THE MEAN INTEGRAL

THESIS

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MASTER OF ARTS

By

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The purpose of this paper is to examine properties of the mean integral. The mean integral is compared with the regular integral. If \([a;b]\) is an interval, \(f\) is quasicontinuous on \([a;b]\) and \(g\) has bounded variation on \([a;b]\), then the mean integral of \(f\) with respect to \(g\) exists on \([a;b]\). The following theorem is proved. If \([a^*;b^*]\) and \([a;b]\) each is an interval and \(h\) is a function from \([a^*;b^*]\) into \(\mathbb{R}\), then the following two statements are equivalent:

1) If \(f\) is a function from \([a;b]\) into \([a^*;b^*]\), \(g\) is a function from \([a;b]\) into \(\mathbb{R}\) with bounded variation, and

\[
\left( m \right) \int_a^b f \, dg \text{ exists, then } \left( m \right) \int_a^b h(f) \, dg \text{ exists.}
\]

2) \(h\) is continuous.
Chapter I

INTRODUCTION

The basic differences in the definition of the regular integral and in the definition of the mean integral are the following. The regular integral \( \int_{a}^{b} f \, dg \) is defined such that \( f \) and \( g \) each is a function from an interval \([a;b]\) into \( \mathbb{R} \). An interpolating function is used for its approximating values. For a subdivision and a given interval in the subdivision, an arbitrary choice is made from the interval. For an approximation on the interval, the \( f \) value at this choosen point is multiplied times the difference of the \( g \) values of the end points. For the mean integral, the approximating values are predetermined. On a given interval, the \( f \) values of the end points are averaged. This average is multiplied times the difference of the \( g \) values of the endpoints.

Chapter II deals with general concepts of real numbered set-valued integrals. Chapter II first deals with fields of a set as a basis for defining subdivisions and refinements. A subdivision (as defined in Chapter 2) can have elements which are not necessarily connected. The "normal" subdivision or partition on the real line has elements which are connected. This "normal" subdivision will be called a
simple subdivision. The mean integral could have been defined using the general concept of a subdivision.

Theorem 3.5 shows that if the regular integral exists, then the mean integral exists and that the value of the two integrals are equal. Theorem 3.7 shows that integration by parts is valid for the mean integral (as it is for the regular integral). Theorem 3.8 is a very simple theorem. Any two functions, h and k, defined on an interval [a;b] with equal values for their end points are such that

\[ \int_a^b h \, dh = \int_a^b k \, dk = \frac{k(b)^2 - k(a)^2}{2}. \]

The values that these two functions take in the interior of the interval has no effect on the values of the integral. Because of the role that the interpolating function plays in the definition of the regular integral, the values associated with the interior of the interval can play a crucial role in determining whether or not the regular integral exists. f could be a function that is dense in the plane and the mean integral of f with respect to f would exist for any interval [p;q]. This example shows a radical difference between mean and regular integrals. Theorem 3.10 shows that some properties of g are transmitted to \( \int_a^b f \, dg \). Theorem 3.9 shows that by restricting the conditions on g (in the integral, \( \int_a^b f \, dg \)), the mean integral starts behaving more like the regular integral. Theorem 3.9 can also be proved
using the regular integral.

Theorem 4.3 states that if \( f \) is a step function and \( g \) has bounded variation on an interval, then \( (m) \int_a^b f dg \) exists. The regular integral is not defined for functions of this type. The mean integral exists because its value is dependent on the end points of intervals. The regular integral does not exist because its value is dependent on being able to arbitrarily choose points on an interval. Where there is a step, there is no consistent approximation for the regular integral for intervals like \([x; x+h]\). Trying to apply Theorem 4.4 to the regular integral might be stated as: If \( f \) is a function from \([a; b]\) into \( \mathbb{R} \), then

1) \( f \) is continuous on \([a; b]\).

2) If \( g \) has bounded variation on \([a; b]\), then \( \int_a^b fdg \) exists.

In light of previous parts of this paragraph, 1) could not be stated as: \( f \) is quasicontinuous on \([a; b]\).

In proving Theorem 5.1, many of the properties of mean integrability are explored. \( h \) is assumed to be a continuous function on the interval \([a^*; b^*]\). For an appropriate simple subdivision of \([a^*; b^*]\), \( h(x) \) could be approximated by \( h(a^*) + \frac{\sum_{q} h(q) - h(p)}{q-p} \max(\min(x-p, q-p), 0) \). By using the linear properties of this approximation, it can be shown that \( \int_a^b h(f) dg \) exists when "\( \int_a^b fdg \) exists" replaces "\( (m) \int_a^b fdg \) exists". Two values of \( f \) are used in the approximating process for the mean integral. The
approximating sums are of the form
\[ \sum h(f(p)) h(f(q)) (g(q) - g(p)) \]. By adding the restriction that
g be non-decreasing, \( f \) resembles a quasicontinuous function.

For an appropriate subdivision of \( D [a;b] \), the values of
\( f(u) \) and \( f(v) \) can be in the same \( \delta \)-neighborhood where \([p;q]\)
is in the subdivision and \( p<u<v<q \) except for a subset \( P \) of \( D \)
which is such that \( \sum_{[s:t] \in P} |g(t) - g(s)| \) is "arbitrarily small".

Thus the definition of uniform continuity for the function \( h \)
can be applied with the range of \( f \) being the domain of \( h \).
CHAPTER II

FIELDS OF SETS, SET FUNCTION INTEGRALS, AND DIFFERENTIAL EQUIVALENCE

Definition 2.1. For a set $U$, a field $F$ of subsets is a set of subsets of $U$ which has the properties that:
1) If $A$ and $B$ each is in $F$, then $A \cup B$ is in $F$.
2) If $A$ is in $F$ and $A$ is not $U$, then $U - A$ is in $F$.

Example 2.2. If $a < b$ and $F^* = \{[p; q) : [p; q) \subset [a; b)\}$, then $F = \{UT : T$ is a finite subset of $F^* \}$ is a field of subsets of $[a; b)$.

Definition 2.3. Given that $U$ is a set, that $F$ is a field of subsets of $U$ and that $V$ is in $F$, then the statement that $D$ is a subdivision of $V$ means $D$ is a finite subcollection of $F$, each element of $D$ is disjoint from any other element of $D$, and $U \cup D = V$.

Definition 2.4. If $a < b$ and $F$ is a field of subsets as in Example 2.2, then $D$ is a simple subdivision of $[a; b)$ means that $D$ is a subdivision of $[a; b)$ and $D \subset F^*$.

Definition 2.5. If $a < b$, then $D$ is a simple subdivision of $[a; b)$ means that $D$ is a simple subdivision of $[a; b)$.

Definition 2.6. Given that $U$ is a set, $F$ is a field of subsets of $U$, $V$ is in $F$ and $D$ is a subdivision of $V$, then the statement that $E$ is a refinement of $D(E << D)$ means that $E$
is a subdivision of \( V \) and if \( I \in D \), then there is a subset of \( E \) which is a subdivision of \( I \).

**Definition 2.7.** If \( a < b \) and \( D \) is subdivision (or a simple subdivision) of either \([a; b]\) or \([a; b)\), then \( E \) is a simple refinement of \( D \) means that \( E \ll D \) and \( E \) is a simple subdivision of \([a; b)\).

**Remark 2.8.** If \( \{D_1, D_2, \ldots, D_n\} \) is a finite sequence of subdivisions of a set \( U \), then there is a common refinement \( E \) of each \( D_i \) in \( \{D_1, \ldots, D_n\} \).

**Definition 2.9.** If \( U \) is a set and \( F \) is a field of subsets of \( U \), then \( \alpha \) is in \( \exp(\mathbb{R})(F) \) means that \( \alpha \) is a function such that if \( I \) is in \( F \), then \( \alpha(I) \subseteq \mathbb{R} \).

**Definition 2.10.** If \( U \) is a set, \( F \) is a field of subsets of \( U \), \( \alpha \) is in \( \exp(\mathbb{R})(F) \), \( V \) is in \( F \) and \( D \) is a subdivision of \( V \), then \( b \) is an \( \alpha \)-function on \( D \) means that \( b \) is a function with domain \( D \) such that if \( I \in D \), then \( b(I) \in \alpha(I) \).

**Remark 2.11.** If \( \alpha \) is in \( \exp(\mathbb{R})(F) \) such that for each \( I \) in \( F \), \( \alpha(I) \) is a singleton, and if \( \alpha(I) = \{x\} \), then \( \alpha(I) \) will also denote \( x \).

**Definition 2.12.** If \( U \) is a set and \( F \) is a field of subsets of \( U \), then \( \alpha \) is integrable on \( U \) (exists) means that \( \alpha \) is in \( \exp(\mathbb{R})(F) \) and that \( \int \alpha \) is in \( \mathbb{R} \) such that if \( \varepsilon > 0 \), then there is a subdivision \( D \) of \( U \) such that if \( E \) is a refinement of \( D \) and \( b \) is an \( \alpha \)-function on \( E \), then
If \( U \) is a set, \( F \) is a field of subsets of \( U \), and \( \alpha \) is in \( \exp(\mathbb{R})(F) \), then the statement that the integral of \( \alpha \) exists on \( U \) means that \( \alpha \) is integrable on \( U \). If \( U \) is a set, \( F \) is a field of subsets of \( U \), \( \alpha \) is in \( \exp(\mathbb{R})(F) \) and the integral of \( \alpha \) exists on \( U \), then there is one and only one \( K \) in \( \mathbb{R} \) such that \( K = \int_U \alpha \).

**Remark 2.13.** If \( \alpha \) and \( \mu \) each is in \( \exp(\mathbb{R})(F) \), then \( (\alpha \mu)(I) = \alpha(I)\mu(I) = \{xy : x \in \alpha(I) \text{ and } y \in \mu(I)\} \).

**Definition 2.14.** If \( U \) is a set and \( F \) is a field of subsets of \( U \), then \( \text{ba}(\mathbb{R})(F) \) denotes the set to which \( \xi \) belongs if and only if \( \xi \) is a function from \( F \) into a bounded subset of \( \mathbb{R} \) such that if \( V_1 \) and \( V_2 \) are disjoint elements of \( F \), then \( \xi(V_1 \cup V_2) = \xi(V_1) + \xi(V_2) \).

**Definition 2.15.** If \( U \) is a set and \( F \) is a field of subsets of \( U \), then \( \text{ba}(\mathbb{R})(F)^+ \) denotes the set to which \( \xi \) belongs if and only if \( \xi \) is a function from \( F \) into \( (x : 0 \leq x) \) such that if \( V_1 \) and \( V_2 \) are disjoint elements in \( F \), then \( \xi(V_1 \cup V_2) = \xi(V_1) + \xi(V_2) \).

**Definition 2.16.** If \( U \) is a set, \( F \) is a field of subsets of \( U \), and \( \mu \) is in \( \text{ba}(\mathbb{R})(F)^+ \), then \( a_\mu^+ \) denotes the set to which \( \xi \) belongs if and only if \( \xi \) is in \( \text{ba}(\mathbb{R})(F)^+ \) and if \( 0 < c \), then there is \( d > 0 \), such that if \( V \) is in \( F \) and \( \mu(V) < d \), then \( \xi(V) < c \).

**Definition 2.17.** If \( U \) is a set, \( F \) is a field of subsets of \( U \) and \( \eta \) is in \( \text{ba}(\mathbb{R})(F) \), then \( a_\eta \) denotes the set...
to which \( \xi \) belongs if and only if \( \xi \) is in \( \text{ba}(\mathbb{R})(F) \) and \( f(\xi) \)
is in \( a^{-}\) where \( \mu = f(\eta) \).

**Definition 2.18.** If \( U \) is a set, \( F \) is a field of subsets of \( U \) and \( \mu \) is in \( \text{ba}(\mathbb{R})(F) \), then \( \text{Lip}_\mu \) denotes the set to which \( \xi \) belongs if and only if \( \xi \) is in \( \text{ba}(\mathbb{R})(F) \) and for some \( k \geq 0 \) and all \( V \) in \( F \), \( |\xi(V)| \leq k|f(\mu(1))| \).

**Remark 2.19.** For the following theorems and definitions, assume that \( U \) is a set and \( F \) is a field of subsets of \( U \).

**Theorem 2.20.** If \( \alpha \) is integrable on \( U \) and \( W \) is in \( F \), then \( \alpha \) is integrable on \( W \).

**Theorem 2.21.** If \( \alpha \) is integrable on \( U \), then \( \int \alpha - f(\alpha) = 0 \).

**Theorem 2.22.** If \( \mu \) is in \( \text{ba}(\mathbb{R})(F) \), then \( \text{Lip}_\mu \subset \alpha \).

**Theorem 2.23.** If \( \mu \) is in \( \text{ba}(\mathbb{R})(F) \), then \( \alpha \mu \) is integrable on \( U \) if and only if \( \alpha / |\mu| \) is integrable on \( U \).

**Theorem 2.24.** If each of \( p, q, r \) and \( s \) is in \( \mathbb{R} \), \( 0 \leq \min(r, s) \) and \( p=0 \) if \( r=0 \) and \( q=0 \) if \( s=0 \), then \( \frac{(p+q)^2}{r+s} \leq \frac{p^2}{r} + \frac{s^2}{s} \).

**Definition 2.25.** If \( \mu \) is in \( \text{ba}(\mathbb{R})(F)^+ \), then \( H^2_\mu \) denotes the set to which \( \xi \) belongs if and only if \( \xi \) is in \( \text{ba}(\mathbb{R})(F) \), for each \( V \) in \( F \), \( \xi(V) = 0 \) if \( \mu(V) = 0 \), and \( \xi^2 \) is integrable on \( \mu \).

**Theorem 2.26.** If \( \mu \) is in \( \text{ba}(\mathbb{R})(F)^+ \) and \( \xi \) is in \( \text{ba}(\mathbb{R})(F) \), then the following three statements are
equivalent:
1) \( \xi \) is in \( H^2_\mu \).
2) For each \( V \) in \( F \), \( \xi(V) = 0 \) if \( \mu(V) = 0 \) and \( \{ \xi(V) \in D < \cup \} \) is bounded.
3) There is an element \( \rho \) in \( ba(\mathbb{R})(F) \) such that for each \( V \) in \( F \), \( \rho(V) \mu(V) - \xi(V)^2 \geq 0 \).

**Theorem 2.27.** If \( \mu \) is in \( ba(\mathbb{R})(F)^+ \) and each of \( \eta \) and \( \rho \) is in \( H^2_\mu \) and each of \( r \) and \( s \) is in \( \mathbb{R} \), then \( r\eta + s\rho \) is in \( H^2_\mu \).

**Theorem 2.28.** If \( \mu \) is in \( ba(\mathbb{R})(F)^+ \) and each of \( \eta \) and \( \rho \) is in \( H^2_\mu \), then \( \eta \rho \) is integrable on \( U \).

**Theorem 2.29.** Suppose that each of \( \alpha \) and \( \beta \) is in \( \exp(\mathbb{R})(F) \), \( V \) is in \( F \) and each of \( \alpha^2 \), \( \beta^2 \), and \( \alpha \beta \) is integrable on \( V \). Then

\[
\left[ \int_{V} |\alpha(I)\beta(I)|^2 \right] \leq \left[ \int_{V} |\alpha(I)|^2 \right] \left[ \int_{V} |\beta(I)|^2 \right].
\]

**Theorem 2.30.** If \( \mu \) is in \( ba(\mathbb{R})(F)^+ \), then \( \text{Lip}_\mu \subset H^2_\mu \).

**Theorem 2.31.** If \( \mu \) is \( ba(\mathbb{R})(F)^+ \) and \( \eta \) is in \( \text{Lip}_\mu \) and \( D < < \{U\} \), then

\[
\sum_{V \in D} \left| \eta(I) - \frac{\eta(V)}{\mu(V)^{1/2}} \mu(I)^{1/2} \right| \leq \left[ \frac{\eta(I)^2}{\mu(I)^{1/2}} - \sum_{U} \frac{\eta(V)^2}{\mu(V)^{1/2}} \right] \left( \mu(U)^{1/2} \right)^{1/2}.
\]

**Theorem 2.32.** If \( \mu \) is in \( ba(\mathbb{R})(F)^+ \) and \( \xi \) is in \( a_\mu \), then

\[
\int_{U} \left| \xi(I) - \frac{\max_{J}(\min \{\xi(J), K_2 \mu(J)\}, K_1 \mu(J))}{\mu(I)} \right| = 0
\]

as \( \min \{K_1, K_2\} \to \infty \), for \( K_1 \leq 0 \leq K_2 \).

**Theorem 2.33.** If \( \mu \) is in \( ba(\mathbb{R})(F)^+ \) and \( \xi \) is in \( a_\mu \), then

\[
\int_{U} \left| \xi(I) - \frac{\xi(V)}{\mu(V)^{1/2}} \mu(I)^{1/2} \right| = 0.
\]
Theorem 2.34. Suppose that $\alpha$ is in $\exp(\mathbb{R})(F)$, $\mu$ is in $\text{ba}(\mathbb{R})(F)^+$, $\alpha \mu$ is integrable on $U$, and $\int \alpha \mu$ is in $\text{ba}(\mathbb{R})(F)$. Then the following two statements are equivalent:

1) $\int \alpha \mu$ is in $\alpha \mu$.
2) $\int \int |\alpha(J) \mu(J) - \alpha(V) \mu(I)| = 0$.

Theorem 2.35. If $V$ is in $F$ and $\alpha$ is in $\exp(\mathbb{R})(F)$, then the following three statements are equivalent:

1) $\int \alpha$ exists.
2) If $c > 0$, then there is a subdivision $D$ of $V$ such that if $E << D$ and $u$ is an $\alpha$-function on $D$ and $v$ is an $\alpha$-function on $E$, then $\left| \sum_{D} u(I) - \sum_{E} v(J) \right| < c$.
3) If $c > 0$, then there is a subdivision $D$ of $V$ such that if $E_1$ and $E_2$ each is a refinement of $D$ and $u$ is an $\alpha$-function on $E_1$ and $v$ is an $\alpha$-function on $E_2$, then $\left| \sum_{E_1} u(I) - \sum_{E_2} v(J) \right| < c$. 
CHAPTER III

COMPARISON BETWEEN MEAN AND REGULAR INTEGRALS

Definition 3.1. If \([a;b]\) is an interval and each of \(f\) and \(g\) is a function from \([a;b]\) into \(\mathbb{R}\), then let the statement that the mean integral of \(f\) with respect to \(g\) exists on \([a;b]\) \((\text{m})\int_a^b fdg \text{ exists})\) mean that \((\text{m})\int_a^b fdg \in \mathbb{R}\) such that if \(\varepsilon > 0\), then there is a simple subdivision of \([a;b]\) such that if \(E\) is a simple refinement of \(D\), then

\[
\left| \frac{1}{2} \sum_{E} \frac{f(p)+f(q)}{2} (g(q)-g(p)) - (\text{m})\int_a^b fdg \right| < \varepsilon.
\]

If \([a;b]\) is an interval, each of \(f\) and \(g\) is a function from \([a;b]\) into \(\mathbb{R}\), and the mean integral of \(f\) with respect to \(g\) exists on \([a;b]\), then there is one and only one \(K\) in \(\mathbb{R}\) such that \(K=(\text{m})\int_a^b fdg\).

Remark 3.2. If \(a<b\) and \(D\) is a simple subdivision of \(([a;b])\), then let \(\sum\int_D fdg\) denote \(\sum_{[p;q] \in D} \frac{f(p)+f(q)}{2} (g(q)-g(p))\).

Definition 3.3. If \([a;b]\) is an interval and \(D\) is a simple subdivision of \([a;b]\), then let the statement that \(v\) is an interpolating function of \(E\) mean that \(v\) is a function with domain \(E\) such that if \(I\) is in \(E\), then \(v(I)\) is in \(I\).

Definition 3.4. If \([a;b]\) is an interval and each of \(f\) and \(g\) is a function from \([a;b]\) into \(\mathbb{R}\), then let the statement that the regular integral of \(f\) with respect to \(g\)
exists on \([a; b]\) \(\int_a^b f dg\) exists) mean that \(\int_a^b f dg\) is in \(\mathbb{R}\) such that if \(\epsilon > 0\), then there is a simple subdivision of \([a; b]\) such that if \(E\) is a simple refinement of \(D\) and \(v\) is an interpolating function of \(E\), then \(\sum_{E} f(v)dg - \int_a^b f dg \ll \epsilon\). If \([a; b]\) is an interval, each of \(f\) and \(g\) is a function from \([a; b]\) into \(\mathbb{R}\), and \(\int_a^b f dg\) exists on \([a; b]\), then there is one and only \(K\) in \(\mathbb{R}\) such that \(K = \int_a^b f dg\).

**Theorem 3.5.** Suppose that \(f\) and \(g\) each is a function from \([a; b]\) into \(\mathbb{R}\) and that \(\int_a^b f dg\) exists. Then \((m)\int_a^b f dg\) exists.

**Proof.** Suppose \(\epsilon > 0\). Then there is a simple subdivision \(D\) of \([a; b]\) such that if \(E\) is a simple refinement of \(D\) and \(v\) is an interpolating function of \(E\), then \(\left| \sum_{E} f(v)dg - \int_a^b f dg \right| \ll \epsilon\). Suppose that \(E\) is a simple refinement of \(D\). Let \(\alpha\) be a function such that if \([p; q) \in E\), then \(\alpha([p; q)) = p\). Let \(\beta\) be a function such that if \([p; q) \in E\), then \(\beta([p; q)) = q\). Let \(\sum_{E} f(\alpha)dg - \int_a^b f dg \ll \epsilon\) and \(\sum_{E} f(\beta)dg - \int_a^b f dg \ll \epsilon\). Therefore, \(\sum_{E} (f(\alpha) + f(\beta))dg - 2\int_a^b f dg \ll 2\epsilon\). \(\sum_{E} f dg - \int_a^b f dg \ll \epsilon\). Therefore, \((m)\int_a^b f dg\) exists and \((m)\int_a^b f dg = \int_a^b f dg\).

**Remark 3.6.** The converse of this theorem is not necessarily true. See Remark 3.12 and Example 3.13.
Theorem 3.7. If \( a < b \) and each of \( f \) and \( g \) is a function from \([a; b]\) into \( \mathbb{R} \) and \( (m)\int_a^b fdg \) exists, then \( (m)\int_a^b gdf \) exists and \( (m)\int_a^b fdg + (m)\int_a^b gdf = fg\big|_a^b \).

Proof. Suppose \( c > 0 \). There is a simple subdivision \( D \) of \([a; b]\) such that if \( E \) is a simple refinement of \( D \), then

\[
\left| \sum_{n=0}^{n-1} fg \right| < c \cdot c = \left| \sum_{n=0}^{n-1} fdg \right| = 
\frac{f(x_0) + f(x_1)}{2} + \frac{g(x_1) - g(x_0)}{1} + \frac{f(x_1) + f(x_2)}{2} + \frac{g(x_2) - g(x_1)}{1} + \ldots +
\]

\[
\frac{f(x_{n-2}) + f(x_{n-1})}{2} + \frac{g(x_{n-1}) - g(x_{n-2})}{1} + \frac{f(x_{n-1}) + f(x_n)}{2} + \frac{g(x_n) - g(x_{n-1})}{1} - \frac{f(x_{n-2})g(x_{n-2})}{2} + \frac{f(x_{n-1})g(x_{n-2})}{2} + \frac{f(x_{n-1})g(x_{n-1})}{2} + \ldots +
\]

\[
\frac{f(x_n)g(x_n)}{2} - \frac{f(x_n)g(x_n)}{2} + \frac{f(x_{n-1})g(x_{n-1})}{2} + \frac{f(x_{n-1})g(x_{n-1})}{2} - \frac{f(x_{n-2})g(x_{n-2})}{2} + \frac{f(x_{n-1})g(x_{n-2})}{2} + \frac{f(x_{n-1})g(x_{n-1})}{2} - \ldots +
\]

\[
(m)\int_a^b fdg = \left| \frac{g(x_0) + g(x_1)}{2} \right| + \ldots + \left| \frac{g(x_1) + g(x_2)}{2} \right| - \left| \frac{g(x_1) + g(x_2)}{2} \right| - \ldots -
\]
\[
g(x_{n-1}) + g(x_n) \quad f(x_n) - f(x_{n-1}) - \left[ g(x_0) f(x_0) - f(x_n) g(x_n) \right] + \nonumber
\]
\[
\frac{1}{2} \left[ \left( g(x_n) f(x_n) - f(x_n) g(x_n) \right) + \right]
\]
\[
(m) \int_a^b f(x) \, dx = \left| \sum g(x) - \left[ f(b) g(b) - g(a) f(a) - (m) \int_a^b f(x) \, dx \right] \right|
\]

Therefore, \( (m) \int_a^b g(x) \, dx \) exists and \( (m) \int_a^b f(x) \, dx = f(b) g(b) - g(a) f(a) - (m) \int_a^b f(x) \, dx \).

Theorem 3.8. If \( a < b \) and \( h \) is a function from \([a; b]\) into \( \mathbb{R} \), then \( (m) \int_a^b h(x) \, dx \) exists.

Proof. Suppose \( c > 0 \). Let \( D \) be a simple subdivision of \([a; b]\). Suppose \( E \) is a simple refinement of \( D \).

\[
\sum_{E} \frac{h(p) + h(q)}{2} (h(q) - h(p)) = \sum_{E} \frac{(h(q))^2 - (h(p))^2}{2} \nonumber
\]
\[
= \frac{(h(b))^2 - (h(a))^2}{2}.
\]
\[
c > \left| \int_a^b h(x) \, dx - \frac{(h(b))^2 - (h(a))^2}{2} \right|. \text{ Therefore, } (m) \int_a^b h(x) \, dx = \frac{(h(b))^2 - (h(a))^2}{2}.
\]

Theorem 3.9. If \( a < b \) and \( g \) is a function from \([a; b]\) into \( \mathbb{R} \), then the following two statements are equivalent:

1) If \( f \) is a function from \([a; b]\) into \( \mathbb{R} \) and \( (m) \int_a^b f(x) \, dx \) exists, then there is a simple subdivision \( D \) of \([a; b]\) such that if \( I \) is in \( D \), then either \( f \) is bounded on \( I \) or \( g \) is constant on \( I \).

2) \( g \) is bounded on \([a; b]\).

Proof. Suppose that statement 1) is true. Suppose \( g \) is not bounded on \([a; b]\). \((m) \int_a^b g(x) \, dx \) exists. There is a simple subdivision \( D \) of \([a; b]\) such that if \( I \) is in \( D \), then either \( g \) is bounded on \( I \) or \( g \) is constant on \( I \). Since there
is an $E \subseteq \mathbb{E}$ where $g$ is neither bounded nor constant, then $g$ must be bounded on $[a;b]$.

Suppose that 2) $g$ is bounded on $[a;b]$ by some number $M$. Suppose that $f$ is a function from $[a;b]$ into $\mathbb{R}$ and $(m) \int_{a}^{b} f \, dg$ exists. There is a simple subdivision $D$ of $[a;b]$ such that if $E$ is a simple refinement of $D$, then $\left| \sum_{E} f dg - (m) \int_{a}^{b} f \, dg \right| < 1$.

Suppose $(p;t) \subseteq D$. Suppose $g$ is not constant on $(p;t)$. There is a $q \in (p;t)$ such that $g(p) \neq g(q)$. Suppose $x \in (p;q)$. Let $E$ be a simple refinement of $D$ such that if $(r;s) \subseteq D$ and $(r;s) \neq (p;q)$, then $(r;s) \subseteq E$ and each of $(p;x)$ and $(x;q)$ is in $E$.

$$2 > \left| \sum_{E} f dg - \sum_{D} f dg \right| = \left| \frac{f(p) + f(x)}{2} (g(x) - g(p)) + \frac{f(x) + f(q)}{2} (g(q) - g(x)) - \frac{f(p) + f(q)}{2} (g(q) - g(p)) \right| \geq \frac{1}{2} \left[ \left| f(x)g(q) - f(x)g(p) \right| - \left| f(p)g(x) \right| - \left| f(q)g(x) \right| - \left| f(p)g(q) \right| - \left| f(q)g(p) \right| \right] \geq \frac{4 + M(|f(p)| + |f(q)|) + |f(p)| + |f(q)|}{|g(q) - g(p)|} > |f(x)|. \text{ Therefore, } f \text{ is bounded on } (p;q). \text{ Statement 2) implies statement 1).}$

Theorem 3.10. Suppose that each of $f$ and $g$ is a function from the interval $[a;b]$ into $\mathbb{R}$ such that $(m) \int_{a}^{b} f \, dg$ exists. Then the following two statements are true:

1) If $g$ has bounded variation on $[a;b]$ and $a \leq c \leq b$, then so
does \((x,(m)\int_c^x fdg): a \leq x \leq b\).

2) If \(a \leq v \leq b\) and \(g\) is continuous at \(v\), then \((x,(m)\int_c^x fdg): a \leq x \leq b\) is continuous at \(v\).

Proof. (1) Let \(c \in [a;b]\). Suppose \(g\) has bounded variation on \([a;b]\) and \(a \leq c \leq b\). \(g\) is bounded on \([a;b]\). There is a simple subdivision \(H\) of \([a;b]\) such that if \(I\) is in \(H\), then either \(f\) is bounded on \(I\) or \(g\) is constant. Let \(M = \max(\sup\{|f(t)|: t \in I\}, I \in H\) and \(f\) is bounded). If, for each \(I \in H\), \(f\) is not bounded on \(I\), then \(g\) is constant on each \(I \in H\) and \((x,(m)\int_c^x fdg): a \leq x \leq b\) has bounded variation. Suppose \(f\) is bounded on some element of \(H\). There is \(Q\) a simple subdivision of \([a;b]\) such that if \(E\) is a simple refinement of \(Q\), then \(|\sum_{E}^{Q}|fdg - (m)\int_{E}^{Q} fdg| < 1\). Let \(D\) be a simple refinement of each of \(H\) and \(Q\). Suppose \(E\) is a simple refinement of \(D\).

\[ \sum_{[r;s) \in E} |(m)\int_r^s fdg - (m)\int_r^s fdg| = \sum_{[r;s) \in E} |(m)\int_r^s fdg| < \left[ \sum_{E} |fdg| \right] + \sum_{[r;s) \in E} |dg| \]

\[ 1 \leq \left[ \sum_{E} |dg| \right] + 1 \leq \sum_{E} |dg| + 1. \]

Therefore, \((x,(m)\int_c^x fdg): a \leq x \leq b\) has bounded variation. Statement 1) is true.

(2) Suppose \(a \leq v \leq b\) and \(g\) is continuous at \(v\). There is a \(\delta > 0\) such that if \(y \in [a;b]\) and \(y \in (v - \delta; v + \delta)\), then \(|g(v) - g(y)| < 1\). Let \(r = \max(a, v - \delta/2)\) and \(s = \min(b, v + \delta/2)\). Then \((m)\int_r^s fdg\) exists and \(g\) is bounded on \([r;s]\).
There is a simple subdivision $H$ of $[r; s]$ such that if $I \in H$, then $f$ is bounded on $I$ or $g$ is constant on $I$. If, for each $I \in H$, $f$ is unbounded on $I$, then $g$ is constant on $I$ and 

\[ \{(x, (m) \int_a^b f dg) : a \leq x \leq b\} \text{ is continuous at } v. \]

Suppose, for some element of $H$, $f$ is bounded. Suppose there is $\epsilon > 0$ such that if $h > 0$, then there is $y \in [a; b]$ such that $y \in (v-h; v+h)$ and $g(v) = g(y)$. If, for each $I \in H$, $f$ is unbounded on $I$, then $g$ is constant on $I$ and 

\[ \left\{ (x, (m) \int_a^b f dg) : a \leq x \leq b \right\} \text{ is continuous at } v. \]

Suppose, for some element of $H$, $f$ is bounded. Suppose there is $\epsilon > 0$ such that if $h > 0$, then there is $y \in [a; b]$ such that $y \in (v-h; v+h)$ and $g(v) = g(y)$. If, for each $I \in H$, $f$ is unbounded on $I$, then $g$ is constant on $I$ and 

\[ \left\{ (x, (m) \int_a^b f dg) : a \leq x \leq b \right\} \text{ is continuous at } v. \]

There is a simple subdivision $Q$ of $[r; s]$ such that if $E$ is a simple refinement of $Q$, then 

\[ \sum_{E} \left| \frac{f(q) + f(p)}{2} (g(q) - g(p)) - (m) \int_a^b f dg \right| < \frac{\epsilon}{2}. \]

Let $D$ be a simple refinement of $Q$ and $E$ be a simple refinement of $H$. Suppose, without loss of generality, that there is a $z$ such that 

\[ [v; z) \in D \text{ and there is } y \in [a; b] \text{ such that } y \in (v; z) \cap (v; v+e) \text{ and } \left| (m) \int_a^b f dg - (m) \int_a^b f dg \right| < \frac{\epsilon}{2}. \]

Let $E$ be a simple refinement of $D$ such that if $I \in D$ and $I \neq [v; z)$, then $I \in E$ and each of $[v; y)$ and $[y; z) \in D$. Then $\epsilon / 2 > \sum_{E} \left| \int_a^b f dg - (m) \int_a^b f dg \right|$ 

\[ \geq \frac{f(v) + f(y)}{2} (g(v) - g(y)) - (m) \int_a^b f dg \right| \leq \frac{\epsilon}{2} \]

\[ + \frac{f(v) + f(y)}{2} (g(v) - g(y)) \left| \frac{\epsilon}{2} + \frac{M \epsilon}{2M} = \epsilon. \]

Since this is
impossible, then if \( \varepsilon > 0 \), then there is \( h > 0 \) such that if \( y \in [a;b] \) and \( y \in (v-h;v+h) \), then \( \left| (m) \int_0^y f \, dg - (m) \int_0^v f \, dg \right| < \varepsilon \).

Theorem 3.11. If each of \( f \) and \( g \) is a function from \([a;b]\) into \( \mathbb{R} \) and \( \int_0^b f \, dg \) exists, then, if \( 0 < \min\{c,d\} \), then there is a simple subdivision \( D \) of \([a;b]\) such that if \( E \) is a simple refinement of \( D \) and \( E_1 = \{ I \in E : |f(x) - f(y)| \geq c \text{ for each } x \text{ and } y \text{ in } I \} \), then \( \sum_{E_1} |4g| < d \).

Proof. Suppose each of \( f \) and \( g \) is a function from \([a;b]\) into \( \mathbb{R} \) and \( \int_0^b f \, dg \) exists. Suppose \( 0 < \min\{c,d\} \). There is a simple subdivision \( Q \) of \([a;b]\) such that if \( E \) is a simple refinement of \( Q \) and \( \gamma \) is an interpolating function on \( E \), then \( \left| \sum_{E} f(\gamma) \, 4g - \int_{-b}^a f \, dg \right| < cd/2 \). Suppose \( E \) is a simple refinement of \( D \). Let \( E_1 = \{ I \in E : |f(x) - f(y)| \geq c \text{ for some } x,y \text{ in } I \} \). Let, if any, each of \( \alpha \) and \( \beta \) be an interpolating function on \( E \) such that if \( I \in E \) such that there exists \( (x,y) \in I \) such that \( |f(x) - f(y)| \geq c \), then \( |f(\alpha(I)) - f(\beta(I))| \geq c \).

Let each of \( u \) and \( v \) be a function from \( E \) into \( \mathbb{R} \) such that if \( (f(\alpha(I)) - f(\beta(I)))(4g) \geq 0 \text{ where } I \in E, \) then \( u(I) = \alpha(I) \) and \( v(I) = \beta(I) \), and if \( (f(\alpha(I)) - f(\beta(I)))(4g) < 0 \text{ where } I \in E, \) then \( u(I) = \beta(I) \) and \( v(I) = \alpha(I) \). Then \( \sum_{E} \left| f(u) \, 4g - \sum_{E} f(v) \, 4g \right| = \sum_{E} |f(u) - f(v)| \, 4g \geq \sum_{E} |f(u) - f(v)| \, 4g \geq c \sum_{E} \left| f(u) \, 4g - f(v) \, 4g \right| = d \sum_{E} \left| 4g \right| \).
Therefore, this theorem is true.

Remark 3.12. Suppose that $f$ is a function from $[0;2]$ into $\mathbb{R}$ such that if $x \in [0;1]$, then $f(x)=0$ and if $x \in (1;2]$, then $f(x)=2$. Then $(m) \int_{a}^{b} f \, df$ exists. Let $D$ be a simple subdivision of $[0;2]$. Suppose that $E$ is a simple refinement of $D$ and of $([0;1],[1;2])$. There is $E_1 = \{I \in E: |f(x)-f(y)| \geq 1 \text{ for some } (x,y) \in I \}$. For this example, $\sum_{E_1} |f(x)-f(y)| \geq 1$. Therefore, the previous theorem would not hold when $\int_{a}^{b} f \, dg$ exists is replaced by $(m) \int_{a}^{b} f \, dg$ exists.

Example 3.13. Let $f$ be the function such that $f(1)=1$ and if $x \neq 1$, then $f(x)=0$. Let $g$ be the function such that $g(x)=0$ if $x < 1$ and $g(x)=2$ if $x \geq 1$. $(m) \int_{0}^{2} f \, dg = 1$.

Example 3.14. Let $f$ be the function such that $f(1)=1$ and if $x \neq 1$, then $f(x)=0$. Let $g$ be the function such that $g(1)=2$ and if $x \neq 1$, then $g(x)=0$. $(m) \int_{0}^{1} f \, dg = 1$ and $(m) \int_{0}^{2} f \, dg = 0$.

Example 3.15. Let $f$ be the function from $[0;1]$ onto $(0,1)$ such that if $x$ is rational, then $f(x)=x$ and if $x$ is irrational, then $f(x)=0$. $(m) \int_{0}^{1} f \, df = \frac{1}{2}$ and $(m) \int_{0}^{3} f \, dg = \frac{5}{2}$. 
Chapter IV

QUASICONTINUITY, BOUNDED VARIATION AND MEAN INTEGRAL EXISTENCE

Definition 4.1. If $a < b$, then the statement that $f$ is a real valued step function on $[a; b]$ means that $f$ is a function whose domain includes $[a; b]$ and whose range is a subset of $\mathbb{R}$ such that there is a simple subdivision $D$ of $[a; b]$ such that if $[p; q)$ is in $D$ and $p < x < y < q$, then $f(x) = f(y)$.

Theorem 4.2. If $a < b$, $f$ is a real valued function and $f$ is quasicontinuous on $[a; b]$ and $0 < c$, then there is a real valued step function $g$ on $[a; b]$ such that if $x$ is in $[a; b]$, then $|f(x) - g(x)| < c$.

Proof. Suppose $c > 0$. For each $x$ in $[a; b]$, let $s(x) > 0$ such that if $y$ is in $(x; x + s(x))$ and is in $[a; b]$, then $|f(x) - f(y)| < c/2$. For each $x$ in $(a; b)$, let $d(x) > 0$ such that if $y$ is in $(x - d(x); x)$ and is in $[a; b]$, then $|f(x) - f(y)| < c/2$. For each $x$ in $(a; b)$, let $r(x) = \min(s(x), d(x))$. Let $r(a) = s(a)$ and $r(b) = d(b)$.

By applying the Heine-Borel theorem, there is a finite subcollection $Q$ of $\{(x - r(x); x + r(x)) : x \in [a; b]\}$ which covers $[a; b]$ and includes $(a - r(a); a + r(a))$ and $(b - r(b); b + r(b))$. Let, for each $I \in Q$, $m(I)$ be such that for some $x \in [a; b]$,
m(I)=x and (x-r(x);x+r(x))=I, and m(a-r(a);a+r(a))=a and 
m(b-r(b);b+r(b))=b. Let P be a simple subdivision of [a;b] such that if [p;q)\in P, then for some I in Q and some J in Q, 
m(I)=p and m(J)=q and for any K in Q, m(K)\in(p;q) and if I\in Q, 
then for some y\in[a;b], either [m(I);y) \in P or [y;m(I))\in P.
For each [m(I);m(J)) in P, either m(I) + r(m(I))\geq m(J) or 
m(I)+r(m(I))\in [m(I);m(J)). Let, for each [m(I);m(J))\in P, 
n(m(I))\in (m(I);m(J)) if m(I)+r(m(I)) \geq m(J) and, otherwise, 
n(m(I)) = m(I)+r(m(I)). Let D be a simple subdivision of 
[a;b] such that K\in D if and only if for some [m(I);m(J))\in P, 
either K = [m(I);n(m(I)) or K = [n(m(I));m(J)). Let g be a 
function from [a;b] into \mathbb{R} such that if [p;q) is in D, then 
f(p)=g(p), f(q)=g(q) and for each x in (p;q), g(x) = 
f((p+q)/2). Then g is a step function. If [p;q) and 
p<x<y<q, then |f(x)-f(y)| < \epsilon.

Theorem 4.3. Suppose a<b, f is a real valued step function and g has bounded variation on [a;b]. Then
\[
\int_{a}^{b} f dg \text{ exists.}
\]

Proof. Let M= \max(|f(x)|: x\in[a;b]). There is a simple 
subdivision D of [a;b] such that if [p;q) is in D and 
p<x<y<q, then f(x)=f(y). Let m be the cardinality of D.
Suppose \epsilon>0. For each I=(p;q) in D, there is \delta_I>0 such that 
if (x;y)\subseteq[a;b], and (x;y)\subseteq(p;p+\delta_I) or (x;y)\subseteq(q-q-;q), then 
|g(x)-g(y)| < \frac{\epsilon}{4mM}. Let \delta = \min(\delta_I: I\in D). Let, for each [p;q) 
in D, x(p)= \min(p+\delta,\frac{2p+q}{3}) and y(p)= \max(q-\delta,\frac{1}{3}(p+2q)), then
\( p < x(p) < y(p) < v \). Let \( Q = \bigcup \{ [p; x(p)), [x(p); y(p)), [y(p); v) \} \).

Then \( Q \) is a simple refinement of \( D \). Suppose that \( E_1 \) is a simple refinement of \( Q \) and \( E_2 \) is a simple refinement of \( Q \).

Let, for each \( [p; q) \in E_1 \), \( u(p) \) be such that \( [p; u(p)) \in E_1 \) and \( v(p) \) be such that \( [v(p); q) \in E_1 \). Let, for each \( [p; q) \in E_2 \), \( w(p) \) be such that \( [p; w(p)) \in E_2 \) and \( z(p) \) be such that \( 1:1 \) Irf(p)+f(x(p)) g(u(p))-g(p) +

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(p) + \sum_{E_2} f(q)+f(x(p)) g(q)-g(v(p)) \]

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(p) + \sum_{E_2} f(q)+f(x(p)) g(q)-g(v(p)) \]

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(w(p)) \]

\[ \sum_{E_2} f(q)+f(x(p)) g(q)-g(z(p)) \]

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(w(p)) \]

\[ \sum_{E_2} f(q)+f(x(p)) g(q)-g(z(p)) \]

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(w(p)) \]

\[ \sum_{E_2} f(q)+f(x(p)) g(q)-g(z(p)) \]

\[ \sum_{E_1} f(p)+f(x(p)) g(u(p))-g(w(p)) \]

\[ \sum_{E_2} f(q)+f(x(p)) g(q)-g(z(p)) \]

Therefore, \((m)\int_a^b f g \text{ exists. This theorem is not necessarily true for regular integrals.}\)

Theorem 4.4. If \( f \) is a function from \([a; b] \) into \( \mathbb{R} \), then the following two statements are equivalent:

1. \( f \) is quasicontinuous on \([a; b] \).
2. If \( g \) is a function from \([a; b] \) into \( \mathbb{R} \) and has bounded
variation on \([a;b]\), then \((m)^{b}_{a}\) exists.

Proof. Suppose that \(f\) is a function from \([a;b]\) into \(\mathbb{R}\).

(1) Suppose that \(f\) is quasicontinuous on \([a;b]\). Suppose that \(g\) is a function from \([a;b]\) into \(\mathbb{R}\) and \(g\) has bounded variation on \([a;b]\). There is an \(M>1\) such that if \(D\) is a simple subdivision of \([a;b]\), then \(\sum |d g| < M\). Suppose \(c>0\).

Let \(h\) be a real valued step function on \([a;b]\) such that if \(a<x<b\), then \(|h(x)-f(x)| < \frac{c}{4M}\). There is a simple subdivision \(Q\) of \([a;b]\) such that if \([p;q]\) is in \(Q\) and \(p<x<y<q\), then \(h(x) = h(y)\). For each \([p;q]\) in \(Q\), let \(1(p)\) be in \([p;q]\).

There is a simple subdivision \(P\) such that if \(E_1\) and \(E_2\) each is a simple refinement of \(P\), then \(\sum_{E_1} \sum_{E_2} h d g - \sum_{E_1} \sum_{E_2} h d g < \frac{c}{4}\).

Let \(D\) be a simple refinement of \(P\) and of \(Q\).

Suppose that each of \(E_1\) and \(E_2\) is a simple refinement of \(D\). Suppose \(j\in\{1, 2\}\).

\[
\sum_{E_j} \sum_{E_j} f d g - \sum_{E_j} \sum_{E_j} h d g = \sum_{E_j} \frac{f(p)+f(q)-h(p)-h(q)}{2} \frac{g(q)-g(p)}{1} \leq \sum_{E_j} \left| \frac{f(p)-h(p)}{2} + \frac{f(q)-h(q)}{2} \right| (g(q)-g(p)) \leq \sum_{E_j} \frac{c}{4M} \sum \left| g(q)-g(p) \right| < \frac{c}{4}.
\]

\[
\sum_{E_1} \sum_{E_2} f d g - \sum_{E_1} \sum_{E_2} h d g = \left| \sum_{E_1} \sum_{E_1} f d g - \sum_{E_1} \sum_{E_1} h d g \right| + \left| \sum_{E_1} \sum_{E_1} h d g + \sum_{E_2} \sum_{E_2} h d g - \sum_{E_2} \sum_{E_2} h d g \right| \leq \sum_{E_1} \sum_{E_1} f d g \leq \sum_{E_1} \sum_{E_1} h d g + \sum_{E_1} \sum_{E_1} h d g + \sum_{E_2} \sum_{E_2} h d g + \sum_{E_2} \sum_{E_2} h d g < c.
\]
Therefore, \((m) \int_{a}^{b} fdg\) exists.

(2) Suppose that if \(g\) is a function from \([a; b]\) into \(\mathbb{R}\) and has bounded variation on \([a; b]\), then \((m) \int_{a}^{b} fdg\) exists.

Suppose there is \(p\) in \([a; b]\) and \(c > 0\) such that if \(d > 0\), then there is \((x, y) \in [a; b]\) such that \(p - d < x \leq y < p\) and \(|f(x) - f(y)| \geq c\).

Let \(g\) be a function with domain \([a; b]\) such that \(g(p) = 2\) and if \(x\) is in \([a; b] - \{p\}\), then \(g(x) = 0\). \(g\) has bounded variation on \([a; b]\) and \((m) \int_{a}^{b} fdg\) exists.

Let \(D\) be a simple subdivision of \([a; b]\) such that if \(E_1\) and \(E_2\) each is a simple refinement of \(D\), then \(\left| \sum_{E_1} fdg - \sum_{E_2} fdg \right| < c\). Let \(e < p\) such that for some \(r\), \([e; r) \in D\) and \(p \in [e; r]\).

There is \((x, y) \in [a; b]\) such that \(e < x < y < p\) and \(|f(x) - f(y)| \geq c\).

Let \(E_1\) be a simple refinement of \(D\) such that if \((w; z) \in D\) and \((w; z) \neq (e; r)\), then \((w; z) \in E_1\) and each of \([e; x)\) and \([x; p)\) is in \(E_1\) and if \(p \neq r\), then \([p; r) \in E_1\). Let \(E_2\) be a simple refinement of \(D\) such that if \((w; z) \in D\) and \((w; z) \neq (e; r)\), then \((w; z) \in E_2\) and each of \([e; y)\) and \((y; p)\) is in \(E_2\) and if \(p \neq r\), then \([p; r) \in E_2\).

\[ c > \left| \sum_{E_1} fdg - \sum_{E_2} fdg \right| = \left| \frac{f(x) + f(e)}{2} \frac{g(x) - g(e)}{1} \right| + \]

\[ \left| \frac{f(p) + f(x)}{2} \frac{g(p) - g(x)}{1} \right| \]

\[ \left| \frac{f(p) + f(y)}{2} \frac{g(p) - g(y)}{1} \right| = \left| \frac{f(p) + f(x)}{2} - \frac{f(p) + f(y)}{2} \right| = \left| f(x) - f(y) \right| \geq c. \]

Since \(c > c\) is not possible, then if \(p \in [a; b]\) and \(c > 0\), then there is \(d > 0\) such that if \((x, y) \in [a; b]\) such that
p-d<x<y<p, then \( |f(x) - f(y)| < c \), and, similarly, if p \( \in [a; b] \) and c>0, then there is d>0 such that if (x,y) \( \in [a; b] \) such that p<x<y<p+d, then \( |f(x) - f(y)| < c \). Therefore, f is quasicontinuous on [a; b].
CHAPTER V

CONTINUITY AND MEAN INTEGRABILITY

PRESERVATION

Theorem 5.1. Suppose that $h$ is a continuous function from $[a^*; b^*]$ into $\mathbb{R}$. Then, if $[a;b]$ is an interval, $g$ is a non decreasing function from $[a; b]$ into $\mathbb{R}$, $f$ is a function from $[a;b]$ into $[a^*; b^*]$ and $(m)\int_a^b f dg$ exists, then

$(m)\int_a^b h(f) dg$ exists.

Proof. Suppose that $[a;b]$ is an interval, $g$ is a non decreasing function from $[a; b]$ into $\mathbb{R}$, $f$ is a function from $[a;b]$ into $[a^*; b^*]$ and $(m)\int_a^b f dg$ exists. Let $M = \sup\{|f(x)| : x \in [a;b]|+1$.

Lemma 5.2. If $x \in [a;b]$ and $g(x) < g(x+)$, then if $\varepsilon > 0$, then there is $\delta > 0$ such that if $\{y_1, y_2\} \subseteq (x; x+\delta) \cap [a; b]$, then $|f(y_1) - f(y_2)| < \varepsilon$.

Proof. Suppose $x \in [a;b]$ and $g(x) < g(x+)$. Let $\varepsilon = g(x+)-g(x)$. Consider the statement that there is an $\varepsilon > 0$ such that if $\delta > 0$, then there is a set $\{y_1, y_2\} \subseteq [a; b] \cap (x; x+\delta)$ such that $|f(y_1) - f(y_2)| \geq \varepsilon$.

There is a $\delta > 0$ such that if $\{y_1, y_2\} \subseteq [a; b] \cap (x; x+\delta)$, then $|g(y_1) - g(y_2)| < \frac{\varepsilon}{4M}$ and such that if $y \in [a; b] \cap (x; x+\delta)$, then
\[ |g(y) - g(x^+)| < \varepsilon/4. \]

Let \( y \in [a;b] \cap (x; x + \delta) \). There is \( D \) a simple refinement of \( \{[x; y]\} \) such that if \( E_1 \) is a simple refinement of \( D \) and \( E_2 \) is a simple refinement \( D \), then
\[
\left| \sum_{E_1} \text{fdg} - \sum_{E_2} \text{fdg} \right| < \frac{\varepsilon \varepsilon}{2}.
\]

Let \( z \in [a;b] \) such that \([x; z] \in D \). There is \((y_1; y_2) \in (x; z) \) such that \(|f(y_1) - f(y_2)| \geq \varepsilon \). Let \( E_1 \) be a simple refinement of \( D \) such that if \([p; q] \in D \) and \([p; q] \neq [x; z] \), then \([p; q] \) is in \( E_1 \) and each of \([x; y_1], [y_1; y_2], \) and \([y_2; z] \) is in \( E_1 \). Let \( E_2 \) be a refinement of \( D \) such that if \([p; q] \in D \) and \([p; q] \neq [x; z] \), then \([p; q] \in E_2 \) and each of \([x; y_2] \) and \([y_2; z] \) is in \( E_2 \).

\[
\varepsilon \varepsilon > \left| \sum_{E_1} \text{fdg} - \sum_{E_2} \text{fdg} \right| = \left| \frac{f(x) + f(y_1)}{2} \right| \frac{g(y_1) - g(x)}{1} + \left| \frac{f(y_2)}{2} \right| \frac{g(y_2) - g(y_1)}{1} - \left| \frac{f(x) + f(y_2)}{2} \right| \frac{g(y_2) - g(x)}{1}.
\]
\[ \frac{1}{2} f(y_1) - f(y_2) \geq \frac{1}{2} |g(y_1) - g(x)| - \frac{\epsilon e}{4}. \]

\[ \frac{3}{8} \epsilon e > \frac{1}{2} \epsilon \left[ |g(x^+) - g(x)| - |g(y_1) - g(x^+)| \right]. \]

\[ \frac{3}{8} \epsilon e + \frac{1}{2} \epsilon |g(y_1) - g(x^+)| > \frac{1}{2} \epsilon |g(x^+) - g(x)| = \frac{1}{2} \epsilon e. \]

\[ \frac{3}{8} \epsilon e + \frac{1}{8} \epsilon e > \frac{1}{2} \epsilon e. \] This is impossible. Therefore, this lemma is true.

Remark 5.3. Let \([a; b) = \cup \) and \( F \) be as defined in Example 2.2. Let, for each \( I \in F, C_I \) be the components of \( I \). Then, for each \( I \in F \) and \( J \in C_I, J \notin F^* \). Let \( \gamma \) be a function such that if \( I \in F \), then \( \gamma(I) = \sum_{[p; q] \in C_I} \frac{f(p) + f(q)}{2} (g(q) - g(p)). \)

Suppose \( c > 0 \). There is a simple subdivision \( D \) of \([a; b]\) such \( E \) is a simple refinement of \( D \), then \( \left| \sum_{E} \text{fdg} - (m) \right| < c. \)

Suppose \( E \) is a refinement of \( D \). There is a simple refinement \( B \) of \( D \) such that \( \sum_{B} \text{fdg} = \sum_{E} \gamma \). Therefore, \( \int_{[a; b]} \gamma = \left( \int_{a}^{b} \text{fdg} \right) / \mu \).

\( (m) \left( \int_{a}^{b} \text{fdg} \right) \). Let \( \mu \) be a function such that if \( I \in F \), then \( \mu(I) = \sum_{[p; q] \in C_I} (g(q) - g(p)). \) Let \( \xi \) be a function such that if \( I \in F \), then \( \xi(I) = \frac{\gamma(I)}{\mu(I)} \) if \( \mu(I) \neq 0 \) and, otherwise, \( \xi(I) = 0 \).
\[ \zeta(I) = 0. \ \int \zeta \mu = \int \gamma. \] Let \( \alpha \) be a function such that if 
\[ [a;b) \cap [a;b) \]
\[ \mu([p;q]) \neq 0, \text{ then } \zeta([p;q]) = \alpha([p;q]). \]
If \([p;q) \subseteq [a;b] \) and
\[ \alpha([p;q]) = 0 \] and \([r;s) \subseteq [p;q) \), then \( \mu([r;s)) = 0 \) and
\[ \alpha([p;q]) \mu([r;s)) = 0. \]

Lemma 5.4 \( \zeta \mu \) is absolutely continuous with respect to 
\( \mu \) and 
\[ \sum \left| \int \zeta \mu - \zeta(V) \mu(I) \right| = 0. \]

If \([p;q) \subseteq [a;b] \), then 
\[ \int -M \mu \leq \int \zeta \mu \leq \int M \mu. \]
\( \zeta \mu \) is in \( \text{Lip}_\mu \) and is absolutely continuous with respect to \( \mu \). By 
Theorem 2.34, this lemma is true.

Lemma 5.5. Suppose \( c > 0, D \) is a simple subdivision of 
\([a;b] \) such that if \( E \) is a simple refinement of \( D, \) then
\[ \sum \int \left| \int \zeta \mu - \alpha(V) \mu(I) \right| < c, \text{ and each of } G \text{ and } H \text{ is a simple } 
E \ V \ I \]
refinement of \( D. \) Then the following two statements are
true:
1) There is \( \{G_Y : Y \in G \text{ and } G_Y \text{ is a simple refinement of } \{Y\} \} \)
such that if, for each \( Y \in G, E_Y \) is a simple refinement of \( G_Y, \)
then 
\[ \sum \sum \left| \int \zeta \mu - \alpha(Y) \mu(I) \right| < c \]
and there is \( \{H_Z : Z \in H \text{ and } H_Z \text{ is a simple refinement of } \{Z\} \} \)
a simple refinement of \( \{Z\} \) such that if, for each \( Z \in H, E_Z \)

is a simple refinement of $H_Z$, then $\sum \sum_{H I \in E_Z} |\int \psi \mu - \alpha(Z) \mu(I)| < c$.

2) There is $L$ a simple refinement of $G$ and of $H$ such that if $E$ is a simple refinement of $L$, then

$$\sum_{I \in E} |\alpha(Y) - \alpha(Z)| \mu(I) < 2c.$$  

Proof. Let $m$ be the cardinality of $H$. For each $W \in H$, there is $H_W$ a simple refinement of $\{W\}$ such that if $E_W$ is a simple refinement of $H_W$, then $\sum |\int \psi \mu - \alpha(W) \mu(I)| < \frac{1}{2m} \left[ c - \sum_{H Z I} |\int \psi \mu - \alpha(Z) \mu(I)| \right]$. $U H_Z$ is a simple refinement of $D$. Suppose, for each $Z \in H$, $E_Z$ is a simple refinement of $H_Z$. If $W \in H$, then

$$\sum_{E_W} |\int \psi \mu - \alpha(W) \mu(I)| < \frac{1}{2m} \left[ c - \sum_{H Z I} |\int \psi \mu - \alpha(Z) \mu(I)| \right] + \sum_{Z \in H} |\int \psi \mu - \alpha(Z) \mu(I)| <$$

$$\sum_{H Z I} |\int \psi \mu - \alpha(Z) \mu(I)| + \sum_{i=1}^{m} \left[ \frac{1}{2m} \left[ c - \sum_{H Z I} |\int \psi \mu - \alpha(Z) \mu(I)| \right] \right] = \frac{c}{2} +$$
Similarly, there is \( (G_Y: Y \in G \text{ and } G_Y \text{ is a simple refinement of } \{Y\}) \) such that if \( Y \in G \) and \( E_Y \) is a simple refinement of \( G_Y \), then \( \sum \sum_{I \in E_Y} \left| \int I \sum_{\mu - \alpha(Y)\mu(I)} \right| < c \). Let \( L \) be a simple refinement of \( \cup G_Y \) and \( L \) be a simple refinement of \( \cup H \).

Suppose \( E \) is a simple refinement of \( L \).

\[
\sum_{I \in H} \left| \int I \sum_{\mu - \alpha(Y)\mu(I)} \right| < 2c. \text{ Therefore, this lemma is true.}
\]

**Lemma 5.6.** If \( \delta > 0 \) and \( [p; q) \subset [a; b) \) such that \( g(q) > g(p) \), then there is \( [x; y) \subset [p; q) \) such that if \( (s; t) \subset (x; y) \), then \( |f(s) - f(t)| < \delta \).

**Proof.** Consider the statement that there is a \( \delta > 0 \) and there is a \( [p; q) \subset [a; b) \) such that \( g(q) > g(p) \) and such that if \( [x; y) \subset [p; q) \), then there is \( (s; t) \subset (x; y) \) such that \( |f(s) - f(t)| \geq \delta \). There is a simple subdivision \( D \) of \( [a; b] \) such that if \( E \) is a simple refinement of \( D \), then

\[
\sum_{E \in V} \sum_{I \in E} \left| \int I \sum_{\mu - \alpha(Y)\mu(I)} \right| < \frac{1}{\delta} \left( g(q) - g(p) \right)(\delta). \text{ Let } H \text{ be a simple refinement of } \cup D \text{ and of a subdivision of } [a; b] \text{ which contains}
\]
\([p; q]\) as an element. For each \((u; v) \in H\), let \(x(u) \in (u; v)\). For each \((u; v) \in H\), either \(g(x(u)) - g(u) > \frac{1}{2}(g(v) - g(u))\) or 
\(g(v) - g(x(u)) > \frac{1}{2}(g(v) - g(u))\). For each \((u; v) \in H\), (1) if 
\(g(x(u)) - g(u) > \frac{1}{2}(g(v) - g(u))\), then let \(J(u) = [u; x(u))\) and 
\(I(u) = [x(u); v)\), and (2) if, otherwise, then let \(J(u) = [x(u); v)\) and \(I(u) = [u; x(u))\). If \((u; v) \in H\), then \(\mu(J(u)) > \frac{1}{2}(g(v) - g(u))\). For each \((u; v) \in H\), let \(s(u)\) and \(t(v)\) be such 
that \(s(u) < t(v)\), \((s(u), t(v)) \in (c; d)\) where \((c; d) = I(u)\), and if 
\(I(u) \subset [p; q]\), then \(|f(s(u)) - f(t(v))| > \delta\).

Let \(G = \{I \in H : I \subset [p; q]\}\). Let \(G_1 = \{[u; s(u)) : (u; v) \in G\}\) \(U\) 
\(\{[s(u); v) : (u; v) \in G\}\). Let \(G_2 = \{[u; t(v)) : (u; v) \in G\}\) \(U\) \(\{[t(v); v) : (u; v) \in G\}\). There is \(E\) which is a simple refinement of \(G_1\) 
and a simple refinement of \(G_2\) such that 
\[
\frac{1}{4}(g(q) - g(p))(\delta) > \sum_{I \in E} |\alpha(Y) - \alpha(Z)| \mu(I) = \\
\sum_{(u; v) \in G} \left[ \sum_{I \in E} |\alpha([u; s(u)) - \alpha([u; t(v))]| \mu(I) \right. \\
+ \sum_{I \in E} |\alpha([s(u); v)) - \alpha([u; t(v))]| \mu(I) \\
+ \sum_{I \in E} |\alpha([s(u); v)) - \alpha([t(v); v))| \mu(I) \left. \right] \]

\(\sum_{I \in E} \left[ \sum_{I \in E} |\alpha([u; s(u)) - \alpha([u; t(v))]| \mu(I) \right. \\
+ \sum_{I \in E} |\alpha([s(u); v)) - \alpha([u; t(v))]| \mu(I) \\
+ \sum_{I \in E} |\alpha([s(u); v)) - \alpha([t(v); v))| \mu(I) \left. \right] \)
\[
\sum_{[u;v)\in G} \left[ \frac{f(u)+f(s(u)) - f(u)+f(t(v))}{2} \mu([u;s(u)]) + \frac{f(s(u))+f(v) - f(u)+f(t(v))}{2} \mu([s(u);t(v)]) + \frac{f(s(u))+f(v) - f(t(v))+f(v)}{2} \mu([t(v);v]) \right]
\]

\[
\sum_{[u;v)\in G} \left[ \frac{f(s(u))-f(t(v))}{2} \mu(J(u)) \right]
\]

\[
\geq \frac{\delta}{2} \left[ \sum_{[u;v)\in G} \mu(J(u)) \right] \geq \frac{\delta}{4} \left[ \sum_{[u;v)\in G} \mu([u;v]) \right] = \frac{\delta}{4} (g(q)-g(p)) \quad \text{This is impossible. Therefore, this lemma is true.}
\]

Remark 5.7. If \( g(b)-g(a)=0 \), then \( \left( \int_a^b h(f) \, dg \right) = 0 \) and \( \left( \int_a^b h(f) \, dg \right) = 0 \). Suppose \( g(b)-g(a) > 0 \). Let, for each \( \delta > 0 \),

\[
S_\delta = \{ [p;q) \subset [a;b) : \text{if } (x;y) \subset (p;q), \text{ then } |f(x)-f(y)| < \delta \}.
\]

Let, for each \( \delta > 0 \), \( T_\delta = \{ Q : Q \text{ is a subset of some simple subdivision of } [a;b] \text{ and } Q \subset S_\delta \} \).

Lemma 5.8. If \( \delta > 0 \) and \( \sigma = \sup \{ E \mu : Q \in T_\delta \} \), then \( \sigma = g(b)-g(a) \).

Proof. Consider the statement that there is a \( \delta > 0 \) such that \( g(b)-g(a) > \sigma = \sup \{ E \mu : Q \in T_\delta \} \). Let \( e = g(b)-g(a)-\sigma \).

There is \( Q \in T_\delta \) such that \( \sigma - E \mu < e/2 \). Let \( P \) be such that
PUQ is a simple refinement of \( ([a;b]) \). Let \( c>0 \) such that 
\[
2c < \frac{5\varepsilon}{8}.
\]
There is a simple subdivision \( D \) of \([a;b] \) such that if \( E \) is a simple refinement of \( D \), then \( \Sigma \int E \{\mu - \alpha(V)\mu(I)\} |< c \).

Let \( L \) be a simple refinement of \( D \) and simple refinement of \( PUQ \). Let, if any, \( L_1 = \{ (p;q) \in L : \) there is \((s;t) \subset (p;q) \) such that \( |f(s) - f(t)| \geq \delta \) and if \((u,v) \subset (p;q) \) is such that \( |f(u) - f(v)| \geq \delta \), then \( \mu([p,u]) + \mu([v;q]) < \frac{1}{2} \mu([p;q]) \). Let, if any, \( L_2 = \{ (p;q) \in L : \) if \((s,t) \subset (p;q) \), then \( |f(s) - f(t)| < \delta \}. \)

Let, if any, \( L_3 = \{ (p;q) \in L : \) there is \((s;t) \subset (p;q) \) such that \( |f(s) - f(t)| \geq \delta \) and \( \mu([p;s]) + \mu([t;q]) > \frac{1}{2} \mu([p;q]) \). \( L = L_1 \cup L_2 \cup L_3 \).

Consider some \( (p;q) \in L_1 \). Let \( u = \sup \{ x \in [p;q) : \) there is \( y \in (x;q) \) such that \( |f(x) - f(y)| \geq \delta \) and \( \mu([p;x]) + \mu([y;q]) < \frac{1}{2} \mu([p;q]) \}). \)

Consider the statement that \( u = q \). For each \( k \in [p;q) \), there is \( x \in (k;q) \) and \( y \in (x;q) \) such that \( |f(x) - f(y)| \geq \delta \) and \( \mu([x;y]) > \frac{1}{2} \mu([p;q]) \). This implies that \( \lim_{y \to q^{-}} g(y) \) does not exist. Since \( g \) is non decreasing, \( u < q \). Also, by the definition of \( u \), \( u > p \).

Consider the statement that if \( k \in (p;u) \), then there is \( (x;y) \subset (k;u) \) such that \( |f(x) - f(y)| \geq \delta \) and \( \mu([p;x]) + \)
\[ \mu([y;q]) < \frac{1}{2} \mu([p;q]). \] This, again, would be impossible since \( g \) is non decreasing. There is \( k \in (p;u) \) such that if \( x \in (k;u) \) such that there is \( y \in (x;q) \) such that \( |f(x) - f(y)| \geq \delta \) and \[ \mu([p;x]) + \mu([y;q]) < \frac{1}{2} \mu([p;q]), \] then \( y \geq u \). Let \( v = \inf(y \in (u;q)): \) there is \( x \in (k;u) \) such that \( |f(x) - f(y)| \geq \delta \) and \[ \mu([p;x]) + \mu([y;q]) < \frac{1}{2} \mu([p;q]). \]

Consider the case that \( u = v \). For each \( c \in (p;u) \) and \( d \in (u;q) \), \[ \mu([c;d]) > \frac{1}{2} \mu([p;q]) \] and \( g(u^+) - g(u^-) \geq \frac{1}{2} \mu([p;q]). \)

Consider case 1a): \( g(u^-) = g(u) \). By Lemma 5.2, there is \( t \in (u;q) \) such that \( [u;t) \) is in \( S_5 \). \[ \mu([u;t]) \geq g(u^+) - g(u) \geq \frac{1}{2} \mu([p;q]). \]

Consider case 1b): \( g(u^-) < g(u) < g(u^+) \). By Lemma 5.2, there is \( s \in (p;u) \) such that \( [s;u) \in S_5 \) and there is \( t \in (u;q) \) such that \( [u;t) \in S_5 \). \[ \mu([s;t]) \geq \frac{1}{2} \mu([p;q]). \]

Consider case 1c): \( g(u) = g(u^+) \). By Lemma 5.2, there is \( s \in (p;u) \) such that \( [s;u) \in S_5 \). \[ \mu([s;u]) \geq \frac{1}{2} \mu([p;q]). \]

Consider the case when \( u < v \). If there is \( (x;y) \in (u;q) \) such that \( |f(x) - f(y)| \geq \delta \), then, by the definition of \( u \), \[ \mu([p;x]) + \mu([y;q]) < \frac{1}{2} \mu([p;q]). \] This a contradiction to the definition of \( L_1 \). \( [u;q) \) is in \( S_5 \).

Consider the case that \( \mu([u;q)) < \frac{1}{2} \mu([p;q)). \) For each
m \in (p; u), there is \( c \in (m; u) \) and \( d \in (u; q) \) such that \( \mu([p; c]) + \mu([d; q]) < \frac{1}{2} \mu([p; q]) \). \( \mu([c; d]) > \frac{1}{2} \mu([p; q]) \). If \( c \in (m; u) \),
\( \mu([c; q]) > \frac{1}{2} \mu([p; q]) \), \( g(q) - g(u) \geq \frac{1}{2} \mu([p; q]) \).

Consider case 2a): \( g(u^-) = g(u) \). \( \frac{1}{2} \mu([p; q]) > \mu([u; q]) \geq \frac{1}{2} \mu([p; q]) \). This is impossible.

Consider case 2b): \( g(u^-) < g(u) < g(u^+) \). By Lemma 5.2, there is \( s \in (p; u) \) such that \( [s; u] \in S_6 \). \( [u; q] \in S_5 \) and \( \mu([s; q]) > \frac{1}{2} \mu([p; q]) \).

Consider case 2c): \( g(u) = g(u^+) \). By Lemma 5.2, there is an \( s \in (p; u) \) such that \( [s; u] \in S_6 \). \( [u; q] \in S_5 \) and \( \mu([s; q]) > \frac{1}{2} \mu([p; q]) \).

Consider case 3): \( \mu([u; q]) \geq \frac{1}{2} \mu([p; q]) \). \( [u; q] \in S_6 \).

Let \( L_4 \) and \( L_5 \) be such that \( L_4 \cup L_5 \ll L_1 \) and the following are true for the six cases:

1) If \([p; q] \in L_1 \) such that case 1a) holds, then there is \([u; t] \) in \( S_5 \) such that \( (u; t) \subset (p; q) \) and \( \mu([u; t]) \geq \frac{1}{2} \mu([p; q]) \) and \([u; t] \in L_4 \) and \([p; u] \in L_5 \) and \([t; q] \in L_5 \).

2) If \([p; q] \in L_1 \) such that case 1b) holds, then there is \([s; u] \) and \([u; t] \) each in \( S_5 \) such that \( (s; t) \subset (p; q) \) and \( \mu([s; t]) \geq \frac{1}{2} \mu([p; q]) \) and \([s; u] \in L_4 \) and \([u; t] \in L_4 \) and \([p; s] \in L_5 \) and \([t; q] \in L_5 \).
3) If \([p; q) \in L_1\) such that case 1c) holds, then there is
\((s; u)\) in \(S_5\) such that \((s; u) \prec [p; q)\) and \(\mu([s; u]) \geq \frac{1}{2} \mu([p; q))\)
and \((s; u) \in L_4\) and \([p; s) \in L_5\) and \((u; q) \in L_5\).

4) If \([p; q) \in L_1\) such that case 2b) holds, then there is
\((s; u)\) such that \([s; u) \prec [p; q)\) each in \(S_5\), \(\mu([s; q)) \geq \frac{1}{2} \mu([p; q))\)
and each of \([s; u)\) and \([u; q) \in L_4\) and \([p; s) \in L_5\).

5) If \([p; q) \in L_1\) such that case 2c) holds, there \((s; u)\) such
that \([s; u)\) and \([u; q)\) each in \(S_5\), \(\mu([s; q)) \geq \frac{1}{2} \mu([p; q))\)
and each of \([s; u)\) and \([u; q)\) is in \(L_4\) and \([p; s) \in L_5\).

6) If \([p; q) \in L_1\) such that case 3) holds, then \((u; q) \in L_4\) and
\([p; u) \in L_5\).

\[ L = L_2 \cup L_3 \cup L_4 \cup L_5. \] Let \(G\) be a simple refinement of \(L\) and \(H\)
be a simple refinement of \(L\) such that if \([p; q) \in L_3\), then
there is \((s; t) \prec (p; q)\) such that \(|f(s) - f(t)| \geq 5\) and such that
\(\mu([p; s)) + \mu([t; q)) \geq \frac{1}{2} \mu([p; q))\), \([p; s)\) and \([s; q)\) each is in
\(G\), \([p; t)\) and \([t; q)\) each is in \(H\) and if \([p; q)\) is in \(L_2 \cup L_4 \cup L_5\),
then \([p; q) \in G\) and \([p; q) \in H\). There is \(E\) is a simple refinement
of \(G\) and of \(H\) such that:

\[ 2c > \sum_{I \in E} |\alpha(Y) - \alpha(Z)| \mu(I). \] Also,
\[ I \in E \]
\[ I \subset Y \subset G \]
\[ I \subset Z \subset H \]
\[ \delta e > \sum_{(p;q) \in L_3} \left( \frac{|f(s)+f(p)|}{2} - \frac{|f(t)+f(p)|}{2} \right) \mu([p;s]) \]

\[ + \left( \frac{|f(s)+f(q)|}{2} - \frac{|f(t)+f(p)|}{2} \right) \mu([s;t]) \]

\[ + \left( \frac{|f(s)+f(q)|}{2} - \frac{|f(t)+f(q)|}{2} \right) \mu([t;q]) \]

\[ \geq \sum_{(p;q) \in L_3} \left( \frac{|f(s)-f(t)|}{2} \right) \mu([p;s]) + \mu([t;q]) \]

\[ \frac{5e}{2} \left( \frac{1}{2} \sum_{[t] \subseteq L_3} \mu \right). \]

Therefore, \( e/2 > \sum \mu \).

Since \( U \cap U_2, \sum \mu \geq \sum \mu \). \( L_2 \) and \( L_4 \) are disjoint.

\[ \sigma - \sum \mu < e/2, \sigma > \sum \mu + \sum \mu, \sigma - \sum \mu < \sigma - \sum \mu, \text{ and, so, } \sum \mu < \]

\[ e/2. \]

Considering how \( L_4 \) and \( L_5 \) were defined, \( \sum \mu \leq \frac{1}{2} \sum \mu \leq \sum \mu \leq \sum \mu \) since \( g(b) - g(a) - \sigma \geq \sum \mu \), \( \sigma \geq \)

\[ \sum \mu > \sigma. \text{ This is impossible. This lemma is true.} \]

Let \( A = (\sup(\|h(f(x))\|: x \in [a;b]) + 1 \). Suppose \( e > 0 \).

Then there is a \( \delta > 0 \) such that if \((x,y) \in [a^*;b^*] \) and \( |x-y| < \)
5, then \(|h(x) - h(y)| < \frac{\varepsilon}{2(g(b) - g(a) + 1)}\). There are subsets of
\(F, P\) and \(Q\), such that \(PUQ\) is a simple refinement of \([a; b]\),
such that \(Q \subseteq S_5\), and such that \(\sum \mu < \frac{\varepsilon}{4A}\). Let \(d = \min\{\frac{1}{4}(q-p): [p; q) \in Q\}\). Let \(G\) and \(H\) be subsets of \(F\) such that \(G \cup H\) is a
simple refinement of \(Q\) such that if \([p; q) \in Q\), then \([p; p+d)\)
and \([q-d; q)\) each is in \(G\) and \([p+d; q-d)\) is in \(H\). Suppose \(E\)
is a simple refinement \(P \cup G \cup H\). Let \(G_1 = \{(p; s) \in E: [p; q) \in Q\}\)
and \(G_2 = \{(t; q) \in E: [p; q) \in Q\}\). Let \(B = \{(s; t) \in E: [s; t) \in G_1 \cup G_2\)
and \([s; t) \in Q\}\).

\[
\sum_{[x; y) \in P \cup G \cup H} \left[ \frac{h(f(x)) + h(f(y))}{2} (g(y) - g(x)) \right] -
\sum_{[u; v) \in E} \left[ \frac{h(f(u)) + h(f(v))}{2} (g(v) - g(u)) \right] =
\sum_{[x; y) \in P} \sum_{[u; v) \subseteq [x; y)} \left[ \frac{h(f(x)) + h(f(y))}{2} -
\frac{h(f(u)) + h(f(v))}{2} \right] (g(v) - g(u)) +
\sum_{[p; q) \in Q} \sum_{[p; v) \subseteq [p; q)} \left[ \frac{h(f(p)) + h(f(p+d))}{2} -
\frac{h(f(p)) + h(f(v))}{2} \right] (g(v) - g(p)) +
\sum_{[u; v) \subseteq [p; q]} \left[ \frac{h(f(p+d)) + h(f(q-d))}{2} -
\frac{h(f(v)) + h(f(u))}{2} \right] (g(v) - g(u)) +
\sum_{[u; q) \subseteq G_2} \left[ \frac{h(f(q-d)) + h(f(q))}{2} -
\frac{h(f(u)) + h(f(q))}{2} \right] (g(q) - g(u)) \leq
\]
\[
\sum_{[x,y] \subseteq P} \frac{h(f(x)) + h(f(y)) - h(f(u)) - h(f(v))}{2} (g(v) - g(u)) + \\
\sum_{[u,v] \subseteq [x,y]} \left[\sum_{[p;q] \subseteq Q} \frac{|h(f(p+d)) - h(f(v))|}{2} (g(v) - g(p)) + \right. \\
\left. \sum_{[u,v] \subseteq [p;q]} \frac{|h(f(q-d)) - h(f(u))|}{2} (g(v) - g(u)) \right] \leq \frac{4A}{2} \sum_{P} \mu([x;y]) + \\
\sum_{[u;q] \subseteq G_2} \left(\frac{\epsilon}{2(g(b) - g(a) + 1)} \mu([p;q]) \right) < \frac{4A}{2} \frac{\epsilon}{4A + \frac{\epsilon}{2}} = \epsilon. \quad \text{Therefore,}
\]

Therefore, \( \int_{a}^{b} h(f) dg \) exists.

**Theorem 5.9.** Suppose that \([a; b]\) and \([a^*; b^*]\) each is an interval, and that \(h\) is a function from \([a^*; b^*]\) into \(\mathbb{R}\), and that if \(f\) is a function from \([a; b]\) into \([a^*; b^*]\), \(g\) is a non-decreasing function from \([a; b]\) into \(\mathbb{R}\), and \(\int_{a}^{b} f dg\) exists, then \(\int_{a}^{b} h(f) dg\) exists. Then \(h\) is continuous on \([a^*; b^*]\).

**Proof.** Suppose that there is a \(\gamma \in [a^*; b^*]\) and an \(\epsilon > 0\) such that if \(\delta > 0\), then there is an \(x \in (\gamma - \delta; \gamma + \delta) \cap [a^*; b^*]\) such that \(|h(x) - h(\gamma)| \geq \epsilon\). Either there is a strictly increasing sequence of points each less than \(\gamma\) and each in \([a^*; b^*]\) such that it converges to \(\gamma\) and if \(\gamma_n\) is in the sequence, then \(|h(\gamma_n) - h(\gamma)| \geq \epsilon\), or there is a strictly decreasing sequence
of points each greater than \( \rho \) and each in \([a^*;b^*]\) such that it converges to \( \rho \) and if \( \rho_n \) is in the sequence, then
\[
|h(\rho_n) - h(\rho)| \geq \epsilon. \] Suppose that there is a strictly decreasing sequence \( \{\rho_n\}_{n=1}^{\infty} \) which converges to \( \rho \) and
\[
|h(\rho_n) - h(\rho)| \geq \epsilon. \]

Let \( f \) be a function from \([a;b]\) into \([a^*;b^*]\) such that \( f(a)=\rho \) and if \( x \in (a;b) \), then \( f(x) = \min(\rho + \max((x-a)\sin\frac{1}{(x-a)}), 0), b^*). \) \( f \) is continuous. Let \( g \) be a function from \([a;b]\) into \( \mathbb{R} \) such that \( g(a)=0 \) and if \( x \in (a;b) \), then \( g(x)=1. \) \( g \) is non decreasing. \( (m) \int_{a}^{b} f \,dg \) exists and \( (m) \int_{a}^{b} h(f) \,dg \) exists.

There is a simple subdivision \( D \) of \([a;b]\) such that if each of \( E_1 \) and \( E_2 \) is a simple refinement of \( D \), then
\[
\left| \sum_{E_1} h(f) \,dg - \sum_{E_2} h(f) \,dg \right| < \epsilon/2.
\]

Let \( p \) be such that \([a;p) \in D \). There is a \( y \in (a;p) \) such that \( f(y) = \rho \). If \( n \in \{1,2,3,\ldots\} \), then \( \rho_n \in (\rho;b^*) \). There is an \( n \in \{1,2,3,\ldots\} \) and \( y_n \in (a;p) \) such that \( f(y_n) = \rho_n \). Let \( E_1 \) be a simple refinement of \( D \) such that if \([w;z) \in D \) and \([w;z) \neq (a;p) \), then \([w;z) \in E_1 \) and each of \([a;y) \) and \([y;p) \) is in \( E_1 \). Let \( E_2 \) then \([w;z) \in E_1 \) and each of \([a;y) \) and \([y;p) \) is in \( E_1 \). Let \( E_2 \) be a simple refinement of \( D \) such that if \([w;z) \in D \) and \([w;z) \neq (a;p) \), then \([w;z) \in E_2 \) and each of \([a;y_n) \) and
\[ [y_n; p) \text{ is in } E_2. \]
\[
\frac{h(f(a)) + h(f(y))}{2} (g(y) - g(a)) - \frac{h(f(a)) + h(f(y_n))}{2} (g(y_n) - g(a))
\]
\[
> \frac{h(\varphi) - h(\varphi_n)(1)}{2} \geq \varepsilon/2. \quad \varepsilon/2 > \varepsilon/2 \text{ is impossible.}
\]

Therefore, this theorem is true.

Theorem 5.10. Suppose that each of \([a^*; b^*]\) and \([a; b]\) is an interval and \(h\) is a function from \([a^*; b^*]\) into \(\mathbb{R}\). Then the following two statements are equivalent:
1) If \(f\) is a function from \([a; b]\) into \([a^*; b^*]\), \(g\) is a function from \([a; b]\) into \(\mathbb{R}\) with bounded variation, and \((m)\int_a^b f dg\) exists, then \((m)\int_a^b h(f) dg\) exists.
2) \(h\) is continuous.

Proof. Statement 1) is a weaker hypothesis than in Theorem 5.9. Therefore, \(h\) is continuous if statement 1) is true.

Suppose \(h\) is continuous. Suppose that \(f\) is a function from \([a; b]\) into \([a^*; b^*]\), \(g\) is a function from \([a; b]\) into \(\mathbb{R}\) with bounded variation, and \((m)\int_a^b f dg\) exists.

Let \(U = [a; b)\) and \(F\) be as defined in Example 2.2. Let \(\gamma, \mu, \varsigma,\) and \(\alpha\) be as defined in Remark 5.3. Let \(j\) be a function from \([a; b]\) into \(\mathbb{R}\) such that \(j(a) = 0\) and if \(x \in (a; b)\), then \(j(x) = \int_{[a; x]} |\mu|\). Let \(\varphi\) be a function such that if \(I \in F, [a; x)\)
then, \( \varphi(I) = \sum_{[p;q) \in C_I} \frac{h(f(q)) + h(f(p))}{2} (j(q) - j(p)) \). Let \( \xi \) be a function such that if \( V \in F \), then \( \xi(V) = ||\mu|| \). Let \( \rho \) be a function such that if \( V \in F \), then \( \rho(V) = \frac{\varphi(V)}{\xi(V)} \) if \( \xi(V) \neq 0 \) and, otherwise, \( \rho(V) = 0 \). Let \( \beta \) be a function such that if \( [p;q) \subset [a;b] \), then \( \beta([p;q)) = \frac{h(f(p)) + h(f(q))}{2} \).

\[
\int_a^b \xi |\mu| = (m) \int_a^b f \, dg. \text{ According to Theorem 2.23, } \int_a^b \xi |\mu| \exists \text{ if and only if } \int_a^b \xi |\mu| \exists. \int_a^b \xi = \int_a^b \xi |\mu|. \text{ According to Theorem 5.1, } (m) \int_a^b h(f) \, dj \exists. \text{ Suppose } \epsilon > 0. \text{ There is a simple subdivision } D \text{ of } [a;b] \text{ such that if } E \text{ is a simple refinement of } D, \text{ then } \left| \sum_{E} \xi \xi - \int_a^b \xi \xi \right| < \epsilon. \text{ Suppose } E \text{ is a simple subdivision of } D. \text{ Then } \epsilon > \left| \sum_{E} \xi \xi - \int_a^b \xi \xi \right| = \left| \sum_{E} (f(q) + f(p)) \right| \left| \left( j(q) - j(p) \right) \right| = \sum_{E} \frac{f(q) + f(p)}{2} (j(q) - j(p)) - \int_a^b \xi \xi |\mu|. \text{ Therefore, } (m) \int_a^b h(f) \, dj = \int_a^b \xi |\mu|. \text{ Suppose } \epsilon > 0. \text{ There is a simple subdivision } A \text{ of } [a;b] \text{ such that if } B \text{ is a simple refinement of } A, \text{ then } \left| \sum_{B} f d j - (m) \int_a^b h(f) \, dj \right| < \epsilon. \text{ Suppose } B \text{ is a refinement of } A. \text{ There is a simple}
refinement \( C \) of \( A \) such that \( \sum_{f \in C} dj = \sum \varphi \). (m) \[ \int_{C}^{b} h(f) \, dj = \int_{a}^{b} \varphi \, da \]

\[ \int_{a; b}^{b} \rho | \mu | = \int_{a; b}^{b} \rho \xi = \int_{a; b}^{b} \varphi \]. According to Theorem 2.23, \( \int_{a; b}^{b} \rho | \mu | \)

exists if and only if \( \int_{a; b}^{b} \rho \mu \) exists. Suppose \( c > 0 \). Then

there is a simple subdivision \( Q \) of \( [a; b) \) such that if \( T \) is a

simple refinement of \( Q \), then \( \left| \sum_{T} \rho \mu - \int_{a; b}^{b} \rho \mu \right| < c \). Suppose \( T \)

is a simple refinement of \( Q \). Then \( c > \left| \sum_{T} \rho \mu - \int_{a; b}^{b} \rho \mu \right| = \left| \sum_{T} \rho \mu - \int_{a; b}^{b} \rho \mu \right| \).

\[ \int_{a; b}^{b} \rho \mu \] = \[ \sum_{T} \frac{h(f(p)) + h(f(q))}{2} (g(q) - g(p)) - \int_{a; b}^{b} \rho \mu \].

(m) \[ \int_{a}^{b} h(f) \, dg = \int_{a; b}^{b} \rho \mu \]. Therefore, this theorem is true.
BIBLIOGRAPHY

