A THEORETICAL INVESTIGATION OF BOUND ROTON PAIRS
IN SUPERFLUID HELIUM-4

DISSERTATION

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By

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The Bogoliubov theory of excitations in superfluid helium is used to study collective modes at zero temperature. A repulsive delta function shell potential is used in the quasiparticle excitation energy spectrum to fit the observed elementary excitation spectrum, except in the plateau region. The linearized equation of motion method is used to obtain the secular equation for a collective mode consisting of a linear combination of one and two free quasiparticles of zero total momentum. It is shown that in this case for high-lying collective modes, vertices involving three quasiparticles cancel, and only vertices involving four quasiparticles are important. A decomposition into various angular momentum states is then made. Bound roton pairs in the angular momentum D-state observed in light-scattering experiments exist only for an attractive coupling between helium atoms in this oversimplified model. Thus, the interaction between particles can be reinterpreted as a phenomenological attractive coupling between quasiparticles, in order to explain the Raman scattering from bound roton pairs in superfluid helium.
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CHAPTER I
INTRODUCTION

At temperatures below 2.17 K (called the \( \lambda \)-point) at saturated vapor pressure of 0.5006 atm, liquid helium \( \text{He}^4 \) shows superfluidity and is called superfluid helium or \( \text{He} \text{ II} \).\(^1\) The elementary excitation energy spectrum of \( \text{He} \text{ II} \) was originally proposed by Landau,\(^2\) and confirmed by neutron scattering experiments by Cowley and Woods,\(^3\) and others. The observed elementary excitation energy spectrum is linear (called phonons) at small momenta, has a dip (called rotons) at momenta (divided by \( \frac{\hbar}{\pi} \)) about \( 1.90^{-1} \text{Å}^{-1} \), and has a plateau (not proposed by Landau) at large momenta. The phonon-roton spectrum can be fitted by the Bogoliubov theory\(^4\) of a weakly interacting Bose gas, if a suitable choice of an effective potential is made.

In 1970 Greytak, Woerner, Yan, and Benjamin\(^5\) found a peak in their Raman scattering data from \( \text{He} \text{ II} \), which they interpreted as due to the existence of bound roton pairs. The analysis of their experimental data is based on a modification of the theory of bound roton pairs proposed by Ruvalds and Zawadowski\(^6\) (RZ) for an an attractive roton-roton interaction in the angular momentum D-state. According to Griffin,\(^7\) the attractive contact potential between rotons
in configuration space used in the RZ theory has no microscopic justification. The motivation of the present work is thus to investigate the existence of bound roton pairs within the framework of the Bogoliubov theory.

The peak in intensity of scattered light observed by Greytak, et al.\textsuperscript{5} corresponds to a photon energy transfer (divided by the Boltzmann constant $k_B$) of $0.37 \pm 0.10$ K less than twice the single roton energy, and to a photon momentum transfer which is essentially negligible compared with the roton momentum. This peak is interpreted as the creation of a bound roton pair in superfluid helium by light scattering. In the dipole approximation of light scattering, the light couples only to the angular momentum D-state ($\ell=2$) of a roton pair. The measured depolarization ratios of scattered light by Greytak and Yan\textsuperscript{8} and by Greytak and Woerner\textsuperscript{9} show that the roton pair is indeed in a D-state.

The Bogoliubov theory\textsuperscript{4} of quasiparticles (called bogolons) in superfluid helium can provide a good fit to the phonon and roton regions of the elementary excitation spectrum, and gives explicit forms for the bogolon interactions so it is used in this work. One aspect of the Bogoliubov theory is that it allows only a predominantly repulsive potential between particles. If a predominantly attractive potential between particles is used in the Bogoliubov theory, an imaginary speed of sound is obtained. According to Woods and
Cowley,\textsuperscript{10} the potential between helium atoms has to be effectively repulsive since the bulk compressibility of liquid helium is positive. Hence it is reasonable to study the possible connection between the RZ theory\textsuperscript{6} and the Bogoliubov theory with a simple potential between the particles. The particle potential model used in this work is the delta function shell potential, which has been used by Kobe\textsuperscript{11} to describe phenomenologically the phonon-roton excitation spectrum in superfluid helium.

By means of Bogoliubov's canonical transformation, the Hamiltonian of liquid helium is expressed in terms of bogolon operators. Following Takano,\textsuperscript{12} we assume the existence of a collective mode in superfluid helium. The collective mode composed of two bogolons is described by the linearized equations of motion method based on the Schrödinger equation. The method automatically leads to the compensation of the lowest order dangerous diagrams which was proposed by Bogoliubov\textsuperscript{13} for determining the coefficients in his canonical transformation. The compensation of dangerous diagrams simplifies the linearized equations of motion, which are described by diagrams. One of the new results in this work is that for high-lying collective modes of zero total momentum, the three-bogolon vertices cancel in the linearized equation of motion. It is important to note that this cancellation is independent of the potential. Without this simplification, the integral equations obtained would be extremely formidable to solve.
Using a delta function shell potential to describe the repulsion between two helium atoms, we derive the secular equations for a collective mode of zero total momentum in various angular momentum states in the new Tamm-Dancoff approximation (NTDA). The NTDA is an approximation which takes into account both bogolon (quasiparticle) pairs and quasihole pairs. If the quasihole pairs in the NTDA are omitted, the Tamm-Dancoff approximation (TDA) is obtained. The solution of the secular equation in the NTDA for this simple potential is an original contribution, since the phenomenological RZ theory considers only the quasiparticle pairs.

The results of our calculations for the collective mode energy of a D-state in terms of the interparticle coupling show that bound roton pairs in the angular momentum D-state observed in light scattering experiments exist in this oversimplified model only if the effective coupling between particles is attractive. Therefore, for future investigation, we suggest the renormalization of bogolon vertices, which hopefully can give an attractive interaction between bogolons in spite of the basically repulsive interaction between particles. A reinterpretation of the interatomic interaction as a phenomenological roton-roton interaction can be made to obtain bound roton pairs.

Extending Fetter's theory of light scattering, we investigate the Raman scattering from the collective mode
(bound roton pair) which is assumed to be due to an attractive roton-roton interaction based on a phenomenological reinterpretation of our theory. Our treatment of Raman scattering from the collective mode is simpler than Iwamoto's theory.\textsuperscript{15} We do not use an average matrix element to calculate the photon cross section as the RZ theory\textsuperscript{6} does. The calculated depolarization ratios are in better agreement with experimental observations than Fetter's more sophisticated theory\textsuperscript{14} which is only for free roton pairs.

Chapter II provides the experimental background with which this work is concerned. For the purpose of later comparison, we discuss quantum hydrodynamic theories of the single quasiparticle energy spectrum, the roton-roton interaction, and bound roton pairs in Chapter III. The Bogoliubov theory of quasiparticles in superfluid helium is reviewed in Chapter IV. The linearized equations of motion and the compensation of dangerous diagrams are derived in Chapter V. The secular equations for a collective mode of zero total momentum in various angular momentum states are obtained in Chapter VI and solved in Chapter VII. The theory of light scattering from He II is reviewed in Chapter VIII. The Raman scattering from bound roton pairs is investigated in Chapter IX. Finally, the dissertation is summarized and conclusions are drawn in Chapter X. For the aid of the reader Appendix A gives a list of the symbols used and their definitions.
CHAPTER II

RELEVANT EXPERIMENTAL BACKGROUND

In order to understand the idea of bound roton pairs in superfluid $\text{He}^4$, some relevant experimental background is given in this chapter. First the phase diagram of $\text{He}^4$ is discussed to show under what conditions superfluid $\text{He}^4$, called He II, exists. Then the spectrum of elementary excitations in He II is discussed, including both the phonon and roton parts of the spectrum. Collective modes at higher energy have been discovered, using inelastic neutron scattering and, more recently, laser light scattering. Particular attention is paid to the light scattering experiments, which apparently indicate the existence of bound roton pairs.

Phase Diagram

Although helium was first liquefied at the low temperature of 4 K by H. Kammerlingh Onnes in 1908, its peculiar behavior was not realized until 1936 when an anomaly was found in the specific heat $C_p$ by Keesom and Keesom. The specific heat curve of liquid $\text{He}^4$ at low temperatures bears some resemblance to the Greek letter $\lambda$ with a discontinuity at 2.17 K at saturated vapor pressure of 0.5006 atm. This temperature is thus named the $\lambda$-point, at which a transition to a new phase of liquid $\text{He}^4$ occurs. The extraordinary phase diagram of $\text{He}^4$ in the pressure-temperature plane shown in Fig. 1 exhibits no triple point, but has instead the $\lambda$-line. At temperature below the $\lambda$-line $\text{He}^4$
exhibits superfluidity, which is the ability to flow through extremely fine channels without showing any viscosity. Below the $\lambda$-temperature liquid $\text{He}^4$ is called He II to distinguish it from He I above the $\lambda$-point, which behaves like an ordinary liquid. At pressures less than about 25 atm, $\text{He}^4$ remains liquid even at absolute zero temperature. The reason why He II does not solidify at extremely low temperatures is that the attractive forces between atoms are weak because of the closed electron shell, and the zero-point oscillation energy is large because of the small atomic mass.

Among the many remarkable properties of He II, superfluidity is perhaps the most dramatic. The experimentally observed values of the viscosity coefficient $\eta$ of He II show quite different character as a function of temperature when measured by different methods, as shown in Fig. 2. On one hand, He II can flow with a value of $\eta$ less than $10^{-11}$ poise through narrow channels about a few microns in diameter. On the other hand, the drag on a cylinder rotating in He II always shows a value of $\eta$ not less than $1.2 \times 10^{-5}$ poise. The viscosity coefficient of He I is about $3 \times 10^{-5}$ poise, however. To explain this and other behavior of He II, a phenomenological two-fluid model which assumes that He II is composed of interpenetrating superfluid and normal fluid components has been quite successful. Landau suggested that the He II system is composed of a background superfluid plus weakly
interacting elementary excitations or quasiparticles associated with the normal fluid.

At constant temperature and entropy, He II supports first sound which is a density or pressure wave. In this case the normal fluid and superfluid components move in phase. At constant density and pressure, however, He II supports a temperature or entropy wave called second sound. In this case the normal fluid and superfluid components move out of phase. The two-fluid model is very successful in explaining these phenomena.

Since a He\(^4\) nucleus consists of two neutrons with opposite spin and two protons with opposite spin, and the two electrons in the atom have opposite spin, the atom can be considered as a boson of spin zero. Consequently London\(^{19}\) suggested that Bose-Einstein condensation plays an important role in liquid He\(^4\). Liquid He\(^4\) is a quantum fluid to which classical statistical mechanics cannot be applied. There is so far no satisfactory microscopic theory for the phase transition of liquid He\(^4\) at the \(\lambda\)-point. The interpretation of various peculiar thermodynamic and transport properties of He II requires an understanding of the elementary excitation energy \(E_K\) versus wave-vector \(\vec{k}\) or momentum \(\vec{\pi}\) in liquid He\(^4\) which should be used in the boson average occupation number

\[
\mathcal{N}_K = \left[ \exp(\beta E_K) - 1 \right]^{-1} \tag{2.1}
\]
where $\beta^{-1} = k_B T$, $k_B$ is Boltzmann's constant, and $T$ is the absolute temperature. When Landau$^2$ tried to relate the experimentally observed specific heat $C_p$ of liquid He$^4$, using the average occupation number $n_k$, he found that the elementary excitation spectrum $E_k$ has to be linear for small momentum, and to have a dip at large momentum, as shown in Fig. 3. The atom energy spectrum $\frac{n^2 k^2}{2m}$ does not fit the thermodynamic data for He$^4$. The strong interatomic interactions must be taken into account to obtain the phonon-roton spectrum in Fig. 3.

Elementary Excitation Spectrum

Cohen and Feynman$^{20}$ suggested that the excitation spectrum of He II could be directly determined by thermal neutron inelastic scattering experiments. The single-excitation dispersion curve obtained from the experimental results of neutron scattering from He II by Cowley and Woods$^4$ is shown in Fig. 3, which confirms the phenomenological phonon-roton spectrum proposed by Landau$^2$ mentioned in the previous section. Earlier experiments were performed by groups in Stockholm,$^{21}$ Chalk River$^{22}$ and Los Alamos.$^{23}$

By means of both a time-of-flight rotating crystal spectrometer and a triple-axis crystal spectrometer, Cowley and Woods$^3$ have studied the inelastic scattering of thermal neutrons from liquid helium at a temperature of 1.1 K, over
a momentum transfer range of $0.2 \text{Å}^{-1} < q < 9.0 \text{Å}^{-1}$ and an energy transfer range of $0 \text{K} < W < 900 \text{K}$. The momentum $q$ is given in wave-number units, since the de Broglie relation states that momentum is $\pi$ times the wave number, where $2\pi\hbar = \hbar$ is the Planck constant. The energy $W$ is given in temperature units, since the energy is $k_B$ times a temperature, where $k_B$ is the Boltzmann constant. The observed energy spectrum consists of two branches. The lower-lying single-excitation curve $E_k$, where $k$ is the magnitude of wave-vector of a single excitation, shown in Fig. 3 agrees quite well with Landau's phonon-roton spectrum. The upper-lying broad collective excitation continuum $W_q$, where $q$ is the magnitude of the total wave-vector, shown in Fig. 4 is composed of two or more excitations.

In neutron scattering with incident wave number $k$ from isotropic He II, the neutron cross section per He$^4$ with the bound atom scattering cross section is given by

$$
\frac{d^2\sigma}{d\Omega dE} = \frac{\sigma}{4\pi k} \frac{k'}{k} S(q, W),
$$

where $d\Omega$ is the acceptance solid angle, $dE$ the energy interval, and $k'$ the wavenumber of the scattered neutron beam. In Eq. (2.2), $q$ is the magnitude of the momentum transfer and $W$ is the energy transfer. The neutrons couple to the density fluctuations, described by the dynamic liquid structure factor $S(q, W)$. It is the Fourier transform of the pair correlation function,

$$
G(\vec{r}, t) = \langle \rho(\vec{r}, t) \rho(0, 0) \rangle,
$$
where the angular brackets denote a thermal average and 
\( \rho(\vec{r},t) \) is the density at position \( \vec{r} \) at time \( t \). The pair correlation function \( G(\vec{r},t) \) is the probability of finding a particle at position \( \vec{r} \) at time \( t \) per unit volume when there is a particle at the origin 0 at time 0 in the liquid. Thus, the two-branch spectrum of He II is directly related to the peaks of the dynamic liquid structure factor \( S(q,W) \) plotted against \( W \) at a given value of \( q \), as shown for example in Fig. 5.

In the lower-lying spectrum in Fig. 3, the linear part of the elementary excitation spectrum \( E_k \) at small momenta \( (k < 0.4 \text{ Å}^{-1}) \) is called phonon region, the hump part (around the local maximum \( k=1.1 \text{ Å}^{-1} \) and \( E_k=13.80 \text{ K} \)) is called the "maxon" region, the dip part (around the local minimum \( k=1.9 \text{ Å}^{-1} \) and \( E_k=8.70 \text{ K} \)) is called the "roton" region, and the nearly flat part at large momenta \( (2.5 \text{ Å}^{-1} < k < 3.6 \text{ Å}^{-1} \) and \( 16.45 \text{ K} < E_k < 18.45 \text{ K} \)) is called the plateau region. The plateau region of \( E_k \) reaches an excitation energy slightly greater than twice the roton energy before disappearing. The existence of a plateau which disappears at higher momentum seems to indicate the decay\(^{26} \) of a single excitation in the plateau region into two lower energy rotons with conservation of energy and momentum. Hence the lifetime of an elementary excitation in He II should be considered as being finite instead of infinite, which is shown experimentally by the finite linewidths in Fig. 5. The elementary excitation \( E_k \) between
maxon and roton regions has negative group velocity, the physical meaning of which is still quite puzzling. The phonons are the usual quantized sound waves, which are density fluctuations.

We still do not have a clear physical picture of the roton which is responsible for much of the peculiar behavior of the He II. Landau\(^2\) called the excitations at large momentum "rotons," because he thought they had something to do with rotational motion of groups of He\(^4\) atoms. Feynman\(^27\) considers the roton as "the ghost of a vanishing vortex ring." According to Feynman and Cohen\(^25\), rotons are small quantized vortex rings, or modified He\(^4\) atoms surrounded by a backflow of other He\(^4\) atoms like smoke rings. Stephen and Yau\(^28\) considered the roton-roton interaction at large distance to be due to the interaction between the dipolar velocity fields created by small quantized vortex rings. At small distances they used a hard-core repulsion between rotons. Most theorists now believe that rotons are dressed density fluctuations of short wavelength (about 0.5 Å). The He II system can support these density fluctuations since, below the \(\lambda\)-point, the thermal de Broglie wavelength of a He\(^4\) atom is larger than the average interatomic spacing. Hence the He II system behaves in many ways like a continuum. The neutron scattering experiment offers almost complete information on elementary excitation in He II, except in the phonon region which, however, can be determined by measurements of the speed of sound in He II.\(^29\)
Collective Excitations

The collective mode observed in He II is shown in Fig. 4, which is obtained from both neutron and photon scattering experiments. The neutron scattering experiments are discussed first.

Neutron Scattering

In the neutron scattering experiment, Cowley and Woods\textsuperscript{3} fixed the momentum transfer and swept the energy transfer to observe the intensities (counts per second) of scattered neutrons. As a function of energy transfer $W_q$, the intensity of scattered neutrons at a fixed momentum transfer $q$ exhibits, in addition to a sharp peak due to the single elementary excitations, a broad maximum at higher energies as shown in Fig. 5. Therefore the collective mode excitation spectrum in Fig. 4 is best described by a curve of the maximum intensity peak, flanked on either side by curves indicating the half-maximum intensity. The observed maximum-intensity peak of scattered neutrons in the collective mode continuum in the range $0.4 \text{ Å}^{-1} < q < 2.3 \text{ Å}^{-1}$ has an energy transfer parallel to the corresponding part of the lower-lying elementary excitation curve $E_k$. The maximum intensity curve of $W_q$ bends upward in the range $2.3 \text{ Å}^{-1} < q < 3.5 \text{ Å}^{-1}$, and is approximately centered at the energy dispersion curve $\pi^2 q^2/2m$ of a free He\textsuperscript{4} atom at large momentum transfer $q > 3.5 \text{ Å}^{-1}$. 
This upward bending may be explained by the hybridization of a pair-excitation with a single-excitation as suggested by Zawadowski, Ruvalds and Solana. Hybridization means the mutual modification of the single- and two-quasiparticle spectra in the region where their energies are equal or nearly equal resulting in the noncrossing of their energies.

In the range $0.9 \, \text{Å}^{-1} < q < 2.5 \, \text{Å}^{-1}$, the lower half-maximum intensity curve in the collective mode $W_q$ reaches the single-excitation curve $E_k$ and is obscured by the latter whose intensity is much stronger. This effect is shown in Fig. 5 for $S(q, W)$. The broadening of collective mode excitations may be, in part, due to the finite lifetimes of the single-excitations which make up the collective mode. However, the relation between collective-mode broadening and single-excitation lifetimes is difficult to determine. The collective mode energy at very small momenta cannot be determined from neutron scattering experiments, but this information can be provided by photon scattering.

**Light Scattering**

In 1968, Halley proposed light scattering as a method of studying the excitation spectrum in liquid helium. In He II, an incident photon can be scattered either by density fluctuations (first sound) or by temperature fluctuations (second sound). The former type of scattering is called Brillouin scattering if only one elementary excitation is created, or called Raman scattering if two or more elementary
excitations are created. The latter type of scattering is known as Rayleigh scattering. The main concern of the present work is the Raman scattering in which a pair of elementary excitations with essentially equal and opposite momentum is created.

The Raman scattering from He II was first experimentally observed in 1969 by Greytak and Yan.\textsuperscript{9} In their experiments, the polarized \(5145\ \text{Å}\) light obtained from an argon laser was incident along the negative x-axis and the scattered light was collected either along the y-axis or along the z-axis. The photon momentum transfer in these experiments may be taken as zero, since it is less than \(10^{-3}\) times the roton momentum. The energy transfer was detected by a double grating spectrometer, and the intensity of scattered light by a cooled photomultiplier.

Their result at temperature 1.16 K is shown in Fig.6, where the sharp peak at about 17 K corresponds to two-roton excitations and the smaller maximum at 36 K corresponds to two excitations in the plateau region. No peak corresponding to two maxon excitations is observed. The depolarization ratios (ratios of the polarized scattered light intensities in various polarization and scattering directions) were also observed which are listed in Table I. Compared with the theory of light scattering\textsuperscript{14} which is reviewed in Chapter VIII, the observed depolarization ratios of the scattered light indicates that the angular momentum of a roton pair is \(\ell =2\), i.e., a D-state.
The detailed shape of the sharp peak at temperature 1.2 K was experimentally studied by Greytak, Woerner, Yan and Benjamin. Using 4880 Å laser light (photon energy $W \approx 2.95 \times 10^4$ K and wave-vector $k \approx 1.29 \times 10^{-3}$ Å$^{-1}$) and a Fabry-Perot spectrometer, they found a peak in the photon cross section from liquid He$^4$ at a scattering angle of 90° at temperature 1.2 K as shown in Fig. 7. The peak occurs at an energy shift of $W_q = 17.022 \pm 0.027$ K from the Brillouin scattering peak, which is less than twice the energy of a single roton at the same temperature ($2 \varepsilon_0 = 17.34 \pm 0.08$ K). The momentum transfer $q(\approx 1.29 \times 10^{-3}$ Å$^{-1}$) is negligible compared with the roton momentum ($\approx 1.9$ Å$^{-1}$). When they correct the peak for instrumental resolution and finite roton linewidth, their experimental result gives evidence for a two-roton bound state with binding energy

$$E_B = 0.37 \pm 0.10 \text{ K}$$

and angular momentum $\ell = 2$. The uncertainty in the binding energy $E_B$ is due to that of a single roton energy $\varepsilon_0 = 8.67 \pm 0.04$ K and its linewidth $\delta \varepsilon_0 = 0.15$ K. The model used in their analysis of the experimental data is discussed in Chapter III. Recently, Cowley$^{32}$ has surveyed the data, and concluded that, if bound roton pairs exist at all, their binding energy is 0.1 K or less. Experiments now being performed by Greytak, et al.$^{33}$ cast strong doubts on the conclusions of Cowley,$^{32}$ however.
CHAPTER III

QUANTUM HYDRODYNAMIC APPROACH

This chapter is on the theoretical background relevant to the theory of bound roton pairs. Quantum hydrodynamics has been used and developed by many authors as a phenomenological approach to the excitation spectrum of He II. However, the foundations of quantum hydrodynamics are questionable, and an alternative approach is presented in the next chapter.

Quantum Hydrodynamic Approach to the Quasiparticle Hamiltonian of He II

Landau\(^2\) originally proposed a quantization of classical hydrodynamics. However, the basic idea of the quantum hydrodynamic approach developed by Sunakawa, Yamasaki and Kebukawa\(^3^4\) is to express the Hamiltonian \(H\) of He II in terms of density and velocity fluctuation operators of the boson system. The Hamiltonian in second quantization is given by\(^3^5\)

\[
H = \frac{\hbar^2}{2m} \int d\vec{r} \vec{\nabla} \gamma^{\dagger}(\vec{r}) \cdot \vec{\nabla} \gamma(\vec{r}) + \frac{1}{2} \iint d\vec{r} d\vec{r}' \gamma^{\dagger}(\vec{r}) \gamma^{\dagger}(\vec{r}') \sqrt{V(\vec{r}-\vec{r}')} \gamma(\vec{r}') \gamma(\vec{r}),
\]

where \(V\) is the potential between two He\(^4\) atoms and \(m\) is the atomic mass. The field annihilation operator,

\[
\gamma(\vec{r}) = \mathcal{N}^{-1/2} \sum_{\vec{k}} a_{\vec{k}} e^{i \vec{k} \cdot \vec{r}},
\]

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describes the annihilation of a boson \( \text{He}^4 \) atom at the point \( \vec{r} \), where \( \mathcal{N} \) is the liquid volume. The annihilation operator \( a^{-\mathbf{k}} \) of a \( \text{He}^4 \) atom with wave-vector \( \vec{k} \) satisfies the usual boson commutation relations:

\[
[\psi(\vec{r}), \psi^+(\vec{r}')] = \delta(\vec{r} - \vec{r}')
\]

and

\[
[\psi(\vec{r}), \psi(\vec{r}')] = 0 = [\psi^+(\vec{r}), \psi^+(\vec{r}')] .
\]

The density operator is defined as

\[
\rho(\vec{r}) = \psi^+(\vec{r}) \psi(\vec{r})
\]

in second quantization, which is equivalent to the density operator

\[
n(\vec{r}) = \sum_{\vec{r}_i=1}^N \delta(\vec{r} - \vec{r}_i)
\]

in the \( N \) particle subspace.\(^{35}\) The density fluctuation operator is defined as

\[
\rho'(\vec{r}) = \rho(\vec{r}) - \mathcal{N} \hat{N},
\]

where

\[
\hat{N} = \int d\vec{r} \psi^+(\vec{r}) \psi(\vec{r}) = \sum_{\vec{k}} a^{+\vec{k}} a_{\vec{k}}
\]
is the total number operator of the He$^4$ atoms. They\textsuperscript{34} define the Fourier component of density operator $\rho(\vec{r})$ as

\begin{equation}
\rho_{\vec{k}} = \hat{N}^{-1/2} \sum_{\vec{k}'} a^{+}_{\vec{k}'+\vec{k}/2} a_{\vec{k}'-\vec{k}/2},
\end{equation}

where, of course, $\hat{N}^{-1/2}$ is meaningful only in the N particle subspace. However, $\rho_{\vec{k}}$ is usually defined as

\begin{equation}
\rho_{\vec{k}} (\text{usual definition}) = \sum_{\vec{k}'} a^{+}_{\vec{k}'+\vec{k}/2} a_{\vec{k}'-\vec{k}/2}
\end{equation}

which differs from their $\rho_{\vec{k}}$ in the factor $\hat{N}^{-1/2}$. The Fourier component of $n(\vec{r})$ is

\begin{equation}
n_{\vec{k}} = \sum_{l=1}^{N} \sum_{\vec{l}} e^{-i\vec{k} \cdot \vec{r}_{l}} = \sum_{\vec{k}'} a^{+}_{\vec{k}'} a_{\vec{k}'} = n_{\vec{k}}^+. \end{equation}

According to Sunakawa, Yamasaki and Kebrkawa,\textsuperscript{34} the velocity field operator,

\begin{equation}
\overrightarrow{v}(\vec{r}) = m^{-1} \hat{N}^{-1/2} \sum_{\vec{k}} \overrightarrow{v}_{\vec{k}} e^{i \vec{k} \cdot \vec{r}},
\end{equation}

is defined in terms of its Fourier component given by the integral equation

\begin{equation}
\overrightarrow{v}_{\vec{k}} = \hat{N}^{-1/2} \left( \sum_{\vec{k}'} \overrightarrow{a}_{\vec{k}'+\vec{k}/2} a^{+}_{\vec{k}'+\vec{k}/2} - \sum_{\vec{k}'+\vec{k}} \rho_{\vec{k}'+\vec{k}} \overrightarrow{v}_{\vec{k}'} \right),
\end{equation}

where $m$ is the mass of a He$^4$ atom. The velocity field operator $\overrightarrow{v}_{\vec{k}}$ is defined in such a way that the simple commutation relations\textsuperscript{34}
\[
[\rho(\vec{r}), \rho(\vec{r}')] = 0,
\]
(3.14)

\[
[\overline{\nu}(\vec{r}), \rho(\vec{r}')] = -(i\hbar/m) \overline{\nabla}_r \delta(\vec{r}-\vec{r}'),
\]
(3.15)

and

\[
[\nu(\vec{r})_k, \nu(\vec{r}')_{j*}] = (i\hbar/m) \delta(\vec{r}-\vec{r}') \rho(\vec{r}')^{-1} [\overline{\nabla}_r \times \overline{\nu}(\vec{r})]_k
\]
(3.16)

are obtained, where \((ijk)\) are cyclic.

It is assumed in this approach that the expansion of the inverse density operator in Eq. (3.7),

\[
\rho(\vec{r})^{-1} = \Omega \hat{N}^{-1} \sum_{n=0}^{\infty} [-\Omega \hat{N}^{-1} \rho(\vec{r})]^n,
\]
(3.17)

is convergent. However, the possible nonexistence of \(\rho(\vec{r})^{-1}\) has been discussed by Kobe and Coomer.36

The field operators can ultimately be eliminated in the Hamiltonian \(H\) in Eq. (3.1) by the insertion of the unit operator,34

\[
\rho(\vec{r}) = \nu(\vec{r}) \rho(\vec{r})^{-1} \nu(\vec{r}) = 1,
\]
(3.18)

in the He Hamiltonian

\[
H = \frac{\hbar^2}{2m} \int d\vec{r} \left[ \overline{\nabla} \nu^*(\vec{r}) \right] \cdot \rho(\vec{r}) \left[ \overline{\nabla} \nu(\vec{r}) \right] \\
+ \frac{i}{\hbar} \int d\vec{r} d\vec{r}' \nu^*(\vec{r}) \nu^*(\vec{r}') V(\vec{r}-\vec{r}') \nu(\vec{r}') \nu(\vec{r}).
\]
(3.19)
The Hamiltonian $H$ in Eq. (3.19) can be expressed now in terms of the operators $\hat{P}_k$ and $\hat{v}_k$. By a linear transformation to the quasiparticle operators $A_k^-$ and $A_k^+$, defined by

$$\hat{P}_k = \lambda_k^{\nu/2} (A_k^- + A_k^+)$$

and

$$\hat{v}_k = (2 \lambda_k^{\nu/2})^{-1/2} \hbar k (A_k^- - A_k^+),$$

the Hamiltonian $H$ can be expressed completely in terms of the quasiparticle operators $A_k^-$ and $A_k^+$. In Eqs. (3.20) and (3.21) the quantity $\lambda_k$ is defined by

$$\lambda_k = k \left( k^2 + c_k^2 \right)^{-1/2},$$

where

$$c_k^2 = \frac{4 N m}{\hbar} V_k,$$

and $N$ is the total number of He$^4$ atoms in the liquid. The Fourier component of the two-body potential $V_k$ is determined by

$$V(r) = \sum_k V_k e^{-i k \cdot r}.$$

On expressing the Hamiltonian $H$ in Eq. (3.19) in terms of the quasiparticle operators defined by Eqs. (3.20) and (3.21), one obtains
\[ H = H_{(0)} + H_{(2)} + H_{(3)} + H_{(4)} + \mathcal{O}(\hat{N}^{-3/2}) \]  

(3.25)

where the subscripts in parentheses denote the number of quasiparticle operators \( A_k^+ \) and/or \( A_k^- \). The \( H_{(0)} \) term gives the ground-state energy of the He II system. The second term,

\[ H_{(2)} = \sum_{k \neq 0} E_k^B A_k^+ A_k^- , \]

(3.26)

is diagonal due to the choice of \( \lambda_k \) in Eq. (3.22). The single-excitation spectrum is

\[ E_k^B = (\hbar^2/2m) k (k^2 + c_k^2)^{1/2} , \]

(3.27)

which is just the Bogoliubov \(^4\) spectrum. However, from the definition of \( c_k \) in Eq. (3.23) it is the total number of He\(^4\) atoms \( N \) which enters the spectrum \( E_k^B \), instead of the number of He\(^4\) atoms in the Bose-Einstein condensate \( N_0 \). This difference is a result of the theory, and not due to a neglect of depletion (the fraction of particles not in the condensate). The two terms \( H_{(3)} \) and \( H_{(4)} \) describe the quasiparticle interactions. It is worth mentioning that the expression in Eq. (3.17) is not unique and the role of Bose-Einstein condensation is not clear in this approach.

The Bogoliubov excitation energy \( E_k^B \) in Eq. (3.27) is the lowest order approximation to the excitation spectrum in He II. Kebukawa, Yamasaki, and Sunakawa\(^{37}\) took the effect of
quasiparticle interactions into account, and obtained a modified excitation spectrum

$$E_k = E_k^B (1 - \eta_k),$$

(3.28)

where the quantity $\eta_k$ reflects the effect of the interactions between quasiparticles. They use a soft core repulsive potential together with a square well attractive potential, and find more than one solution for $\eta_k$ to their equation. Their lower energy branch in Eq. (3.28) is in good agreement with the experimental single-excitation spectrum obtained by neutron scattering, as shown in Fig. 3. However, it is difficult to understand the significance of their other, higher energy, solutions of Eq. (3.28), which is essentially a single quasiparticle spectrum.

Theory of the Roton-Roton Interaction

In this section we review the approach used by Rajagopal, Bagchi and Ruvalds to calculate the roton-roton interaction from the quantum hydrodynamics of Sunakawa, et al., using the Hamiltonian of Eq. (3.25). They consider the scattering amplitude $\langle f | T | i \rangle$, where the initial state $|i\rangle$ is a pair of quasiparticles

$$|i\rangle = A^+_k A^+_{k'} |0\rangle,$$

(3.29)

and the final state $|f\rangle$ is another pair of quasiparticles

$$|f\rangle = A^+_k A^+_{k+\vec{q}} A^+_{k' - \vec{q}} |0\rangle.$$

(3.30)
The quasiparticle vacuum state $|0\rangle$ is defined by
\[ A_k |0\rangle = 0. \tag{3.31} \]

The effective interaction $T$ which describes the quasiparticle scattering processes is chosen to be first order in $H(4)$ and second order in $H(3)$:
\[ T = H(4) + H(3) \left[ (E_k^B + E_{k'}^B) - \sum_{\vec{p} \neq 0} E_p^B A_p^+ A_p \right] H(3). \tag{3.32} \]

The quasiparticle-pair scattering amplitude $\langle \Phi | T | \Omega \rangle$ depends not only on the total momentum $\overline{q}$ of the two quasiparticles, but also on the relative relation between individual momenta $\overline{k}$ and $\overline{k}'$. They cast the amplitude in the form
\[ \langle \Phi | T | \Omega \rangle = \sum_{n=1}^{k} \sum_{\overline{k}, \overline{k}'} \langle \Phi | g_4^{(n)}(\overline{k}, \overline{k}', \overline{q}) \rangle \]
\[ + \sum_{n=1}^{5} \sum_{\overline{k}, \overline{k}'} \langle \Phi | g_3^{(n)}(\overline{k}, \overline{k}') D^{(n)}(\overline{k}, \overline{k}', \overline{q}) g_3^{(n)}(\overline{k}, \overline{k}') \rangle, \tag{3.33} \]

where $g_4^{(n)}$, $g_3^{(n)}$ and $D^{(n)}$ are vertex functions, and $D^{(n)}$ are propagators. According to some typical calculations of theirs, the scattering amplitude $\langle \Phi | T | \Omega \rangle$ is negative, which indicates that the effective coupling $T$ between rotons is attractive. Hence the existence of a bound roton-pair is possible. However, their calculated value for $T$ is too large.

In order to reduce the value of the roton-roton coupling $T$, they propose a method to renormalize (calculate higher-order corrections to) their theory. They suggest that Dyson's
equation\textsuperscript{38} may be used to renormalize the propagators $D^{(n)}$, and the Bethe-Salpeter equation\textsuperscript{38} to renormalize the vertex functions $g_{4}^{(n)}$. However the effect of renormalization on the roton-roton coupling is quite difficult to calculate accurately, but they state that the coupling is reduced.

Phenomenological Theory of Bound Roton Pairs

Based on the model Hamiltonian

$$
H = \sum_{\vec{k}} E_{\vec{k}} \, a_{\vec{k}}^{+} a_{\vec{k}} + \frac{1}{2} g_{4} \langle d\vec{x} \psi_{\vec{x}}^{\prime}(\vec{x}) \psi_{\vec{x}'}^{\prime}(\vec{x}) \psi_{\vec{x}}(\vec{x}) \psi_{\vec{x}'}(\vec{x}) \rangle,
$$

(3.34)

Ruvalds and Zawadowski\textsuperscript{6} and Zawadowski, Ruvalds and Solana\textsuperscript{30} have presented a phenomenological theory of bound roton pairs. They identify the quantity $E_{\vec{k}}$ in Eq. (3.34) as the experimentally observed elementary excitation spectrum in He II, so $a_{\vec{k}}^{+}$ and $a_{\vec{k}}$ are the quasiparticle creation and annihilation operator, respectively. The second term in Eq. (3.34) is the phenomenological interaction Hamiltonian between quasiparticles. In configuration space the interactions between the quasiparticles is taken to be an attractive contact potential $g_{4} \delta(\vec{x} - \vec{x}')$, where $g_{4} < 0$. By this choice they insure a bound roton pair in the S-state. However rotons are not highly localized excitations, and the observed bound roton pair is a D-state\textsuperscript{5}.

The annihilation field operator $\psi_{\vec{x}}^{\prime}(\vec{x})$ which Zawadowski, Ruvalds and Solana\textsuperscript{30} use to describe the annihilation of a quasiparticle is

$$
\psi_{\vec{x}}^{\prime}(\vec{x}) = (2\pi)^{-3/2} \int d\vec{k} \, a_{\vec{k}}^{\dagger} e^{i \vec{k} \cdot \vec{x}} N_{0}(\tau)^{1/2},
$$

(3.35)
where \( x = (\vec{x}, t) \), \( a_{\vec{k}}^-(t) \) is the Bose quasiparticle destruction operator in the Heisenberg representation, and \( N_0(T) \) is the temperature-dependent number of \( \text{He}^4 \) atoms in the zero momentum condensate. According to Griffin, the field operator \( \varphi'([x]) \) in the Hamiltonian of Eq. (3.34) actually describes the density fluctuation \( \rho'(x) \) of particles in the sense of quantum hydrodynamics in Eq. (3.7).

The bare single-quasiparticle Green's function is assumed to be of the form

\[
G_1(\vec{k}, \omega) = \left[ \omega - E_k + i \Gamma / 2 \right]^{-1}
\]  (3.36)

in momentum space, where the quasiparticle energy \( E_k \) is approximated by parabolic forms near the energy local maximum and minimum, and \( \Gamma \) is the phenomenologically determined roton line width. The two-quasiparticle Green's function is calculated approximately from the Bethe-Salpeter equation to obtain

\[
G_2(\vec{k}, \omega) = 2 F(\vec{k}, \omega) \left[ 1 - G_1(\vec{k}, \omega) \right]^{-1}
\]  (3.37)

in momentum space. The expansion of the denominator gives a geometrical series, which is the sum of loop diagrams. The loop function \( F(\vec{k}, \omega) \) is defined by

\[
F(\vec{k}, \omega) = \frac{i}{\hbar} \int \frac{d\vec{k}' \omega'}{(2\pi)^4} G_1(\vec{k}' - \vec{k}, \omega' - \omega) G_1(\vec{k}', \omega').
\]  (3.38)
The factor of 2 in Eq. (3.37) for $G_2(k,W)$ is due to ordering of field operator pairs. The two-quasiparticle density of states in momentum space which takes the final state interaction into account is given by \(^6\)

$$
\rho_2(k',W) = -\frac{i}{\hbar} \frac{i}{\hbar} \text{Im} \ G_2(k',W),
$$

(3.39)

where the factor of 4 arises because of the ordering of field operator pairs and the indistinguishability of quasiparticles. In this theory the cross section for photon scattering from He II is assumed to be proportional to $\rho_2(k,W)$.

For two-excitation processes, the value of $W$ for which $\rho_2(k,W)$ is a maximum for each fixed value of $k$ should give the two-excitation spectrum $W_k$. For an attractive interaction $g_4 < 0$, the $\rho_2(k=0,W)$ exhibits a peak at less than twice the roton energy, which shows the existence of a two-roton bound state. The coupling constant $g_4$ is chosen so that the calculated energy is in agreement with the experimentally observed energy $W_k$. However, Zawadowski, et al. \(^3\), \(^6\) obtain only a bound roton pair in the S-state.

To obtain a bound roton-pair in the $\ell=2$ angular momentum state (D-state), Greytak, et al. \(^5\) replace the coupling constant $g_4$ by

$$
g_4^{(\ell=2)} = 5 g P_2(k_0 \cdot k_0'),
$$

(3.40)

where $P_2$ is the second order Legendre polynomial, and $k_0 \cdot k_0'$ is the cosine of the angle between the scattered rotons. They \(^5\)
use the Bethe-Salpeter equation in the RZ theory, and obtain the expression

\[ \rho_2(\mathbf{k}=0, W) = 2\pi A E_B \frac{1}{r} \delta(W - 2E_0 + E_B) \]

\[ + A (W - 2E_0)^{1/2} (W - 2E_0 + E_B)^{-1} \delta(W - 2E_0), \]

(3.41)

where the binding energy of a bound roton pair is \( E_B = A^2 g^2 \), \( A = (k_0/2\pi)^2 \mu^{1/2} \hbar^{-1} \), \( k_0 \) is the wave vector and \( \mu \) is the effective mass of the roton with energy \( E_0 \). Equation (3.41) has a peak at \( W = 2E_0 - E_B \) and a continuum above \( 2E_0 \).

Since the cross section of photon is proportional to \( \rho_2(\mathbf{k}=0, W) \), the experimental result of light scattering gives evidence for a two-roton bound D-state in He II as discussed in Chapter II. Greytak, et al.\(^5\) obtain the good fit to their scattered intensity shown in Fig. 7 when the coupling constant

\[ g = 1.2 \times 10^{-39} \text{ cm}^3 \text{ erg} = 8.7 \, \AA^3 \text{ K}, \]

(3.42)

a roton linewidth of 0.15 K, and the instrumental resolution of 0.17 K are used. The binding energy they obtain is \( E_B = 0.37 \pm 0.10 \text{ K} \) for the D-state bound roton pair.

However, Cowley\(^32\) has recently surveyed the data of Greytak, et al.,\(^5\) and Pike and Vaughan,\(^39\) taking the interaction between the excitations and the light into account. He concludes that when roton linewidth and resolution broadening are properly
taken into account, the bound roton pairs, if they exist at all, have a binding energy of less than 0.1 K. This situation is discussed further in Chapter IX.
CHAPTER IV

BOGOLIUBOV'S CANONICAL TRANSFORMATION

In this chapter, we first review Bogoliubov's canonical transformation which transforms the system of He\(^4\) particles to Bogoliubov quasiparticles, or, as they are called here, bogolons. Then the free single-bogolon energy spectrum is discussed. The bogolon energy spectrum is obtained by using a spherical \(\delta\)-function shell potential between the He\(^4\) atoms. It is then empirically fit to the observed phonon-roton spectrum of He II. Finally the role of the bogolon interactions is considered.

Transformation to Bogolons

Consider a system of \(N\) He\(^4\) atoms (bosons) which is described by the Hamiltonian in second quantization

\[
H = \int d\mathbf{x} \, \mathcal{F}^\dagger(x) \left( -\frac{i}{2m} \nabla^2 - \mu \right) \mathcal{F}(x) + \frac{1}{2} \int d\mathbf{x} \, d\mathbf{x}' \, \mathcal{F}^\dagger(x) \mathcal{F}^\dagger(x') V(|x-x'|) \mathcal{F}(x') \mathcal{F}(x), \tag{4.1}
\]

where \(\mu\) is the chemical potential, \(m\) the He\(^4\) atomic mass, and \(V(r)\) the two-body potential between two He\(^4\) atoms separated by the distance \(r\). The field annihilation operator \(\mathcal{F}(\mathbf{x})\) for a particle at the point \(\mathbf{x}\) can be expanded in plane waves,

\[
\mathcal{F}(\mathbf{x}) = \sum_k a_k \mathcal{F}_k(\mathbf{x}), \tag{4.2}
\]
in terms of the annihilation operator $a_{-\vec{k}}$ for a particle with momentum $\vec{k}$. The plane wave $\mathcal{Y}_{\vec{k}}(\vec{x})$ is defined as

$$\mathcal{Y}_{\vec{k}}(\vec{x}) = \Omega^{1/2} e^{\imath \vec{k} \cdot \vec{x}}, \quad (4.3)$$

where $\Omega$ is the volume of the He II system. The creation operator $a_{\vec{k}}^+$ and the annihilation operator $a_{-\vec{k}}$ for particles with momentum $\vec{k}$ satisfy the usual boson commutation relations,

$$[a_{-\vec{k}}, a_{\vec{k}'}^+] = \delta_{\vec{k}, \vec{k}'} \quad (4.4)$$

and

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a_{\vec{k}}^+, a_{\vec{k}'}^+]. \quad (4.5)$$

The field operator in Eq. (4.2) can be substituted into the Hamiltonian in Eq. (4.1) to obtain the usual form in momentum space

$$H = \sum_i \epsilon_i a_i^+ a_i + \frac{1}{2} \sum_{1234} \langle 12 | V | 34 \rangle a_i^+ a_j^+ a_j a_i \quad (4.6)$$

where $H_0$ and $H_1$ are the kinetic and potential energies, respectively. The kinetic energy minus the chemical potential of a particle with momentum $\langle 1 \rangle = \vec{k}_1$ is

$$\epsilon_i = k_i^2 / 2m - \mu. \quad (4.7)$$

The symmetrized matrix element of the local potential $V$ is

$$\langle 12 | V | 34 \rangle = \frac{\imath}{2} \int d\vec{x}_1 d\vec{x}_2 \mathcal{Y}_{\vec{k}_1}^* \mathcal{Y}_{\vec{k}_2}^* \mathcal{Y}_{\vec{k}_3} \mathcal{Y}_{\vec{k}_4} V(\vec{x}_1 - \vec{x}_2) \times \left[ \mathcal{Y}_{\vec{k}_3}(\vec{x}_2) \mathcal{Y}_{\vec{k}_4}(\vec{x}_1) + \mathcal{Y}_{\vec{k}_4}(\vec{x}_2) \mathcal{Y}_{\vec{k}_3}(\vec{x}_1) \right]. \quad (4.8)$$
where (1) = \kappa_1, (2) = \kappa_2, etc. When Eq. (4.3) is substituted into Eq. (4.8), and the integrals are performed, the result is

\[
\langle 12 | V/3 \mathbf{4} \rangle = \langle 12 \rangle \mathbf{4} \left( V_{1\mathbf{k}_1 - \mathbf{k}_2} + V_{1\mathbf{k}_1 - \mathbf{k}_3} \right)
\times \mathbf{S}_{\mathbf{k}_1} \left( \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 \right),
\]

(4.9)

where the Kronecker delta \( \mathbf{S}_{\mathbf{k}_1} \) insures momentum conservation. The Fourier component \( V_{\mathbf{k}} \) of the binary (two-body) potential \( V(r) \) is defined as

\[
V_{\mathbf{k}} = \int d\mathbf{r} e^{-i \mathbf{k} \cdot \mathbf{r}} V(r).
\]

(4.10)

In order to obtain an elementary excitation spectrum \( E_k \) from the Hamiltonian in Eq. (4.6), a canonical transformation can be made which transforms the Bose particle (He\(^4\) atom) operators \( a_{\mathbf{k}} \) and \( a_{\mathbf{k}}^+ \) that describe the bare He\(^4\) atoms into Bose quasiparticle (bogolon) operators \( f_{\mathbf{k}} \) and \( f_{\mathbf{k}}^+ \). In the limit of small depletion the bogolon operators approximate density fluctuations. Otherwise they can be regarded in some sense as dressed density fluctuations, as long as the zero-momentum state is macroscopically occupied.

The bogolons approximate the low-lying energy levels of the system as a whole. Hopefully, the energy of the bogolons can give a good approximation to the energy spectrum of the elementary excitations, as observed, for example, in inelastic neutron scattering.\(^3\) The linear Bogoliubov transformation\(^11\)
to bogolons can be made on the Hamiltonian to diagonalize it partially. The quantities \( \mathcal{J}_0 \), \( u_k \) and \( v_k \) are real coefficients. In order for the bogolon annihilation operator \( \gamma_k \) and bogolon creation operator \( \gamma_k^+ \) to satisfy the boson commutation relations in Eqs. (4.4) and (4.5), the transformation coefficients \( u_k \) and \( v_k \) in Eq. (4.11) must satisfy

\[
\begin{align*}
    u_k^2 - v_k^2 &= 1, \\
    u_k &= u_{-k}, \\
    v_k &= v_{-k}.
\end{align*}
\]

The condensate amplitude \( \mathcal{J}_0 \) in Eq. (4.11) is required to treat the Bose-Einstein condensation in the zero-momentum state properly. When the transformation of Eq. (4.11) is made on the Hamiltonian in Eq. (4.6) and all the operators are normal ordered (creation operators on the left and annihilation operators on the right), the Hamiltonian in Eq. (4.6) becomes

\[
\mathcal{H} = \sum_{j, k} \mathcal{H}_{jk},
\]

where \( j, k = 0, 1, 2, 3, 4 \) and \( j+k = 0, 1, 2, 3, 4 \). Each term in the Hamiltonian of Eq. (4.13) has the form

\[
\mathcal{H}_{jk} = \sum_{i, z, \cdots, j+k} \mathcal{H}_{ijk}(i, z, \cdots, j+k) \gamma_i^+ \cdots \gamma_j^+ \gamma_{j+1} \cdots \gamma_{j+k},
\]
with \( j \) creation operators and \( k \) annihilation operators, and the sum is over all momenta. The Hamiltonian is Hermitian, so
\[
H_{jk} = H_{kj}^\dagger. \tag{4.15}
\]

The coefficients \( h_{jk} \) in Eq. (4.14), which are given in Table II of Ref. 42, are repeated here in Table II for completeness. They are symmetric with respect to the interchange of the first \( j \) momenta, and with respect to the last \( k \) momenta.

**Bogolon Energy**

The coefficient of \( \gamma_k^T/\gamma_k \) in the transformed Hamiltonian in Eq. (4.13),
\[
h_{11}(k,k) = U_k (\omega_k^2 + \nu_k^2) + 2 \Delta_k u_k \nu_k, \tag{4.16}
\]
is the bare single-bogolon energy. The Fourier transform of the Hartree-Fock energy \( U_k \) is
\[
U_k \equiv \hbar^2 / 2 m \mathcal{M} + \omega^{-1} V_0 \left( \omega_0^2 + \sum_{k'} \nu_{k'}^2 \right) \\
+ \omega^{-1} \nu_k \omega_0^2 + \omega^{-1} \sum_{k'} V_{k+k'} u_{k'} \nu_{k'}, \tag{4.17}
\]
and the Fourier transform of the pair potential \( \Delta_k \) is
\[
\Delta_k \equiv V_k \omega^{-1} \omega_0^2 + \omega^{-1} \sum_{k'} V_{k+k'} u_{k'} \nu_{k'}. \tag{4.18}
\]
The ground-state energy of the system is $H_{00}$. The bogolon interaction terms are $H_{01}, H_{20}, H_{30}, H_{21}, H_{40}, H_{31}, H_{22}$ and their Hermitian conjugates in Table II.

We chose $u_k$ and $v_k$ to minimize the ground-state energy $H_{00}$. Lagrangian multipliers $\Lambda_k$ are convenient to impose the constraints in Eq. (4.12), and we seek the $u_k$ and $v_k$ such that

$$H_{00} - \sum_k \Lambda_k (u_k^2 - v_k^2 - 1) = \text{minimum.}$$

(4.19)

On differentiation with respect to $u_k$ and $v_k$, and elimination of the Lagrangian multipliers, the condition

$$2 \mu_k u_k v_k + \Delta_k (u_k^2 + v_k^2) = 0$$

(4.20)

is obtained. Equation (4.20) results in the simplification of the Hamiltonian, since

$$h_{20} = h_{02} = 0,$$

(4.21)

which is the compensation of the lowest order dangerous diagrams of Bogoliubov.

Equations (4.12) and (4.20) can be solved together for the coefficients $u_k$ and $v_k$:

$$u_k^2 = \left( \mu_k / 2 E_k \right) + \frac{1}{2},$$

(4.22)

$$v_k^2 = \left( \mu_k / 2 E_k \right) - \frac{1}{2},$$

(4.23)

and

$$u_k v_k = -\Delta_k / 2 E_k,$$

(4.24)
where

\[ E_K \equiv (U_K - \Delta_K) / 2 \equiv \frac{1}{2} \mathcal{E}_K. \] (4.25)

When Eqs. (4.22)-(4.24) are substituted into Eq. (4.16) for the free bogolon energy \( h_{11}(\vec{k}, \vec{k}) \), we obtain

\[ h_{11}(\vec{k}, \vec{k}) = (U_K^2 - \Delta_K^2) / 2 = E_K. \] (4.26)

Using Eq. (4.26), one can cast the Eqs. (4.22) and (4.23) in the forms

\[ U_K^2 = \left[ (E_K^2 + \Delta_K^2) / 2 E_K \right] + \frac{1}{2}, \] (4.27)

and

\[ V_K^2 = \left[ (E_K^2 + \Delta_K^2) / 2 E_K \right] - \frac{1}{2}, \] (4.28)

which, together with Eq. (4.24) for \( u_k v_k \), are useful in the practical evaluations of the values of \( u_k \) and \( v_k \) when a starting approximate expression for the free bogolon energy spectrum \( E_K \) is used. The substitution for \( U_K \) is valid, since on physical grounds \( U_K \) should always be positive. When Eq. (4.24) is substituted into Eq. (4.18), the nonlinear integral equation for the pair potential
\[ \Delta_k = \sum' \varphi_0^2 V_k - \sum' \sum_{k+k'} V_{k+k'} \Delta_{k'} / 2 \epsilon_{k'} \]  

(4.29)

is obtained.

The determination of the nonzero condensate amplitude \( \varphi_0 \) is now discussed. If one chooses \( \varphi_0 \) to minimize the ground-state energy \( H_{00} \), the condition

\[ \partial H_{00} / \partial \varphi_0 = 0 \]  

(4.30)

determines the chemical potential \( \mu \) to be

\[ \mu = v_0 \sum' \omega' N + \sum' \sum_{k} V_k \left( v_k^2 + u_k v_k \right) \]  

(4.31)

if \( \varphi_0 \neq 0 \). However when this expression for \( \mu \) is substituted into Eqs. (4.17) and (4.25), the energy at \( k = 0 \) is

\[ E_0 = 2 \varphi_0 \left( - \sum' \omega' \sum_{k} V_k u_k v_k \right)^{1/2} \]  

(4.32)

namely an energy gap is obtained. This gap was first obtained by another method by Girardeau and Arnowitt.\(^4\)\(^3\) On the other hand, a theorem of Hugenholtz and Pines\(^4\)\(^4\) states that \( E_k \) should vanish as \( k \to 0 \). Experimentally, no gap is observed.\(^3\)

Finally the coefficient \( \varphi_0^2 \) is determined by the total number \( N \) of He\(^4\) atoms in the liquid,

\[ N = \sum_{k} \langle a_k^+ a_k \rangle = \varphi_0^2 + \sum_{k} v_k^2 \]  

(4.33)
where the expectation value is with respect to the bogolon vacuum state. Hence the free bogolon energy $E_k$ in Eq. (4.25) should theoretically be calculated in a self-consistent way from the three coupled integral equations in Eqs. (4.17), (4.29) and (4.33) for $U_k$, $\Delta_k$ and $\phi_0$, respectively, where Eq. (4.31) should be used in Eq. (4.17). However, to avoid the unphysical energy gap in Eq. (4.32) some modifications can be made.\textsuperscript{i}

Phenomenological Modification of the Bogolon Energy

To make the unphysical energy gap $E_0$ vanish, the chemical potential $\mu$ can be phenomenologically redefined as

$$\mu = \mu_0 (\phi_0^2 + \sum_{k'} \nu_{k'}^2) \Omega^{-1} + \sum_k \nu_k (\nu_k^2 - \nu_k' \nu_{k'})$$

(4.34)

which is only a change of sign in the last term in Eq. (4.31). When Eq. (4.34) is substituted into Eq. (4.17) and the result substituted into Eq. (4.25), the free bogolon energy spectrum $E_k$ is given by

$$E_k = \left[ T_k (T_k + 2 \Delta k) \right]^{1/2}$$

(4.35)

where the single-particle energy $T_k$ is dressed by Hartree-Fock-like terms,

$$T_k = k^2/2m + \Omega^{-1} \sum_{k'} \left( V_{k+k'} - V_{k'} \right)$$

$$\times \left( \nu_{k'}^2 - \nu_{k'} \nu_{k'} \right)$$

(4.36)
Now we discuss the phonon nature of the free bogolon energy spectrum $E_k$ in Eq. (4.35) as $k \to 0$. When $k \to 0$, we obtain the linear phonon spectrum

$$E_k = (2 \Delta_o T_K)^{\nu_2} \equiv s k$$

from Eq. (4.35), where $s$ is the speed of sound and the dressed single-particle energy $T_k$ is given by the expansion

$$T_k = \frac{k^2}{2m} + \frac{k^2}{(2\pi)^2} \int_0^\infty dk' k'^2 \left[ \frac{1}{2} \frac{d^2 \alpha_k'}{dk'^2} + \frac{12k'^2}{k^2} \frac{d^2 \alpha_k'}{dk'^2} \right] (v_k'^2 - u_k'^2 v_k') + O(k^2)$$

$$\equiv \frac{k^2}{2m^*},$$

which also defines the effective mass $m^*$. Therefore the speed of sound is

$$s = (\Delta_o / m^*)^{\nu_2}.$$ 

Delta Function Shell Potential

In order to simplify the numerical calculations, a simple effective potential model can be used for the interaction between the He$^4$ atoms. We choose the spherical $\delta$-function shell potential

$$V(r) = \lambda (4\pi a^2)^{-1} \delta(r-a),$$

where $r$ is the interatomic separation, $a$ is the repulsive shell diameter, and $\lambda > 0$ the repulsive interaction strength. Although the potential chosen is unrealistic (because it has no attractive part), it gives an energy spectrum that can be used to fit
the experimental one. With this model potential, the matrix element in Eq. (4.9) has the explicit form

\[ \langle \vec{k}, \vec{k}' | V | \vec{k}, \vec{k}' \rangle = \lambda (2 \Omega)^{1/2} \delta_{kr} (\vec{k}_i + \vec{k}_z - \vec{k}_y - \vec{k}_z') \times \left[ j_0 (1/k - 1/k') + j_0 (1/k' - 1/k') \right], \]  

(4.41)

where \( j_0(x) = \sin(x)/x \) is the spherical Bessel function of zero order. The advantage of this potential model is that the interaction in the integral equation for \( \Delta_k \) in Eq. (4.29) is separable, since the angular average is

\[ \int \frac{d\vec{k}'}{4\pi} V_{\vec{k} + \vec{k}'} = \lambda j_0 (ka) j_0 (k'a). \]  

(4.42)

From Eq. (4.10) the Fourier transform of the potential is

\[ V_K = \lambda j_0 (ka). \]  

(4.43)

The solution of Eq. (4.29) is

\[ \Delta_K = \Delta_0 j_0 (ka), \]  

(4.44)

where

\[ \Delta_0 = \lambda j_0 - \lambda \Delta_0 (2\pi a)^3 \int d\vec{x} \sin^2(x)/2x^2 E_x, \]  

(4.45)

and \( x = ka \). The energy spectrum \( E_k \) in Eq. (4.35) is therefore

\[ E_x = \left[ T_x (T_x + 2\Delta_0 \sin(x)/x) \right]^{1/2}, \]  

(4.46)
where

$$T_x = \beta x^2 + A \left( \frac{\sin(\alpha)}{\alpha} - 1 \right)$$  \hspace{1cm} (4.47)

and the two quantities $\beta$ and $A$ are defined by

$$\beta \equiv (2\pi a^2)^{-1}$$  \hspace{1cm} (4.48)

and

$$A \equiv \lambda(2\pi a^3)^{-1}\int d\vec{x} (\nu_x^2 - u_x \nu_x) \frac{\sin(\alpha)}{\alpha},$$  \hspace{1cm} (4.49)

respectively. The particle density $\rho = N/\Omega$ is determined by dividing Eq. (4.33) by the volume $\Omega$ to give

$$\rho = \rho_0 + (2\pi a)^{-3}\int d\vec{x} \nu_x^2,$$  \hspace{1cm} (4.50)

where $\rho_0 = \rho_{0}^2/\Omega$ is the condensate density.

**Empirical Fit to Experimental Excitation Spectrum**

The purpose of the present work is to study, from the theoretical point of view, the interaction of two bogolons in the $\ell$ angular momentum state with a $\delta$-function shell potential between the He$^4$ atoms. We are not primarily concerned here with the self-consistent calculation of the free bogolon energy spectrum $E_k$ from the three coupled integral equations in Eqs. (4.17), (4.29) and (4.33). Instead, an empirical fit of a modified free bogolon energy spectrum$^{11}$ to the experimental spectrum is used in our work.
For small $x$, Eq. (4.47) for $T_x$ can be approximated by

$$T_x \approx \beta (1 - A/6 \beta) x^2,$$  \hspace{1cm} (4.51)

by expanding the sine. For large $x$, Eq. (4.47) shows that $T_x$ is free-particle like. Like Larsson and Otnes, we make the effective mass approximation

$$T_x = \beta^* x^2,$$  \hspace{1cm} (4.52)

where $\beta^* = \beta (1 - A/6 \beta)$ for all values of $x$. From Eq. (4.47) the effective mass is

$$m^* = m (1 - A/6 \beta)^{-1}.$$  \hspace{1cm} (4.53)

Hence Eq. (4.46) for $E_x$ becomes

$$E_x = \beta^* \frac{x}{\sqrt{x^2 + y \sin(x)/x}}.$$

where

$$y = 2 \Delta / \beta^*.$$  \hspace{1cm} (4.55)

Equation (4.54) for the single-excitation energy spectrum $E_x$ is phenomenologically determined to fit the experimental data, except for the plateau region for $k > 2.2$ Å⁻¹. The parameter values

$$\beta^* = 1.13 K, \quad y = 67.1 \quad \text{and} \quad a = 2.047 \text{Å}.$$

(4.56)
were used by Larsson and Otnes\textsuperscript{48} to fit their experimental data at temperature 1.44 K and are used in the numerical calculations in the latter parts of this work. Using the above parameter values, the values of $E_k$ for different $k$ are listed in Table III. The data of some critical points on the elementary excitation curve in Eq. (4.54) are:

$$k = 1.11 \text{Å}^{-1}, \quad E = 13.5189 \text{K}$$ \hspace{1cm} (4.57)

for the maxon, and

$$k = 1.90 \text{Å}^{-1}, \quad E = 8.09866 \text{K}$$ \hspace{1cm} (4.58)

for the roton. The experimentally observed plateau region is not fit by Eq. (4.54) which tends to the free particle energy spectrum for large $k$. The crossing of $E_k$ with a collective mode of energy $W = 2 E_{\text{roton}} - 0.37 \text{K}$ occurs at

$$k = 2.32 \text{Å}^{-1}, \quad W = E = 15.82732 \text{K}$$ \hspace{1cm} (4.59)

where 0.37 K is the observed binding energy of a roton pair in a bound D-state.\textsuperscript{5}

Equation (4.45) can be solved for the condensate density

$$\rho_o = (\beta^*/2) \left[ \lambda^{-1} + (4 \pi^2 a^2)^{-1} \int_0^\infty dx \sin^2(x) / E_x \right]$$

$$= 37.9 \left[ \lambda^{-1} + 9.726 \times 10^{-4} \right]$$ \hspace{1cm} (4.60)
with $\rho_0$ in atoms $\text{Å}^{-3}$ and $\lambda > 0$ in $\text{Å}^3$. The density

$$\rho = \rho_0 + 3.7 \times 10^{-2}$$

is obtained from Eq. (4.50), where $\rho$ and $\rho_0$ are in atoms $\text{Å}^{-3}$. Equations (4.60) and (4.61) give $\rho_0 = 3.686 \times 10^{-2}$ atoms $\text{Å}^{-3}$, $\rho \geq 7.386 \times 10^{-2}$ atoms $\text{Å}^{-3}$, and the depletion $(\rho - \rho_0)/\rho \leq 50\%$. The experimental values are $\rho = 2.19 \times 10^{-2}$ atoms $\text{Å}^{-3}$, $\rho_0 = (5.256 \pm 2.19) \times 10^{-3}$ atoms $\text{Å}^{-3}$ and $(\rho - \rho_0)/\rho = 97.6\%$. The calculated number density is at least 2.4 times greater than its experimental value. For a simple arrangement of cubes 2.047 $\text{Å}$ on a side, the calculated number density is 0.116 atoms $\text{Å}^{-3}$. This value is greater than 0.074 atoms $\text{Å}^{-3}$ calculated above, which is greater than the experimental density of 0.0219 atoms $\text{Å}^{-3}$.

Bogolon Interactions

The bogolon interactions are described in a complicated way by all the terms $H_{21}$, $H_{30}$, $H_{40}$, $H_{31}$, $H_{22}$ and their Hermitian conjugates in the bogolon Hamiltonian $H$ in Eq. (4.13) and Table II. These interaction terms can renormalize free bogolons, and are responsible for the formation of collective modes. It is, however, too difficult to take all of these interaction terms into account in a practical calculation. The method of normal modes was used by Takano,\textsuperscript{12} who generalized work by Tamm\textsuperscript{51} and Dancoff.\textsuperscript{52} This method is used here to treat the collective modes with the $\delta$-function shell potential in
Eq. (4.40) in Chapter VI. However, in the next chapter the method of normal modes is discussed in general.
CHAPTER V

LINEARIZED EQUATIONS OF MOTION

This chapter is devoted to deriving the basic equations for the normal modes of the He II system, applicable to an arbitrary potential. The secular equation for the energy of a bound bogolon pair can be obtained more easily if a model potential separable in momentum space is used. The collective excitations are the quanta (like phonons) associated with collective motions (like sound waves) of all the He$^4$ atoms in the liquid as a whole. In this sense the liquid He$^4$ behaves like a giant molecule, and can exhibit quantum phenomena on a macroscopic scale. As particular kinds of collective excitations, the ones consisting only of single- and two-bogolon excitations may be used to describe the two-branch spectrum of He II. In principle we can take into account the hybridization effect suggested by Ruvalds and Zawadowski. Hybridization is the mutual modification of the single- and two-bogolon spectra in the region where their energies are equal or nearly so, which results in the noncrossing of their energies.

Equation of Motion

Following Takano, we consider the eigenmodes of a collective excitation creation operator for momentum $\overline{q}$ of the form
\[
B_q^\dagger = B_{10} + B_{01} + B_{20} + B_{02} + B_{11}
\]
\[
= \varphi (\bar{\theta}) \gamma_{\bar{k}}^\dagger + \chi (\bar{\theta}) \gamma_{\bar{k}} + \frac{1}{2} \sum_k \left[ \delta (\bar{k} + \bar{\theta}, -\bar{k}) \gamma_{\bar{k} + \bar{\theta}}^\dagger \gamma_{\bar{k}} + \gamma (\bar{k} + \bar{\theta}, \bar{k}) \gamma_{\bar{k} + \bar{\theta}}^\dagger \gamma_{\bar{k}} \right],
\] (5.1)

which consists of one bogolon (quasiparticle) \( \gamma_{\bar{k}}^\dagger \), one quasi-hole (quasiparticle hole) \( \gamma_{-\bar{k}} \), two bogolons \( \gamma_{\bar{k} + \bar{\theta}}^\dagger \gamma_{-\bar{k}} \), two quasiholes \( \gamma_{-\bar{k} - \bar{\theta}} \gamma_{\bar{k}} \) and the pair of one bogolon and one quasi-hole \( \gamma_{\bar{k} + \bar{\theta}}^\dagger \gamma_{-\bar{k}} \). Takano did not include the term \( \gamma_{\bar{k} + \bar{\theta}}^\dagger \gamma_{-\bar{k}} \) in his collective excitation operator. Its effect is shown later to be unimportant. In principle we could include an arbitrary number of operators in Eq. (5.1).

The coefficients \( \varphi, \chi, \xi, \gamma \) and \( \zeta \) are determined from the equation of motion. If \( B_q^\dagger \left| 0 \right\rangle \) were an exact eigenmode of the system, it would have an exact excitation energy \( \omega_q \). However it is not, so the \( B_q^\dagger \) is assumed to satisfy the eigenmode equation
\[
[H, B_q^\dagger] = \omega_q B_q^\dagger + \text{small terms},
\] (5.2)

where the excitation energy is
\[
\omega_q = \varepsilon_{\bar{q}} - \varepsilon^{(0)}.
\] (5.3)

The small terms are corrections due to the state \( B_q^\dagger \left| 0 \right\rangle \) being only an approximate eigenstate, and are assumed negligible. The
true ground-state vector $|0\rangle$ of the system satisfies the Schrödinger equation

$$
H|0\rangle = \mathcal{E}^{(0)}|0\rangle,
$$

(5.4)

where $\mathcal{E}^{(0)}$ is the true eigenenergy. The state $B_q^+|0\rangle$ with one collective excitation is assumed to be an eigenstate with the eigenenergy $\mathcal{E}_q$ determined by the Schrödinger equation

$$
H B_q^+|0\rangle = \mathcal{E}_q B_q^+|0\rangle.
$$

(5.5)

For simplicity, it is assumed that in a homogeneous system the excitation energy $W_q$ is independent of the momentum direction $q$. The explicit expression for part of the commutator $[H,B_q^+]$ is given in Appendix B. The reason the small terms are added in the eigenmode equation (5.2) is that the eigenoperator $B_q^+$ constructed from only a finite number of terms in Eq. (5.1) cannot exactly satisfy Eq. (5.5). However, the eigenoperator $B_q^+$ in Eq. (5.1) is an Ansatz which is justified by experiments which show two branches in the excitation spectrum of He II, as discussed in Chapter II.

The coefficients $\varphi, \kappa, \xi, \gamma$ and $\zeta$ in Eq. (5.1) for the eigenoperator $B_q^+$ are probability amplitudes. The probability amplitude that only a single bogolon $f_q^+$ is present in the collective mode is

$$
\langle 01 | f_{\bar{q}}^+ B_{\bar{q}}^+ | 0 \rangle \approx \varphi(\bar{q}),
$$

(5.6)
while the probability amplitude that only a single quasihole $\gamma_{-\overline{q}}$ is present is

$$\langle 0 | \gamma_{-\overline{q}}^{+} B_{\overline{q}}^{+} | 0 \rangle \approx \zeta (-\overline{g}) ,$$

where $| 0 \rangle$ denotes the true ground state, instead of the bogolon vacuum state. Consequently, the annihilation operator acting on it does not give zero,

$$\gamma_{-\overline{q}} | 0 \rangle \neq 0 .$$

The probability amplitude that a pair of bogolons $\gamma_{k+\overline{q}}^{+} \gamma_{-\overline{k}}^{+}$ is present in the collective mode is

$$\langle 0 | \gamma_{-\overline{k}}^{+} \gamma_{k+\overline{q}}^{+} B_{\overline{q}}^{+} | 0 \rangle \approx \zeta (k+\overline{g}, -\overline{k}) ,$$

while the probability amplitude that two quasiholes $\gamma_{-k-\overline{q}}^{+} \gamma_{k}^{+}$ are present is

$$\langle 0 | \gamma_{k}^{+} \gamma_{-k-\overline{q}}^{+} B_{\overline{q}}^{+} | 0 \rangle \approx \zeta (-k-\overline{g}, \overline{k}) .$$

The probability amplitude that a bogolon $\gamma_{k+\overline{q}}^{+}$ and a quasihole $\gamma_{k}^{+}$ are present is

$$\langle 0 | \gamma_{k}^{+} \gamma_{k+\overline{q}}^{+} B_{\overline{q}}^{+} | 0 \rangle \approx \zeta (k+\overline{g}, \overline{k}) .$$

Thus, $\gamma_{-\overline{q}} | 0 \rangle$ and $\gamma_{-k-\overline{q}} \gamma_{k} | 0 \rangle$ are quasihole state with negative energy.
The term $H_{21}$ in the system Hamiltonian $H$ in Eq. (4.13) describes the decay of a single bogolon into two, and the Hermitian conjugate describes the converse process. Therefore the eigenoperator $B_{q}^{\dagger}$, which contains one- and two-bogolon operators, also describes the hybridization of the two-excitation spectrum with the single-excitation spectrum. Consequently, the eigenvalues $W_q$ should give both branches of the renormalized two-branch energy spectrum in He II.

Now we assume that the small terms on the right-hand side of Eq. (5.2) are negligible. Then we can equate the coefficients of $f^\dagger, f, r^\dagger r^\dagger, r r, r^\dagger r$ and the unit operator on both sides of Eq. (5.2) and obtain the linearized equations of motion for the coefficients $\varphi, \alpha, \xi, \gamma$ and $\zeta$, respectively. For generality, all of $h_{jk}$ terms in the bogolon Hamiltonian $H$ in Eq. (4.13) are retained in the linearized equations of motion. Depending on how the coefficients $\psi_0$ and $u_k$ (or $v_k$) in the canonical transformation of Eq. (4.11) are chosen, some of the $h_{jk}$ terms may vanish.

The normalization condition on the coefficients in Eq. (5.1) can be obtained from the commutation relation

$$[B_{\bar{q}}, B_{\bar{q}'}^{\dagger}] = \delta_{\bar{q}, \bar{q}'} \quad (5.12)$$

and the orthogonality condition from

$$[B_{\bar{q}}, B_{\bar{q}'}] = 0 = [B_{\bar{q}}^{\dagger}, B_{\bar{q}'}^{\dagger}] \quad (5.13)$$
For instance, the normalization condition requires
\[ |\psi(\vec{q})|^2 - |\chi(-\vec{q})|^2 + \frac{1}{2} \sum_{\vec{r}} \left[ |\gamma(\vec{r}+\vec{q}, -\vec{r})|^2 - |\gamma(-\vec{r}-\vec{q}, \vec{r})|^2 \right] = 0 \] (5.14)

which, however, is not actually used in the present work.

In order to have concise physical pictures of the mathematically complicated linearized equations of motion which are derived, a diagram technique can be used. In the diagram dictionary shown in Fig. 8, we denote (1) a collective excitation by a wiggly line with an arrow, (2) the creation of a collective excitation either by a neutron or by a photon by a cross, (3) a free bogolon by a solid line with an arrow, (4) the vertex function describing bogolon interactions by a dot with lines in and out, and (5) a probability amplitude by a block with the appropriate number of lines in and out. Diagrams of this type have not to our knowledge been used before in this context.

**Linearized Equations of Motion**

If Eq. (4.13) for \( H \) is substituted into Eq. (5.2) and the coefficients of \( \gamma_{\vec{q}}^+ \) equated, the equation for \( \psi \), the single-bogolon amplitude, is obtained. This equation is
\[
\psi(\vec{q}) = (W_q - E_q)\left\{ -2 h_{20}(\vec{q}, -\vec{q}) \chi(-\vec{q}) + h_{01}(0) \gamma(\vec{q}, 0) \\
+ \sum_{\vec{r}} h_{12}(\vec{q}, \vec{r}+\vec{q}, -\vec{r}) \gamma(\vec{r}+\vec{q}, -\vec{r}) - 3 \sum_{\vec{r}} h_{30}(-\vec{r}-\vec{q}, \vec{r}, \vec{q}) \gamma(-\vec{r}-\vec{q}, \vec{r}) \right\} \\
- \frac{1}{2} h_{10}(0) \gamma(\vec{q}, 0) \right\},
\] (5.15)
the structure of which is shown in Fig. 9. The energy denominator \((W_q - E_q)\) is the incoming collective excitation energy \(W_q\) minus the outgoing bogolon energy \(E_q\). Equation (5.15) contains five different bogolon scatterings described by the vertex functions \(h_{20}, h_{01}, h_{12}, h_{30}\) and \(h_{10}\).

The equation for \(\mathcal{X}\), the single-quasihole amplitude, is obtained by equating the coefficients of \(\int_{-q}\) in Eq. (5.2), and is

\[
\mathcal{X}(\vec{q}, -\vec{q}) = (W_q + E_q) \left\{ \sum_{\vec{k}} h_{02}(\vec{k}, \vec{q}, -\vec{q}) \mathcal{V}(\vec{q}) + \sum_{\vec{k}} h_{03}(\vec{k}, \vec{k} + \vec{q}, \vec{q}) \mathcal{V}(\vec{k} + \vec{q}, \vec{k}) \right.
- \sum_{\vec{k}} h_{10}(\vec{k}, \vec{q}, -\vec{q}) \sum_{\vec{k}} h_{21}(\vec{k}, -\vec{k} + \vec{q}, \vec{q}) \mathcal{V}(\vec{k}, -\vec{k} + \vec{q}, \vec{k})
+ \frac{1}{2} h_{01}(\vec{k}, \vec{k} + \vec{q}, \vec{q}) \mathcal{V}(\vec{k} + \vec{q}, \vec{k}) \right. \Bigg\}
\]

(5.16)

the structure of which is shown in Fig. 10. The quasihole energy, \(-E_q\), is negative, since the energy denominators in Eq. (5.16) is \(W_q + E_q = W_q - (-E_q)\). The collective excitation annihilates with a quasihole which already exists in the system.

The equation for \(\mathcal{Y}\), the two-bogolon amplitude, is obtained by equating the coefficients of \(\int_{k+q} + \int_{k-}\) in Eq. (5.2), and is

\[
\mathcal{Y}(\vec{k} + \vec{q}, \vec{k}) = (W_q - E_k - E_{k+\vec{q}}) \left\{ \sum_{\vec{k}} h_{30}(\vec{k} + \vec{q}, -\vec{k}, -\vec{q}) \mathcal{V}(\vec{q}) 
- 6 \sum_{\vec{k}} h_{21}(\vec{k}, \vec{k} + \vec{q}, -\vec{k}, -\vec{q}) \mathcal{V}(\vec{k} + \vec{q}, \vec{k})
+ 2 \sum_{\vec{k}} h_{22}(\vec{k} + \vec{q}, -\vec{k}, \vec{k} + \vec{q}, -\vec{k}) \mathcal{V}(\vec{k} + \vec{q}, -\vec{k})
- 12 \sum_{\vec{k}} h_{40}(\vec{k} + \vec{q}, -\vec{k}, -\vec{k} + \vec{q}, \vec{k}) \mathcal{V}(\vec{k} + \vec{q}, \vec{k})
- 2 \sum_{\vec{k}} h_{20}(\vec{k} + \vec{q}, -\vec{k}) \mathcal{V}(\vec{k} + \vec{q}, \vec{k}) \right\}
\]

(5.17)
which is shown diagrammatically in Fig. 11. The direct scattering of a bogolon-pair is described by the vertex function \( h_{22} \). The \( h_{21} \) term is responsible for the possible hybridization of the pair excitation with a single-excitation. The other three terms \( h_{40}, h_{30} \), and \( h_{20} \) may be of little importance in a first approximation.

The equation for \( \gamma \), the two-quasihole amplitude, is obtained by equating the coefficients of \( \gamma_{-\vec{q},\vec{k}} \gamma_{\vec{k}} \), and is

\[
\gamma(-\vec{q},\vec{k}) = (W_0 + E_\vec{k} + E_{\vec{k}+\vec{q}})^{-1} \left\{ 6 h_{03}(\vec{k},-\vec{q},\vec{q}) \gamma(\vec{q}) \\
-2 h_{12}(\vec{q},\vec{k},-\vec{q},-\vec{q}) \gamma(-\vec{q}) \\
+12 \sum_{\vec{k}'} h_{04}(\vec{k}'+\vec{q},-\vec{k}',\vec{k},-\vec{q}) \gamma(\vec{k}'+\vec{q},-\vec{k}') \\
-2 \sum_{\vec{k}'} h_{22}(\vec{k}',-\vec{q}',\vec{k},-\vec{q}) \gamma(-\vec{q}',\vec{k}') \\
+2 h_{02}(\vec{k}+\vec{q},-\vec{q}) \gamma(\vec{k}+\vec{q},\vec{k}) \right\},
\]

which is shown diagrammatically in Fig. 12. Equation (5.18) shows five possible processes describing how two quasiholes can be destroyed by a collective mode.

The equation for \( \zeta \), the bogolon and quasihole amplitude, is obtained by equating the coefficients of \( \zeta_{\vec{k}+\vec{q},\vec{k}} \), and is

\[
\zeta(\vec{k}+\vec{q},\vec{k}) = (W_0 + E_\vec{k} - E_{\vec{k}+\vec{q}})^{-1} \left\{ 4 h_{12}(\vec{k}+\vec{q},\vec{k},\vec{q}) \gamma(\vec{q}) \\
-4 h_{21}(\vec{k}+\vec{q},\vec{q},\vec{k}) \gamma(-\vec{q}) \\
+4 h_{02}(\vec{k},\vec{k}) \gamma(\vec{k}+\vec{q},-\vec{k}) \\
+6 \sum_{\vec{k}'} h_{13}(\vec{k}+\vec{q},\vec{k}'+\vec{q},-\vec{k}',\vec{k}) \gamma(\vec{k}'+\vec{q},-\vec{k}') \\
-4 h_{20}(\vec{k}+\vec{q},\vec{q},\vec{k}) \gamma(-\vec{q},\vec{k}) \\
-6 \sum_{\vec{k}'} h_{31}(-\vec{k}',\vec{k}',\vec{k}+\vec{q},\vec{k}) \gamma(-\vec{q},\vec{k}') \right\},
\]

(5.19)
which is shown diagrammatically in Fig. 13. Equation (5.19) describes how a quasihole can be destroyed by the collective excitation and a quasiparticle created.

Compensation of Lowest Order Dangerous Diagrams

The vanishing of the coefficient of the unit operator on the left-hand side of Eq. (5.2) gives

\[ h_{01}(\bar{g}) \gamma(\bar{g}) - h_{10}(-\bar{g}) \gamma(-\bar{g}) + \sum_k \left[ h_{02}(k, \bar{g}, -k) \gamma(k, \bar{g}, -k) \right] = 0, \]  

(5.20)

which is shown in Fig. 14. The obvious solution to this equation is to choose the coefficient \( \gamma_0 \) and \( u_k \) (or \( v_k \)) in the canonical transformation in Eq. (4.11) such that

\[ h_{01}(\gamma) = h_{10}^*(\gamma) = 0 \]

(5.21)

and

\[ h_{20}(k, -k) = h_{02}^*(-k, k) = 0. \]

(5.22)

These two equations are the "compensation of lowest order dangerous diagrams" (CLODD) for a boson system.\(^{42}\) Equation (5.21) is equivalent to minimizing the unperturbed ground-state energy \( H_{00} \) in Eq. (4.13) with respect to \( \gamma_0 \), and Eq. (5.22) is equivalent to minimizing the unperturbed ground-state energy with respect to \( u_k \) and \( v_k \). It is interesting to
conjecture that if the operator $B_{-q}$ in Eq. (5.1) had been composed of terms involving more than two operators, correction to Eq. (5.12) and (5.22) would have been obtained which would correspond to Bogoliubov's principle of compensation of dangerous diagrams (PCDD). This conjecture is an avenue of future research.

When Eqs. (5.21) and (5.22) are imposed on the equations for the probability amplitudes, considerable simplification is obtained. Equation (5.15) for $\mathcal{G}(q)$ becomes

$$\mathcal{G}(q) = (W_q - E_q) \sum_k \left[ h_{12}(\vec{q}, k, -\vec{k}) \mathcal{G}(k, \vec{k}, -\vec{k}) \right]$$

$$- 3 h_{30}(\vec{k}, \vec{k}, \vec{k}) \gamma(-\vec{k}, \vec{k}, \vec{k}),$$

which is shown graphically in Fig. 15. Equation (5.16) for $\mathcal{X}(-q)$ becomes

$$\mathcal{X}(-q) = (W_q + E_q) \sum_k \left[ 3 h_{03}(\vec{k}, \vec{k}, \vec{k}) \mathcal{G}(k, \vec{k}, -\vec{k}) \right]$$

$$- h_{21}(\vec{k}, \vec{k}, \vec{k}) \gamma(-\vec{k}, \vec{k}, \vec{k}),$$

which is shown graphically in Fig. 16. These two equations no longer involve $\mathcal{G}$ and $\mathcal{X}$ on the right-hand side. Thus to obtain $\mathcal{G}(q)$ and $\mathcal{X}(-q)$ in this approximation, it is only necessary to know $\mathcal{G}$ and $\gamma$.

Equations (5.17)-(5.19) do not simplify as much as Eqs. (5.15) and (5.16) when Eqs. (5.21) and (5.22) are used. Equation (5.17) simplifies by the omission of the last term.
involving $h_20$, while Eq. (5.18) simplifies by the omission of the last term involving $h_02$. These two equations now are uncoupled from $\Psi$ in Eq. (5.19). Equation (5.19) simplifies by the omission of the third and fifth terms, involving $h_02$ and $h_20$, respectively. It involves only $\xi$ and $\gamma$, and is therefore uncoupled from the other equations.

The quantities $\Psi(\bar{q})$ and $\mathcal{X}(-\bar{q})$ can easily be eliminated in Eqs. (5.17) and (5.18) by substituting Eqs. (5.23) and (5.24) in for them. However, this procedure will not be pursued here to decouple partially the four equations. The study of these coupled equations for finite $\bar{q}$ leads to Takano's result that for small $\bar{q}$, at least one solution for the collective excitation spectrum is phonon-like with no gap of the type found in Eq. (4.32), and first found by Girardeau and Arnowitt. However, Takano did not proceed further to obtain an expression for the speed of sound or even show that the phonon spectrum lead smoothly into the bogolon spectrum for large values of $\bar{q}$. Thus there is room for further work here. The present formalism is considerably simplified over Takano's, since we have used the symmetry properties of the vertex functions $h_{jk}$. The diagrams, which are introduced here for the linearized equation of motion method, aid in the visualization of the extremely complicated equations. The formalism can be extended by considering a $B_{\bar{q}^T}$ in Eq. (5.1) with an arbitrary number of operators.
The equations for $\mathcal{P}(q)$ and $\mathcal{Z}(-q)$ in Eqs. (5.23) and (5.24), respectively, dress the lower branch of the excitation phonon-roton spectrum in He II. The upper branch is in principle determined from Eqs. (5.17) and (5.18) for $\mathcal{P}$ and $\mathcal{Z}$, respectively. It is extremely difficult to convert these formidable equations into the algebraic equations which give the secular equation for $W_q$ if a realistic potential $V(r)$ is used.

Equations (5.17) and (5.18) simplify considerably if only collective modes of zero total momentum are considered, as shown in the next section.

Collective Mode with Zero Momentum

In light scattering, the photon momentum transfer $\bar{q}$ can be essentially regarded as zero as discussed in Chapter II. In this case, Eqs. (5.17) and (5.18) can be further simplified. From Table II we have

\[
\begin{align*}
\tag{5.25}
\mathcal{P}(\bar{k},-\bar{k},0) &= \mathcal{P}_0 \left[ (\langle \bar{k},0|V|\bar{k},0 \rangle + \langle -\bar{k},0|V|-\bar{k},0 \rangle ) (u_0 + v_0) u_k v_k \\
&\quad + \langle \bar{k},-\bar{k}|V|0,0 \rangle (u_0 u_k^2 + v_0 v_k^2) \right] \\
\tag{5.26}
3 \mathcal{P}_3(\bar{k},-\bar{k},0) &= \mathcal{P}_0 \left[ (\langle \bar{k},0|V|\bar{k},0 \rangle + \langle -\bar{k},0|V|-\bar{k},0 \rangle ) (u_0 + v_0) u_k v_k \\
&\quad + \langle \bar{k},-\bar{k}|V|0,0 \rangle (u_0 v_k^2 + v_0 u_k^2) \right].
\end{align*}
\]
Hence we obtain

\[
\begin{align*}
\bar{h}_{21}(\vec{k},-\vec{k},0) & = 3\bar{h}_{30}(\vec{k},-\vec{k},0) + \bar{\varphi}_0 <\vec{k},-\vec{k}|\psi_0,0>(u_0 + v_0)^{-1} \\
\end{align*}
\]

(5.27)

where the relation in Eq. (4.12) has been used. Assuming \( E_0 \approx 0 \) in Eqs. (4.22) and (4.23), we get

\[
\bar{h}_{21}(\vec{k},-\vec{k},0) \approx 3\bar{h}_{30}(\vec{k},-\vec{k},0)
\]

(5.28)

since \( u_0 + v_0 \approx \infty \) from Eqs. (4.22) and (4.23). For a high-lying collective mode with energy \( W_0 \gg E_0 \approx 0 \), Eqs. (5.23) and (5.24) give

\[
\bar{\varphi}(0) = \bar{\chi}(0) = W_0^{-1} \sum_k \bar{h}_{21}(\vec{k},-\vec{k},0)\left[ \frac{\bar{\xi}(\vec{k},-\vec{k})}{\bar{\zeta}(\vec{k},-\vec{k})} - \bar{\eta}(\vec{k},-\vec{k}) \right]
\]

(5.29)

if Eq. (5.28) is used. Consequently, because of Eq. (5.28) for a collective mode with zero momentum, the two terms \( \bar{h}_{21} \) and \( \bar{h}_{30} \) cancel each other in Eq. (5.17) for \( \bar{\xi}(\vec{k},-\vec{k}) \) and Eq. (5.18) for \( \bar{\eta}(\vec{k},-\vec{k}) \). This cancellation is an important new result. When Eqs. (5.21) and (5.22) for the compensation of lowest order dangerous diagrams are used, Eq. (5.17) for \( \bar{\xi} \) reduces to

\[
\bar{\xi}(\vec{k},-\vec{k}) = 2(W_0 - 2E_k)^{-1} \left\{ \sum_{k'} \bar{h}_{22}(\vec{k},-\vec{k},\vec{k'},-\vec{k'}) \bar{\xi}(\vec{k'},-\vec{k'}) \\
- 6\sum_{k'} \bar{h}_{k0}(\vec{k},-\vec{k},\vec{k'},-\vec{k'}) \bar{\gamma}(\vec{k'},-\vec{k'}) \right\}
\]

(5.30)
and the equation in Eq. (5.18) for \( \eta \) reduces to

\[
\eta(\vec{k}, \vec{k}') = 2(\omega_0 + 2E_K)^{-1} \left[ 6 \sum_{\vec{k}, \vec{k}'} h_{04}(\vec{k}, -\vec{k}', \vec{k}, -\vec{k}') \xi(\vec{k}, -\vec{k}') - \sum_{\vec{k}, \vec{k}'} h_{22}(\vec{k}, -\vec{k}', \vec{k}, -\vec{k}') \xi(\vec{k}, -\vec{k}') \right] .
\] (5.31)

The two coupled equations in Eqs. (5.30) and (5.31) for \( \xi \) and \( \eta \), respectively, are the basic equations of the present work and are shown in Fig. 17. In the next chapter they are solved for the special case of a \( \delta \) function shell potential, for which they simplify considerably.
CHAPTER VI

COLLECTIVE MODE WITH ZERO MOMENTUM IN THE
NEW TAMM - DANCOFF APPROXIMATION

In this chapter, using the spherical $\delta$-function shell potential in Eq. (4.37), we derive the secular equation for a collective mode of zero momentum in various angular momentum states in the new Tamm-Dancoff approximation (NTDA)\textsuperscript{12} in Eqs. (5.30) and (5.31). By neglecting the two-quasihole amplitude $\gamma$, the results reduce to the corresponding secular equations in the Tamm-Dancoff approximation (TDA).\textsuperscript{15}

New Tamm-Dancoff Approximation for Two Bogolons

In light scattering, we can approximately set the photon momentum transfer $\tilde{q}$ to be zero as discussed in Chapter II. Now a two-bogolon state is characterized by its angular momentum $l$ which is a good quantum number since the total momentum $\tilde{q}$ of the two bogolons is taken as zero.\textsuperscript{15}

In this section we study the possibility of existence of two-bogolon bound state with zero momentum in the NTDA. For $\tilde{q}=0$, we may use Eq. (5.30) which is

\begin{equation}
\psi_1(\vec{k},-\vec{k}) = (\omega_0 - 2E_{k})^{-1}\left\{ \Omega (2\pi^3)^{-2} \int d\vec{k}' h_{22}(\vec{k},-\vec{k},\vec{k}',-\vec{k}') \psi_1(\vec{k}',-\vec{k}') - 3 \Omega (2\pi^3)^{-2} \int d\vec{k}' h_{40}(\vec{k},-\vec{k},\vec{k}',-\vec{k}') \gamma(\vec{k}',-\vec{k}') \right\},
\end{equation}

(6.1)
and Eq. (5.31) which is

\[
\eta(\vec{k}, -\vec{k}) = (W_0 + 2E_k)^{-1} \left[ -\Omega (4\pi^2)^{1/2} \int d\vec{k}' \ h_{22}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}') \eta(\vec{k}', -\vec{k}') \right.
\]

\[
+ 3\Omega (2\pi^3)^{1/2} \int d\vec{k}' \ h_{40}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}') \xi(\vec{k}', -\vec{k}') \right].
\]  

(6.2)

From Table II, the vertex function \( h_{22} \) for the spherical function shell potential of Eq. (4.37) is given by

\[
h_{22}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}') = \lambda \Omega^{-1} \left\{ \frac{1}{4} \left[ j_0(y_0) + j_0(y_v) \right] (uu' + vv')^2 \right\}
\]  

(6.3)

and the vertex function \( h_{40} \) is given by

\[
h_{40}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}') = \lambda \Omega^{-1} \left\{ \frac{1}{4} \left[ j_0(y_0) + j_0(y_v) \right] (uu' + vv')^2 \right\}
\]  

(6.4)

where \( u = u_0 \), \( v = v_0 \), \( u' = u_0' \), \( v' = v_0' \), \( y_0 = |\vec{k} \pm \vec{k}'| \) and \( j_0(x) = \sin x / x \).

Equations (6.1) and (6.2) have the mathematical structure

\[
\xi(\vec{k}, \vec{k}') = \int d\vec{k}'' d\vec{k}''' \left[ b_{11} \xi(\vec{k}''', \vec{k}) + b_{12} \xi(\vec{k}'', \vec{k}') \right]
\]  

(6.5)

and
where $b_{ij} = b_{ij}(k,k',\mu)$ and $\mu = k \cdot k'$. To obtain the angular momentum characteristics of a two-bogolon state, we now expand $\xi$ and $\gamma$ in terms of spherical harmonics $Y^m_\ell$: 

$$\xi(\vec{k},-\vec{k}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \xi_{\ell m}(k) Y^m_\ell(\hat{k}),$$

and 

$$\gamma(\vec{k},-\vec{k}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \gamma_{\ell m}(k) Y^m_\ell(\hat{k}).$$

The $b_{ij}$ may also be expanded: 

$$b_{ij}(k,k',\mu) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (2\ell+1)^{-1}$$

$$\times \int b_{ij,\ell}(k,k') Y^m_\ell(\hat{k}) Y^m_\ell(\hat{k}'),$$

where the components $b_{ij,\ell}$ are given by 

$$b_{ij,\ell}(k,k') = (\ell+1/2) \int d\mu P^\ell_\mu(\mu) b_{ij}(k,k',\mu)$$
and $P_l$ are Legendre polynomials. Equating coefficients of $\gamma_l^m$ on both sides of Eqs. (6.5) and (6.6), we get the coupled integral equations

\[
\xi_{l,m}(k) = 2\pi\int_0^\infty dk' \int d\mu P_l(m)
\]
\[
\times [b_{11}\xi_{l,m}(k') + b_{12}\gamma_{l,m}(k')]
\]

and

\[
\gamma_{l,m}(k) = 2\pi\int_0^\infty dk' \int d\mu P_l(m)
\]
\[
\times [b_{21}\xi_{l,m}(k') + b_{22}\gamma_{l,m}(k')]
\]

for various angular momentum states $l$. The components $\xi_{l,m}$ and $\gamma_{l,m}$ of the undetermined probability amplitudes $\xi$ and $\gamma$ are obviously independent of subscript $m$ and will be replaced by $\xi_l$ and $\gamma_l$ hereafter. Since the bogolons with which we deal are bosons, Eq. (5.1) requires that

\[
\xi(R, -R) = \xi(-R, R),
\]

and

\[
\gamma(R, -R) = \gamma(-R, R),
\]

so from Eqs. (6.7) and (6.8) only state of even $l$ are permitted. Substituting the explicit expressions for $b_{ij}$ from Eqs. (6.3) and (6.4), Eq. (6.11) becomes
\[
\xi_{\vec{k}, \vec{\beta}} = \lambda (8\pi^2a^2)^\prime (W_0 - 2E_x)^\prime [4uv\beta_e \int dx' x'^2 u' v' \xi_{\vec{k}, \vec{\beta}} \\
+ u^2 \int dx' x'^2 u'^2 \beta_e \xi_{\vec{k}, \vec{\beta}} + v^2 \int dx' x'^2 v'^2 \beta_e \xi_{\vec{k}, \vec{\beta}} \\
+ 2uv \int dx' x'^2 u' v' \beta_e \xi_{\vec{k}, \vec{\beta}} - 4uv\beta_e \int dx' x'^2 u' v' \eta_{\vec{k}, \vec{\beta}} \\
- u^2 \int dx' x'^2 u'^2 \beta_e \eta_{\vec{k}, \vec{\beta}} - v^2 \int dx' x'^2 v'^2 \beta_e \eta_{\vec{k}, \vec{\beta}} \\
- 2uv \int dx' x'^2 u' v' \beta_e \eta_{\vec{k}, \vec{\beta}} ]
\] (6.15)

and Eq. (6.12) becomes
\[
\eta_{\vec{k}, \vec{\beta}} = \lambda (8\pi^2a^2)^\prime (W_0 + 2E_x)^\prime [4uv\beta_e \int dx' x'^2 u' v' \xi_{\vec{k}, \vec{\beta}} \\
+ u^2 \int dx' x'^2 u'^2 \beta_e \xi_{\vec{k}, \vec{\beta}} + v^2 \int dx' x'^2 v'^2 \beta_e \xi_{\vec{k}, \vec{\beta}} \\
+ 2uv \int dx' x'^2 u' v' \beta_e \xi_{\vec{k}, \vec{\beta}} - 4uv\beta_e \int dx' x'^2 u' v' \eta_{\vec{k}, \vec{\beta}} \\
- u^2 \int dx' x'^2 u'^2 \beta_e \eta_{\vec{k}, \vec{\beta}} - v^2 \int dx' x'^2 v'^2 \beta_e \eta_{\vec{k}, \vec{\beta}} \\
- 2uv \int dx' x'^2 u' v' \beta_e \eta_{\vec{k}, \vec{\beta}} ]
\] (6.16)

In Eqs. (6.15) and (6.16), the integration range of \( x' \) is from 0 to \( \infty \), and we define \( x=ka, x'=k'a, \mu=kk' \), \( u=u_x, v=v_x \), \( u'=u_x' \), \( v'=v_x' \), \( \xi_{\vec{k}, \vec{\beta}} = \xi_{\vec{k}}(x) \), \( \eta_{\vec{k}, \vec{\beta}} = \eta_{\vec{k}}(x) \), \( \xi'_{\vec{k}, \vec{\beta}} = \xi'_{\vec{k}}(x') \), and \( \eta'_{\vec{k}, \vec{\beta}} = \eta'_{\vec{k}}(x') \). The constant \( \beta_{\vec{2}} \) is defined by
\[
\beta_{\vec{k}} \equiv \int_{-1}^{1} p_{\vec{k}}(\mu) d\mu = 2 \delta_{\vec{k},0} \]
(6.17)

and the function \( \beta_{\vec{k}}'(k,k') \) is defined by
\[
\beta_{\vec{k}}' \equiv \int_{-1}^{1} p_{\vec{k}}(\mu)[j_y(y,\alpha)+j_y(y,\alpha)] d\mu,
\]
(6.18)
where \( j_0(x) = \frac{\sin x}{x} \) and \( \gamma_{\pm} = |k \pm k'| = (k^2 \pm 2 k k' + k'^2)^{\frac{1}{2}} \).

The integral in Eq. (6.18) can be evaluated explicitly by making use of the expansion of the spherical Bessel function of zero order\(^{55}\)

\[
\hat{j}_0(qa) = \sum_{n=0}^{\infty} (2n+1) j_n(ka) j_n(k'a) \hat{P}_n(\mu),
\]

(6.19)

where \( j_n \) is the spherical Bessel function of order \( n \). When Eq. (6.19) is substituted into Eq. (6.18), and the orthogonality properties of the Legendre polynomials \( P_n \) are used, the result is

\[
\beta_{\ell} (k, k') = 4 j_{\ell}(x) j_{\ell}(x')
\]

(6.20)

for even \( \ell \), and zero for odd \( \ell \). For odd values of \( \ell \), both \( \beta_{\ell} \) and \( \beta'_{\ell} \) vanish which is consistent with the bogolons being bosons.

In the following sections, we will solve Eqs. (6.15) and (6.16) for even values of \( \ell \).

**S-State of Collective Mode**

For an S-state \( (\ell = 0) \) of a bogolon pair with zero momentum in the new Tamm-Dancoff (NTD) approximation, the integral equations in Eqs. (6.15) and (6.16) become

\[
\begin{align*}
\xi_0 = & (2c)^2 (W_0 - 2EX - 2) [2uv(uv, \xi_0)_{\ell} + j_0 u^2(u^2, \xi_0)_{\ell} + j_0 v^2(u^2, \xi_0)_{\ell} + j_0 u^2(u^2, \xi_0)_{\ell} - 2uv(uv, \xi_0)_{\ell}] \\
+ & 2uv(uv, \xi_0)_{\ell} + j_0 u^2(u^2, \xi_0)_{\ell} + j_0 v^2(u^2, \xi_0)_{\ell} - 2uv(uv, \xi_0)_{\ell} - j_0 u^2(u^2, \xi_0)_{\ell} - j_0 v^2(u^2, \xi_0)_{\ell} - 2j_0 uv(uv, \xi_0)_{\ell}
\end{align*}
\]

(6.21)
and

\[
\gamma_0 = (2c)^{-1} (W_0 + 2E_x)^{\frac{1}{2}} \left[ 2uv (uv, \xi_0) + j_0 u^2 (u^2, \xi_0) + j_0 u^2 (v^2, \xi_0) - 2u \nu (uv, \xi_0) - j_0 u^2 (u^2, \xi_0) + j_0 u^2 (v^2, \xi_0) - 2j_0 uv (uv, \xi_0) \right],
\]

(6.22)

on using Eq. (6.20), where \( c = \pi^2 a^3 / \lambda \), \( x = ka \), \( u = u_x \), \( v = v_x \),

\( \xi_0 = \xi_0 (x) \), \( \gamma_0 = \gamma_0 (x) \), \( j_0 = j_0 (x) \), \( (uv, \xi_0) = \int_0^\infty dx x^2 uv \xi_0 \),

\( (u^2, \xi_0) = \int_0^\infty dx^2 j_0 u^2 \xi_0 \), etc. After converting the functions \( \xi_0 \) and \( \gamma_0 \) on the left-hand sides of Eqs. (6.21) and (6.22) into various integrals \( (uv, \xi_0) \), \( (u^2, \xi_0) \), etc. which appear on the right-hand sides, we obtain a set of eight simultaneous algebraic equations (see Appendix C). For a nontrivial solution for the eight integrals on the right-hand sides of Eqs. (6.21) and (6.22) to exist, the 8 x 8 determinant of their coefficients, which can be reduced to a 4 x 4 determinant, must vanish:

\[
\begin{vmatrix}
S_1 - C & S_2 & S_3 & S_4 \\
S_2 & S_5 - 2c & S_6 & S_7 \\
S_3 & S_6 & S_8 - 2c & S_9 \\
S_4 & S_7 & S_9 & S_6 - C
\end{vmatrix} = 0,
\]

(6.23)
where the quantities $s_n$ are defined by

\[
\begin{align*}
S_1 &= \int dx \left( g_- c_0 g_+ \right) (u v)^2, \\
S_2 &= \int dx j_0 \left( g_- u^2 - c_0 g_+ v^2 \right) u v, \\
S_3 &= \int dx j_0 \left( g_- v^2 - c_0 g_+ u^2 \right) u v, \\
S_4 &= \int dx j_0 \left( g_- - c_0 g_+ \right) (u v)^2, \\
S_5 &= \int dx j_0 \left( g_- u^4 - c_0 g_+ v^4 \right), \\
S_6 &= \int dx j_0 \left( g_- v^4 - c_0 g_+ u^4 \right), \\
S_7 &= \int dx j_0 \left( g_- v^2 - c_0 g_+ u^2 \right) u v, \\
S_8 &= \int dx j_0 \left( g_- u^2 - c_0 g_+ v^2 \right) u v.
\end{align*}
\]

and

\[
S_9 \equiv \int dx j_0 \left( g_- v^2 - c_0 g_+ u^2 \right) u v.
\]  
(6.24)

Here $u = u_x$, $v = v_x$, $j_0 = j_0(x)$, $g_\pm = x^2 (W_0 \mp 2E_x)^{-1}$, and $c_0 = 1$ in the NTDA and $c_0 = 0$ in the TDA.

The secular equation in Eq. (6.23) contains two unknowns, the total energy $W_0$ of a bogolon pair with essentially zero total momentum in an angular momentum $S$-state and the effective interaction strength $\lambda = \pi^2 a^3/c$ between two He$^4$ atoms. An $S$-state is, however, not coupled to light scattering in the dipole approximation. It is investigated here for the sake of comparison with the calculations of RZ.\(^6\)
D-State of Collective Mode

For a bogolon pair in an angular momentum $\ell$ state ($\ell=2,4,6,\ldots$), the integral equations in Eqs. (6.15) and (6.16) reduce to

\[
\xi_\ell = (2c)^2 \left( w_0 - 2E_x \right) \int J_\ell \left[ u^2 (u^2 \xi_\ell) + v^2 (v^2 \xi_\ell) \\
+ 2uv (uv \xi_\ell) - u^2 (v^2 \eta_\ell) - v^2 (u^2 \eta_\ell) \\
- 2uv (uv, \eta_\ell) \right] (6.25)
\]

and

\[
\eta_\ell = (2c)^2 \left( w_0 + 2E_x \right) \int J_\ell \left[ u^2 (u^2 \xi_\ell) + v^2 (u^2 \xi_\ell) \\
+ 2uv (uv \xi_\ell) - u^2 (v^2 \eta_\ell) - v^2 (v^2 \eta_\ell) \\
- 2uv (uv, \eta_\ell) \right] (6.26)
\]

where $\ell=2,4,6,\ldots$, $j_\ell = j_\ell(x)$, $\xi_\ell = \xi_\ell(x)$, $\eta_\ell = \eta_\ell(x)$, $(u^2, \xi_\ell)$ = \int_0^\infty dx x^2 j_\ell u^2 \xi_\ell$, etc. When Eqs. (6.25) and (6.26) are converted to algebraic equations (see Appendix D), and non-trivial solutions are sought, the secular equation is the vanishing of a $6 \times 6$ determinant which can be reduced to a $3 \times 3$ determinant. The result is

\[
\begin{vmatrix}
  d_1 - 2c & d_2 & d_3 \\
  d_2 & d_4 - 2c & d_5 \\
  d_3 & d_5 & d_2 - c
\end{vmatrix} = 0,
\]

(6.27)
where the quantities $d_n$ are defined by
\[
d_1' \equiv d_1' - c_0 d_1'' = \int \delta x \delta z \left( g_- u^k - c_0 g_+ \nu^k \right),
\]
\[
d_2' \equiv d_2' - c_0 d_2'' = \int \delta x \delta z \left( g_- - c_0 g_+ \right) (\nu v)^2,
\]
\[
d_3' \equiv d_3' - c_0 d_3'' = \int \delta x \delta z \left( g_- u^2 - c_0 g_+ \nu^2 \right) \nu v,
\]
\[
d_4' \equiv d_4' - c_0 d_4'' = \int \delta x \delta z \left( g_- \nu^4 - c_0 g_+ u^4 \right),
\]
and
\[
d_5' \equiv d_5' - c_0 d_5'' = \int \delta x \delta z \left( g_- \nu^2 - c_0 g_+ u^2 \right) \nu v.
\]

For a D-state of two bogolons, which is coupled to light scattering, we let $\ell=2$ in Eqs. (6.25) - (6.28). The numerical solutions to the secular equations in Eqs. (6.23) and (6.27) with $\ell=2$ are presented in the next chapter.
CHAPTER VII

RESULTS FOR COLLECTIVE MODES AND DISCUSSION

Using the energy spectrum for bogolons in Eq. (4.54) fitted to the observed spectrum, and various values for the binding energy ($E_B=W_0-2E_{roton}$) of a bound roton pair, the secular equation in Eqs. (6.23) and (6.27) are solved on an I.B.M. 360/50 computer for the interaction strength $\lambda$ between two He$^4$ atoms. The results of these calculations are presented and discussed in this chapter.

Calculations

There are nine different types of integrals $s_n$ in the secular equation in Eq. (6.23) for a bogolon pair in an S-state and five $d_n$ in the secular equation in Eq. (6.27) with $\ell=2$ for a bogolon pair in a D-state. Both $s_n$ and $d_n$ are principal value integrals whose singularity $x_S=k_Sa$ for a given value of binding energy $E_B$ lies in the phonon region, and whose integration range of $x=ka$ is from 0 to $\infty$. The principal value integrals $s_n$ and $d_n$, if they exist, should be independent of the half-width parameter $\eta a$ of the integration gap, $(x_S-\eta a, x_S+\eta a)$, whose middle point $x_S$ is the singularity. The upper limit $x_U=k_Ua$ of each integral should be sufficiently large to insure
that the error due to the finite integration range is negligible. On the other hand, the lower limit \(x_A=k_{\text{p}}a\) should be sufficiently small to insure convergence, also. Since the experimentally observed angular momentum state of a roton pair is a D-state, our main concern is with the integrals \(d_n\). The existence of the principal value integrals \(d_n\) is studied in Appendix E. In our numerical integrations, the values \(\gamma = 1 \times 10^{-4} \text{Å}^{-1}\), \(k_{\text{p}}=1 \times 10^{-2} \text{Å}^{-1}\) and \(k_{u}=41.11 \text{Å}^{-1}\) are used in the trapezoidal rule for approximating areas.\(^{56}\)

As discussed in Appendix E, the principal value integrals \(d_n\) do exist. For the binding energy \(E_B\) of a roton pair in a bound D-state, Greytak, et al.\(^5\) obtained a binding energy in the range \(0.27 \leq E_B \leq 0.47 \text{K}\), and Cowley\(^{32}\) suggested the range \(0 \leq E_B \leq 0.1 \text{K}\). In order to locate the singularity of the middle point of the integration gap, we use \(E_B\) of five or six significant figures to cover these two binding energy ranges. The integration results are listed in Table IV for an S-state and in Table V for a D-state, where the contributions from the phonon region (from \(1 \times 10^{-2} \text{Å}^{-1}\) to the maxon at \(1.11 \text{Å}^{-1}\)) and the roton region (from the maxon at \(1.11 \text{Å}^{-1}\) to \(41.11 \text{Å}^{-1}\)) are explicitly specified. Tables IV and V are discussed in the next two sections.

The secular equations in Eqs. (6.23) and (6.27) with \(\ell=2\) then can be solved for the interaction strength \(\lambda = \pi^2a^3/c\) between two He\(^4\) atoms. For a given binding energy, there are
four roots $\lambda_n^{(S)}$, $n=1,2,3,4$, of the secular equation for an S-state, and three roots $\lambda_n^{(D)}$, $n=1,2,3$, for a D-state. The results are listed in Tables VI and VII for an S-state and a D-state, respectively. These results are discussed later in this chapter.

Contribution to the Integrals for the S-State

Now we discuss the contributions to the integrals $s_n$ from the phonon and roton regions listed in Table IV, where the integrals $s_n$ for the phonon part are calculated with the half-width parameter $\eta = 1 \times 10^{-4} \text{Å}^{-1}$. These integrals could be calculated more accurately by the method derived in Appendix E.

On physical grounds, the main contribution to each integral $s_n$ or $d_n$ should come from the roton region because the total energy $W_0$ of a bogolon pair in an angular momentum D-state observed in light scattering experiments is just slightly less than twice the energy of a single roton.

As a specific example, let us take a look at the integral $s_6$ in Table IV. Except for the two cases of $E_B = 0.37142 \text{K}$ and $0.10072 \text{K}$ in the NTDA, the phonon region dominates in $s_6$ in all other cases in Table IV. It is unphysical that the phonon contribution tends to be due to low lying phonons with small energy. In order to avoid this unphysical result, a cutoff momentum less than the roton momentum could be used in performing the integrations. However, there is no justification for a cutoff momentum, nor a good criterion available for its location.
As an opposite example, this unphysical situation does not occur in the $s_4$ corresponding to $E_B = 0.47042 \, K$ in both the NTDA and the TDA, where the phonon region contributes only 1.9% and 8.8%, respectively. However to calculate the particle coupling strength $\lambda$ the secular equation in Eq. (6.23) involves all $s_n$'s.

Our discussion of an S-state is only for comparison with the RZ theory, since in the dipole approximation light couples only to a D-state ($Q=2$).

Contributions to the Integrals for the D-State

The values of the principal value integrals $d_n$, listed in Table V with $\eta = 1 \times 10^{-4} \, \text{Å}^{-1}$, for a bogolon pair with zero total momentum in an angular momentum D-state are reliable as discussed in Appendix E and accurate to about 0.01% as estimated in Appendix E. Table V also shows that the main contributions to all of the five integrals $d_n$ come from the roton region in both the NTDA and TDA, corresponding to the nine trial values of binding energy between 0.01 K and 4 K. Hence the calculated values of $d_n$ are acceptable both mathematically and physically, and the validity of the NTDA and the TDA used in the present work is justified for a D-state.

Effective Interaction Strength between Two He$^4$ Atoms

Substituting the values of $s_n$ and $d_n$ listed in Tables IV and V into Eqs. (6.23) and (6.27) and expanding the determinants, we solved the secular equations for the effective interaction
strength $\lambda = \pi^2 a^3/c$ between two He$^4$ atoms, corresponding to various binding energies. The results are listed in Table VI for an S-state and in Table VII for a D-state.

Table VI shows that a roton pair can exit in a bound S-state with a repulsive or attractive $\lambda$, but Table VII shows that it can exist in a bound D-state only with an attractive $\lambda$. In two-excitation processes, the light scattering (in the dipole approximation) creates only a bogolon pair of essentially zero total momentum (compared with the roton momentum) in an angular momentum D-state ($\ell=2$).

According to Greytak, et al., the binding energy of a roton pair is $0.37 \pm 0.10$ K. With reference to Cowley's calculation, the binding energy, if it exists at all, should be $0.1$ K or less. The binding energy $0.37 \pm 0.10$ K is based on a phenomenological theory of the attractive roton-roton interaction in Eq. (3.41), proposed by Zawadowski, et al. Based upon Griffin's comment, the foundation of this phenomenological theory is not justified by any microscopic theory. There is thus a great need for justification of the phenomenological theory from first principles. The motivation of the present research is to seek such a justification. However, the conclusion to be drawn from Table VII is that for the simple $\delta$ function shell potential between the atoms, a bound D-state cannot form for repulsive coupling. The bound D-state forms only for attractive coupling between the atoms, which is not realistic and contradicts the Bogoliubov theory.
In the Bogoliubov theory\textsuperscript{4} which is used in the present work, \( \lambda \) has to be repulsive, or an imaginary speed of sound \( s \) in Eq. (4.39) is obtained. The lower bound of the repulsive \( \lambda \) may be determined from the Morse dipole-dipole (MDD) potential. Using MDD-1 and MDD-2 potentials\textsuperscript{57} in Eq. (4.10) for \( k=0 \), Goble\textsuperscript{58} obtains for the volume integral of the potential

\[
V_{0}^{(MDD-1)} = 7.659 \times 10^{5} \text{ A}^{3} \text{ K}
\]  

and

\[
V_{0}^{(MDD-2)} = 6.161 \times 10^{5} \text{ A}^{3} \text{ K},
\]

respectively. Therefore the positive value of \( \lambda \) should be equal or greater than either \( V_{0}^{(MDD-1)} \) or \( V_{0}^{(MDD-2)} \) which are of order \( 10^{5} \text{ A}^{3} \text{ K} \).

\( \lambda \) can also be regarded as a pseudopotential, which can be determined by fitting the speed of sound \( s \) in Eq. (4.39) in the simple Bogoliubov theory where \( \Delta_{0} = \lambda \rho_{0} \). This could be used to estimate \( \lambda \) also.

Unfortunately, for a bound roton pair in a D-state to exist, the results of our calculations in Table VII require the effective interatomic potential \( \lambda \) to be attractive, which is not allowed by the elementary Bogoliubov theory. The possibility that the bogolon-bogolon interaction \( h_{22} \) could be attractive for a repulsive \( \lambda \) due to the functions \( u_{k} \) and \( v_{k} \) in Eq. (6.3) is ruled out. Although the functions \( u_{k} \) and \( v_{k} \) in the bogolon interaction terms \( H_{21}, H_{30}, H_{40}, H_{31}, H_{22} \) and their Hermitian conjugates
in Eq. (4.13) may modify the bare vertices \( h_{22} \) and \( h_{40} \) for the bogolon interactions in such a way that the interaction between rotons is effectively attractive for a realistic particle potential, it is however extremely difficult to solve the integral equations involving all of these renormalization terms. According to Rajagopal, et al.,\(^{38}\) whose theory is discussed in Chapter III, the quasiparticle interaction terms have an important role in renormalizing (modifying) the interaction vertex. For instance, in the present work the bare vertex \( h_{22} \) may be replaced by the renormalized vertex \( \tilde{h}_{22} \) shown in Fig. 18. The effect of this renormalization could be to give the essentially attractive roton-roton interaction, which is required for a bound D-state.

Comparison with Other Theories

For a bound roton pair with binding energy about 0.37 K in a D-state in the NTDA, one of our three solutions in Table VII for the interatomic coupling is

\[
\lambda_{1}^{(D)} \quad (E_B = 0.37/4.2 \text{ K})
\]

\[
= -8.3 \times 10^{-39} \text{ cm}^3 \text{ erg},
\]

which of course contradicts our assumption that \( \lambda \) is positive in Eq. (4.39). On the other hand, corresponding to \( E_B = 0.37 \text{ K} \), Greytak, et al.,\(^{5}\) obtain a roton-roton coupling constant

\[
5g = -4.35 \times 10^{-39} \text{ cm}^3 \text{ erg}
\]
for the D-state in Eq. (3.40) which is based on the theory of Zawadowski, et al., Both the roton-roton coupling constant $5g$ that Greytak, et al. determined and the particle-particle coupling constant $\lambda_1^{(D)}$ that we calculated are to fit the experimental binding energy 0.37 K.

Equation (3.37) is the basic equation for the two-quasiparticle Green's function used by Zawadowski, et al. Greytak et al. replace $g_4$ for $q=0$ by the quantity $g_4^{(l=2)}$ in Eq. (3.40). Their basic equation is simply the sum of an infinite number of loop diagrams with the phenomenological vertex function $g_4^{(l=2)}$ playing a role similar to our vertex function $h_{22}$. Both $g_4^{(l=2)}$ and $h_{22}$ describe the quasiparticle scattering. The function factor in our simple potential model in Eq. (4.40) relates the particle-particle coupling $\lambda$ directly to the bogolon-bogolon vertex function $h_{22}$ in Eq. (6.3). Since the function $u_k$ in Eq. (6.3) is of order unity and $v_k$ is much smaller, it is not surprising that our interatomic $\lambda_1^{(D)}$ in Eq. (7.3) is comparable with their $5g$ in Eq. (7.4).

Corresponding to the smaller binding energy about 0.1 K, one of our solutions in Table VII is

$$\lambda_1^{(D)} (E_B = 0.10072 K) = -3.98 \, \text{A}^3 \text{K} = -5.49 \times 10^{-39} \text{cm}^3 \text{erg},$$

which is still negative and unacceptable in the Bogoliubov theory.
Therefore, we conclude that vertex renormalization as shown in Fig. 18, or a more realistic potential with an attractive part, or both, have to be included in a more complete theory to give bound roton pairs. Recent experiments\textsuperscript{59} have definitely verified the existence of bound roton pairs with a binding energy of about 0.37 K, which have overcome the objections previously raised by Cowley.\textsuperscript{32}
CHAPTER VIII

THEORY OF LIGHT SCATTERING

Theories of light scattering from liquid helium have been proposed by Halley, Stephen, Iwamoto, Fetter, and Cowley. In this chapter we discuss, following Fetter, the quantum theory of light scattering from He II with the creation of either a phonon or two noninteracting excitations. The Hamiltonian for the system of helium and the radiation field is first discussed, and then the cross section for light scattering is obtained. Brillouin scattering from phonons and Raman scattering from two free rotons are each discussed. Finally, the density of roton pair states is reviewed.

Hamiltonian for the Helium and the Radiation Field

According to Fetter, the total Hamiltonian of liquid helium plus the electromagnetic radiation field is composed of three terms:

\[ H_{\text{tot}} = H + H_{\text{rad}} + H' \]  

(8.1)

where \( H \) is the Hamiltonian of the interacting He\(^4\) atoms in the liquid given in Eq. (4.13), \( H_{\text{rad}} \) is the Hamiltonian of the free radiation field, and \( H' \) is the interaction between the He\(^4\) and the radiation.
Hamiltonian for Radiation Field

The radiation field at the position \( \vec{r} \) and time \( t \) in the Coulomb or radiation gauge is described by the vector potential\(^6^2 \)

\[
\overrightarrow{A}(\vec{r},t) = \sum_{\vec{q} s} \left( \frac{\hbar c^2}{\omega_{\vec{q} s} \mathcal{N}'} \right)^{1/2} e^{i \vec{q} \cdot \vec{r}} \times \left[ \hat{b}_{\vec{q} s}(t) \hat{\mathcal{E}}_{\vec{q} s} + \hat{b}_{\vec{q} s}^{+} s(t) \hat{\mathcal{E}}_{-\vec{q}, s} \right],
\]  

(8.2)

where \( \mathcal{N}' \) is the quantization volume of the radiation field, \( \vec{q} \) and \( s \) are respectively the wave-vector and polarization of a photon, \( \hat{\mathcal{E}}_{\vec{q} s} \) is the unit polarization vector transverse to \( \vec{q} \), and \( \omega_{\vec{q}} = c \omega_{\vec{q}} \) is the photon energy. The annihilation operator for a photon in the mode \( \vec{q}, s \) in the Heisenberg picture is

\[
\hat{b}_{\vec{q} s} (t) = \hat{b}_{\vec{q} s} e^{-i \omega_{\vec{q}} t},
\]  

(8.3)

The creation operator \( \hat{b}^{+}_{\vec{q} s} \) and the annihilation operator \( \hat{b}_{\vec{q} s} \) for the photons together satisfy the usual boson commutation relations,

\[
\left[ \hat{b}_{\vec{q} s}, \hat{b}^{+}_{\vec{q}' s'} \right] = 0,
\]  

(8.4)

\[
\left[ \hat{b}_{\vec{q} s}, \hat{b}_{\vec{q}' s'} \right] = \delta_{\vec{q}, \vec{q}'} \delta_{s, s'},
\]  

(8.5)

and

\[
\left[ \hat{b}^{+}_{\vec{q} s}, \hat{b}^{+}_{\vec{q}' s'} \right] = 0.
\]  

(8.6)

The free electric field in this gauge is

\[
\overrightarrow{E}_0 (\vec{r},t) = -\frac{i}{c} \frac{\partial \overrightarrow{A}(\vec{r},t)}{\partial t} \\
= i \sum_{\vec{q} s} \left( \frac{2\pi \hbar \omega_{\vec{q} s}}{\mathcal{N}'} \right)^{1/2} e^{i \vec{q} \cdot \vec{r}} \left[ \hat{b}_{\vec{q} s}(t) \hat{\mathcal{E}}_{\vec{q} s} - \hat{b}^{+}_{\vec{q} s} s(t) \hat{\mathcal{E}}_{-\vec{q}, s} \right],
\]  

(8.7)
and the magnetic field can be calculated from $\mathbf{B} = \nabla \times \mathbf{A}$. The energy density for the electromagnetic field can be written in terms of $E^2 + B^2$. The Hamiltonian density for the free radiation field is obtained by expressing the energy density in terms of the quantized electric and magnetic field operators, and consequently in terms of the photon creation and annihilation operators. When this expression is integrated over all space, the Hamiltonian of the free radiation field is \(^6^2\)

$$H_{rad} = \sum_{\mathbf{q}} \hbar \omega_{\mathbf{q}} b^\dagger_{\mathbf{q},s} b_{\mathbf{q},s}. \quad (8.8)$$

**Interaction Hamiltonian**

The interaction Hamiltonian is given by \(^6^3\)

$$H' = -\frac{1}{2} \int d\mathbf{r} \mathbf{P}(\mathbf{r}) \cdot \mathbf{E}_0(\mathbf{r}), \quad (8.9)$$

where $\mathbf{P}$ is the polarization of liquid helium. The polarization $\mathbf{P}(\mathbf{r})$ is given by

$$\mathbf{P}(\mathbf{r}) = \alpha n(\mathbf{r}) \left[ E_0(\mathbf{r}) + E_D(\mathbf{r}) \right], \quad (8.10)$$

where $\alpha$ is the He\(^4\) atomic polarizability, and the operator

$$n(\mathbf{r}) = \sum_{L=1}^{N} \delta(\mathbf{r} - \mathbf{r}_L^2) \quad (8.11)$$
is the number density of He\(^4\) atoms in the N-particle subspace. The induced dipole field \(\overline{E}_D\) at \(\overline{r}_i\) is

\[
\overline{E}_D(\overline{r}_i, t) = \sum_{j=1}^{N} \frac{r_{ij}}{s^5} \left[ 3 \overline{p}_j \cdot \overline{r}_{ij} \overline{r}_{ij} - k_j^2 \overline{p}_j \right],
\]

(8.12)

where \(\overline{p}_j\) is the polarization at \(\overline{r}_j\) and \(\overline{r}_{ij} = \overline{r}_i - \overline{r}_j\). Assuming \(\overline{p}_j \approx \alpha \overline{E}_0(\overline{r}_j)\), we have

\[
\overline{E}_D(\overline{r}_i, t) = \sum_{j=1}^{N} \frac{r_{ij}}{s^5} \left[ \alpha \overline{E}_0(\overline{r}_j, t) \cdot \overline{r}_{ij} \overline{r}_{ij} - k_j^2 \alpha \overline{E}_0(\overline{r}_j, t) \right].
\]

(8.13)

This can be written using \(n(\overline{r})\) as

\[
\overline{E}_D(\overline{r}, t) = \int d\overline{r}' f(|\overline{r}-\overline{r}'|) |\overline{r}-\overline{r}'|^{-5} \left[ 3 \alpha n(\overline{r}) \overline{E}_0(\overline{r}', t) \cdot (\overline{r}-\overline{r}') (\overline{r}-\overline{r}') 
- (\overline{r}-\overline{r}')^2 \alpha n(\overline{r}') \overline{E}_0(\overline{r}', t) \right].
\]

(8.14)

The function \(f(|\overline{r}-\overline{r}'|)\) is a cutoff function which is introduced to prevent a divergent self-interaction. It is taken to be a unit step function \(\Theta(|\overline{r}-\overline{r}'|-a)\) by Stephen\(^6\) who used \(a = 1.7\) Å. Fetter\(^{14}\) used \(a = 1.7\) Å and 1.8 Å, and Cowley\(^{32}\) used \(a = 1.57\) Å. However a larger value of the cutoff seems to be more appropriate since the distance of closest approach of two He\(^4\) atoms is \(~2.5\) Å because of the strong repulsive core. Baeriswyl\(^{64}\) uses the pair correlation function, but there is no fundamental justification for this choice. This point is discussed more in Chapter IX.
The interaction Hamiltonian now can be written as two terms,

\[ H' = H_1' + H_2', \quad (8.15) \]

where the first term,

\[ H_1' = -\frac{1}{2} \int d\vec{r} \alpha n(\vec{r}) \vec{E}_0(\vec{r},t)^2, \quad (8.16) \]

is the direct interaction Hamiltonian. The second term

\[ H_2' = -\frac{1}{2} \int d\vec{r} \alpha n(\vec{r}) \vec{E}_D(\vec{r},t) \cdot \vec{E}_0(\vec{r},t) \quad (8.17) \]

is the indirect interaction Hamiltonian, which can be written as

\[ H_2' = \frac{1}{2} \alpha^2 \int d\vec{r} d\vec{r}' n(\vec{r}) n(\vec{r}') \]

\[ \times \sum_{\lambda_\mu=1}^3 E_0(\vec{r},t)_{\lambda} \tau(\vec{r}-\vec{r}')_{\lambda_\mu} E_0(\vec{r}',t)_{\mu}, \quad (8.18) \]

on using Eq. (8.14). The cartesian tensor operator \( \tau_{\lambda\mu} \) is defined as

\[ \tau_{\lambda\mu} = r^{-5} (r^2 \delta_{\lambda\mu} - 3 x_\lambda x_\mu) f(r), \quad (8.19) \]

and can be decomposed into three spherical tensor operators \( \tau(\ell) \), \( \ell = 1,2,3 \). The operator is traceless,

\[ \tau^{(0)} = \sum_{\lambda=1}^3 \tau_{\lambda\lambda} = 0, \quad (8.20) \]
all of the components of the first rank spherical tensor $T^{(1)}$ vanish,

$$T^{(1)}_{\lambda} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) = 0, \quad \lambda \mu \nu \text{ cyclic},$$

so the only part of $T(\mathbf{r})$ that contributes is $T_m^{(2)}$, $m = -2, -1, \ldots, 2$. Thus the interaction of light with the helium excitations characterized by $T_{\lambda \mu}$ is entirely D-like, namely $\ell = 2$. By substituting Eq. (8.7) into Eq. (8.18), $H_2'$ can be expressed explicitly in terms of the photon operators.

The total Hamiltonian in Eq. (8.1) can now be written as

$$H_{tot} = (H + H_{rad}) + (H_1' + H_2'),$$

where $H_{rad}$, $H_1'$ and $H_2'$ are expressible in terms of photon operators $b_{qs}$ and $b_{qs}^\dagger$.

Light Scattering Cross Section in Helium

Light Scattering as a probe of He II has been used by Greytak, et al.,\textsuperscript{5,8,9} and Pike and Vaughan.\textsuperscript{39} In addition to the Brillouin scattering from phonons, the Raman scattering from a pair of rotons has been especially interesting. In this section the cross section of light scattering in liquid helium is obtained, following Fetter.\textsuperscript{14} It is defined per unit length of path as
\[
\frac{1}{\sqrt{2}} \frac{d^2 \sigma_{q'}^s}{d \Omega_{q'} d \Omega'_q} \equiv (d\Omega_{q'} d\Omega'_q)^{-1} \times \left\{ \text{the fraction of incident photons that are scattered into a solid angle } d\Omega_{q'} \text{ with polarization } s' \text{ and frequency between } \omega_{q'} \text{ and } \omega_{q'} + d\omega_{q'} \text{ per unit path length in the scattering sample of He II with volume } \Omega \right\} (8.23)
\]

The initial state of the radiation field with \( N_{q^s} \) photons of mode \( q^s \) and the He II medium in state \( |i\rangle \) of energy \( E_i \) is

\[
|\vec{\bar{\xi}}_i\rangle = |N_{q^s}\rangle |i\rangle, \quad (8.24)
\]

and the final state of the radiation field with \( N_{q^s} - 1 \) photons of mode \( q^s \) and one photon of mode \( q's' \) with the He II medium in the state \( |f\rangle \) with energy \( E_f \) is

\[
|\vec{\bar{\xi}}_f\rangle = b_{\bar{q}^s}^\dagger |N_{\bar{q}^s} - 1\rangle |f\rangle. \quad (8.25)
\]

Both \( |\vec{\bar{\xi}}_i\rangle \) and \( |\vec{\bar{\xi}}_f\rangle \) are eigenstates of the unperturbed Hamiltonian \( \hat{H} + \hat{H}_{\text{rad}} \) in Eq. (8.22). The matrix element of the direct photon interaction Hamiltonian \( H_1' \) between the initial and final states is

\[
M_i \equiv \langle \vec{\bar{\xi}}_f | b_{\bar{q}^s}' s' H_1' | \vec{\bar{\xi}}_i \rangle, \quad (8.26)
\]
which becomes\textsuperscript{14} on the substitution of Eq. (8.16),
\[
\mathcal{M}_1 = -\alpha (2\pi /\Omega') (\omega_{g'} \omega_{g''} N_{g''} / 2) \frac{1}{2} \times \hat{E}_{g''} \cdot \hat{E}_{g'} \langle f | n_{g''} - g'' | \alpha \rangle .
\] (8.27)

The Fourier component $n_{\vec{k}}^-$ of the density operator $n(\vec{r})$ in Eq. (8.11) is given by
\[
n_{\vec{k}}^- = \sum_{\xi=1}^{N} e^{-i \vec{k} \cdot \vec{r}_{\xi}} .
\] (8.28)

The matrix element of the indirect photon interaction Hamiltonian $H_2'$ is
\[
\mathcal{M}_2 = \langle \vec{r}_f | b_{g''}^* \mathcal{H}_2' | \vec{r}_z \rangle ,
\] (8.29)

which becomes\textsuperscript{14} on the substitution of Eq. (8.18),
\[
\mathcal{M}_2 = \frac{1}{2} \alpha^2 (\omega_{g} \omega_{g'} N_{g} / 2) \frac{1}{2} \times \hat{E}_{g} \cdot \hat{E}_{g'} \langle f | n_{g} - g | \alpha \rangle \times \sum_{\lambda \mu} (\hat{E}_{g''}^\dagger)_{\lambda} (\hat{E}_{g'}^\dagger)_{\mu} \left[ T(\vec{k} \vec{r}_{\xi}) \lambda \mu + T(\vec{k} \vec{r}_{\xi}') \lambda \mu \right] .
\] (8.30)

Using Eq. (4.10), the Fourier component of $T(\vec{r})_{\lambda \mu}$ in Eq. (8.19),
\[
T(\vec{k})_{\lambda \mu} = (\epsilon_{\lambda \mu} - 3 \hat{k}_{\lambda} \hat{k}_{\mu}) T(k),
\] (8.31)

is obtained, where
\[
T(k) = -4\pi \int_{0}^{\infty} dr j_2(kr) f(r) r^{-1} .
\] (8.32)
Equation (8.31) is obtained from the expansion of a plane wave in spherical harmonics,
\[ e^{-i \mathbf{k} \cdot \mathbf{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m'=-l}^{l} (-i)^{l} j_{l}^{*}(kr) Y_{l}^{m'}(\hat{r}) Y_{l'}^{m'}(\hat{r}) \]  \hspace{1cm} (8.33)
which gives
\[ \int d\mathbf{r} e^{-i \mathbf{k} \cdot \mathbf{r}} Y_{l}^{m} \gamma f(r) \]
\[ = -4\pi j_{l}(kr) \int_{0}^{\infty} dr j_{l}^{*}(kr) f(r) r^{2} \] \hspace{1cm} (8.34)
If \( f(r) = \theta (r - a) \), where \( \theta \) is the unit step function, one has
\[ T(k) = -4\pi j_{l}(ka) / ka \] \hspace{1cm} (8.35)
which is used by Stephen, Fetter, and Cowley.

The density operator in Eq. (8.11) has the Fourier expansion
\[ n(\mathbf{r}) = \int \frac{d\mathbf{k}}{2\pi} \sum_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{r}} n_{\mathbf{k}} \] \hspace{1cm} (8.36)
and the tensor operator in Eq. (8.19) also has the Fourier expansion
\[ T(\mathbf{r}) \lambda_{\mu} = \int \frac{d\mathbf{k}}{2\pi} \sum_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{r}} T(\mathbf{k}) \lambda_{\mu} \] \hspace{1cm} (8.37)

The cross section defined in Eq. (8.23) can be written as a product of terms
\[ \frac{1}{\sqrt{2}} \frac{d^{2} \sigma_{S}}{d\omega_{S} d\Omega_{S}} = (\rho d\omega_{S} \rho d\Omega_{S})^{-1} ABCD^{-1} \] \hspace{1cm} (8.38)
The number of photon states between $q'$ and $q' + dq'$ is

$$A \equiv 4\pi \lambda^2 dq' \left[ (2\pi)^3 / \mathcal{Q}' \right]^{-1},$$

(8.39)

where $\mathcal{Q}'$ is the volume in which the radiation field is quantized, and $A$ is the same as the number of states between $\omega_q'$ and $\omega_q' + d\omega_q'$ since $\omega_q' = c q'$. The ratio of effective solid angle $d\Omega q'$ to whole solid angle $4\pi$ is

$$B \equiv d\Omega q' (4\pi)^{-1}.$$

(8.40)

The transition probability per unit time between states $|\Xi_i\rangle$ and $|\Xi_f\rangle$ is given by Fermi's Golden Rule:

$$\hat{P}_{fi} = (2\pi/\hbar) |M_1 + M_2|^2 \delta(E_i^* + \hbar \omega_{q_i} - E_f^* - \hbar \omega_{q_f}),$$

(8.41)

where the $\delta$-function insures conservation of energy. Since the experiment does not detect the initial He II state $|\Xi_i\rangle$, an average over the initial states of He II is necessary, which gives

$$C_f = \sum_i P_{i} \hat{P}_{fi}$$

(8.42)

as the average rate of transition to the final state $|f\rangle$. The probability of the initial state $|i\rangle$ is

$$P_{i} = e^{-\beta E_i} \left( \sum_j e^{-\beta E_j} \right)^{-1},$$

(8.43)
which is the Gibbs canonical probability distribution of He II at temperature \( T = (k_B \beta)^{-1} \). The experiment does not detect the final state \( |f\rangle \) of the He II, either, so a sum over all possible final states \( |f\rangle \) must be made, which gives

\[
C = \sum_f C_f = \sum_{fi} P_i \hat{P}_{fi} \tag{8.44}
\]

as the transition rate. The flux of the \( N_{qs} \) incident photons with momentum \( \vec{q} \) and polarization \( s \) is

\[
D = c N^q_s S \tag{8.45}
\]

When Eqs. (8.39) to (8.45) are substituted into the cross section in Eq. (8.38), it takes the form

\[
\frac{1}{J^2} \frac{d^2 \rho}{d\Omega_d d\Omega_q} = \left( \frac{\omega_p^2}{2\pi c} \right)^2 \frac{\omega_p^2}{N_{qs}} \sum_{fi} P_i |M_f + M_k|^2 \\
\times \delta \left( \omega_q - \omega_{q'} - \frac{E_f - E_i}{\hbar} \right). \tag{8.46}
\]

In a two-excitation process, the cross section in Eq. (8.46) should be proportional to the fourth power of the atomic polarizability \( \alpha \), so the matrix element \( M_1 \) in Eq. (8.27) for the direct process should not appear. Hence from Eq. (8.30) the created two-excitation state should be in the angular momentum D-state. The energy transfer of the photons is \( W = \omega_q - \omega_{q'} \), and the momentum transfer of the photons is \( |\vec{Q}| = /\vec{q} - \vec{q}'/ \sim 10^{-3} \text{Å}^{-1} \). When the two excitations created have energy \( E_{k+\vec{Q}} \) and \( E_{-\vec{k}} \), the energy difference in Eq. (8.46) is \( E_f - E_i = E_{k+\vec{Q}} + E_{-\vec{k}} \approx 2E_k \).
Brillouin Scattering

Following Fetter, we now consider the process in which a phonon of momentum \( \vec{k} \) is created in He II by light scattering. In this case the approximation \( M_1 + M_2 \approx M_1 \) can be made in Eq. (8.46). The photon cross section thus becomes

\[
\frac{1}{\mathcal{N}} \frac{d^2 \sigma}{d \omega d \Omega} = n c^2 \left( \frac{\omega_p}{\omega q} \right)^3 \left( \hat{E}_{\vec{q} s} \cdot \hat{E}_{\vec{q} s'} \right)^2 \times S(\vec{q} - \vec{q}', \omega q - \omega q'),
\]

(8.47)

where \( n = N/\Omega \) is the number density of \( \text{He}_4 \) atoms, \( \vec{k} = \vec{q} - \vec{q}' \) is the momentum transfer and \( \omega = \omega_q - \omega q' \) is the energy transfer.

The dynamic liquid structure factor \( S(\vec{k}, \omega) \) is defined by

\[
S(\vec{k}, \omega) = N^f \sum_{\vec{r} f} \left| \langle f | n_{\vec{r}}^f i | i \rangle \right|^2 S(\omega - E_f / \hbar),
\]

(8.48)

where \( E_f \) is the energy of the He II state \( |f\rangle \). The static liquid structure factor is defined as

\[
S(\vec{k}) = \int_{-\infty}^{\infty} S(\vec{k}, \omega) d\omega.
\]

(8.49)

In the long-wavelength limit, Eq. (8.49) for the static liquid structure factor takes the form (see Appendix F)

\[
S(\vec{k}) \approx k_B T / ms^2,
\]

(8.50)

where \( k_B \) is Boltzmann's constant and \( s \) is the speed of first sound. From Eq. (8.47) the integrated Brillouin intensity is
Thus given by

\[
\frac{1}{\sqrt{2}} \frac{d\sigma^{\gamma \rho}_{\hat{q}' \hat{q}}}{d\Omega_{\hat{q}' \hat{q}}} = n a^2 \frac{\omega_q \omega_{q'}}{c^2} \left( \hat{\epsilon}_{\hat{q}s} \cdot \hat{\epsilon}_{\hat{q}'s'} \right)^2 \frac{k_B T}{m \sigma^2}. \tag{8.51}
\]

If the final polarization \(s'\) is not detected, the integrated Brillouin intensity is

\[
\frac{1}{\sqrt{2}} \frac{d\sigma^{\gamma \rho}_{\hat{q}' \hat{q}}}{d\Omega_{\hat{q}' \hat{q}}} = n a^2 \frac{\omega_q \omega_{q'}}{c^2} \left[ 1 - \left( \hat{\epsilon}_{\hat{q}s} \cdot \hat{\epsilon}' \right)^2 \right] \frac{k_B T}{m \sigma^2}, \tag{8.52}
\]

on summing over both polarization states. In the radiation gauge, the polarization vectors are orthogonal to the direction of propagation, which gives the relation

\[
\sum_{s'} \left( \hat{\epsilon}_{\hat{q}s} \cdot \hat{\epsilon}_{\hat{q}'s'} \right)^2 + \left( \hat{\epsilon}_{\hat{q}s} \cdot \hat{\epsilon}' \right)^2 = 1, \tag{8.53}
\]

used in Eq. (8.52).

**Raman Scattering from Free Roton Pairs**

In this section, we consider the creation of a free roton pair with total momentum \(q = q - q' \approx 10^{-3} \text{ Å}^{-1}\) (\(\approx 0\) compared with the roton momentum 1.9 Å\(^{-1}\)) in He II by light scattering at zero temperature.

For \(k \neq 0\), the approximation to the Fourier component of the number density (see Appendix G)

\[
n_{\vec{k}} \approx \zeta + \nu_{\vec{k}} \left( \gamma_{\vec{k}} + \gamma_{-\vec{k}} \right) \tag{8.54}
\]
should be valid if the depletion is not too large. The static liquid structure factor is

\[ N_S(\bar{k}) = \langle 0 | n_{\bar{k}} n_{-\bar{k}} | 0 \rangle \cong \langle \Phi_0 | n_{\bar{k}} n_{-\bar{k}} | \Phi_0 \rangle \]

\[ = q_0^2 (u_{\bar{k}} + v_{\bar{k}})^2 = N \bar{Z}(\bar{k}), \]  \hspace{1cm} (8.55)\n
where \( |0\rangle \) is the true ground-state vector of the He II system, \( |\Phi_0\rangle \) is the bogolon vacuum and \( \bar{Z}(\bar{k}) \) is the single bogolon contribution to the static liquid structure factor \( S(\bar{k}) \). The multi-excitation part of \( S(\bar{k}) \) is neglected in this approximation.

The initial and final states of the He II medium are

\[ |1\rangle = |\Phi_0\rangle \]  \hspace{1cm} (8.56)\n
and

\[ |f\rangle = \gamma_{\bar{k}}^t \gamma_{-\bar{k}}^t |\Phi_0\rangle \]  \hspace{1cm} (8.57)\n
respectively, where use has been made of the relation \( \bar{Q} = q = q' \approx 0 \). Using Eqs. (8.54) and (8.55), we obtain the matrix element

\[ M_z = \left[ 2 \pi \hbar \alpha^2 (w_{\bar{k}} w_{\bar{k}'})^{1/2} N_{\bar{s}}^{1/2} N \bar{Z}(\bar{k}) / \bar{s}' \right] \]

\[ \times 2 \sum_{\lambda \mu} (\hat{E}_{\bar{s}} s)_\lambda (\hat{E}_{\bar{s}'} s')_\mu T(\bar{k})_{\lambda \mu} \]  \hspace{1cm} (8.58)\n
from Eq. (8.30). In Eq. (8.58) we have set \( \bar{k} \pm \bar{q} \approx \bar{k} \), since \( q \sim 10^{-3} \text{Å}^{-1} \ll k \sim 1 \text{Å}^{-1} \) for most of \( k \). Neglecting \( M_1 \), the cross section at zero temperature in Eq. (8.46) for two-quasiparticle (2QP) scattering now takes the form
\[
\frac{1}{\sqrt{2}} \frac{d^2 \sigma_{\xi}^2}{d \omega_\xi d \omega_\xi'} \sum_{k} \tilde{z}(k)^2 T(k)^2 \\
\times [\hat{\xi} \cdot \hat{\xi}' - 3(\hat{\xi} \cdot \hat{k})^2] \delta(\omega_\xi - \omega_\xi' - 2E_k/h),
\]
(8.59)

where \( \hat{\xi} = \frac{\hat{k}}{Q} \) and \( \hat{\xi}' = \frac{\hat{k}'}{Q'} \). Using angular averages 
\(<\ldots> = \int \frac{d^2 k}{4\pi} \ldots \), one has

\[
<\hat{R}_{li} \hat{R}_{lj} > = \frac{1}{3} \delta_{ij}
\]
(8.60)

and

\[
<\hat{R}_{li} \hat{R}_{lj} \hat{R}_{lk} \hat{R}_{lm} > = \frac{1}{15} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl})
\]
(8.61)

where \( \hat{k} = \frac{\hat{k}}{||\hat{k}||} \) is a unit vector. Thus we have the angular average

\[
< T(\hat{k}) \lambda_{\mu'} T(\hat{k}'') \lambda_{\nu'} >=
\]
(8.62)

from Eq. (8.31). Therefore the cross section in Eq. (8.59)

becomes

\[
\frac{1}{\sqrt{2}} \frac{d^2 \sigma_{\xi}^2}{d \omega_\xi d \omega_\xi'} \sum_{k} \frac{\omega_\xi \omega_\xi'}{c^2} \left[ 3 + (\hat{\xi} \cdot \hat{\xi}')^2 \right] \\
\times \int_0^\infty dk k^2 \tilde{z}(k)^2 T(k)^2 \delta(\omega_\xi - \omega_\xi' - 2E_k/h).
\]
(8.63)
In the experiments of Woerner and Greytak the laser beam was incident in the y direction (\( \hat{q} = \hat{y} \)) with polarization in the x direction (\( \hat{\varepsilon}_{q} = \hat{x} \)). The polarization intensity is defined as

\[
I_{\alpha}(\beta) = \left[ \frac{1}{\sqrt{2}} \frac{d^{2} \sigma_{\alpha}^{\beta}}{d\Omega_{q}^{\prime}} \right]_{\hat{q} = \hat{y}, \hat{\varepsilon}_{q} = \hat{x}} \tag{8.64}
\]

where \( \alpha \) and \( \beta \) denote the direction of the final polarization \( \hat{\varepsilon}_{q^\prime} \), and the scattered beam direction \( \hat{q}^\prime \), respectively.

Hence we have the following depolarization ratios

\[
\frac{I_{y}(z)}{I_{x}(z)} = 3/4, \tag{8.65}
\]

\[
\frac{I_{y}(x)}{I_{z}(x)} = 1, \tag{8.66}
\]

and

\[
\frac{I_{y}(x) + I_{z}(x)}{I_{x}(z) + I_{y}(z)} = \frac{6}{7}, \tag{8.67}
\]

which were first derived by Stephen for free roton pairs, and listed in Table I.

By summing Eq. (8.63) over the final polarizations \( s' \), and integrating over the peak centered at \( k_{0} \), the integrated cross section due to noninteracting roton pairs is

\[
\left[ \frac{1}{\sqrt{2}} \frac{d\sigma}{d\Omega_{q}^{\prime}} \right]_{\text{noninteracting roton pairs}} = n^{2} \lambda^{4} (\omega_{q} / c)^{4} \pi^{-2} \left[ k_{0} \varepsilon(k_{0}) \gamma(k_{0}) \right]^{2} \Delta k
\]

\[
\times 5^{-1} \left[ 1 - (\hat{\varepsilon} \cdot \hat{\varepsilon}')^{2} \right], \tag{8.68}
\]
where the factor of \([7 - (\mathbf{q} \cdot \mathbf{q'})^2]\) is obtained from Eq. (8.53), and the width \(\Delta k\) of the peak in momentum space is defined by

\[
\int d(\omega_{q} - \omega_{q'}) dk [K_{\mathbf{k}}(\mathbf{k}) T(k)]^2 \delta(\omega_{q} - \omega_{q'} - 2 E_k / \hbar) \nabla \cdot \mathbf{k}
\]

\[= \Delta k \left[K_{\mathbf{k}} Z(k_0) T(k_0)\right]^2. \quad (8.69)
\]

If Eq. (8.68) is divided by the corresponding Brillouin cross section in Eq. (8.52) for \(\mathbf{q} \cdot \mathbf{q'} = 0\) and the \(T(k)\) in Eq. (8.35) corresponding to a sharp cutoff function, the result is

\[
\frac{I(\text{Raman})}{I(\text{Brillouin})} = (112/5^2) N a^2 Z(k_0)^2 \int_{k_0} (k_0 a)^2 d^2 a \times (m s^2 / k_B T) \Delta k \quad (8.70)
\]

for free roton pairs. This ratio is rather sensitive to the value of the cutoff \(a\). The corresponding ratio for bound roton pairs is investigated in the next chapter.

Density of Roton Pair States

In this section we discuss the relation between the photon cross section and the density of roton pair states. If an average squared matrix element \(\langle |M_2|^2 \rangle\) is used in Eq. (8.46) for the photon cross section in a two-excitation process,\(^6\) and the sum over all possible final states \(|f\rangle\) is converted to an integral, the result is

\[
\frac{1}{\sqrt{2}} d^2 \omega_{\mathbf{q'}} d\omega_{\mathbf{q'}} \frac{2 \alpha^p}{\omega_{\mathbf{q'}}} \langle |M_2|^2 \rangle \int_{\alpha=0}^{\alpha=0} \langle \tilde{f}^{10} | \tilde{a} = 0, W \rangle. \quad (8.71)
\]
The noninteracting density of two-excitation states with total momentum $\vec{q} = \vec{q} - \vec{q}' \neq 0$ is given by

$$\rho^0_2(\vec{q} = 0, W) = \frac{1}{2} \int \frac{d\vec{k}}{(2\pi)^3} \delta (W - E_K - E_{-K}), \quad (8.72)$$

where $E_K = E_{-K}$ is the elementary excitation spectrum and the factor of $1/2$ is due to the indistinguishability of the two rotons. If the interaction between quasiparticles is taken into account, the density of noninteracting two-excitation states $\rho_2^{(0)}(\vec{Q}, W)$ in Eq. (8.71) is replaced by the density of interacting two-excitation states $\rho_2(\vec{Q}, W)$. The type of interaction and the calculation of $\rho_2$ in the RZ theory are discussed in Chapter III. Consequently, the photon cross section in Eq. (8.46) is proportional to $\rho_2(\vec{Q}, W)$. The intensity peak of scattered light from He II can be obtained from the density of two-quasiparticle states $\rho_2(\vec{Q}, W)$ in Eq. (3.41), which is used by Greytak, et al. to interpret their experimental results.\(^5\)

However, the replacement of $|M_2|^2$ by $\langle |M_2|^2 \rangle$ is equivalent to neglecting the coupling between photons and quasiparticles, which according to Cowley\(^3\) is important to the explanation of the peak shape of the scattered light. In the next chapter the coupling of the light to bound roton pairs is discussed.
CHAPTER IX

LIGHT SCATTERING FROM BOUND ROTON PAIRS

In the previous chapter only light scattering from free quasiparticles is considered. In this chapter, it is assumed that bound roton pairs, or more precisely collective modes of two quasiparticles, do in fact form in He II, and their contribution to the photon cross section is calculated. Our treatment in this chapter is new, because no one has included the coupling of bound roton pairs to the light before.

Photon Cross Section

If a collective mode (bound roton pair) of zero total momentum, angular momentum \( \ell \), and magnetic quantum number \( m \) exists in He II, its state (unnormalized) can be described microscopically by a linear superposition of free quasiparticle pairs,

\[
\left| B_{\ell m} \right| \Xi_0 \rangle = \frac{1}{2} \sum_{\vec{k}} \zeta_{\ell m}(\vec{k}) \gamma_{\ell m}^{*}\langle \vec{k} \rangle \gamma_{-\ell m}^{*}\langle -\vec{k} \rangle \Xi_0 \rangle,
\]

(9.1)

where \( \zeta_{\ell m}(\vec{k}) \gamma_{\ell m}(\vec{k}) \) is the (unnormalized) probability amplitude of finding free quasiparticles with momenta \( \vec{k} \) and \( -\vec{k} \) in the collective mode. Equation (9.1) is the simplest special case of Eq. (5.1). Because of the boson commutation relations, only states of even \( \ell \) contribute. The amplitude \( \zeta_{\ell m} \) is determined by the method of normal modes in Eq. (5.2),
\[ [H, B_{\pm m}^+] = W_0 B_{\pm m}^+ \]  

where \( H \) is the Hamiltonian for the He II system in Eq. (4.13) and \( W_0 \) is the excitation energy of the bound quasiparticle (roton) pair.

If the light scattering excites a collective mode in the He II, the initial state is the quasiparticle vacuum,

\[ \left| \Phi_0 \right> = \left| \Phi_0 \right> \]  

and the final state, which has an excitation energy \( W_0 \), is given by

\[ \left| f \right> = \left< \Phi_0 \right| B_{\pm m} B_{\pm m}^+ \left| \Phi_0 \right>^{-1/2} B_{\pm m}^+ \left| \Phi_0 \right> \]  

Since the dipole interaction \( T(\kappa) \) in Eq. (8.31) is equivalent to a spherical tensor \( T_m^{(2)} \), \( m = -2, -1, \ldots, 2 \), of rank two as shown in Chapter VIII, only the D-state \( (\ell = 2) \) contributes in the matrix element \( M_2 \) in Eq. (8.30),

\[
\begin{align*}
M_{2, zm} &= 4 \pi a^2 (\omega \xi \omega') (N_{\xi 5})^{1/2} (2\pi \hbar /\Omega') N \left< \Phi_0 \right| B_{zm} B_{zm}^+ \left| \Phi_0 \right>^{-1/2} \\
& \quad \times (2\pi)^{-3} \int dk \frac{k^2}{4\pi^2} \mathcal{T}_2(k) \mathcal{T}_2(k)^* \mathcal{E}(k) T(k) \\
& \quad \times \left< \frac{\tilde{r}_{2,m}(k)}{2} \right>[\mathcal{E} \cdot \tilde{r} - 3(\mathcal{E} \cdot \tilde{r})(\mathcal{E} \cdot \tilde{r})] \right>,
\end{align*}
\]  

\[
(9.5)
\]
where the notation is the same as in Chapter VIII. The normalization constant squared is

\[
\langle \Phi_0 | B_{2m} B_{2m}^\dagger | \Phi_0 \rangle = \sqrt{2} \left[ 2 (z \pi)^3 \right]^{-1} \int_0^\infty dk \ k^2 / |\xi_{2m}(k)|^2 ,
\]

from Eq. (9.1).

Now the cross section is

\[
\frac{1}{\sqrt{2}} \frac{d^2 \sigma_{5'}}{d\omega_0' d\omega_0} = \left( \frac{Q'}{2 \pi \hbar} \right)^2 \frac{\omega_0'^2}{c^4 \Omega \ N_{\xi_5}} \times \sum_{m=-2}^{2} |M_{2m}|^2 \delta(\omega_0 - \omega_0' - W_0)
\]

from Eq. (8.46). We assume that the amplitude \( \xi_{2m} \) is independent of the magnetic quantum number m. Using the relation (see Appendix H)

\[
4\pi \sum_{m=-2}^{2} \left| \langle \xi_{2m}(k) [\hat{E} \cdot \hat{E}' - 3 (\hat{E} \cdot \hat{R})(\hat{E}' \cdot \hat{R})] \rangle \right|^2
\]

\[
= \frac{1}{5} \left[ 3 + (\hat{E} \cdot \hat{E}')^2 \right],
\]

we obtain

\[
\frac{1}{\sqrt{2}} \frac{d^2 \sigma_{5'}}{d\omega_0' d\omega_0} = \kappa^2 \omega_0' / c \left( \right)^4 \pi^{-2} \times \int_0^\infty dk \ k^2 \ |\xi_{2m}(k)|^2 \left[ \int_0^\infty dk \ k^2 / |\xi_{2m}(k)|^2 \right]^{-1} \times \frac{1}{5} \left[ 3 + (\hat{E} \cdot \hat{E}')^2 \right] \delta(\omega_0 - \omega_0' - W_0)
\]

(9.9)
from Eq. (9.7), which has the same polarization factor as for noninteracting roton pairs in Eq. (8.63). As Table I shows there is no way of distinguishing between noninteracting and bound roton pairs on the basis of their depolarization of the scattered light. However a distinction of the peak shapes between them is possible, since for the roton pair with an attractive interaction there is a contribution from the bound state and also from the continuum states which are neglected here.

In order to evaluate the integrals in the cross section of Eq. (9.9), it is necessary to solve Eq. (9.2) for the probability amplitudes in the bound pair. For a simple model based on a reinterpretation of our results discussed in Chapter VII, this solution is given in the next section.

**Roton-Roton Interaction**

If the effective interaction between rotons is attractive, a bound roton pair forms, as discussed in Chapter VII. In this section a phenomenological approach to the roton-roton interaction is taken, motivated by the results of Chapter VII.

If the Bogoliubov Hamiltonian $H$ in Eq. (4.13) and the collective mode operator $B_{\xi_m}$ in Eq. (9.1) are substituted into Eq. (9.2), the integral equation in Eq. (6.11) without $\zeta_2'$,

$$\zeta_2(k) \propto \Omega (2\pi^2)^{-1} \int_0^\infty d\kappa'\kappa'^2 \int d\mu P_2(\mu) h_{22}(\kappa,-\kappa,\kappa',-\kappa')$$

$$\times (\omega_0 - 2E_\kappa)^{-1} \zeta_2(\kappa'),$$

(9.10)

is obtained.
The scattering of two quasiparticles in the state $\vec{k}$ and $-\vec{k}$ to the state $\vec{k}'$ and $-\vec{k}'$ is described by the vertex function $h_{22}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}')$. The roton-roton potential in configuration space is now phenomenologically taken to be a $\delta$ function shell

$$V_{rr}(r) = g_r \left(4\pi b^2\right)^{-1} \delta(r-b), \quad (9.11)$$

where $g_r$ is the strength of the roton-roton interaction and $b$ is its range. This potential has the advantage of giving bound D-states, as well as S-states, which is not the case for the attractive $\delta$ function contact potential used in the RZ theory.\(^6\) In the phenomenological treatment here the vertex function is

$$h_{22}(\vec{k}, -\vec{k}, \vec{k}', -\vec{k}') = \frac{1}{2} \langle -\vec{k}, -\vec{k}' | V_{rr} | \vec{k}, \vec{k}' \rangle = g_r \left(4\pi b^2\right)^{-1} \left[j_0(g_r b) + j_0(g_r b)\right], \quad (9.12)$$

which is the same form as Eq. (6.3) with $u=1$ and $v=0$. When Eq. (9.12) is substituted into Eq. (9.10), and the integrations performed, the result is

$$\varphi_2(k) = g_r \left(2\pi^2\right)^{-1} \left(w_0 - 2E_k\right)^{-1} j_2(kb) A_2, \quad (9.13)$$

where

$$A_2 = \int_0^\infty dk k^2 j_2(kb) \varphi_2(k) \quad (9.14)$$

is a functional of $\varphi_2$. 
The integral equation in Eq. (9.13) can be converted into an algebraic equation,

\[ A_2 = \frac{g_s}{(2\pi^2)} \int_0^\infty \frac{dk (W_0 - 2E_k)^{-1} k^2 j_z^2(kb)}{A_2}, \]  

(9.15)

by multiplying by \( k^2 j_z^2(kb)(W_0 - 2E_k)^{-1} \) and integrating over \( k \). If \( A_2 \neq 0 \) the secular equation,

\[ g_t^{-1} = \frac{2\pi^2}{(2\pi^2)} \int_0^\infty \frac{dk k^2 j_z^2(kb)(W_0 - 2E_k)^{-1}}{A_2}, \]  

(9.16)

is obtained for the energy \( W_0 \) of the bound state. From the discussion in Chapter VII concerning Table V, the dominant contribution in Eq. (9.16) comes from the vicinity of the roton minimum where \( 2E_k - W_0 = \text{minimum} \). Therefore, only the roton region, where the energy is given by

\[ E_k = E_0 + \hbar^2 (k - k_0)^2 / 2\mu_r, \]  

(9.17)

is used in the evaluation of the integral in Eq. (9.16). The energy of the roton minimum is \( E_0 = 8.67 \, \text{K} \), the momentum at the roton minimum is \( k_0 = 1.9 \, \text{Å}^{-1} \), and the effective mass of the roton is \( \mu_r = 0.16 \, \text{m}_{\text{He}} \).³ When Eq. (9.17) is substituted into Eq. (9.16), and the integral is approximated, the effective coupling strength in the D-state is given in terms of the D-state binding energy \( E_B = 2E_0 - W_0 = 0.34 \, \text{K} \) by

\[ g_t j_z^2(k_0 b) \]

\[ = -2\pi k_0^{-2} \left[ 2E_B \hbar^2 k_0^2 / 2\mu_r \right]^{1/2} \]

(9.18)

\[ = -1.2 \times 10^{-37} \, \text{cm}^2 \, \text{erg} = -8.8 \, \text{Å}^3 \, \text{K}, \]  

(9.19)
which is the same value obtained by Greytak, et al.⁵ The coupling constant \( g_r \) is a rather sensitive function of \( b \), the range of the roton-roton interaction. If Eq. (9.18) is solved for the binding energy, the result is

\[
E_B = \left[ g_r j_2^2(k_0 b) \right]^{\frac{1}{2}} k_0^4 m \left( \frac{2\pi \hbar}{\theta} \right)^{-2}.
\]  

(9.20)

However, the expression for the binding energy in Eq. (3.41) is

\[
E_B = \left[ g (2\pi)^{-1} \right]^{\frac{1}{2}} k_0^4 m \left( \frac{2\pi \hbar}{\theta} \right)^{-2}.
\]  

(9.21)

For our purpose, the form of the probability amplitude in Eq. (9.13) can be used in evaluating the cross section in Eq. (9.9).

Photon Cross Section for the Bound Roton Pair in A Simple Model

The simple model for roton-roton interactions in Eq. (9.11) is used in this section to evaluate the integrals in Eq. (9.9). When Eq. (9.13) is substituted into Eq. (9.9), the principal value integrals would have to be evaluated numerically. However, since the major contribution to the integrals comes from the roton region as in Eq. (9.16), Eq. (9.17) can be used for the energy and the numerator in the integrand can be evaluated at the roton minimum. When this procedure is followed the term \( j_2^2(k_0 b) \) involving the range of the roton-roton interaction cancels out, and the cross section in Eq. (9.9) becomes
If the polarization of the scattered photon is not measured, the $s'$ is summed over both its values. Integrating the result over the $\delta$ function peak, we obtain

$$\frac{1}{\sqrt{2}} \frac{d\sigma}{d\Omega_q} = \frac{e^2}{\hbar^2} \left( \frac{\omega_q}{c} \right)^4 \pi^{-2} \left[ k_0 \tilde{Z}(k_0) T(k_0) \right]^2 \left( \mu_r E_B \right)^{1/2} \left( \frac{2\pi}{\hbar} \right) \times \delta(\omega_q - \omega_q - W_0) \int \frac{1}{5} \left[ 3 + \left( \mathbf{E} \cdot \mathbf{E}' \right)^2 \right]. \quad (9.22)$$

Now we compare this cross section with the one for free roton pairs obtained by Fetter,\textsuperscript{14} neglecting the additional terms due to the convolution approximation for which there is little experimental justification. The ratio of Eqs. (9.23) and (8.68) gives the ratio of the intensities

$$\frac{I_{\text{(bound roton pairs)}}}{I_{\text{(noninteracting roton pairs)}}} = \left[ \frac{2 \pi^2 E_B}{\mu_r \hbar^2 (\Delta k)^2} \right]^{1/2}. \quad (9.24)$$

When the values of $E_B = 0.34$ K and $\Delta k = 0.44 \text{ Å}^{-1}$ are substituted into Eq. (9.24) the ratio is 0.96. Thus, compared to the noninteracting roton case most of the intensity in the interacting case is in the peak.

The ratio of the total intensity of the two roton Raman peak to the Brillouin phonon peak has been measured by Woerner
At 1.2 K they obtain \( \frac{I(\text{Raman})}{I(\text{Brillouin})} = (3.40 \pm 0.08) \times 10^{-4} \). If Eq. (9.23) is divided by the corresponding Brillouin cross section in Eq. (8.52) for \( \hat{\epsilon} \cdot \hat{q'} = 0 \) and the \( T(k) \) in Eq. (8.35) corresponding to a sharp cutoff function, the result is

\[
\frac{I(\text{Raman})}{I(\text{Brillouin})} = \left( \frac{112/5}{2\pi} \right) \frac{d}{\alpha_2} Z(k_0)^2 \frac{j_1(k_0 a)}{k_0 a} a^2 \left( \frac{m s^2}{k_B T} \right)^{1/2} x \left[ \frac{2E_B}{2\mu_r / \hbar^2 k_0^2} \right]^{1/2}.
\]

(9.25)

This ratio is rather sensitive to the value of the cutoff \( \alpha \) used to eliminate the divergent self-induced energy. Values of the ratio for different values of \( \alpha \) between 0 and 10.0 Å are given in Table VIII, where the values \( m=1.144 \times 10^{-24} \text{ gm}, \ s=240 \text{ m/sec}, \ T=1.2 \text{ K}, \ E_B=0.34 \text{ K}, \ k_0=1.9 \text{ Å}^{-1}, \ \mu_r=0.16 \text{ m}, \ Z(k_0)=0.9, \ \alpha=0.2 \text{ Å}^3 \) and \( n=0.0219 \text{ atoms Å}^{-3} \) are used.

Table VIII shows that \( \alpha \approx 2.15 \text{ Å} \) and 3.35 Å fit the experimental ratio of \( (3.40 \pm 0.08) \times 10^{-4} \) very well. The cutoff 2.75 Å is about the distance of closest approach 2.7 Å of the atoms in liquid helium. The interatomic spacing is 3.57 Å. Stephen and Fetter use about half the interatomic spacing 1.7 Å, whereas Cowley finds that his fit to the observed peak shape by a noninteracting roton pair model is best for \( \alpha=1.57 \text{ Å} \).

However from Table VIII these values give a ratio of \( I_R/I_B \) which is too large by an order of magnitude. The sharp cutoff function is not very realistic and contains an adjustable parameter. The theory should be refined to make it more realistic.
CHAPTER X

CONCLUSION

There is no doubt about the existence of bound roton pairs in superfluid helium. A phenomenological theory of bound roton pairs has been proposed by Ruvalds and Zawadowski (RZ). The RZ theory assumes a delta function contact interaction between rotons in configuration space without any microscopic justification. The purpose of this work is to study whether or not such a microscopic justification can be offered by the elementary Bogoliubov theory of superfluid helium when a delta function shell interaction between particles in configuration space is used.

In this dissertation, the relevant experimental and theoretical backgrounds are first reviewed. Then the Bogoliubov theory of a weakly interacting boson system and the linearized equations of motion for the collective modes in superfluid helium are discussed. The secular equations for the energy of a collective mode (with zero total momentum in various angular momentum states) in terms of the particle-particle interaction are derived, solved and compared with the RZ theory. Finally, the theory of light scattering from noninteracting quasi-particles is discussed and our theory of light scattering from bound roton pairs is presented.
On the basis of the simple $\delta$ function shell potential between atoms used here, bound roton pairs in superfluid helium can exist if the particle-particle interaction is effectively attractive. This result contradicts the Bogoliubov theory, in which the particle-particle potential must be repulsive. Thus the model used here is too simple to explain the observed phenomena, but gives some insight into a microscopic approach to the problem. If a more realistic particle-particle potential of a repulsive soft core together with an attractive well is used in the present theory, there would be an infinite number of integral types in the secular equations resulting in great mathematical difficulty.

The original results obtained in this research are (1) the cancellation of the three-bogolon vertices in the linearized equations of motion for high-lying collective modes in Chapter V which is independent of the particle-particle interaction model, (2) the solution of the secular equation in the new Tamm-Dancoff approximation, which takes into account both bogolon pairs and quasihole pairs, for the delta function and (3) the inclusion of the coupling of bound roton pairs to the light in the theory of light scattering from liquid helium in Chapter IX.

For future research, we suggest (1) the renormalization (modification or dressing) of the bare vertex functions which hopefully can give the effectively attractive roton-roton interaction required by the bound state, (2) the use of a more
realistic particle-particle potential, (3) the inclusion of more terms in the collective mode operator $B_{q}^{+}$, (4) the hybridization\textsuperscript{30} of a pair-excitation with a single-excitation, and (5) a more realistic theory of light scattering from liquid helium.
APPENDIX A

SYMBOLS USED

The symbols used in this work, their definitions and the equation numbers in which they first appear are given in this appendix.

TABLE IX
SYMBOLS USED

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<th>Equation Number</th>
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<td>Average occupation number</td>
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<td>$k_B$</td>
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<td>Absolute temperature</td>
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<td>$\beta$</td>
<td>$(k_B T)^{-1}$</td>
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<tr>
<td>$d_n$</td>
<td>Integrals for a D-state</td>
<td>Eq. (6.27)</td>
</tr>
<tr>
<td>$\vec{A}$</td>
<td>Vector potential</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$s$</td>
<td>Photon polarization</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of light</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$\mathcal{S'}$</td>
<td>Quantization volume of the radiation field</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$b_{qs}(t)$</td>
<td>Photon annihilation operator</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$b_{qs}(t)^+$</td>
<td>Photon creation operator</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$\vec{\mathcal{E}}_{qs}$</td>
<td>Unit polarization vector</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$\omega_q$</td>
<td>Photon energy</td>
<td>Eq. (8.2)</td>
</tr>
<tr>
<td>$\overline{E}_0$</td>
<td>Free electric field</td>
<td>Eq. (8.7)</td>
</tr>
<tr>
<td>$\mathcal{P}$</td>
<td>Polarization</td>
<td>Eq. (8.9)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>The He$^4$ atomic polarizability</td>
<td>Eq. (8.10)</td>
</tr>
<tr>
<td>$\overline{E}_D$</td>
<td>Induced dipole field</td>
<td>Eq. (8.10)</td>
</tr>
<tr>
<td>$f$</td>
<td>Cutoff function</td>
<td>Eq. (8.14)</td>
</tr>
<tr>
<td>$T_{\gamma\mu}$</td>
<td>Tensor operator</td>
<td>Eq. (8.18)</td>
</tr>
<tr>
<td>$</td>
<td>\overline{\mathcal{E}}_i\rangle$</td>
<td>Initial state of the liquid helium plus the radiation field</td>
</tr>
<tr>
<td>$</td>
<td>N_{\overline{q}s}\rangle$</td>
<td>State of the radiation field</td>
</tr>
<tr>
<td>$</td>
<td>\overline{\mathcal{E}}_f\rangle$</td>
<td>Final state of the liquid helium plus the radiation field</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Matrix element of the direct interaction</td>
<td>Eq. (8.26)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td>Equation Number</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Matrix element of the indirect interaction</td>
<td>Eq. (8.29)</td>
</tr>
<tr>
<td>$P_1$</td>
<td>Gibbs canonical probability distribution</td>
<td>Eq. (8.43)</td>
</tr>
<tr>
<td>$S(k)$</td>
<td>Static liquid structure factor</td>
<td>Eq. (8.49)</td>
</tr>
<tr>
<td>$\mathbf{1}_{0}\rangle$</td>
<td>True ground-state vector</td>
<td>Eq. (8.55)</td>
</tr>
<tr>
<td>$\mathbf{1}_{\Phi_0}\rangle$</td>
<td>Bogolon vacuum state</td>
<td>Eq. (8.55)</td>
</tr>
<tr>
<td>$Z(k)$</td>
<td>Single bogolon contribution to $S(\mathbf{k})$</td>
<td>Eq. (8.55)</td>
</tr>
<tr>
<td>$\langle \cdots \rangle$</td>
<td>Angular average</td>
<td>Eq. (8.60)</td>
</tr>
<tr>
<td>$I_{\alpha\beta}$</td>
<td>Intensity in direction $\beta$ with polarization direction $\alpha$</td>
<td>Eq. (8.64)</td>
</tr>
<tr>
<td>$k_0$</td>
<td>Roton momentum</td>
<td>Eq. (8.68)</td>
</tr>
<tr>
<td>$\Delta k$</td>
<td>Peak width</td>
<td>Eq. (8.68)</td>
</tr>
<tr>
<td>$I$</td>
<td>Intensity of scattered light</td>
<td>Eq. (8.70)</td>
</tr>
<tr>
<td>$B_{\ell n}$</td>
<td>Collective mode creation operator</td>
<td>Eq. (9.1)</td>
</tr>
<tr>
<td>$V_{rr}$</td>
<td>Roton-roton potential</td>
<td>Eq. (9.11)</td>
</tr>
<tr>
<td>$g_r$</td>
<td>Strength of the roton-roton interaction</td>
<td>Eq. (9.11)</td>
</tr>
<tr>
<td>$b$</td>
<td>Range of the roton-roton interaction</td>
<td>Eq. (9.11)</td>
</tr>
</tbody>
</table>
APPENDIX B

THE COMMUTATOR $[H, B_q^+]$

The calculation of the commutator $[H, B_q^+]$ in Eq. (5.2) is simple and straightforward. As an example, we show $[H_{13}, B_{20}]$ in this appendix, where

$$H_{13} = \sum_{I_2 \neq 34} h_{13}(1, 2, 3, 4)\gamma_1^{+} \gamma_2 \gamma_3 \gamma_4$$

from Eq. (4.14), and

$$B_{20} = \frac{i}{2} \sum_{k} \frac{k}{k} \xi(k+q, -k)\gamma_1^{+} \gamma_2 \gamma_3 \gamma_4$$

from Eq. (5.1). Denoting $\xi \equiv \xi(k+q, -k)$, we have

$$[H_{13}, B_{20}]$$

$$= \frac{i}{2} \sum_{I_2 \neq 34} h_{13}(1, 2, 3, 4)\xi [\gamma_1^{+} \gamma_2 \gamma_3 \gamma_4, \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2].$$

Using Wick's theorem which states that the normal ordering of a product of operators can be obtained from all the possible contractions of that product, one obtains

$$[\gamma_1^{+} \gamma_2 \gamma_3 \gamma_4, \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2]$$

$$= \gamma_1^{+} \gamma_2 \gamma_3 \gamma_4 \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2 - \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2 \gamma_1^{+} \gamma_2 \gamma_3 \gamma_4$$

$$= \gamma_1^{+} \gamma_2 \gamma_3 \gamma_4 \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2 + \gamma_1^{+} \gamma_2 \gamma_3 \gamma_4 \gamma_2^{+} \gamma_3 \gamma_4^{+} \gamma_2.$$
where the dots mean normal ordering. Hence the first term and the last term cancel. Since the pairing or contraction is defined as

\[ \gamma^+_{14} /_{\kappa+\bar{\kappa}} = \langle \Phi_0 | \gamma^+_{14} /_{\kappa+\bar{\kappa}} | \Phi_0 \rangle = \delta_{\kappa+\bar{\kappa}}, \]  

etc., one gets

\[ \{ \gamma^+_{14}, \gamma^+_{13} \} /_{\kappa+\bar{\kappa}} = \delta_{\kappa+\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]

\[ + \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \delta_2,\kappa+\bar{\kappa} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \gamma^+ /_\kappa, \gamma^+ /_{\bar{\kappa}} \]
Since $h_{13}(1,2,3,4)$ is symmetric with respect to the interchange of 2, 3 and 4, Eq. (B3) reduces to

$$\left[ H_{13}, B_{20} \right]$$

$$= 3 \sum_{123 \bar{k}} h_{13} (1,2,3,\bar{k} + \vec{p}) \zeta (\vec{k} + \vec{p}, -\vec{k}) \gamma_1^+ \gamma_2^+ \gamma_3^+ \gamma_3$$

$$+ 3 \sum_{12 \bar{k}} h_{13} (1,2,\bar{k} + \vec{p}, -\vec{k}) \zeta (\vec{k} + \vec{p}, -\vec{k}) \gamma_1^+ \gamma_2$$.

(B7)

The other commutators in Eq. (5.2) are done in a similar manner. Only the terms involving one or two bogolon operators are retained in our approximation.
The integral equations in Eqs. (6.21) and (6.22) can be converted into the eight simultaneous algebraic equations

\[
\begin{pmatrix}
1' - c & 2' & 3' & 4' & 1' & 3' & 2' & 4' \\
2' & 5' - 2c & 6' & 7' & 2' & 6' & 5' & 7' \\
3' & 6' & 8' - 2c & 9' & 3' & 8' & 6' & 9' \\
4' & 7' & 9' & 6' - c & 4' & 9' & 7' & 6' \\
1'' & 2'' & 3'' & 4'' & 1'' + c & 3'' & 2'' & 4'' \\
3'' & 6'' & 8'' & 9'' & 3'' & 8'' + 2c & 6'' & 9'' \\
2'' & 5'' & 6'' & 7'' & 2'' & 6'' & 5'' + 2c & 7'' \\
4'' & 7'' & 9'' & 6'' & 4'' & 9'' & 7'' & 6'' + c
\end{pmatrix}
\begin{pmatrix}
\mathbf{S}_1 \\
\mathbf{S}_2 \\
\mathbf{S}_3 \\
\mathbf{S}_4 \\
\mathbf{S}_5 \\
\mathbf{S}_6 \\
\mathbf{S}_7 \\
\mathbf{S}_8
\end{pmatrix}
= \begin{pmatrix}
2(u\nu, \bar{\nu}_0) \\
(u^2, \bar{\nu}_0) \\
(u^2, \bar{\nu}_0) \\
2(u\nu, \bar{\nu}_0) \\
2(u\nu, \bar{\nu}_0) \\
-2(u\nu, \bar{\nu}_0) \\
-2(u\nu, \bar{\nu}_0) \\
-2(u\nu, \bar{\nu}_0)
\end{pmatrix}
\]

\[ \equiv \mathbf{S} = 0 \tag{C1} \]

where \( n' \equiv s_{n'} \) and \( n'' \equiv s_{n''} \). Hence the secular equation for an S-state is

\[ \det \mathbf{S} = 0 \tag{C2} \]

By eliminating the non-diagonal elements in the last four columns in \( \det \mathbf{S} \), Eq. (6.23) is obtained.
APPENDIX D

THE SECULAR EQUATION FOR A D-STATE

The integral equations in Eqs. (6.25) and (6.26) can be converted into the six simultaneous algebraic equations ($\ell \neq 0$)

\[
\begin{pmatrix}
1' - 2c & 2' & 3' & 1' & 2' & 3' \\
2' & 4' - 2c & 5' & 2' & 4' & 5' \\
3' & 5' & 2' - c & 3' & 5' & 2' \\
2'' & 4'' & 5'' & 2'' & 4'' + 2c & 5'' \\
1'' & 2'' & 3'' & 1'' + 2c & 2'' & 3'' \\
3'' & 5'' & 2'' & 3'' & 5'' & 2'' + c
\end{pmatrix}
\begin{pmatrix}
\xi_{0,1} \\
\xi_{0,2} \\
\xi_{0,3} \\
\xi_{1,1} \\
\xi_{1,2} \\
\xi_{1,3}
\end{pmatrix}
\begin{pmatrix}
-u^2, \gamma_2 \\
\gamma_2, \xi_2 \\
4u\gamma_2, \gamma_2 \\
-u^2, \gamma_2 \\
2u\gamma_2, \gamma_2 \\
-2u\gamma_2, \gamma_2
\end{pmatrix}
\]

\[\equiv \Delta \mathbf{D} = 0 \ , \quad (D1)\]

where $n' \equiv d_n'$ and $n'' \equiv d_n''$. Hence the secular equation for an $\ell$-state ($\ell = 2, 4, 6, \cdots$) is

\[\det \Delta = 0 \ . \quad (D2)\]

By eliminating the non-diagonal elements in the last three columns in $\det \Delta$, Eq. (6.27) is obtained. For a D-state, let $\ell = 2$ in Eq. (6.27).
In order to investigate the existence of the principal value integrals $s_n$ and $d_n$ in Eqs. (6.24) and (6.28), respectively, we consider the integral
\[
I = \left( \int_{k_2' a}^{(k_s-\gamma) a} + \int_{(k_s+\gamma) a}^{k_4' a} \right) \frac{N_x}{W_0 - 2E_x} \, dx,
\]
(E1)

where $N_x$ is a function of $x=ka$. The singularity $x_s=ksa$ is determined by the solution to $W_0 - 2E_x=0$. The values of $k_s-k'_s=k_u-k_s>\gamma$ are sufficiently small to insure the validity of a Taylor expansion of $N_x$ and $E_x$ about the singularity. Using the variable $y=x-x_s$, we have
\[
I = \left( \int_{-L}^{-\epsilon} + \int_{\epsilon}^{L} \right) \left[ \sum_{n=0}^{\infty} \frac{1}{n!} N^{(n)} y^n \right] \\
\times \left[ W_0 - 2 \sum_{n=0}^{\infty} \frac{1}{n!} E^{(n)} y^n \right]^{-1} dy,
\]
(E2)

where $L=(k_s-k'_s)a$, $\epsilon=\eta a$, $N^{(n)}=(d^nN/dy^n)_{y=0}$ and $E^{(n)}=(d^nE/dy^n)_{y=0}$.

Therefore we obtain
\[
I = -\frac{1}{2} \frac{N}{E'} \int y^{-1} dy + \left( \frac{1}{4} \frac{N E''}{E'} - \frac{1}{2} \frac{N'}{E'} \right) \int dy \\
+ \left( \frac{1}{12} \frac{N E^{(iv)}}{E'} - \frac{1}{8} \frac{N E''}{E'} + \frac{1}{4} \frac{N E''}{E'} - \frac{1}{4} \frac{N''}{E'} \right) \int y^2 dy \\
+ \left( \frac{1}{12} \frac{N E^{(iv)}}{E'} - \frac{1}{12} \frac{N E''}{E'} - \frac{1}{12} \left( \frac{N''}{E'} \right)^2 \right) \int y^3 dy + \cdots
\]
(E3)
where
\[ \int \equiv \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{\mathcal{L}}. \] (E4)

In Tables IV and V the integrals \( s_n \) and \( d_n \) are calculated numerically with \( \gamma = 10^{-4} \text{ Å}^{-1} \). The error due to this integration gap can be estimated from Eq. (E3) by letting \( L = (10^{-4} \text{ Å}^{-1}) \)
\[ a = 4.094 \times 10^{-4}. \] Corresponding to \( E_B = 0.37142 \text{ K} \), the corrections to the values in Tables IV and V are listed in Table X in this appendix. For this binding energy, the phonon part of \( s_9 \) (in the NTDA) in Table IV is \(-9.40 \times 10^{-3} \text{ Å}^{-1} \text{ K}^{-1}\) and the correction to it is \(-5.934 \times 10^{-5} \text{ Å}^{-1} \text{ K}^{-1}\) in Table XI, which means an error of 0.6% in the phonon part of \( s_9 \). For the same binding energy, the phonon part of \( d_2 \) (in the NTDA) in Table V is \(-9.5 \times 10^{-3} \text{ Å}^{-1} \text{ K}^{-1}\) and the correction to it is \(-1.157 \times 10^{-6} \text{ Å}^{-1} \text{ K}^{-1}\) in Table XI, which means an error of 0.01% in the phonon part of \( d_2 \).
### TABLE X

THE GAP DEPENDENCE OF THE PRINCIPAL VALUE INTEGRALS*

<table>
<thead>
<tr>
<th>$\eta \times 10^4$ Å</th>
<th>11</th>
<th>9</th>
<th>7</th>
<th>5</th>
<th>3</th>
<th>1</th>
<th>Extrapolation</th>
<th>Calculation**</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-1.307</td>
<td>-1.534</td>
<td>-1.770</td>
<td>-2.024</td>
<td>-2.342</td>
<td>-3.065</td>
<td>-3.6</td>
<td>-2.660</td>
</tr>
<tr>
<td>3</td>
<td>1.779</td>
<td>2.087</td>
<td>2.407</td>
<td>2.750</td>
<td>3.173</td>
<td>4.111</td>
<td>4.8</td>
<td>3.078</td>
</tr>
<tr>
<td>4</td>
<td>-0.6932</td>
<td>-0.8145</td>
<td>-0.9414</td>
<td>-1.080</td>
<td>-1.257</td>
<td>-1.686</td>
<td>-2.0</td>
<td>-1.415</td>
</tr>
<tr>
<td>5</td>
<td>0.9546</td>
<td>1.121</td>
<td>1.295</td>
<td>1.482</td>
<td>1.720</td>
<td>2.278</td>
<td>2.8</td>
<td>1.603</td>
</tr>
</tbody>
</table>

\[ * \int \eta \left( \frac{1}{k_e'} \right)^{\frac{\kappa - \gamma}{a}} \left( \frac{1}{k_u'} \right)^{\frac{\kappa u}{a}} \int_{k_e'}^{k_u'} f_n \, dx \times (10^5 \text{ Å K}) \], where $E_B = 0.37142$ K, $k_a = 0.4453$ Å$^{-1}$, $k_{a}' = 0.4430$ Å$^{-1}$, $k_{u}' = 0.4476$ Å$^{-1}$ and $f_n$ are the integrands defined in Eq. (6.28) with $\mathcal{E} = \frac{1}{2}$ and $c_0 = 0$.

**Calculated by the method derived in App. E.**
TABLE XI
THE PRINCIPAL VALUE INTEGRALS<sup>a</sup>

<table>
<thead>
<tr>
<th>Type&lt;sup&gt;b&lt;/sup&gt;</th>
<th>N&lt;sup&gt;c&lt;/sup&gt;</th>
<th>N'&lt;sup&gt;d&lt;/sup&gt;</th>
<th>I in Å&lt;sup&gt;-1&lt;/sup&gt; K&lt;sup&gt;-1&lt;/sup&gt;&lt;sup&gt;e&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>3.583</td>
<td>-1.285</td>
<td>2.145 x 10⁻⁶</td>
</tr>
<tr>
<td>s₂</td>
<td>-3.944</td>
<td>1.510</td>
<td>-5.123 x 10⁻⁶</td>
</tr>
<tr>
<td>s₃</td>
<td>-2.448</td>
<td>2.399</td>
<td>-4.554 x 10⁻⁵</td>
</tr>
<tr>
<td>s₄</td>
<td>3.107</td>
<td>-2.115</td>
<td>3.087 x 10⁻⁵</td>
</tr>
<tr>
<td>s₅</td>
<td>4.341</td>
<td>-1.764</td>
<td>8.594 x 10⁻⁶</td>
</tr>
<tr>
<td>s₆</td>
<td>2.695</td>
<td>-2.702</td>
<td>5.190 x 10⁻⁵</td>
</tr>
<tr>
<td>s₇</td>
<td>-3.420</td>
<td>2.408</td>
<td>-3.630 x 10⁻⁵</td>
</tr>
<tr>
<td>s₈</td>
<td>1.672</td>
<td>-2.677</td>
<td>6.120 x 10⁻⁵</td>
</tr>
<tr>
<td>s₉</td>
<td>-2.123</td>
<td>2.765</td>
<td>-5.934 x 10⁻⁵</td>
</tr>
<tr>
<td>d₁</td>
<td>1.572 x 10⁻²</td>
<td>6.859 x 10⁻²</td>
<td>-3.336 x 10⁻⁶</td>
</tr>
<tr>
<td>d₂</td>
<td>9.757 x 10⁻³</td>
<td>3.661 x 10⁻²</td>
<td>-1.157 x 10⁻⁶</td>
</tr>
<tr>
<td>d₃</td>
<td>1.238 x 10⁻²</td>
<td>-5.037 x 10⁻²</td>
<td>1.338 x 10⁻⁶</td>
</tr>
<tr>
<td>d₄</td>
<td>6.055 x 10⁻³</td>
<td>1.919 x 10⁻²</td>
<td>-6.153 x 10⁻⁷</td>
</tr>
<tr>
<td>d₅</td>
<td>7.687 x 10⁻³</td>
<td>-2.665 x 10⁻²</td>
<td>6.971 x 10⁻⁷</td>
</tr>
</tbody>
</table>

<sup>a</sup> Based on Eq. (E6) in App. E with E<sub>B</sub>=0.37142 K and L=4.094 x 10⁻⁴.

<sup>b</sup> sₙ and dₙ are defined by Eqs. (6.24) and (6.28) with c<sub>0</sub>=0, respectively.

<sup>c</sup>,<sup>d</sup> For definitions of N and N', see App. E.
APPENDIX F

STATIC LIQUID STRUCTURE FACTOR IN THE LONG WAVELENGTH LIMIT

The static liquid structure factor $S(k)$ has been studied for He II by Feynman and Cohen. In this appendix we review its form in the long wavelength limit. Using the normal coordinates

$$n_k = \int d\vec{r} \ e^{i\vec{k} \cdot \vec{r}} \rho(\vec{r}),$$  \hspace{1cm} (F1)

where $\rho(\vec{r})$ is the density of the liquid helium, the energy of the liquid can be written as $E = \sum_k E_k$, where $E_k$ takes the harmonic oscillator form

$$E_k = \frac{1}{2} m_k (\hat{n}_k^2 + \omega_k^2 n_k^2 n_k^2).$$  \hspace{1cm} (F2)

In Eq. (F2), $\omega_k = sk$ is the phonon spectrum and $m_k$ is a quantity to be determined.

By the virial theorem on the thermal average of the energy of a harmonic oscillator,

$$\frac{1}{2} m_k \langle \hat{n}_k^2 \rangle = \frac{1}{2} m_k \omega_k^2 \langle n_k^2 n_k^2 \rangle = \frac{1}{2} \langle E_k \rangle.$$  \hspace{1cm} (F3)
one can express $S(k)$ in terms of $\langle E_k \rangle$, namely

$$S(k) = N^{-1} \langle n_k n_k^* \rangle = \left( N m_k \omega_k^2 \right)^{-1} \langle E_k \rangle.$$  \hspace{1cm} (F4)

In Eq. (F4), $N$ is the total number of particles in the liquid. Using the energy levels

$$E_{k,n} = (n + \frac{1}{2}) \hbar \omega_k,$$  \hspace{1cm} (F5)

the partition function is

$$Z_k = \sum_{n=0}^{\infty} e^{-\beta E_{k,n}} = e^{-\beta \hbar \omega_k / 2} (1 - e^{-\beta \hbar \omega_k})^{-1},$$  \hspace{1cm} (F6)

and the average energy is

$$\langle E_k \rangle = -\frac{\partial}{\partial \beta} \ln Z_k = \frac{1}{2} \hbar \omega_k \coth \left( \frac{1}{2} \beta \hbar \omega_k \right).$$  \hspace{1cm} (F7)

From Eqs. (F4) and (F7), one obtains

$$S(k) = \hbar \left( 2 N m_k \omega_k \right)^{-1} \coth \left( \frac{\beta \hbar \omega_k}{2} \right),$$  \hspace{1cm} (F8)

which reduces to

$$S(k) \propto \left( \beta N m_k s^2 k^2 \right)^{-1}$$  \hspace{1cm} (F9)

as $k \to 0$, and to

$$S(k) \propto \frac{\hbar}{2 N m_k s k}$$  \hspace{1cm} (F10)

as $T \to 0$. 
Now we determine the quantity $m_k$. By the f-sum rule,\footnote{72}

$$\int_0^\infty \omega S(k, \omega) d\omega = Nk^2 \hbar / 2m,$$\hspace{1cm} (F11)

one has

$$\sum_n \omega_n <\phi|n\phi^+|n\phi|\phi> = Nk^2 \hbar / 2m,$$\hspace{1cm} (F12)

where $\{|n\rangle\}$ is a complete set of energy eigenstates. In the case where only one state exhausts the f-sum rule, Eq. (F12) can be written as

$$S(k) = Nk^2 \hbar / 2m,$$\hspace{1cm} (F13)

which can be solved for $S(k)$ and substituted into Eq. (F10) to obtain

$$m_k = m / Nk^2.$$\hspace{1cm} (F14)

Substituting this quantity for $m_k$ into Eq. (F9), one has

$$S(k) \propto k_B T / m s^2,$$\hspace{1cm} (F15)

which is the same as Eq. (8.50).
APPENDIX G

NUMBER DENSITY

In this appendix we express the number density \( n_k \) in terms of bogolon operators \( \gamma_k \) and \( \gamma_k^\dagger \). Equation (4.2) for \( \gamma(\vec{r}) \) gives for the density operator

\[
\gamma^\dagger(\vec{r}) \gamma(\vec{r}) = \mathcal{S}^{-1} \sum_k e^{i \vec{k} \cdot \vec{r}} \left( \sum_{k'} a_{k' \vec{R}}^\dagger a_{k' \vec{R}} \right), \tag{G1}
\]

and Eq. (4.11) for \( n(\vec{r}) \) gives

\[
n(\vec{r}) = \mathcal{S}^{-1} \sum_{\vec{R}} \sum_{l=1}^N \int \frac{d\vec{k}}{(2\pi)^3/2} \ e^{i \vec{k} \cdot (\vec{r} - \vec{R})} = \mathcal{S}^{-1} \sum_{\vec{R}} e^{i \vec{K} \cdot \vec{F}} n_{\vec{K}}, \tag{G2}
\]

where

\[
n_{\vec{K}} = \sum_{\vec{R}} e^{-i \vec{K} \cdot \vec{R}}. \tag{G3}
\]

In the \( N \)-particle subspace, we have \( n(\vec{r}) = \gamma^\dagger(\vec{r}) \gamma(\vec{r}) \), namely

\[
n_{\vec{K}} = \sum_{\vec{R'}} \sum_{\vec{R}} a_{\vec{R} \vec{K}}^\dagger a_{\vec{R'} \vec{K}} = n_{\vec{K}}^+, \tag{G4}
\]

where \( a_{\vec{K}} \) and \( a_{\vec{K}}^\dagger \) are He\(^4\) atom operators. In terms of bogolon operators, \( n_{\vec{K}} \) becomes

\[
n_{\vec{K}} = (g_0^2 + \sum_{\vec{R}} v_{\vec{K}, \vec{R}}^2) s_{\vec{K}, 0} + g_0 (u_{\vec{K}} + v_{\vec{K}}) (\gamma_{\vec{K}} + \gamma_{\vec{K}}^+) + \sum_{\vec{R}} \left[ u_{\vec{K}} v_{\vec{K} \vec{R}} \gamma_{\vec{K} \vec{R}}^\dagger \gamma_{\vec{K}}^+ + u_{\vec{K} \vec{R}} v_{\vec{K} \vec{R}}^\dagger \gamma_{\vec{K} \vec{R}}^+ \gamma_{\vec{K} \vec{R}}^\dagger + u_{\vec{K} \vec{R}}^\dagger v_{\vec{K} \vec{R}}^\dagger \gamma_{\vec{K}}^+ \gamma_{\vec{K} \vec{R}}^\dagger \right], \tag{G5}
\]

127
on substituting Eq. (4.11) into Eq. (G4) and normal ordering the bogolon operators.

For $k \neq 0$, the approximation

$$n_K \approx g_o (u_K + v_K) \left( \frac{1}{2} + \frac{1}{2} \right)$$

should be valid if the depletion is not too large. In this approximation, the single-bogolon state

$$\left\langle \mathcal{F}_0 | n_{-K} \frac{1}{2} n_{-K} \frac{1}{2} \left| \mathcal{F}_0 \right\rangle \right. \approx \left. \left\langle \mathcal{F}_0 | n_{-K} \frac{1}{2} n_{-K} \frac{1}{2} \right| \mathcal{F}_0 \right\rangle^{-1/2} \frac{1}{2} n_{-K} \frac{1}{2} \left| \mathcal{F}_0 \right\rangle$$

is approximately equal to a state with a single density fluctuation, where $\left| \mathcal{F}_0 \right\rangle$ is the bogolon vacuum. The two-bogolon state

$$\left\langle \mathcal{F}_0 | n_{-K} \frac{1}{2} n_{-K} \frac{1}{2} \left| \mathcal{F}_0 \right\rangle \right. \approx \left. \left\langle \mathcal{F}_0 | n_{-K} \frac{1}{2} n_{-K} \frac{1}{2} \right| \mathcal{F}_0 \right\rangle^{-1/2} \frac{1}{2} n_{-K} \frac{1}{2} \left| \mathcal{F}_0 \right\rangle$$

is approximately equal to a state of two density fluctuations.
APPENDIX H

THE ANGULAR AVERAGE

In this appendix we calculate the quantity

\[ J = 4\pi \sum_{m=-2}^{2} \left| \langle Y_2^m (\hat{K}) | \hat{\mathbf{E}} \cdot \hat{\mathbf{E'}} - 3 (\hat{\mathbf{E}} \cdot \hat{\mathbf{K}})(\hat{\mathbf{E'}} \cdot \hat{\mathbf{K}}) \rangle \right|^2 \]

\[ = 36\pi \sum_{m=-2}^{2} \left| \hat{\mathbf{E}}_\mu \hat{\mathbf{E'}}_\nu \langle Y_2^m (\hat{\mathbf{K}}) \hat{\mathbf{K}}_\mu \hat{\mathbf{K}}_\nu \rangle \right|^2 \]  \hspace{1cm} (H1)

where the angular average is \( \langle \cdots \rangle = (4\pi)^{-1} \int \cdots \), which occurs in the derivation of Eq. (9.7). The spherical harmonics are

\[ Y_2^0 (\hat{\mathbf{K}}) = \left( \frac{5}{16\pi} \right)^{1/2} (3 \hat{\mathbf{K}}_x \hat{\mathbf{K}}_y - 1) \]  \hspace{1cm} (H2)

\[ Y_2^\pm (\hat{\mathbf{K}}) = \pm \left( \frac{15}{8\pi} \right)^{1/2} (\hat{\mathbf{K}}_x \pm \hat{\mathbf{K}}_y) \hat{\mathbf{K}}_z \]  \hspace{1cm} (H3)

and

\[ Y_2^\pm (\hat{\mathbf{K}}) = \left( \frac{15}{32\pi} \right)^{1/2} (\hat{\mathbf{K}}_x \pm \hat{\mathbf{K}}_y)^2 \]  \hspace{1cm} (H4)

in terms of the cartesian components \( \hat{\mathbf{K}}_\mu, \mu = x, y, z \), of the unit vector \( \hat{\mathbf{k}} \). Using Eqs. (8.60) and (8.61), we have

\[ \langle Y_2^0 (\hat{\mathbf{K}}) \hat{\mathbf{K}}_\mu \hat{\mathbf{K}}_\nu \rangle \hat{\mathbf{E}}_\mu \hat{\mathbf{E'}}_\nu \]

\[ = \left( \frac{5}{16\pi} \right)^{1/2} \frac{7}{15} (3 \hat{\mathbf{E}}_z \hat{\mathbf{E'}}_z - \hat{\mathbf{E}} \cdot \hat{\mathbf{E'}}) \]  \hspace{1cm} (H5)
Substituting Eqs. (H5), (H6) and (H7) into Eq. (H1), we obtain

\[ J = \frac{l^2}{\beta} \left[ 3 + (\hat{\mathbf{E}} \cdot \hat{\mathbf{E}}')^2 \right], \tag{H8} \]

which is the same polarization factor as obtained in Eq. (8.63) for noninteracting roton pairs.
**TABLE I**

DEPOLARIZATION RATIOS FOR RAMAN SCATTERING

<table>
<thead>
<tr>
<th>Ratio(a)</th>
<th>Experimental(b)</th>
<th>Fetter(c)</th>
<th>Stephen(d)</th>
<th>Present Work(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_y(z)/I_x(z))</td>
<td>0.740 ± 0.020</td>
<td>0.69</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>(I_y(x)/I_z(x))</td>
<td>1.001 ± 0.020</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>([I_y(x)+I_z(x)]/[I_x(z)+I_y(z)])</td>
<td>0.845 ± 0.015</td>
<td>0.82</td>
<td>0.857</td>
<td>0.857</td>
</tr>
</tbody>
</table>

\(a\)\(I_\alpha(\beta)\) is the intensity of scattered light in direction \(\beta\) with polarization direction \(\alpha\). For incident light \(\beta=-\gamma\) and \(\alpha=x\).

\(b\)From Ref. 68.

\(c\)From Ref. 14 for free roton pairs.

\(d\)From Ref. 61 for free roton pairs.

\(e\)For bound roton pairs.
TABLE II
THE COEFFICIENTS IN THE BOGOLON HAMILTONIAN*

\[ H_{00} = -\psi_0^2 + \Sigma(e_1 - \mu)\psi_1^2 + \frac{1}{2}\langle 00|V|00\rangle\psi_0^4 + \psi_0^2\Sigma\langle 00|V|-11\rangle u_1 v_1 + 2\psi_0^2\Sigma\langle 01|V|10\rangle v_1^2 + \frac{1}{2}\Sigma \langle 1 - 1|V|-22\rangle u_1 v_1 u_2 v_2 + \Sigma\langle 12|V|21\rangle v_1^2 v_2^2. \]

\[ h_{01} = [-\mu + \Sigma \langle 1 - 1|V|00\rangle u_1 v_1 + 2\Sigma\langle 01|V|10\rangle v_1^2 + \langle 00|V|00\rangle\psi_0^2]\phi_0(u_0 + v_0). \]

\[ h_{11}(1,1) = U_1(u_1^2 + v_1^2) + \Delta_1 u_1 v_1. \]

\[ h_{20}(1,-1) = \frac{1}{2}[U_1(2u_1 v_1 + \Delta_1(u_1^2 + v_1^2))]. \]

where

\[ \Delta_1 = \Sigma\langle 2 - 2|V|1 - 1\rangle(u_2 v_2 + \psi_0^2) \]

and

\[ U_1 = e_1 - \mu + 2\Sigma\langle 12|V|21\rangle(v_2^2 + \psi_0^2). \]

\[ h_{30}(123) = (\psi_0/3)\langle 12|V|30\rangle u_1 u_2 v_3 + (2\leftrightarrow 3) + (1\leftrightarrow 3) + (u\leftrightarrow v). \]

\[ h_{21}(123) = \psi_0(\langle 1 - 3|V|-20\rangle v_1 u_2 u_3 + (1\leftrightarrow 2)) + \langle 12|V|30\rangle v_1 v_2 v_3 \]

\[ + (u\leftrightarrow v). \]

\[ h_{22}(1234) = \frac{1}{2} \langle 12|V|34\rangle u_1 u_2 u_3 u_4 + \langle 1 - 3|V|-24\rangle u_1 v_2 v_3 u_4 \]

\[ + (3\leftrightarrow 4). \]

\[ h_{31}(1234) = \frac{1}{3} \langle 12|V|34\rangle u_1 u_2 v_3 u_4 + (2\leftrightarrow 3) + (1\leftrightarrow 3) + (u\leftrightarrow v). \]

\[ h_{40}(1234) = \frac{2}{4!} \langle 12|V|34\rangle u_1 u_2 v_3 v_4 + (2\leftrightarrow 3) + (2\leftrightarrow 4) + (u\leftrightarrow v). \]

*Taken from Table II of Ref. 42. For definitions see Eqs. (4.13) and (4.14).
### TABLE III

**MOMENTUM DEPENDENCE OF THE ELEMENTARY EXCITATION ENERGY SPECTRUM**

<table>
<thead>
<tr>
<th>k in Å⁻¹</th>
<th>E_k(expt) in K*</th>
<th>E_k in K**</th>
<th>Deviation</th>
<th>k in Å⁻¹</th>
<th>E_k(expt) in K*</th>
<th>E_k in K**</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.70 ± 0.5</td>
<td>3.741</td>
<td>1.10</td>
<td>1.9</td>
<td>8.70 ± 0.20</td>
<td>8.099</td>
<td>-6.91</td>
</tr>
<tr>
<td>0.3</td>
<td>5.65 ± 0.20</td>
<td>5.523</td>
<td>-2.25</td>
<td>2.0</td>
<td>8.95 ± 0.20</td>
<td>8.538</td>
<td>-4.60</td>
</tr>
<tr>
<td>0.4</td>
<td>7.40 ± 0.20</td>
<td>7.198</td>
<td>-2.73</td>
<td>2.1</td>
<td>10.00 ± 0.20</td>
<td>9.939</td>
<td>-0.61</td>
</tr>
<tr>
<td>0.5</td>
<td>9.15 ± 0.20</td>
<td>8.734</td>
<td>-4.55</td>
<td>2.2</td>
<td>11.65 ± 0.25</td>
<td>12.15</td>
<td>4.29</td>
</tr>
<tr>
<td>0.6</td>
<td>10.75 ± 0.20</td>
<td>10.10</td>
<td>-6.05</td>
<td>2.3</td>
<td>13.55 ± 0.25</td>
<td>14.97</td>
<td>10.5</td>
</tr>
<tr>
<td>0.7</td>
<td>11.75 ± 0.20</td>
<td>11.27</td>
<td>-4.09</td>
<td>2.4</td>
<td>15.50 ± 0.30</td>
<td>18.20</td>
<td>17.4</td>
</tr>
<tr>
<td>0.8</td>
<td>12.65 ± 0.20</td>
<td>12.21</td>
<td>-3.48</td>
<td>2.5</td>
<td>16.45 ± 0.5</td>
<td>21.74</td>
<td>32.2</td>
</tr>
<tr>
<td>0.9</td>
<td>13.15 ± 0.20</td>
<td>12.91</td>
<td>-1.83</td>
<td>2.6</td>
<td>17.0 ± 0.5</td>
<td>25.51</td>
<td>50.1</td>
</tr>
<tr>
<td>1.0</td>
<td>13.55 ± 0.25</td>
<td>13.35</td>
<td>-1.48</td>
<td>2.7</td>
<td>17.3 ± 0.5</td>
<td>29.55</td>
<td>70.2</td>
</tr>
<tr>
<td>1.1</td>
<td>13.80 ± 0.30</td>
<td>13.52</td>
<td>-2.03</td>
<td>2.8</td>
<td>17.5 ± 0.5</td>
<td>33.48</td>
<td>91.3</td>
</tr>
<tr>
<td>1.2</td>
<td>13.75 ± 0.25</td>
<td>13.40</td>
<td>-2.55</td>
<td>2.9</td>
<td>17.7 ± 0.6</td>
<td>37.59</td>
<td>122</td>
</tr>
<tr>
<td>1.3</td>
<td>13.50 ± 0.25</td>
<td>13.02</td>
<td>-3.56</td>
<td>3.0</td>
<td>17.85 ± 0.8</td>
<td>41.73</td>
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<tr>
<td>1.4</td>
<td>12.95 ± 0.20</td>
<td>12.57</td>
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<td>3.1</td>
<td>18.00 ± 0.8</td>
<td>45.87</td>
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<td>1.5</td>
<td>12.20 ± 0.20</td>
<td>11.50</td>
<td>-5.74</td>
<td>3.2</td>
<td>18.15 ± 0.8</td>
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<td>1.6</td>
<td>11.20 ± 0.20</td>
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<td>3.3</td>
<td>18.30 ± 0.8</td>
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<td>3.4</td>
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<tr>
<td>1.8</td>
<td>9.25 ± 0.20</td>
<td>8.499</td>
<td>-8.12</td>
<td>3.5</td>
<td>18.40 ± 1.4</td>
<td>61.95</td>
<td>237</td>
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</table>

*From Ref. 3 at 1.1 K.

**Empirical formula of Eqs. (4.54) and (4.56).
<table>
<thead>
<tr>
<th>$E_B$ in K</th>
<th>n</th>
<th>NTD Approximation**</th>
<th>TD Approximation***</th>
</tr>
</thead>
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<td>Roton*</td>
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<td>-2.183</td>
</tr>
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<td>- 0.067</td>
<td>1.108</td>
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<td></td>
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<td>- 0.830 x 10^{-1}</td>
<td>2.020 x 10^{-1}</td>
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<tr>
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<td>4</td>
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<td>3.755 x 10^{-1}</td>
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<td>1.445 x 10^{-1}</td>
<td>-6.285 x 10^{-1}</td>
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<td>-1.883 x 10^{-1}</td>
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<td>-3.25 x 10^{-2}</td>
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<tr>
<td>E_B in K</td>
<td>n</td>
<td>NTD Approximation**</td>
<td>TD Approximation***</td>
</tr>
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<td>----</td>
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</table>

<p>| 0.10072 | 1  | -0.214  | -5.137 | 0.025  | -5.035 |
|         | 2  | 0.122   | 2.569  | 0.049  | 2.563  |
|         | 3  | 0.322 x 10^-1 | 3.875 x 10^-1 | 2.197 x 10^-1 | 3.184 x 10^-1 |
|         | 4  | -0.424 x 10^-1 | 8.959 x 10^-1 | 1.628 x 10^-1 | 8.807 x 10^-1 |
|         | 5  | -0.162  | -1.352 | 0.075  | -1.352 |
|         | 6  | 0.617 x 10^-1 | -1.661 x 10^-1 | 2.443 x 10^-1 | -1.621 x 10^-1 |
|         | 7  | -0.314 x 10^-1 | -4.459 x 10^-1 | 1.882 x 10^-1 | -4.450 x 10^-1 |
|         | 8  | 0.378 x 10^-2 | -9.694 x 10^-2 | 26.98 x 10^-2 | -2.06 x 10^-2 |
|         | 9  | 4.940 x 10^-2 | -6.542 x 10^-2 | 26.68 x 10^-2 | -5.620 x 10^-2 |</p>
<table>
<thead>
<tr>
<th>$E_B$ in K</th>
<th>$n$</th>
<th>( s_n ) in ( \text{Å}^{-1} \text{K}^{-1} )</th>
</tr>
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<tr>
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<td>-1.874 x 10^{-1}</td>
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</table>

*The \( s_n \) are defined in Eq. (6.24). For calculations, see Chapter VII.

**The new Tamm-Dancoff approximation (NTDA) is given in Eq. (6.24) with \( c_0 = 1 \).

***The Tamm-Dancoff approximation (TDA) is given in Eq. (6.24) with \( c_0 = 0 \).
<table>
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<tr>
<th>$E_B$ in K</th>
<th>$n$</th>
<th>NTD Approximation**</th>
<th>TD Approximation***</th>
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<td>Roton*</td>
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<td>(n)</td>
<td>(d_n) in (\text{Å}^{-1} \text{K}^{-1})</td>
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**NTD Approximation**

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**TD Approximation**

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*The $d_n$ are defined in Eq. (6.28). For calculations, see Chapter VII.

**The new Tamm-Dancoff approximation (NTDA) is given in Eq. (6.28) with $c_0=1$.

***The Tamm-Dancoff approximation (TDA) is given in Eq. (6.28) with $c_0=0$. 
TABLE VI
THE INTERACTION STRENGTH FOR S-STATE*

<table>
<thead>
<tr>
<th>$E_B$ in K</th>
<th>NTD Approximation</th>
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<th>3</th>
<th>4</th>
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</table>

*Values given are $\lambda_n^{(s)} = \pi^2a_3/c$ in units of $KA^3$ given in Eq. (6.23). Calculated by means of the subroutine "MAOIO: POLYNOMIAL ROOT SOLUTION" in the Computer Center of North Texas State University.
TABLE VI—Continued

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TABLE VII

THE INTERACTION STRENGTH FOR D-STATE*

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<th>TD Approximation</th>
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*Values given are $\chi^{(D)}_n = \pi^2 a^3/c$ in units of K Å³ given in Eq. (6.27).

Calculated by means of the subroutine "MAOIO: POLYNOMIAL ROOT SOLUTION" in the Computer Center of North Texas State University.
<table>
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<th>a in Å</th>
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*$I_R/I_B$ is defined in Eq. (9.25).*
Fig. 1--Pressure-temperature phase diagram of He\textsuperscript{4} from the first book in Ref. 1.
Fig. 2--The apparent viscosity coefficient $\eta$ of liquid He$^4$ as measured by flow through narrow channels (curve 1), and by rotating cylinder viscometer (curve 2) from the first book in Ref. 1. Above the $\lambda$-point, the two methods give essentially the same results.
Fig. 3--The elementary excitation spectrum of He II proposed by Landau (curve 1), the elementary excitation spectrum obtained by neutron scattering at 1.1 K (dotted curve) from Ref. 3, the spectrum calculated by means of quantum hydrodynamics (curve 2) from Ref. 34, and the empirically fitted spectrum (curve 3) from Ref. 11.
Fig. 4--The elementary excitation spectrum of He II (curve 1) and the collective mode (curve 2 flanked by dashed lines) obtained by neutron scattering at 1.1 K from Ref. 3. The collective mode with essentially zero total momentum (the point marked by x) is obtained by light scattering from Ref. 5.
Fig. 5--The dynamic liquid structure factor $S(q,W)$ of He$^4$ plotted against energy $W$ for momentum $q \approx 1.57 \, \text{Å}^{-1}$ at 1.1 K from Ref. 3. The dashed line denotes the energy at which there is a change of scale.
Fig. 6--The Raman spectrum of liquid He\textsuperscript{4} at 1.16 K obtained by Greytak and Yan in Ref. 8. The scattering angle is 90° in a direction parallel to the electric field vector of the incident light, and the instrumental width is 4.2 K.
Fig. 7--The Raman spectrum (solid curve R) and the Brillouin peak (peak B with reduction factor 0.003) of liquid He$^4$ at 1.2 K obtained by Greytak, et al. in Ref. 5. The dotted curve corresponds to a bound roton pair model and the broken line corresponds to a noninteracting roton pair. The instrumental resolution is 0.17 K. The energy of two free rotons is $2\varepsilon_0$. 
A collective mode of momentum $\bar{q}$

The creation of a collective excitation by neutron or photon scattering

A bogolon (or quasihole) of momentum $\bar{q}$

Fig. 8--The diagram dictionary for the linearized equation of motion.
Fig. 9--The graphical equation for \( \mathcal{G} \) in Eq. (5.15).
Fig. 10--The graphical equation for $\chi$ in Eq. (5.16).
Fig. 11--The graphical equation for $\xi$ in Eq. (5.17).
Fig. 12--The graphical equation for $\gamma$ in Eq. (5.18).
Fig. 13--The graphical equation for $\zeta$ in Eq. (5.19).
Fig. 14--The graphical equation for the compensation of the lowest order dangerous diagrams in Eq. (5.20).
Fig. 15--The graphical equation for $\mathcal{F}$ in Eq. (5.23).
Fig. 16--The graphical equation for $Z$ in Eq. (5.24).
Fig. 17--The two coupled equations in Eqs. (5.30) and (5.31) for $\xi$ and $\eta$. 
Fig. 18--The renormalized vertex $h_{22}$ for quasiparticle scattering.
REFERENCES AND NOTES


33. T.J. Greytak, private communication.


41. See Ref. 40, p. 12.
49. See, e.g., J. Wilks in Ref. 1, p. 666.
53. By equating the coefficients of the c-numbers on both sides of the commutation relation, Eq. (5.14) is obtained.
56. The subroutine "DQTFE" in the Computer Center of North Texas State University.
58. G.W. Goble, private communication.
59. T.J. Greytak, private communication.


