THE TORUS DOES NOT HAVE A
HYPERBOLIC STRUCTURE

THESIS

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By

Joe R. Butler, B.S.
Denton, Texas
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I define several basic topics from Algebraic Topology, including fundamental group and universal covering space. I show that the quotient space of the universal cover of a space is isomorphic to the space.

Then, I define the hyperbolic plane, including its metric and show what the "straight" lines are in the plane and what the isometries are on the plane.

Finally, I define a hyperbolic surface, show that the two hole torus is a hyperbolic surface, the hyperbolic plane is a universal cover for any hyperbolic surface, and the quotient space of the universal cover of a surface to the group of automorphisms on the covering space is equivalent to the original surface. I use these results to arrive at the final conclusion.
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Section I

The Fundamental Group

The Fundamental Group of a topological space $X$ at a point $x_0 \in X$, denoted by $\pi(X, x_0)$ is the group consisting of the homotopy classes of loops based at $x_0$. A loop based at $x_0$ is a continuous map $\sigma : [0, 1] \to X$ where $\sigma(0) = \sigma(1) = x_0$. In this group two loops $\sigma_1 \sim \sigma_2$, that is homotopic, if one loop can be deformed onto the other loop in a continuous way. That is, there exists a continuous function $f : [0, 1] \times [0, 1] \to X$ where

$$f(t, 0) = \sigma_1(t), f(t, 1) = \sigma_2(t),$$

and

$$f(0, s) = f(1, s) = x_0.$$  

It is not hard to see that these classes actually form a group under the following operation. $[\sigma_1][\sigma_2] = [\sigma]$ where

$$\sigma(t) = \begin{cases} \sigma_2(2t) & 0 \leq t \leq \frac{1}{2} \\ \sigma_1(2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The identity element would be the class of loops homotopic to the constant map $\sigma(t) = x_0$ for all $t \in [0, 1]$. The inverse of a class of loops $[\sigma]$ would be the class of
loops homotopic to the loop $\sigma'$ where $\sigma'$ is the continuous loop $[0,1] \to X$ along the same path as $\sigma$ except in the opposite direction, i.e.

$$\sigma'(t) = \sigma(1-t),$$

where $t \in [0,1]$. One may wonder if $\pi(X, x) \simeq \pi(X, y)$ for any $x, y \in X$. Well, in general this is not true, but if the space is path connected, then the groups are isomorphic. This is very important to remember, because otherwise the fundamental group of a space would depend on the point $x_0$ in the space that the loops are centered and there would not be a single $\pi$ of a space. Note, if $p : X \to Y$ is continuous and $p(x) = y$, then $p$ induces a homomorphism $P : \pi(X, x) \to \pi(Y, y)$ for $x \in X$ and $y \in Y$ defined by $P([\sigma]) = [p(\sigma)]$ for $\sigma$ any loop in $X$ based at $X$.

**Th. 1.1.** If $X$ is path connected then $\pi(X, x) \simeq \pi(X, y)$ for all $x, y \in X$.

**PF:** Let $[\sigma] \in \pi(X, x)$. Since $X$ is path connected $\exists$ a path $\gamma : [0,1] \to X$ where $\gamma(0) = y$ and $\gamma(1) = x$. Consider $f : \pi(X, x) \to \pi(X, y)$ by $f([\sigma]) = [\gamma^{-1}\sigma\gamma]$.

![Fig 1.2](image)

First $F$ is well defined since if $\sigma_1 \sim \sigma_2$ then $\gamma^{-1}\sigma_1\gamma \sim \gamma^{-1}\sigma_2\gamma$. Since

$$f([\sigma_1][\sigma_2]) = [\gamma^{-1}\sigma_1\sigma_2\gamma]$$

$$= [\gamma^{-1}\sigma_1\gamma\gamma^{-1}\sigma_2\gamma]$$

$$= [\gamma^{-1}\sigma_1\gamma][\gamma^{-1}\sigma_2\gamma],$$

$f$ is a group homomorphism. Since $f$ is 1-1 and onto, $\pi(X, x) \simeq \pi(X, y)$. Note that the isomorphism does depend on the path $\gamma$ chosen.
Now one can denote the fundamental group of a space $X$ that is path connected simply as $\pi(X)$ since the group does not depend on the point at which it is centered in the space.

Ex. 1) $\pi(\mathbb{R}^2) \simeq \{e\}$. This is not hard to see, because clearly any loop in the plane can be deformed to the constant loop.

Ex. 2) $\pi(S^1) \simeq \mathbb{Z}$ where $S^1$ is the unit circle and $\mathbb{Z}$ is the integers. Consider a point $x_0 \in S^1$, any loop $\sigma$ centered at $x_0$ can be related to an integer called the winding number of a loop. The winding number of a loop $\sigma$ is the sum of the number of times $\sigma$ wraps around $S^1$ in the counter clockwise direction minus the number of times $\sigma$ wraps around $S^1$ in the clockwise direction. In order to have a more intuitive feel for the winding number, one can consider a loop in $S^1$ to be a string wrapped (mapped) around a nail, and pulling both ends of the string taunt (a continuous deformation of the loop). The number of times the string is still wrapped around the nail is the winding number for the loop.

It will be shown later that any two loops in $S^1$ are homotopic if and only if they have the same winding number. Thus, $[\sigma]$ would be the class of all loops centered at $x_0$ with the same winding number as $\sigma$. 
Th. 1.2. If $\pi(X, x) \simeq G_x$ and $\pi(Y, y) \simeq G_y$, then $\pi[X \times Y, (x, y)] \simeq G_x \times G_y$.

PF: Consider the projection functions $p_x : X \times Y \to X$ and $p_y : X \times Y \to Y$, and the homomorphisms $P_x$ and $P_y$ induced by $p_x$ and $p_y$. Let

$$F : \pi_1[X \times Y, (x, y)] \to G_x \times G_y$$

defined by $F([\sigma]) = P_x([\sigma]) \times P_y([\sigma])$. $F$ is a group homomorphism since $P_x$ and $P_y$ are homomorphisms. It can shown that $F$ is also 1-1 and onto, $F$ is an isomorphism.

Thus, since $\pi(S^1) \simeq \mathbb{Z}$ and $T = S^1 \times S^1$, then $\pi_1(T) \simeq \mathbb{Z} \times \mathbb{Z}$. Hence $\pi(T)$ is isomorphic to a free abelian group on two generators. To get a better feel for this, consider the loops $a, b$ in $T$ in Fig. 1.5.

Note that $a$ and $b$ are not null homotopic (homotopic to the constant map) and are not homotopic to each other. By looking at an alternative view of the torus as the square with opposite sides identified, it can easily be seen that $a$ and $b$ commute from the fact that $aba^{-1}b^{-1}$ is null homotopic.

Thus $a$ and $b$ generate $\pi(T)$. 
Section II

Covering Spaces

For the rest of the paper, all of the spaces will be path connected and locally path connected (each point has a path connected neighborhood). The covering space of a space $X$ is a pair consisting of a space $\tilde{X}$ and a continuous map $p : \tilde{X} \rightarrow X$ such that each $x \in X$ has a path connected neighborhood $U$ (called elementary), such that each path component of $p^{-1}(U)$ is a homeomorphism onto $U$ by $p$. The fiber of a point $x \in X$ is the set $p^{-1}(x)$, i.e. the set of all $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = x$. Note that since for each point in the fiber of $x$, there is an elementary neighborhood that contains the point, which implies that the fiber of a point is discrete, i.e. for each fiber point one can find a neighborhood that contains no other points of the fiber.

![Fig. 2.1](image)

Ex.1) $(\mathbb{R}, p(t))$, where $p(t) = (\cos(2\pi t), \sin(2\pi t))$, is a covering space for $S^1$.

![Fig. 2.2](image)
Ex.2) \((\mathbb{R} \times \mathbb{R}, p \times p)\), where \((p \times p)(x, y) = (p(x), p(y))\) with \(p(x) = (\cos(2\pi x), \sin(2\pi x))\) is the covering space for the torus.

![Diagram](image)

**Fig. 2.3**

Each topological space that is connected, locally path connected, and semi-locally simply connected, i.e. every \(x \in X\) has a neighborhood such that the homomorphism \(\pi(U, x) \to \pi(X, x)\) is trivial, has a special covering space called a universal cover. A universal cover of a space is a covering space that is simply connected, i.e. \(\pi(\tilde{X}) = \{e\}\). Since \(\mathbb{R}\) and \(\mathbb{R}^2\) are both simply connected, both are examples of universal covering spaces for \(S^1\) and the torus respectively.

One can see how points go back and forth between a space and a covering space using the continuous map \(p\) and \(p^{-1}\). Now since \(p\) is continuous, it will also send a path to a path, and it does preserve homotopic paths. Notice that a path in \(\tilde{X}\) with endpoints in the fiber of a point \(x_0 \in X\) will map down to a loop in \(X\) based at \(x_0\).

![Diagram](image)

**Fig. 2.4**

The next two theorems deal with what is called the lifting of loops in \(X\) to \(\tilde{X}\).

Note that a loop in \(X\) based at \(x_0 \in X\) will lift to a path (not necessarily a loop) in \(\tilde{X}\) and that the endpoints of the path will be points in the fiber of \(x_0\).
TH. 2.1. If \( \sigma \) is a loop in \( X \) centered at \( x \in X \) and \( \tilde{x} \) is in the fiber of \( x \), then \( \exists \) a unique path \( \tilde{\sigma} \) in \( \tilde{X} \) such that \( \tilde{\sigma}(0) = \tilde{x} \) and \( \rho \tilde{\sigma} = \sigma \).

PF: By the definition of a covering space, for each \( x \in X \) one can find an elementary neighborhood \( U \) of \( x \), so one can cover \( \sigma \) with a family of elementary neighborhoods \( \{U_\alpha\} \). Since \( \sigma : [0,1] \to X \) is continuous and \([0,1]\) is compact, the image of \( \sigma \) is compact. So one can get a finite cover of the image of \( \sigma \) from this infinite cover. Now lift this finite set of elementary neighborhoods starting with the elementary neighborhood about \( x \) to a neighborhood in \( \tilde{X} \) about \( \tilde{x} \) and then continuing to the next neighborhood in the covering of \( \sigma \) and lifting it into \( \tilde{X} \) onto the end of the previous neighborhood lifted as shown in Fig. 2.5.

\[
\begin{array}{c}
\tilde{X} \\
\tilde{\sigma} \\
\rho \\
X
\end{array}
\]

Fig. 2.5

Continue this process for all the neighborhoods in the covering of \( \sigma \). This gives a path \( \tilde{\sigma} \) in \( \tilde{X} \) such that \( \rho \tilde{\sigma} = \sigma \). The fact that \( \tilde{\sigma} \) was built from the elementary neighborhoods and \([0,1]\) is connected ensures its uniqueness. \( \diamond \)

The next theorem shows that if two loops in \( X \) are homotopic, then the unique lifts of both loops that begin at the same point in \( \tilde{X} \) are also homotopic.

TH. 2.2. Let \( \sigma_1, \sigma_2 \) be homotopic loops in \( X \) based at \( x_0 \in X \) and let \( \tilde{\sigma}_1, \tilde{\sigma}_2 \) be the paths in \( \tilde{X} \) such that \( \rho \tilde{\sigma}_1 = \sigma_1 \), \( \rho \tilde{\sigma}_2 = \sigma_2 \), and \( \tilde{\sigma}_1(0) = \tilde{\sigma}_2(0) \), then \( \tilde{\sigma}_1 \sim \tilde{\sigma}_2 \).

The proof is done by showing that the endpoints of the unique lifts (that begin at the same point) of two nearby homotopic loops are in the same basic neighborhood and thus are the same point. The loops will be nearby enough to
ensure that these lifts are homotopic.

**PF:** Let \( \sigma_1 \sim \sigma_2 \) in \( X \). By Th. 2.1, there exists unique lifts \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \) such that \( \tilde{\sigma}_1(0) = \tilde{\sigma}_2(0) = \tilde{x} \), where \( p(\tilde{x}) = x \). Now construct a homotopy between \( \tilde{\sigma}_1 \) and \( \tilde{\sigma}_2 \). Since \( \sigma_1 \sim \sigma_2 \) there exists a continuous function \( F : [0,1] \times [0,1] \to X \) such that \( F(t,0) = \sigma_1(t) \), \( F(t,1) = \sigma_2 \), and \( F(0,t) = F(1,t) = x_0 \). Note that for any \( a \in [0,1] \), then \( F(t,a) \) is a loop in \( X \) that is homotopic to \( \sigma_1 \) and \( \sigma_2 \). Denote the loop \( F(t,a) \) by \( \sigma_a \). By the process of Th. 2.1, one can cover \( \sigma_a \) by a finite number of elementary neighborhoods, \( \{U_a_i\}_i \). Let \( U_a = \cup U_a_i \), and let \( V_a = F^{-1}(U_a) \). Now \( V_a \) covers the preimage of \( U_a \), and inside \( V_a \) one can find a sufficiently small open strip \( V_a' \) that contains the line \( y = a \).

![Fig. 2.6](https://example.com/fig26.png)

One can do this for each \( a \in [0,1] \), and thus form a cover for \([0,1] \times [0,1] \). Since the unit square is compact one can find a finite cover. Each strip \( V_a' \) in the finite cover would map to a thin strip in \( X \) about the loop \( \sigma_a \) and be entirely contained in the finite cover of \( \sigma_a \) consisting of elementary neighborhoods in \( X \). Now consider the lift of the thin strip about \( \tilde{\sigma}_a \), the lifts of all the loops in the thin strip would begin and end at the same points in \( \tilde{X} \)(their endpoint is in the same elementary neighborhood).

![Fig. 2.7](https://example.com/fig27.png)
Thus they are homotopic. Since this is an open cover (meaning adjacent open neighborhoods overlap) and on the overlaps the lifts of the loops must agree (again they are in the elementary neighborhoods), and since one can do this with any set in the finite cover of $[0,1] \times [0,1]$, one can see that the lifts beginning at $\tilde{x}$ of all the loops in $X$ that are images of horizontal lines in $[0,1] \times [0,1]$ must end at the same point and thus be homotopic. Thus $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$. "

The following argument proves that $\pi(S^1) \simeq \mathbb{Z}$ which was stated in Section I. Consider the cover $(\mathbb{R}, p(t))$ of $S^1$ as stated above. Consider the fundamental group of $S^1$ to be based at the point $(1,0)$. Let $[\sigma] \in \pi(S^1, (1,0))$. Consider $\tilde{\sigma}$, the unique lift of $\sigma$ such that $\tilde{\sigma}(0) = 0$. Now $\tilde{\sigma}$ is a path in $\mathbb{R}$ such that $\tilde{\sigma}(1) = n$ where $n \in \mathbb{Z}$. This is because the endpoint of $\tilde{\sigma}$ has to be a fiber point of $(1,0)$. Define $F : \pi_1(S^1, (1,0)) \to \mathbb{Z}$ by $F([\sigma]) = n$ where $n = \tilde{\sigma}(1)$ and $\tilde{\sigma}(0) = 0$. By theorems 2.1 and 2.2, $F$ is well defined. Now show $F$ is an isomorphism. Since $F$ is clearly 1-1 and onto, all that is left to show is that $F$ is a homomorphism. Consider $[\sigma_1]$ and $[\sigma_2]$ in $\pi_1(S^1, (1,0))$, where $F([\sigma_1]) = n_1$ and $F([\sigma_2]) = n_2$. Let $[\sigma] = [\sigma_1][\sigma_2]$, such that $\sigma = \sigma_1 \sigma_2$ that is

$$
\sigma(t) = \begin{cases} 
\sigma_2(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\sigma_1(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
$$

Thus

$$
\tilde{\sigma}(t) = \begin{cases} 
\tilde{\sigma}_2(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
n_2 + \tilde{\sigma}_1(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
$$

Hence $F([\sigma]) = \tilde{\sigma}(1) = n_2 + \tilde{\sigma}_1(1) = n_2 + n_1 = F([\sigma_2]) + F([\sigma_1])$. Thus $F$ is a homomorphism.

Let $(\widetilde{X}, p)$ be a covering space for $X$, and let $\phi : \widetilde{X} \to \widetilde{X}$ be a homeomor-
phism, such that the following diagram commutes,

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\phi} & \tilde{X} \\
p \downarrow & \searrow & \downarrow p \\
X
\end{array}
\]

then \(\phi\) is an automorphism. The simplest example of an automorphism is the identity map, but in general the automorphism could be much more complicated. It will be important later to note that since the above diagram must commute that an automorphism must map fiber points to fiber points, i.e. if \(p(\tilde{x}) = x\), then \(p(\phi(\tilde{x})) = x\).

Now clearly the set of all automorphisms on a covering space forms a group under the operation of composition of maps. The identity element is be the identity map, and since automorphisms are isomorphisms, the inverses exist. We call this group the group of automorphisms on a covering space, and denote it by \(A(\tilde{X}, p)\).

**Th. 2.3.** Let \((\tilde{X}, p)\) be a universal covering space of \(X\), then \(A(\tilde{X}, p) \simeq \pi(X)\).

Note that an automorphism can be viewed basically as an operation over the fiber of a point. Now fix a point \(x \in X\) and consider the fundamental group of \(X\) as being based at that point, then one only needs to see what the automorphism does to the fiber of \(x\).

**PF:** Let \(\sigma\) be a loop in \(X\) based at \(x_0 \in X\), and \(\tilde{x}_0\) be a point in the fiber of \(x_0\), then by Th. 2.1 \(\exists\) a unique path \(\tilde{\sigma}\) in \(\tilde{X}\) such that \(p\tilde{\sigma} = \sigma\) and \(\tilde{\sigma}(0) = \tilde{x}_0\). In Th. 2.2 it was shown that for any loop \(\sigma' \in [\sigma]\) if \(\tilde{\sigma}'(0) = \tilde{\sigma}(0)\) then the paths \(\tilde{\sigma}'\) and \(\tilde{\sigma}\) are homotopic. Thus \(\tilde{\sigma}'(1) = \tilde{\sigma}(1) = \tilde{x}\), where \(\tilde{x}\) is in the fiber of \(x\). Let \(\tilde{x} = \tilde{x}_0 \cdot [\sigma]\) be the unique end point of \([\tilde{\sigma}]\) where \(\tilde{\sigma}(0) = \tilde{x}_0\). Now define \(F : \pi(X) \to A(\tilde{X}, p)\) for
$x_0 \in X$ choose $\sigma$ to represent $[\sigma]$ based at $x$, then for $\tilde{x}_0 \in p^{-1}(x)$, define $F([\sigma]) = \phi$ where $\phi$ is the automorphism that sends $\tilde{x}_0$ to the other endpoint of $\tilde{\sigma}$ in $p^{-1}$. By Theorems 2.1 and 2.2 $F$ is well defined, 1-1, and onto.

Now to show that $F$ is a homomorphism and thus an isomorphism. Let $[\sigma_1]$ and $[\sigma_2]$ be elements of $\pi(X, x)$ with $F([\sigma_1]) = \phi_1$ and $F([\sigma_2]) = \phi_2$. Then $F([\sigma_1][\sigma_2]) = F([\sigma_1\sigma_2]) = \phi$ where $\phi(\tilde{x}_0) = \tilde{x}_0 \cdot [\sigma_1\sigma_2] = \tilde{x}$, and $\tilde{x}$ is the terminal point of $\tilde{\sigma}_1\tilde{\sigma}_2$. But $\tilde{\sigma}_1\tilde{\sigma}_2 = \tilde{\sigma}_1\tilde{\sigma}_2$ where $\tilde{\sigma}_1(0) = x_0$ and $\tilde{\sigma}_2(1) = \tilde{\sigma}_1(0)$. Thus, if we let $\tilde{\sigma}_2(1) = \tilde{z}_2$, then we get

$$
\phi(\tilde{x}_0) = \tilde{x}_0 \cdot [\sigma_1\sigma_2]
= \tilde{x}_0 \cdot ([\sigma_1][\sigma_2])
= \tilde{x}_2 \cdot [\sigma_1] \tilde{x}_0 \cdot [\sigma_2]
= \phi_1(\phi_2(\tilde{x}_0)),
$$

where $\phi_1$ and $\phi_2$ corresponds to $\sigma_1$ and $\sigma_2$ respectively. Thus $\phi = \phi_1 \circ \phi_2 \circ \phi$.

At this point some important results dealing with universal covering spaces have been reached. However, when a universal covering space for a space $X$ was defined, there were several conditions on $X$ given (connected, locally path connected, and semilocally simply connected) so that it would have a universal cover. Now it would be appropriate to show why those conditions are necessary in order to ensure that a space has a universal cover. Existence of a universal covering space $\tilde{X}$ of a space $X$ can be shown by constructing ($\tilde{X}, p$) in a special way from the existing space $X$. For a space $X$ satisfying these conditions and $x_0 \in X$, let $\tilde{X}$ be the set of all equivalence classes of paths in $X$ with initial point $x_0$. Let $p : \tilde{X} \to X$ be the map that sends a path class $\gamma \in \tilde{X}$ to a point $x \in X$ by $p(\gamma) = x$ where $x$ is the terminal point of $\gamma$. Since $X$ is connected and locally path connected, it is path connected, which implies that every point $x \in X$ is the endpoint of some path.
and since if two paths are going to be equivalent they must at least have the same endpoint, each point in $X$ is the endpoint of at least one path class. Thus $p$ is onto. Now that $\tilde{X}$ and $p$ have been defined, a topology can be put on $\tilde{X}$. Since $X$ is semilocally simply connected one can form a basis for $X$ of neighborhoods $U$ such that the homomorphism $\pi(U) \rightarrow \pi(X)$ is trivial. This implies that every loop in $U$ is homotopic, which implies for any two points $x, y$ in $U$, all paths in $U$ starting at $x$ and ending at $y$ are path homotopic. Thus, define a basic set in $\tilde{X}$ in this way, for any path class $\gamma$ in $\tilde{X}$ with $p(\gamma)$ in the basis neighborhood $U_\gamma$, let $\tilde{U}_\gamma$ be the set containing all of the path classes $\tilde{\gamma}$ where $\tilde{\gamma} = \gamma' \gamma$ with $\gamma'$ is a path class in $U_\gamma$ and has initial point $p(\gamma)$.

All such basic sets can be shown to form a basis for $\tilde{X}$. Note that the neighborhoods in $X$ that were used to form the basis for $X$, are elementary neighborhoods.

To show that $p$ is continuous, one can show that if $U$ is open in $X$, then $p^{-1}(U)$ is open in $\tilde{X}$. All that is left to do to show that $\tilde{X}$ is a universal cover is to show that $\tilde{X}$ is simply connected. First note that for any cover of a space the homomorphism $P: \pi(\tilde{X}, \tilde{x}_0) \rightarrow \pi(X, x_0)$ induced by $p$ is 1-1 ($p(\tilde{x}_0) = x_0$). This is because:

1) if $\tilde{\sigma}$ is a loop in $\tilde{X}$, then $p\tilde{\sigma}$ is a loop in $X$,

2) if $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$ in $\tilde{X}$, then $p\tilde{\sigma}_1 \sim p\tilde{\sigma}_2$ and,

3) Th. 2.2.
Thus, all that is left to be shown is that $P(\pi(\tilde{X}, \tilde{x}_0)) = \{1\}$ (the identity element), then $\pi(\tilde{X}) = 1$, and hence $\tilde{X}$ would be simply connected. Choose any $[\sigma] \in \pi(X, x_0)$ (which is also a point in $\tilde{X}$) and consider the following path in $\tilde{X}$, $\tilde{\sigma} : [0, 1] \to \tilde{X}$ by $\tilde{\sigma}(s) = [\sigma(st)]$ for $t \in [0, 1]$. Note that $\tilde{\sigma}(0) = [\sigma(0 \cdot t)] = \tilde{x}_0$ (it is just the equivalence class of constant paths based at $x_0$) and $\tilde{\sigma}(1) = [\sigma(1 \cdot t)] = [\sigma(t)]$ (the point in $\tilde{X}$ we started with). Thus $\tilde{\sigma}$ is a path in $\tilde{X}$ that connects $\tilde{x}_0$ and $[\sigma]$ (one can show that $\tilde{\sigma}$ is continuous using the open base neighborhoods from above). Then the endpoint of the actual lift of $\sigma$ is $\tilde{x}_0 \cdot [\sigma] = [\sigma]$. Thus, since these equivalence classes of loops in $X$ lift to equivalence classes of loops in $\tilde{X}$, then $\tilde{x}_0 \cdot [\sigma] = \tilde{x}_0$, which is true iff $[\sigma] = 1$. Thus the image of $P(\pi(\tilde{X}, \tilde{x}_0)) = \{1\}$, and since $P$ is 1-1, $\pi(\tilde{X}) = 1$. Hence $\tilde{X}$ is simply connected.
Section III

The Hyperbolic Plane

In this section I wish to define the hyperbolic plane \( \mathbb{H}^2 \) and get some useful properties of it. Define \( \mathbb{H}^2 = \{ z : z \in \mathbb{C} \text{ and } \text{Im}(z) > 0 \} \), \( \text{Im}(z) \) is the imaginary part of \( z \). The \( x \)-axis union infinity is the boundary of our model and is identified with the circle at infinity, \( S^1_\infty \).

Now that I have a definition for the hyperbolic plane, I need a metric on it. The hyperbolic metric for this model is derived from the differential

\[
\frac{ds}{y},
\]

where \( ds \) is the differential euclidian distance and \( y \) is the imaginary part of a complex number. The first question one may ask is how is this integral going to give me a distance between any two points \( z_1, z_2 \in \mathbb{H}^2 \). I define my hyperbolic metric to be,

\[
\rho(z_1, z_2) = \inf_{\{\gamma\}} \int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt,
\]

where \( \gamma : [0, 1] \to \mathbb{H}^2 \) is continuously differentiable and \( \gamma(0) = z_1 \) and \( \gamma(1) = z_2 \). In general \( [0,1] \) could be any closed interval. Note that this would mean that the distance between some point \( z_1 \) and another point \( z_2 \) as \( \text{Im}(z_2) \) goes to 0 goes to \( \infty \), in other words

\[
\lim_{\text{Im}(z_2) \to 0} \rho(z_1, z_2) = \infty.
\]

At this point we want to consider possible isometries of \( \mathbb{H}^2 \).
Consider the Mobius transformations,

\[ T(z) = \frac{az + b}{cz + d}, \]

where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc > 0 \). For a general Mobius transformation \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \). The added restrictions that \( a, b, c, d \in \mathbb{R} \) and \( ad - bc > 0 \) are to insure that the real line is mapped onto the real line and that the upper half-plane is mapped onto itself. Later in this section it will be important to note that the composition of two Mobius transformations and the inverse of a Mobius transformation are Mobius transformations, and thus form a group under composition. Now it can easily be shown that these transformations are 1-1 and onto, so all that is left to show is that these transformations preserve the metric. Let

\[ T(z) = \frac{az + b}{cz + d} \]

be a Mobius transformation and \( \gamma \) be any continuously differentiable curve between \( z_1 \) and \( z_2 \), then

\[
\rho(T(z_1), T(z_2)) = \inf_{\{\gamma\}} \int_0^1 \frac{|T'(\gamma(t))|}{\text{Im}[T(\gamma(t))]} \, dt = \inf_{\{\gamma\}} \int_0^1 \frac{|T'(\gamma(t))||\gamma'(t)|}{\text{Im}[T(\gamma(t))]} \, dt,
\]

and since

\[
\frac{|T'(z)|}{\text{Im}(T(z))} = \frac{1}{\text{Im}(z)},
\]

then

\[
\inf_{\{\gamma\}} \int_0^1 \frac{|T'(\gamma(t))||\gamma'(t)|}{\text{Im}[T(\gamma(t))]} \, dt = \inf_{\{\gamma\}} \int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt
\]

\[
= \rho(z_1, z_2).
\]

Thus the Mobius transformations are isometries on \( \mathbb{H}^2 \) with this metric.
The next property that is needed concerning these transformations deals with a principal called orientation preserving. An orientation on a circle (by circle it is meant vertical lines, which can be viewed as a circle with infinite radius, and circles by the regular definition of a circle) is any ordered set of three points on the circle. A Mobius transformation \( T \) is called orientation preserving if any three ordered points \((z_1, z_2, z_3)\) on a circle, \(C\), are mapped to another circle, \(T(C)\), and the right side of \(C\) (that is the right side with respect to the orientation on \(C\)) is mapped to the right side of \(T(C)\) with respect to the orientation given by \((T(z_1), T(z_2), T(z_3))\).

This definition works because three distinct points determine a unique circle and Mobius transformations map circles to circles and interiors of circles to either the interior of a circle or the exterior. The right side of a circle is either its interior or exterior depending on the orientation.

At this point it would be useful to know what a “straight” line, the shortest path between two points, would be in \( \mathbb{H}^2 \). A straight line in \( \mathbb{H}^2 \) is called a geodesic, and by using the metric and the properties of the orientation preserving isometries to find the geodesics of \( \mathbb{H}^2 \).

Claim: the geodesics of \( \mathbb{H}^2 \) are all vertical lines or segments and all half circles that intersect the x-axis at right angles. First, show that the shortest path between two points on the y-axis is a vertical line.

Let \( p_i \) and \( q_i \) be in \( \mathbb{H}^2 \). Then

\[
\rho(p_i, q_i) = \inf_{\{\gamma\}} \int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt.
\]
If \( \gamma \) were a vertical line between \( p_i \) and \( q_i \), then let \( \gamma(t) = iY(t) \), where \( Y(t) = \text{Im}(\gamma(t)) \). Thus

\[
\int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt = \int_0^1 \frac{|Y'(t)|}{Y(t)} \, dt
\]

\[
= \ln(Y(1)) - \ln(Y(0))
\]

\[
= \ln(p_i) - \ln(q_i)
\]

\[
= \ln\left(\frac{p}{q}\right).
\]

Assume a vertical line is not the shortest path between \( p_i \) and \( q_i \). Then there exists a \( \gamma_0 \) that is not a vertical line and the line integral along \( \gamma_0 \) is less than the line integral on the vertical line between \( p_i \) and \( q_i \). Note that since \( \gamma_0 \) is not a vertical line then there is some \( (a, b) \subset [0, 1] \), such that the real part of \( \gamma_0(t) \neq 0 \) for all \( t \in (a, b) \). Let \( \gamma_0(t) = X(t) + iY(t) \) where \( X(t) \) and \( Y(t) \) are real functions. Thus

\[
\ln\left(\frac{p}{q}\right) > \int_0^1 \frac{|\gamma_0(t)|}{\text{Im}(\gamma_0(t))} \, dt
\]

\[
= \int_0^1 \frac{|X'(t) + iY'(t)|}{Y(t)} \, dt
\]

\[
= \int_0^1 \frac{\sqrt{X'(t)^2 + Y'(t)^2}}{Y(t)} \, dt
\]

\[
\geq \int_0^1 \frac{|Y'(t)|}{Y(t)} \, dt
\]

\[
= \ln(Y(1)) - \ln(Y(0))
\]

\[
= \ln(p_i) - \ln(q_i)
\]

\[
= \ln\left(\frac{p}{q}\right).
\]

Thus, by contradiction, \( \gamma_0 \) is a vertical line, and hence vertical lines are geodesics.

Since Mobius transformations map circles to circles and preserves angles and are orientation preserving isometries, then all circles that intersect the x-axis at
right angles are geodesics. Also, given any two circles in the plane, there exists a Mobius transformation that maps one to the other. Note that given any two points in $\mathbb{H}^2$ there is a unique circle that intersects the x-axis that contains the two points, and thus, the circles that intersect the x-axis at right angles represent all of the geodesics.

It has already been shown that $\rho(pi, qi) = \log(p/q)$, and from that, one can derive the following for any $z, w \in \mathbb{H}^2$:

$$\sinh\left(\frac{1}{2} \rho(z, w)\right) = \frac{|z - w|}{2(\text{Im}[z]\text{Im}[w])^{1/2}}.$$

This result will be needed later in the paper.

It is often useful to look at the Mobius transformations as matrices. The set of Mobius transformations can be represented precisely by the group

$$PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}/ \pm 1,$$

where the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

represents the transformation $z \rightarrow \frac{az + b}{cz + d}$.

Claim: $PSL_2(\mathbb{R})$ is isomorphic to the set of all orientation preserving isometries. Since $PSL_2(\mathbb{R})$ represents the set of all Mobius transformations and each Mobius transformation is an orientation preserving isometry, then all one needs to show is that if $\phi$ is an orientation preserving isometry then it is a Mobius transformation.

Let $\phi$ be an orientation preserving isometry and $\phi(i) = z$. Then there exists a Mobius transformation $T$ such that $T\phi$ maps the positive imaginary axis onto itself, i.e. $T(\phi(ki)) = ci$, where $c > 0$, for all $k > 0$. By changing $T$ by a certain multiple, one can make $T(z) = i$, and then by applying the inversion isometry onto
T, if needed, one can get \( T\phi \) to map \((0, 1)\) onto itself and \((1, \infty)\) onto itself. Thus, since \( T\phi \) is an orientation preserving isometry, it must fix the positive imaginary axis.

Let \( z = x + yi \) and \( T(\phi(z)) = u + vi \), then

\[
\rho(x + yi, ki) = \rho(T(\phi(z)), T(\phi(ki))) = \rho(u + vi, ki)
\]

Recall that

\[
\sinh\left(\frac{1}{2} \rho(z, w)\right) = \frac{|z - w|}{2(\text{Im}[z]\text{Im}[w])^{\frac{1}{2}}}
\]

Applying that result here, one can show that

\[
(x^2 + (y - k)^2)v = (u^2 + (v - k)^2)y \quad \text{for all} \quad k > 0 \Rightarrow y = v \quad \text{and} \quad x^2 = u^2.
\]

Thus \( T(\phi(z)) = z \) or \( T(\phi(z)) = -\bar{z} \). Since \( T \) and \( \phi \) are orientation preserving isometries, then \( T\phi \) is an orientation preserving isometry as well. However, \( T(\phi(z)) = -\bar{z} \) is not an orientation preserving. Thus \( \phi(z) = T^{-1}(z) \), which implies that \( \phi \) is a Mobius transformation.

At this point, one can classify orientation preserving isometries according to the number of fixed points each has. Consider the following:

\[
T(z) = \frac{az + b}{cz + d} = z \Rightarrow cz^2 + (d - a)z - b = 0
\]

\[
\Rightarrow z = \frac{a - d \pm \sqrt{(d - a)^2 + 4cb}}{2c}.
\]

Thus a non-trivial Mobius transformation fixes at most two points. Note that if \( T(z) \) fixed more than two points, then it must be the identity transformation. Using the number of fixed points of an isometry one can place an isometry in one of the
the following three categories:

1) If \(((d - a)^2 + 4cb < 0)\), then \(T(z)\) has one fixed point in \(\mathbb{H}^2\) and is called elliptic.

2) If \(((d - a)^2 + 4cb = 0)\), then \(T(z)\) has one real fixed point and is called parabolic.

3) If \(((d - a)^2 + 4cb > 0)\), then \(T(z)\) has two real fixed points and is called hyperbolic.

So far in this paper the upper half-plane model has been used for the hyperbolic plane, however there is another useful and commonly used model called the Poincaré Disk. In this model the hyperbolic plane is identified with the interior of the unit disk in \(\mathbb{R}^2\). The boundary of the disk is identified with \(S^1_{\infty}\). The isometry from the Poincaré Disk to the upper half-plane is the Mobius transformation

\[ T(z) = \frac{z + i}{-z + i}. \]

This isometry maps \(S^1_{\infty}\) to the real axis and gives geodesics in this model that are the arcs of circles that intersect the disk and intersect \(S^1_{\infty}\) at right angles.

The final result needed about \(\mathbb{H}^2\) has to do with finding a formula for the hyperbolic area of a convex hyperbolic polygon. First find such a result for a hyperbolic triangle, and then for a general convex hyperbolic n-gon.
TH. 3.1. For any hyperbolic triangle with angles $\alpha$, $\beta$, and $\gamma$, the hyperbolic area of the triangle is $\pi - (\alpha + \beta + \gamma)$.

To measure angles in the hyperbolic plane, one measures the Euclidean angle between the Euclidean tangents of the intersecting geodesics at the point of intersection.

PF: Consider the h-triangle with angles $\alpha$, $\beta$, and $\gamma$ in Figure 3.3,

$$\text{Fig. 3.3}$$

where $\gamma$ is the angle with a measure of $0^\circ$ and the vertices $v_\alpha$ and $v_\beta$ are on a geodesic semicircle with a radius of 1. Then the area of the h-triangle is the integral

$$\int_{\cos(\pi - \alpha)}^{\cos \beta} \left[ \int_0^\infty \frac{dy}{y^2} \right] dx = \int_{\cos(\pi - \alpha)}^{\cos \beta} \frac{dx}{(1 - x^2)^{\frac{1}{2}}}$$

$$= -[\cos -1(\cos \beta) - \cos -1(\cos(\pi - \alpha))]$$

$$= \pi - (\alpha + \beta).$$

Since any two h-triangles is the difference of two such triangles, one can get the general result. In order to see that any two h-triangles is the difference of two such triangles consider the triangles in Fig. 3.4,

$$\text{Fig. 3.4}$$

then apply an isometry that maps $w$ to $\infty$, which gives me the following.
Thus the h-area is $\pi - (\alpha + \beta + \phi_2) - (\pi - (\phi_1 + \phi_2))$, and since $\phi_1 = \pi - \gamma$, the h-area is $\pi - (\alpha + \beta + \gamma)$.\hfill\blackslug

Theorem 3.1 leads to the following corollary.

**COROLLARY 3.2.** For any convex hyperbolic $n$-gon with angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, the hyperbolic area of the $n$-gon is $(n - 2)\pi - (\alpha_1 + \ldots + \alpha_n)$.

The following proof will be easier to visualize if the Poincaré disk model for $\mathbb{H}^2$ is used.

**PF:** Let $G$ be a convex hyperbolic $n$-gon. Since $G$ is convex, and dissect $G$ into $n$ h-triangles in the following way.

Thus $h\text{-area}(G) = n\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_n + \beta_1 + \beta_2 + \ldots + \beta_n)$. Since $\beta_1 + \beta_2 + \ldots + \beta_n = 2\pi$, then $h\text{-area}(G) = (n - 2)\pi - (\alpha_1 + \alpha_2 + \ldots + \alpha_n)$.\hfill\blackslug
Section IV

Hyperbolic Surfaces

A hyperbolic surface is a surface, F, with a hyperbolic structure on it determined by what is called an orientation preserving atlas of charts. A chart is a map \( \phi_\alpha \) paired with an open set \( U_\alpha \subseteq F \), such that \( \phi_\alpha \) maps \( U_\alpha \) homeomorphically into \( \mathbb{H}^2 \). Also, the \( U_\alpha \) cover \( F \), i.e. \( \cup U_\alpha = F \). This being a hyperbolic atlas of charts gives the added restriction that for any pair of charts \( (\phi_\alpha, U_\alpha), (\phi_\beta, U_\beta) \), then

\[
\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)
\]

is an orientation preserving isometry on \( \mathbb{H}^2 \) restricted to \( \phi_\alpha(U_\alpha \cap U_\beta) \).

Fig. 4.1

An example of a hyperbolic surface is \( T^2 \), the two hole Torus.

Fig. 4.2

In order to show that \( T^2 \) has a hyperbolic structure, one can identify it with an octagon with the noted identifications made.
This view of $T^2$ is found in the same manner as the alternative view of the Torus used in Section I. Now consider this same construction using geodesics in the Poincaré disk,

and recall that the area of the h-octagon is $6\pi - (\text{the angle sum})$. By adjusting the size of the h-octagon, its h-area can be made to be exactly $4\pi$ (if all the vertex points were on the circle at infinity, the h-area would be $6\pi$, and if one shrank it down very small, then the h-area would be very small). Then by using orientation preserving isometries to glue together the sides that were identified, one gets a set of charts for the edges. Since the angle sum is $2\pi$ and the angles are preserved by the isometries, a small open neighborhood containing a vertex would map down to an elementary neighborhood on the surface about the image of the vertex as a local homeomorphism giving a chart for the vertices.
The reason this construction does not work for the torus is that the h-area of the h-square with opposite sides identified (to which the torus would have been identified) is less than $2\pi$.

The following results concern surfaces and their universal covers.

**Th. 4.1.** $\mathbb{H}^2$ is a universal cover for any compact hyperbolic surface.

**PF:** First show that $\mathbb{H}^2$ is isometric to any complete, connected, simply connected hyperbolic surface, and since any universal cover of a hyperbolic surface is itself a hyperbolic surface, then $\mathbb{H}^2$ is a universal cover for $F$.

Define a map $f$ from $\mathbb{H}^2$ to the surface, and define a map $f'$ from the surface to $\mathbb{H}^2$ such that $f' \circ f$ is the identity on $\mathbb{H}^2$ and $f \circ f'$ is the identity on the surface.

Let $S$ be a complete, connected, simply connected hyperbolic surface and choose $x \in S$ such that $x$ is in the chart $U$ and $\phi(U) = \tilde{x}$, the origin of $\mathbb{H}^2$. Choose any point $\tilde{y} \in \mathbb{H}^2$. There exists a unique geodesic $\gamma$ in $\mathbb{H}^2$ that contains $\tilde{x}$ and $\tilde{y}$. Extend the geodesic $\phi^{-1}(\phi(U) \cap \gamma)$ in $S$ to a complete geodesic. Define $f(\tilde{y}) = y$ where $y$ is the point on the extended geodesic in $S$ such that $\text{dist}(x, y) = \text{dist}(\tilde{x}, \tilde{y})$, with in the same direction from $x$ that $\tilde{y}$ is from $\tilde{x}$.

Consider $x$ as above and let $y \in S$. Choose any path $\sigma$ in $S$ from $x$ to $y$. Cover $\sigma$ with a finite number of neighborhood charts in the same way that paths were covered in Section II. Let $U_{i_1 \leq i \leq i_n}$ be the cover. Choose points $x_i$ on $\sigma$ such that the interval $[x_i, x_{i+1}] \subset U_i$. If $\phi_1$ and $\phi_2$ agree on $U_1 \cap U_2$, good. If they do not agree, then since $\phi_1$ and $\phi_2$ are from a hyperbolic chart $\phi_2\phi_1^{-1}$ is the restriction of an orientation preserving isometry, $g_1$, to $\phi_1(U_1 \cap U_2)$. Thus $\phi_2 = g_1\phi_1$. Replace $\phi_2$ with $g_1^{-1}\phi_2$. Now the maps agree on $U_1 \cap U_2$. Assume that by using this process $\phi_j = \phi_{j-1}$ agree on $U_j \cap U_{j-i}$ for $j = 1, ..., i$ and assume that $\phi_{i+1} \neq \phi_i$ on $U_{i+1} \cap U_i$. By the same process, one can replace $\phi_{i+1}$ with $g_1^{-1}\phi_{i+1}$ and then the maps agree
on $U_{i+1} \cap U_i$. Thus by induction on $i$ this procedure can be done for any $1 \leq i \leq n$.

Set $f'(y) = \phi_n(y)$. Since $f'$ is defined using the open chart neighborhoods, it is a local isometry and $f'|_{U_i} = \phi_i$.

Since one can show that the point $f'(y)$ only depends on the path $\sigma$, and since $S$ is simply connected and homotopic paths define the same value for $f'(y)$, then $f'$ is well defined.

Note $f' \circ f$ is the identity on $\mathbb{H}^2$. Also, since one can show that the image of $f$ is both open and closed in $S$ and $S$ is connected, then $f \circ f'$ is the identity on $S$. Thus $S$ and $\mathbb{H}^2$ are isometric. Hence $\mathbb{H}^2$ is a universal cover for any hyperbolic surface. $\diamond$

**Th. 4.2.** Let $X$ be a surface and $(\tilde{X}, p)$ be a universal cover for $X$, then

$$X = \tilde{X}/A(\tilde{X}, p).$$

**PF:** First one shows that $\tilde{X}/A(\tilde{X}, p)$ is a surface, and then show that it is equivalent to $X$.

Define $\pi : \tilde{X} \to \tilde{X}/A(\tilde{X}, p)$ by $\pi(\tilde{x}) = [\tilde{x}]$, and use $\pi$ to topologize $\tilde{X}/A(\tilde{X})$. Define $U \subset \tilde{X}/A(\tilde{x}, p)$ to be open if $\pi^{-1}(U)$ is open in $\tilde{X}$. Then if one chooses $U$ to be a sufficiently small neighborhood of some point $\tilde{x} \in \tilde{X}$ that represents $[\tilde{x}] \in \tilde{X}/A(\tilde{X}, p)$, such that there are no indentifications being made in $U$ and restrict $\pi$ to $U$, then $\pi|_U : U \to \pi(U)$ is a homeomorphism between an open neighborhood of
[x] and U. Also, if p is restricted to U, then p is a homeomorphism between an open neighborhood of x, U, and an open neighborhood of a point x = p(x) ∈ X, p(U).

Thus, since X is a surface, then one can identify p(U) with a small disk, D², and get a homeomorphism from the disk to π⁻¹(U) ⊂ ∼ X / A(X, p). Hence ∼ X / A(X, p) is a surface.

Define f : ∼ X / A(X, p) → X by f([x]) = p(x). Thus f is 1-1 and onto. One can show that f is also continuous. Hence ∼ X / A(X, p) = X.

By Th. 4.1,

\[ F \simeq \mathbb{H}² / A(F, p). \]

Note that A(F, p) is a set of orientation preserving isometries on \( \mathbb{H}² \), and recall that A(F, p) ~ π(F). Hence

\[ F \simeq \mathbb{H}² / \pi(F). \]

Now the machinery is built to show that the torus does not have a hyperbolic structure.

Assume the torus has a hyperbolic structure, then T ~ \( \mathbb{H}² / \mathbb{Z} \times \mathbb{Z} \). Let α, β be the loops that generate \( \pi(T) \). Since \( \mathbb{H}² \) is hyperbolic, α and β each must fix two points on the x-axis, and act freely and discretely on \( \mathbb{H}² \). Assume α(x) = x and α(y) = y. Then, since α and β commute,

\[ \alpha(\beta(x)) = \beta(\alpha(x)) = \beta(x). \]

Thus α fixes β(x). Similarly α fixes β(y). Since α is hyperbolic, it can only fix two points. Thus either β(x) = x and β(y) = y or β(x) = y and β(y) = x. In either case, both α and β map the unique geodesic between x and y onto itself. Now let z be any point on the geodesic and consider the orbit of z. Since α and β act isometrically on all of \( \mathbb{H}² \), then the orbit of z will be on the geodesic. Since α and β
act freely and discretely on $\mathbb{H}^2$, then around each point of the orbit of $z$ one can find a shortest distance to the next point of the orbit of $z$ on the geodesic. By using that one, can define a tree where the vertex points of the tree are elements of the orbit of $z$ and the edges are the segments between two adjacent vertices. Clearly, one can see that because $\alpha$ and $\beta$ act freely and discretely on the geodesic and because of how the tree is defined, then $\alpha$ and $\beta$ act freely on the tree. However, due to Bass-Serre Theorem (which states that only free groups can act freely on a tree) and since $\mathbb{Z} \times \mathbb{Z}$ is not a free group, then there is a contradiction to the assumption that the torus has a hyperbolic structure. Thus, the torus does not have a hyperbolic structure.
BIBLIOGRAPHY


