SOME PROPERTIES OF NOETHERIAN RINGS

THESIS

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This paper is an investigation of several basic properties of noetherian rings. Chapter I gives a brief introduction, statements of definitions, and statements of theorems without proof. Some of the main results in the study of noetherian rings are proved in Chapter II. These results include proofs of the equivalence of the maximal condition, the ascending chain condition, and that every ideal is finitely generated. Some other results are that if a ring R is noetherian, then R[x] is noetherian, and that if every prime ideal of a ring R is finitely generated, then R is noetherian.

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CHAPTER I

INTRODUCTORY CONCEPTS

This thesis investigates some of the properties of Noetherian rings. The definitions and basic theorems which are assumed are stated in this chapter. For proofs of these theorems see [Z and S]. The structure and properties of Noetherian rings are developed in Chapter II.

It is assumed that the reader is familiar with the basic properties of commutative rings and ideals in commutative rings. All rings considered in this thesis are commutative rings with a unity. Addition of ring elements is denoted by +. Multiplication of elements is denoted by $\cdot$, although the symbol will be omitted except when needed for clarity. The additive identity is denoted by 0 and the unity (multiplicative identity) is denoted by 1. Set containment is denoted by $\subseteq$, with proper containment denoted by $<$.

A nonempty subset N of a ring R is called an _ideal of R_ provided x,y in N and r in R imply x-y in N and rx in N.

An element b in a ring R is called a _zero divisor_ if there exists a non-zero element c in R such that $bc = 0$; b is called a _proper zero divisor_ if b is a zero divisor and $b \neq 0$.

A ring is called an <u>integral domain</u> (or simply a <u>domain</u>) if it contains a unity $1 \neq 0$ and contains no proper zero divisors.

A ring R (with a unity $1 \neq 0$) is called a <u>field</u> if for each non-zero element b in R, there exists an element c in R such that $bc = 1$.

Let D be a domain and let $S = \{(a,b) | a,b \varepsilon D \text{ and } b \neq 0\}$. Then (a,b) and (c,d) in S are <u>equivalent</u>, written $(a,b) \sim (c,d)$, if and only if $ad = bc$. It follows that $\sim$ forms an equivalence relation on S, and thus defines a partition of S. Now for $(a,b) \varepsilon S$, let

$$\frac{a}{b} = ab^{-1} = \{(x,y) \ \varepsilon \ S | (x,y) \sim (a,b)\}.$$

Then let $K = \{\frac{a}{b} | (a,b) \ \varepsilon \ S\}$ and define two binary operations such that if $\frac{a}{b}, \frac{c}{d} \ \varepsilon \ K$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \text{ and } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Then $\{K; +, \cdot\}$ is a field which is called the <u>quotient field of D</u>.

A subring R of a field F is called a <u>maximal proper subring</u> of F if and only if $(0) < R < F$ and if there does not exist a subring R' of F such that $R < R' < F$. Equivalently, R is a maximal proper subring of F if and only if $(0) < R < F$ and if there exists a subring R' of F such that $R \subseteq R' < F$ then $R = R'$.

An ideal A in a ring R is called a proper ideal if it is not the zero ideal and not R itself; i.e., if $(0) < A < R$. A is said to be a prime ideal in R if $c,d \in R$ and $cd \in A$ implies that either $c \in A$ or $d \in A$. If A is an ideal in R, then A is maximal in R provided there does not exist an ideal $B \in R$ such that $A < B < R$; i.e., A is maximal in R if B is an ideal in R such that $A < B < R$ implies $B = R$. An ideal A in a ring R is called a principal ideal if there exists an element b in R so that $A = \{br \mid r \in R\}$. The element b is called the generator of the ideal A, and A is denoted by (b). An ideal A in R which is generated by several elements $a_1, a_2, \ldots, a_n \in R$, denoted by $A = (a_1, a_2, \ldots, a_n)$, consists of all finite sums of the form $\sum_i r_i a_i$ where $r_i \in R$. The ideal $A = (a_1, a_2, \ldots, a_n)$ is called a finitely generated ideal.

If A and B are ideals in R, then the sum $A + B$ is defined by $A + B = \{a+b \mid a \in A \text{ and } b \in B\}$, and the product $A \cdot B$ is defined by $A \cdot B = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in A,\ b_i \in B, \text{ and } n = 1,2,\ldots\}$. The sum $A + B$ and product $A \cdot B$ are both ideals in R. It is easily seen that $A,B \subset A + B$ and $A,B \supset A \cdot B$.

If A is an ideal in R, then the radical of A, denoted by $\sqrt{A}$, is the set $\{x \in R \mid x^n \in A$ for some positive integer $n\}$. The $\sqrt{A}$ is itself an ideal in R which contains A. The quotient of A by B is defined as

$$A:B = \{x \in R \mid xB \subset A\}.$$

Theorem 1.1: If A is an ideal in R, then $\sqrt{A} = \bigcap_{\alpha} P_{\alpha}$ where the intersection is taken over all the prime ideals in R which contain A.

It follows immediately from the above theorem that if P is a prime ideal in R, then $\sqrt{P} = P$.

An ideal Q in a ring R is primary if and only if $a, b \in R$, $ab \in Q$ and $a \notin Q$ implies $b^m \in Q$ for some positive integer m.

Theorem 1.2: Let Q be a primary ideal in a ring R. If $P = \sqrt{Q}$, then P is a prime ideal. Moreover, if $ab \in Q$ and $a \notin Q$, then $b \in P$. Also, if A and B are ideals in R such that $Ab \subseteq Q$ and $A \not\subseteq Q$, then $B \subseteq P$.

Theorem 1.3: Let Q and P be ideals in a ring R. Then Q is primary and P is its radical if and only if the following conditions are satisfied:

(1) $Q \subseteq P$;

(2) If $b \in p$, then $b^m \in Q$ for some positive integer m;

(3) If $ab \in Q$ and $a \notin Q$, then $b \in P$.

Note: (3) is equivalent to

(3') If $ab \in Q$ and $b \notin P$, then $a \in Q$.

Let A be an ideal in a ring R. Then a prime ideal P in R is said to be a minimal prime ideal belonging to A if $A \subseteq P$ and there is no prime ideal P' in R such that $A \subseteq P' < P$.

<u>Theorem 1.4</u>: If A is an ideal in a ring R and $P \in R$ a prime ideal such that $A \subseteq P$, then $A \subseteq \sqrt{A} \subseteq P$; i.e., an ideal and its radical are contained in precisely the same prime ideals.

It follows from 1.4 that if Q is a primary ideal in R, then $\sqrt{Q} = P$ is the only minimal prime ideal belonging to Q.

<u>Theorem 1.5</u>: If A is an ideal in a ring R, then the set $R/A = \{x+A \mid x \in R\}$ with addition and multiplication defined by $[x+A] + [y+A] = (x+y) + A$ and $[x+A] \cdot [y+A] = (xy) + A$ where $x,y \in R$ is a ring, called the <u>residue class ring of R by A</u>.

A non-empty subset S of a ring R is a <u>multiplicative system in R</u> if and only if $0 \notin S$, and $a,b \in S$ implies that $a \cdot b \in S$. Let S be a multiplicative system in D, then

$$D_S = \{\tfrac{a}{b} \mid a,b \in D, \text{ and } b \in S\}$$

is called the <u>quotient ring of D with respect to the multiplicative system S</u>. If P is a proper prime ideal of D, then $D \setminus P$ defined by

$$D \setminus P = \{x \in D \mid x \notin P\}$$

is a multiplicative system in D, and $D_{D \setminus P}$ is usually denoted simply as D_P so that

$$D_P = \{\tfrac{a}{b} \mid a,b \in D, \text{ and } b \notin P\}.$$

If A is an ideal of D, then AD_S defined by

$$AD_S = \{ \sum_{i=1}^{m} a_i b_i \mid a_i \in A,\ b_i \in D_S, \text{ and } m \in J_+\}$$

is called the <u>extension of A to D_S</u> (or <u>A extended to D_S</u>). If B is an ideal of D_S, then $B \cap D$ is called the <u>contraction of B in D</u> (or <u>B contracted to D</u>).

Let R be a ring and x an indeterminate. Then $R[x] = \{ \sum_{i=0}^{n} a_i x^i \mid a_i \in R \text{ and } n \in J_+$, the positive integers$\}$.

<u>Theorem 1.6</u>: If R is a domain, then R[x] is a domain.

<u>Theorem 1.7</u>: If A, B, and C are ideals in a commutative ring R, then $A{:}BC = (A{:}B){:}C$.

<u>Theorem 1.8</u>: If A and B are ideals in a commutative ring R, then $A{:}B^{n+1} = (A{:}B^n){:}B = (A{:}B){:}B^n$ for any $n \in J_+$, the positive integers.

<u>Theorem 1.9</u>: If A and B are ideals in a commutative ring R, then $A{:}B = R$ if and only if $B \subseteq A$.

<u>Theorem 1.10</u>: If A and B are ideals in a commutative ring R, then $A{:}B = A{:}(A+B)$.

<u>Theorem 1.11</u>: If A and $\{B_i\}$ are ideals in a commutative ring R, then $A{:} \sum_{i=1}^{m} B_i = \bigcap_{i=1}^{m} (A{:}B_i)$.

<u>Theorem 1.12</u>: If A and B_i are ideals in a commutative ring R, then $(\bigcap_{i=1}^{m} A_i){:}B = \bigcap_{i=1}^{m} (A_i{:}B)$.

<u>Theorem 1.13</u>: If A and B are ideals in a commutative ring R, then $\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

<u>Theorem 1.14</u>: If A and B are ideals in a commutative ring R, then $\sqrt{A+B} = \sqrt{\sqrt{A} + \sqrt{B}} \supseteq \sqrt{A} + \sqrt{B}$.

Theorem 1.15: If A and B are ideals in a commutative ring R and if $A^k \subset B$ for some positive integer k, then $\sqrt{A} \subset \sqrt{B}$.

Theorem 1.16: If A is an ideal in a commutative ring R, then $\sqrt{\sqrt{A}} = \sqrt{A}$.

Theorem 1.17: If an ideal A of a ring R has $\{a_1, a_2, \ldots, a_k\}$ as a basis, then A^n has $\{\prod_{i=1}^{k} a_i^{n_i} \mid \sum_{i=1}^{k} n_i = n$ and $0 \leq n_i \leq n\}$ as a basis.

CHAPTER II

NOETHERIAN RINGS

Definition 2.1: A ring satisfies the ascending chain condition for ideals iff given any sequence of ideals $A_1, A_2, \ldots$ of R with $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$, there exists an integer n such that $A_m = A_n$ for all $m \geq n$.

Definition 2.2: An ideal A is said to be finitely generated if there exists an n such that $A = (a_1, a_2, \ldots, a_n)$.

Definition 2.3: A ring R is said to satisfy the maximum condition iff every non-empty set of ideals of R, partially ordered by inclusion, has a maximal element.

Theorem 2.1: Let R be a ring. The following three statements are equivalent:

(1) R satisfies the ascending chain condition.

(2) R satisfies the maximum condition.

(3) Every ideal of R is finitely generated.

Proof: Show that (1) implies (2). Suppose the maximum condition does not hold. Then there exists a non-empty set of ideals, partially ordered by inclusion such that it does not have a maximum element. Therefore there exists A_1, an ideal in R such that A_1 is not maximal. Hence there is an A_2 such that $A_1 < A_2$. Now A_2 cannot be maximal. Thus there exists A_3 such that $A_1 < A_2 < A_3$ and A_3 is not maximal.

Therefore there is A_n such that $A_1 < A_2 < \ldots < A_n$ and A_n is not maximal. Hence there is A_{n+1} such that $A_1 < A_2 < \ldots < A_n < A_{n+1} < \ldots$ and thus a contradiction to the ascending chain condition. Show that (2) implies (3). Suppose the maximum condition holds. Show A is finitely generated. Let A be an ideal in R. Now consider (0). Now $(0) \subset A$ and either $(0) = A$ or $(0) < A$. If $(0) = A$, then we are finished; otherwise there exists $0 \neq a_1 \in A$ such that $(a_1) \subset A$. Now either $(a_1) = A$ or $(a_1) < A$. If $(a_1) = A$, then we are finished and A is finitely generated. Suppose $(a_1) \neq A$. Then there is $a_2 \in A\setminus(a_1)$ and $(a_1,a_2) \subset A$. Either $(a_1,a_2) = A$ or $(a_1,a_2) < A$. If $(a_1,a_2) = A$, then we are done and A is finitely generated. This process leads us to an ascending chain of ideals $(a_1) < (a_1,a_2) < (a_1,a_2,a_3) < \ldots$, $a_i \in A$. The maximum condition assures us that there is a maximal element in the set. Say $(a_1,a_2,\ldots,a_n)$. Moreover $(a_1,a_2,\ldots,a_n) = A$, for if $(a_1,a_2,\ldots,a_n) \neq A$, then there exists an $a \in A\setminus(a_1,a_2,\ldots,a_n)$ and $(a_1,a_2,\ldots,a_n) < (a_1,a_2,\ldots,a_n,a)$, contradiction. Thus A is finitely generated by $\{a_1,a_2,\ldots,a_n\}$. Show that (3) implies (1). Suppose A is finitely generated and there exists an ascending chain of ideals such that $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ and $\bigcup_{i=1}^{\infty} A_i = A$. Show that the chain terminates. Since A is finitely generated $A = (a_1,a_2,\ldots,a_k)$. Now this means that $a_1 \in A_{i_1}$, $a_2 \in A_{i_2}, \ldots, a_k \in A_{i_k}$. Let $n = \max(i_{1_j}, i_{2_j}, \ldots, i_k)$.

Then $A_{i_1} \subset A_n$, $A_{i_2} \subset A_n, \ldots, A_{i_k} \subset A_n$. Thus $(a_1, a_2, \ldots, a_k) \subset A_n$. Hence $A \subset A_n$, and hence $\bigcup_{i=1}^{\infty} A_i \subset A_n$. Now $A_n \subset A_m$ for all $m \geq n$ and $A_m \subset A_n$ for all m. Thus $A_m = A_n$ for all $m \geq n$. And the ascending chain condition holds.

Definition 2.4: A ring R is noetherian iff R satisfies any of the three conditions of Theorem 2.1.

Theorem 2.2: If R is a noetherian ring and A is an ideal in R, then there exists a positive integer n such that $(\sqrt{A})^n \subset A$.

Proof: Suppose A is an ideal in R. Then $\sqrt{A} = (a_1, a_2, \ldots, a_m)$ since every ideal in R is finitely generated. So for each $a_i \in \sqrt{A}$ there exists $n_i \in J_+$ such that $a_i^{n_i} \in A$. Let $n = n_1 + n_2 + \ldots + n_m$. Now $(\sqrt{A})^n = (a_1, a_2, \ldots, a_m)^n$. So $\{a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_m^{k_m}\}$ is a basis for $(\sqrt{A})^n$, $0 \leq k_i \leq n$, $k_1 + k_2 + \ldots + k_m = n$. Thus $n_1 + n_2 + \ldots + n_m = n = k_1 + k_2 + \ldots + k_m$. Thus there is $1 \leq j \leq m$ such that $k_j \geq n_j$. For if not, then $k_1 + k_2 + \ldots + k_m < n$. Hence $a_j^{k_j} \in A$. Therefore $a_1^{k_1} \cdot a_2^{k_2} \cdot \ldots \cdot a_m^{k_m} \in A$ since $a_i^{k_i} \in R$ for all i. Hence every generator of $(\sqrt{A})^n$ is in A. Thus $(\sqrt{A})^n \subset A$.

Theorem 2.3: Let R be a noetherian ring and Q primary in R. Also suppose A,B are ideals in R such that $AB \subset Q$, then $A \subset Q$ or $B^n \subset Q$ for some positive integer n.

Proof: Suppose A,B are ideals in R such that $AB \subset Q$. Suppose $A \not\subset Q$. Now $B = (b_1, b_2, \ldots, b_t)$. There is an $a \in A \backslash Q$.

Let $b_i \in \{b_1,b_2,\ldots,b_t\}$. Now $ab_i \in Q$, thus since $a \notin Q$, $b_i^{n_i} \in Q$ for some $n_i \in J_+$. Also $b_1^{n_1},\ldots,b_t^{n_t} \in Q$. Let $n_1+n_2+\ldots+n_t = n$. Claim $B^n \subset Q$. Now $B^n = (b_1,b_2,\ldots,b_t)^n$. So $\{b_1^{s_1}\cdot b_2^{s_2}\cdot\ldots\cdot b_t^{s_t} \mid s_i$ are nonnegative integers and $s_1+s_2+\ldots+s_t = n\}$ is a basis for B^n. Let $b_1^{s_1}\cdot b_2^{s_2}\cdot\ldots\cdot b_t^{s_t}$ be any one of the generators of B^n. Since $s_1+s_2+\ldots+s_t = n = n_1+n_2+\ldots+n_t$ there is $1 \le j \le t$ so that $s_j \ge n_j$. For if not, then $s_1+s_2+\ldots+s_t < n$, a contradiction. Then $b_j^{s_j} = b_j^{n_j}b_j^{m_j}$ where $m_j = s_j-n_j \ge 0$. Thus $b_j^{s_j} \in Q$ and it follows that $b_1^{s_1}b_2^{s_2}\ldots b_t^{s_t} \in Q$. That is every generator of B_n is in Q. Thus $B^n \subset Q$.

Theorem 2.4: If Q is a primary ideal of a noetherian ring R and if A and B are ideals of R such that $AB \subset Q$, then either $A \subset Q$ or $(\sqrt{B})^n \subset Q$ for some positive integer n.

Proof: Let Q be a primary ideal of a noetherian ring R. And let A and B be ideals of R such that $AB \subset Q$. Suppose $A \not\subset Q$. By Theorem 2 there exists an $m \in J_+$ such that $(\sqrt{B})^m \subset B$. Thus by Lemma 1 there is an $n \in J_+$ such that $B^n \subset Q$. Hence $[(\sqrt{B})^m]^n \subset B^n \subset Q$. And therefore $(\sqrt{B})^{mn} \subset Q$.

Theorem 2.5: If R is a noetherian ring, then any homomorphic image of R is noetherian.

Proof: Let f be a homomorphism such that $f:R \to R'$ where $R' = \{f(r) \mid r \in R\}$. Let $A_1' \subset A_2' \subset \ldots \subset A_n' \subset \ldots$ be an ascending chain of ideals in R'. Then $f^{-1}(A_1') \subset f^{-1}(A_2') \subset \ldots \subset f^{-1}(A_n') \subset \ldots$ where $f^{-1}(A_i')$ are ideals in R and we have an ascending chain of ideals in R. There is an $n \in \mathbb{N}$

such that $f^{-1}(A'_m) = f^{-1}(A'_n)$ for all $n \geq m$. And $A'_m = f(f^{-1}(A'_m)) = f(f^{-1}(A'_n)) = A'_n$ for all $n \geq m$. Thus R' is noetherian.

Theorem 2.6: If A is an ideal of a noetherian ring R, then the residue class ring R/A is noetherian.

Proof: R/A is the homomorphic image of R under the natural map f defined by $f(r) = r+A$ for all $r \in R$. Thus R/A is noetherian by Theorem 2.5.

Theorem 2.7: If A, B, and C are ideals of a ring R such that (1) $B \subset C$, (2) $B \cap A = C \cap A$, and (3) $B/A = C/A$, then $B = C$.

Proof: Let $c \in C$. Now $B/A = \{b+A | b \in B\}$ and $C/A = \{c+A | c \in C\}$. Since $B/A = C/A$, there exists some $b \in B$ such that $b+A = c+A$. Hence $c-b = a \in A$ for some $a \in A$. Now $b \in C$ since $b \in B \subset C$. Thus $c-b = a \in C$. Hence $a \in A \cap C = A \cap B$ and $a \in B$. So $c = a+b \in B$. Therefore $C \subset B$ and $B = C$.

Theorem 2.8: Let A be an ideal in a ring R. If A and R/A are both noetherian rings, then R is also a noetherian ring.

Proof: Suppose A is an ideal in R, and let $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ be any ascending chain of ideals in R. Now $A_1 \cap A \subset A_2 \cap A \subset \ldots \subset A_n \cap A \subset \ldots$ is an ascending chain of ideals in A, since $A_i \cap A$ for each $i \in J_+$ is an ideal in A. Thus there is an $r \in J_+$ such that $A_n \cap A = A_r \cap A$, for all $n \geq r$. Now consider $A_1/A \subset A_2/A \subset \ldots \subset A_n/A \subset \ldots$ an ascending chain of ideals in R/A. There is an $s \in J_+$ such that $A_n/A = A_s/A$ for all $n \geq s$. Let $M = \max\{s,r\}$. For all

For all $n \geq m$ $A_n \cap A = A_m \cap A$, $A_n/A = A_m/A$, and $A_m \subseteq A_n$. Thus by Theorem 2.7 $A_n = A_m$ for all $n \geq m$. Thus R is noetherian.

Theorem 2.9: If A, an ideal of a ring R, is not prime, then there exist B and C ideals in R such that $A < B$, $A < C$, and $BC \subseteq A$.

Proof: Suppose A is an ideal of a ring R and is not prime. Since A is not prime there exist $b \notin A$ and $c \notin A$ such that $bc \in A$. Let $b, c \in R$ such that b and $c \notin A$. Then $A < A+(b)$ and $A < A+(c)$. Let $A+(b) = B$, and let $A+(c) = C$. Now $A < B$ and $A < C$ and $[A+(b)][A+(c)] = A^2+(b)A+(c)A+(b)(c)$ $= A^2+(b)A+(c)A+(bc)$. Now $A^2 \subseteq A$, $(b)A \subseteq A$, $(c)A \subseteq A$, and $(bc) \subseteq A$, since $bc \in A$. Thus $BC \subseteq A$.

Theorem 2.10: Every ideal in a noetherian ring R contains a product of prime ideals.

Proof: Let $S = \{A,$ ideal in $R \mid A$ does not contain a product of primes$\}$. Now claim S is empty. Suppose S is not empty. Since R is noetherian, S has a maximal element. Call it P. Now P is not prime, for if it were prime, then it would contain a product of primes. Thus there exist ideals B and C in R such that $P < B$, $P < C$, and $BC \subseteq P$. Also $B \notin S$ and $C \notin S$. Hence each of B and C contains a product of prime ideals. Thus BC contains a product of prime ideals. Thus P contains a product of prime ideals, a contradiction to the supposition. Hence S is empty.

Theorem 2.11: If R is noetherian, then R[x] is noetherian.

<u>Proof</u>: Suppose $(0) \neq A$ is an ideal in $R[x]$. For each $n \geq 0$ let $A_n = \{r \in R \mid r=0 \text{ or } rx^n+\ldots+a_1x+a_0 \in A\}$. We note that A_n is an ideal in R and $A_n \subset A_{n+1}$. Suppose $a,b \in A_n$ and $c \in R$; then there exist $f_1(x)$, $f_2(x) \in A$ such that $f_1(x) = ax^n+\ldots+a_1x+a_0$ and $f_2(x) = bx^n+\ldots+b_1+b_0$. Moreover, $f_1(x)-f_2(x) = (a-b)x^n+\ldots+(a_1-b_1)x+(a_0-b_0) \in A$. Hence, either $a-b = 0$ is in A_n or $a-b \neq 0$ is the leading coefficient of a polynomial in A of degree n and again $a-b \in A_n$. Also $cf_1(x) = (ca)x^n+\ldots+ca_1x+ca_0$ is a polynomial in A of degree n with leading coefficient ca. Thus $ca \in A_n$ and A_n is an ideal in R. Now $A_n \subset A_{n+1}$ follows from the fact that if $b \in A_n$, then either $b = 0$ and is in A_{n+1} or b is the leading coefficient of a polynomial $f(x)$ in A of degree n, and hence is the leading coefficient of the polynomial $xf(x)$ in A of degree $n+1$. Thus $b \in A_{n+1}$ and $A_n \subset A_{n+1}$.

Since R is noetherian there is a positive integer t such that $A_t = A_n$ for all $n \geq t$. Moreover, each A_n is finitely generated, say $A_n = (r_{n_1}, r_{n_2}, \ldots, r_{n_{j_n}})$. For each r_{n_j}, $0 \leq n \leq t$, $1 \leq j \leq i_n$, let $f_{n_j}(x)$ be a polynomial in A of degree n with leading coefficient r_{n_j}. Let B be the ideal generated by the polynomials $f_{n_j}(x)$. That is, $B = (\{f_{n_j}(x) \mid 0 \leq n \leq t,\ 1 \leq j \leq j_n\}) = (f_{0_1}(x), \ldots, f_{0_{i_n}}(x), \ldots, f_{t_1}(x), \ldots, f_{t_{i_n}}(x))$. We prove $A = B$, and hence A is finitely generated. It is clear that $B \subset A$, since the $f_{n_j}(x)$ are in A. Let $g(x)$ be any polynomial in A of degree $k \geq 0$. We prove $g(x)$ is in B by induction on k the degree of $g(x)$.

For $k = 0$, $g(x) = g_0$ and, since g_0 is the leading coefficient of a polynomial in A of degree zero, it follows that $g(x) = g_0$ is in A_0. Now $g(x) = g_0 = \sum_{j=1}^{i_n} s_j r_{0_j}$ $= \sum_{j=1}^{i_n} s_j f_{0_j}(x) \in B$ where $s_j \in R$, $1 \leq j \leq i_n$. Thus B contains all polynomials in A of degree zero.

Now suppose B contains all polynomials in A of degree less than k and let g(x) be any polynomial in A of degree k, say $g(x) = rx^k + \ldots + g_1(x) + g_0$, $r \neq 0$. For $k \leq t$, $r \in A_k$, hence $r = \sum_{j=1}^{i_n} s_j r_{k_j}$ for some $s_j \in R$. Consider $h(x)$ $= \sum_{j=1}^{i_n} s_j f_{k_j}(x) \in B \subset A$. Now h(x) has leading coefficient r and is of degree k. And

$$\begin{aligned} h(x) = \sum_{j=1}^{i_n} s_j f_{k_j}(x) &= s_1 f_{k_1}(x) + \ldots + s_{i_n} f_{k_{i_n}}(x) \\ &= s_1(r_{k_1} x^k + \ldots + c_1) + \ldots + s_{i_n}(r_{k_{i_n}} s^k + \ldots + c_{i_n}) \\ &= (s_1 r_{k_1} + \ldots + s_{i_n} c_i) x^k + \ldots + (s_1 r_{k_1} + \ldots + s_{i_n} c_{i_n}) \\ &= rx^k + \ldots + h_0. \end{aligned}$$

Thus $w(x) = g(x) - h(x) - (rx^k + \ldots + g_0) - (rx^k + \ldots + h_0)$ is a polynomial in A of degree less than k and hence by the induction hypothesis is in B.

Otherwise, if $k > t$, then $r \in A_k = A_t$ and $r = \sum_{j=1}^{i_n} s_j r_{t_j}$ for some $j \in R$. Consider $m(x) = \sum_{j=1}^{i_n} s_j x^{k-t} \cdot f_t \cdot g(x) \in B \subset A$. Now

$$m(x) = \sum_{j=1}^{i_n} s_j x^{k-t} f_{t_j}(x)$$

$$= s_1x^{k-t}(r_{t_1}x^t+\ldots+d_1)+\ldots+s_{i_n}(r_{t_{i_n}}x^t+\ldots+d_{i_n})$$

$$= (s_1r_{t_1}+\ldots+s_{i_n}r_{t_{i_n}})x^k+\ldots+s_{i_n}d_{i_n}$$

$$= rx^k+\ldots+m_0.$$

Thus $n(x) = g(x)-m(x) = (rx^k+\ldots+g_0)-(rx^k+\ldots+m_0)$ is a polynomial in A of degree less than k and hence is in B. Hence $g(x) = n(x)+m(x)$ is in B and $A \subseteq B$. Thus $A = B$.

Definition 2.5: Let A be an ideal of a ring R. Then A is irreducible if and only if it is not a finite intersection of ideals of R properly containing A; otherwise A is reducible.

Theorem 2.12: Every ideal in a noetherian ring is a finite intersection of irreducible ideals.

Proof: Let F be the set of all ideals of R which are not finite intersections of irreducible ideals. Suppose $F \neq 0$. Then there exists an ideal B which is a maximal element in F. Let E be any ideal of R which properly contains B; then $E \notin F$. Hence E is a finite intersection of irreducible ideals. Since $B \in F$, B is not irreducible. Thus $B = A_1 \cap A_2 \cap \ldots \cap A_k$, and $B < A_i$, for $1 \leq i \leq k$. Now $A_1 = \bigcap_{i=1}^{n_1} A1_i$ where $A1_i$ is irreducible, $A_2 = \bigcap_{i=1}^{n_2} A2_i$ where $A2_i$ is irreducible, ..., and $A_k = \bigcap_{i=1}^{n_k} Ak_i$ where Ak_i is irreducible. Thus B is a finite intersection of irreducible ideals, a contradiction. Hence $F = \emptyset$ and the theorem follows.

Theorem 2.13: If R is a noetherian ring, then every irreducible ideal of R is primary.

Proof: We prove that if A is an ideal of R and A is not primary, then A is reducible. Since A is not primary there exist $a,b \in R$ such that $ab \in A$, $b \notin A$ and no power of a is in A.

Consider $A:(a) \subset A:(a^2) \subset \ldots \subset A:(a^n) \subset \ldots$ which is an ascending chain of ideals of R. For if $x \in A:(a^i)$ then $x(a^i) \subset A$ and $xa^i \in A$. Now $xa^i \cdot a \in A$ implies $xa^{i+1} \in A$ which means that $x(a^{i+1}) \subset A$. Thus $x \in A:(a^{i+1})$. Hence there exists an integer n such that $A:(x^n) = A:(a^{n+1})$.

We prove $A = [A+(a^n)] \cap [A+(b)]$ where $A < A+(a^n)$ and $A < A+(b)$. Clearly $A \subset [A+(a^n)] \cap [A+(b)]$. Let $x \in [A+(a^n)] \cap [A+(b)]$; then $x = a_1+r_1a^n = a_2+r_2b$, for $a_1,a_2 \in A$, $r_1,r_2 \in R$. Thus $r_1a^n = a_2-a_1+r_2b$ and $r_1a^{n+1} = a[a_2-a_1+r_2b] = [a_2-a_1]a+r_2[ab] \in A$. Thus $r_1 \in A:(a^{n+1}) = A:a^n$. Hence $r_1a^n \in A$ and $x = a_1+r_1a^n \in A$. Therefore $A = [A+(a^n)] \cap [A+(b)]$ where $A < A+(a^n)$ and $A < A+(b)$ since $a^n \notin A$ and $b \notin A$. Thus A is reducible.

Theorem 2.14: Every ideal in a noetherian ring can be represented as a finite intersection of primary ideals.

Proof: The theorem follows from Theorems 2.12 and 2.13.

Definition 2.6: A representation $A = \bigcap_{i=1}^{n} Q_i$, where the Q_i are primary ideals, is called a primary representation of the ideal A. The Q_i are called the primary components of A and $\sqrt{Q_i}$ are called the associated prime ideals of A.

Definition 2.7: A primary representation $A = \bigcap_{i=1}^{n} Q_i$ is said to be irredundant if it satisfies the following conditions:

(1) no Q_i contains the intersection of the other primary components, i.e. $\bigcap_{j\neq i} Q_j \not\subset Q_i$ for $i=1,2,\ldots,n$.

(2) $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$.

Theorem 2.15: Every ideal in a noetherian ring has a finite irredundant representation.

Proof: Suppose $A = \bigcap_{i=1}^{n} Q_i$ is a primary representation. Let Q_i' be the intersection of all those primary components which have the same associated prime. That is, if $\sqrt{Q_{i_1}} = \sqrt{Q_{i_2}} = \ldots = \sqrt{Q_{i_r}}$, then take $Q_i' = Q_{i_1} \cap Q_{i_2} \cap \ldots \cap Q_{i_r}$. Now Q_{i_r}' is primary and $\sqrt{Q_i'} = \sqrt{Q_{i_r}}$ and $A = \bigcap_i Q_i'$. In this way we make the associated primes distinct. Next delete one at a time those ideals Q_i' which contain the intersection of the remaining ones.

Definition 2.8: A minimal element in the family of associated prime ideals of an ideal A of a ring R is called an isolated prime ideal of A.

Theorem 2.16: Let R be a ring, A an ideal of R such that A has a finite irredundant primary representation $A = \bigcap_{i=1}^{n} Q_i$, and let $P_i = \sqrt{Q_i}$. A prime ideal P of R contains A if and only if P contains some P_i. Thus, the isolated prime ideals of A are the minimal elements of the family of prime ideals of R which contain A.

Proof: If $P \supset P_i$ for some i, then $P \supset P_i \supset Q_i \supset \bigcap_{i=1}^{n} Q_i = A$. Conversely if $P \supset A = \bigcap_{i=1}^{n} Q_i \supset \prod_{i=1}^{n} Q_i$, then $P \supset Q_i$ for some i. Thus, $P \supset \sqrt{Q_i} = P_i$.

Let $S = \{P \mid P$ is a minimal element of the family of all primes containing $A\}$. P_i is an isolated prime of A if and only if $P_i \in S$. Suppose $P_i \in S$. Then P_i does not properly contain any prime ideal P containing A. Hence, $P_i \not\supset P_j$ for all $j \neq i$ and P_i is isolated.

Now suppose P_i is an isolated prime of A. Then $P_i \supseteq A$ and P_i contains no P_j for which $i \neq j$. We claim $P_i \in S$. Suppose there is a $P \in S$ such that $P_i > P$. Then $P_i > P \supseteq A$. Hence $P \supseteq P_j$ for some $j \neq i$. And $P_i \supseteq P_j$, a contradiction. Therefore P_i is in S.

Theorem 2.17: Suppose that an ideal A in a ring R has a finite irredundant primary representation, $A = \bigcap_{i=1}^{n} Q_i$, and let P be any prime ideal in R. Then $P = \sqrt{Q_i}$ for some i if and only if there exists an element $a \in R$ such that $a \notin A$ and $P = \sqrt{A:(a)}$.

Proof: Suppose $P = \sqrt{Q_i}$ for some i and $A = \bigcap_{k=1}^{n} Q_k$ $= (\bigcap_{k\neq i} Q_k) \cap Q_i$. Since A is irredundant $Q_i \not\supseteq \bigcap_{k\neq i} Q_k$. Thus there is an $a \in \bigcap_{k\neq i} Q_k$ such that $a \notin Q_i$. That is, $a \in \bigcap_{k\neq i} Q_k$, $a \notin A$. Now prove $A:(a)$ is primary and P is its radical.

Since $A:(a) = \{b \in R \mid b(a) \subseteq A\}$, $a[A:(a)] \subseteq A \subseteq Q_i$, $a \notin Q_i$. Hence $A:(a) \subseteq \sqrt{Q_i} = P$. Now $aQ_i \subseteq (\bigcap_{k\neq i} Q_k) \cap Q_i = A$. Thus $Q_i \subseteq A:(a)$ and hence $P = \sqrt{Q_i} \subseteq \sqrt{A:(a)}$. Therefore, if $x \in P$, then $x \in \sqrt{A:(a)}$ and $x^n \in A:(a)$ for some $n \in J_+$. Now suppose $bc \in A:(a)$, $b \notin P = \sqrt{Q_i}$, then prove $c \in A:(a)$. So $[ac]b = a[bc] \in a[A:(a)] \subseteq A \subseteq Q_i$, $b \notin P$ imply $ac \in Q_i$. Also $ac \in (a) \subseteq \bigcap_{k\neq i} Q_k$. Thus $ac \in (\bigcap_{k\neq i} Q_k) \cap Q_i = A$ and $c \in A:(a)$. Hence $A:(a)$ is primary and P is its radical.

Suppose there is an $a \notin A$ and $P = \sqrt{A:(a)}$. Now $A:(a) = (\bigcap_{i=1}^{n} Q_i):(a) = \bigcap_{i=1}^{n} [Q_i:(a)]$. And $P = \sqrt{A:(a)} = \bigcap_{i=1}^{n} \sqrt{Q_i:(a)}$. If $a \in Q_i$, then for every $y \in R$, $ya \in Q_i$, and $y \in Q_i:a$. Thus $R \subset Q_i:(a)$ which implies that $R = Q_i:(a)$. Also $Q_i:(a) \subset \sqrt{Q_i:(a)} \subset R = Q_i:(a)$. Therefore $\sqrt{Q_i:(a)} = R$. Suppose $a \notin Q_i$. We claim $\sqrt{Q_i:(a)} = \sqrt{Q_i}$. Now $Q_i \subset Q_i:(a)$ is always true. So $a[Q_i:(a)] \subset Q_i$, and $a \notin Q_i$ imply that $Q_i:(a) \subset \sqrt{Q_i} = P$. Thus $Q_i \subset Q_i:(a) \subset \sqrt{Q_i}$, and hence $\sqrt{Q_i} \subset \sqrt{Q_i:(a)} \subset \sqrt{\sqrt{Q_i}} = \sqrt{Q_i}$. Therefore $\sqrt{Q_i:(a)} = \sqrt{Q_i}$. Hence $P = \sqrt{Q_{i1}} \cap \sqrt{Q_{i2}} \cap \ldots \cap \sqrt{Q_{in}}$ where $\{Q_{in}\}$ is the subset of $\{Q_i\}$ with $a \notin Q_{ir}$. That is $P = P_{i1} \cap P_{i2} \cap \ldots \cap P_{ir} \supset P_{i1} \cdot P_{i2} \cdot \ldots \cdot P_{ir}$. Thus $P \supset P_{ij}$ for some j, but $P \subset P_{ir}$ for all r. Hence $P = P_{ij}$ for some P_{ij}. That is $P = \sqrt{Q_i}$ for some i.

<u>Theorem 2.18</u>: If $A = \bigcap_{i=1}^{n} Q_i = \bigcap_{i=1}^{m} Q_i'$ are two different irredundant representations, then $n = m$ and $\sqrt{Q_i} = \sqrt{Q_j'}$ in some order.

<u>Proof</u>: Let $P_i = \sqrt{Q_i}$ for each i. There exists an $a \in R$ such that $a \notin A$ and $A:(a)$ is primary for P_i. Thus $P_i = \sqrt{Q_j'} = P_j'$ for some j. Hence $\{P_i\} \subset \{P_j'\}$ and similarly $\{P_j'\} \subset \{P_i\}$. Therefore $\{P_i\} = \{P_j\}$ and they have the same number of elements. Thus $m = n$. We are now guaranteed that primes in one irredundant representation are the primes in the other irredundant representation, possibly in a different arrangement.

<u>Example</u>: Let F be a field and $F[x,y]$ the polynomial ring in two indeterminates. Now $(x^2,xy) = (x^2,xy,y^2) \cap (x)$

and $(x^2,xy) = (x^2,y) \cap (x)$. There we have (x^2,xy) having two different finite irredundant primary representations. However, $\sqrt{(x^2,xy,y^2)} = (x,y)$ and $\sqrt{(x^2,y)} = (x,y)$. Of course $\sqrt{(x)} = (x)$. Furthermore, for every positive integer n, $(x^2,xy,y^n) \cap (x)$ is an irredundant primary representation for (x^2,xy). Moreover, if $(x^2,xy) = \bigcap_{i=1}^{n} Q_i$ is any finite irredundant primary representation, then $n = 2$ and the associated prime ideals of (x^2,xy) are (x) and (x,y). We also have (1) (x^2,y) is primary but is not a prime power, and (2) (x^2,xy) has prime radical (x), but is not primary.

<u>Theorem 2.19</u>: If A is an ideal of a ring R with a finite irredundant primary representation $A = \bigcap_{i=1}^{n} Q_i$, then A is semi-prime if and only if each Q_i is a prime ideal of R.

<u>Proof</u>: An ideal A is semi-prime if and only if $A = \sqrt{A}$. Suppose Q_i are prime ideals. Let $x \in \sqrt{A}$. Then $x^n \in A = \bigcap_{i=1}^{n} Q_i$, and thus $x^n \in Q_i$ for all i. Since the Q_i are prime, $x \in Q_i$ for all i. Therefore $x \in \bigcap_{i=1}^{n} Q_i = A$. Hence $\sqrt{A} \subset A$, and since $A \subset \sqrt{A}$ is always true, $\sqrt{A} = A$. Thus A is semi-prime. Now suppose A is semi-prime. Thus $A = \sqrt{A} = \sqrt{\bigcap_{i=1}^{n} Q_i}$ $= \bigcap_{i=1}^{n} \sqrt{Q_i} = \bigcap_{i=1}^{n} P_i$. We claim $\bigcap_{i=1}^{n} P_i$ is an irredundant representation. Suppose $\bigcap_{i=1}^{n} P_i$ is not an irredundant representation. Then there exists a $j \in J_+$ such that $P_j \supset \bigcap_{i \neq j} P_i$. Then $A = \sqrt{A} = \bigcap_{i \neq j} P_i = \bigcap_{i \neq j} \sqrt{Q_i} \supset \bigcap_{i \neq j} Q_i \supset A$ which implies $A = \bigcap_{i \neq j} Q_i$, a contradiction to irredundancy of the Q_i. Thus the P_i's representation is irredundant.

Let j be an arbitrary element in $\{1,2,\ldots,n\}$. Let $a \in \sqrt{Q_j}$. Since $\bigcap_{i\neq j}\sqrt{Q_i} \neq \bigcap_{i=1}^{n}\sqrt{Q_i}$, there exists $b \in \bigcap_{i\neq j}\sqrt{Q_i}$, such that $b \notin \bigcap_{i=1}^{n}\sqrt{Q_i} = A$. Now $ab \in \sqrt{Q_j} \cap (\bigcap_{i\neq j}\sqrt{Q_i}) = A \subset Q_j$, which implies that $a \in Q_j$, and $b \notin \sqrt{Q_j}$. Thus $\sqrt{Q_j} \subset Q_j$. Hence $Q_j = \sqrt{Q_j}$, and since Q_j is primary $\sqrt{Q_j}$ is prime. Therefore Q_j is prime.

Theorem 2.20: Let A and B be ideals of a ring R, with A finitely generated. If AB = A, then there exists an element $b \in B$ such that $[1-b]A = (0)$.

Proof: Suppose $A = (a_1,a_2,\ldots,a_n) = (a_1,a_2,\ldots,a_n,0)$. Let $A_1 = (a_i,a_{i+1},\ldots,a_n,0)$ so that $A_1 = A$, $A_n = (a_n,0)$, and $A_{n+1} = (0)$.

We prove by induction on i the existence of an element $b_i \in B$ such that $[1-b_i]A \subset A_i$ for $i=1,2,\ldots,n+1$. In particular $b_{n+1} = b$. For $i = 1$, $A = A_1$ and $b_1 = 0$. Thus $[1-b]A = [1-0]A = 1A = A = A_1 \subset A_1$. Hence true for $i=1$. Now suppose there is a $b_i \in B$ such that $[1-b_j]A \subset A_i$. Then since $AB = A$, $[1-b_i]A = [1-b_i]AB \subset A_iB = (a_i,a_{i+1},\ldots,a_n,0)B$. Thus $[1-b_i]a_i \in [1-b_i]A_i \subset [(a_i)+(a_{i+1})+\ldots+(a_n+(0)]B$ $= a_iB+a_{i+1}B+\ldots+a_nB$, and hence $[1-b_i]a_i = \sum_{k=i}^{n} b_{i_k}a_k$, where $b_{i_k} \in B$ and $a_k \in A_i$, for $i \leq k \leq n$. Now $[1-b_i-b_{ii}]a_i$ $= \sum_{k=i+1}^{n} b_{i_k}a_k \in A_{i+1}$. Let $1-b_{i+1} = [1-b_i][1-b_i-b_{ii}]$. Then $b_{i+1} \in B$. Therefore $[1-b_{i+1}]A = [1-b_i][1-b_i-b_{ii}]A$ $= [1-b_i-b_{ii}][(a_i)+A_{i+1}] = [1-b_i-b_{ii}]([1-b_i]A) \subset [1-b_i-b_{ii}]A_i$ $= [1-b_i-b_{ii}][(a_i)+A_{i+1}] = [1-b_i-b_{ii}]a_i+[1-b_i-b_{ii}]A_{i+1}$ $\subset A_{i+1}+A_{i+1} = A_{i+1}$. Hence $[1-b_{n+1}]A = A_{n+1} = (0)$.

<u>Theorem 2.21</u>: If A is a proper ideal in a noetherian ring R, then $\bigcap_{n=1}^{\infty} A^n = \{r\varepsilon R \mid [1-a]\cdot r = 0 \text{ for some } a\varepsilon A\}$.

<u>Proof</u>: Let $r \varepsilon \{r\varepsilon R \mid [1-a]\cdot r = 0 \text{ for some } a\varepsilon A\}$. Then $r = ar = a^2r = a^3r = \ldots = a^nr = \ldots$. That is $r \varepsilon A^n$ for all $n \varepsilon J_+$, thus $r \varepsilon \bigcap_{n=1}^{\infty} A^n$.

Let $B = \bigcap_{n=1}^{\infty} A^n$ and consider AB. Now $AB = \bigcap_{i=1}^{n} Q_i$ where $\sqrt{Q_i} = P_i$, and AB is irredundant. Prove AB = B. It is sufficient to show $B \subset AB$ since $AB \subset B$ is always true. So we need to show $B \subset Q_i$, $i=1,2,\ldots,n$. Since $AB \subset Q_i$, $i=1,2,\ldots,n$, either $B \subset Q_i$ or $(\sqrt{A})^n \subset Q_i$ for some $n \varepsilon J_+$. But if $(\sqrt{A})^n \subset Q_i$, then $B \subset A^n \subset (\sqrt{A})^n \subset Q_i$. Thus $B \subset Q_i$, $i=1,2,\ldots,n$, and $B \subset \bigcap_{i=1}^{n} Q_i = AB$. Hence AB = B. By the previous theorem there is an $a \varepsilon A$ such that $(1-a)B = (0)$. Therefore if $b \varepsilon B$, then $(1-a)b = 0$. Thus $b \varepsilon \{r\varepsilon R \mid [1-a]r = 0 \text{ for some } a \varepsilon A\}$. Hence $\bigcap_{n=1}^{\infty} A^n = \{r\varepsilon R \mid [1-a]r = 0 \text{ for some } a\varepsilon A\}$.

<u>Theorem 2.22</u>: Let A be a proper ideal of a noetherian ring R. Then $\bigcap_{n=1}^{\infty} A^n = (0)$ if and only if no element of the set $1 - A = \{1-a \mid a\varepsilon A\}$ is a zero divisor in R.

<u>Proof</u>: We need to show that if there is a $z \varepsilon 1-A$ such that z is a zero divisor then $\bigcap_{n=1}^{\infty} A^n \neq (0)$. Let $1-a = z \varepsilon 1-A$ be a zero divisor in R. Then there is an $r \neq 0$ and $r \varepsilon R$ such that $[1-a]r = 0$. Thus $r \varepsilon \{r\varepsilon R \mid [1-a]r = 0 \text{ for some } a \varepsilon A\} = \bigcap_{n=1}^{\infty} A^n$. Hence $\bigcap_{n=1}^{\infty} A^n \neq (0)$. Conversely if no element in 1-A is a zero divisor, then $\bigcap_{n=1}^{\infty} A^n = \{r\varepsilon R \mid [1-a]r = 0\} = \{0\} = (0)$.

<u>Theorem 2.23</u>: Let A and B be ideals in a noetherian ring R. Then $A^n \subset B$ for some $n \varepsilon J_+$ if and only if $\sqrt{A} \subset \sqrt{B}$.

<u>Proof</u>: Suppose $\sqrt{A} \subset \sqrt{B}$. Now $A^n \subset (\sqrt{A})^n \subset \sqrt{A} \subset \sqrt{B}$. Thus $A^n \subset \sqrt{B}$. So there exists a k such that $(\sqrt{B})^k \subset B$. Thus $A^{nk} \subset (\sqrt{B})^k \subset B$.

Suppose $A^n \subset B$. Let $y \varepsilon \sqrt{A}$. Then there exists a t such that $y^t \varepsilon A$. Thus $y^{tn} \varepsilon A^n \subset B$. Hence $y^{tn} \subset B$ and $y \varepsilon \sqrt{B}$. Therefore $\sqrt{A} \subset \sqrt{B}$.

<u>Theorem 2.24</u>: If every prime ideal of a ring R is finitely generated, then R is noetherian. That is, every ideal of R is finitely generated.

<u>Proof</u>: We prove the contrapositive. That is, if there is an ideal of R that is not finitely generated, then there is a prime ideal of R which is not finitely generated. Suppose there exists an ideal of R that is not finitely generated. The set S of ideals which are not finitely generated is nonempty. Appealing to Zorn's Lemma, S must contain a maximal element. Call it I. We claim I is a prime ideal. Suppose to the contrary that I is not a prime ideal. There exist a and $b \varepsilon R$ such that $ab \varepsilon I$, $a \not\varepsilon I$, and $b \not\varepsilon I$.

Now I+(a) is finitely generated, since $I < I+(a)$. Let $I+(a) = (i_1+r_1a, i_2+r_2a, \ldots, a_n+r_na)$, with $i_j \varepsilon I$ and $r_j \varepsilon R$. Let $J = I:(a) = \{y \varepsilon R | ya \varepsilon I\}$. Now $Ja \subset I$. For if $x \varepsilon Ja$, then $x = ba$ for some $b \varepsilon J$; then $x = ba \varepsilon I$. Also

$I < I:(a)$, since $b \in I:(a)$ and $b \notin I$. Thus $J = I:(a)$ is finitely generated and hence Ja is finitely generated.

We prove $I = (i_1, i_2, \ldots, i_n)+Ja$. That $(i_1, i_2, \ldots, i_n)+Ja \subset I$ is clear since $i_i \in I$ and $Ja \subset I$. Suppose $z \in I$; then $z \in I+(a)$ and $z = u_1(i_1+r_1a)+\ldots+u_n(i_n+r_na)$ $= u_1i_1+\ldots+u_ni_n+(u_1r_1+\ldots+u_nr_n)a$, $u_i \in R$. Thus $(u_1r_1+,,,+u_nr_n)a = z-(u_1i_1+\ldots+u_ni_n) \in I$, where $u_1r_1+\ldots+u_nr_n \in J$. Hence $(u_1r_1+\ldots+u_nr_n)a \in Ja$. Therefore $z = u_1i_1+\ldots+u_ni_n+(u_1r_1+\ldots+u_nr_n)a \in (i_1, i_2, \ldots, i_n)+Ja$. It follows that $I \subset (i_1, i_2, \ldots, i_n)+Ja$ and thus $I = (i_1, i_2, \ldots, i_n)+Ja$ is finitely generated, a contradiction. Thus I is a prime ideal.

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