
STRENGTH OF A FLAT CIRCULAR MEMBRANE

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APPENDIX C

THE EFFECTS OF A CENTRAL RIGID INSERT ON THE STRENGTH OF A FLAT CIRCULAR MEMBRANE

I. INTRODUCTION

In this appendix the tensile instability loads (burst pressures) are determined for flat, circular, membrane plates under uniform pressure loading. The plates have a fixed outer edge and a centrally located rigid circular insert as shown in Fig. C-1. The solution was obtained for an incompressible, linear isotropic strain hardening material obeying Tresca's failure criterion and its associated flow law and having negligible elastic strains.

It is shown that an assumed spherical deformation mode leads to the exact solution. Based on this mode a statically admissible solution is first established and then shown to be kinematically admissible.

For convenience the nomenclature is listed in the next section. The presentation of the solution is followed by a discussion of the computational procedures and presentation of the results.

Fig. C-1 ANNULAR PLATE WITH RIGID INSERT

II. NOMENCLATURE

t	=	thickness of deformed plate
t_0	=	initial plate thickness
b	=	support radius
c	=	insert radius
a	=	radius of curvature of sphere
h	=	displacement of insert
η	=	t/t_0
ξ	=	c/b
$\bar{\xi}$	=	a/b
$\bar{h}$	=	h/b
p	=	pressure
σ	=	stress
σ_t	=	meridional stress
σ_c	=	circumferential stress
σ_0	=	yield stress
C	=	rate of strain hardening
ϵ	=	natural strain
τ	=	"time" parameter
α	=	$pb/2\sigma_0 t_0$
H	=	C/σ_0
ϵ_n	=	normal strain
ϵ_c	=	circumferential strain
ϵ_t	=	meridional strain
r	=	distance of a material point from axis of symmetry
$\bar{r}$	=	r/b
y	=	distance of a material point from the support circle
$\bar{y}$	=	y/b

III. SOLUTION

A spherical mode of deformation with uniform thickness is assumed as shown in Fig. C-2. A statically admissible solution is constructed and its kinematic admissibility is established for both small (less than hemispherical proportions) and large deformations.

A. Statically Admissible Solution

If the plate deforms into a sphere the condition of incompressibility for the plate as a whole reduces to

$$\frac{t}{t_0} = \frac{b^2 - c^2}{2ah} \quad (C-1)$$

in which

- t = thickness of deformed plate
- t_0 = initial plate thickness
- b = support radius
- c = insert radius
- a = radius of curvature of sphere
- h = displacement of insert

Equation C-1 may be put in the form

$$\eta = \frac{1 - \bar{c}^2}{2\bar{a}\bar{h}} \quad (C-2)$$

by the introduction of the dimensionless variables

$$\begin{aligned} \eta &= t/t_0 \\ \bar{c} &= c/b \\ \bar{a} &= a/b \\ \bar{h} &= h/b \end{aligned}$$

The stress σ in the sphere is related to the applied pressure p by the equilibrium relation

$$\sigma = \frac{pa}{2t} \quad (C-3)$$

Fig. C-2 DEFORMED CONFIGURATION OF PLATE

A material which strain hardens linearly and isotropically and which satisfies Tresca's yield condition has characteristics illustrated in Fig. C-3 for uniaxial and plane stress states. The symbols in the figure have the following definitions:

- σ_t = meridional stress
- σ_c = circumferential stress
- σ_0 = initial yield stress
- C = "rate" of strain hardening
- ϵ = natural strain
- τ = a suitable "time" parameter

The "normal" (i.e., perpendicular to the shell) stress is negligible and therefore the stress-strain relation for plane stress given in Fig. C-3a can be used, that is

$$\sigma = \sigma_0 + C \int_0^{\tau} \max \left| \frac{d\epsilon}{d\tau} \right| d\tau \quad (C-4)$$

The yield condition states that if the stress condition is such that, in a plot of the stress point in the σ_c , σ_t stress space (Fig. C-3a), the point lies inside the yield hexagon, then no plastic flow occurs. If the stress point lies on the hexagon and the stress rates are such that the point is moving outward across a side, then plastic flow occurs and the yield hexagon expands uniformly with motion of the stress point according to Eq. (C-4). For spherical deformation, $\sigma = \sigma_t = \sigma_c$, and the stress point for the entire shell always is at corner B' (Fig. C-3) of the yield hexagon. The flow law for this corner requires that the circumferential and meridional strain "rates" $d\epsilon_c/d\tau$ and $d\epsilon_t/d\tau$ be non-negative, i.e.,

$$\dot{\epsilon}_c \geq 0, \quad \dot{\epsilon}_t \geq 0 \quad (C-5)$$

where the dots denote differentiation with respect to the "time" parameter. Since the material is incompressible it follows from Eq. (C-5) that the integrand in Eq. (C-4) for corner B' can be written in terms of the normal

$$\sigma = \sigma_0 + C \int_0^{\epsilon} \max \left| \frac{d\epsilon}{d\tau} \right| d\tau$$

(a) Plane Stress

(b) Uniaxial Stress

Fig. C-3 LINEAR ISOTROPIC STRAIN HARDENING CHARACTERISTICS

strain "rate" as follows:

$$\max |\dot{\epsilon}| = |\dot{\epsilon}_n| = \left| \frac{\dot{\eta}}{\eta} \right| \quad (C-6)$$

The stress may now be expressed as

$$\sigma = \sigma_0 - C \log \eta \quad (C-7)$$

The combination of Eqs. (C-3) and C-7) yield the following expression for the dimensionless load parameter α :

$$\alpha = \frac{\eta}{a} (1 - H \log \eta) \quad (C-8)$$

in which

$$\alpha = \frac{p b}{2 \sigma_0 t_0}$$

$$H = C/\sigma_0$$

From the geometry of Fig. C-2 it can be shown that

$$\bar{a} = \frac{1}{2\bar{h}} \sqrt{(1 - \bar{h}^2 - \bar{c}^2)^2 + 4\bar{h}^2} \quad (C-9)$$

Equations (C-2), C-8) and (C-9) can be combined to relate the load parameter α to the material, geometry and displacement parameters H , $\bar{c}$ and $\bar{h}$ respectively.

$$\alpha = \frac{2\bar{h}(1 - \bar{c}^2)}{(1 - \bar{h}^2 - \bar{c}^2)^2 + 4\bar{h}^2} \left\{ 1 - H \log \left[\frac{1 - \bar{c}^2}{\sqrt{(1 - \bar{h}^2 - \bar{c}^2)^2 + 4\bar{h}^2}} \right] \right\} \quad (C-10)$$

The procedure for computing the maximum value of α (dimensionless burst pressure) with respect to $\bar{h}$ is discussed in Section IV.

B. Kinematic Admissibility

In order for the preceding solution to be the exact solution within the framework of the theory, it is necessary that the circumferential and meridional strain "rates" associated with the solution satisfy Eq. (C-5)

$$\dot{\epsilon}_c > 0 \quad , \quad \dot{\epsilon}_t > 0 \quad (C-5)$$

1. "Small" Deformations

The term small deformation is defined here to mean any magnitude of deformation which has not reached hemispherical proportions, that is, for $\bar{h} < \sqrt{1-\bar{c}^2}$. The circumferential strain rate will be examined first. It is given by the relation

$$\dot{\epsilon}_c = \frac{\dot{\bar{r}}}{\bar{r}} \quad (C-11)$$

in which $\bar{r} = r/b$ (see Fig. C-2).

A convenient timelike parameter for this problem was chosen to be $\bar{h}$ so that the dots denote differentiation with respect to $\bar{h}$. In order to determine the sign of $\dot{\epsilon}_c$, expressions for $\dot{\bar{y}}$ ($\bar{y} = y/b$, see Fig. C-2) and $\dot{\bar{a}}$ will be obtained first.

The condition of incompressibility for the material contained between the outer support and a generic circumferential ring located at a distance r from the axis of symmetry is given by

$$\bar{a} \bar{y} \eta = \text{constant} \quad (C-12)$$

The condition of incompressibility for the total volume was stated in Eq. (C-2) as

$$\bar{a} \eta = \frac{1-\bar{c}^2}{2\bar{h}} \quad (C-12)$$

The desired expression for $\dot{\bar{y}}$ is obtained by combining these relations and differentiating to get

$$\dot{\bar{y}} = \frac{\dot{\bar{y}}}{\bar{h}} \quad (C-13)$$

The equation for $\dot{\bar{a}}$ can be derived by differentiating the geometrical relation (see Fig. C-2)

$$\bar{h} = \sqrt{\bar{a}^2 - \bar{c}^2} - \sqrt{\bar{a}^2 - 1} \quad (C-14)$$

to obtain

$$\dot{\bar{a}} = \frac{-\sqrt{\bar{a}^2 - 1} \sqrt{\bar{a}^2 - \bar{c}^2}}{\bar{h} \bar{a}} \quad (C-15)$$

A relationship containing $\dot{\bar{r}}$ can be introduced by differentiating and rearranging the geometrical relation

$$\bar{y} = \sqrt{\bar{a}^2 - \bar{r}^2} - \sqrt{\bar{a}^2 - 1} \quad (C-16)$$

The result is

$$\frac{\bar{r} \dot{\bar{r}}}{\sqrt{\bar{a}^2 - \bar{r}^2}} = -\bar{a} \dot{\bar{a}} \left[\frac{1}{\sqrt{\bar{a}^2 - 1}} - \frac{1}{\sqrt{\bar{a}^2 - \bar{r}^2}} \right] - \dot{\bar{y}} \quad (C-17)$$

Equations (C-13), (C-15) and (C-17) may be solved for $\dot{\bar{r}}$, which when substituted in Eq. (C-11) gives

$$\dot{\epsilon}_c = \frac{\dot{\bar{r}}}{\bar{r}} = \frac{\dot{\bar{y}}}{\bar{h} \bar{r}^2} (\bar{h} - \bar{y}) \geq 0 \quad (C-18)$$

Thus, the circumferential strain rate is seen to satisfy the flow rule, Eq. (C-5). Now all that remains is to demonstrate that $\dot{\epsilon}_t \geq 0$. This may be accomplished by noting that from the condition of incompressibility,

$$\dot{\epsilon}_t = -\dot{\epsilon}_n - \dot{\epsilon}_c \quad (C-19)$$

First, it will be shown that $-\dot{\epsilon}_n$ is positive. Then, since $\dot{\epsilon}_c \geq 0$

from Eq. (C-18), it will be sufficient to subtract the maximum value of $\dot{\epsilon}_c$ to establish the non-negative nature of $\dot{\epsilon}_t$.

By use of Eqs. (C-2) and (C-9), it can be shown that

$$\dot{\epsilon}_n = \frac{\dot{\eta}}{\eta} = \frac{-2\bar{h}(1+\bar{h}^2+\bar{c}^2)}{(1-\bar{h}^2-\bar{c}^2)^2+4\bar{h}^2} \leq 0 \quad (C-20)$$

The maximum value of $\dot{\epsilon}_c$ as a function of $\bar{y}$ can be found by differentiating Eq. (C-18) with respect to $\bar{y}$ and equating the results to zero and using Eq. (C-16). The result is

$$\dot{\epsilon}_{c_{max}} = \frac{\bar{h}}{1+\bar{h}^2+2\bar{c}+\bar{c}^2} \quad (C-21)$$

The requirement that $\dot{\epsilon}_t$ be non-negative may be written

$$\dot{\epsilon}_t \geq \dot{\epsilon}_n \left[-1 - \frac{\dot{\epsilon}_{c_{max}}}{\dot{\epsilon}_n} \right] \geq 0 \quad (C-22)$$

Hence, we must have

$$-\frac{\dot{\epsilon}_{c_{max}}}{\dot{\epsilon}_n} = \frac{(1-\bar{c}^2)+2\bar{h}^2(1+\bar{c}^2)+\bar{h}^4}{2[(1+\bar{c}^2)(1+\bar{c})^2+2\bar{h}^2(1+\bar{c}+\bar{c}^2)+\bar{h}^4]} \leq 1 \quad (C-23)$$

which is easily seen to be satisfied for all permissible ranges of $\bar{c}$ and $\bar{h}$. Therefore the proposed solution is valid in the range $0 \leq \bar{h} \leq \sqrt{1-\bar{c}^2}$.

2. "Large" Deformations

For $\sqrt{1-\bar{c}^2} < \bar{h}$ the checking procedure parallels that used for small deformations. However, Eqs. (C-14) through (C-18) must be replaced by the following in their respective order:

$$\bar{h} = \sqrt{\bar{x}^2 - \bar{c}^2} + \sqrt{\bar{x}^2 - 1} \quad (C-24)$$

$$\frac{\dot{a}}{a} = \frac{\sqrt{\bar{a}^2 - 1} \sqrt{\bar{a}^2 - \bar{e}^2}}{\bar{h} \bar{a}} \quad (C-25)$$

$$\bar{y} = \sqrt{\bar{a}^2 - 1} \pm \sqrt{\bar{a}^2 - \bar{e}^2} \quad (C-26)$$

$$\pm \frac{\bar{r} \dot{\bar{r}}}{\sqrt{\bar{a}^2 - \bar{e}^2}} = \bar{a} \dot{\bar{a}} \left[\frac{1}{\sqrt{\bar{a}^2 - 1}} \pm \frac{1}{\sqrt{\bar{a}^2 - \bar{e}^2}} \right] - \dot{\bar{y}} \quad (C-27)$$

$$\begin{cases} \dot{\epsilon}_c = \frac{\bar{y}}{\bar{h} \bar{r}^2} (\bar{h} - \bar{y}) \geq 0 & (\sqrt{\bar{a}^2 - 1} \leq \bar{y} \leq \bar{h}) \end{cases} \quad (C-28a)$$

$$\begin{cases} \dot{\epsilon}_c = \frac{\bar{y}}{\bar{h} \bar{r}^2} \left(\bar{y} + \frac{1 - \bar{e}^2}{\bar{h}} \right) \geq 0 & (0 \leq \bar{y} \leq \sqrt{\bar{a}^2 - 1}) \end{cases} \quad (C-28b)$$

Where the dual signs appear in Eqs. (C-26) and (C-27), the plus sign corresponds to the upper hemispherical portion (i. e., between the "equator" and the rigid insert) and the minus sign corresponds to the lower hemispherical portion. The value of $\bar{y}$ for the "equator" is

$$\bar{y} = \sqrt{\bar{a}^2 - 1} \quad (C-29)$$

and as seen in Eq. (C-28), the equations for $\dot{\epsilon}_c$ are different for the two regions. The value of $\dot{\epsilon}_c$ is non-negative for both regions and the maximum value occurs in the upper region and is the same as the corresponding expression for small deformations, that is

$$\dot{\epsilon}_{c_{max}} = \frac{\bar{h}}{1 + \bar{h}^2 + 2\bar{e} + \bar{e}^2} \quad (C-21)$$

The normal strain rate $\dot{\epsilon}_n$ is the same as given in Eq. (C-20) for small deformations. Therefore, by the same reasoning as before, the meridional strain rate $\dot{\epsilon}_t$ is non-negative and the solution is valid for all values of $\bar{h}$.

IV. COMPUTATIONAL PROCEDURES

For the range of insert sizes $0 \leq \bar{c} < 1$ the dimensionless burst pressure $\alpha_{\max}$ was determined as a function of H and $\bar{c}$ by merely plotting α as a function of $\bar{h}$ for given values of H and $\bar{c}$ using Eq. (C-10).

The determination of the limiting value of $\alpha_{\max}$ as $\bar{c}$ approaches unity, $\bar{h}_{\max}$ approaches zero. Since the expression for α becomes an indeterminate form for $\bar{c} = 1$ and $\bar{h} = 0$, it was necessary to use a limiting process to determine $\alpha_{\max}$ for $\bar{c} = 1$. To describe this technique it will be convenient to refer to a plot of $\bar{h}_{\max}$ vs. $\bar{c}$ for a given value of H as shown in Fig. (C-4). The solid line shows qualitatively the relation between $\bar{h}_{\max}$ and $\bar{c}$ for a typical value of H . Theoretically, if an explicit analytical expression were known for $\bar{h}_{\max} = f(\bar{c})$, the value of $\alpha_{\max}$ could be computed by substituting this function into Eq. (C-10) for α and taking the limit as $\bar{c}$ approaches unity. However, as mentioned earlier this analytical relation could not be obtained, therefore, an infinity of linear "paths" containing the point $\bar{h}_{\max} = 0$ and $\bar{c} = 1$ in the $\bar{h}_{\max} - \bar{c}$ plane were considered. Several of these are shown in Fig. (C-4) as dotted lines. This is a one-parameter family of paths with the slope, s , as the characterizing parameter. For a particular value of s , say $s = s^*$, the linear path will be tangent to the actual path at the critical point. Therefore, if the value of s^* could be determined, then it would be a simple matter to compute $\alpha_{\max}$.

Fig. C-4 MAXIMUM DISPLACEMENT VS. INSERT SIZE

The equation of the linear path lines is

$$\bar{h}_{max} = s(1-\bar{z}) \quad (C-30)$$

When this is substituted into Eq. (C-10) for α and the limit is taken as $\bar{z}$ approaches unity, the result is

$$\alpha|_{\bar{z}=1} = \frac{s}{1+s^2} \left[1 - H \log \frac{1}{\sqrt{1+s^2}} \right] \quad (C-31)$$

This quantity will attain its maximum value for $s = s^*$. By setting the derivative of this with respect to s equal to zero the following relation is obtained.

$$\frac{1}{\sqrt{1+s^2}} = \exp \left[\frac{-1 + (1-H)s^2}{(s^2-1)H} \right] \quad (C-32)$$

For a given value of H , this may be solved numerically for s^* which can then be used in Eq. (C-31) to determine the corresponding value of α_{max} .

The above procedures were used to construct graphs illustrating the effect of rate of strain hardening and insert size on the burst pressure for the ranges $0 \leq \bar{z} \leq 1$ and $0 \leq H \leq 8$. The results are presented in the following section.

V. RESULTS

This section contains graphs illustrating the effect of strain hardening and insert size on the burst pressure, load-deformation relations, and the thickness reduction at burst.

The effect of strain hardening on the burst pressure is shown in Fig. (C-5). Burst pressure increases with increasing strain hardening in a nearly linear fashion for the entire range considered. Figure (C-6) illustrates the relative insensitivity of the burst pressure to changes in insert

size in the range $0 \leq \bar{r} \leq 0.5$. It should be noted that the absolute maximum value of burst pressure occurs for a zero insert radius for low strain hardening rates and at some non-zero insert size for higher rates of hardening. Approximate analytical expressions for dimensionless burst pressure as a function of $\bar{r}$ and H were derived and are given in Fig. (C-7) where they are compared with actual computed values for the ranges $0 \leq H \leq 8$ and $0 \leq \bar{r} \leq 0.6$. For these values the approximations give fairly accurate results. For values of $\bar{r} > 1$ the approximate relations give results which are in error by an appreciable amount. The linear approximation was used in an attempt to gain overall accuracy by use of one very simple expression. The bilinear approximation represents a simple refinement of the linear one.

Load-displacement relations are plotted in Fig. (C-8) through (C-15). The first four of these show the effect of insert size on the shape of the curves for given values of H . The last four illustrate the effect of H for given values of $\bar{r}$.

The value of the ratio of final thickness at burst to initial thickness as a function of $\bar{r}$ and H is presented in Fig. (C-16). The percentage reductions in thickness at burst range from 26% for low rates of hardening and small size inserts to 66% for high rates of hardening and large size inserts.

Other observations which can be made from an examination of the graphs are that the burst pressure is proportional to the initial plate thickness, approximately inversely proportional to the plate radius and approximately a linearly increasing function of the yield stress.

Fig. C-5 EFFECT OF STRAIN HARDENING ON BURST PRESSURE

Fig. C-6 EFFECT OF INSERT SIZE ON BURST PRESSURE

Fig. C-7 APPROXIMATE RELATIONS FOR BURST PRESSURE FOR $0 \leq \tau \leq 0.6$

Fig. C-8 LOAD-DISPLACEMENT CURVES FOR $H=2$

Fig. C-9 LOAD-DISPLACEMENT CURVES FOR $H = 4$

Fig. C-10 LOAD-DISPLACEMENT CURVES FOR $H = 6$

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Fig. C-11 LOAD-DISPLACEMENT CURVES FOR $H = 8$

Fig. C-12 LOAD-DISPLACEMENT CURVES FOR $c = 0$

Fig. C-13 LOAD-DISPLACEMENT CURVES FOR $c = 0.3$

Fig. C-14 LOAD-DISPLACEMENT CURVES FOR $c = 0.6$

Fig. C-15 LOAD-DISPLACEMENT CURVES FOR $c = 0.8$

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Fig. C-16 REDUCTION IN PLATE THICKNESS AT BURST

END