TAUBERIAN THEOREMS FOR CERTAIN REGULAR PROCESSES

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In 1943 R. C. Buck showed that a sequence x is convergent if some regular matrix sums every subsequence of x. Thus, for example, if every subsequence of x is Cesàro summable, then x is actually convergent. Buck's result was quite surprising, since research in summability theory up to that time gave no hint of such a remarkable theorem. The appearance of Buck's result in the Bulletin of the American Mathematical Society created immediate interest and has prompted considerable research which has taken the following directions: (i) to study regular matrix transformations in order to shed light on Buck's theorem, (ii) to extend Buck's theorem, (iii) to obtain analogs of Buck's theorem for sequence spaces other than the space of convergent sequences, and (iv) to obtain analogs of Buck's theorem involving processes other than subsequencing, such as stretching. The purpose of the present paper is to contribute to all facets of the problem, particularly to (i), (iii), and (iv).

In 1944 R. P. Agnew obtained a result closely related to Buck's theorem. Given a bounded sequence x and a regular

matrix A, Agnew was able to demonstrate the existence of a subsequence y of x such that each limit point of x is a limit point of Ay. Recently, J. A. Fridy has obtained a theorem similar to Buck's in which "subsequence" is replaced with "rearrangement." In addition, he has characterized ℓ by showing that $x \in \ell$ if there is a sum preserving ℓ-ℓ matrix that transforms every rearrangement of x into ℓ.

In 1970 I. J. Maddox obtained what might be considered as the ultimate improvement of Buck's theorem. He considered a matrix A which summed every subsequence of a divergent sequence x and showed that A must be Schur. Since the class of Schur matrices is disjoint from the class of regular matrices, Buck's theorem follows as a corollary. The second and third chapters of this paper contain theorems which follow the pattern established by Maddox. In the second chapter an analog is proved in which "subsequence" is replaced with "rearrangement." The third chapter deals with absolute summability, and a theorem is obtained which has Fridy's characterization of ℓ as a corollary. This theorem shows that if x is in c_0 but not in ℓ and the matrix A transforms every rearrangement of x into ℓ, then A is not sum-preserving ℓ-ℓ. In addition, the following question proposed by J. A. Fridy is answered in the affirmative. Is a null sequence x necessarily in ℓ in case there is a sum-preserving ℓ-ℓ matrix A such that Ay is in ℓ for every subsequence y of x?

In 1958 F. K. Keogh and G. M. Petersen were able to extend Buck's result by showing that x is convergent if some regular matrix A sums a set of subsequences of x which is of the second category. The fourth chapter of this paper contains analogs to this theorem in which the requirement of regularity is weakened somewhat. In addition, the sequence space ℓ, as well as c, is investigated, and rearrangements as well as subsequences are considered. Typical of the results in Chapter IV are theorems which show that a sequence x is convergent if there exists a non-Schur matrix A with convergent columns that sums a set of subsequences (rearrangements) which is of the second category.

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CHAPTER I

INTRODUCTION

In 1943 R. C. Buck showed that a sequence x is convergent if some regular matrix sums every subsequence of x. Thus, for example, if every subsequence of x is Cesàro summable, then x is actually convergent. Buck's result was quite surprising, since research in summability theory up to that time gave no hint of such a remarkable theorem. The appearance of Buck's result in the Bulletin of the American Mathematical Society (3) created immediate interest and has prompted considerable research which has taken the following directions: (i) to study regular matrix transformations in order to shed light on Buck's theorem, (ii) to extend Buck's theorem, (iii) to obtain analogs of Buck's theorem for sequence spaces other than the space of convergent sequences, and (iv) to obtain analogs of Buck's theorem involving processes other than subsequencing, such as stretching. The purpose of the present paper is to contribute to all facets of the problem, particularly to (i), (iii), and (iv).

One of the major contributions in the study of sequence spaces through matrix maps is the Silverman-Toeplitz (2, 14, 16) characterization of regular matrices which was obtained in 1911 (Theorem 1.2). In 1921 H. Steinhaus (15) made use of this characterization in showing that no regular matrix transforms m (the space of all bounded complex sequences) into c (the space of all convergent complex sequences). This result of Steinhaus was the major tool used by Buck in obtaining his characterization of c. In 1944 R. P. Agnew (1) obtained a result closely related to Buck's theorem. Given a bounded sequence x and a regular matrix A, Agnew was able to demonstrate the existence of a subsequence y of x such that each limit point of x is a limit point of Ay (Theorem 1.4). Thus, in the case of bounded sequences, Agnew's theorem includes Buck's.

Results similar to those of Buck and Agnew have been obtained in which stretchings or rearrangements, rather than subsequences, have been considered. In 1973 D. F. Dawson (5, p. 456) showed that there exists no analog to Buck's theorem in which c is replaced by BV (the space of all sequences of bounded variation). But he was able to obtain characterizations of c, BV, and other spaces by proving analogs to Buck's theorem replacing "subsequence" with "stretching" (5, p. 457). Recently, J. A. Fridy (8) has obtained a theorem similar to Buck's in which "subsequence" is replaced with "rearrangement." In addition, he has characterized ℓ (the space of all complex sequences x such that $\sum_{q=1}^{\infty} |x_q| < \infty$) by showing that $x \in \ell$ if there is a sum preserving ℓ-ℓ matrix (Definition 1.4) that transforms every rearrangement of x into ℓ.

In 1970 I. J. Maddox (11) obtained what might be considered as the ultimate improvement of Buck's theorem. He considered a matrix A which summed every subsequence of a divergent sequence x and showed that A must be Schur (Definition 1.3, Theorem 1.6). Since the class of Schur matrices is disjoint from the class of regular matrices, Buck's theorem follows as a corollary. Recently, Dawson (6) has obtained an analog to this result of Maddox involving stretchings. The second and third chapters of this paper contain theorems which follow the pattern established by Maddox and Dawson. In the second chapter an analog is proved in which "subsequence" is replaced with "rearrangement" (Theorem 2.3). The third chapter deals with absolute summability, and a theorem is obtained which has Fridy's characterization of ℓ as a corollary. This theorem shows that if x is in c_0 (the space of all null complex sequences) but not in ℓ and the matrix A transforms every rearrangement of x into ℓ, then A is not sum-preserving ℓ-ℓ (Theorem 3.2). In addition, the following question proposed by J. A. Fridy (8, p. 9) is answered in the affirmative. Is a null sequence x necessarily in ℓ in case there is a sum-preserving ℓ-ℓ matrix A such that Ay is in ℓ for every subsequence y of x? (Theorem 3.1).

In the study of sequence spaces in analysis, topological structures are often supplied. Hence equipping the space of all subsequences (rearrangements) of a sequence x with a topology is natural. Thus in 1958 F. K. Keogh and G. M. Petersen (9) were able to extend Buck's result by showing that x is convergent if some regular matrix A sums a set of subsequences of x which is of the second category. The fourth chapter of this paper contains analogs to this theorem in which the requirement of regularity is weakened somewhat. In addition, the sequence space ℓ, as well as c, is investigated, and rearrangements as well as subsequences are considered. Typical of the results in Chapter IV are theorems which show that a sequence x is convergent if there exists a non-Schur matrix A with convergent columns that sums a set of subsequences (rearrangements) which is of the second category (Theorem 4.1, Theorem 4.3).

The following notation conventions will hold throughout this paper:

1. s represents the set of all complex sequences,
2. m represents the set of all bounded complex sequences,
3. c represents the set of all convergent complex sequences,
4. c_o represents the set of all null complex sequences,
5. cs represents the set of all complex sequences x such that $\sum_{q=1}^{\infty} x_q$ converges,

6. ℓ represents the set of all complex sequences x such that $\sum_{q=1}^{\infty} |x_q| < \infty$,

7. if $x \in s$ and y is a subsequence of x, then $x \setminus y$ represents the subsequence of x such that x_q is a term of $x \setminus y$ if and only if x_q is not a term of y.

The following definitions and theorems will be utilized in subsequent chapters:

Definition 1.1. Let A be a matrix with entries a_{pq} $(p = 1,2,3,\ldots;\ q = 1,2,3,\ldots)$; then

1. A is row finite if for each row p there exists $N_p > 0$ such that $q_{pq} = 0$ for every $q > N_p$;
2. A is the identity matrix if $a_{pp} = 1$, $p = 1,2,3,\ldots$; $a_{pq} = 0$ otherwise;
3. Ax is the sequence $(\sum_{q=1}^{\infty} a_{pq}x_q)_{p=1}^{\infty}$;
4. A sums the sequence x if $Ax \in c$.

Theorem 1.1. If A is a matrix, then $Ax \in c_0$ for every $x \in m$ if and only if $(\sum_{q=1}^{\infty} |a_{pq}|)_{p=1}^{\infty} \in c$.

Definition 1.2. The matrix A is regular if $Ax = y \in c$ for every $x \in c$ and $\lim_q x_q = \lim_q y_q$.

Theorem 1.2. The matrix A is regular if and only if

1. $\lim_p a_{pq} = 0$ for $q = 1,2,3,\ldots$;
2. $\lim_p \sum_{q=1}^{\infty} a_{pq} = 1$; and

3. There exists $M > 0$ such that $\sum_{q=1}^{\infty} |a_{pq}| < M$,

$p = 1,2,3,\ldots$. (2, 14, 16).

Theorem 1.3. The sequence x is convergent if there exists a regular matrix A that sums every subsequence of x. (3, 4).

Theorem 1.4. If x is bounded and A is regular, then there exists a subsequence y of x such that every limit point of x is a limit point of Ay. (1).

Definition 1.3. The matrix A is Schur if A sums every element of m.

Theorem 1.5. The matrix A is Schur if and only if

1. $\mathrm{Lim}_p a_{pq} = a_q$ for $q = 1,2,3,\ldots$; and

2. $\mathrm{Lim}_p \sum_{q=1}^{\infty} |a_{pq}| = \sum_{q=1}^{\infty} |a_q|$. (13).

Theorem 1.6. If x is divergent and A is a matrix such that A sums every subsequence of x, then A is Schur (11).

Theorem 1.7. The sequence x is convergent if there exists a matrix A satisfying the first two properties of regularity (Theorem 1.2) which sums every stretching of x (6, p. 457).

Theorem 1.8. If x is a sequence having a finite limit point and A is a matrix satisfying the first two properties of regularity, then there exists an increasing sequence of positive integers $(p_1, p_2, p_3, \ldots)$ and a subsequence y of x such that every finite limit point of x is a limit point of $(\sum_{q=1}^{\infty} a_{p_i q} y_q)_{i=1}^{\infty}$. (5, p. 458).

Theorem 1.9. If x is divergent and A is a matrix which sums every stretching of x, then there exists N such that

1. $\mathrm{Lim}_p a_{pq} = a_q$, $q > N$,

2. $\sum_{q=N+1}^{\infty} a_q$ converges, and

3. $\mathrm{Lim}_p \sum_{q=N+1}^{\infty} (a_{pq} - a_q) = 0$. (6).

Definition 1.4. The matrix A is called an ℓ-ℓ matrix provided Ax is in ℓ whenever x is in ℓ. If, in addition, $\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_{pq} x_q = \sum_{q=1}^{\infty} x_q$, whenever x is in ℓ, then A is a sum-preserving ℓ-ℓ matrix (8, p. 6).

Theorem 1.10. The matrix A is ℓ-ℓ if and only if there exists $M > 0$ such that $\sum_{p=1}^{\infty} |a_{pq}| < M$ for $q = 1,2,3,\ldots$. (7, 10, 12).

Theorem 1.11. The matrix A is a sum-preserving ℓ-ℓ matrix if and only if A is an ℓ-ℓ matrix and $\sum_{p=1}^{\infty} a_{pq} = 1$ for $q = 1,2,3,\ldots$. (8).

Theorem 1.12. The null sequence x is in ℓ if and only if there exists a sum-preserving ℓ-ℓ matrix A such that $Ay \in \ell$ for every rearrangement y of x (8, p. 7).

<u>Definition 1.5</u>. A topological space is called a Baire space if the intersection of every sequence of dense open sets is dense (17, p. 178).

<u>Theorem 1.13</u>. Every complete semimetric space is a Baire space (17, p. 178).

<u>Definition 1.6</u>. Let X be a topological space and K be a subset of X; then

1. K is nowhere dense in X if the interior of $\overline{K}$ is empty;
2. K is of the first category in X if $K = \sum_{n=1}^{\infty} K_n$, where each K_n is nowhere dense in X;
3. K is of the second category in X if K is not of the first category in X (17, p. 179).

<u>Theorem 1.14</u>. A topological space X is a Baire space if and only if each nonempty open set is of the second category in X (17, p. 179).

<u>Theorem 1.15</u>. A Baire space is of the second category in itself (17, p. 180).

<u>Theorem 1.16</u>. A G_δ in a complete semimetric space must be a Baire space (17, p. 183).

CHAPTER I BIBLIOGRAPHY

1. Agnew, R. P., "Summability of Subsequences," Bulletin of the American Mathematical Society, 50(1944), 596-598.

2. Bohr, H., "On the Generalization of a Known Convergence Theorem," Nyt Tidskrift for Matematik, (B) 20(1909), 1-4.

3. Buck, R. C., "A Note of Subsequences," Bulletin of the American Mathematical Society, 49(1943), 898-899.

4. ______, "An Addendum to 'A Note on Subsequences'," Proceedings of the American Mathematical Society, 7(1956), 1074-1975.

5. Dawson, D. F., "Summability of Subsequences and Stretchings of Sequences," Pacific Journal of Mathematics, Vol. 44, No. 2 (1973), 455-460.

6. ______, "A Tauberian Theorem for Stretchings," Journal of the London Mathematical Society, (to appear).

7. Fridy, J. A., "A Note on Absolute Summability," Proceedings of the American Mathematical Society, 20(1969), 285-286.

8. ______, "Summability of Rearrangements of Sequences," unpublished paper read before the Annual Meeting of the American Mathematical Society, Washington, D. C., January 23, 1975.

9. Keogh, F. R., and Petersen, G. M., "A Universal Tauberian Theorem," Journal of the London Mathematical Society, 33(1958), 121-123.

10. Knopp, K. and Lorentz, G. G., "Beiträge zur Absolutem Limitierung," Archiv der Mathematik, 2(1949), 10-16.

11. Maddox, I. J., "A Tauberian Theorem for Subsequences," Bulletin of the London Mathematical Society, 2(1970), 63-65.

12. Mears, F. M., "Absolute Regularity and Norlund Mean," Annals of Mathematics, 38(1937), 594-601.

13. Shur, I., "Über Linear Transformationen in der nheorie die Unendlich Reihen," Journal für die Reine und Angewandte Mathematik, 151(1921), 79-111.

14. Silverman, L. L., "On the Definition of the Sum of a Divergent Series," unpublished thesis, University of Missouri Studies, Mathematical Series, Vol. 1, No. 1, 1913.

15. Steinhaus, H., "Some Remarks on the Generalization of the Notion of Limit," Prace Matematyczno Fizyczne, 22(1921), 121-134.

16. Toeplitz, O., "Über Allgemeine Lineare Mittelbildungen," Prace Matematyczno Fizyczne, 22(1911), 113-119.

17. Wilansky, A., Topology for Analysis, Waltham, Mass., Ginn and Company, 1970.

CHAPTER II

SUMMABILITY OF REARRANGEMENTS

The sequence y is called a rearrangement of the sequence x provided that there is a one-to-one function π from the positive integers onto themselves such that for each k, $y_k = x_{\pi(k)}$. (9, p. 1). This chapter contains analogs to Theorems 1.3 and 1.4 in which rearrangements rather than subsequences are considered. These results are then generalized by proving a theorem similar to Theorem 1.6 in which "subsequence" is replaced with "rearrangement." Many of the results in this chapter are also paralleled by findings of D. F. Dawson (7, 8) involving stretchings. In addition, Theorem 2.1 and Corollary 2.2 closely resemble results of J. A. Fridy (9), differing in that they do not presume the third property of regularity (Theorem 1.2).

Lemma 2.1. Let x be a sequence, y be a subsequence of x, and A be a matrix such that both $\lim_p a_{pq} = 0$ for $q = 1,2,3,\ldots$ and $\lim_q a_{pq} = 0$ for $p = 1,2,3,\ldots$. If Ay exists, then there is a rearrangement r of x and an increasing sequence of positive integers $(p_1,p_2,p_3,\ldots)$ such that each limit point of Ay (finite or infinite) is a limit point of

$$\left(\sum_{p=1}^{\infty} a_{p_i q} r_q\right)_{i=1}^{\infty}.$$

Proof. Suppose first that Ay has a finite limit point. Using the separability of the complex plane, let $(u_1, u_2, u_3, \ldots)$ be a sequence of numbers such that each u_i is a finite limit point of Ay and each finite limit point of Ay is either one of the u_i or a limit point of the u_i. Rewrite the sequence $(u_1; u_1, u_2; u_1, u_2, u_3, \ldots)$ as $(v_1, v_2, v_3, \ldots)$. Suppose that Ay also has a subsequence that diverges to infinity. Let $x \setminus y = (z_1, z_3, z_5, \ldots)$ and $p_1 > 0$ such that

$$\left| \sum_{q=1}^{\infty} a_{p_1 q} y_q - v_1 \right| < \frac{1}{2}.$$

Let $N_1 > 0$ such that $|a_{p_1 n_1} y_{n_1}| < \frac{1}{4}$ and $|a_{p_1 n_1}| < \frac{1}{8|z_1|+1}$.

Let $r_i = y_i$ for $i = 1, 2, \ldots, N_1 - 1$; $r_{N_1} = z_1$; and $z_2 = y_{N_1}$.

Let $p_2 > p_1$ be chosen such that $|a_{p_2 N_1}| < \frac{1}{2}$, $|a_{p_2 N_1} y_{N_1}| < \frac{1}{4}$, and

$$\left| \sum_{q=1}^{\infty} a_{p_2 q} y_q \right| > 101.$$

Let $N_2 > N_1$ be chosen such that

$$|a_{p_2 N_2} y_{N_2}| < \frac{1}{8},$$

$$|a_{p_2 N_2} z_2| < \frac{1}{16},$$

$$|a_{p_1 N_2} y_{N_2}| < \frac{1}{16},$$

and

$$|a_{p_1 N_2} z_2| < \frac{1}{32} .$$

Let $r_i = y_i$ for $i = N_1 + 1, \ldots, N_2 - 1$; $r_{N_2} = z_2$; and $z_4 = y_{N_2}$.

This process may be continued so that $|\sum_{q=1}^{\infty} a_{p_i q} r_q| > \frac{i}{2}(100)$ when i is even and $|\sum_{q=1}^{\infty} a_{p_i q} r_q - v_j| < \frac{1}{2^{i-1}}$ when i is odd, $i = 2j-1$. Therefore each limit point of Ay is a limit point of $(\sum_{q=1}^{\infty} a_{p_i q} r_q)_{i=1}^{\infty}$. This argument may be modified depending on the types of limit points (finite or infinite) in question.

Theorem 2.1. If x is a sequence having a finite limit point and A is a matrix satisfying the first two properties of regularity, then there exist a rearrangement y of x and an increasing sequence of positive integers $(p_1, p_2, p_3, \ldots)$ such that each finite limit point of x is a limit point of

$(\sum_{q=1}^{\infty} a_{p_i q} y_q)_{i=1}^{\infty}$.

Proof. By Theorem 1.8 there exist a subsequence y of x and an increasing sequence of positive integers $(P_1, p_2, p_3, \ldots)$ such that each finite limit point of x is a limit point of

$(\sum_{q=1}^{\infty} a_{p_i q} y_q)_{i=1}^{\infty}$. But by Lemma 2.1 there exist a rearrangement r of x and a subsequence $(p_1', p_2', p_3', \ldots)$ of $(p_1, p_2, p_3, \ldots)$ such that each finite limit point of $(\sum_{q=1}^{\infty} a_{p_i q} y_q)_{i=1}^{\infty}$ is a limit point of $(\sum_{q=1}^{\infty} a_{p_i' q} r_q)_{i=1}^{\infty}$.

Corollary 2.1. A sequence x diverges to ∞ if and only if there exists a matrix A satisfying the first two properties of regularity such that Ay diverges for every rearrangement y of x.

Proof. The identity matrix suffices for necessity. For sufficiency suppose that x has a bounded subsequence y with finite limit point L. By Theorem 2.1 there exist an increasing sequence of positive integers $(p_1, p_2, p_3, \ldots)$ and a rearrangement r of y such that L is a limit point of $(\sum_{q=1}^{\infty} a_{p_i q} r_q)_{i=1}^{\infty}$. Let $z = x \setminus r$ and $w = (r_1, z_1, r_2, z_2, \ldots)$. By Lemma 2.1 there exists an increasing sequence of positive integers $(p_1', p_2', p_3', \ldots)$ and a rearrangement t of w (hence t is also a rearrangement of x) such that L is a limit point of $(\sum_{q=1}^{\infty} a_{p_i' q} t_q)_{i=1}^{\infty}$, a contradiction.

Theorem 2.2. If A is a row finite matrix satisfying the first two properties of regularity and x is a sequence, then there exists a rearrangement y of x such that every limit point of x (finite or infinite) is a limit point of Ay.

Proof. If x is bounded, then the theorem follows from Theorem 2.1. Suppose x is unbounded and y is a subsequence of x that diverges to infinity. Let $z = x \setminus y$. By Theorem 2.1 there exists a rearrangement w of z such that each finite limit point of z (and thus of x) is a limit point of Aw. For $p = 1, 2, 3, \ldots$ let a_{pk_p} be the last nonzero element of the p-th row. Making use of the separability of the complex plane, let $(u_1,u_2,u_3,\ldots)$ be a sequence such that each u_i is a finite limit point of x and each finite limit point of x is either one of the u_i or a limit point of the u_i. Let $(u_1;u_1,u_2;\ u_1,u_2,u_3,\ldots) = (v_1,v_2,v_3,\ldots)$ and $p_1 > 0$ such that $|\sum_{q=1}^{\infty} a_{p_1 q} w_q - v_1| < \frac{1}{2}$. Making use of the first property of regularity, choose $t_1 > p_1$ such that $k_{t_1} > k_{p_1}$. Let $r_q = w_q$ for $q = 1,2,\ldots,k_{t_1} - 1$ and choose $r_{k_{t_1}}$ from $x\backslash y$ such that $|\sum_{q=1}^{k_{t_1}} a_{t_1 q} r_q| > 2$. Again making use of the first property of regularity, choose $p_2 > t_1$ such that $|\sum_{q=1}^{k_{t_1}} a_{p_2 q} r_q + \sum_{q=k_{t_1}+1}^{\infty} a_{p_2 q} w_q - v_2| < \frac{1}{4}$. Let $t_2 > p_2$ such that $k_{t_2} > k_{p_2}$. Let $r_q = w_q$ for $q = k_{t_1}+1,\ldots,k_{t_2} - 1$ and choose $r_{k_{t_2}}$ from $x\backslash(r_{k_{t_1}},y_1,y_2,y_3,\ldots)$ such that $|\sum_{q=1}^{k_{t_2}} a_{t_2 q} r_q| > 4$. This process may be continued defining a rearrangement r of a subsequence of x such that each limit point of x (finite or infinite) is a limit point of Ar. Therefore by Lemma 2.1 there exists a rearrangement r' of x such that every limit point of x is a limit point of Ar'.

<u>Corollary 2.2</u>. A sequence x converges if and only if there exists a matrix A with the first two properties of reglarity such that A sums every rearrangement of x.

Proof. The identity matrix suffices for necessity.

For sufficiency note that x cannot be unbounded, for if that were the case A would have to be row finite, and by Theorem 2.2 there would exist a rearrangement r of x such that Ar would have a infinite limit point, a contradiction. But if x is bounded, then by Theorem 2.1 there exist a rearrangement r of x and an increasing sequence of positive integers $(p_1, p_2, p_3, \ldots)$ such that each limit point of x is a limit point of $\left(\sum_{q=1}^{\infty} a_{p_i q} r_q\right)_{i=1}^{\infty}$. Thus x must have but one limit point and therefore must be convergent.

Corollary 2.3. A sequence x is bounded if and only if there exists a matrix A satisfying the first two properties of regularity such that Ay is bounded for every rearrangement y of x.

Proof. The identity matrix suffices for necessity.

For sufficiency suppose x is not bounded. Then A must be row finite or else it is easy to construct a rearrangement y of x such that Ay fails to exist. Thus by Theorem 2.2 there exists a rearrangement y of x such that Ay has an infinite limit point, a contradiction. Hence the proof is complete.

Professor A. Wilansky of Lehigh University has pointed out in a private communication that Theorem 2.3 may also be approached by utilizing results obtained by G. Bennett and

N. J. Kalton. (32) In addition, it should be noted that Corollary 2.2 follows directly from Theorem 2.3.

Lemma 2.2. If x is divergent and a is a sequence such that $\sum_{q=1}^{\infty} a_q y_q$ exists whenever y is a rearrangement of x, then $a \in \ell$.

Proof. If x is unbounded, then clearly a is eventually zero and hence in ℓ. Suppose x consists of only two elements $t_1 \neq t_2 \neq 0$, and that $a \notin \ell$. Then a must be a null sequence since otherwise there exists a rearrangement y of x such that $\lim_q |a_q y_q| \neq 0$. If $a \notin cs$, then there exists $\varepsilon > 0$ such that if $N > 0$, then there exist $m > n \geq N$ such that $|\sum_{q=n}^{m} a_q| > \varepsilon$. Thus a rearrangement y could be chosen such that if $N > 0$, then there would exist $m > n \geq N$ such that

$$|\sum_{q=n}^{m} a_q y_q| > |t_2| \varepsilon > 0,$$

a contradiction. Hence $a \in cs$. Let $N > 0$ such that if $m > n \geq N$ then

$$|\sum_{q=n}^{m} a_q| < \tfrac{1}{2} |t_2|^{-1}.$$

But $a \notin \ell$, therefore given $M \geq N$ there exist $m > n \geq M$ and $(a_q^{(1)})_{q=n}^{m}$ such that

$$|\sum_{q=n}^{m} a_q^{(1)}| > |t_1 - t_2|^{-1},$$

where either $a_q^{(1)} = 0$ or $a_q^{(1)} = a_q$ for $q = n, n+1, \ldots, m$.

Define $(a_q^{(2)})_{q=n}^m$ such that $a_q^{(2)} = 0$ if $a_q^{(1)} = a_q$ and $a_q^{(2)} = a_q$ otherwise. Then

$$\sum_{q=n}^m a_q = \sum_{q=n}^m a_q^{(1)} + \sum_{q=n}^m a_q^{(2)}$$

and

$$\left| \sum_{q=n}^m a_q^{(1)} t_1 + \sum_{q=n}^m a_q^{(2)} t_2 \right|$$

$$\geq \left| \sum_{q=n}^m a_q^{(1)} \right| \left| t_1 - t_2 \right|$$

$$- |t_2| \left| \sum_{q=n}^m a_q \right| > \frac{1}{2}.$$

Hence a rearrangement y of x may be constructed such that $\sum_{q=1}^{\infty} a_q y_q$ does not converge, a contradiction. Since a is null the Lemma follows in the more general case.

Theorem 2.3. If x is a divergent sequence and A is a matrix that sums every rearrangement of x, then A is Schur.

Proof. Suppose x is not bounded. A must be row finite or else it is easy to construct a rearrangement y of x such that Ay' fails to exist. Also it is clear that all but a finite number of columns of A are zero columns since otherwise a rearrangement y of x can be constructed so that Ay is unbounded. Let q* be fixed and $q' \neq q^*$ be a zero column of A. Let $y_{q^*} = y_{q'}$ be two terms of x and y be a rearrangement of x with y_{q^*} and $y_{q'}$ so defined. Let $z_q = y_q$ if $q \neq q^*, q'$; $z_{q^*} = y_{q'}$; and $z_{q'} = y_{q^*}$. Then Ay and Az are convergent;

therefore so is $A(y-z) = (a_{pq*}(y_{q*}-y_{q'}))_{p=1}^{\infty}$. Hence $(a_{pq*})_{p=1}^{\infty}$ converges and A is Schur.

Suppose x is bounded. Let $L_1 \neq 0$ and L_2 be two distinct limit points of x and note that by Lemma 2.2 each row of A is in ℓ. If any one single column of A converges, then by an argument similar to that used in the unbounded case above every column of A converges. Suppose that the q* column of A fails to converge. Then there exists $\varepsilon > 0$ such that if $N > 0$, then there exist $m > n \geq N$ such that $|a_{nq*}-a_{mq*}| > \varepsilon$. Let $(p_1, p_1', p_2, p_2', \ldots)$ be an increasing sequence of positive integers such that $|a_{p_i q*}-a_{p_i' q*}| > \varepsilon$ for each i. Let B be the matrix such that $b_{rs} = (a_{p_r s}-a_{p_r' s})$ for $r = 1,2,3,\ldots;$ $s = 1,2,3,\ldots$. Then B has the property that By is null whenever y is a rearrangement of x. Also, $|b_{pq*}| > \varepsilon$ for each p, and $\lim_p (b_{pq*}-b_{pq}) = 0$ for each q by an argument similar to that used in the unbounded case above. In addition, since each row of A is in ℓ so is each row of B. Suppose that $(y_1,\ldots,y_k)$ has been determined, where each y_i is a term of x. Choose m so large that $(y_{k+1},\ldots,y_m)$ may be chosen from $x\backslash(y_1,\ldots,y_k)$ such that $|\sum_{q=1}^{m} b_{pq}y_q| > \frac{4|L_1|}{|L_1-L_2|} + 2$ for some $p > k$. Let $N > m$ such that $\sum_{q=N}^{\infty} |b_{pq}| < \frac{1}{\sup_n |x_n|}$. Let $y_{n+1} = x_i$, where $i = \min\{j : x_j \in x\backslash(y_1,\ldots,y_m)\}$. Suppose that

$$\lambda_1 = |\sum_{q=1}^{m} b_{pq}y_q + L_1 \sum_{q=m+1}^{N} b_{pq}| \le 2.$$

Then

$$2 \ge |\sum_{q=1}^{m} b_{pq}y_q| - |L_1|\,|\sum_{q=m+1}^{N} b_{pq}|$$

and

$$|\sum_{q=m+1}^{N} b_{pq}| \ge \frac{|\sum_{q=1}^{m} b_{pq}y_q| - 2}{|L_1|} > \frac{4}{|L_1 - L_2|} .$$

Therefore

$$\lambda_2 = |\sum_{q=1}^{m} b_{pq}y_q + L_2 \sum_{q=m+1}^{N} b_{pq}|$$

$$\ge |L_1 - L_2|\;|\sum_{q=m+1}^{N} b_{pq}|$$

$$- |\sum_{q=1}^{m} b_{pq}y_q + L_1 \sum_{q=m+1}^{N} b_{pq}| > 2.$$

Hence one of λ_1 or λ_2 is greater than 2, and $(y_{m+1},\ldots,y_N)$ may be chosen from $x\backslash(y_1,\ldots,y_m,y_{N+1})$ so that irregardless of the manner in which $(y_{N+2},y_{N+3},\ldots)$ is selected from $x\backslash(y_1,\ldots,y_{N+1})$, $|\sum_{q=1}^{\infty} b_{pq}y_q| > 1$. This contradicts the fact that By must be null. Therefore each column of A is convergent.

Suppose that there exists a subsequence y of x such that A does not sum y. Then By is not null and must have a non-zero limit point. Since B has null rows and columns, then by Lemma 2.1, there exists a rearrangement r of x such that each limit point of By is a limit point of Br. But Br is null, a contradiction. Hence A sums each subsequence y of x and by Theorem 1.6, A is Schur.

Corollary 2.4. If x is divergent and A is a matrix such that Ay is null for every rearrangement y of x, then A transforms all bounded sequences into null sequences.

Proof. Suppose x is unbounded. By an argument similar to that used in Theorem 2.3 each column of A must converge, and all but a finite number of columns of A are zero columns. But Ay is null for each rearrangement y of x, therefore every column of A is null, and by Theorem 1.1, A transforms all bounded sequences into c_o.

Suppose x is bounded. By Theorem 2.3 A is Schur; therefore all columns of A converge. But by an argument similar to that used in Theorem 2.3 all columns of A must have a common limit to which they converge. Since A is Schur this limit must be zero, and by Theorem 1.1, A transforms all bounded sequences into c_o.

CHAPTER II BIBLIOGRAPHY

1. Agnew, R. P., "On Rearrangements of Series," Bulletin of the American Mathematical Society, 46(1940), 797-799.

2. __________, "Permutations Preserving Convergence of Series," Proceedings of the American Mathematical Society, 6(1958), 563-564.

3. __________, "Summability of Subsequences," Bulletin of the American Mathematical Society, 50(1944), 596-598.

4. Bennett, G. and Kalton, N. J., "Inclusion Theorems for K-Spaces," Canadian Journal of Mathematics, 25, No. 3(1973), 511-524.

5. Buck, R. C., "A Note on Subsequences," Bulletin of the American Mathematical Society, 49(1943), 898-899.

6. __________, "An Addendum to 'A Note on Subsequences'," Proceedings of the American Mathematical Society, 7(1956), 1074-1075.

7. Dawson, D. F., "Summability of Subsequences and Stretchings of Sequences," Pacific Journal of Mathematics, Vol. 44, No. 2(1973), 455-460.

8. __________, "A Tauberian Theorem for Stretchings," Journal of the London Mathematical Society (to appear).

9. Fridy, J. A., "Summability of Rearrangements of Sequences," unpublished paper read before the Annual Meeting of the American Mathematical Society, Washington, D. C., January 23, 1975.

10. Maddox, I. J., "A Tauberian Theorem for Subsequences," Bulletin of the London Mathematical Society, 2(1970), 63-65.

CHAPTER III

ABSOLUTE SUMMABILITY

In this chapter results are proved which follow the pattern established by Theorems 1.6, 1.9, and 2.3 but which are concerned with the characterization of ℓ rather than c. J. A. Fridy (2, p. 585) has given an example of a non-zero constant sequence x and a sum-preserving ℓ-ℓ matrix A such that $Ay \in \ell$ for every subsequence (rearrangement) y of x. Therefore, in this chapter interest is restricted to null sequences. Corollary 2.1 was first proposed in a slightly different form by Fridy (2, p. 585) in 1970 and was later stated by him as a formal proposition in 1974 (3, p. 9). Corollary 2.2 is a result previously obtained by Fridy (3, p. 7).

Lemma 3.1. Suppose x and a are sequences such that $\sum_{q=1}^{\infty} a_q y_q$ converges for every subsequence y of x. If $\varepsilon > 0$, then there exist $M > 0$ and a strictly increasing function $\delta: I^+ \to I^+$ such that if $m \geq M$, then $|\sum_{q=m}^{\infty} a_q y_q| \leq \varepsilon$ for every subsequence $(y_q)_{q=m}^{\infty}$ of $(x_q)_{q=\delta(m)}^{\infty}$.

Proof. Suppose the conclusion is false. Let $\delta_1(i) = i$ for $i = 1,2,3,\ldots$; $M_1 = 1$; and $(y_q^{(1)})_{q=m}^{\infty}$ be a subsequence of

$(x_q)_{q=\delta_1(m)}^{\infty}$, $m \geq 1$, such that $|\sum_{q=m}^{\infty} a_q y_q^{(1)}| > \varepsilon$. Let $M_1' > m$ such that $|\sum_{q=m}^{k} a_q y_q^{(1)}| > \varepsilon$ for every $k \geq M_1' - 1$. Let $\delta_2(i) = \delta_1(i)$ if $i < m$ and $\delta_2(i) = q$ where $y_i^{(1)} = x_q$ otherwise. Let $M_2 \geq M_1'$ and $(y_q^{(2)})_{q=M_2}^{\infty}$ be a subsequence of $(x_i)_{i=\delta_2(M_2)}^{\infty}$ such that $|\sum_{q=M_2}^{\infty} a_q y_q^{(2)}| > \varepsilon$. Let $y_q = x_q$ if $q < m$ and $y_q = y_q^{(1)}$ if $m \leq q < M_2$. Proceeding as above, let $M_2' > M_2$ such that $|\sum_{q=M_2}^{k} a_q y_q^{(2)}| > \varepsilon$ for every $k \geq M_2' - 1$. Let $\delta_3(i) = \delta_2(i)$ if $i < M_2$ and $\delta_3(i) = q$ where $y_i^{(2)} = x_q$ otherwise. Let $M_3 \geq M_2'$ and $(y_q^{(3)})_{q=M_3}^{\infty}$ be a subsequence of $(x_i)_{i=\delta_3(M_3)}^{\infty}$ such that $|\sum_{q=M_3}^{\infty} a_q y_q^{(3)}| > \varepsilon$. Define $y_q = y_q^{(2)}$ if $M_2 \leq q < M_3$. This process may be continued, defining a subsequence y of x for which $\sum_{q=1}^{\infty} a_q y_q$ fails to converge, a contradiction.

<u>Lemma 3.2</u>. If x is a null sequence not in ℓ and a is a nonnull convergent sequence, then there exists a subsequence y of x such that $\lim_t |\sum_{q=1}^{t} y_q| = \infty$ and $(\sum_{q=1}^{n} a_q y_q)_{n=1}^{\infty}$ is not bounded.

Proof. The lemma is clear if both a and x are real sequences. Let $a_q = a_q^{(1)} + ia_q^{(2)}$ and $x_q = x_q^{(1)} + ix_q^{(2)}$ for $q = 1,2,3,\ldots$. Suppose $x \notin \ell$ and there exists a subsequence $y^{(1)}$ of $x^{(1)}$ such that $y_q^{(1)} > 0$ for each q and $\sum_{q=1}^{\infty} y_q^{(1)} = +\infty$.

Let y be the subsequence of x determined by $y^{(1)}$. Clearly

$\lim_t |\sum_{q=1}^{t} y_q| = \infty$. Also for each q

$$a_q y_q = (a_q^{(1)} y_q^{(1)} - a_q^{(2)} y_q^{(2)}) + i(a_q^{(1)} y_q^{(2)} + a_q^{(2)} y_q^{(1)}).$$

Consider the following special cases:

i.) Suppose $\lim_q a_q^{(1)} = a^{(1)} > 0$, $\lim_q a_q^{(2)} = a^{(2)} > 0$,

and $(\sum_{q=1}^{n} (a_q^{(1)} y_q^{(1)} - a_q^{(2)} y_q^{(2)}))_{n=1}^{\infty}$ is bounded. Since

$\lim_n \sum_{q=1}^{n} a_q^{(1)} y_q^{(1)} = +\infty$, it follows that $\lim_n \sum_{q=1}^{n} y_q^{(2)} = +\infty$.

Therefore $\lim_n \sum_{q=1}^{n} (a_q^{(1)} y_q^{(2)} + a_q^{(2)} y_q^{(1)}) = +\infty$ and $(\sum_{q=1}^{n} a_q y_q)_{n=1}^{\infty}$ is not bounded.

ii.) Suppose that $\lim_q a_q^{(1)} = a^{(1)} > 0$, $\lim_q a_q^{(2)} = a^{(2)} < 0$,

and $(\sum_{q=1}^{n} (a_q^{(1)} y_q^{(1)} - a_q^{(2)} y_q^{(2)}))_{n=1}^{\infty}$ is bounded. Since

$\lim_n \sum_{q=1}^{n} a_q^{(1)} y_q^{(1)} = +\infty$ it follows that $\lim_n \sum_{q=1}^{n} y_q^{(2)} = -\infty$.

Therefore $\lim_n \sum_{q=1}^{n} (a_q^{(1)} y_q^{(2)} + a_q^{(2)} y_q^{(1)}) = -\infty$ and $(\sum_{q=1}^{n} a_q y_q)_{n=1}^{\infty}$ is not bounded.

iii.) Suppose $\lim_q a_q^{(1)} = a^{(1)} > 0$, $\lim_q a_q^{(2)} = a^{(2)} = 0$,

$\lambda_1 = (\sum_{q=1}^{n} (a_q^{(1)} y_q^{(1)} - a_q^{(2)} y_q^{(2)}))_{n=1}^{\infty} \in m$, and

$\lambda_2 = (\sum_{q=1}^{n} (a_q^{(1)} y_q^{(2)} + a_q^{(2)} y_q^{(1)}))_{n=1}^{\infty} \in m$. Therefore both

$\lambda_1 + \lambda_2$ and $\lambda_1 - \lambda_2$ are bounded, and it follows that both

$$\lambda_3 = (\sum_{q=1}^{n} [(a_q^{(1)} + a_q^{(2)}) y_q^{(1)} + (a_q^{(1)} - a_q^{(2)}) y_q^{(2)}])_{n=1}^{\infty} \in m$$

and

$$\lambda_4 = (\sum_{q=1}^{n} [(a_q^{(1)} - a_q^{(2)}) y_q^{(1)} - (a_q^{(1)} + a_q^{(2)}) y_q^{(2)}])_{n=1}^{\infty} \in m.$$

But $\lim_q (a_q^{(1)} + a_q^{(2)}) = a^{(1)} > 0$, therefore

$$\lim_n \sum_{q=1}^{n} (a_q^{(1)} + a_q^{(2)}) y_q^{(1)} = +\infty.$$

Also, $\lim_q (a_q^{(1)} - a_q^{(2)}) = a^{(1)} > 0$, therefore

$$\lim_n \sum_{q=1}^{n} y_q^{(2)} = -\infty.$$

But this contradicts the fact that $\lambda_4 \in m$. Hence one of λ_1 or λ_2 is not bounded, thus $(\sum_{q=1}^{n} a_q y_q)_{n=1}^{\infty}$ is not bounded.

Clearly each remaining case can be reduced to one of the above three cases, and the lemma is proved.

Theorem 3.1. Let x be a null sequence not in ℓ, and suppose A is a matrix such that $Ay \in \ell$ for every subsequence y of x. Then i.) $\sum_{p=1}^{\infty} |a_{pq}| < \infty$ for $q = 1,2,3,\ldots$; and

ii.) if $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$, then $L = 0$.

Proof. To show i.), let k be fixed and $j > i > k$ such that $x_i \neq x_j$. Let y be the subsequence of x such that $y_q = x_q$ for $q = 1,2,\ldots,k-1$; $y_k = x_i$; and $y_{k+t} = x_{j+t}$ for $t = 1,2,3,\ldots$. Let z be the subsequence of x such that $z_k = x_j$ and $z_q = y_q$ otherwise. Then

$$\infty > \sum_{p=1}^{\infty} \left| \sum_{q=1}^{\infty} a_{pq} y_q - \sum_{q=1}^{\infty} a_{pq} z_q \right| = |x_i - x_j| \sum_{p=1}^{\infty} |a_{pk}|.$$

Therefore $\sum_{p=1}^{\infty} |a_{pk}| < \infty$.

Suppose $\lim_q \sum_{p=1}^{\infty} a_{pq} = L$ and $L \neq 0$. Let $(y_1,\ldots,y_{M-1})$ be a subsequence of x with $y_{M-1} = x_r$. Since $x \notin \ell$ there exists a subsequence $(w_q)_{q=M}^{\infty}$ of $(x_q)_{q=r+1}^{\infty}$ such that $\lim_t \left| \sum_{q=M}^{t} w_q \right| = \infty$. By Lemma 3.2 there exists a subsequence $(z_q)_{q=M}^{\infty}$ of $(w_q)_{q=M}^{\infty}$ such that $\lim_t \left| \sum_{q=M}^{t} z_q \right| = \infty$ and $\limsup_t \left| \sum_{q=M}^{t} z_q \sum_{p=1}^{\infty} a_{pq} \right| = \infty$. Choose $k > M$ such that $\left| \sum_{q=M}^{k} z_q \sum_{p=1}^{\infty} a_{pq} \right| > M + \sum_{q=1}^{M-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| + 3$. Let $K > 0$ such that $\left| \sum_{p=K+1}^{\infty} a_{pq} \right| < \frac{1}{k(|z_q|+1)}$ for $q = m,\ldots,k$. By Lemma 3.1, letting $\varepsilon = \frac{1}{K}$, there exist N_p' and δ_p' for $1 \le p \le K$, such that if $N = \max\{N_1', \ldots, N_K', k+2\}$ and $\delta(i) = \max\{\delta_p'(i) : p = 1, \ldots, K\}$, then $\sum_{p=1}^{k} |\sum_{q=N}^{\infty} a_{pq} v_q| < 1$ for every subsequence $(v_q)_{q=N}^{\infty}$ of $(x_q)_{q=\delta(N)}^{\infty}$. Let $y_q = z_q$ for $M \le q \le k$, and choose $(y_{k+1}, \ldots, y_{N-1})$ a subsequence of $(x_q)_{q=\delta(N)}^{\infty}$ such that $\sum_{q=k+1}^{N-1} |y_q| \sum_{p=1}^{\infty} |a_{pq}| < 1$. Note that the first N-1 terms of a fixed sequence y have now been determined. If y* is any subsequence of x that agrees with y for these first N-1 terms, then

$$\sum_{p=1}^{K} |\sum_{q=1}^{\infty} a_{pq} y_q^*| \ge |\sum_{q=M}^{k} y_q^* \sum_{p=1}^{K} a_{pq}| - \sum_{q=1}^{M-1} |y_q^*| \sum_{p=1}^{K} |a_{pq}|$$

$$- \sum_{q=k+1}^{N-1} |y_q^*| \sum_{p=1}^{k} |a_{pq}| - \sum_{p=1}^{K} |\sum_{q=N}^{\infty} a_{pq} y_q^*|$$

$$> |\sum_{q=M}^{k} y_q^* \sum_{p=1}^{\infty} a_{pq}| - \sum_{q=M}^{k} |y_q^*| \, |\sum_{p=K+1}^{\infty} a_{pq}|$$

$$- \sum_{q=1}^{M-1} |y_q^*| \sum_{p=1}^{K} |a_{pq}| - 2$$

$$> M.$$

This process for defining terms of y may be continued so that if $T > 0$, then there exists $M \ge T$ and $K > 0$ such that

$$\sum_{p=1}^{K} |\sum_{q=1}^{\infty} a_{pq} y_q| > M.$$

Thus a subsequence y of x can be constructed such that $Ay \notin \ell$, a contradiction.

Corollary 3.1. A null sequence x is in ℓ if and only if there exists a sum-preserving ℓ-ℓ matrix A such that $Ay \in \ell$ for every subsequence y of x.

Proof. The identity matrix suffices for necessity. By Definition 1.4 if A is a sum-preserving ℓ-ℓ matrix, then $\lim_q \sum_{p=1}^{\infty} a_{pq} = 1$. Hence by Theorem 3.2, x must be in ℓ.

Theorem 3.2. If x is a null sequence not in ℓ and A is a matrix such that $Ay \in \ell$ for every rearrangement y of x, then $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.

Proof. Let $x_n \neq x_m$ be nonzero elements of x. Suppose the first column of A is not in ℓ. Let $q > 1$ and y be a rearrangement of x with $y_1 = x_n$ and $y_q = x_m$. Let z be the rearrangement of x such that $z_1 = x_m$, $z_q = x_n$, and $z_q = y_q$ otherwise. Then $|x_n - x_m| \sum_{p=1}^{\infty} | a_{p1} - a_{pq}| = \sum_{p=1}^{\infty} | \sum_{q=1}^{\infty} a_{pq} y_q - \sum_{q=1}^{\infty} a_{pq} z_q| < \infty$. Therefore $\sum_{p=1}^{\infty} |a_{p1} - a_{pq}| < \infty$ for $q = 2,3,4,\ldots$. Since $\sum_{p=1}^{\infty} |a_{p1}| = \infty$, it now follows that $\sum_{p=1}^{\infty} |a_{pq}| = \infty$ for $q \geq 2$. Suppose a permutation $(r_1, \ldots, r_M)$ of M terms of x has been chosen such that $\sum_{q=1}^{M} r_q \neq 0$. Suppose $N > 0$. If

$\lambda = \sum_{p=1}^{\infty} | \sum_{q=1}^{M} a_{pq} r_q | < \infty$, then

$$\infty > \lambda + \sum_{q=2}^{M} |r_q| \sum_{p=1}^{\infty} |a_{p(q-1)} - a_{pq}|$$
$$\geq |\sum_{q=1}^{M} r_q| \sum_{p=1}^{\infty} |a_{p1}|,$$

a contradiction. Therefore $\lambda = \infty$ and there exists $K > N$ such that $\sum_{p=N}^{K} | \sum_{q=1}^{M} a_{pq} r_q | > 2$. Let $i = \min\{q : x_q \in x \backslash (r_1, \ldots, r_m)\}$.

J. A. Fridy (3, p. 6) has shown that each row of A is null.

Therefore there exists $T > M$ such that $|x_i| \sum_{p=1}^{K} |a_{pT}| < 2^{-(M+1)}$.

Let $r_T = x_i$ and $(r_{M+1}, \ldots, r_{T-1})$ be a subsequence of $x \backslash (r_1, \ldots, r_M, r_T)$ such that $\sum_{p=1}^{K} \sum_{q=M+1}^{T-1} |a_{pq}| |r_q| < 2^{-(M+2)}$.

Then

$$\sum_{p=N}^{K} | \sum_{q=1}^{T} a_{pq} r_q | \geq \sum_{p=N}^{K} | \sum_{q=1}^{M} a_{pq} r_q |$$
$$- \sum_{p=N}^{K} \sum_{q=M+1}^{T-1} |a_{pq} r_q| - |r_T| \sum_{p=N}^{K} |a_{pT}|$$
$$> 2 - 2^{-(M+1)} - 2^{-(M+2)} > 1.$$

But this process may be continued. Therefore there exists a rearrangement r of x such that if $L > 0$, then there exist $K > N \geq L$ such that $\sum_{p=N}^{K} | \sum_{q=1}^{\infty} a_{pq} r_q | > 1$, a contradiction.

Hence each column of A is in ℓ.

Now suppose there exists $\varepsilon > 0$ such that if $N > 0$, then there exists $q > N$ such that $\sum_{p=1}^{\infty} |a_{pq}| > \varepsilon$. Let $z \in \ell$ be a subsequence of x that includes all zero terms of x. Let $j_1 = \min\{q : x_q \in x \setminus z\}$. Let $N_1 > 0$ such that $\sum_{p=1}^{\infty} |a_{pN_1}| > \varepsilon$. Let $r_{N_1} = x_{j_1}$, $r_{N_1+1} = z_1$, and $(r_1, \ldots, r_{N_1-1})$ be a subsequence of z such that $\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < \frac{1}{2}$. Let $M_1 > 0$ such that $\sum_{p=1}^{M_1} |a_{pN_1}| > \frac{\varepsilon}{2}$ and $|r_{N_1}| \sum_{p=M_1+1}^{\infty} |a_{pN_1}| < \frac{1}{4}$. Let $j_2 = \min\{q : x_q \in x \setminus (r_1, z_1, z_2, \ldots)\}$ and $i_2 = \min\{q : z_q \in z \setminus (z_1, r_1, \ldots, r_{N_1-1})\}$. Since each row of A is null, there exists $N_2 > N_1+1$ such that $\sum_{p=M_1+1}^{\infty} |a_{pN_2}| > \frac{\varepsilon}{2}$ and $|x_{j_2}| \sum_{p=1}^{M_1} |a_{pN_2}| < \frac{1}{8}$. Let $r_{N_2} = x_{j_2}$, $r_{N_2+1} = z_{i_2}$, and $(r_{N_1+2}, \ldots, r_{N_2-1})$ be a subsequence of $z \setminus (z_1, r_1, \ldots, r_{N_1-1}, r_{N_2+1})$ such that

$$\sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{\infty} |a_{pq}| < \frac{1}{16} .$$

Let $M_2 > M_1$ such that $\sum_{p=M_1+1}^{M_2} |a_{pN_2}| > \frac{\varepsilon}{2}$ and $|r_{N_2}| \sum_{p=M_2+1}^{\infty} |a_{pN_2}| < \frac{1}{32}$. This selection process may be continued so that if k is fixed, then

$$\sum_{p=1}^{M_k} \left| \sum_{q=1}^{\infty} a_{pq} r_q \right| \geq \Big(\sum_{p=1}^{M_1} |a_{pN_1} r_{N_1}| - \sum_{p=1}^{M_1} \left| \sum_{q=1}^{N_1 - 1} a_{pq} r_q \right|$$

$$- \sum_{p=1}^{M_1} \left| \sum_{q=N_1+1}^{N_2 - 1} a_{pq} r_q \right| - \sum_{p=1}^{M_1} |a_{pN_2} r_{N_2}| - \ldots \Big)$$

$$+ \Big(\sum_{p=M_1+1}^{M_2} |a_{pN_2} r_{N_2}| - \sum_{p=M_1+1}^{M_2} |a_{pN_1} r_{N_1}|$$

$$- \sum_{p=M_1+1}^{M_2} \left| \sum_{q=1}^{N_1-1} a_{pq} r_q \right| - \sum_{p=M_1+1}^{M_2} \left| \sum_{q=N_1+1}^{N_2-1} a_{pq} r_q \right| - \ldots \Big) + \ldots$$

$$+ \Big(\sum_{p=M_{k-1}+1}^{M_k} |a_{pN_k} r_{N_k}| - \sum_{p=M_{k-1}+1}^{M_k} \left| \sum_{q=1}^{N_1-1} a_{pq} r_q \right| - \ldots \Big)$$

$$= \Big(\sum_{p=1}^{M_1} |a_{pN_1} r_{N_1}| + \ldots + \sum_{p=M_{k-1}+1}^{M_k} |a_{pN_k} r_{N_k}| \Big)$$

$$- \Big(\sum_{q=1}^{N_1-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + \sum_{p=M_1+1}^{M_k} |a_{pN_1} r_{N_1}|$$

$$+ \sum_{q=N_1+1}^{N_2-1} |r_q| \sum_{p=1}^{M_k} |a_{pq}| + \sum_{p=M_2+1}^{M_k} |a_{pN_2} r_{N_2}| + \ldots \Big)$$

$$\geq \frac{\varepsilon}{2} \sum_{i=1}^{k} |r_{N_i}| - 1.$$

But r has been selected so that $\lim_k \sum_{i=1}^{k} |r_{N_i}| = \infty$. Therefore $Ar \notin \ell$, a contradiction. Hence $\lim_q \sum_{p=1}^{\infty} |a_{pq}| = 0$.

Corollary 3.2. The null sequence x is in ℓ if and only if there exists a sum-preserving ℓ-ℓ matrix A such that $Ay \in \ell$ for every rearrangement y of x.

Proof. The identity matrix suffices for necessity. By Definition 1.4 if A is a sum-preserving ℓ-ℓ matrix, then $\lim_q \sum_{p=1}^{\infty} a_{pq} = 1$. Hence by Theorem 3.2, x must be in ℓ.

Example 3.1. By Theorem 3.2 a matrix A that maps all rearrangements of a sequence $x \in c_o \backslash \ell$ into ℓ must be an ℓ-ℓ matrix. But Theorem 3.1 gives little insight into the question of whether A must be ℓ-ℓ if it maps all subsequences of x into ℓ. The following example shows that A need not be ℓ-ℓ in this case. Let $x_n = \frac{1}{n}$ for $n = 1,2,3,\ldots$; $a_{qq} = q^{\frac{1}{3}}$ for $q = 1,8,27,64,\ldots$; and $a_{pq} = 0$ otherwise. If y is a subsequence of x and $Ay = z$, then $|z_q| < q^{-\frac{2}{3}}$ for $q = 1,8,27,\ldots$ and $z_q = 0$ otherwise. Thus $z \in \ell$, but clearly $x \in c_o \backslash \ell$ and A is not ℓ-ℓ.

The pattern established by Theorem 1.6, Theorem 2.3, and Corollary 2.4 might cause one to suspect that if A maps all subsequences (rearrangements) of a sequence $x \in c_o \backslash \ell$ into ℓ, then $Ay \in \ell$ for every $y \in cs$. The following example shows that this is not true in the case of subsequences, and a slight alteration of this example shows it also fails for rearrangements. Let $x_n = \frac{1}{n}$ for $n = 1,2,3,\ldots$; $a_{pq} = \frac{(-1)^q}{q2^p}$

for $p \geq 1$, $q \geq 1$. Then $Ay \in \ell$ for every subsequence y of x since $|a_{pq}y_q| \leq (\frac{1}{2^p})q^{-2}$. Let $z_1 = -1$ and $z_q = (-1)^q(2)$ for $2 \leq q \leq n_1$, where n_1 is the least positive integer such that $\sum_{q=2}^{n_1} (\frac{1}{q} + \frac{1}{q+1}) > 1$. Let $z_{n_1+1} = (-1)^{n_1+1}(\frac{3}{2})$ and $z_q = (-1)^q$ for $n_1+2 \leq q \leq n_2$, where n_2 is the least positive integer such that $\sum_{q=n_1+2}^{n_2} (\frac{1}{q} + \frac{1}{q+1}) > 2$. Let $z_{n_2+1} = (-1)^{n_2+1}(\frac{3}{4})$ and $z_q = (-1)^q(\frac{1}{2})$ for $n_2+2 \leq q \leq n_3$, where n_3 is the least positive integer such that $\sum_{q=n_2+2}^{n_3} (\frac{1}{q} + \frac{1}{q+1}) > 4$. Continue this process defining the sequence z such that $\sum_{q=1}^{\infty} z_q = 0$. Using summation by parts,

$$\begin{aligned}
\lim_n \sum_{q=1}^{n} a_{1q}z_q &= \lim_n [(a_{11}-a_{12})z_1 \\
&+ (a_{12}-a_{13})(z_1+z_2) + \ldots \\
&+ a_{1n} \sum_{q=1}^{n} z_q] \\
&= \tfrac{1}{2}[(1)\{(1+\tfrac{1}{2})+\ldots+|\tfrac{1}{n_1} + \tfrac{1}{n_1+1}|\} \\
&+ \tfrac{1}{2}\{|\tfrac{1}{n_1+1} + \tfrac{1}{n_1+2}| +\ldots+ |\tfrac{1}{n_2} + \tfrac{1}{n_2+1}|\}+\ldots] \\
&> \tfrac{1}{2}[1(1) + \tfrac{1}{2}(2) + \tfrac{1}{4}(4) + \ldots] = \infty,
\end{aligned}$$

since $\lim_n a_{1n} \sum_{q=1}^{n} z_q = 0$. Therefore $Az \notin \ell$.

CHAPTER III BIBLIOGRAPHY

1. Dawson, D. F., "Summability of Subsequences and Stretchings of Sequences," _Pacific Journal of Mathematics_, Vol. 44, No. 2(1973), 455-460.

2. Fridy, J. A., "Properties of Absolute Summability Matrices," _Proceedings of the American Mathematical Society_, 24(1970), 583-585.

3. ____________, "Summability of Rearrangements of Sequences," unpublished paper read before the Annual Meeting of the American Mathematical Society, Washington, D. C., January 23, 1975.

4. Maddox, I. J., "A Tauberian Theorem for Subsequences," _Bulletin of the London Mathematical Society_, 2(1970), 63-65.

CHAPTER IV

SUMMABILITY OF CERTAIN CATEGORY TWO CLASSES

The set of all subsequences (rearrangements) of a sequence can be thought of as a metric space which is of the second category in itself (1, 2, 4). It then becomes natural to ask questions concerning subsets of these spaces and to describe these subsets in terms of sets which are of the first or second category. In this chapter analogs to Theorem 1.3 and Corollaries 2.2, 3.1, and 3.2 are considered in which the requirement that "every" subsequence (rearrangement) of a given sequence satisfy certain conditions is replaced with the requirement that "a set of subsequences (rearrangements) of the second category" fulfill the same conditions. A result similar to Theorem 3.1 has been obtained by F. R. Keogh and G. M. Petersen (4). (Rather than require A to be non-Schur with convergent columns, they demanded that A be regular.)

Let x be an arbitrary sequence. A |-| map of the set of all subsequences of x on the interval $0 < t \leq 1$ can be obtained as follows. Let $t = .a_1a_2a_3\ldots$ be the infinite dyadic expansion of a point t of the interval. Corresponding to this point select the subsequence y of x as follows: retain x_n if $a_n = 1$ and drop it otherwise (2). The meanings of terms such as "a dense set of subsequences of x" and "a set of subsequences of x of the second (first) category" are now evident.

Theorem 4.1. Suppose A is a non-Schur matrix with convergent columns. If A sums a set of subsequences of x of the second category, then x is convergent.

Proof. Following Keogh and Petersen (4), it will be shown that if x is divergent, then A sums a set of subsequences of x of the first category. Suppose x is bounded. Also suppose that there exists row p such that $(a_{pq})_{q=1}^{\infty} \notin cs$. Then there exists $\varepsilon > 0$ such that if $N > 0$, then there exists $m > n \geq N$ such that $|\sum_{q=n}^{m} a_{pq}| > \varepsilon$. Let L be a nonzero limit point of x. For $N = 1,2,3,\ldots$, let $E_N = \{y : y$ is a subsequence of x and there exist $m > n \geq N$ such that $|\sum_{q=n}^{m} a_{pq}y_q| > \frac{|L|\varepsilon}{2}\}$. Then for each N, E_N is both open and dense. Also, if y is a subsequence of x such that $Ay \in c$, then there exists $N > 0$ such that $y \notin E_n$ for $n > N$. Therefore the set of all subsequences of x which A sums is of the first category.

Suppose now that $(a_{pq})_{q=1}^{\infty} \in cs$ for each p and that there exists a row p such that $(a_{pq})_{q=1}^{\infty} \notin \ell$. Let $L_1 \neq 0$ and L_2

be two distinct limit points of x. For N = 1,2,3,..., let E_N = {y:y is a subsequence of x and there exist $m > n \geq N$ such that $|\sum_{q=n}^{m} a_{pq} y_q| > 1$}. Clearly each E_N is open. Suppose a finite subsequence $(y_1,\ldots,y_{k-1})$ of x is given. Choose $n > N + k$ such that if $m > n$, then

$$|\sum_{q=n}^{m} a_{pq}| < \frac{1}{(|L_2|+1)} .$$

Choose a subsequence $(a_{pq}^{(1)})_{q=n}^{m}$ such that $|\sum_{q=n}^{m} a_{pq}^{(1)}| > \frac{2}{|L_1 - L_2|}$ and either $a_{pq}^{(1)} = a_{pq}$ or $a_{pq}^{(1)} = 0$ for $n \leq q \leq m$. Let $(a_{pq}^{(2)})_{q=n}^{m}$ be defined such that $a_{pq}^{(2)} = 0$ if $a_{pq}^{(1)} = a_{pq}$ and $a_{pq}^{(2)} = a_{pq}$ otherwise. Then

$$\sum_{q=n}^{m} a_{pq} = \sum_{q=n}^{m} a_{pq}^{(1)} + \sum_{q=n}^{m} a_{pq}^{(2)}$$

and

$$|\sum_{q=n}^{m} L_1 a_{pq}^{(1)} + \sum_{q=n}^{m} L_2 a_{pq}^{(2)}| \geq |L_1 - L_2| \ |\sum_{q=n}^{m} a_{pq}^{(1)}| - |L_2| \ |\sum_{q=n}^{m} a_{pq}| > 1.$$

Hence each E_N is dense. But each A-summable sequence of x is in $\bigcup_{N=1}^{\infty} \sim E_N$. Therefore the set of subsequences of x which A sums is of the first category.

Suppose now that $(a_{pq})_{q=1}^{\infty} \in \ell$ for each p. Since $x \in m$, by a familiar argument (3) A may be assumed to be row finite.

Let $w = (L_1, L_2, L_1, L_2, \ldots)$. By Theorem 1.6 there exists a subsequence z of w such that $Az \notin c$. Therefore there exists $\varepsilon > 0$ such that if $N > 0$, then there exist $m > n \geq N$ such that $|\sum_{q=1}^{\infty} a_{nq} z_q - \sum_{q=1}^{\infty} a_{mq} z_q| > \varepsilon$. For $N = 1, 2, 3, \ldots$, let

$E_N = \{y : y$ is a subsequence of x such that there exist $m > n \geq N$ such that $|\sum_{q=1}^{\infty} a_{nq} y_q - \sum_{q=1}^{\infty} a_{mq} y_q| > \frac{\varepsilon}{2}\}$. Since A is row finite, each E_N is open. Suppose that a finite subsequence $(y_1, \ldots, y_{k-1})$ of x is given. Let $N > 0$. Since the columns of A converge, there exist $m > n \geq N$ such that $|\sum_{q=k}^{\infty} a_{nq} z_q - \sum_{q=k}^{\infty} a_{mq} z_q| > \frac{3\varepsilon}{4}$. But A is row finite and each z_q is either L_1 or L_2. Therefore $(y_k, \ldots, y_t)$ can be chosen such that $(y_1, \ldots, y_t)$ is a subsequence of x and $|\sum_{q=1}^{t} a_{nq} y_q - \sum_{q=1}^{t} a_{mq} y_q| > \frac{\varepsilon}{2}$, where $t = \max\{q : |a_{nq}| + |a_{mq}| > 0\}$. Let $(y_{t+1}, y_{t+2}, \ldots)$ be chosen such that y is a subsequence of x. Then $y \in E_N$, and E_N is dense. But each A-summable subsequence of x is in $\bigcup_{N=1}^{\infty} \sim E_N$, therefore the set of subsequences of x which A sums is of the first category.

Suppose now that x is unbounded and A is row finite. For $N = 1, 2, 3, \ldots$, let $S_N = \{y : y$ is a subsequence of x and there exists $m \geq N$ such that $|\sum_{q=1}^{\infty} a_{mq} y_q| > N\}$. Each S_N is open since A is row finite. Let $(y_1,\ldots,y_k)$ be a subsequence of x. Since A is not Schur there exist $m \geq N$ and $n > k$ such that $a_{mn} \neq 0$ and $a_{mq} = 0$ if $q > n$. Choose $(y_{k+1},\ldots,y_n,\ldots)$ such that y is a subsequence of x and $|\sum_{q=1}^{\infty} a_{mq}y_q| = |\sum_{q=1}^{n} a_{mq}y_q| > N$. Thus $y \in S_N$ and S_N is dense. But each A-summable sequence of x is in $\bigcup_{N=1}^{\infty} \sim S_N$, therefore the set of subsequences of x which A sums is of the first category.

Suppose x is unbounded and row p of A has an infinite number of nonzero entries. For N = 1,2,3,..., let $S_N = \{y{:}y$ is a subsequence of x and there exists $n \geq N$ such that $|a_{pn}y_n| > 1\}$. Each S_N is both open and dense, and each A-summable subsequence y of x is in $\bigcup_{N=1}^{\infty} \sim S_N$. Therefore the set of subsequences of x which A sums is of the first category.

Example 4.1. The following example illustrates the necessity of the requirement in Theorem 4.1 that all columns of A be convergent. A similar argument shows the necessity of the same requirement in Theorem 4.3. Let $a_{p1} = (-1)^p$ for $p \geq 1$ and $a_{pq} = 0$ otherwise. Then A is non-Schur. Let $x = (0,1,0,1,\ldots)$ and $T = \{y{:}y$ is a subsequence of x and $y_1 = x_1\}$. Then T is open, therefore by Theorem 1.14, T is of the second category. But clearly x is divergent.

Theorem 4.2. Suppose A is a matrix with the following three properties:

i.) $\sum_{p=1}^{\infty} |a_{pq}| < \infty$ for $q = 1,2,3,\ldots$;

ii.) $\lim_q \sum_{p=1}^{\infty} a_{pq} = L \neq 0$; and

iii.) $\sum_{q=1}^{\infty} |a_{pq}| < \infty$ for $p = 1,2,3,\ldots$.

The null sequence x is in ℓ if A maps a set of subsequences of x of the second category into ℓ.

Proof. Let x be a null sequence not in ℓ. Since x is bounded and each row of A is in ℓ, by a familiar argument (3) A may as well be assumed to be row finite. Let $K = \{y: y$ is a subsequence of x and $Ay \notin \ell\}$. Let $(y_1,\ldots,y_n)$ be a finite subsequence of x with $y_n = x_m$. Let B be the submatrix of A consisting of columns $n+1,n+2,\ldots$ of A. Let $z = (x_q)_{q=m+1}^{\infty}$. Since B satisfies i.) and ii.), by Theorem 3.1 there exists a subsequence w of z such that $Bw \notin \ell$. Let $y = (y_1,\ldots,y_n,w_1,w_2,\ldots)$. Since $\sum_{q=1}^{n} |y_q| \sum_{p=1}^{\infty} |a_{pq}| < \infty$, then $Ay \notin \ell$ and K is dense in the set of all subsequences of x. For $N = 1,2,3,\ldots$, let $S_N = \{y: y$ is a subsequence of x and there exists $m > N$ such that $\sum_{p=N}^{m} |\sum_{q=1}^{\infty} a_{pq}y_q| > 1\}$. Each S_N contains K and therefore is dense, but A is row finite, therefore each S_N is also open. If z is a subsequence of x such that $Az \in \ell$, then $z \in \bigcup_{N=1}^{\infty} \sim S_N$, therefore the set of all such subsequences is of the first category. This completes the proof of Theorem 4.2.

Let E denote the space in which a point t is a permutation $(t_1, t_2, t_3, \ldots)$ of the positive integers, and the distance between two points $t = (t_1, t_2, \ldots)$ and $s = (s_1, s_2, \ldots)$ is given by the Fréchet formula

$$(t,s) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\frac{|t_n - s_n|}{1 + |t_n - s_n|} \right].$$

Let E* be the space in which a point is a sequence of positive integers not necessarily a permutation of all positive integers, and the distance between two points is given by the above Fréchet formula. A 1-1 map pairing E with the space of all rearrangements of a fixed sequence x is therefore evident. R. P. Agnew (1) has shown that E is a G_δ in the complete space E*. Therefore by Theorems 1.16 and 1.15, E is of the second category in itself. It is now natural to speak of a set of rearrangements of a sequence x being "dense" or "of the second (first) category" in the space of all rearrangements of x.

Theorem 4.3. Suppose A is a non-Schur matrix with convergent columns. If A sums a set of rearrangements of x of the second category, then x is convergent.

Proof. Following Keogh and Petersen (4), let x be divergent. Suppose there exist a row p and a rearrangement y of x such that $(a_{pq}y_q)_{q=1}^{\infty} \notin cs$. Then there exists $\varepsilon > 0$ such that if $N > 0$, then there exist $m > n \geq N$ such that $|\sum_{q=n}^{m} a_{pq}y_q| > \varepsilon$. For $N = 1,2,3,\ldots$, let $B_N = \{z : z$ is a rearrangement of x and there exists $m > n \geq N$ such that $|\sum_{q=n}^{m} a_{pq}z_q| > \varepsilon\}$. Clearly each B_N is open. Let $(z_1,\ldots,z_k)$ be a permutation of a finite number of elements of x. Let $i = \max\{j : y_j = z_t$ for some $1 \leq t \leq k\}$ and choose $(z_{k+1},\ldots,z_i)$ from $(y_1,\ldots,y_i)\backslash(z_1,\ldots,z_k)$ in any order. Let $z_q = y_q$ for $q > i$. Then there exist $m > n \geq (i+N)$ such that $|\sum_{q=N}^{m} a_{pq}z_q| = |\sum_{q=n}^{m} a_{pq}y_q| > \varepsilon$. Therefore each B_N is dense, and since each A-summable rearrangement of x is in $\bigcup_{N=1}^{\infty} \sim B_N$, the set of all such rearrangements is of the first category.

Suppose that for each row p and for each rearrangement y of x that $(a_{pq}y_q)_{q=1}^{\infty} \in cs$. Then by Lemma 2.2 each row of A is in ℓ. Since x is bounded, by a familiar argument (3) A may as well be assumed to be row finite. By Theorem 2.3 there exist a rearrangement y of x and $\delta > 0$ such that if $N > 0$, then there exist $m > n \geq N$ such that

$$\left|\sum_{q=1}^{\infty} a_{nq}y_q - \sum_{q=1}^{\infty} a_{mq}y_q\right| > \delta.$$

For $N = 1,2,3,\ldots$ let $D_N = \{z:z$ is a rearrangement of x and there exist $m > n \geq N$ such that $|\sum_{q=1}^{\infty} a_{nq}z_q - \sum_{q=1}^{\infty} a_{mq}z_q| > \frac{\delta}{2}\}$. Since A is row finite, each D_N is open. Let $(z_1,\ldots,z_k)$ be a permutation of a finite number of elements of x. Let $i = \max\{j:y_j = z_t$ for some $1 \leq t \leq k\}$ and choose $(z_{k+1},\ldots,z_i)$ from $(y_1,\ldots y_i)\setminus(z_1,\ldots z_k)$ in any order. Let $z_q = t_q$ for $q > i$. Then since each column of A is convergent, there exist $m > n \geq N$ such that

$$|\sum_{q=1}^{\infty} a_{nq}z_q - \sum_{q=1}^{\infty} a_{mq}z_q|$$

$$> |\sum_{q=1}^{\infty} a_{nq}y_q - \sum_{q=1}^{\infty} a_{mq}y_q| - \frac{\delta}{2} > \frac{\delta}{2} .$$

Thus each D_N is dense. Since the set of A-summable rearrangements of x is contained in $\bigcup_{N=1}^{\infty} \sim D_N$, it is of the first category.

In the case where x is unbounded, the proof follows as in Theorem 4.1 with only slight alterations.

Lemma 4.1. Suppose each of x and a is a sequence such that $(a_q y_q)_{q=1}^{\infty} \in cs$ for every rearrangement y of x. If $\varepsilon > 0$, then there exists $N > 0$ such that if $n \geq N$ and $(y_q)_{q=n}^{\infty}$ is a rearrangement of $(x_q)_{q=n}^{\infty}$, then $|\sum_{q=n}^{\infty} a_q y_q| \leq \varepsilon$.

Proof. Suppose the Lemma is false and $(z_1,\ldots,z_k)$ is a permutation of k terms of x. Let $t = \max\{i:z_q = x_i$ for some $1 \le q \le k\}$ and choose $(z_{k+1},\ldots,z_t)$ from $(x_1,\ldots,x_t)\backslash(z_1,\ldots,z_k)$ in any order. Let $n \ge t$ such that there exists a rearrangement $(y_q)_{q=n}^{\infty}$ of $(x_q)_{q=n}^{\infty}$ with $|\sum_{q=n}^{\infty} a_q y_q| > \varepsilon$. Let $m > n$ such that $|\sum_{q=n}^{m} a_q y_q| > \varepsilon$. Let $z_q = x_q$ if $t < q < n$ and $z_q = y_q$ if $n \le q \le m$. This process may be continued, defining a rearrangement z of x such that $(a_q z_q)_{q=1}^{\infty} \notin cs$, a contradiction.

Theorem 4.4. Let x be a null sequence and A be a matrix such that each column of A is in ℓ and $\lim_q \sum_{p=1}^{\infty} |a_{pq}| \neq 0$. If there exists a set of rearrangements of x of the second category which A maps into ℓ, then x is in ℓ.

Proof. Let x be a null sequence not in ℓ. If there exists a row p and a rearrangement y of x such that $(a_{pq}y_q)_{q=1}^{\infty} \notin cs$, then by an argument similar to that used in Theorem 4.3, the set of rearrangements of x which A maps into ℓ is of the first category.

Suppose whenever y is a rearrangement of x that $(a_{pq}y_q)_{q=1}^{\infty} \in cs$ for each p. For $N = 1,2,3,\ldots$, let $E_N = \{z : z$ is a rearrangement of x such that there exists $n > N$ such that $\sum_{p=N}^{n} |\sum_{q=1}^{\infty} a_{pq}z_q| > 1\}$. By Lemma 4.1 each E_N is open. Suppose $(y_1,\ldots,y_k)$ is a permutation of k terms of x. Then $x\backslash(y_1,\ldots,y_k)$ is not in ℓ. Let B be the submatrix of A

consisting of the columns k+1,k+2,k+3,... of A. By Theorem 3.2, since $\lim_q \sum_{p=1}^{\infty} |b_{pq}| \neq 0$, there exists a rearrangement $(y_{k+1}, y_{k+2}, \ldots)$ of $x \backslash (y_1, \ldots, y_k)$ such that $\sum_{p=1}^{\infty} | \sum_{q=k+1}^{\infty} a_{pq} y_q | = \infty$. Let $n > N$ such that $\sum_{p=N}^{n} | \sum_{q=K+1}^{\infty} a_{pq} y_q | > 1 + \sum_{q=1}^{k} |y_q| \sum_{p=1}^{\infty} |a_{pq}|$. Thus $y \in E_N$, and E_N is dense. But any rearrangement y of x with the property that $A_y \in \ell$ is in $\bigcup_{N=1}^{\infty} \sim E_N$, therefore the class of all rearrangements with A maps into ℓ is of the first category.

Example 4.3. The following example illustrates the necessity of the requirement "each column of A is in ℓ" in Theorem 4.4. For an arbitrary $n > 0$ let $A_{pn} = 1$ for each p and $a_{pq} = 0$ otherwise. Let $x = (0,1,\frac{1}{2},\frac{1}{3},\ldots)$. Then $K = \{y{:}y$ is a rearrangement of x, $y_q = \frac{1}{q}$ if $q < n$, and $y_n = 0\}$ is open, and therefore by Theorem 1.14 is of the second category. But $Ay \in \ell$ whenever $y \in k$. Thus A maps a set of rearrangements of the second category into ℓ, yet clearly x is not in ℓ.

CHAPTER IV BIBLIOGRAPHY

1. Agnew, R. P., "On Rearrangements of Series," Bulletin of the American Mathematical Society, 46(1940), 797-799.

2. Buck, R. C. and Pollard, H., "Convergence and Summability Properties of Subsequences," Bulletin of the American Mathematical Society, 49(1943), 924-931.

3. Goffman, C. and Petersen, G. M., "Submethods of Regular Matrix Summability Methods," Canadian Journal of Mathematics, 8(1956), 40-46.

4. Keogh, F. R. and Petersen, G. M., "A Universal Tauberian Theorem," Journal of the London Mathematical Society, 33(1958), 121-123.

5. Wilansky, A., Topology for Analysis, Ginn and Company, Waltham, Mass., 1970.

BIBLIOGRAPHY

Books

Cooke, R. G., Infinite Matrices and Sequence Spaces, Macmillan Publishing Company, London, 1950.

Day, M. M., Normed Linear Spaces, 3rd ed., Springer-Verlag, New York, 1973.

Hardy, G. H., Divergent Series, Clarendon Press, Oxford, 1949.

Knopp, Konrad, Theory and Application of Infinite Series, Hafner Publishing Company, New York, 1971.

Powell, R. E., and Shah, S. M., Summability Theory and Applications, Van Nostrand Reinhold Company, London, 1972.

Wilansky, A., Functional Analysis, Blaisdell Publishing Company, New York, 1973.

____________, Topology for Analysis, Ginn and Company, Waltham, Mass., 1970.

Articles

Agnew, R. P., "Methods of Summability Which Evaluate Sequences of Zeros and Ones Summable C_1," American Journal of Mathematics, 70(1948), 75-81.

____________, "On Rearrangements of Series," Bulletin of the American Mathematical Society, 46(1940), 797-799.

____________, "Permutations Preserving Convergence of Series," Proceedings of the American Mathematical Society, 6(1955), 563-564.

____________, "A Simple Sufficient Condition That a Method of Summability Be Stronger Than Convergence," Bulletin of the American Mathematical Society, 52(1946), 128-132.

__________, "Subseries of Series Which Are Not Absolutely Convergent," Bulletin of the American Mathematical Society, 53(1947), 118-120.

__________, "Summability of Subsequences," Bulletin of the American Mathematical Society, 50(1944), 596-598.

Bennett, G. and Kalton, N. J., "Inclusion Theorems for K-Spaces," Canadian Journal of Mathematics, 25, No. 3(1973), 511-524.

Bohr, H., "On the Generalization of a Known Convergence Theorem," Nyt Tidskrift for Mathematik, (B) 20(1909), 1-4.

Buck, R. C., "Limit Points of Subsequences," Bulletin of the American Mathematical Society, 50(1944), 395-397.

__________, "A Note on Subsequences," Bulletin of the American Mathematical Society, 49(1943), 898-899.

__________, "An Addendum to 'A Note on Subsequences'," Proceedings of the American Mathematical Society, 7(1956), 1074-1075.

__________ and Pollard, H., "Convergence and Summability Properties of Subsequences," Bulletin of the American Mathematical Society, 49(1943), 924-931.

Dawson, D. F., "Matrix Summability of Divergent Sequences," Portugaliae Mathematica, 33(1974), 113-116.

__________, "Summability of Subsequences and Other Regular Transformations of a Sequence," Bollettino della Unione Matematica Italiana (4), 8(1973), 449-455.

__________, "Summability of Subsequences and Stretching of Sequences," Pacific Journal of Mathematics, 44, No. 2(1973), 455-460.

Fridy, J. A., "A Note on Absolute Summability," Proceedings of the American Mathematical Society, 20(1969), 285-286.

__________, "Properties of Absolute Summability Matrices," Proceedings of the American Mathematical Society, 24(1970, 583-585.

Gaier, D., "Limitierung Gestreckter Folgen," Publication of the Ramanujan Institute, No. 1 (1969), 223-234.

Goffman, Casper and Petersen, G. M., "Submethods of Regular Matrix Summability Methods," Canadian Journal of Mathematics, 8(1956), 40-46.

Keogh, F. R. and Petersen, G. M., "A Generalized Tauberian Theorem," Canadian Journal of Mathematics, 10(1958), 111-114.

__________, "A Universal Tauberian Theorem," Journal of the London Mathematical Society, 33(1958), 121-123.

Knopp, K. and Lorentz, G. G., "Beitrage zur Absolutem Limitierung," Archiv der Mathematik, 2(1949), 10-16.

Maddox, I. J., "A Tauberian Theorem for Subsequences," Bulletin of the London Mathematical Society, 2(1970), 63-65.

Mears, F. M., "Absolute Regularity and Norlund Mean," Annals of Mathematics, 38(1937), 594-601.

Schur, I., "Über Linear Transformationen in der Theorie die Unendlich Reihen," Journal fur die Reine und Angewandte Mathematik, 151(1921), 79-111.

Steinhaus, H., "Some Remarks on the Generalization of the Notion of Limit," Prace Matematyczno Fizycne, 22 (1921), 121-134.

Toeplitz, O., "Uber Allgemeine Lineare Mittelbildungen," Prace Matematyczno Fizycne, 22(1911), 113-119.

Unpublished Materials

Dawson, D. F., "A Tauberian Theorem for Stretchings," Journal of the London Mathematical Society (to appear).

Fridy, J. A., "Summability of Rearrangements of Sequences," unpublished paper read before the Annual Meeting of the American Mathematical Society, Washington, D. C., January 23, 1975.

Silverman, L. L., "On the Definition of the Sum of a Divergent Series," unpublished thesis, University of Missouri Studies, Mathematical Series, Vol. 1, No. 1, 1913.