DECOHERENCE, MASTER EQUATIONS FOR OPEN QUANTUM SYSTEMS, AND

THE SUBORDINATION THEORY

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This thesis addresses the problem of a form of anomalous decoherence that sheds light into the spectroscopy of blinking quantum dots.

The system studied is a two-state system, interacting with an external environment that has the effect of establishing an interaction between the two states, via a coherence generating coupling, called inphasing. The collisions with the environment produce also decoherence, named dephasing. Decoherence is interpreted as the entanglement of the coherent superposition of these two states with the environment.

The joint action of inphasing and dephasing generates a Markov master equation statistically equivalent to a random walker jumping from one state to the other. This model can be used to describe intermittent fluorescence, as a sequence of "light on" and "light off" states.

The experiments on blinking quantum dots indicate that the sojourn times are distributed with an inverse power law. Thus, a proposal to turn the model for Poisson fluorescence intermittency into a model for non-Poisson fluorescence intermittency is made. The collision-like interaction of the two-state system with the environment is assumed to takes place at random times rather than at regular times. The time distance between one collision and the next is given by a distribution, called the subordination distribution. If the subordination distribution is exponential, a sequence of collisions yielding no persistence is turned into a sequence of "light on" and "light off" states with significant persistence. If the subordination function is an inverse power law the sequel of "light on" and "light off" states becomes equivalent to the experimental sequences.

Different conditions are considered, ranging from predominant inphasing to predominant dephasing. When dephasing is predominant the sequel of "light on" and "light off" states in the time asymptotic limit becomes an inverse power law. If the predominant dephasing involves a time scale much larger than the minimum time scale accessible to the experimental observation, thereby generating persistence, the resulting distribution becomes a Mittag-Leffler function. If dephasing is predominant, in addition to the inverse power law distribution of "light off" and "light on" time duration, a strong correlation between "light on" and "light off" state is predicted.

ACKNOWLEDGMENTS

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CHAPTER 1

INTRODUCTION

This thesis is devoted to the treatment of the theoretical problems emerging from the spectroscopy of blinking quantum dots and single molecules [1, 2, 3, 4]. The main property emerging from these experimental results is the non-Poisson nature of the times of sojourn in the "light on" and "light off" states. This means that these waiting time distributions are not exponential functions: they are inverse power law functions with an index $\mu < 2$. A typical value considered by the theoretical work on this subject [5] is $\mu = 1.5$. It is also important to notice that, according to the statistical analysis recently made by two research groups [6, 7], the experimental sequences of "light on" and "light off" state obey renewal theory. This means that after a jump from the "light on" to the "light off" state, there is no memory left of the earlier sojourn times.

There is no quantum mechanical treatment available that might derive these important results from a prime principle perspective. This thesis does not aim at such an ambitious purpose. Rather I plan to adopt a backward approach, similar to that adopted years ago by Cook and Kimble [8] to account for the experimental results of Dehmelt [9]. Also these experiments had to do with a problem of intermittent fluorescence. In this case the distributions of waiting times in the "light on" and "light off" states are exponential, and the quantum mechanical treatment of Cook and Kimble account for this experimental result by building up a Pauli master equation, which is statistically equivalent to the observed process. In other words, a random walker jumping back and forth from a "light on" to a "light off"

state, is adequately described by a Pauli master equation, and, this important equation can be derived in principle from a rigorous quantum mechanical treatment, such as the Zwanzig projection approach, described in this thesis, supplemented by the Markov approximation.

In other words, according to the experimental results on blinking quantum dots and single molecule spectroscopy the random walker jumps from the one to the other state with a non-Poisson distribution of sojourn times in both states. This means that the survival probability, namely, the population difference between the "light on" and the "light off" state, must decay as an inverse power law. According to the philosophy adopted by Cook and Kimble I have, therefore, to build up a quantum master equation fitting this main requirement.

It is expected that this master equation is not Markovian. Therefore, I am forced to address another important issue. This has to do with the constraint of keeping non-negative the diagonal density matrix elements. This is a property that the known master equations, except for the Lindblad equation, do not fit.

The main idea that I adopt in this thesis serves the purpose of realizing both requests at the same time. I move from the Pauli master equation, which can be derived from the Lindblad master equation, being thus compatible with a quantum mechanical derivation. The events described by this equation occur in a natural time scale. This means that the distance between one event and the next is fixed. This assumption is not quite realistic. If these events correspond to abrupt and almost instantaneous collisions, it is more realistic to assume that the time distance between one event and the next fluctuates. I see, however, that in the case where the time distance between one even and the next is given by an exponential distribution, the resulting master equation remains Markovian, and the survival probability undergoes an exponential decay, although with a larger relaxation time. Thus,

I make the assumption that the time distance between one event and the next is not an exponential, and is described by an inverse power law.

This way of proceedings yields a master equation with an infinite memory without conflicting with the request of maintaining non-negative the diagonal density matrix elements.

To make this treatment compatible with a quantum mechanical origin, I derive the Pauli master equation describing the process occurring in the natural time scale from the joint action of a inphasing and a dephasing term. This means that I consider two non degenerate quantum state, with a quantum coupling producing a linear superposition of both states with expansion coefficients that are harmonic function of time with frequency V. This is the inphasing term. The phase correlation established by the inphasing terms is annihilated through an exponential relaxation, with time constant τ_m. This is the dephasing term. I consider three possible physical conditions: (a) The dephasing term is predominant, and the resulting Pauli master equation corresponds to the quantum Zeno effect.

(b) The inphasing term is predominant

(c) The inphasing time scale is comparable to the dephasing time scale.

The treatment that I apply to move from the natural to the experimental time scale is general and the three earlier conditions can be derived from it as special limiting conditions. In the conclusions of this thesis I shall discuss the properties of the corresponding master equations and I will make conjecture on what would be the corresponding behavior of the blinking quantum dots.

This research work also analyzes the decoherence that affects a quantum bit when it inevitably interacts with the environment. I explain what a quantum bit is and show some models that describe decoherence with the traditional tools of quantum mechanics. Also the process of entanglement between the quantum bit and the environment is widely discussed.

Then, the entanglement measure is considered in both the case of pure and of mixed states of a bipartite quantum system. The entanglement of formation of mixed quantum states is defined and discussed. Then, I show some results about the connection between the entanglement of formation and a nonextensive entropy indicator. Some attempts have been made in order to get maximally entangled states, useful for the teleportation, from the Jaynes principle, by maximizing the nonextensive entropy. Here, I show some limits of this technique. Then, I show some methods that help to study the time evolution of open quantum systems. I consider the situation of one half spin interacting with different baths and explain the way of building up the master equation that describes the time evolution of the reduced statistical density matrix of the spin system. Decoherence is analyzed in depth and I show conditions where the relaxation is not exponential, namely, a stretched exponential might appear, depending on the features of the external bath.

1.1 Q-bit and Decoherence

In this introductory Section I discuss the basic concept of decoherence. According to quantum mechanics the time evolution of the state ket $|\Psi(t)\rangle$, is driven by the Schrodinger equation:

$$\imath\hbar\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle, \tag{1.1}$$

the operator H, acting on the Hilbert space of the ket states is the Hamiltonian of the system. Once the initial state ket $|\Psi(0)\rangle$ is known, the time evolution is fixed by the Eq. (1.1).

Here, I focus my attention on the so called quantum bit. The spin state of a one half spin

particle is a perfect example of a quantum bit. Recent research is dedicating great attention to quantum bit in order to realize dreams like the quantum computer. Unfortunately a quantum bit is destroyed by environment fluctuations. Thus, one of the main issues of this areas is to protect the quantum bit from the influence of the environment. Let us analyze this process which is called decoherence using the postulates of quantum mechanics.

The quantum bit is represented by the following expression:

$$|s\rangle = \alpha|+\rangle + \beta|-\rangle, \tag{1.2}$$

$$|\alpha|^2 + |\beta|^2 = 1.$$

The state kets $|+\rangle$ and $|-\rangle$ can be considered as the eigenstates of the spin operator S_x, the x component of the spin of a one half spin particle respectively to the eigenvalues $\frac{1}{2}$ and $\left(-\frac{1}{2}\right)$. Let us discuss the meaning of Eq. (1.2) according to quantum mechanics . If we make a measurement of the average value of the x-component of the spin on a collection of identical states $|s\rangle$, we get as result respectively $\frac{1}{2}$ with probability $|\alpha|^2$ and $\left(-\frac{1}{2}\right)$ with probability $|\beta|^2$,

$$\langle s|S_x|s\rangle = |\alpha|^2 \left(\frac{1}{2}\right) + |\beta|^2 \left(-\frac{1}{2}\right) \tag{1.3}$$

and the system collapses respectively into $|+\rangle$ or $|-\rangle$. This behavior is named von Neumann postulate of measurement.

1.2   The Environment

The research work on magnetic resonance, for instance, shows that a one half nuclear spin, which might become as a possible candidate of q-bit, undergoes a decoherent process

due to the influence of the environment. The effect of the environment on the q-bit is the "entanglement" with the q-bit. Consequently, the environment destroys the purity of the quantum state. This process takes place as follows: initially the "universe" is described by the state $|\Psi_T(0)\rangle$ which is the tensor product of the q-bit state $|s\rangle$ and the state $|E\rangle$ which represent the environment:

$$\Psi_T(0)\rangle = (\alpha|+\rangle + \beta|-\rangle)\,|E\rangle. \tag{1.4}$$

Then, the environment entangles with the q-bit, destroying it, so that, at the end of the process, the universe is described by the ket state $|\Psi_T(\infty)\rangle$ given by the following form:

$$|\Psi_T(\infty)\rangle = \alpha|+\rangle|E_+\rangle + \beta|-\rangle|E_-\rangle. \tag{1.5}$$

The ket states $|E_+\rangle$ and $|E_-\rangle$ result to be orthogonal so that the q-bit finally is no more a pure state and it is described by the reduced density matrix $\rho_{\mathrm{red}}(\infty)$ given by the following expression:

$$\rho_{\mathrm{red}}(\infty) = |\alpha|^2\,|+\rangle\langle+| + |\beta|^2\,|-\rangle\langle-|. \tag{1.6}$$

The transition from the state $|\Psi_T(0)\rangle$ to the state $|\Psi_T(\infty)\rangle$ is called entanglement between the environment and the system of interest. As to the system of interest, the transition from the pure state $|s\rangle$ to the reduced statistical density matrix $\rho_{\mathrm{red}}(\infty)$ is the process named decoherence. I am going to build up an hamiltonian system which reproduces this effect. First, I want to give some basic ideas of the entanglement.

1.3   The Entanglement

In this Section I plan to explain what the entanglement of formation is. The historical origin of this fascinating subject is settled in the teleportation theory [10]. Since 1993 some research work has been done in this direction and a proper theory has been developed. I make a small report of what teleportation is and how entanglement is related to it.

A sender, which in literature is named Alice, can make experiments on two particles and communicate with a distant receiver, Bob, by using a classical channel, for example a telephone. Alice wants to "send" the unknown information of a spin state $|\phi\rangle$, of a one-half spin particle, to an identical particle in the laboratory of Bob. In Ref. [10] there is a long description of the quantum measurement that Alice has to perform on her couple of particles in order to make Bob's particle collapse into the state $|\phi\rangle$; also, at the end of the experiment, Alice's particle is not any more in the quantum state $|\phi\rangle$. In Ref. [10] it is shown how this process does not violate any relativistic law, since it is necessary that Alice informs Bob of the results of her measurement by using a classical channel, in order to realize the teleportation. The process does not violate the No-cloning theorem, which states that you cannot reproduce a quantum state without destroying the original one; in fact, Alice's particle at the end of the process is not anymore described by the state $|\phi\rangle$.

The quantum property that, in principle, permits to realize the quantum teleportation is the Einstein-Podolsky-Rosen correlation. In order to explain what this property is all about, let us consider the spin Hilbert space of two one-half spin particles. Instead of using the canonical base,

$$\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}\,, \tag{1.7}$$

I consider the Bell basis, given by the following set of four states:

$$|\Psi^{(\pm)}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle \pm |-\rangle|+\rangle \right), \tag{1.8}$$

$$|\Phi^{(\pm)}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|+\rangle \pm |-\rangle|-\rangle \right). \tag{1.9}$$

According to the principles of quantum mechanics, if two particle are prepared in a state described by, for example, $|\Psi^{(+)}\rangle$, one particle influences the other, no matter how far they are apart. By making a measurement, for example, of S_x, the probability of finding the first particle in the state $|+\rangle$ is $\frac{1}{2}$, same as the probability of finding the particle in $|-\rangle$, but once the state of the first particle is detected, it is absolutely certain that the second particle's spin points in the opposite direction. This effect is also named EPR paradox, and it descend from the postulate of quantum mechanics. These long range correlations are responsible for the teleportation of the q-bit.

In Ref. [10] it is shown how teleportation can be performed by using each state of the Bell basis set.

Now the entanglement strength is nothing but a measurement of how a bipartite quantum state is "close" to the possibility of being used to realize the teleportation.

A pure state of a pair of quantum systems is called entangled if it is not factorizable, an example of this condition being the states belonging to the Bell basis set. Obviously those states correspond to the maximum entanglement, which will have an entanglement strength of unity. In the case of a bipartite pure state $|\psi\rangle$, the measurement of the entanglement is defined as the entropy of the reduced density matrix, obtained by tracing the total density matrix $|\Psi\rangle\langle\Psi|$ on the Hilbert space of one of the particles. The logarithm is performed in the base 2. It can be shown that the result does not depend on which Hilbert space you

trace over. According to this technique, the entanglement of the bipartite quantum state $|\Psi\rangle$ is given by the following expression:

$$E\left(|\Psi\rangle\right) = -\text{Tr}\left\{\rho_A \log_2 \rho_A\right\} = -\text{Tr}\left\{\rho_B \log_2 \rho_B\right\}, \tag{1.10}$$

where the reduced density matrix ρ_A is given by

$$\rho_A = \text{Tr}_{H_A}|\Psi\rangle\langle\Psi| \tag{1.11}$$

and ρ_B has similar meaning. Of course Eq. (1.10) means that both ρ_A and ρ_B must be diagonalized and the evaluation of the logarithm is applied to the corresponding eigenvalues. The entanglement ranges between 0 and 1; it can be easily checked that the Bell basis states have maximum entanglement corresponding to unity, while any factorized state has vanishing entanglement:

$$E\left(|\Psi_A\rangle|\Psi_B\rangle\right) = 0. \tag{1.12}$$

The entanglement measure for pure states has some basic properties [11] that I am going to enunciate. The entanglement of independent bipartite system is additive. The entanglement is conserved under any unitary transformation that can be expressed as a product of unitary operators acting on the separate subsystems:

$$E\left(U_A \times U_B|\Psi\rangle\langle\Psi|U_A^\dagger \times U_B^\dagger\right) = E\left(|\Psi\rangle\langle\Psi|\right). \tag{1.13}$$

This property is quite important, the physical meaning being th at both Alice and Bob cannot increase the degree of entanglement by operating on their particles, and the entanglement stored in the bipartite state cannot be increased by making local transformation.

The expectation value of the entanglement cannot be increased by local non unitary transformations: if a bipartite pure state $|\Psi\rangle$ is subjected to a local non unitary operation, for example by Alice, which makes the system collapse in the state $|\Psi_j\rangle$ with probability p_j, the expectation value of the entanglement, $\sum_j p_j E\left(|\Psi_j\rangle\right)$, is no greater than the initial entanglement $E\left(|\Psi\rangle\right)$.

Entanglement can be concentrated and diluted with unit asymptotic efficiency. It means that, for any two bipartite states $|\Psi\rangle$ and $|\Psi'\rangle$, if Alice and Bob share n identical systems in the state $|\Psi\rangle$, they can use local operations and one-way classical communication to prepare m identical systems in the state $|\Psi'\rangle$ and the ratio $\frac{n}{m}$ approaches the quantity $\frac{E\left(|\Psi_{(n)}\rangle\right)}{E\left|\Psi'_{(m)}\right\rangle}$, where $|\Psi_{(n)}\rangle$ is given by the following expression:

$$|\Psi_{(n)}\rangle = |\Psi\rangle|\Psi\rangle\ldots|\Psi\rangle_{\text{n times}}, \tag{1.14}$$

and $|\Psi'_{(m)}\rangle$ is given by

$$|\Psi'_{(m)}\rangle = |\Psi'\rangle|\Psi'\rangle\ldots|\Psi'\rangle_{\text{m times}}. \tag{1.15}$$

The fidelity $\left|\langle\Psi_{(n)}|\Psi'_{(m)}\rangle\right|^2$ approaches unity and the probability of failure vanishes in the limit of large n There are also transformation, named purification protocols, which act on several couples of particles and permit to increase the entanglement of some couples despite the entanglement of the others.

Those are the basic rules describing the entanglement measure in the case of pure states.

Let us consider, now, the case of mixed states, that is a statistical superposition of pure states. The entanglement measurement still has some unsettled problems. There are many proposals in this sense, but, the most claimed measurement is the entanglement of formation. The main issue is that you have to deal with the statistical mixture and the teleportation is

performed by using the maximally entangled pure states. So, if you work with a mixed state you can see if in some way it is possible to find a procedure of extracting the maximally entangled states. The entanglement of formation is a measure that quantify the resources stored in the bipartite mixed states useful to create a maximally entangled state.

The entanglement of formation of a bipartite mixed state ρ is evaluated as follows. The first step consists of considering the set Ω of all the pure-state decompositions of the mixed state ρ, that is all the ensembles $\{p_i, |\Psi_i\rangle, i = 1, 2, 3, \dots\}$ such that the following equality holds true:

$$\rho = \sum_i p_i \, |\Psi_i\rangle\langle\Psi_i| \, . \tag{1.16}$$

Then, the minimum average entanglement of all the pure states of the composition must be evaluated:

$$E_F\left(\rho\right) = \min_\Omega \sum_i p_i E\left(|\Psi_i\rangle\right) . \tag{1.17}$$

From the definition it results that, in a sense, the entanglement of formation is the minimum entanglement stored in the mixed state, since the procedure consists of evaluating the minimum among all the possible decompositions of the mixed state in a statistical mixture of pure states.

In the case of a couple of one half spin particles the entanglement of formation can be evaluated analytically. This procedure is shown in Ref. [12]. I summarize here the steps necessary to evaluate it without giving the demonstration. The first step consists in building up $\tilde{\rho}$, the so called spin-flipped state of ρ. For a pure state $|\Psi\rangle$ the spin-flipped state $|\tilde{\Psi}\rangle$ is given by the following equality:

$$|\tilde{\psi}\rangle = \sigma_y|\psi\rangle^*, \tag{1.18}$$

where σ_y is the Pauli operator expressed in the basis of the eigenstates of σ_z, and the complex

conjugation is expressed in this base too. The spin flip operation can be easily extended to mixed states:

$$\tilde{\rho} = (\sigma_y \times \sigma_y) \cdot \rho^* \cdot (\sigma_y \times \sigma_y). \qquad (1.19)$$

At this stage I have to define the "concurrence" $C(\rho)$, given by the following expression:

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \qquad (1.20)$$

where the λ_i' s are the square roots of the eigenvalues of the non-hermitian matrix $\rho \cdot \tilde{\rho}$, numbered in decreasing order. In Ref. [12] it is demonstrated that these λ_i's are not negative numbers and the concurrence ranges from 0 to unity. The entanglement is a monotonic strictly increasing function of the concurrence, described by the following law:

$$E_F(|\Psi\rangle) = h\left(\frac{1 + \sqrt{1 - C(|\Psi\rangle\langle\Psi|)}}{2}\right), \qquad (1.21)$$

where the function $h(x)$ is defined by the following expression:

$$h(x) \equiv -x \log_2(x) - (1 - x) \log_2(1 - x). \qquad (1.22)$$

This procedure works in the case of pure state, too. In this case the entanglement of formation coincides with the entanglement measure defined by Eq. (1.10). It can be demonstrated that this analytic treatment fulfills the properties I have mentioned before. From Eq. (1.19) it can be easily checked that the entanglement of formation is invariant under the local unitary transformation $U_A \times U_B$. These descriptions of the basic properties of the entanglement measure will help in the following in order to establish a possible connection with some entropy indicators.

1.4 Nonextensive Thermodynamics and Entanglement

In this subsection I want to show some research work that has been done in order to establish a connection between thermodynamics and entanglement. Lots of research work has been devoted to study the thermodynamical behavior of the entanglement by using non-extensive entropy indicators. So it is necessary to make a quick introduction to it.

Let us consider a quantum state described by the statistical density matrix ρ. The entropy indicator $S_q(\rho)$ used is given by the following formula:

$$S_q(\rho) = \text{Tr} \frac{\rho - \rho^q}{q - 1}, \tag{1.23}$$

where the parameter q is a positive real number. Of course Eq. (1.23) is referred to the diagonal form of ρ.

It is easy to prove that, in the limit $q \to 1$, $S_q(\rho)$ tends to the von Neumann entropy:

$$S(\rho) = -\text{Tr}\rho \log(\rho). \tag{1.24}$$

The entropy indicator $S_q(\rho)$ is named non-extensive since it does not fit the additivity condition. Let us consider two statistically independent systems A and B. Since they do not interact the statistical density matrix of the whole system has the factorized form $\rho_A \times \rho_B$. The von Neumann entropy is of course additive:

$$S(\rho_A \times \rho_B) = S(\rho_A) + S(\rho_B), \tag{1.25}$$

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while the non-extensive entropy indicator S_q has the following property:

$$S_q \left(\rho_A \times \rho_B \right) = S_q \left(\rho_A \right) + S_q \left(\rho_B \right) + \left(1 - q \right) S_q \left(\rho_A \right) S_q \left(\rho_B \right). \tag{1.26}$$

In the case $0 < q < 1$, the entropy indicator is superadditive, while, in the case q is greater than unity, it is subadditive. Due to this property this entropy indicator has been largely used in order to describe systems where long range correlations appear.

One of the first attempts in order to build up a thermodynamics of entanglement has been made by the authors of Ref. [13] by using the Jaynes principle of maximum entropy with constraints. The authors consider a system of a couple of one half spin particles, and maximize the non extensive entropy indicator with the constraint on the q-generalized average value of $\hat{B}$, the Bell-CHSH. observable:

$$\hat{B} = \sqrt{2} \left(\sigma_{A\,x} \times \sigma_{B\,x} + \sigma_{A\,y} \times \sigma_{B\,y} \right). \tag{1.27}$$

The Bell-CHSH observable can also be expressed in the Bell basis set:

$$\hat{B} = 2\sqrt{2} \left(\left| \Phi^{(+)} \right\rangle \langle \Phi^{(+)} \right| - \left| \Psi^{(-)} \right\rangle \langle \Psi^{(-)} \right| \right). \tag{1.28}$$

At this stage it is convenient to define the q-expectation value $\langle \hat{Q} \rangle_q$ of a generic observable Q, given by the following form:

$$\langle \hat{Q} \rangle_q = \frac{\mathrm{Tr} \hat{\rho}^q Q}{\mathrm{Tr} \rho^q}. \tag{1.29}$$

The authors of Ref. [13] maximize S_q with the following constraints:

$$\langle \hat{B} \rangle_q = b_q \tag{1.30}$$

and

$$\langle \hat{B}^2 \rangle_q = \sigma_q^2. \tag{1.31}$$

From Eq. (1.28) the following inequalities can be easily checked:

$$0 \leq b_q \leq 2\sqrt{2}, \quad \sigma_q^2 \leq 8 \tag{1.32}$$

The authors use the Lagrange multipliers theorem to find out the quantum state corresponding to the maximum non-extensive entropy indicator which fits the constraints given by Eqs. (1.30) and (1.31). After some algebra they obtain the following form:

$$\rho_{\text{max}} = \frac{1}{(Z_q)^{\left(1-\frac{1}{q}\right)}} \left\{ \left(\frac{8-\sigma_q^2}{16}\right)^{\frac{1}{q}} \left(|\Phi^{(-)}\rangle\langle\Phi^{(-)}| - |\Psi^{(+)}\rangle\langle\Psi^{(+)}|\right) + \right.$$
$$\left(\frac{\sigma_q^2 + 2\sqrt{2}b_q}{16}\right)^{\frac{1}{q}} |\Phi^{(+)}\rangle\langle\Phi^{(+)}| +$$
$$\left. \left(\frac{\sigma_q^2 - 2\sqrt{2}b_q}{16}\right)^{\frac{1}{q}} |\Psi^{(-)}\rangle\langle\Psi^{(-)}| \right\}. \tag{1.33}$$

The quantity Z_q is given by the following equality:

$$(Z_q)^{\left(1-\frac{1}{q}\right)} = 2\left(\frac{8-\sigma_q^2}{16}\right)^{\frac{1}{q}} + \left(\frac{\sigma_q^2 - 2\sqrt{2}b_q}{16}\right)^{\frac{1}{q}} + \left(\frac{\sigma_q^2 + 2\sqrt{2}b_q}{16}\right)^{\frac{1}{q}}. \tag{1.34}$$

In the limit $q \to \infty$, the statistical density matrix ρ_{max} tends to $\frac{1}{4}I_A \times I_B$.

The authors of Ref. [13] study the entanglement stored in ρ_{max}, the simplest way is to evaluate the concurrence since in this case the spin flip operation leaves the statistical density matrix unchanged. This way the condition of non vanishing entanglement turns out

to be:

$$\left(\frac{\sigma_q^2 + 2\sqrt{2}b_q}{16}\right)^{\frac{1}{q}} > \frac{1}{2} \qquad (1.35)$$

So, if the inequality (1.35) is satisfied, the Jaynes principle of maximizing the nonextensive entropy indicator gives mixed states storing a certain amount of entanglement. Let us choose $b_q = 2\sqrt{2}$, the value of σ_q is forced to be equal to 8 and, as result of the Jaynes principle, the state $\rho_{\max}$ turns out to be $|\Phi^{(+)}\rangle\langle\Phi^{(+)}|$. This is an extremely interesting result, since the state $|\Phi^{(+)}\rangle$ is one of the precious bipartite states that can be used to realize the teleportation of a q-bit. This is the result of the research performed in Ref. [13].

The summary that I have made about the connection between entanglement and Jaynes principle applied to the non extensive entropy indicators is important. It helps to understand a part of my research work which is the argument of Ref. [14]. Here, a completely different perspective from the one presented in Ref. [13] is shown.

I am considering the spin state of a couple of one half spin particles, for which the most general expression for a mixed state ρ is given by the following expression:

$$\rho = \frac{1}{16}\left\{I \times I + \left(\sum_{i=1}^{3} r_i \sigma_i\right) \times I + I \times \left(\sum_{i=1}^{3} s_i \sigma_i\right) + \sum_{i=1}^{3}\sum_{j=1}^{3} t_{i,j}\sigma_i \times \sigma_j\right\}, \qquad (1.36)$$

where the symbol I denotes the identity operator in the spin space of each particle. The parameters r_i, s_j and $t_{i,j}$ are real numbers. It is extremely difficult to deal with the general form of ρ, for this reason I focus my attention on the set $\mathcal{I}$, defined as follows:

$$\mathcal{I} = \{\rho : C(\rho) = 2P_m - 1 > 0\}. \qquad (1.37)$$

this set contains the states whose concurrence is a given function of the maximum eigenvalue

of the statistical density matrix: $C\left(\rho\right) = 2P_m - 1$.

I stress that, since $\text{Tr}\rho = 1$, there is a unique eigenvalue greater than $\frac{1}{2}$ and the following inequality

$$0 \le \frac{P_j}{P_m} < 1 \tag{1.38}$$

holds true for every $j = 1, 2$ and 3. Here, P_j denotes each one of the eigenvalues of ρ different from the maximum one.

The physical conditions of interest for the field of quantum teleportation are described by state which belong to the set $\mathcal{I}$. First of all, since the concurrence is a monotonically strictly increasing function of the entanglement, every state belonging to $\mathcal{I}$ has positive entanglement, thus, it is not factorizable:

$$E_F\left(\rho\right) > 0, \quad \forall \rho \in \mathcal{I}. \tag{1.39}$$

Then, it is also easy to prove that every mixed state fulfilling the equality

$$\rho \cdot \tilde{\rho} = \rho^2, \tag{1.40}$$

belongs to the set $\mathcal{I}$, under the additional condition that the corresponding statistical density matrix has an eigenvalue greater than $\frac{1}{2}$. In fact, according to the prescription given by Eqs. (1.19) and (1.20), if Eq. (1.40) is fulfilled and the maximum eigenvalue is greater than $\frac{1}{2}$, the bipartite state belongs to the set $\mathcal{I}$.

Every quantum state whose density matrix is diagonal in the Bell basis and is at least partially entangled, belongs to the set $\mathcal{I}$.

It is important to notice that every mixed state can be turned into a diagonal form in

the Bell basis set by random bilateral rotations. A detailed explanation of this operation is given in Ref. [10]. These authors consider the purification of a pair of Werner states, showing how it is possible to increase the entanglement of one pair of particles while losing the entanglement stored in the other couple.

Let us show now how, in the set $\mathcal{I}$, it is possible to detect a change of the entanglement using the non extensive entropy indicator S_q. Let us consider that, due to a purification protocol, for example, a transition from initial state ρ_i to the final state ρ_f occurs. The following discussion is devoted to show how it is possible to built up an entropy indicator sensitive to ΔE_F. In order to solve this problem, I consider the case where both the initial and final states belong to the set $\mathcal{I}$. I will show that there does exists an entropy index $Q(\rho_i, \rho_f)$ such that, for $q > Q(\rho_i, \rho_f)$ the nonextensive entropy becomes a strictly decreasing monotonic function of the entanglement of formation: that is, the condition $\Delta E_F > 0$ yields $\Delta S_q < 0$, while the condition $\Delta E_F < 0$ yields $\Delta S_q > 0$.

The way to build up this critical entropy index is quite long. The first step consists of considering two auxiliary states: $\rho_B^{(1)}$ and $\rho_B^{(2)}$, which are equivalent to ρ_i and ρ_f, respectively, as far as the entanglement and the entropy are concerned.

Let $P_m^{(1)}$ be the maximum and $P_1^{(1)}$, $P_2^{(1)}$ and $P_3^{(1)}$ the other eigenvalues of the statistical density matrix ρ_1, and let $P_m^{(2)}$ be the maximum and $P_1^{(2)}$, $P_2^{(2)}$ and $P_3^{(2)}$ be the other eigenvalues of the statistical density matrix ρ_2 set in decreasing order. The auxiliary states are defined as follows:

$$\rho_B^{(1)} \equiv P_m^{(1)}|e_m\rangle\langle e_m| + \sum_{j=1}^{3} P_j^{(1)}|e_j\rangle\langle e_j| \tag{1.41}$$

and

$$\rho_B^{(2)} \equiv P_m^{(2)}|e_m\rangle\langle e_m| + \sum_{j=1}^{3} P_j^{(2)}|e_j\rangle\langle e_j|, \tag{1.42}$$

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where the set $\{|e_m\rangle, |e_1\rangle, |e_2\rangle, |e_3\rangle\}$ is the Bell basis set defined by Eqs. (1.8) and (1.9), no matter the order the quantum states are selected. It can be easily checked that the quantum states $\rho_B^{(1)}$ and $\rho_B^{(2)}$ have the following properties:

$$E_F(\rho_1) = E_F\left(\rho_B^{(1)}\right), \quad E_F(\rho_2) = E_F\left(\rho_B^{(2)}\right), \tag{1.43}$$

$$S_q(\rho_1) = S_q\left(\rho_B^{(1)}\right), \quad S_q(\rho_2) = S_q\left(\rho_B^{(2)}\right). \tag{1.44}$$

At this stage I define the transformation $\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right]$, through the following equality:

$$\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right]\left(\rho_B^{(1)}\right) \equiv P_m(\xi)|e_m\rangle\langle e_m| + \sum_{j=1}^{3} P_j(\xi)|e_j\rangle\langle e_j|. \tag{1.45}$$

The function $P_m(\xi)$ is defined by

$$P_m(\xi) \equiv P_m^{(1)} + \xi\left(P_m^{(2)} - P_m^{(1)}\right), \tag{1.46}$$

and the functions $P_j(\xi)$, with j running from 1 to 3, are defined by the following equalities:

$$P_j(\xi) \equiv P_j^{(1)} + \xi\left(P_j^{(2)} - P_j^{(1)}\right). \tag{1.47}$$

The transformation Ξ_ξ has the following properties:

(a) the quantum state $\Xi_\xi\left[\rho_B^{(1)}, \rho_B^{(2)}\right]\left(\rho_B^{(1)}\right)$ belongs to the set $\mathcal{I}$ for every value of the parameter ξ belonging to the interval $[0, 1]$,

(b) $\Xi_0\left[\rho_B^{(1)}, \rho_B^{(2)}\right]\left(\rho_B^{(1)}\right) = \rho_B^{(1)}$,

(c) $\Xi_1\left[\rho_B^{(1)}, \rho_B^{(2)}\right]\left(\rho_B^{(1)}\right) = \rho_B^{(2)}$.

It can be also easily checked that the maximum eigenvalue of the state $\Xi_\xi \left[\rho_B^{(1)}, \rho_B^{(2)} \right] \left(\rho_B^{(1)} \right)$ is $P_m(\xi)$, for every value of ξ belonging to the interval $[0,1]$. Since both ρ_i and ρ_f belong to the set $\mathcal{I}$, for every value of ξ belonging to $[0,1]$, the quantity $\frac{dP_m}{d\xi}$ is positive if $\Delta E_F > 0$. Conversely, in the case where $\Delta E_F < 0$, the quantity $\frac{dP_m}{d\xi}$ is negative.

Now, let us consider the quantity $\frac{d}{d\xi} S_q \left(\Xi_\xi \left[\rho_B^{(1)}, \rho_B^{(2)} \right] \left(\rho_B^{(1)} \right) \right)$, that for simplicity I denote as $\frac{d}{d\xi} S_q(\xi)$. By using Eq. (1.23), I get the following form:

$$\frac{d}{d\xi} S_q \left(\Xi_\xi \left[\rho_B^{(1)}, \rho_B^{(2)} \right] \left(\rho_B^{(1)} \right) \right) = -\frac{q P_m^{q-1}}{q-1} \frac{dP_m}{d\xi} \left\{ 1 + \sum_{j=1}^{3} \left(\frac{dP_m}{d\xi} \right)^{-1} \left(\frac{P_j}{P_m} \right)^{q-1} \frac{dP_j}{d\xi} \right\}. \quad (1.48)$$

From Eqs. (1.46) and (1.47) I obtain that the eigenvalue $P_m(\xi)$ is the greatest one, so, the ratio $\frac{P_j(\xi)}{P_m(\xi)}$ belongs to the interval $[0,1[$. Thus, I get the following property:

$$\lim_{q \to \infty} \left\{ 1 + \sum_{j=1}^{3} \left(\frac{dP_m}{d\xi} \right)^{-1} \left(\frac{P_j}{P_m} \right)^{q-1} \frac{dP_j}{d\xi} \right\} = 1. \quad (1.49)$$

This means that, for *enough large* q, the following relation holds true:

$$\mathrm{sgn} \left(\frac{d}{d\xi} E_F(\xi) \right) = -\mathrm{sgn} \left(\frac{d}{d\xi} S_q(\xi) \right), \quad (1.50)$$

where $\frac{d}{d\xi} E_F(\xi)$ denotes the function $\frac{d}{d\xi} E_F \left(\Xi_\xi \left[\rho_B^{(1)}, \rho_B^{(2)} \right] \left(\rho_B^{(1)} \right) \right)$. The physical meaning of Eq. (1.50) is quite important: due to the ξ-transformation, an increase of the entanglement of formation is associated with a decrease of the non extensive entropy, provided that the entropy index q is large enough.

Now I am going to show how this critical index can be built up. Let us consider the case $\Delta E_F < 0$, first. In this case, obviously $P_m^{(2)}$ is less than $P_m^{(1)}$, so that $\frac{dP_m}{d\xi}$ is negative. The

entropy index $q^\star(\xi)$, such that for $q > q^\star(\xi)$, the quantity $\frac{dS_q}{d\xi}$ is negative, is built up as follows:

$$q^\star(\xi) \equiv \max\left\{1, \alpha_1^\star(\xi), \alpha_2^\star(\xi), \alpha_3^\star(\xi)\right\}. \tag{1.51}$$

The parameters $\alpha_j^\star(\xi)$ are defined in such a way that, in the case $q > q^\star(\xi)$, the following inequality holds true:

$$\left\{1 + \sum_{j=1}^{3} \left(\frac{dP_m}{d\xi}\right)^{-1} \left(\frac{P_j}{P_m}\right)^{q-1} \frac{dP_j}{d\xi}\right\} > 0. \tag{1.52}$$

The choice of such a parameter is not unique, so I will adopt the following prescription: if both $P_j(\xi)$ and $\frac{dP_j}{d\xi}$ are positive, I set

$$\alpha_j^\star(\xi) \equiv 1 + \left(\log\left(\frac{P_m(\xi)}{P_j(\xi)}\right)\right)^{-1} \log\left(3\left|\frac{dP_m}{d\xi}\right|^{-1} \frac{dP_j}{d\xi}\right); \tag{1.53}$$

otherwise I set $\alpha_j^\star = 1$. This prescription realizes the following inequality:

$$\left|\frac{dP_m}{d\xi}\right|^{-1} \left(\frac{P_j(\xi)}{P_m(\xi)}\right)^{q-1} \frac{dP_j}{d\xi} < \frac{1}{3}, \tag{1.54}$$

which makes the inequality (1.52) true. By plugging the functions $P_m(\xi)$ and $P_j(\xi)$, given by Eqs. (1.46) and (1.47), respectively, in Eq. (1.53), I obtain the following prescription for the evaluation of $\alpha_j(\xi)$: if the constraints

$$P_j^{(2)}(\xi) > P_j^{(1)}(\xi), \quad P_j^{(1)}(\xi) + \xi\left(P_j^{(2)}(\xi) - P_j^{(1)}(\xi)\right), \tag{1.55}$$

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with $j = 1, 2, 3$, hold true, I get

$$\alpha_j^\star(\xi) = 1 + \frac{\log\left(3\frac{P_J^{(1)} - P_J^{(2)}}{P_m^{(2)} - P_m^{(1)}}\right)}{\log\left(\frac{P_m^{(1)}(\xi) + \xi \ P_m^{(2)}(\xi) - P_m^{(1)}(\xi)}{P_j^{(1)}(\xi) + \xi \ P_j^{(2)}(\xi) - P_j^{(1)}(\xi)}\right)}. \tag{1.56}$$

If, at least one of the constraints given by Eq. (1.55) does not hold true, I set the function $\alpha_j^\star(\xi)$ equal to unity. Thus, the entropy index $q(\xi)$ has the following property: for $q > q^\star(\xi)$, the condition $\frac{d}{d\xi}E_F(\xi) < 0$ yields $\frac{d}{d\xi}S_q(\xi) > 0$.

Due to Eq. (1.44), the entropy change $(S_q(\rho_f) - S_q(\rho_i))$ is identical to the quantity $\left(S_q\left(\rho_B^{(2)}\right) - S_q\left(\rho_B^{(1)}\right)\right)$, which is nothing but the entropy change due to the transformation Ξ_ξ, when the parameter ξ continuously goes from 0 to unity. So, the entropy change can be evaluated as follows:

$$\Delta S_q = \int_0^1 \frac{d}{d\xi}S_q(\xi)\,d\xi. \tag{1.57}$$

On the basis of these results, I define the function $Q^\star[0, 1]$ as follows:

$$Q^\star[0, 1] = \sup_{\xi \in [0,1]}\{q^\star(\xi)\}. \tag{1.58}$$

This way, for every entropy index q greater than $Q^\star[0, 1]$, the function $\frac{d}{d\xi}S_q(\xi)$ is positive for every value of ξ belonging to the interval $[0, 1]$. Thus, the quantity ΔS_q, given by Eq. (1.57) is *positive*.

The entropy index $Q^{(\star)}[0, 1]$ could in principle diverge due to the presence of the following term:

$$\frac{P_m^{(1)} + \xi\left(P_m^{(2)} - P_m^{(1)}\right)}{P_j^{(1)} + \xi\left(P_j^{(2)} - P_j^{(1)}\right)}, \tag{1.59}$$

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which I denote with the symbol $\gamma(\xi)$. The divergence would be given by $\gamma \to 1^+$. The function $\gamma(\xi)$ is either increasing (decreasing) or constant function of ξ, depending on wether the quantity $\left(P_j^{(1)} P_m^{(2)} - P_m^{(1)} P_j^{(2)} \right)$ is positive (negative) or vanishing. So, the minimum value of $\gamma(\xi)$ is $\gamma(0) = P_m^{(1)}/P_j^{(1)}$ in the case of $\frac{d\gamma}{d\xi} > 0$, and $\gamma(1) = P_m^{(2)}/P_j^{(2)}$ in the case of $\frac{d\gamma}{d\xi} < 0$. The condition $\frac{d\gamma}{d\xi} = 0$ gives the same value for the two minima. From these properties the following inequality descends:

$$
Q^\star[0,1] \leq \max_{j=1,2,3} \left[1 + \left[\log \left(\min_{j=1,2,3} \left\{ \frac{P_m^{(1)}}{P_j^{(1)}}, \frac{P_m^{(2)}}{P_j^{(2)}} \right\} \right) \right]^{-1} \right.
$$
$$
\left. \left| \log \left(3 \left| \frac{P_j^{(2)} - P_j^{(1)}}{P_m^{(1)} - P_m^{(2)}} \right| \right) \right| \right]. \tag{1.60}
$$

At this stage, for better clarity, it is useful to make a summary of what has been done. I have considered a set $\mathcal{I}$, composed by both mixed and pure states of $S_{1/2}^{(1)} \times S_{1/2}^{(2)}$. I consider a physical transformation whose initial and final states, ρ_1 and ρ_2, respectively, belong to the set $\mathcal{I}$ and for which the entanglement of formation decreases: $\Delta E_f < 0$. It is possible to build up an entropy index $Q^\star[0,1]$ such that, for every q greater than $Q^\star[0,1]$, the corresponding entropy change is positive, $\Delta S_q > 0$. This critical value has been built up considering the virtual states $\rho_B^{(1)}$ and $\rho_B^{(2)}$, equivalent to ρ_i and ρ_f, respectively, as far as the entanglement of formation and entropy are concerned. Then, I create a virtual state $\Xi_\xi \left[\rho_B^{(1)}, \rho_B^{(2)} \right] \left(\rho_B^{(1)} \right)$, which depends on the parameter ξ belonging to the interval $[0,1]$. For every value of ξ the virtual state belongs to the set $\mathcal{I}$ and its entanglement of formation is a strictly decreasing monotonic function of ξ. This way the effective entropy change, ΔS_q, is equal to the entropy change which occurs when the parameter goes from 0 to 1. Each infinitesimal change $dS_q(\xi)$, corresponding to $d\xi$, an infinitesimal change of the parameter ξ, is positive if an entropy index q greater than a critical value $q^*(\xi)$ is chosen. By selecting the supremum of the set of values

that $q^*(\xi)$ assumes for every ξ belonging to $[0, 1]$, I build up a critical index $Q^\star[0, 1]$ with the desired property.

Now I consider the case where a positive change of the entanglement of formation occurs, $\Delta E_F > 0$, and the initial and final states, ρ_1 and ρ_2, respectively, belong to the set $\mathcal{I}$. In this situation, too, it is possible to build up a critical index $Q^{\star\star}[0, 1]$ such that, if $q > Q^{\star\star}[0, 1]$, the corresponding entropy change is negative, $\Delta S_q < 0$.

The way to build up $Q^{\star\star}[0, 1]$ is almost the same as the case of decreasing entanglement. The auxiliary states $\rho_B^{(1)}$ and $\rho_B^{(2)}$ are the same as those given in Eqs. (1.41) and (1.41). The transformation Ξ_ξ is the same as the one described by Eqs. (1.45), (1.46) and (1.47). The only difference is that, due to the entanglement increase, $P_m^{(2)} > P_m^{(1)}$, and consequently the quantity $\frac{d}{d\xi}P_m(\xi)$ is poitive. As in the previous case, for $j = 1, 2$ and 3, I define a quantity $\alpha_j^{\star\star}(\xi)$ in the following way: if both the conditions $P_j(\xi) > 0$ and $\frac{dP_j}{d\xi} > 0$ apply, I set

$$\alpha_j^{\star\star}(\xi) \equiv 1 + \left[\log\left(\frac{P_m}{P_j}\right)\right]^{-1} \log\left[3\left(\frac{dP_m}{d\xi}\right)^{-1}\left|\frac{dP_j}{d\xi}\right|\right]; \qquad (1.61)$$

if at least one of these condition does not apply, I set $\alpha_j^{\star\star}(\xi) \equiv 1$. Then, I define also the critical entropy index $q^{\star\star}(\xi)$ by the following form:

$$q^{\star\star}(\xi) \equiv \max\left\{1, \alpha_1^{\star\star}(\xi), \alpha_2^{\star\star}(\xi), \alpha_3^{\star\star}(\xi)\right\}. \qquad (1.62)$$

This way, in the case $q > q^{\star\star}(\xi)$, the inequality (1.52) is fulfilled and, due to Eq. (1.48), the condition $\frac{d}{d\xi}S_q(\xi) < 0$ is recovered.

It is possible to simplify the expressions of the functions $\alpha_j^{\star\star}(\xi)$ by using Eqs (1.46) and (1.47). If both the constraints $P_j^{(1)}(\xi) > P_j^{(2)}(\xi)$ and $P_j^{(1)} + \xi\left(P_j^{(2)} - P_j^{(1)}\right) > 0$ hold true, I

get

$$\alpha_j^{\star\star}(\xi) = 1 + \frac{\log\left(3\frac{P_J^{(1)} - P_J^{(2)}}{P_m^{(2)} - P_m^{(1)}}\right)}{\log\left(\frac{P_m^{(1)}(\xi) + \xi \ \ P_m^{(2)}(\xi) - P_m^{(1)}(\xi)}{P_j^{(1)}(\xi) + \xi \ \ P_j^{(2)}(\xi) - P_j^{(1)}(\xi)}\right)}, \tag{1.63}$$

while if at least one of these constraints is not fulfilled, the function $\alpha_j^{\star\star}(\xi)$ is set equal to the unity. At this stage I am equipped to define the critical entropy index $Q^{**}[0,1]$ by the following expression:

$$Q^{\star\star}[0,1] = \sup_{\xi \in [0,1]} \{q^{\star\star}(\xi)\}. \tag{1.64}$$

This way, the condition $q > Q^{\star\star}[0,1]$ yields $\frac{d}{d\xi}S_q(\xi) < 0$ for every value of ξ belonging to the interval $[0,1]$. Since the entropy change can be expressed by using Eq. (1.57), by choosing values of the entropy index such that $q > Q^{\star\star}[0,1]$, the condition $\Delta E_F > 0$ yields $\Delta S_q < 0$.

The critical index $Q^{\star\star}[0,1]$ turns out to be finite. Also in this case the term which could make $Q^{\star\star}[0,1]$ diverge is the one given by Eq. (1.59). Adopting the same procedure as in the previous case, I get the following inequality:

$$Q^{\star\star}[0,1] \leq \max_{j=1,2,3}\left[1 + \left[\log\left(\min_{j=1,2,3}\left\{\frac{P_m^{(1)}}{P_j^{(1)}}, \frac{P_m^{(2)}}{P_j^{(2)}}\right\}\right)\right]^{-1}\right.$$
$$\left.\left|\log\left(3\left|\frac{P_j^{(1)} - P_j^{(2)}}{P_m^{(2)} - P_m^{(1)}}\right|\right)\right|\right]. \tag{1.65}$$

Thus, the critical index $Q^{\star\star}[0,1]$ turns out to be finite. Finally, I can define the critical value $Q(\rho_i, \rho_f)$. It is given by the following form:

$$Q(\rho_i, \rho_f) = \max\{Q^{\star}[0,1], Q^{\star\star}[0,1]\}. \tag{1.66}$$

On the base of the properties shown earlier, the critical value $Q(\rho_i, \rho_f)$ turns out to be

finite. Then, for any initial state, ρ_1, and final state, ρ_2, belonging to the set $\mathcal{I}$, with different entanglements of formation, $E_F(\rho_1) \neq E_F(\rho_2)$, the nonextensive entropy change, ΔS_q, is negative or positive according to whether $\Delta E_F > 0$ or $\Delta E_F < 0$, respectively.

Now I consider the reverse property. I imagine a transformation where both the initial and the final states belong to the set $\mathcal{I}$ and the nonextensive entropy indicator *asymptotically changes* and the entanglement of formation *changes*, too. It means that there exists an entropy index value $Q^{(S)}$, such that, for $q > Q^{(S)}$, the nonextensive entropy changes:

$$S_q(\rho_1) \neq S_q(\rho_2), \quad \forall q > Q^{(S)}. \tag{1.67}$$

The additional assumption that the entanglement of formation changes is crucial: in fact, in the set $\mathcal{I}$, it depends only on P_m, the maximum value of the statistical density matrix describing the quantum state. So, the condition $E_F(\rho_1) \neq E_F(\rho_2)$ yields $P_m^{(1)} \neq P_m^{(2)}$.

Let us assume that the nonextensive entropy asymptotically decreases:

$$S_q(\rho_1) > S_q(\rho_2); \tag{1.68}$$

thus, for every $q > Q^{(s)}$, the following inequality holds true:

$$\left(\frac{P_m^{(2)}}{P_m^{(1)}}\right)^q > 1 + \sum_{j=1}^{3}\left[\left(\frac{P_j^{(1)}}{P_m^{(1)}}\right)^q - \left(\frac{P_j^{(2)}}{P_m^{(1)}}\right)^q\right]. \tag{1.69}$$

Since the entanglement of formation changes, the quantity $\frac{P_m^{(2)}}{P_m^{(1)}}$ is different from unity. Then, due to the inequality (1.38), both the ratio $\frac{P_j^{(1)}}{P_m^{(1)}}$ and $\frac{P_j^{(2)}}{P_m^{(1)}}$ are less than unity. Now, since the inequality (1.69) holds true in the limit $q \to \infty$, the ratio $\frac{P_m^{(2)}}{P_m^{(1)}}$ is forced to be greater than unity. If it is not so, the quantity $\left(\frac{P_m^{(f)}}{P_m^{(i)}}\right)^q$ would vanish, and, since the quantity

$\sum_{j=1}^{3} \left[\left(\frac{P_j^{(i)}}{P_m^{(i)}} \right)^q - \left(\frac{P_j^{(f)}}{P_m^{(i)}} \right)^q \right]$ vanishes, in the limit $q \to +\infty$, and the inequality (1.69) would not be fulfilled anymore. Since I am considering quantum states belonging to the set $\mathcal{I}$, the condition $\frac{P_m^{(2)}}{P_m^{(1)}} > 1$ yields $\Delta E_F > 0$.

I stress that the additional condition that the entanglement changes is crucial for the demonstration. Since the nonextensive entropy is a functional of three independent parameters, it is easy to imagine cases where the entropy changes and the entanglement remains constant. Anyway, if, in the set $\mathcal{I}$, the condition that the entanglement changes is fulfilled, the asymptotic change of the nonextensive entropy implies an opposite change of the entanglement of formation.

At this stage I show a result [14] that turns out to interesting, if it is compared to the one of Ref. [13]. Let us consider a subset $\mathcal{I}'$ of the set $\mathcal{I}$, composed of a *finite* number of states, with *different* values of the entanglement of formation. In the set $\mathcal{I}'$, the nonextensive entropy indicator S_q turns out to be equivalent to the inverse of the entanglement of formation E_F, provided that $q > Q_{\mathcal{I}'}$, defined by the following expression:

$$Q_{\mathcal{I}'} \equiv \max \{ Q(\rho_k, \rho_l), \forall \rho_k, \rho_l \in \mathcal{I}', k \neq l \} \tag{1.70}$$

This result shows a quite different point of view from that of Ref. [13]. In fact, in the set $\mathcal{I}'$, the maximum entangled state corresponds to the *minimum* value of the nonextensive entropy indicator S_q, provided that $q > Q_{\mathcal{I}'}$. This is a more direct and efficient way of detecting the entangled states than the Jaynes principle applied to the non extensive entropy shown in Ref.[13].

The non extensive entropy indicator seems to have lots of limitation if we try to use it in order to create a thermodynamics of the entanglement. This is the conclusion that I get

from my research work.

1.5 A Hamiltonian Model for Decoherence

In this Section I go back to the problem of *decoherence* introduced in Section 1.2. I show an hamiltonian picture that describes the process of entanglement between the environment and the q-bit. Again, I name the state ket that, initially, describes the environment, as $|E\rangle$ and I imagine that it interacts with the spin of interest through the following hamiltonian:

$$H = g\,\sigma_x x + H_B; \tag{1.71}$$

the operator σ_x is the Pauli operator and acts on the basis set of the system of interest in the following way:

$$\sigma_x|\pm\rangle = \pm|\pm\rangle, \tag{1.72}$$

while H_B is the hamiltonian which drives the environment, sometimes also named "bath", and x is a bath observable.

At this stage I apply the usual laws of quantum mechanics and evaluate the time evolution of the q-bit. Initially, the q-bit is "factorized" from the state ket of the environment. It is convenient to work in the interaction picture:

$$|\Psi_T^{(I)}(t)\rangle = \exp\left\{\frac{\imath}{\hbar}\,H_B\,t\right\}|\Psi_T(t)\rangle, \tag{1.73}$$

$$A^{(I)}(t) = \exp\left\{\frac{\imath}{\hbar}\,H_B\,t\right\}A_S \exp\left\{-\frac{\imath}{\hbar}\,H_B\,t\right\}. \tag{1.74}$$

The symbol A_S refers to any operator in the Schrodinger picture. In the interaction picture,

the Schrodinger equation, referred to the hamiltonian operator $H = H_{int} + H_0$, gives the following equation:

$$i\hbar\frac{d}{dt}|\Psi_T^{(I)}(t)\rangle = H_{int}^{(I)}(t)|\Psi_T^{(I)}(t)\rangle. \tag{1.75}$$

The solution of Eq. (1.75) is given by the following expression:

$$|\Psi_T^{(I)}(t)\rangle = \overleftarrow{\exp}\left\{-\frac{i}{\hbar}\int_0^t H_{int}^{(I)}(t')\,dt'\right\}|\Psi_T(0)\rangle, \tag{1.76}$$

the symbol $\overleftarrow{\exp}$ denotes the time ordered exponential:

$$\overleftarrow{\exp}\left\{\int_0^t A(t')\,dt'\right\} = 1 + \sum_{n=1}^{\infty}\int_0^t A(t_1)\,dt_1\int_0^{t_1} A(t_2)\,dt_2\ldots\int_0^{t_{n-1}} A(t_n)\,dt_n. \tag{1.77}$$

In the case where the commutator $[A(t_1), A(t_2)]$ vanishes, the following equality holds true:

$$\int_0^t A(t_1)\,dt_1\int_0^{t_1} A(t_2)\,dt_2\ldots\int_0^{t_{n-1}} A(t_n) = \frac{1}{n!}\left(\int_0^t A(t')\,dt'\right)^n, \tag{1.78}$$

so that the time ordered exponential turns into the ordinary exponential operator.

Now I apply these techniques to the model described by Eq. (1.71). After some easy algebra I get the following expression for time evolution of the total system:

$$|\Psi_T^{(I)}(t)\rangle = \overleftarrow{\exp}\left\{-i\frac{g}{\hbar}\sigma_x\int_0^t x^{(I)}(t')\,dt'\right\}|\Psi_T(0)\rangle. \tag{1.79}$$

The operator $x(t)$ is defined by:

$$x^{(I)}(t) = \exp\left\{\frac{i}{\hbar}H_B t\right\}x\exp\left\{-\frac{i}{\hbar}H_B t\right\}. \tag{1.80}$$

The state ket of the whole system at time t, $|\Psi_T(t)\rangle$, can be written in the following form:

$$|\Psi_T(t)\rangle = \alpha|+\rangle|E_+(t)\rangle + \beta|-\rangle|E_-(t)\rangle, \tag{1.81}$$

and the states $|E_\pm(t)\rangle$ are given by the following expression:

$$|E_\pm(t)\rangle = \exp\left\{-\frac{\imath}{\hbar}H_B t\right\}\overleftarrow{\exp}\left\{\mp\imath\frac{g}{\hbar}\int_0^t x^{(I)}(t')\,dt'\right\}|E\rangle. \tag{1.82}$$

At this stage, I am going to face another fundamental issue of quantum mechanics, which is how to describe the system of interest, that is, the q-bit, starting from the state ket of the whole system described by the Eq. (1.81). Ordinary quantum mechanics suggests the operation of tracing the total density matrix over the degrees of freedom of the environment:

$$\rho_{red}(t) = \text{Tr}_{\{B\}}|\Psi_T(t)\rangle\langle\Psi_T(t)| = \sum_i\langle u_i|\Psi_T(t)\rangle\langle\Psi_T(t)|u_i\rangle, \tag{1.83}$$

where $\{|u_i\rangle, i = 1, 2...\}$ is a complete basis set of the Hilbert space of the bath:

$$\sum_i|u_i\rangle\langle u_i| = I_B, \tag{1.84}$$

where I_B is the identity operator acting on the Hilbert space of the bath. After some easy algebra, I get the following expression for the reduced density matrix of the q-bit:

$$\rho_{red}(t) = |\alpha|^2|+\rangle\langle+| + |\beta|^2|-\rangle\langle-| +$$
$$\alpha\beta^*|+\rangle\langle-|\langle E_-(t)|E_+(t)\rangle + \alpha^*\beta|-\rangle\langle+|\langle E_-(t)|E_+(t)\rangle. \tag{1.85}$$

At this stage, I assume that $x(t)$ behaves like a classical variable, so that the commutator

$[x(t_1), x(t_2)]$ vanishes; thus, the time ordered exponential can be replaced by the usual exponential operator. In this way the calculation can be easily done:

$$\langle E_-(t)|E_+(t)\rangle = \langle E_+(t)|E_-(t)\rangle^* = \langle E| \exp\left\{-2\imath \frac{g}{\hbar} \int_0^t x^{(I)}(t')\,dt'\right\} |E\rangle. \tag{1.86}$$

So, the off-diagonal elements of the reduced density matrix are

$$\langle +|\rho_{red}(t)|-\rangle = \langle +|\rho_{red}(t)|-\rangle^* = \alpha\beta^* \exp\left\{-2\imath \frac{g}{\hbar} \int_0^t x^{(I)}(t')\,dt'\right\}. \tag{1.87}$$

I assume that the bath is at equilibrium, so that the variable $y(t)$, defined by:

$$y(t) \equiv \int_0^t x^{(I)}(t')\,dt', \tag{1.88}$$

can be considered gaussian, that is the probability distribution $p(y,t)$ is given by:

$$p(y,t) = \frac{1}{\sqrt{4\pi\langle y^2\rangle}} \exp\left\{-\frac{y^2}{4\langle y^2\rangle}\right\}. \tag{1.89}$$

Thus, the following properties [15] hold true:

$$\langle y(t)\rangle = 0, \tag{1.90}$$

$$\langle y(t_1) y(t_2) y(t_3)\rangle = 0, \tag{1.91}$$

$$\langle y(t_1) y(t_2) y(t_3) y(t_4)\rangle = \langle y(t_1) y(t_2)\rangle\langle y(t_3) y(t_4)\rangle +$$
$$\langle y(t_1) y(t_3)\rangle\langle y(t_2) y(t_4)\rangle + \langle y(t_1) y(t_4)\rangle\langle y(t_2) y(t_3)\rangle. \tag{1.92}$$

Using the properties of gaussian integrals, the result can be easily generalized in the following way:

$$\langle y^{2n+1} \rangle = 0, \tag{1.93}$$

$$\langle y^{2n} \rangle = (2n-1)!! \quad \langle y^2 \rangle^n, \tag{1.94}$$

where n is any natural number. Under the same assumption that the bath is at equilibrium, I can now evaluate the following average value:

$$C(k) = \langle \exp\{\imath ky\} \rangle_{eq} = \int_{-\infty}^{\infty} \exp\{\imath ky\}\, p(y,t) dy \tag{1.95}$$

by considering the Taylor series expansion of the function $C(k)$ about $k = 0$. It is given by the following form:

$$C(k) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(C^{(n)}(k) \right)_{k=0} k^n. \tag{1.96}$$

The quantity $\left(C^{(n)}(k) \right)_{k=0}$ can be evaluated from the expression

$$\left(C^{(n)}(k) \right)_{k=0} = \int_{-\infty}^{\infty} (\imath y)^n\, p(y,t) dy = \imath^n \langle y^n(t) \rangle_{eq}. \tag{1.97}$$

Due to Eq. (1.94), the following equalities hold true:

$$C(k) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\imath)^{2n} (2n-1)!! \quad \langle y^2 \rangle^n k^{2n} = \exp\left\{ -\frac{k^2}{2} \langle y^2 \rangle \right\}. \tag{1.98}$$

Since y is assumed to be a gaussian variable, the following relation [15] holds true:

$$\langle \exp\{-\imath\, k\, y\} \rangle = \exp\left\{ -\frac{k^2}{2} \langle y \rangle^2 \right\}. \tag{1.99}$$

The quantity $\langle y^2(t)\rangle$ can be easily evaluated by using some basic techniques of statistical physics [15]:

$$\langle y^2(t)\rangle = \int_0^t dt' \int_0^t dt'' \langle x(t') x(t'')\rangle, \qquad (1.100)$$

I make the assumption that $x(t')$ and $x(t'')$ are uncorrelated; this means

$$\langle x(t') x(t'')\rangle = 2\langle x^2\rangle \tau_D \delta(|t - t'|) \qquad (1.101)$$

and, since the process is stationary, the quantity $\langle x^2(t)\rangle$ is independent of time. This way I get the following form for $\langle y^2(t)\rangle$:

$$\langle y^2(t)\rangle = 2\langle x^2\rangle \tau_D t, \qquad (1.102)$$

so that Eq. (1.86) can be turned into the following expression:

$$\langle E_-(t)|E_+(t)\rangle = \exp\left\{-4\frac{g^2}{\hbar^2}\langle x^2\rangle \tau_D\, t\right\}. \qquad (1.103)$$

At this stage I can evaluate the time evolution of the off-diagonal elements of the reduced density matrix:

$$\langle +|\rho_{red}(t)|-\rangle = \alpha\beta^* \exp\left\{-4\frac{g^2}{\hbar^2}\langle x^2\rangle \tau_D\, t\right\}. \qquad (1.104)$$

This approach explains why decoherence is a spontaneous process. As times goes on, due to expression (1.86) the states $|E_+(t)\rangle$ and $|E_-(t)\rangle$ tend to be orthogonal and the off diagonal elements of the reduced density matrix tend to vanish with an exponential law, so that the initial pure state is completely destroyed by the process of entangling with the environment. I plan to analyze decoherence induced by different baths and some possible ways to keep it

under control in order to protect the q-bit. I can already observe that, in situations where $y(t)$ does not obey ordinary statistics, for example $\langle y^2(t)\rangle \sim t^\alpha$, with $\alpha < 1$, the process of decoherence is slower and, in a sense, the q-bit is in a "safer" condition than in the ordinary case.

Let us make a more general treatment. The bath is still considered at equilibrium. Again, I make the assumption that the variable $y(t)$ is classical and described by a gaussian density probability distribution. So, Eq. (1.99) still holds true, but I imagine a more general picture than the one described by Eq. (1.101). I consider that the process is stationary, so that $\langle x^2(t)\rangle$ is independent on time, and the correlation function $\Phi_x(t, t')$, defined by

$$\Phi_x(t, t') = \frac{\langle x(t) x(t')\rangle}{\langle x^2\rangle}, \tag{1.105}$$

is just a function of $(t - t')$. I am going to analyze how the decoherence takes place in processes where the correlation function is different from a delta function. The quantity $\langle y^2(t)\rangle$ is given by:

$$\langle y^2(t)\rangle = \langle x^2\rangle \int_0^t dt' \int_0^t dt'' \Phi_x(|t' - t''|) =$$
$$\langle x^2\rangle \int_0^t dt' \int_0^{t'} dt'' \Phi_x(|t' - t''|) + \langle x^2\rangle \int_0^t dt' \int_{t'}^t dt'' \Phi_x(|t' - t''|) =$$
$$2\langle x^2\rangle \int_0^t dt' \int_0^{t'} dt'' \Phi_x(|t' - t''|). \tag{1.106}$$

By making a change of variable, I get the following expression:

$$\langle y^2(t)\rangle = 2\langle x^2\rangle \int_0^t dt' \int_0^{t'} dt'' \Phi_x(t''). \tag{1.107}$$

The latter equation is quite interesting; in fact, it helps to evaluate the asymptotic behavior for $t \to \infty$, by calculating the second time derivative of both the members of Eq. (1.107):

$$\frac{d^2}{dt^2} \langle y^2(t) \rangle = 2 \langle x^2 \rangle \Phi_x (t'') . \tag{1.108}$$

Now let us consider the asymptotic condition $t \to \infty$. The correlation function has the following behavior:

$$\Phi_x (t) \sim \frac{a}{t^\xi} . \tag{1.109}$$

In order to keep $\langle y^2(t) \rangle$ finite for large times, the parameter ξ must be greater than unity; in fact, by integrating twice both the members of Eq. (1.108), I get the following asymptotic form:

$$\langle y^2(t) \rangle \sim \frac{2a \langle x^2 \rangle}{(2 - \xi)(1 - \xi)} t^{2-\xi} . \tag{1.110}$$

I stress that, in order to get diffusion, the parameter ξ must be less that 2, so, according to Eq. (1.109), since $\langle y^2(t) \rangle$ is positive, the parameter a must be negative. This means that the correlation function has asymptotically a negative tail. These results are quite interesting. In fact, going back to Eqs. (1.87) and (1.99), the time evolution of the off-diagonal elements of the reduced density matrix is given by

$$\langle +|\rho_{red}(t)|- \rangle = \alpha \beta^* \exp \left\{ -4 \frac{g^2}{\hbar^2} \eta \, t^\alpha \right\} , \tag{1.111}$$

where the parameters α and η are defined by

$$\alpha = 2 - \xi , \quad \eta = \frac{a \langle x^2 \rangle}{(2 - \xi)(1 - \xi)} . \tag{1.112}$$

The parameter α is less than unity, so the decoherence is described, asymptotically, by a stretched exponential function of time, which decays more slowly than the ordinary exponential. This means that the q-bit is destroyed in a longer time. So, when the q-bit interacts with a bath through the interaction hamiltonian $g\sigma_x x$, the bath can be assumed to be at equilibrium, so that x can be considered to be a classical gaussian variable and the correlation function $\Phi_x(t)$ is asymptotically described by a power law (1.109).

1.6   Bath of Spins

In this Section I analyze the decoherence in the case the spin of interest interacts with a bath of spins. I imagine that the q-bit collides, for a "small" time interval τ_{int}, with one spin a time. This perspective has been already adopted in the literature [16]. I assume that, during the time interval τ_{int}, the time evolution of the system is driven by the following hamiltonian:

$$H_{int} = g\,\sigma_x \Sigma_y, \qquad (1.113)$$

where Σ_y is the Pauli operator acting on the spins of the bath:

$$\Sigma_y|\pm\rangle_y = \pm|\pm\rangle_y. \qquad (1.114)$$

Let us consider the case where the system is prepared in the quantum state described by the expression (1.4) and that each spin of the bath is in the quantum state $|+\rangle_z$, so that the environment $|E\rangle$ is described by the following expression:

$$|E\rangle = |+^{(1)}\rangle_z|+^{(2)}\rangle_z \ldots |+^{(n)}\rangle_z \ldots . \qquad (1.115)$$

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The calculation can be easily done by considering the following relation:

$$|+\rangle_z = \frac{1}{\sqrt{2}} \left(|+\rangle_y + |-\rangle_y \right). \tag{1.116}$$

Throughout this subject I consider a unit system where $\hbar = 1$. Since the q-bit interacts with only one spin of the bath, the time evolution of the system, during the first time interval τ_{int}, is given by the following expression:

$$|\Psi_T(t)\rangle = \exp\left\{ -\imath\, g\, \sigma_x \Sigma_y^{(1)}\, t \right\} \left(\alpha|+\rangle + \beta|-\rangle \right) |+^{(1)}\rangle_z |+^{(2)}\rangle_z \dots. \tag{1.117}$$

Using Eq. (1.116) I can easily evaluate the time evolution of the system, which has the following form:

$$|\Psi_T(t)\rangle = \alpha|+\rangle|E_+(t)\rangle + \beta|-\rangle|E_-(t)\rangle, \tag{1.118}$$

where

$$|E_\pm(t)\rangle = \exp\left\{ \mp\imath\, g\, \Sigma_y^{(1)}\, t \right\} |+^{(1)}\rangle_z |+^{(2)}\rangle_z \dots. \tag{1.119}$$

The latter procedure can be easily generalized in order to find out the time evolution at time t, belonging to the interval $](n-1)\tau_{\text{int}}, n\tau_{\text{int}}]$:

$$|E_\pm(t)\rangle = \exp\left\{ \mp\imath\, g\, \Sigma_y^{(1)}\, \tau_{\text{int}} \right\} |+^{(1)}\rangle_z \exp\left\{ \mp\imath\, g\, \Sigma_y^{(2)}\, \tau_{\text{int}} \right\} |+^{(2)}\rangle_z \dots$$

$$\dots \exp\left\{ \mp\imath\, g\, \Sigma_y^{(n)}\, (t - (n-1)\,\tau_{\text{int}}) \right\} |+^{(n)}\rangle_z \dots \tag{1.120}$$

The spin of interest is described by the reduced density matrix obtained by tracing the total density matrix over the Hilbert space of the bath, which is exactly the same procedure as the one used in Eq. (1.83). The q-bit is so described, at any time instant t belonging to the

interval $](n-1)\tau_{\text{int}}, n\tau_{\text{int}}]$, by Eq. (1.85), where the natural number n represents the number of interactions that the q-bit has had with a spin of the bath. At this stage the quantities $\langle E_+(t)|E_-(t)\rangle$ and $\langle E_-(t)|E_+(t)\rangle$ can be easily evaluated:

$$\langle E_+(t)|E_-(t)\rangle = \langle E_+(t)|E_-(t)\rangle^* =$$

$$(\cos(2g\tau_{\text{int}}))^{n-1} \cos(2g(t-(n-1)\tau_{\text{int}})). \tag{1.121}$$

At the time instants $t = n\tau_{\text{int}}$ the off-diagonal elements of the reduced density matrix are given by

$$\langle +|\rho_{red}(n\tau_{\text{int}})|-\rangle = \alpha\beta^* (\cos(2g\tau_{\text{int}}))^n. \tag{1.122}$$

If $g\tau_{\text{int}} \ll 1$, the quantity $\cos(2g\tau_{\text{int}})$ is positive and less that unity, so that the decoherence is described by the following expression:

$$\langle +|\rho_{red}(t)|-\rangle = \alpha\beta^* \exp\left\{-|\ln(\cos(2g\tau_{\text{int}}))|\frac{t}{\tau_{\text{int}}}\right\}, \tag{1.123}$$

which shows an exponential relaxation. A good approximation of Eq. (1.122) is given by the following form:

$$\langle +|\rho_{red}(t)|-\rangle \simeq \alpha\beta^* \exp\left\{-2g^2\tau_{\text{int}}t\right\}, \tag{1.124}$$

and the time constant $1/(2g^2\tau_{\text{int}})$ gives an estimation of the interval after which the coherence is lost. In this picture I am assuming that the experimental time scale is so large compared to the time interval τ_{int}, that the time t, equal to $(n\tau_{\text{int}})$, can be treated as a continuous variable.

Now I want to analyze another model, slightly different from the previous one, which will be helpful for the future discussions. I imagine that the interactions between the q-bit

and each spin of the bath are separated by a time interval T_{NI}, where the label stands for non-interaction time. I will study the decoherence in the case where the non-interaction time fluctuates randomly. The non-interaction time is assumed to be much larger than τ_{int}, in such a way that the time interval T, between the beginning of an interaction and the next one, is well approximated by T_{NI}:

$$T = (\tau_{\text{int}} + T_{NI}) \simeq T_{NI}. \tag{1.125}$$

Since the system is assumed to be frozen during the interval T_{NI}, at the time instant $t = nT$, the off-diagonal element of the reduced density matrix is still described by Eq. (1.122), which means:

$$\langle + | \rho_{red}(nT) | - \rangle = \alpha\beta^* \left(\cos\left(2g\tau_{\text{int}}\right) \right)^n. \tag{1.126}$$

An alternative form is given by the following expression:

$$\langle + | \rho_{red}(t) | - \rangle = \alpha\beta^* \exp\left\{ -\left| \ln\left(\cos\left(2g\tau_{\text{int}}\right) \right) \right| \frac{t}{T_{NI}} \right\} \tag{1.127}$$

and approximated by

$$\langle + | \rho_{red}(t) | - \rangle \simeq \alpha\beta^* \exp\left\{ -2 \frac{g^2 \tau_{\text{int}}^2}{T_{NI}} t \right\}. \tag{1.128}$$

In this model, too, I assume that t is a continuous variable. The decoherence is, thus, described by an exponential law. These results will be helpful for future discussions. I intend to discuss the case where the time interval T_{NI} fluctuates randomly and see how decoherence changes.

CHAPTER 2

THE MASTER EQUATION

This Chapter is devoted to analyzing the procedure of building up quantum master equations that describe the time evolution of open quantum systems, that is, systems interacting with an external environment characterized by so many degrees of freedom that it is impossible to evaluate the time evolution of the total system.

I have already shown an example where I have considered a q-bit interacting with a bath of spins. Here, the time evolution of the reduced density matrix is evaluated from the Schrodinger equation by tracing, after each interaction, the total density matrix over the degrees of freedom of the environment. Let us see in detail the way the master equation emerges.

2.1   Open Quantum Systems

Let us consider an open quantum system interacting with an external bath. Let $\rho_T(0)$ be the statistical density matrix describing the total system at time $t = 0$. I assume that, initially, the two systems are statistically independent, which means that $\rho_T(0)$ is the product of the density matrices describing each system:

$$\rho_T(0) = \rho(0) \cdot \rho_{\text{eq}}(B).$$ (2.1)

$\rho(0)$ and $\rho_B(0)$ are the density matrixes of the system of interest and the bath, respectively, at time $t = 0$. The whole system is driven by a Hamiltonian H_T, given by the following

form:

$$H_T = H_S + V + H_B. \tag{2.2}$$

The terms H_S and H_B are the Hamilton operators of the system of interest and of the bath, respectively, while V is the interaction term. According to the Schrodinger equation, the time evolution of the statistical density matrix of the whole system is given by:

$$\rho_T(t) = \exp\{-\imath H_T t\}\,\rho_T(0)\exp\{\imath H_T t\}. \tag{2.3}$$

The system of interest, at the time instant t, is described by the reduced density matrix, obtained by tracing over the Hilbert space of the bath:

$$\rho(t) = \mathrm{Tr}_{\{B\}}\left(\exp\{-\imath H_T t\}\,\rho(0)\cdot\rho_{\mathrm{eq}}(B)\exp\{\imath H_T t\}\right). \tag{2.4}$$

Since the operator H_B is hermitian, there exists an orthonormal complete set $\{|v_s\rangle, s = 1, 2, \dots\}$ of eigenvectors of H_B, so that $\rho_{\mathrm{eq}}(B)$ can be expressed in the following form:

$$\rho_{\mathrm{eq}}(B) = \sum_s p_s|v_s\rangle\langle v_s|, \tag{2.5}$$

where $|v_s\rangle$ is the eigenvector of H_B to the eigenvalue λ_s:

$$H_B|v_s\rangle = \lambda_s|v_s\rangle, \quad s = 1, 2\dots \tag{2.6}$$

and λ_s, the eigenvalues of H_B, are real numbers. For every $s = 1, 2, \ldots$, the commutator $[H_B, |v_s\rangle\langle v_s|]$ vanishes and the Liouvillian operator $\mathcal{L}_B$, defined by:

$$\mathcal{L}_B [\cdot] = -\imath [H_B, \cdot], \tag{2.7}$$

has the following property:

$$\mathcal{L}_B \rho_{\mathrm{eq}}(B) = 0. \tag{2.8}$$

By using Eq. (2.5) and the completeness of the set $\{|v_s\rangle, \quad s = 1, 2, \ldots\}$, I can write $\rho(t)$ in the following form:

$$\rho(t) = \sum_s \sum_{s'} p_s \langle v_s| \exp\{-\imath H_T t\} |v_{s'}\rangle \rho_0 \langle v_{s'}| \exp\{\imath H_T t\} |v_s\rangle. \tag{2.9}$$

In order to make the notation easier let us define the operator $A_{s,s'}$ through the following expression:

$$A_{s,s'} \equiv \langle v_s| \exp\{-\imath H_T t\} |v_{s'}\rangle, \tag{2.10}$$

this way the expression for $\rho(t)$ is much simpler:

$$\rho(t) = \sum_s \sum_{s'} A_{s,s'} \rho(0) A_{s,s'}^\dagger. \tag{2.11}$$

The operators $A_{s,s'}$ are called Kraus operators and the transformation given by Eq. (2.9) is named operator sum representation [18]. Now, I make the assumption that, as an observer interested to the dynamics of the system of interest, I am not allowed to make observations at a time scale smaller than τ, which means that I evaluate times $t = n\tau$, for every natural number n. The authors of reference [18] name τ the coarse-grained time.

Let us make the assumption that the bath has only two states: the ground state $|0\rangle$,

$$\mathcal{L}_B |0\rangle\langle 0| = 0, \tag{2.12}$$

and the excited state $|1\rangle$,

$$\mathcal{L}_B |1\rangle\langle 1| = -\gamma |1\rangle\langle 1|. \tag{2.13}$$

Let us consider the case where the interaction term, V, has the following form:

$$V = T\Lambda^\dagger + T^\dagger \Lambda, \tag{2.14}$$

where Λ is given by

$$\Lambda = |0\rangle\langle 1|. \tag{2.15}$$

The trace operation can be easily performed:

$$\mathrm{Tr}_{\{B\}} \Lambda^\dagger \rho_{\mathrm{eq}}(B) = 0. \tag{2.16}$$

Since $\langle \Lambda\Lambda(t)\rangle = \langle \Lambda\Lambda\rangle \exp(-\gamma t)$, I can set $\tau = \frac{1}{\gamma}$. At this stage, the reduced density matrix of the spin of interest can be evaluated at each step n. I imagine that, by making the trace operation at the time instant $n\tau$, the system of interest decouples itself from the bath, which is supposed to go back to equilibrium, so that, at each step, the total density matrix is factorized. Then the system evolves according to Eq. (2.3), where of course $\rho_T(0)$ is replaced by $\rho_T(n)$. The following formula will be helpful:

$$\exp\{\alpha A\} B \exp\{-\alpha A\} = B + \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \frac{\alpha^3}{3!} [A, [A, [A, B]]] + \ldots \tag{2.17}$$

The time evolution for the reduced density matrix is then

$$\rho_{n+1} = \rho_n - \imath\tau \operatorname{Tr}_{\{B\}}\left[H_T, \rho_n \cdot \rho_{\text{eq}}(B)\right] - \frac{\tau^2}{2}\operatorname{Tr}_{\{B\}}\left[H_T, \left[H_T, \rho_n \cdot \rho_{\text{eq}}(B)\right]\right] + \ldots \quad (2.18)$$

After some algebra, I get the following form:

$$\rho_{n+1} - \rho_n = -\imath\left[H_S, \rho_n\right] - \frac{\tau^2}{2}\left(\left[H_S, \left[H_S, \rho_n\right]\right] + \langle \Lambda\Lambda^\dagger \rangle \left\{T^\dagger T\rho_n - 2T\rho_n T^\dagger + \rho_n T^\dagger T\right\}\right). \quad (2.19)$$

whose time derivative is defined by

$$\dot{\rho}(t) = \lim_{\tau \to 0} \frac{\rho_{n+1} - \rho_n}{\tau}, \quad (2.20)$$

thus leading to the result

$$\dot{\rho}(t) = -\imath\left[H_S, \rho(t)\right] + L\rho(t)L^\dagger - \frac{1}{2}L^\dagger L\rho(t) - \frac{1}{2}\rho(t)L^\dagger L. \quad (2.21)$$

The operator L is defined by:

$$L \equiv \tau T \langle \Lambda\Lambda^\dagger \rangle, \quad (2.22)$$

where I have used the van Hove assumption in order to avoid troubles in the limit of vanishing τ. These techniques are deeply discussed in the book of Ref. [17]

In the more general case where the operator V is given by:

$$V = \sum_{n=1}^{N}\left(T_n\Lambda_n^\dagger + T_n^\dagger\Lambda_n\right), \quad (2.23)$$

I get the following master equation:

$$\dot{\rho}(t) = -\imath\left[H_S, \rho(t)\right] + \sum_{n=1}^{N'} \left(L_n \rho(t) L_n^\dagger - \frac{1}{2} L_n^\dagger L_n \rho(t) - \frac{1}{2} \rho(t) L_n^\dagger L_n \right). \qquad (2.24)$$

The form of Eq. (2.24) is named the Lindblad equation [19], which has the important property of keeping the positivity of the density matrix during the time evolution, if the number N' is less than N^2, where N is the dimension of the density matrix [20, 21]. The Lindblad equation typically emerges from the model just considered, for example a q-bit interacting with a bath through a term $k\sigma_x x$. The time evolution of the reduced density matrix is well described by a master equation with the following Lindblad structure:

$$\frac{d}{dt}\rho(t) = -\imath\frac{\Delta}{2}\left[\sigma_z, \rho(t)\right] + k\left(\sigma_x\rho(t)\sigma_x - \rho(t)\right). \qquad (2.25)$$

I will further discuss this point.

Let us go back to the arguments of Section 1.6, considering the same model and making some additional assumptions about the bath. I am going to build up the time evolution of the system with the usual method of imagining to observe the system at time step n, which corresponds to the instant $n\tau$. This way, at each step, the reduced density matrix is given by

$$\rho_{n+1} = \text{Tr}_{\{B\}}\left[U_{(n,\,n+1)}\left(\rho_n \cdot \rho_{\text{eq}}\left(B\right)\right)U_{(n,\,n+1)}^\dagger\right], \qquad (2.26)$$

where $U_{(n,\,n+1)}$ is the time evolution operator from the instant $(n\tau)$ to the instant $(n+1)\tau$. I assume that the interaction with the bath, described by the hamiltonian $(g\sigma_x\eta)$, takes place at regular times $n\tau$, so fast that the decoherence, described by Eqs. (1.81), (1.85) and (1.86), takes place almost instantaneously. During the interval τ, the time evolution of the spin is

driven by the hamiltonian $H_S = -\frac{\Delta}{2}\sigma_z$; thus, starting from the initial condition $|+\rangle_x$, the time evolution of the spin, $|s(t)\rangle$, is given by the following form:

$$|s(t)\rangle = \exp\{-\imath H_S t\}|+\rangle_z = \cos\left(\frac{\Delta}{2}t\right)|+\rangle + \imath \sin\left(\frac{\Delta}{2}t\right)|-\rangle. \tag{2.27}$$

At the instant τ the process of entanglement between the spin and the bath takes place and the total system $|\Psi_T(\tau)\rangle$ is described by the following expression:

$$|\Psi_T(\tau)\rangle = \cos\left(\frac{\Delta}{2}t\right)|+\rangle|\eta_+\rangle + \imath \sin\left(\frac{\Delta}{2}t\right)|-\rangle|\eta_-\rangle. \tag{2.28}$$

Due to fact that the bath is infinitely fast, the decoherence takes place instantaneously, which means

$$\langle\eta_+|\eta_-\rangle = 0. \tag{2.29}$$

Thus, the reduced density matrix turns out to be

$$\rho(\tau) = \left[\cos\left(\frac{\Delta}{2}t\right)\right]^2|+\rangle\langle+| + \left[\sin\left(\frac{\Delta}{2}t\right)\right]^2|-\rangle\langle-|. \tag{2.30}$$

In the following steps the time evolution can be evaluated in the same way, although of course the initial conditions are different. Let us consider the reduced density matrix corresponding to the instant $n\tau$, due to Eq. (2.29), which has the following form:

$$\rho_n = p_1(n)|+\rangle\langle+| + p_2(n)|-\rangle\langle-|. \tag{2.31}$$

Then the time evolution is driven by the hamiltonian H_S, until the next interaction with the

bath takes place. So ρ_{n+1} is described by the following form:

$$\rho_{n+1} = \exp\left\{\imath\frac{\Delta}{2}\sigma_z\right\}\rho_n \exp\left\{-\imath\frac{\Delta}{2}\sigma_z\right\}, \tag{2.32}$$

and the calculations can be easily done using the following equality:

$$|\pm\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right). \tag{2.33}$$

Also the following formula about the Pauli spin operators is useful:

$$\exp\left\{-\imath\frac{\phi}{2}\sigma \cdot \hat{\phi}\right\} = I \cos\left(\frac{\phi}{2}\right) - \imath \sin\left(\frac{\phi}{2}\right)\left(\sigma \cdot \hat{\phi}\right). \tag{2.34}$$

Due to the fact that decoherence takes place at each step, ρ_n can be written in the following way:

$$\rho_n = p_1(n)\,|+\rangle\langle+| + p_2(n)\,|-\rangle\langle-|. \tag{2.35}$$

For simplicity, I fix $p \equiv p_1(1) = \left[\cos\left(\frac{\Delta}{2}\tau\right)\right]^2$ and $q \equiv p_2(1) = \left[\sin\left(\frac{\Delta}{2}\tau\right)\right]^2$. This way I get the following expressions:

$$\rho_{n+1} = (p_1(n)p + p_2(n)q)\,|+\rangle\langle+| + (p_1(n)q + p_2(n)p)\,|-\rangle\langle-|, \tag{2.36}$$

$$p_1(n+1) - p_1(n) = -q\left(p_1(n) - p_2(n)\right), \tag{2.37}$$

and

$$p_2(n+1) - p_2(n) = -q\left(p_2(n) - p_1(n)\right). \tag{2.38}$$

I apply the same technique as the one used in Eq. (2.20), then imagining τ so small as to

be able to make the transit to continuous time, and I get the following system for $p_1(t)$ and $p_2(t)$:

$$\dot{p}_1(t) = -\gamma \left(p_1(t) - p_2(t) \right), \tag{2.39}$$

$$\dot{p}_2(t) = -\gamma \left(p_2(t) - p_1(t) \right), \tag{2.40}$$

$$\gamma = \frac{1}{\tau} \left[\sin \left(\frac{\Delta \tau}{2} \right) \right]^2. \tag{2.41}$$

In the case where $\Delta \tau << 1$, I get the following simpler expression: $\gamma \simeq \frac{\Delta^2 \tau}{4}$.

I have shown how an infinitely fast bath acts on a q-bit and how to build up an appropriate master equation. At this stage I am equipped to motivate the appearance of the Lindblad structure shown in Eq. (2.25). Let us analyze it in detail by using the following notation:

$$\rho_{i,j}(t) \equiv \langle i | \rho(t) | j \rangle, \tag{2.42}$$

where the indexes i and j assume the values 1 and 2. The value 1 is related to the state ket $|+\rangle$ while the value 2 is related to the state ket $|-\rangle$.

The Lindblad form of Eq. (2.25) is equivalent to the following system of differential equations:

$$\dot{\rho}_{1,1}(t) = -i \frac{\Delta}{2} \left(\rho_{1,2}(t) - \rho_{2,1}(t) \right), \tag{2.43}$$

$$\dot{\rho}_{1,2}(t) = -i \frac{\Delta}{2} \left(\rho_{1,1}(t) - \rho_{2,2}(t) \right) - 2k \rho_{1,2}(t), \tag{2.44}$$

$$\dot{\rho}_{2,1}(t) = -i \frac{\Delta}{2} \left(\rho_{2,2}(t) - \rho_{1,1}(t) \right) - 2k \rho_{2,1}(t), \tag{2.45}$$

$$\dot{\rho}_{2,2}(t) = -i \frac{\Delta}{2} \left(\rho_{2,1}(t) - \rho_{1,2}(t) \right), \tag{2.46}$$

Although this system of equations can be easily solved, I limit myself to showing how to

recover the system of Eqs. (2.39) and (2.40). Since the bath is supposed to be infinitely fast, I assume that both $\dot{\rho}_{1,2}(t)$ and $\dot{\rho}_{2,1}(t)$ vanish. With this assumption the following equalities hold true:

$$\rho_{1,2}(t) = -i\frac{\Delta}{4k}\left(\rho_{1,1}(t) - \rho_{2,2}(t)\right), \tag{2.47}$$

and

$$\rho_{2,1}(t) = -i\frac{\Delta}{4k}\left(\rho_{2,2}(t) - \rho_{1,1}(t)\right). \tag{2.48}$$

By plugging Eqs. (2.47) and (2.48) in Eqs. (2.43) and (2.46), I get the following system:

$$\dot{\rho}_{1,1}(t) = -\frac{\Delta^2}{4k}\left(\rho_{1,1}(t) - \rho_{2,2}(t)\right), \tag{2.49}$$

$$\dot{\rho}_{2,2}(t) = -\frac{\Delta^2}{4k}\left(\rho_{2,2}(t) - \rho_{1,1}(t)\right). \tag{2.50}$$

This method is named the Smoluchowsky approximation [24]. By choosing $k = \frac{1}{\tau}$ the system of Eqs. (2.39) and (2.40) is recovered and the equivalence between the two pictures is shown. Still, there are some differences: in the former case, the q-bit interacts abruptly with the bath at each time instant $n\tau$, while in the latter case the system keeps on interacting with the bath continuously in time. By making the assumption that the bath produces *decoherence* instantaneously the results are equivalent.

2.2   The Zwanzig Projective Method.

This Section is devoted to illustrate the main tools of the Zwanzig projection method [25] which is an extremely powerful way to build up a master equation for the time evolution of the system of interest. In the following I will use the same notation as the one adopted in the previous Section.

Let us introduce the projection operator P acting on ρ_T, the statistical density matrix of the whole system, defined by the following expression:

$$P\rho_T(t) = \text{Tr}_{\{B\}}\{\rho_T(t)\} \cdot \rho_{\text{eq}}(B).$$ (2.51)

Due to the fact that the trace operation acts on the Hilbert space of the bath, and, since $\text{Tr}_{\{B\}}\rho_{\text{eq}}(B)$ is equal to unity, it can be easily checked that the operator P is a projector:

$$P^2\rho_T(t) = P\rho(t)\rho_{\text{eq}}(B) = \rho(t)\rho_{\text{eq}}(B)\left(\text{Tr}_{\{B\}}\rho_{\text{eq}}(B)\right) = \rho(t)\rho_{\text{eq}}(B) = P\rho_T(t).$$ (2.52)

Let us introduce the operator Q, defined by:

$$Q = I - P.$$ (2.53)

It easy to check that the operator Q is a projector too, $Q^2 = Q$, and both P and Q commute with the time derivative operator $\frac{\partial}{\partial t}$:

$$\left[\frac{\partial}{\partial t}, Q\right] = \left[\frac{\partial}{\partial t}, P\right] = 0.$$ (2.54)

Let us assume that the total system is described by the following master equation:

$$\frac{\partial}{\partial t}\rho_T(t) = L\rho_T(t),$$ (2.55)

I simplify the notation, through the following definitions:

$$\rho_1(t) = P\rho_T(t)$$ (2.56)

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and

$$\rho_2(t) = Q\rho_T(t). \tag{2.57}$$

This way, I get the following forms:

$$\frac{\partial}{\partial t}\rho_1(t) = L\left(\rho_1(t) + \rho_2(t)\right) \tag{2.58}$$

and

$$\frac{\partial}{\partial t}\rho_2(t) = L\left(\rho_1(t) + \rho_2(t)\right). \tag{2.59}$$

The solution of Eq. (2.59) is given by the following expression:

$$\rho_2(t) = \int_0^t Q\,L\exp\left\{QL\left(t - t'\right)\right\}\rho_1\left(t'\right)dt' + \exp\left\{Q\,Lt\right\}\rho_2(0). \tag{2.60}$$

By plugging Eq. (2.60) into Eq. (2.62), I get the following equation:

$$\frac{\partial}{\partial t}\rho_1(t) = \int_0^t PLQ\,L\exp\left\{Q\,L\left(t - t'\right)\right\}P\rho_1\left(t'\right)dt' + P\,L\exp\left\{Q\,Lt\right\}\rho_2(0), \tag{2.61}$$

which can also be written in the following way:

$$\frac{\partial}{\partial t}P\rho_T(t) = \int_0^t PLQ\,L\exp\left\{Q\,L\left(t - t'\right)\right\}P\rho_T\left(t'\right)dt' + P\,L\exp\left\{Q\,Lt\right\}Q\rho_T(0). \tag{2.62}$$

The term $\left(P\,L\exp\left\{Q\,Lt\right\}Q\rho_T(0)\right)$ vanishes by assuming that the initial condition is described by Eq. (2.1).

This formalism is quite powerful since it permits one to build up a master equation for the system of interest which turns out to be non-Markovian. The total system is described by a Markovian master equation while the system of interest is driven by a master equation

which has memory; this change is of course due to the contraction.

Let us apply the projective method to the case where the system is described by the hamiltonian given by Eq. (2.2), where the interaction term V is given by the following expression:

$$V = \sum_\alpha S_\alpha B_\alpha. \tag{2.63}$$

The operator S_α acts on the Hilbert space of the spin, while the operator B_α acts on the Hilbert space of the bath. I will make the following assumption:

$$\text{Tr}_{\{B\}} B_\alpha = 0 \tag{2.64}$$

for every value of the natural index α. At this stage I change notation, by defining $\rho(t)$ in the following way:

$$\rho(t) = P\rho_T(t), \tag{2.65}$$

and by using Eq. (2.62) I end up with the form:

$$\frac{d}{dt}\rho(t) = L_s\rho(t) + \int_0^t \langle L_I \exp\{QLt'\} L_I \rangle \rho(t - t')\, dt'. \tag{2.66}$$

A second-order treatment in the interaction V can be made, so that $\rho(t - t')$ can be approximated by the form $\exp\{-L_S t'\}\rho(t)$. The averaging operation is defined by the following expression:

$$\langle A \rangle = \text{Tr}_{\{B\}} A\, \rho_{\text{eq}}(B). \tag{2.67}$$

After some algebra I can recover the following result:

$$QL_0 L_I \rho_{\text{eq}}(B) = L_0 L_I \rho_{\text{eq}}(B), \tag{2.68}$$

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where L_S is the Liouvillian operator associated with H_S:

$$L_S\left(\cdot\right) = -\imath\left[H_S, \cdot\right]. \tag{2.69}$$

The same formalism holds true for both L_I and L_0. The superoperator L_0 is related to the hamiltonian operator $H_0 = H_S + V$. Thus, Eq. (2.66) is equivalent to the following form:

$$\frac{d}{dt}\rho(t) = L_s\rho(t) + \int_0^t \langle L_I \exp\left\{L_0 t'\right\} L_I \exp\left\{-L_0 t'\right\}\rangle dt' \rho\left(t\right). \tag{2.70}$$

I define $C\left(\tau\right)$ in the following way:

$$C\left(\tau\right) \equiv \mathrm{Tr}_{\{B\}} L_I \exp\left\{L_0 t'\right\} L_I \exp\left\{-L_0 t'\right\} \rho_{\mathrm{eq}}\left(B\right), \tag{2.71}$$

and thus, I get the following master equation:

$$\frac{d}{dt}\rho(t) = L_s\rho(t) + \left(\int_0^t C\left(\tau\right) d\tau\right)\rho\left(t\right). \tag{2.72}$$

At this stage I use the Redfield approximation in order to explore the asymptotic behavior, it consists of replacing the superior extremum of integration with infinity:

$$R\left(\tau\right) = \int_0^\infty C\left(\tau\right) d\tau. \tag{2.73}$$

This way I get the following master equation:

$$\frac{d}{dt}\rho(t) = L_s\rho(t) + R\rho\left(t\right). \tag{2.74}$$

I can give a better estimation of the superoperator R by using the special form of the interaction hamiltonian given by Eq. (2.63). After some algebra I obtain the following equality:

$$R\rho(t) = \sum_\alpha \left(T_\alpha \rho(t) S_\alpha + S_\alpha^\dagger \rho(t) T_\alpha^\dagger - S_\alpha T_\alpha \rho(t) - \rho(t) T_\alpha^\dagger S_\alpha^\dagger \right), \qquad (2.75)$$

where T_α is given by

$$T_\alpha = \sum_\alpha \int_0^\infty d\tau C_{\alpha,\beta}(\tau) \exp\{-\imath H_S \tau\} S_\beta \exp\{-\imath H_S \tau\} \qquad (2.76)$$

and $C_{\alpha,\beta}$ is defined by the following expression:

$$C_{\alpha,\beta} \equiv \mathrm{Tr}_{\{B\}} \rho_{\mathrm{eq}}(B) \exp\{\imath H_B \tau\} B_\alpha \exp\{-\imath H_B \tau\} B_\beta. \qquad (2.77)$$

It can be proved that the master equation built up in this preserves both the hermeticity and keeps the trace of $\rho(t)$ equal to unity at every time:

$$\rho(t) = \rho^\dagger(t), \quad \mathrm{Tr}\rho(t) = 1. \qquad (2.78)$$

Sometimes I can face situations where the operators S_α and B_α are not hermitian, and, of course, the interaction hamiltonian must fulfill the hermeticity condition. I can proceed by building up the following operators: S_α', S_α'', B_α' and B_α'', defined by:

$$S_\alpha' \equiv \frac{S_\alpha + S_\alpha^\dagger}{2}, \quad S_\alpha'' \equiv \frac{S_\alpha - S_\alpha^\dagger}{2} \qquad (2.79)$$

$$B_\alpha' \equiv \frac{B_\alpha + B_\alpha^\dagger}{2}, \quad B_\alpha'' \equiv \frac{B_\alpha - B_\alpha^\dagger}{2}. \qquad (2.80)$$

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I notice that the operators S'_α and B'_α are hermitian:

$$B'_\alpha = B'^\dagger_\alpha, \quad S'_\alpha = S'^\dagger_\alpha, \tag{2.81}$$

while the operators S''_α and B''_α are antihermitian:

$$B''_\alpha = -B''^\dagger_\alpha, \quad S''_\alpha = -S''^\dagger_\alpha. \tag{2.82}$$

By using this formalism, the interaction hamiltonian V is given by the following form:

$$V = \sum_\alpha \left(S'_\alpha B'_{\alpha\alpha} + S''_{\alpha\alpha} B''_\alpha \right). \tag{2.83}$$

In the case where the bath coupling operators are hermitian, the correlation functions $C_{\alpha,\beta}$, given by Eq. (2.77), have the following property:

$$C^\dagger_{\alpha,\beta}(\tau) = C_{\beta,\alpha}(-\tau). \tag{2.84}$$

Now let us consider Eq. (2.76); it is evident that the superoperator T_α gives memory to the master equation (2.74), so I can assume that, in the case of an infinitely fast bath, the memory is completely destroyed, which means that $C_{\alpha,\beta}$ must be proportional to a delta function:

$$C_{\alpha,\beta}(\tau) = 2D_{\alpha,\beta}\delta(\tau). \tag{2.85}$$

Let us assume that the operators under discussion are hermitian, so that the following relation holds true:

$$D_{\alpha,\beta} = D^*_{\beta,\alpha}. \tag{2.86}$$

Then, I consider the matrix $\mathbf{D}$, defined by

$$[D]_{\alpha,\beta} \equiv D_{\alpha,\beta}; \tag{2.87}$$

it is obviously hermitian, so that there exists a unitary matrix $\mathbf{U}$ that diagonalizes $\mathbf{D}$:

$$\mathbf{D} = \mathbf{U}^\dagger \cdot \mathbf{d} \cdot \mathbf{U}, \tag{2.88}$$

where $\mathbf{d}$ is the diagonal matrix. At this stage I can define a superoperator L_α in the following way:

$$L_\alpha \equiv \sqrt{[\mathbf{d}]_{\alpha,\alpha}} \sum_\beta [\mathbf{U}]_{\alpha,\beta} \, S_\beta. \tag{2.89}$$

Going through some algebra, I find the following form for Eq. (2.74):

$$\frac{d}{dt}\rho(t) = L_s\rho(t) + \sum_\alpha \left(2L_\alpha\rho(t)L_\alpha^\dagger - L_\alpha^\dagger L_\alpha\rho(t) - \rho(t)L_\alpha^\dagger L_\alpha\right). \tag{2.90}$$

I want now to show how this treatment is consistent, by analyzing the following interaction Hamiltonian:

$$V = g\sigma_x x, \tag{2.91}$$

by using the approach of the Redfield equation given by Eqs. (2.74), (2.75), (2.76) and (2.85). This interaction hamiltonian has been widely considered in Section 1.2.

From Eq. (2.77) I evaluate the following form:

$$C(\tau) = g^2 \mathrm{Tr}_{\{B\}} \rho_{\mathrm{eq}}(B) \exp\{\imath H_B \tau\} \, x \exp\{-\imath H_B \tau\} \, x, \tag{2.92}$$

I make the same approximation of an infinitely fast bath:

$$C\left(\tau\right) = 2D\delta\left(\tau\right). \tag{2.93}$$

Since there is only one Lindblad operator, there is no need to use any subscript. From Eqs. (2.76) and (2.77) I get the following expression:

$$D = g^2 \langle x \rangle^2(0)\tau_D, \tag{2.94}$$

where τ_D is defined by

$$\tau_D = \int_0^\infty \frac{\langle x(t')x(0) \rangle}{\langle x^2 \rangle} dt'. \tag{2.95}$$

The resulting master equation is given by the following form:

$$\frac{d}{dt}\rho(t) = L_S\rho(t) - g^2 \langle x^2 \rangle \tau_D \left[\sigma_x, [\sigma_x, \rho(t)]\right]. \tag{2.96}$$

In the case where no external magnetic field is applied, the term H_S vanishes and Eq. (2.96) is simplified:

$$\frac{d}{dt}\rho(t) = -g^2 \langle x^2 \rangle \tau_D \left[\sigma_x, [\sigma_x, \rho(t)]\right]. \tag{2.97}$$

The case of one half spin is even simpler; since $\sigma_x^2 = I$, I recover the following form:

$$\frac{d}{dt}\rho(t) = -g^2 \langle x^2 \rangle \tau_D \left(\rho(t) - \sigma_x\rho(t)\sigma_x\right). \tag{2.98}$$

I notice that I end up again with the master equation (2.25) of Section 2.1. From Eq. (2.98) I can evaluate how the off-diagonal elements evolve in time. The corresponding differential

equation is given by:

$$\frac{d}{dt}\langle+|\rho(t)|-\rangle = -4g^2\langle x^2\rangle\tau_D\langle+|\rho(t)|-\rangle, \tag{2.99}$$

which corresponds to an exponential relaxation:

$$\langle+|\rho(t)|-\rangle = \langle+|\rho(0)|-\rangle \exp\left\{-4g^2\langle x^2\rangle\tau_D t\right\}. \tag{2.100}$$

Thus, I recover complete agreement with the previous models.

This long treatment shows how the Zwanzig projective method, making calculation at the second order approximation, gives a Lindblad master equation. This approach to the building up of the master equation will be helpful in the following discussion.

CHAPTER 3

THE SUBORDINATION APPROACH

In this Chapter I explain the subordination approach. This technique traces back to the continuous time random walk, invented by Montroll and Weiss [26], who imagined a generalization of the random walk prescription. The ordinary one-dimensional random walker makes jumps forward and backward at any natural time instant n. The key idea in the subordination approach is to imagine that n is an internal time and that the experimental time t is related to n by a stochastic prescription. Let $t(n)$ be the function expressing the dependence of the external time on the internal one. The time interval $\tau(n) = t(n) - t(n-1)$ is randomly selected from a distribution $\psi_S(\tau)$, named the subordination distribution. I will illustrate the main tools of the continuous time random walk, and then show the subordination technique applied to both classical and quantum processes. The latter procedure will be helpful when I build up a master equation for open quantum systems interacting with an external environment, or, equivalently, subjected to a measurement process, at random times.

3.1   Tools of the Continuous Time Random Walk and the Subordination Technique

The ordinary one-dimensional random walker makes jumps forward or backwards, at any instants of a natural time n. Let $p_n(x)$ be the probability density functions; this means that, after n jumps, the probability of finding the walker in a position between x and $x + dx$ is given by $p_n(x)dx$. Let us name $p(x)$ the function $p_1(x)$, which is the density probability

associated with the position x after one step, and let us assume that, initially, the walker is in the position $x = 0$. The function $p_{n+1}(x)$ is related to $p_n(x)$ by the following equality:

$$p_{n+1}(x) = \int_0^\infty p_n(y)p(x-y)dy. \tag{3.1}$$

In this relation, the random walker is assumed to be in a position between y and $y + dy$ at the nth step and jump to a position between x and $x + dx$ in one step; of course an integration over all the possible positions y is performed. In the continuous time random walk, the time interval τ between one step and the next one is supposed to be a random variable distributed according to the function $\psi(\tau)$, named the waiting time distribution. The survival probability function, $\Psi(t)$, is related to $\psi(t)$ by the following equation:

$$\Psi(t) = \int_t^\infty \psi(\tau)\, d\tau, \tag{3.2}$$

which specifies the probability that the time interval between two consecutive jumps is greater than t. Due to the definition of the waiting time distribution, the following equality holds true:

$$\psi_n(t) = \int_0^t \psi(t-t')\,\psi_{n-1}(t')\, dt', \tag{3.3}$$

and, in the Laplace space, it turns out to be

$$\hat{\psi}_n(u) = \left(\hat{\psi}(u)\right)^n. \tag{3.4}$$

Let us assume that, at time $t = 0$, the random walker is in the position $r = 0$. The density probability, $p(r,t)$, of finding the walker in a position between r and $(r + dr)$ at the instant

t, is related to $p_n(r)$ through the following equation:

$$p(r,t) = \sum_{n=0}^{\infty} p_n(r)\psi_n(\tau)\,\Psi(t-\tau)\,d\tau. \tag{3.5}$$

The random walker is supposed to stay in the position $r = 0$ at the time instant $t = 0$; this constraint is set by the following equalities:

$$\psi_0(t) = \delta(t) \tag{3.6}$$

and

$$p_0(r) = \delta_{r,0}. \tag{3.7}$$

The Laplace transform of Eq. (3.8) has a simple form:

$$\hat{p}(r,u) = \frac{1 - \hat{\psi}(u)}{u} \sum_{n=0}^{\infty} p_n(r)\left(\hat{\psi}(u)\right)^n. \tag{3.8}$$

The transition from the internal time scale n to the external time scale t is performed by imagining that the regular time step $\Delta n = 1$ is randomly changed. This procedure is the basis of the subordination technique.

This idea is extended also to the case of a continuous internal time n. This approach [27] is quite powerful, since it allows to express the formal solution of a non-Markovian Fokker-Plank equation in terms of the solution of a Markovian equation with the same Fokker-Plank operator. Since this technique will be used in this research work, I am going show it in details.

Let us consider the following non-Markovian Fokker-Plank equation:

$$\frac{\partial}{\partial t} P(x,t) = \int_0^t \Phi(t - t')\, LP(x,t')\, dt', \tag{3.9}$$

where $\Phi(t)$ is the memory kernel and L is a linear operator acting on the variable x. I have already shown how this structure emerges from the Zwanzig projective method.

The formal solution of Eq. (3.9) can be expressed by the following integral transformation:

$$P(x,t) = \int_0^\infty p(x,n)T\,(n,t)\,dn, \qquad (3.10)$$

where the function $p\,(x,n)$ is the solution of the following Markov equation:

$$\frac{\partial}{\partial n}p(x,n) = L\,p(x,n). \qquad (3.11)$$

It is important to notice that the Fokker-Plank operator L, appearing in Eq. (3.11), is the same as that of Eq. (3.9). The function $T(n,t)$ describes the transformation from the time scale n to the time scale t. The method works if $\hat{T}(u,t)$, the Laplace transform with respect to t of $T(n,t)$,

$$\hat{T}(n,u) \equiv \int_0^\infty \exp(-ut)T(n,t)dt, \qquad (3.12)$$

has the following form:

$$\hat{T}(n,u) = \frac{1}{\hat{\Phi}(u)}\exp\left(-n\frac{u}{\hat{\Phi}(u)}\right). \qquad (3.13)$$

In order to demonstrate this property let us evaluate the Laplace transform of $p(x,t)$ by using the integral decomposition given by Eq. (3.10):

$$\hat{P}(x,u) = \int_0^\infty dt\exp\left(-ut\right)P(x,t) =$$
$$\int_0^\infty dt\exp(-ut)\int_0^\infty dnp(x,n)T(n,t) = \int_0^\infty dnp(x,n)\hat{T}(n,u). \qquad (3.14)$$

By plugging Eq. (3.13) for $\hat{T}(n, u)$ in Eq. (3.14), the following form for $\hat{P}(x, u)$ is obtained:

$$\hat{P}(x, u) = \frac{1}{\hat{\Phi}(u)} \hat{p}\left(x, \frac{u}{\hat{\Phi}(u)}\right). \tag{3.15}$$

At this stage the Laplace transform of both sides of Eq. (3.9) can be evaluated:

$$u\hat{P}(x, u) - P(x, 0) = \hat{\Phi}(u)L\,\hat{P}(x, u). \tag{3.16}$$

Then, by substituting Eq. (3.15) for $\hat{P}(x, u)$ in Eq. (3.16), I get the following equality:

$$\frac{u}{\hat{\Phi}(u)} \hat{p}\left(x, \frac{u}{\hat{\Phi}(u)}\right) - P(x, 0) = L\,\hat{p}\left(x, \frac{u}{\hat{\Phi}(u)}\right). \tag{3.17}$$

The latter equation is exactly the Laplace transform of Eq. (3.11) in the variable $\frac{u}{\hat{\Phi}(u)}$, if these two processes are subject to the same initial condition:

$$P(x, 0) = p(x, 0). \tag{3.18}$$

Thus, once the solution of Eq. (3.11) is known, the formal solution of Eq. (3.9) is known, too, in the Laplace space.

The subordination technique has been applied to the classical density probability distribution. A similar technique can also be applied to master equations related to quantum systems. This will be the subject of the next Section.

3.2 The Continuous Time Quantum Random Walk

In this Section I am going to show some new techniques that will help to build up a master equations for open quantum systems interacting with an external environment.

These results belong to the recent work of Ref. [28] and prepare the ground for the original results of this research work.

In Section 2.1 I have shown some examples of master equations for quantum systems interacting with an external environment; the usual form is described by the following equation:

$$\frac{d}{dt}\rho(t) = -\imath \left[H, \rho(t) \right] + \frac{1}{\tau_0} \mathcal{L}_0 \rho(t). \tag{3.19}$$

The constant τ_0 is the characteristic time scale of the irreversible dynamics caused by the interaction with the external bath, and $\mathcal{L}_0$ has the form of a Lindblad superoperator:

$$\mathcal{L}_0[\cdot] = \sum_i \left(\left[A_i, \cdot A_i^\dagger \right] + \left[A_i \cdot, A_i^\dagger \right] \right). \tag{3.20}$$

The author of Ref. [28] imagines that the interaction between the system of interest and the environment takes place randomly and is described by the action of a superoperator on the reduced density matrix. The bath is assumed to be infinitely fast. This picture is a generalization of the continuous time random walk to quantum systems.

The effect of the environment, at each interaction, can be described by a completely positive superoperator Γ, given by the following expression:

$$\Gamma[\rho] = \sum_i C_i \rho C_i^\dagger, \tag{3.21}$$

where the operators C_i obey the closure condition:

$$\sum_i C_i^\dagger C_i = I. \qquad (3.22)$$

The operator Γ acts on the reduced density matrix at the instants $t_1 < t_2 < \cdots < t_n$, and the interval $\tau_i = t_i - t_{i-1}$ is selected randomly from a waiting time distribution $\psi(\tau)$. It is convenient to work in the interaction picture. This way, the average evolution of the reduced density matrix is given by the following expression:

$$\rho(t) = \sum_{n=0}^{\infty} P_n(t)\Gamma^n\left[\rho(0)\right], \qquad (3.23)$$

where $P_n(t)$ denotes the probability that n interactions have occurred, the last one exactly at the instant t. At each instant the probability functions $P_n(t)$ obey the normalization constraint:

$$\sum_{n=0}^{\infty} P_n(t) = 1, \qquad (3.24)$$

and the probability, $P_0(t)$, of having no interactions until the instant t is given by

$$P_0(t) = \int_t^{\infty} \psi(\tau)\, d\tau. \qquad (3.25)$$

The probability of having n interaction in the time t can be expressed by the following recursion relation:

$$P_n(t) = \int_0^t \psi(t - t')\, P_{n-1}(t')\, dt'. \qquad (3.26)$$

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It is convenient to move to the Laplace space:

$$\hat{P}_n(u) = \left(\hat{\psi}(u)\right)^n \hat{P}_0(u) = \frac{1 - \hat{\psi}(u)}{u} \left(\hat{\psi}(u)\right)^n ; \qquad (3.27)$$

this way Eq. (3.23) is turned into the following form:

$$\hat{\rho}(u) = \sum_{n=0}^{\infty} \frac{1 - \hat{\psi}(u)}{u} \left(\hat{\psi}(u)\right)^n \Gamma^n \rho(0), \qquad (3.28)$$

which is equivalent to the following expression:

$$\hat{\rho}(u) = \frac{1 - \hat{\psi}(u)}{u} \frac{1}{1 - \hat{\psi}(u)\Gamma} \rho(0). \qquad (3.29)$$

I have made the assumption that the norm of the operator $\hat{\psi}(u)\Gamma$ is less than unity so that the series converges to the latter expression. It can be easily checked that the expression of $\hat{\rho}(u)$, given by Eq. (3.29), fulfills the following master equation:

$$u\hat{\rho}(u) - \rho(0) = \hat{\Phi}(u)\mathcal{L}\left[\hat{\rho}(u)\right], \qquad (3.30)$$

where the superoperator $\mathcal{L}$ is defined by

$$\mathcal{L}\left[\hat{\rho}(u)\right] = \Gamma\left[\hat{\rho}(u)\right] - \hat{\rho}(u) \qquad (3.31)$$

and $\hat{\Phi}(u)$ is given by

$$\hat{\Phi}(u) = \frac{u\hat{\psi}(u)}{1 - \hat{\psi}(u)}. \qquad (3.32)$$

This equation is quite important; in fact, it relates the memory kernel $\Phi(t)$ to the waiting

time distribution $\psi(t)$ in the Laplace transform space. Notice that this equality descends from Eq. (3.23) and from the tools of the continuous time random walk.

By making the anti-Laplace transform of Eq. (3.30), it is possible to obtain the master equation corresponding to the continuous time quantum random walk picture:

$$\frac{d}{dt}\rho(t) = \int_0^t \Phi\left(t - t'\right) \mathcal{L}\left[\rho\left(t'\right)\right] dt'.$$

(3.33)

The operator $\mathcal{L}$ can be expressed in the Lindblad form given by Eq. (3.20).

This result is quite interesting: the continuous time random picture gives a non-Markovian master equation with a Lindblad form operator.

At this stage I want to show how it is possible to recover the results of the subordination technique applied to classical processes that has been described in Section 3.1.

Starting from Eq. (3.30), the function $\hat{\rho}(u)$ can be expressed in the following way:

$$\hat{\rho}(u) = \frac{1}{u - \hat{\Phi}(u)\mathcal{L}}\rho(0).$$

(3.34)

The inverse of the operator $\left(u - \hat{\Phi}(u)\mathcal{L}\left[\cdot\right]\right)$ can be expressed as follows:

$$\frac{1}{u - \hat{\Phi}(u)\mathcal{L}\left[\cdot\right]} = \int_0^\infty \exp\left\{-u\tau' + \hat{\Phi}(u)\tau'\mathcal{L}[\cdot]\right\} d\tau'.$$

(3.35)

By making the change of variable $\tau = \hat{\Phi}(u)\tau'$, Eq. (3.34) can be written in the following way:

$$\hat{\rho}(u) = \int_0^\infty d\tau \frac{1}{\hat{\Phi}(u)} \exp\left\{-\tau\frac{u}{\hat{\Phi}(u)}\right\} \exp\left\{\tau\mathcal{L}\right\} \rho(0).$$

(3.36)

Now, let us introduce the function $\mathcal{T}(t, \tau)$, whose t-Laplace transform is given by

$$\hat{\mathcal{T}}(u, \tau) = \frac{1}{\hat{\Phi}(u)} \exp\left\{-\tau \frac{u}{\hat{\Phi}(u)}\right\};\qquad(3.37)$$

this way $\hat{\rho}(u)$ is given by:

$$\hat{\rho}(u) = \int_0^\infty d\tau \hat{\mathcal{T}}(u, \tau) \exp\{\tau \mathcal{L}\} \rho(0),\qquad(3.38)$$

whose anti-Laplace transform is

$$\rho(t) = \int_0^\infty d\tau \mathcal{T}(t, \tau) \rho_M(\tau).\qquad(3.39)$$

The function $\rho_M(\tau)$ represents the time evolution of the statistical density matrix driven by a Liouvillian identical to the superoperator $\mathcal{L}$, starting from the initial condition $\rho(0)$. This means that $\rho_M(\tau)$ fulfills the following master equation:

$$\frac{d}{d\tau}\rho_M(\tau) = \mathcal{L}\rho_M(\tau).\qquad(3.40)$$

This procedure can be compared to the classical subordination prescription described in Section 3.1; these two treatments turn out to be perfectly equivalent.

Let us evaluate the general solution of Eq. (3.40). In principle, it can be solved by evaluating the eigenoperators K_i of the superoperator $\mathcal{L}$:

$$\mathcal{L}[K_i] = \lambda_i K_i.\qquad(3.41)$$

The solution is given by the following equation:

$$\rho_M(\tau) = \sum_i c_i \exp\{-\lambda_i \tau\} K_i. \quad (3.42)$$

The following closure condition, involving the operator K_j and the dual operator $\check{K}_i$, holds true:

$$\text{Tr}\{\check{K}_i K_j\} = \lambda_i \delta_{ij}; \quad (3.43)$$

this way, the coefficients c_i turn out to be

$$c_i = \text{Tr}\{K_i \rho_M(0)\}. \quad (3.44)$$

I remind the reader that the dual operator $\check{K}$ is defined by the following equality:

$$\check{\mathcal{L}} \check{K}_i = \lambda_i \check{K}_i, \quad (3.45)$$

where $\check{\mathcal{L}}$ is evaluated through the following relation:

$$\text{Tr}\{K\mathcal{L}[\rho]\} = \text{Tr}\{\rho\check{\mathcal{L}}[K]\}. \quad (3.46)$$

The final solution is given by

$$\rho(t) = \sum_i c_i K_i \int_0^\infty d\tau\, \mathcal{T}(t,\tau) \exp\{-\lambda_i \tau\}, \quad (3.47)$$

where the functions $\varphi_i(t)$, defined by

$$\varphi_i(t) \equiv \int_0^\infty d\tau \, \mathcal{T}(t, \tau) \exp\{-\lambda_i \tau\}, \tag{3.48}$$

can be evaluated in the Laplace space by using Eq. (3.37). This way, after some algebra, the function $\hat{\varphi}_i(u)$ turns out to be

$$\hat{\varphi}_i(u) = \frac{1}{u + \lambda_i \hat{\Phi}(u)}. \tag{3.49}$$

Since $\varphi_i(0)$ is equal to unity, Eq. (3.49) can be written in the following way:

$$u\hat{\varphi}_i(u) - 1 = -\lambda_i \hat{\Phi}(u) \hat{\varphi}_i(u). \tag{3.50}$$

The latter equation is the Laplace transform of the following non-Markovian equation:

$$\frac{d}{dt}\varphi_i(t) = -\lambda_i \int_0^t \Phi(t - t') \, \varphi_i(t') \, dt'. \tag{3.51}$$

Once the functions $\varphi_i(t)$ are evaluated, Eq. (3.33) can formally be solved and the solution is given by

$$\rho(t) = \sum_i c_i \varphi_i(t) K_i. \tag{3.52}$$

From this treatment it emerges that this technique gives results in the Laplace transform space; see for example Eq. (3.46). Since I am going to consider the case of a non-Poissonian distribution, I have to prepare the ground to detect the asymptotic behavior of some special functions in the Laplace transform space. This argument will be discussed in the next Section.

3.3   Tauberian Theorem and the Emergence of the Mittag-Leffler Function

The first part of this Section is devoted to the Tauberian theorem, which will be used to detect the asymptotic behavior of functions once their Laplace transforms are known. The second part deals with the definition and the properties of the Mittag-Leffler function.

The literature on the Tauberian theorem is extremely wide. I will report here just some basic properties; for a more detailed treatment I refer the reader to the books of Weiss [29] and Korevaar [30]. The Tauberian theorem deals with slowly varying functions. So, first, I introduce this basic concept. A function $L(x)$ is said to be a slowly varying function at $x = \infty$ if, for every positive constant c, the following equality is fulfilled:

$$\frac{L(cx)}{L(x)} = 1. \tag{3.53}$$

For example, the function $\log(x)$ is slowly varying at $x = \infty$, while a power law, x^α, is not. The most general form of a slowly varying function is given by the following expression:

$$L(x) = h(x) \exp\left\{ \int_0^t \frac{g(v)}{v} dv \right\}, \tag{3.54}$$

with the condition that $h(x)$ tends, for $x \to \infty$, to a positive limit, h_∞, and the function $g(v)$ vanishes in the limit $v \to \infty$. At this stage the Tauberian theorem can be enunciated.

Let $\hat{f}(u)$ be the Laplace transform of the function $f(t)$. Let us assume that in the limit $u \to 0$, the function $\hat{f}(u)$ behaves in the following way:

$$\hat{f}(u) \sim \frac{1}{u^\alpha} L\left(\frac{1}{u}\right), \tag{3.55}$$

where $\alpha > 0$ and $L(x)$ is a slowly varying function at $x \to +\infty$. The Tauberian theorem

establishes a relation between the function $F(t)$, defined by the following expression:

$$F(t) = \int_0^t f\left(t'\right) dt', \tag{3.56}$$

and the Laplace transform $\hat{f}(u)$; it states that, in the limit $t \to +\infty$, the function $F(t)$ behaves in the following way:

$$F(t) \sim \frac{t^\alpha L(t)}{\Gamma\left(1 + \alpha\right)}. \tag{3.57}$$

Usually I will deal with functions whose Laplace transforms behave for vanishing u, like a power law: $\hat{f}(u) \sim \frac{1}{u^\alpha}$. In this case it is possible to integrate Eq. (3.57) and get the following behavior: $f(t) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in the limit $t \to +\infty$. The theorem holds true also for the condition $t \to 0$, which corresponds to $u \to \infty$.

I anticipate that the Tauberian theorem will help to evaluate the asymptotic behavior of the anti-Laplace transform of functions like $1/\left(u + \hat{\Phi}(u)\right)$, where $\hat{\Phi}(u)$ is the Laplace transform of the memory kernel. Since I am going to consider kernels which asymptotically behave like a power law, I am going to deal with a special function which will be fundamental for this research work. This function is named the Mittag-Leffler function. At this stage it is necessary to make a short summary of the properties of the Mittag-Leffler function.

The Mittag-Leffler function [31] plays quite an important role in modern physics. It emerges as a solution of fractional integral equations, in the fractional generalization of kinetic equation, random walks, Levy flights and superdiffusive transport. It has the interesting property of interpolating between an exponential law and a power-law behavior. Since its emergence will be the main result of this research work, I am going to show its properties and analyze possible ways of detecting it.

The Mittag-Leffler function, $E_\alpha(z)$, is defined for every positive value of the parameter

α by the following series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}. \tag{3.58}$$

For particular values of the parameter α, the Mittag-Leffler function has a simple form:

$$E_0(z) = \frac{1}{1 - z}, \tag{3.59}$$

$$E_1(z) = \exp\{z\}, \tag{3.60}$$

$$E_2(z) = \cosh\left(\sqrt{z}\right), \tag{3.61}$$

$$E_3(z) = \frac{1}{3}\left(\exp\left\{z^{\frac{1}{3}} + 2\exp\left\{-\frac{z^{\frac{1}{3}}}{2}\right\}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{4}}\right)\right\}\right), \tag{3.62}$$

$$E_4(z) = \frac{1}{2}\left\{\cos\left(z^{\frac{1}{4}}\right) + \cosh\left(\frac{1}{4}\right)\right\}, \tag{3.63}$$

and

$$E_{\frac{1}{2}}(z) = \exp\left\{z^2\left(1 + \operatorname{erf}(z)\right)\right\}. \tag{3.64}$$

The asymptotic behavior is quite important; for $z \to \infty$ it is given by the following expression:

$$E_\alpha(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1 - n\alpha)} + \operatorname{O}\left(|z|^{-N}\right), \tag{3.65}$$

provided that

$$|\arg(-z)| < \left(1 - \frac{\alpha}{2}\right)\pi. \tag{3.66}$$

So, as first approximation, for $z \to \infty$, I can use the following expression:

$$E_\alpha(z) \simeq -\frac{1}{z\Gamma(1 - \alpha)}. \tag{3.67}$$

The Laplace transform of $E_\alpha(z)$ is given by:

$$\mathcal{L}\left[E_\alpha(-\gamma t^\alpha)\right](u) = \frac{1}{u + \gamma u^{1-\alpha}}, \tag{3.68}$$

provided that $\Re(u) > |\gamma|^{\frac{1}{\alpha}}$. Notice that in the case $u \to 0$, the right hand side of Eq. (3.68) has the following behavior:

$$\frac{1}{u + \gamma u^{1-\alpha}} \sim \frac{1}{\gamma u^{1-\alpha}}, \tag{3.69}$$

where the symbol $\sim$ has the following meaning:

$$A(x) \sim B(x) \Leftrightarrow \lim \frac{A(x)}{B(x)} = 1. \tag{3.70}$$

In the limit $t \to 0$, the Mittag-Leffler function $E_\alpha\left(-\gamma t^\alpha\right)$ has the following behavior:

$$E_\alpha\left(-\gamma t^\alpha\right) \simeq \exp\left\{-\gamma t^\alpha\right\}. \tag{3.71}$$

The function $\left(\gamma u^{1-\alpha}\right)^{-1}$ is the Laplace transform of $\frac{t^{-\alpha}}{\gamma\Gamma(1-\alpha)}$, which, according to the expression (3.67), is exactly the asymptotic form of the Mittag-Leffler function in the limit $t \to +\infty$. So, the quantity $\frac{1}{u+\gamma u^{1-\alpha}}$ shows the presence of the "tail" of $E_\alpha\left(-\gamma t^\alpha\right)$ in the Laplace space, also in the case of vanishing u.

In the following, I am going to deal with the inverse Laplace transform of functions $\hat{f}(u)$, given by

$$\hat{f}(u) = \frac{1}{u - \lambda\hat{\Phi}(u)}, \tag{3.72}$$

where the function $\hat{\Phi}(u)$ is related to $\hat{\psi}(u)$, the Laplace transform of the waiting time distribution, by Eq. (3.32). For this reason, at this stage, it is important to make a deep treatment

of the asymptotic properties of the inverse Laplace transform of $\hat{f}(u)$.

It will be necessary to know the analytical form of the function $\hat{\psi}(u)$, when the function $\psi(t)$ is the power law given by the following equation:

$$\psi(\tau) = (\mu - 1)\frac{T^{\mu-1}}{(\tau + T)^\mu}. \tag{3.73}$$

This Laplace transform has been evaluated in Ref. [33], to be

$$\hat{\psi}(u) = (\mu - 1)\frac{\Gamma(1 - \mu)}{(uT)^{1-\mu}}\left(e^{uT} - E_{\mu-1}^{uT}\right). \tag{3.74}$$

The function $E_{\mu-1}^{uT}$, introduced in Ref. [35], is given by

$$E_{\mu-1}^{uT} = (uT)^{1-\mu}\sum_{k=0}^{\infty}\frac{(uT)^k}{\Gamma(k + 2 - \mu)}. \tag{3.75}$$

For $u \to 0$, the function $\hat{\psi}(u)$ has the following asymptotic form:

$$\hat{\psi}(u) = 1 - \Gamma(2 - \mu)(uT)^{\mu-1} + o\left((uT)^{\mu-1}\right). \tag{3.76}$$

In the case where the quantity (uT) vanishes, as a first approximation, the third term on the right hand side of Eq. (3.76) can be neglected, because

$$\lim_{s\to 0}\frac{o\left(s^{\mu-1}\right)}{s^{\mu-1}} = 0. \tag{3.77}$$

Since μ is supposed to be greater than 1 and smaller than 2, from Eq. (3.76) the following

expression for $\hat{\Phi}(u)$ can be derived:

$$\hat{\Phi}(u) \simeq \frac{u^{2-\mu}}{T^{\mu-1}\Gamma(2-\mu)}. \tag{3.78}$$

By using Eq. (3.72), I get the following form of $\hat{f}(u)$:

$$\hat{f}(u) \simeq \frac{1}{u - \frac{\lambda}{T^{\mu-1}\Gamma(2-\mu)}u^{2-\mu}}. \tag{3.79}$$

By using Eq. (3.68), it is possible to identify Eq. (3.79) with the Laplace transform of $E_{\mu-1}\left(\frac{\lambda}{T^{\mu-1}\Gamma(2-\mu)}t^{\mu-1}\right)$.

At this stage it is convenient to use the Tauberian theorem in order to evaluate the asymptotic behavior of $f(t)$ in the limit $t \to +\infty$. The function $\hat{f}(u)$, in the limit $u \to 0$, has the following behavior:

$$\hat{f}(u) \sim \frac{-\Gamma(2-\mu)T^{\mu-1}}{\lambda}u^{\mu-2}. \tag{3.80}$$

According to Tauberian theorem, this yields in the limit $t \to +\infty$:

$$\left[\mathcal{L}^{-1}\left[\hat{f}(u)\right]\right](t) \sim \frac{-T^{\mu-1}}{\lambda}\frac{1}{t^{\mu-1}}. \tag{3.81}$$

Due to Eqs. (3.79) and (3.81) I conjecture that the function $f(t)$ is closely related to a Mittag-Leffler function. This idea is confirmed by the following arguments: by substituing Eq. (3.32) into Eq. (3.72) I can make a better approximation of $f(t)$. This way I can derive

the following, more accurate, expression:

$$\hat{f}(u) = \left(\frac{1}{1+\lambda}\right) \left(\frac{1}{u - \frac{\lambda u^{2-\mu}}{\Gamma(2-\mu)(1+\lambda)T^{\mu-1}}}\right) A(u), \tag{3.82}$$

where the quantity $A(u)$, defined by

$$A(u) \equiv \left(\frac{1 - \frac{o\left((uT)^{\mu-1}\right)}{\Gamma(2-\mu)(uT)^{\mu-1}}}{1 - \frac{o((uT)^{\mu-1})(1+\lambda)}{\Gamma(2-\mu)(uT)^{\mu-1}(1+\lambda)-\lambda}}\right), \tag{3.83}$$

and tends to unity in the limit of vanishing (uT). So, in the case $uT << 1$, which is equivalent to considering time $t >> T$, the quantity $A(u)$ approaches unity and $\hat{f}(u)$ is accurately approximated by the following expression:

$$\frac{1}{(1+\lambda)} \frac{1}{\left(u - \frac{\lambda}{\Gamma(2-\mu)(1+\lambda)T^{\mu-1}} u^{2-\mu}\right)}, \tag{3.84}$$

which is the Laplace transform of $\frac{1}{1+\lambda} E_{\mu-1}\left(\frac{\lambda}{\Gamma(2-\mu)(1+\lambda)T^{\mu-1}} t^{\mu-1}\right)$, under the condition that the following inequality holds true:

$$\Re(u) > \frac{1}{T} \left|\frac{\lambda}{\Gamma(2-\mu)(1+\lambda)}\right|^{\frac{1}{\mu-1}}. \tag{3.85}$$

As expected, the functions $\frac{1}{1+\lambda} E_{\mu-1}\left(\frac{\lambda}{\Gamma(2-\mu)(1+\lambda)T^{\mu-1}} t^{\mu-1}\right)$, which is the anti-Laplace transform of the expression (3.84), and $E_{\mu-1}\left(\frac{\lambda}{T^{\mu-1}\Gamma(2-\mu)} t^{\mu-1}\right)$, which is the anti-Laplace transform of the expression (3.79), have asymptotically the same tail.

The physical meaning of the inequality (3.85) is that the times t under discussion are

very large but yet fitting the condition

$$t << T \left| \Gamma \left(2 - \mu \right) \left(1 + \frac{1}{\lambda} \right) \right|^{\frac{1}{\mu - 1}}. \tag{3.86}$$

This means that, if the parameters T, τ_m, V and μ are selected in such a way as to fit the condition

$$\left| \Gamma \left(2 - \mu \right) \left(1 + \frac{1}{\lambda} \right) \right|^{\frac{1}{\mu - 1}} >> 1, \tag{3.87}$$

it is possible to find time windows, which I refer to as Mittag-Leffler time windows, defined by

$$T << t << T \left| \Gamma \left(2 - \mu \right) \left(1 + \frac{1}{\lambda} \right) \right|^{\frac{1}{\mu - 1}}, \tag{3.88}$$

where the Mittag-Leffler function appears. In this case the inverse Laplace transform of $\hat{f}(u)$ becomes

$$\frac{1}{1 + \lambda} E_{\mu - 1} \left(\frac{\lambda}{\Gamma \left(2 - \mu \right) \left(1 + \lambda \right) T^{\mu - 1}} t^{\mu - 1} \right). \tag{3.89}$$

In order to study the case where $t >> T$ and the inequality (3.87) is not fulfilled, I go back to Eqs. (3.80), (3.81) and (3.70).

It has to be pointed out that, in the case $u \to 0$, if $\alpha > 0$,

$$\frac{1}{u + \gamma u^{1 - \alpha}} \sim \frac{1}{\gamma u^{1 - \alpha}}. \tag{3.90}$$

This expression corresponds to the time asymptotic limit of the Laplace transform of $\frac{t^{-\alpha}}{\gamma \Gamma(1 - \alpha)}$, which is the fat tail of $E_\alpha \left(-\gamma t^\alpha \right)$.

In this heuristic treatment, I am making the widely used assumption [29] that the behavior of $f(t)$, *at large times*, is mainly determined by the values that its Laplace transform,

$\hat{f}(u)$, assumes when u is *small*. For values of the variable u such that $uT << 1$, the function $\hat{f}(u)$ is well approximated by the Laplace transform of the Mittag -Leffler function. So, in this approximation, I find *times*, $t >> T$, where $f(t)$ is well approximated by a Mittag-Leffler function. This is the spirit of this approach and of these calculations

These tools are sufficient to start our investigation on *decoherence*. I anticipate that some relaxations different from the exponential law will emerge in this work.

At this stage it is possible to analyze the processes described in Section 1.6 by using the subordination approach. The model of the q-bit interacting with one spin a time results in the following time evolution of the off-diagonal element of the reduced density matrix describing the q-bit:

$$\langle+|\rho_{red}(\tau)|-\rangle = \langle+|\rho_{red}(0)|-\rangle \exp\{-K\tau\}, \tag{3.91}$$

where the constant K is given by:

$$K = \frac{|\ln\cos(2g\tau_{\text{int}})|}{T_{NI}}. \tag{3.92}$$

Notice that I have replaced the time variable t with τ. The reason will be clear soon. I imagine that T_{NI} fluctuates so that I can use the subordination theory explained in Section 3.1, applied to the density matrix. This means that I am going to use the transformation (3.47) in order to establish a connection between the solution of Eq. (3.33) and Eq. (3.40). As result of this transformation, the master equation

$$\frac{d}{d\tau}\langle+|\rho_{red}^{(M)}(\tau)|-\rangle = -K\langle+|\rho_{red}^{(M)}(\tau)|-\rangle, \tag{3.93}$$

is turned into the following non-Markovian master equation:

$$\frac{d}{dt}\langle+|\rho_{red}(t)|-\rangle = -K\int_0^t \Phi(t-t')\langle+|\rho_{red}(t')|-\rangle dt'. \tag{3.94}$$

The memory kernel $\Phi(t)$ is described by Eq. (3.32).

Let us consider the ordinary case, that is, a subordination distribution given by an exponential law: $\psi(t) = \lambda\exp\{-\lambda t\}$. Since its Laplace transform is given by

$$\hat{\psi}(u) = \frac{\lambda}{\lambda+u}, \tag{3.95}$$

the corresponding memory kernel in the Laplace space is given by $\hat{\Phi}_{\exp}(u) = \lambda$, which means $\Phi_{\exp}(t) = \lambda\delta(t)$. After some algebra, I get the following time evolution for $\langle+|\rho_{red}(t)|-\rangle$:

$$\langle+|\rho_{red}(t)|-\rangle = \langle+|\rho_{red}(0)|-\rangle\exp\{-K\lambda t\}. \tag{3.96}$$

It is interesting to notice how the subordination procedure, in the case of an exponential subordination function, namely the Poisson case, does not change the form of the exponential decay, just the characteristic time of the decay from $1/K$ to $1/(\lambda K)$.

I am going to consider the case where the subordination distribution is the power law given by Eq. (3.73). In order to apply the subordination technique and detect how decoherence takes place, it is necessary to evaluate the corresponding memory kernel. The analytical form of the Laplace transform of the subordination distribution given by Eq. (3.73) has been evaluated by Bologna *et al.* in Ref. [33], to be

$$\hat{\psi}(u) = (\mu-1)\frac{\Gamma(1-\mu)}{(uT)^{1-\mu}}\left(e^{uT} - E_{\mu-1}^{uT}\right). \tag{3.97}$$

The function $E_{\mu-1}^{uT}$, introduced in Ref. [35], is given by:

$$E_{\mu-1}^{uT} = (uT)^{1-\mu} \sum_{k=0}^{\infty} \frac{(uT)^k}{\Gamma(k+2-\mu)}. \tag{3.98}$$

For $u \to 0$, the function $\hat{\psi}(u)$ has the following asymptotic form:

$$\hat{\psi}(u) = 1 - \Gamma(2-\mu)(uT)^{\mu-1} + o\left((uT)^{\mu-1}\right). \tag{3.99}$$

In the case where the quantity (uT) vanishes, as a first approximation, I can neglect the second term on the right hand side of Eq. (3.99), because

$$\lim_{s \to 0} \frac{o\left(s^{\mu-1}\right)}{s^{\mu-1}} = 0. \tag{3.100}$$

Since μ is greater than 1 and smaller than 2, from Eq. (3.99), in this approximation, it is possible to derive the following expression for $\hat{\Phi}(u)$:

$$\hat{\Phi}(u) \simeq \frac{u^{2-\mu}}{T^{\mu-1}\Gamma(2-\mu)}. \tag{3.101}$$

It is convenient to evaluate the Laplace transform of both sides of Eq. (3.94) using Eq. (3.101); this way I get the following equality:

$$\mathcal{L}\left[\langle+|\rho_{red}(t)|-\rangle\right](u) = \frac{\langle+|\rho_{red}(0)|-\rangle}{u + K\hat{\Phi}(u)} \simeq \frac{\langle+|\rho_{red}(0)|-\rangle}{u + \frac{K}{T^{\mu-1}\Gamma(2-\mu)}u^{2-\mu}}. \tag{3.102}$$

By using Eq. (3.68), it is possible to recognize in Eq. (3.102) the Laplace transform of a

Mittag-Leffler function:

$$\langle +|\rho_{red}\left(t\right)|-\rangle \simeq E_{\mu-1}\left(-\frac{K}{T^{\mu-1}\Gamma\left(2-\mu\right)}t^{\mu-1}\right).\tag{3.103}$$

At this stage I can conclude that, in the limit $t \to +\infty$, the decoherence takes place in time as a power low:

$$\langle +|\rho_{red}\left(t\right)|-\rangle \simeq \frac{T_{NI}T^{\mu-1}}{|\ln\left(2g\tau\right)|\,t^{\mu-1}}, \quad t \to +\infty.\tag{3.104}$$

This is quite interesting. The bath of spins, described in Section 1.6, in the case where T_{NI} fluctuates and the corresponding subordination distribution is a power law, produces a *decoherence* described by a power law. The departure from the Poisson case is characterized by a transition from an exponential to a power law relaxation.

Now I consider the issue of dimensionality. Eqs. (3.33) and (3.40), as well as Eqs. (3.9) and (3.11), have a pathology. The same observation can be made for Eq. (3.103), where the argument of the Mittag-Leffler function has the dimension of the inverse of time, while it should be dimensionless. The origin of this problem is in the dimension of the memory kernel $\Phi(t)$. Since its Laplace transform has the dimension of the inverse of time, it has the dimension of the inverse of time to the power of 2. By assuming that the dimensional check turns out to be correct in both sides of Eq. (3.11), I have to conclude that the dimensional check turns out to be wrong in Eq. (3.9). This problem can be solved by imagining that all the quantities that usually have the dimension of a time are *dimensionless* and are multiple of a unit time τ_u, which is the minimum possible value that any of these quantities can have. Then, this unit time is set equal to unity: $\tau_u = 1$.

At this stage let us consider the idea of applying the subordination technique to the master equation (4.45). As already seen in Section 3.1, the subordination technique is applied to

the probability density, and turns a Markov equation (3.11) into a non-Markovian one (3.9), with a given memory kernel, by making the integral transformation (3.10). This is quite interesting because it can be used to mimic the experimental data of blinking quantum dots. This will be the argument of the next Chapter.

CHAPTER 4

SEARCHING FOR A QUANTUM MASTER EQUATION FOR BLINKING QUANTUM

DOTS

This last Chapter is devoted to illustrating the original results of my research work. I have tried to solve the theoretical issue of building up a master equation compatible with the experimental data about blinking quantum dots. The intermittent nature of fluorescent light, which the experimental data show, can be converted into a distribution of "light on" and "light off" states, which turns out to be non-Poisson. Thus, it is possible to evaluate the functions $p_1(t)$ and $p_2(t)$, corresponding to the probability of being in a condition of "light on" and "light off", respectively. The leading idea of my research work is to consider a two-level system, whose orthonormal state kets, $|1\rangle$ and $|2\rangle$, correspond to the "light on" and "light off" condition, and build up a quantum master equation for $\rho(t)$, the statistical density matrix describing the time evolution of the two-level system. If the model works, the functions $\langle 1|\rho(t)|1\rangle$ and $\langle 2|\rho(t)|2\rangle$, associated with $p_1(t)$ and $p_2(t)$, respectively, must fit the power law behavior given by the experimental data. Also, if there is any kind of correlation between $p_1(t)$ and $p_2(t)$, the off-diagonal elements of the statistical density matrix must be different from 0. So, I explore the idea of considering a two level-system whose time evolution is the superposition of both an in-phasing and a de-phasing process. I will explain clearly what these two processes consist of.

The quantum master equation is built up in order to explore different conditions, moving from a predominant inphasing to a predominant dephasing action. The latter case must reproduce the Zeno effect. In Section 2.1 I have shown a model of a one half spin interacting

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with an infinitely fast bath. The resulting master equation, named the Pauli master equation, is described by Eqs. (2.39) and (2.40). I expect to recover this structure for the time evolution of the diagonal elements of the statistical density matrix. Then, I am going to apply the subordination technique and investigate the asymptotic behavior in the case of a non-Poissonian subordination distribution. I anticipate that, in a condition where the Zeno effect appears, the subordination operation generates a relaxation described asymptotically by a Mittag- Leffler function. At the same time, a predominant inphasing process is converted, through the subordination technique, into an incoherent process again, but with no Mittag-Leffler signature. The quantum master equation is non-Markovian and the positivity of the density matrix is recovered. I plan to discuss in detail these results.

4.1 The Quantum Zeno Effect

Let us consider a one half spin whose time evolution, $|u(t)\rangle$, is given by the following expression:

$$|\Psi(t)\rangle = \cos\left(\frac{\Delta}{2}t\right)|+\rangle - \imath\sin\left(\frac{\Delta}{2}t\right)|-\rangle, \tag{4.1}$$

and let us assume that, at any regular time interval τ_m, a measurement of the observable S_x is performed. The system collapses either into the state $|+\rangle$ with probability p_1, given by the quantity $\cos^2\left(\frac{\Delta}{2}t\right)$, or into the state $|-\rangle$, with probability p_2, given by $\sin^2\left(\frac{\Delta}{2}t\right)$. If the measurement process is fast, i.e., for $\Delta\tau_m << 1$, the quantity p_1 is well approximated by the expression $\left(1 - \frac{\Delta^2\tau_m^2}{8}\right)$ and is much greater than p_2, approximated by the quantity $\left(\frac{\Delta^2\tau_m^2}{4}\right)$. Thus, the rate of collapsing into $|-\rangle$ is $\mathcal{R} = \frac{\Delta^2\tau_m}{2}$ and the survival probability $\Psi(t)$ is given by the following expression:

$$\Psi(t) = \exp\left\{-\frac{\Delta^2\tau_m}{4}t\right\}. \tag{4.2}$$

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The survival probability is defined by the following equality:

$$\Psi(t) = \int_t^\infty \psi(t')\, dt', \tag{4.3}$$

and thus the waiting time distribution, $\psi(t)$, is related to the survival probability, $\Psi(t)$, by the following equation:

$$\psi(t) = -\frac{d}{dt}\Psi(t), \tag{4.4}$$

and it turns out to be exponential:

$$\psi(t) = \frac{\Delta^2 \tau_m}{4} \exp\left\{-\frac{\Delta^2 \tau_m}{4} t\right\}. \tag{4.5}$$

In the limit of vanishing τ_m, both the waiting time distribution and the survival probability tend to decay more and more slowly. This means that the system collapses into the initial state with a higher probability as the measurement process speeds up. This behavior is named the quantum Zeno effect [34].

The measurement process is equivalent to an interaction with an infinitely fast bath, and in Section 2.1 I have shown how this process can be described by Eqs. (2.39), (2.40) and (2.41). Thus, it should be possible to recover the Zeno effect from the following system of differential equations:

$$\dot{p}_1(t) = -\gamma\left(p_1(t) - p_2(t)\right) \tag{4.6}$$

and

$$\dot{p}_2(t) = -\gamma\left(p_2(t) - p_1(t)\right). \tag{4.7}$$

I recall that the parameter γ is given by

$$\gamma = \frac{1}{\tau_m}\left[\sin\left(\frac{\Delta\tau}{2}\right)\right]^2 \tag{4.8}$$

and that in the case where $\Delta\tau_m << 1$, the parameter γ is well approximated by the following expression: $\gamma \simeq \frac{\Delta^2\tau_m}{4}$. In order to make the exposition clear, I remind the reader that the system of Eqs. (4.6) and (4.7) describes a one half spin interacting with an infinitely fast bath at any regular time interval τ_m. It is possible to recover the quantum Zeno effect from this picture by evaluating the waiting time distribution in the case that the measurement process is much faster compared to the order of the time scale of the evolution of the system: $1/\gamma >> \tau_m$. Let us make a more general treatment imagining that the number of sites is arbitrarily large. The master equation that drives the system is of the following form:

$$\frac{d}{dt}p_i(t) = \sum_{j=1}^{N} K_{i,j}p_j(t), \tag{4.9}$$

where the probability functions $p_i(t)$ fulfill the following constraint:

$$\sum_{i=1}^{N} p_i(t) = 1. \tag{4.10}$$

Let us define the column vector $\mathbf{p}(t)$, whose j-th element is $p_j(t)$; this way Eq. (4.9) can be written in the following matrix form:

$$\frac{d}{dt}\mathbf{p}(t) = \mathbf{K} \cdot \mathbf{p}(t), \tag{4.11}$$

where the matrix $\mathbf{K}$ is defined by the following equality:

$$[\mathbf{K}]_{i,j} = K_{i,j}. \tag{4.12}$$

Now let us adopt the prescriptions of the continuous time random walk picture given in Section 3.2. Here, a jump is equivalent to the collapse of the system due to the measurement process performed at regular time intervals τ_m. In this way the following map can be built up:

$$\mathbf{p}(n+1) = \mathbf{M} \cdot \mathbf{p}(n), \tag{4.13}$$

which leads to the following iterative form:

$$\mathbf{p}(n) = \mathbf{M}^n \cdot \mathbf{p}(0). \tag{4.14}$$

At this stage, let us define the function $\Phi_t(n)$, the probability that n collapses have occurred in a time interval t,

$$\Phi_t(n) = \int_0^t dt' \psi_n(t') \Psi(t - t'), \tag{4.15}$$

where $\psi_n(t')$ is the probability that n collapses occur, the last of them exactly at time t'. The survival probability $\Psi(t - t')$ is the probability of having no collapse between the instant t' and t. The continuous time random walk prescription yields

$$p_i(t) = \sum_{n=0}^{\infty} \Phi_t(n) p_i(n), \tag{4.16}$$

where the quantity $p_i(n)$ is the probability that the system collapses into the i-th state after n measurements, while $p_i(t)$ is the probability that the system collapses in the i-th state at

the time t. For a given time t and a given state i, all the possible numbers of jumps are considered, from 0 to infinity. The Laplace transform of both sides of Eq. (4.16) can be easily evaluated:

$$\hat{\mathbf{p}}(u) = \sum_{n=0}^{\infty} \frac{1 - \hat{\psi}(u)}{u} \left(\hat{\psi}(u) \right)^n \mathbf{M}^n \cdot \mathbf{p}(0). \qquad (4.17)$$

The quantity $\left| \hat{\psi}(u) \right|$ is less then unity; thus, the series converges and Eq. (4.17) can be written in the following form:

$$\hat{\mathbf{p}}(u) = \frac{1}{u \left(\mathbf{I} - \frac{\hat{\psi}(u)}{\mathbf{I} - \hat{\psi}(u)} (\mathbf{M} - \mathbf{I}) \right)} \cdot \mathbf{p}(0). \qquad (4.18)$$

The latter equation can be compared with the Laplace transform of Eq. (4.11), which is given by the following equality:

$$u\hat{\mathbf{p}}(u) - \mathbf{p}(0) = \mathbf{K} \cdot \hat{\mathbf{p}}(u), \qquad (4.19)$$

thus, I get

$$\hat{\mathbf{p}}(u) = \frac{1}{u\mathbf{I} - \mathbf{K}} \cdot \mathbf{p}(0). \qquad (4.20)$$

The only way both Eqs. (4.18) and (4.20) hold true is if $\mathbf{K}$ is related to $\mathbf{A}$ by the following equality:

$$\mathbf{K} = \frac{u\hat{\psi}(u)}{1 - \hat{\psi}(u)} (\mathbf{M} - \mathbf{I}), \qquad (4.21)$$

which means

$$\frac{u\hat{\psi}(u)}{1 - \hat{\psi}(u)} = \frac{1}{T}. \qquad (4.22)$$

The latter equation can be easily solved:

$$\hat{\psi}(u) = \frac{1}{1 + Tu}, \tag{4.23}$$

giving

$$\psi(t) = \frac{1}{T} \exp\left\{-\frac{t}{T}\right\}. \tag{4.24}$$

In order to evaluate T, I consider the most general non-Markovian equation

$$\frac{d}{dt}\mathbf{p}(t) = \int_0^t \Phi(t - t') \mathbf{K} \cdot \mathbf{p}(t') \, dt', \tag{4.25}$$

and evaluate the Laplace transform which is given by

$$\hat{\mathbf{p}}(u) = \frac{1}{u\mathbf{I} - \hat{\Phi}(u)\mathbf{K}} \cdot \mathbf{p}(0). \tag{4.26}$$

Then, I compare Eqs. (4.26) and (4.18); this way I recover the following form for the Laplace transform of the memory kernel:

$$\hat{\Phi}(u) = \frac{u\hat{\psi}(u)}{1 - \hat{\psi}(u)}. \tag{4.27}$$

Now, in order to obtain the system of Eqs. (4.6) and (4.7), I have to choose the following memory kernel:

$$\Phi(t) = \gamma\delta(t), \tag{4.28}$$

and by using Eq. (4.27) the parameter $\frac{1}{T}$ turns out to be equal to γ. So, the waiting time distribution is given by:

$$\psi(t) = \gamma \exp\left\{-\gamma t\right\}. \tag{4.29}$$

The quantum Zeno effect can be recovered by observing that, in the case of a measurement process, the parameter γ is proportional to τ_m, the time interval between two measurements. Thus, increasing the frequency of measurement forces γ to vanish, i.e., the waiting time distribution decays more and more slowly in time and the system tends to collapse into the same state with higher and higher probability. So, the quantum Zeno effect is recovered. Also, I have shown how a Pauli master equation gives an exponential waiting time distribution.

I apply, now, the subordination technique to the system of Eqs. (4.6) and (4.7).

Let us move to the picture of the "light on" and "light off" states. Let $p_1(t)$ and $p_2(t)$ denote the probabilities for the system to be in the state $|1\rangle$, "light on", and $|2\rangle$, "light off", respectively. In the following, I am going to apply the Sokolov subordination prescription. As usual, the symbol n denotes the internal time. The system of Eqs. (4.6) and (4.7) can be set in the following matrix form:

$$\frac{d}{dn}\mathbf{p}(n) = -\gamma\mathbf{M}\cdot\mathbf{p}(n),\tag{4.30}$$

where the matrix $\mathbf{M}$ is given by

$$\mathbf{M} \equiv \begin{bmatrix} 1, & -1 \\ -1, & 1 \end{bmatrix}\tag{4.31}$$

and the column vector $\mathbf{p}(n)$ is defined by

$$\mathbf{p}(n) = \begin{bmatrix} p_1(n) \\ p_2(n) \end{bmatrix}.\tag{4.32}$$

According to Eq. (3.9), the subordination prescription yields the following master equation in the external time scale t:

$$\frac{d}{dt}\mathbf{P}(t) = -\gamma \int_0^t \Phi\left(t - t'\right) \mathbf{M} \cdot \mathbf{P}\left(t'\right) dt',$$
(4.33)

where $\mathbf{P}(t)$, by definition, is related to $\mathbf{p}(t)$ by the integral transformation of Eq. (3.10), which means

$$\mathbf{P}(x, t) = \int_0^\infty \mathbf{p}(x, n) T\left(n, t\right) dn,$$
(4.34)

and, according to Eq. (3.14), in the Laplace transform space, the following equality holds true:

$$\hat{\mathbf{P}}(x, u) = \frac{1}{\hat{\Phi}(u)} \hat{\mathbf{p}}\left(x, \frac{u}{\hat{\Phi}(u)}\right).$$
(4.35)

Now, I will use an exponential subordination function, $\psi_S(t) = \lambda \exp\left\{-\lambda t\right\}$, associated to ordinary Poisson statistics. The Laplace transform of the memory kernel can be evaluated by using Eqs. (3.95) and (3.32), with the result $\hat{\Phi}(u) = \lambda$. This way, the Laplace transform of $T(n, t)$ is $\frac{1}{\lambda} \exp\left\{-n\frac{u}{\lambda}\right\}$, and thus

$$T(n, t) = \delta\left(t - \frac{n}{\lambda}\right).$$
(4.36)

By using Eqs. (3.10) and (4.27), the master equation corresponding to the external time scale, t, turns out to be Markovian:

$$\frac{d}{dt}\mathbf{P}(t) = -\gamma\lambda\mathbf{M} \cdot \mathbf{P}(t).$$
(4.37)

Note that both λ and t are dimensionless variables. I set the condition $\lambda << 1$. The

times t of the distribution of Eq. (4.29) are of the order of $1/\lambda$ and correspond to a process of time dilation. This change has the effect of increasing the relaxation time from the value $1/\gamma$ to the much larger value $1/(\gamma\lambda)$. This property is quite important because it has the role of emphasizing a fact observed, for instance, in Ref. [36], but frequently overlooked. An ordinary diffusion process, which is usually generated by a coin-tossing prescription, can also emerge from a condition with persistence, namely, a case where the occurrence of either heads or tails persists for an extended amount of time. In fact, the survival probability

$$\Pi(t) = p_1(t) - p_2(t) \tag{4.38}$$

obeys the exponential prescription

$$\Pi(t) = \exp(-2\gamma\lambda t)\Pi(0). \tag{4.39}$$

A further property of fundamental importance for the study of blinking quantum dots, is the distribution of sojourn times in the "light on" and "light off" states, here assumed to be identical and denoted with the symbol $\psi(t)$. Note that according to Ref. [37], the survival probability is not directly related to $\psi(t)$, but to another distribution, called $\psi^*(t)$, through the following simple expression:

$$\Pi(t) = \int_t^\infty dt' \psi^*(t'). \tag{4.40}$$

This waiting time distribution $\psi^*(t)$ is denoted as *theoretical* and is distinct from $\psi(t)$, which is called *experimental*, being determined by recording the alternate sequence of "light on" and "light off" states. The theoretical distribution $\psi^*(\tau)$ can be used to split the time axis

into a sequence of time intervals, the first beginning at $t = 0$ and ending at $t = \tau_1$, the second beginning at $t = \tau_1$, and ending at $t = \tau_1 + \tau_2$, and so on. After realizing this repartition of the time axis, I rest on the coin tossing procedure to assign to any time interval, of length τ_i, either the sign $+$ or the sign $-$. The resulting sequence generates the waiting time distribution $\psi(\tau)$. It is evident that $\psi(\tau)$ does not coincide with $\psi^*(\tau)$, given the fact that the coin tossing prescription might assign the same sign to two or more consecutive time intervals, determined by $\psi^*(\tau)$. It is known [37, 38] that

$$\hat{\psi}^*(u) = 2 \frac{\hat{\psi}(u)}{1 + \hat{\psi}(u)}, \tag{4.41}$$

where $\hat{\psi}^*(u)$ and $\hat{\psi}(u)$ denote the Laplace transforms of $\psi^*(t)$ and $\psi(t)$, respectively. In the case under discussion here Eqs. (4.39) and (4.40) yield

$$\psi^*(t) = \exp\left\{-2\lambda\gamma t\right\}. \tag{4.42}$$

Using Eq. (4.41) I obtain

$$\psi(t) = \exp\left\{-\lambda\gamma t\right\}. \tag{4.43}$$

The important information emerging from this equation is that, whatever the value of γ is, including also the case of minimum persistence where $\gamma = 1$, the process can be made as persistent as I wish by reducing the parameter λ. In conclusion, I make the assumption that there exists a minimum time value τ_u, which is considered to be the unit time, $\tau_u = 1$. The natural time is a multiple integer of this time. The condition of no persistence corresponds to adopting the coin tossing prescription at any step of this natural time scale. The coin tossing procedure serves the purpose of deciding whether the system has to be located in

the "light on" or "light off" state. This unit time coin-tossing procedure is described by the master equation

$$\frac{d}{dn}\mathbf{p}(n) = -\mathbf{M} \cdot \mathbf{p}(n). \tag{4.44}$$

The time distance between one time step and the next is the minimum possible time, τ_u. As a consequence of this assumption, the parameter γ of Eq. (4.30) is dimensionless. According to the illustrative example that produced Eq. (4.37), Eq. (4.30) can be thought of as being the result of a subordination procedure with $\psi_S(t) = \gamma \exp\{-\gamma t\}$, with $\gamma << 1$. I anticipate that the emergence of the Mittag-Leffler relaxation is a consequence of operating a further subordination, through a $\psi_S(t)$ with an inverse power law form, on a natural process described by Eq. (4.30) with $\gamma << 1$. Thus, the Mittag-Leffler relaxation will be proven to emerge from the consecutive application of two subordination prescriptions, the first with $\psi_S(t) = \gamma \exp\{-\gamma t\}$, which has the effect of turning Eq. (4.44) into Eq. (4.30), and the second with the subordination function $\psi_S(t)$ of Eq. (3.73) applied to Eq. (4.30), which has the effect of generating a generalized master equation, characterized by the Mittag-Leffler relaxation. Notice that in this work the subordination will be operated with the distribution of Eq. (3.73) on a process taking place in the natural time scale, not necessarily in the form of Eq. (4.30). The leading process can also be totally coherent.

4.2 A Master Equation Describing both an Inphasing and a Dephasing Process

In this Section I consider a quantum master equation describing a two-level system whose time evolution is the superposition of both an inphasing and a dephasing process. I imagine that this process takes place in the internal time scale n and I prepare the ground to apply the subordination technique and move to the external time scale t. In the following,

I refer to the set $\{|1\rangle, |2\rangle\}$ as the basis set of the state kets of a two-level system. In the natural time scale n, the process here under study is described by the following quantum master equation:

$$\frac{d}{dn}\rho(n) = L_T\rho(n) \equiv -\imath[H, \rho(n)] + \frac{1}{\tau_m}L_D\rho(n). \qquad (4.45)$$

The first term corresponds to the Hamiltonian

$$H = V(|1\rangle\langle 2| + |2\rangle\langle 1|). \qquad (4.46)$$

The operator H is responsible for the inphasing process. In order to explain what it is, let us assume, for example, that the system is prepared in the state $|1\rangle$. According to the Schrodinger equation, the Hamiltonian H gives the following time evolution:

$$|u(n)\rangle = |1\rangle \cos(Vn) - \imath|2\rangle \sin(Vn). \qquad (4.47)$$

The state ket $|u(n)\rangle$ is the superposition of both the state kets $|1\rangle$ and $|2\rangle$. This means that the off-diagonal elements of $|u(n)\rangle\langle u(n)|$, the corresponding statistical density matrix, are restored even if they vanish at $t = 0$. Of course, if the system is initially in the state ket $|2\rangle$, the time evolution results, again, in a superposition of both the two state kets. For this reason, following Capek and Bok [39], the term H is named the *inphasing* term. The second operator on the right hand side of Eq. (4.45), L_D, has the structure of a Lindblad operator. In Section 2.1 I have shown how the Lindblad operator mimics the effect of the external environment or of a measurement process on the system of interest. The explicit form of this Lindblad operator is given by

$$L_D\rho(n) = [A\rho(n), A^\dagger] + [A, \rho(n)A^\dagger], \qquad (4.48)$$

where the operator A is defined by the following equality:

$$A = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|). \tag{4.49}$$

The operator L_D acts only on the off-diagonal element of the density matrix in the following way:

$$L_D\rho(n) = -|1\rangle\langle 2|\langle 1|\rho(n)|2\rangle - |2\rangle\langle 1|\langle 2|\rho(n)|1\rangle; \tag{4.50}$$

also, the presence of the factor $\frac{1}{\tau_m}$ in Eq. (4.45) makes the off-diagonal elements of the density matrix $\rho(t)$ decay exponentially in time with a decay rate given by $1/\tau_m$. In order to make it clear, let us study the time evolution by adopting the following notation:

$$\rho_{i,j}(n) \equiv \langle i|\rho(n)|j\rangle, \tag{4.51}$$

for every i and j equal to 1 or 2.

I set $\tau_u = 1$. Consequently, the quantities τ_m, $\frac{1}{V}$, T, n and t are dimensionless and $\tau_u = 1$ is the minimum possible value that any of these quantities can have. From Eq. (4.45) the following equations descend:

$$\frac{d}{dn}\rho_{1,1}(n) = iV\left(\rho_{1,2}(n) - \rho_{2,1}(n)\right), \tag{4.52}$$

$$\frac{d}{dn}\rho_{1,2}(n) = iV\left(\rho_{1,1}(n) - \rho_{2,2}(n)\right) - \frac{1}{\tau_m}\rho_{1,2}(n), \tag{4.53}$$

$$\frac{d}{dn}\rho_{2,1}(n) = -iV\left(\rho_{1,1}(n) - \rho_{2,2}(n)\right) - \frac{1}{\tau_m}\rho_{2,1}(n) \tag{4.54}$$

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and

$$\frac{d}{d\tau}\rho_{2,2}(n) = -\imath V \left(\rho_{1,2}(n) - \rho_{2,1}(n)\right). \tag{4.55}$$

I am going to show how the quantum Zeno effect can be recovered as a particular case of this general picture, by considering certain values of the parameters V and τ_m.

4.3 The Quantum Zeno Effect as a Particular Case

This Section is devoted to showing the emergence of the quantum Zeno effect from the general treatment. With this purpose in mind, I am going to consider the limiting case of vanishing τ_m, namely

$$V\tau_m << 1. \tag{4.56}$$

In this case, since $V << \frac{1}{\tau_m}$, Eqs. (4.53) and (4.54) can be approximated by the following forms:

$$\frac{d}{dn}\rho_{1,2}(n) = -\frac{1}{\tau_m}\rho_{1,2}(n), \tag{4.57}$$

and

$$\frac{d}{dn}\rho_{2,1}(n) = \frac{1}{\tau_m}\rho_{2,1}(n), \tag{4.58}$$

respectively. This means that the off-diagonal elements of the density matrix $\rho(n)$ decay exponentially in time with a decay rate given by $1/\tau_m$. Thus, the Smoluchowsky approximation [24] can be applied: due to the fast decay of $\rho_{1,2}(n)$ and $\rho_{2,1}(n)$, their time derivative can be set equal to zero: $\dot{\rho}_{1,2}(n) = 0$ and $\dot{\rho}_{2,1}(n) = 0$. Thus, I get the following expressions:

$$\rho_{1,2}(n) = \imath V \tau_m \left(\rho_{1,1}(n) - \rho_{2,2}(n)\right), \tag{4.59}$$

and

$$\rho_{2,1}(n) = -\imath V \tau_m \left(\rho_{1,1}(n) - \rho_{2,2}(n) \right). \tag{4.60}$$

By substituing Eqs. (4.59) and (4.60) into Eqs. (4.52) and (4.55), I get the following system of differential equations:

$$\frac{d}{dn}\rho_{1,1}(n) = -2V^2\tau_m \left(\rho_{1,1}(n) - \rho_{2,2}(n) \right), \tag{4.61}$$

and

$$\frac{d}{dn}\rho_{2,2}(n) = -2V^2\tau_m \left(\rho_{2,2}(n) - \rho_{1,1}(n) \right). \tag{4.62}$$

The diagonal elements of the statistical density matrix, $\rho_{1,1}(n)$ and $\rho_{2,2}(n)$, are identified with the probability for the particle to be in the state $|1\rangle$ and $|2\rangle$, respectively. Thus, I will use the following notation:

$$p_1(n) \equiv \rho_{1,1}(n), \tag{4.63}$$

and

$$p_2(n) \equiv \rho_{2,2}(n). \tag{4.64}$$

This way, I get

$$\dot{p}_1(n) = -\gamma_S \left(p_1(n) - p_2(n) \right), \tag{4.65}$$

$$\dot{p}_2(n) = -\gamma_S \left(p_2(n) - p_1(n) \right), \tag{4.66}$$

where

$$\gamma_S = 2V^2\tau_m. \tag{4.67}$$

This system is formally identical to the one given by Eq. (4.30), where the factor γ is replaced

by γ_S. The label S stands for Smoluchowsky approximation. If the system is initially in the state $|1\rangle$, the function $\Pi(n)$, defined by

$$\Pi(n) = p_1(n) - p_2(n), \tag{4.68}$$

is the survival probability associated to the state $|1\rangle$. According to the system of Eqs. (4.65) and (4.66), the survival probability is the solution of the following differential equation:

$$\dot{\Pi}(n) = -2\gamma_S \Pi(n), \tag{4.69}$$

which gives

$$\Pi(n) = \Pi(0) \exp\{-2\gamma_S n\}. \tag{4.70}$$

From the point of view of the wave function $|u(t)\rangle$, I get that, due to the measurement process, the system collapses into either $|1\rangle$ or $|2\rangle$, with a Poissonian waiting time distribution [40].

In Section 4.1 I have shown how the distribution of the times of sojourn in either the state $|1\rangle$ or $|2\rangle$ is exponential:

$$\psi(n) = \gamma_S \exp\{-\gamma_S n\}. \tag{4.71}$$

This distribution corresponds to the experimental observation of "light on" and "light off" states. Dehmelt and co-workers [41, 42, 43] produced a condition of intermittent fluorescence that can be easily turned into a symbolic time series, with the symbol $+$ denoting the fluorescent state, the "light on" state, and the symbol $-$ denoting the so called shelved

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state, with no fluorescence occurring, the "light off" state. Notice that the theoretical interpretation proposed by Cook and Kimble [8] implies that the "light on" and "light off" states are characterized by a Poissonian distribution of sojourn times. Here, the distribution of the "light on" and "light off" states are assumed to be the same, both of them described by $\psi(n)$. With this limitation in mind, I can conclude that the theory of Ref. [40], yielding Eq. (4.71), fits the theoretical conclusions of the work of Ref. [8]. I have explained in Section 4.1 that keeping V constant and decreasing τ_m has the effect of making the relaxation of $\Pi(n)$ slower. It is, again, the Zeno effect, which means that the increasing of the measurement frequency has the effect of slowing down the motion of the observed process. There exists therefore, according to Ref. [40], an intimate connection between the phenomenon of intermittent fluorescence and the Zeno effect. The recent experimental observation of blinking quantum dots [3, 44] can be considered to be an extension to the non-Poissonian case of the earlier experimental results of Dehmelt. These new materials, under the influence of an exciting radiation field yield intermittent fluorescence, namely a sequel of "light on" and "light off" states, with a distribution of sojourn times that have the same time asymptotic properties as Eq. (3.73) with $\mu < 2$. Thus, it is interesting to discuss the possibility of generating a non-Markovian quantum equation that might be compatible with the experimental observation of an alternate sequence of "light on" and "light off" states, with a non-Poissonian time distribution. It is evident that the adoption of the function of Eq. (3.73) for the subordination process must yield the same generalized master equation as that of Eq. (4.33). Let us now show the emergence of the Mittag-Leffler function from this picture. The generalized master equation of Eq. (4.33), as a non-stationary master equation, was derived by the authors of Ref. [38]. However, they did not discuss the emergence of the Mittag-Leffler function, an issue to which I now devote my attention. Using Eq. (4.40) and Eq. (4.41) I can express

the Laplace transform of $\Pi(t)$, $\hat{\Pi}(u)$, as

$$\hat{\Pi}(u) = \frac{(1 - \hat{\psi}(u))}{u(1 + \hat{\psi}(u))}. \tag{4.72}$$

This form relates the survival probability $\Pi(t)$ to the experimental property $\psi(t)$. On the other hand, according to the generalized master equation of Eq. (4.33), I get [38]

$$\hat{\Pi}(u) = \frac{1}{1 + 2\hat{\Phi}(u)} = \frac{1 - \hat{\psi}_S(u)}{u\left(1 + (2\gamma - 1)\hat{\psi}_S(u)\right)}. \tag{4.73}$$

To obtain this result I have also used Eq. (3.32). Through this relation I express the survival probability as a function of the subordination distribution $\psi_S(t)$. Thus, comparing Eq. (4.73) to Eq. (4.72) I connect $\psi(t)$ to $\psi_S(t)$. I see that $\gamma = 1$ yields:

$$\psi(t) = \psi_S(t). \tag{4.74}$$

Let us discuss this important result. I am using for the leading process the Pauli master equation of Eq. (4.44), which is for a process with no persistence, namely, a physical condition where the random walker sojourns in a given state, either $|1\rangle$ or $|2\rangle$, for only one or a few more time steps, according to a coin tossing prescription. If I apply the subordination procedure using this process as a leading process, I obtain a generalized master equation, corresponding to a waiting time distribution $\psi(t)$ identical to the subordination function $\psi_S(t)$. This is the reason why, having in mind the blinking quantum dots, I am adopting for the subordination function $\psi_S(t)$ the form of Eq. (3.73). It would be attractive to prove that the results of the blinking quantum dots experiment are derived from the subordination to the Dehmelt process. As earlier discussed, this corresponds to the case where the dimensionless parameter

γ fits the condition $\gamma << 1$. In this case, the condition of Eq. (4.74) is violated. However, it is possible to prove that $\psi(t)$ has the same time asymptotic properties as $\psi_S(t)$, thereby justifying also in this case the choice of Eq. (3.73). Let us discuss this case in detail. Let us set

$$\epsilon \equiv \frac{\left(1 - \hat{\psi}_S(u)\right)}{\left[1 + (2\gamma - 1)\hat{\psi}_S(u)\right]}. \tag{4.75}$$

By comparing Eq. (4.73) to Eq. (4.72) I have

$$\frac{1 - \hat{\psi}(u)}{1 + \hat{\psi}(u)} = \epsilon, \tag{4.76}$$

which yields

$$\hat{\psi}(u) = \frac{1 - \epsilon}{1 + \epsilon}. \tag{4.77}$$

Notice that both $\psi(t)$ and $\psi_S(t)$ are normalized distribution. Thus, the following equalities hold true:

$$\hat{\psi}(0) = \hat{\psi}_S(0) = 1. \tag{4.78}$$

I am considering the limiting condition $u \to 0$. Thus, I have

$$\epsilon << 1, \tag{4.79}$$

which makes it possible to replace Eq. (4.77) with

$$\hat{\psi}(u) \approx 1 - 2\epsilon. \tag{4.80}$$

Note that the analytical form of Eq. (3.73) yields [37]

$$\psi_S(u) = 1 - c_S u^{\mu-1}, \quad u \to 0, \quad \mu < 2, \tag{4.81}$$

with

$$c_S \equiv \Gamma(2 - \mu) T^{\mu-1}. \tag{4.82}$$

Using Eq. (4.80) and the asymptotic condition for ϵ of Eq. (4.75), we get

$$\hat{\psi}(u) \approx 1 - c u^{\mu-1}, \tag{4.83}$$

with

$$c \equiv \frac{c_S}{\gamma}, \tag{4.84}$$

thereby proving the earlier statement that $\psi(t)$ has the same time asymptotic properties as $\psi_S(t)$. On the basis of this important conclusion, I would be tempted to assign to the experimental function $\psi(t)$ the same analytical form as Eq. (3.73), fitting the constraint of Eq. (4.84). This means

$$\psi(\tau) = (\mu - 1) \frac{T_{exp}^{\mu-1}}{(\tau + T_{exp})^{\mu}}, \tag{4.85}$$

with

$$T_{exp} \equiv \frac{T}{\gamma^{\frac{1}{(\mu-1)}}}. \tag{4.86}$$

Due to the fact that $\gamma << 1$ and $T_{exp} >> T >> \tau_u = 1$, this analytical form would imply a much more extended regime of transition from microscopic dynamics to the time asymptotic condition described by the inverse power law form of Eq. (4.85). This is not quite incorrect.

However, in the large time region defined by

$$t_u < t < T_{exp}. \tag{4.87}$$

another time asymptotic expression emerges. To discuss this important property let us go back to

$$\hat{\Pi}(u) = \frac{\Pi(0)}{u + 2\gamma\hat{\Phi}(u)}. \tag{4.88}$$

By using Eq. (3.32) and the approximated expression of Eq. (4.81) for $\hat{\psi}_S(u)$, I get

$$\hat{\Pi}(u) \approx \frac{1}{u(1 - 2\gamma) + \frac{2\gamma}{c_S}u^{2-\mu}}. \tag{4.89}$$

Using the condition $\gamma << 1$ and the explicit expression for c_S of Eq.(4.82), I arrive at

$$\hat{\Pi}(u) \simeq \frac{1}{u + \omega_\mu^{\mu-1}u^{2-\mu}}, \tag{4.90}$$

where ω_μ is the frequency defined by

$$\omega_\mu \equiv \frac{1}{T}\left(\frac{2\gamma}{\Gamma(2 - \mu)}\right)^{1/(\mu-1)}. \tag{4.91}$$

It is straightforward to prove that for $t >> T_{exp}$, u becomes negligible compared to $\omega_\mu^{\mu-1}u^{2-\mu}$. Thus, Eq. (4.90) becomes indistinguishable from an ordinary inverse power law. However, in the wide time region of Eq.(4.87), establishing the condition for the emergence of the Mittag-Leffler relaxation, this is no more so, and $\Pi(t)$ is approximated well by

$$\Pi(t) \simeq E_{\mu-1}\left(-(\omega_\mu t)^{\mu-1}\right), \tag{4.92}$$

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a stretched exponential. A more rigorous derivation of this result will be given in the following by using the techniques illustrated in Section 3.3. In conclusion, the joint action of the dephasing and inphasing processes has the effect of producing the Zeno effect, if the dephasing is much faster than the inphasing process. The subordination procedure applied to the Zeno condition would produce a generalized master equation generating an extensive Mittag-Leffler relaxation. The Mittag-Leffler relaxation is proven therefore to be generated by a twofold application of the subordination operation. I move from the Pauli master equation of Eq. (4.44) to the Pauli master equation of Eq. (4.30) with the subordination function $\psi_S(t) = \gamma \exp\{-\gamma t\}$. Then, I apply the subordination procedure to Eq. (4.30) with the subordination function of Eq. (3.73). This yields a generalized master equation characterized by Mittag-Leffler relaxation. If I apply the inverse power law subordination procedure directly to Eq. (4.30), or, equivalently, I set $\gamma = 1$, the Mittag-Leffler relaxation is replaced by an inverse power law relaxation process.

4.4   The Solution of the General Case

This Section is devoted to studying the general case described by Eq. (4.45), and for this purpose I set the system of Eqs. (4.52), (4.53), (4.54) and (4.55) in the following compact form:

$$\frac{d}{dn}\mathbf{r}(n) = \mathbf{\Lambda} \cdot \mathbf{r}(n), \tag{4.93}$$

where $\mathbf{r}(n)$ and $\mathbf{\Lambda}$ are defined by the following expressions:

$$\mathbf{r}(n) \equiv \begin{bmatrix} \rho_{1,1}(n) \\ \rho_{1,2}(n) \\ \rho_{2,1}(n) \\ \rho_{2,2}(n) \end{bmatrix} \tag{4.94}$$

and

$$\mathbf{\Lambda} \equiv \begin{bmatrix} 0 & \imath V & -\imath V & 0 \\ \imath V & -\frac{1}{\tau_m} & 0 & -\imath V \\ -\imath V & 0 & -\frac{1}{\tau_m} & \imath V \\ 0 & -\imath V & \imath V & 0, \end{bmatrix}, \tag{4.95}$$

respectively.

The eigenvalues of $\mathbf{\Lambda}$ are

$$\lambda_1 = 0, \tag{4.96}$$

$$\lambda_2 = -\frac{1}{\tau_m}, \tag{4.97}$$

$$\lambda_3 = -\frac{1 + \sqrt{\Delta}}{2\tau_m} \tag{4.98}$$

and

$$\lambda_4 = \frac{\sqrt{\Delta} - 1}{2\tau_m}, \tag{4.99}$$

with Δ defined by

$$\Delta \equiv 1 - 16V^2\tau_m^2. \tag{4.100}$$

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The corresponding (non-normalized) eigenvectors are :

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.101}$$

$$\mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \tag{4.102}$$

$$\mathbf{v}^{(3)} = \begin{bmatrix} -1 \\ -\frac{\imath}{4V\tau_m}(1 + \sqrt{\Delta}) \\ \frac{\imath}{4V\tau_m}(1 + \sqrt{\Delta}) \\ 1 \end{bmatrix} \tag{4.103}$$

and

$$\mathbf{v}^{(4)} = \begin{bmatrix} -1 \\ \frac{\imath}{4V\tau_m}(\sqrt{\Delta} - 1) \\ \frac{\imath}{4V\tau_m}(1 - \sqrt{\Delta}) \\ 1 \end{bmatrix}. \tag{4.104}$$

In the case

$$\Delta \neq 0, \tag{4.105}$$

the eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}$ and $\mathbf{v}^{(4)}$, are linearly independent. In the following, I will

refer to $\mathbf{r}_j$ as the j-th element of the column-vector $\mathbf{r}$. So, I am in a position to evaluate the solution of Eq. (4.93), which is given by:

$$\mathbf{r}(n) = \sum_{i=1}^{4} \mu_i e^{\lambda_i n} \mathbf{v}^{(i)}. \tag{4.106}$$

The parameter μ_1 is given by

$$\mu_1 = \frac{1}{2}. \tag{4.107}$$

The parameters μ_i, with $i \neq 1$, depend on the initial conditions; in fact, I get the following relations:

$$\mu_2 = \frac{\rho_{1,2}(0) + \rho_{2,1}(0)}{2}, \tag{4.108}$$

$$\mu_3 = \frac{(\rho_{1,1}(0) - \rho_{2,2}(0))\,(1 - \sqrt{\Delta}) + 4\imath V \tau_m \,(\rho_{1,2}(0) - \rho_{2,1}(0))}{4\sqrt{\Delta}} \tag{4.109}$$

and

$$\mu_4 = \frac{1 + \sqrt{\Delta}}{4\sqrt{\Delta}} \,(\rho_{2,2}(0) - \rho_{1,1}(0)) + \frac{\imath V \tau_m}{\sqrt{\Delta}} \,(\rho_{2,1}(0) - \rho_{1,2}(0)). \tag{4.110}$$

In the case where $\Delta \neq 0$, the solution of the problem is given by Eq. (4.106), supplemented by Eqs. (4.107), (4.108), (4.109) and (4.110). The same solution, namely, the solution of Eq. (4.93) can also be expressed with the following more compact notation:

$$\mathbf{r}(n) = \mathbf{U} \cdot \begin{bmatrix} e^{\lambda_1} & 0 & 0 & 0 \\ 0 & e^{\lambda_2} & 0 & 0 \\ 0 & 0 & e^{\lambda_3} & 0 \\ 0 & 0 & 0 & e^{\lambda_4} \end{bmatrix} \cdot \mathbf{U}^{-1} \cdot \mathbf{r}(0),, \tag{4.111}$$

where the matrix $\mathbf{U}$ is built up with the following procedure: the j-th column is given by

the j-th eigenvector of $\mathbf{\Lambda}$. A detailed treatment of these problems is given in Ref. [45].

In the case where $\Delta = 0$, the solution of Eq. (4.93) is built up with a different procedure from the one given in Eq.(4.106). In fact, it is described by the following equality:

$$\mathbf{r}(n) = \nu_1 \mathbf{v}^{(1)} + \nu_2 \mathbf{v}^{(2)} e^{-\frac{n}{\tau_m}} + \nu_3 e^{-\frac{n}{2\tau_m}} \left(n \mathbf{w}^{(3)} + \mathbf{w}^{(4)} \right). \tag{4.112}$$

The symbol $\mathbf{W}$ denotes the matrix $\mathbf{\Lambda}$ in this particular case. The eigenvalues of $\mathbf{W}$ are : 0, $\left(-\frac{1}{\tau_m}\right)$ and $\left(-\frac{1}{2\tau_m}\right)$; the degeneracy of $\left(-\frac{1}{2\tau_m}\right)$ is 2. The vector $\mathbf{w}^{(3)}$ is the eigenvector of $\mathbf{W}$ corresponding the eigenvalue $\left(-\frac{1}{2\tau_m}\right)$,

$$\mathbf{w}^{(3)} = \begin{bmatrix} -1 \\ -\imath \\ \imath \\ 1 \end{bmatrix}, \tag{4.113}$$

and the vector $\mathbf{w}^{(4)}$ is obtained from the solution of the following equation:

$$\left(\mathbf{W} + \frac{1}{2\tau_m} \mathbf{I} \right) \mathbf{w}^{(4)} = \mathbf{w}^{(3)}. \tag{4.114}$$

The parameters ν_i are given by:

$$\nu_1 = \frac{1}{2}, \tag{4.115}$$

$$\nu_2 = \frac{1}{2} \left(\rho_{1,2}(0) + \rho_{2,1}(0) \right) \tag{4.116}$$

and

$$\nu_3 = \frac{1}{4\tau_m} \left(\rho_{2,2}(0) - \rho_{1,1}(0) + \imath \left(\rho_{2,1}(0) - \rho_{1,2}(0) \right) \right). \tag{4.117}$$

The vector $\mathbf{w}^{(4)}$ has the following form:

$$\mathbf{w}^{(4)} = 2\tau_m \begin{bmatrix} \frac{\rho_{2,2}(0)-\rho_{1,1}(0)}{\rho_{1,1}(0)-\rho_{2,2}(0)+i(\rho_{1,2}(0)-\rho_{2,1}(0))} \\ \frac{\rho_{2,1}(0)-\rho_{1,2}(0)}{\rho_{1,1}(0)-\rho_{2,2}(0)+i(\rho_{1,2}(0)-\rho_{2,1}(0))} \\ \frac{\rho_{1,2}(0)-\rho_{2,1}(0)}{\rho_{1,1}(0)-\rho_{2,2}(0)+i(\rho_{1,2}(0)-\rho_{2,1}(0))} \\ \frac{\rho_{1,1}(0)-\rho_{2,2}(0)}{\rho_{1,1}(0)-\rho_{2,2}(0)+i(\rho_{1,2}(0)-\rho_{2,1}(0))} \end{bmatrix} . \tag{4.118}$$

Again, a detailed explanation of the way to solve Eq. (4.93) in case where $\Delta = 0$ can be found in Ref. [45]. This treatment completes the solution of Eq. (4.45), and gives the form of $\mathbf{r}(n)$ for every value of the parameters, as well.

The general solution of this Section recovers the Pauli master equation of Eq. (4.30) in the limiting case $V\tau_m << 1$. In fact, under this condition I obtain the following extremal values:

$$\lambda_2 \to -\infty, \quad \lambda_3 \to -\infty, \quad \lambda_4 \to 0 \tag{4.119}$$

and

$$\mu_3 \mathbf{v}^{(3)} \to \frac{\rho_{1,2}(0) - \rho_{2,1}(0)}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mu_4 \mathbf{v}^{(4)} \to \frac{\rho_{2,2}(0) - \rho_{1,1}(0)}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} . \tag{4.120}$$

It is straightforward to prove that these expressions lead to the Markov equation of Eq. (4.30).

In the limiting case of $\tau_m \to +\infty$, namely, when $V\tau_m >> 1$, the effect of the dephasing operator L_D in Eq. (4.45) vanishes, and this picture describes the action of the inphasing

process alone. In this case, I obtain

$$\lambda_2 \to 0, \quad \lambda_3 \to -2\imath V, \quad \lambda_4 \to 2\imath V \qquad (4.121)$$

and

$$\mu_3 \mathbf{v}^{(3)} \to \frac{\rho_{2,2}(0) - \rho_{1,1}(0) + \rho_{1,2}(0) - \rho_{2,1}(0)}{4} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad (4.122)$$

$$\mu_4 \mathbf{v}^{(4)} \to \frac{\rho_{2,2}(0) - \rho_{1,1}(0) + \rho_{2,1}(0) - \rho_{1,2}(0)}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} . (4.123)$$

These two limiting conditions will have to be kept in mind in Section 4.5 to understand the physical properties after operating the subordination procedure.

4.5 Subordination in Action without the Zeno Effect Assumption

At this stage, I am equipped to study a more general master equation than the one of Eq. (4.30). I intend to use the prescriptions of Section 3.1 in order to get a generalized master equation of the following form:

$$\frac{d}{dt}\mathbf{R}(t) = \int_0^t \Phi(t - t')\mathbf{\Lambda} \cdot \mathbf{R}(t')dt'. \qquad (4.124)$$

It is evident that $\mathbf{R}(t)$ describes the effect of the subordination to the leading dynamics of $\mathbf{r}(n)$, driven by Eq. (4.93). According to the subordination prescription, the memory kernel $\Phi(t)$ is related to the experimental waiting-time distribution $\psi(t)$ of Eq. (3.73) by means of Eq. (3.32), with the following choice of μ:

$$1 < \mu < 2. \tag{4.125}$$

This important relation, adapted to the general case here under discussion, becomes

$$\hat{\mathbf{R}}_j(u) = \frac{1}{\hat{\Phi}(u)} \hat{\mathbf{r}}_j \left(\frac{u}{\hat{\Phi}(u)} \right), \tag{4.126}$$

with the initial condition

$$\mathbf{R}(0) = \mathbf{r}(0). \tag{4.127}$$

From Eq. (4.106) I get the following form for the solution:

$$\mathbf{r}_j(n) = \sum_{i=1}^{4} \mu_i e^{\lambda_i n} \left[\mathbf{v}^{(i)} \right]_j, \tag{4.128}$$

whose Laplace transform, $\hat{\mathbf{r}}_j(s)$, is given by:

$$\hat{\mathbf{r}}_j(s) = \sum_{i=1}^{4} \mu_i \frac{1}{s - \lambda_i} \left[\mathbf{v}^{(i)} \right]_j. \tag{4.129}$$

Thus, going back to Eq. (4.126), I can write $\hat{\mathbf{R}}_j(u)$ in the following way:

$$\hat{\mathbf{R}}_j(u) = \sum_{i=1}^{4} \frac{\mu_i}{u - \lambda_i \hat{\Phi}(u)} \left[\mathbf{v}^{(i)} \right]_j = \frac{\mu_1}{u} \left[\mathbf{v}^{(1)} \right]_j + \sum_{i=2}^{4} \mu_i \hat{f}_i(u) \left[\mathbf{v}^{(i)} \right]_j. \tag{4.130}$$

In Eq. (4.130), the functions $\hat{f}_i(u)$ are defined by the following expressions:

$$\hat{f}_i(u) \equiv \frac{1}{u - \lambda_i \hat{\Phi}(u)}. \tag{4.131}$$

Now I have to introduce some new techniques that are quite powerful in the detection, not only of the Mittag-Leffler function, but also of the time window where it appears.

At this stage I am equipped to evaluate the asymptotic behavior of $\mathbf{R}_j(t)$ in the limit $t \to \infty$, by studying $\hat{\mathbf{R}}_j(u)$ in the limit $u \to 0$. In fact, using the methods explained in Section 3.3, I am able to evaluate the time asymptotic behavior of $f_i(t)$, from the inverse Laplace transform of $\hat{f}_i(u)$ for $u \to 0$. To realize the condition for the emergence of the Mittag-Leffler function, I have to consider the case where the parameters T, τ_m, V and μ have such values as to yield the following condition:

$$\left| \Gamma\left(2 - \mu\right)\left(1 + \frac{1}{\lambda_i}\right) \right|^{\frac{1}{\mu-1}} >> 1. \tag{4.132}$$

This permits the creation of the following Mittag-Leffler time window:

$$T << t << T \left| \Gamma\left(2 - \mu\right)\left(1 + \frac{1}{\lambda_i}\right) \right|^{\frac{1}{\mu-1}}, \tag{4.133}$$

within which, according to the results obtained in Section 3.3, the Mittag-Leffler function appears, insofar as the function $f_i(t)$ becomes

$$\frac{1}{1 + \lambda_i} E_{\mu-1}\left(\frac{\lambda_i t^{\mu-1}}{\Gamma\left(2 - \mu\right)\left(1 + \lambda_i\right) T^{\mu-1}}\right). \tag{4.134}$$

If this window does not exist, that is, if the inequality (4.132) is not fulfilled, the time-asymptotic condition $t \gg T$ can only generate the inverse power law behavior:

$$f_i(t) \sim \frac{-T^{\mu-1}}{\lambda_i} \frac{1}{t^{\mu-1}}. \tag{4.135}$$

In this case, for any i ranging from 2 to 4, the time asymptotic form of $f_i(t)$ gets the power-law structure of Eq.(4.135), which coincides with the "tail" of the Mittag-Leffler function, emerging in the final time-asymptotic limit from the expression (4.134). I will evaluate the time asymptotic physics produced by subordination using Eq. (4.135) and the explicit expressions for the eigenvalues λ_i, given by Eqs. (4.96), (4.97), (4.98) and (4.99).

Using these arguments, finally, I am able to build up an expression that describes the functions $\mathbf{R}_j(t)$ in the time asymptotic limit $t \gg T$:

$$\mathbf{R}_j(t) \simeq \mu_1 \left[\mathbf{v}^{(1)} \right]_j + \sum_{i=2}^{4} \mu_i f_i(t) \left[\mathbf{v}^{(i)} \right]_j. \tag{4.136}$$

According to the earlier discussion, the functions $f_i(t)$ are determined by the following prescription: if the inequality (4.132) applies, in the time-interval (4.133) the function $f_i(t)$ approaches the Mittag-Leffler function described by the expression (4.134); for larger times, the function $f_i(t)$ tends to the power law given by Eq. (4.135). If, on the contrary, I explore the time asymptotic condition $t \gg T$, and I consider values of the parameters such that the inequality (4.132) is not fulfilled, then $f_i(t)$ assumes the inverse power law form given by Eq. (4.135) without crossing the intermediate time region, where the stretched exponential appears.

4.5.1 The Case where the Zeno Effect Condition Applies

Let us consider, now, the case of vanishing τ_m, that is, $V\tau_m << 1$. In this condition, if my discussion is consistent, I have to recover the results of Section 4.2. As a double check, let us prove that it is so. Using Eqs. (4.119), (4.133), (4.134) and (4.135), which describe the eigenvalues λ_i in the case of vanishing τ_m, I obtain that the time size of the Mittag-Leffler window vanishes for $i = 2$ and $i = 3$. However, at the same time the functions $f_2(t)$ and $f_3(t)$ tend to vanish. In fact, considering Eqs. (4.134) and (4.135), since both λ_2 and λ_3 diverge, I can easily see that both $f_2(t)$ and $f_3(t)$ vanish. For $i = 4$, due to the fact that λ_4 tends to vanish, the Mittag-Leffler time-window of the inequality (4.133) grows and the argument of the Mittag-Leffler function, given by expression (4.134), tends to vanish, so that a stretched exponential-like behavior appears in $f_4(t)$. These arguments justify the following expression for the asymptotic behavior of $\mathbf{R}(t)$:

$$
\mathbf{R}(t) \rightarrow \frac{1}{2} \begin{bmatrix} 1 + (\rho_{1,1}(0) - \rho_{2,2}(0))\, f_4(t) \\ 0 \\ 0 \\ 1 - (\rho_{1,1}(0) - \rho_{2,2}(0))\, f_4(t) \end{bmatrix}. \tag{4.137}
$$

This expression is in complete agreement with Eq. (4.92). In fact, I have earlier demonstrated that, in the case $V\tau_m << 1$, the quantities $[\mathbf{R}(t)]_2$ and $[\mathbf{R}(t)]_3$, which, respectively, are the results of the subordination of both $\rho_{1,2}(t)$ and $\rho_{2,1}(t)$, vanish, while $([\mathbf{R}(t)]_1 - [\mathbf{R}(t)]_4)$, the result of the subordination of $(\rho_{1,1}(n) - \rho_{2,2}(n))$, becomes identical to $\Pi(t)$, as expressed by Eq. (4.92). This proves that the result of Section 4.2 is recovered in the proper limit.

Let us consider, now, the case $V\tau_m >> 1$. This condition, to be correct, is expected to

degenerate in the case without dephasing, for which $L_D = 0$, studied in the next subsection. As expected, after going through the same algebra as in the general case, I see that in the limit $\tau_m \to +\infty$ I recover the relations of Eqs. (4.121), (4.122) and (4.123), in accordance with the result of the next subsection that applies to the case $L_D = 0$.

4.5.2 No Dephasing Process in Action

Let us now evaluate the case where no dephasing operates. I have to study, therefore, the time evolution of $\mathbf{R}^{(inph)}(t)$, this symbol denoting the result of the subordination procedure, when only the inphasing process is active. Substituing the results given in Eqs. (4.121), (4.122) and (4.123) into Eq. (4.130) and going through the same calculations as those done in the earlier cases, I get for $R_{i,j}^{(inph)}(t)$ the following time asymptotic expression:

$$R_{1,1}^{(inph)}(t) \sim \frac{1}{2} + \frac{\imath}{4V}\left(\frac{T}{t}\right)^{\mu-1}(\rho_{1,2}(0) - \rho_{2,1}(0)), \tag{4.138}$$

$$R_{1,2}^{(inph)}(t) \sim \frac{(\rho_{1,2}(0) + \rho_{2,1}(0))}{2} + \frac{\imath}{4V}\left(\frac{T}{t}\right)^{\mu-1}(\rho_{1,1}(0) - \rho_{2,2}(0)), \tag{4.139}$$

$$R_{2,1}^{(inph)}(t) \sim \frac{(\rho_{1,2}(0) + \rho_{2,1}(0))}{2} - \frac{\imath}{4V}\left(\frac{T}{t}\right)^{\mu-1}(\rho_{1,1}(0) - \rho_{2,2}(0)), \tag{4.140}$$

$$R_{2,2}^{(inph)}(t) \sim \frac{1}{2} - \frac{\imath}{4V}\left(\frac{T}{t}\right)^{\mu-1}(\rho_{1,2}(0) - \rho_{2,1}(0)). \tag{4.141}$$

These results are in agreement with those we find in the limit of $\tau_m \to +\infty$ for Eq. (4.136). In fact, according to Eq. (4.121), the size of the Mittag-Leffler time windows corresponding to $i = 2$ increases with increasing τ_m, but the argument of the Mittag-Leffler function (3.89) vanishes, so that $f_2(t)$ approaches the value of unity for times $t >> T$. For $i = 3$ and $i = 4$, on the contrary, the inequality (4.132) does not hold true, the Mittag-Leffler

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time windows disappear and the functions $f_3(t)$ and $f_4(t)$ tend to a power law given by expression (4.135). Thus, in the limiting case of $\tau_m \to +\infty$, the case studied in the earlier subsection recovers the structure of $\mathbf{R}^{(inph)}$.

4.5.3   The Final Time-asymptotic Condition

Whatever the values of the parameters τ_m, V, T and μ are, in the limit of $t \to +\infty$ the final asymptotic expression for $\mathbf{R}_j(t)$ is given by the following expression:

$$\mathbf{R}_j(t) \sim \mu_1 \left[\mathbf{v}^{(1)}\right]_j - \frac{1}{t^{\mu-1}} \sum_{i=2}^{4} \frac{\mu_i T^{\mu-1}}{\lambda_i} \left[\mathbf{v}^{(i)}\right]_j . \tag{4.142}$$

From Eq. (4.142) it is possible to see that the subordinated density matrix tends to the following form:

$$\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.143}$$

This means that the subordination completely destroys the coherence given by the inphasing process. Let us denote by $R_{i,j}(t)$ the density matrix elements $\rho_{i,j}(n)$ after operating the subordination procedure. In the case where $\Delta \neq 0$ and for $t \to +\infty$, we get from Eq. (4.142) the following result:

$$R_{1,1}(t) \sim \frac{1}{2} + \left(\frac{T}{t}\right)^{\mu-1} \left(\frac{\rho_{1,1}(0) - \rho_{2,2}(0)}{8V^2 \tau_m} + i\frac{(\rho_{1,2}(0) - \rho_{2,1}(0))}{4V}\right), \tag{4.144}$$

$$R_{1,2}(t) \sim \frac{1}{2} \left(\frac{T}{t}\right)^{\mu-1} \left(\tau_m \left(\rho_{1,2}(0) + \rho_{2,1}(0)\right) + i\frac{\rho_{1,1}(0) - \rho_{2,2}(0)}{2V}\right), \tag{4.145}$$

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$$R_{2,1}(t) \sim \frac{1}{2} \left(\frac{T}{t}\right)^{\mu-1} \left(\tau_m \left(\rho_{1,2}(0) + \rho_{2,1}(0)\right) - \imath \frac{\rho_{1,1}(0) - \rho_{2,2}(0)}{2V}\right), \tag{4.146}$$

$$R_{2,2}(t) \sim \frac{1}{2} - \left(\frac{T}{t}\right)^{\mu-1} \left(\frac{\rho_{1,1}(0) - \rho_{2,2}(0)}{8V^2\tau_m} + \imath \frac{(\rho_{1,2}(0) - \rho_{2,1}(0))}{4V}\right). \tag{4.147}$$

Again, the relaxation process is characterized by an inverse power law with index $\mu - 1$. In order to make our analysis complete, I have to consider also the last missing case, $\Delta = 0$; this will be the argument of the next subsection.

4.5.4   The Case where $\Delta = 0$

Here, I analyze the case where $\Delta = 0$, that is, $V = \frac{1}{4\tau_m}$. The Laplace transform of $\mathbf{r}(n)$ given by Eq. (4.112) is:

$$\hat{\mathbf{r}}_{\Delta=0}(u) = \nu_1 \mathbf{v}^{(1)} \frac{1}{u} + \nu_2 \mathbf{v}^{(2)} \frac{1}{u + \frac{1}{\tau_m}} + \nu_3 \mathbf{w}^{(4)} \frac{1}{u + \frac{1}{2\tau_m}} + \nu_3 \mathbf{w}^{(3)} \frac{1}{\left(u + \frac{1}{2\tau_m}\right)^2}. \tag{4.148}$$

Let us evaluate the asymptotic behavior of the subordination of the forth term of Eq. (4.148). Using Eqs. (3.32), (4.126), (3.70), (3.99) and (3.100), in the limiting case of $u \to 0$, we obtain

$$\frac{1}{\hat{\Phi}(u)} \frac{1}{\left(\frac{u}{\hat{\Phi}(u)} + \frac{1}{2\tau_m}\right)^2} \sim 4\tau_m^2 \Gamma\left(2 - \mu\right) T^{\mu-1} u^{\mu-2}. \tag{4.149}$$

By using Eqs. (4.131), (3.80), (3.81) and (4.142), I get

$$\mathbf{R}_{\Delta=0}(t) \sim \nu_1 \mathbf{v}^{(1)} + \left(\frac{T}{t}\right)^{\mu-1} \tau_m \left(\nu_2 \mathbf{v}^{(2)} + 4\tau_m \nu_3 \mathbf{w}^{(3)} + 2\nu_3 \mathbf{w}^{(4)}\right). \tag{4.150}$$

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Now, I am in position to evaluate, also for the case where $\Delta = 0$, the asymptotic behavior of the elements $R_{ij}(t)$, which are the results of the subordinations of $\rho_{ij}(t)$, for every i, j equal to 1, 2, 3 and 4. In case where $t \to +\infty$, I obtain the following form:

$$R_{1,1}^{(\Delta=0)}(t) \sim \frac{1}{2} + \tau_m \left(\frac{T}{t}\right)^{\mu-1} \left(2\left(\rho_{1,1}(0) - \rho_{2,2}(0)\right) + \imath \left(\rho_{1,2}(0) - \rho_{2,1}(0)\right)\right), \qquad (4.151)$$

$$R_{1,2}^{(\Delta=0)}(t) \sim \frac{\tau_m}{2} \left(\frac{T}{t}\right)^{\mu-1} \left(\rho_{1,2}(0) + \rho_{2,1}(0) + 2\imath \left(\rho_{1,1}(0) - \rho_{2,2}(0)\right)\right), \qquad (4.152)$$

$$R_{2,1}^{(\Delta=0)}(t) \sim \frac{\tau_m}{2} \left(\frac{T}{t}\right)^{\mu-1} \left(\rho_{1,2}(0) + \rho_{2,1}(0) - 2\imath \left(\rho_{1,1}(0) - \rho_{2,2}(0)\right)\right), \qquad (4.153)$$

$$R_{2,2}^{(\Delta=0)}(t) \sim \frac{1}{2} - \tau_m \left(\frac{T}{t}\right)^{\mu-1} \left(2\left(\rho_{1,1}(0) - \rho_{2,2}(0)\right) + \imath \left(\rho_{1,2}(0) - \rho_{2,1}(0)\right)\right), \qquad (4.154)$$

This completes the analysis of the subordination of the inphasing and dephasing process, described by Eq. (4.45), for every value of the parameters τ_m, V, T and μ.

The set of Eqs. (4.144), (4.145), (4.146) and (4.147), in the case where $\Delta \neq 0$, and the set of Eqs. (4.151), (4.152), (4.153) and (4.154), in case where $\Delta = 0$, are important. These equations prove that in the time-asymptotic case of $t \to +\infty$, the subordination of the statistical density matrix elements $\rho_{ij}(n)$ generates a power law decay, $1/t^{\mu-1}$, implying that the random choice of times $t(n)$ destroys the coherence of the inphasing process generated by the inphasing Hamiltonian H.

The results of this work might help the research in the field of blinking quantum dots

[5, 3, 44]. The intermittent nature of fluorescent light in the case of blinking quantum dots is reminiscent of the quantum jumps processes discussed in the earlier papers of Refs. [8, 41, 42, 43], and on the basis of this observation, I have explored the possibility of deriving the intermittent fluorescence of blinking quantum dots from subordination to the physical picture that Cook and Kimble [8] established to account for the experimental results of Dehmelt and his co-workers [41, 42, 43]. I have shown that their experimental persistency is derived from an elementary Pauli master equation, with no persistence, by means of an exponential distribution of subordinating times. By operating a second subordination, with the subordination function of Eq. (3.73), I obtain for the survival probability $\Pi(t)$ a Mittag-Leffler function relaxation, and the resulting "light on" and "light off" states are uncorrelated. Let us denote this as the physical condition $No.1$.

There exist two additional conditions that might explain the physics of blinking quantum dots without involving the subordination to the persistency model of Cook and Kimble. Let us denote them as conditions $No.2$ and $No.3$. Condition $No.2$ is obtained through subordination procedure, with the subordinating function $\psi_S(t)$ of Eq. (3.73), directly applied to the Pauli master equation of Eq. (4.44), namely, to a process with no persistence. The physical consequences of this assumption are similar to those of condition $No.1$, insofar as the resulting "light on" and "light off" states are uncorrelated. There is no Mittag-Leffler relaxation though, a property that should make it easy to distinguish this condition from the earlier $No.2$ condition. Finally, condition $No.3$ is based on the subordination procedure applied to a leading physical condition where the decoherence process is not fast. In this case, the survival probability $\Pi(t)$ undergoes an inverse power law relaxation with index $\mu - 1$, again with no sign of Mittag-Leffler behavior, as in the case of condition $No.2$. However, in this final condition, there is an evident correlation between "light on" and "light off"

states. Preliminary results [7] seem to suggest condition $No.2$ as the proper representation of blinking quantum dots intermittent fluorescence. However, to either rule out or confirm condition $No.3$ it is necessary to do additional analysis along the lines suggested by the illuminating paper by Flomenbom and Klafter [46]. These authors discuss how to relate correlated trajectories of "light on" and "light off" signals to an underlying kinetic scheme, and it would be interesting to discuss the connection between the kinetic and the quantum source of correlation.

Finally, I want to make a last remark to shed light on the meaning of subordination to a coherent process. In this case the statistical density matrix subordinated to the inphasing process is

$$\rho(t) = \sum_{n=0}^{\infty} \int_0^t dt' \psi_S^{(n)}(t') \Psi_S(t - t') \rho(n), \qquad (4.155)$$

where $\rho(n)$ describes the time evolution of the statistical density matrix in the natural time scale due to the inphasing operator:

$$\rho(n) = (I - \imath H^x)^n \rho(0), \qquad (4.156)$$

where the superoperator H^x is defined by the following equality:

$$H^x \rho = [H, \rho]. \qquad (4.157)$$

Note that $\psi_S^{(n)}(t)$ indicates the probability that n random drawings from the distribution $\psi_S(t)$ of Eq. (3.73) took place, the last occurring exactly at time t, while the symbol $\Psi_S(t)$ expresses the probability that during the time interval t no drawing occurred. The physical meaning of Eq. (4.155) is that the coherent time evolution is frozen into a state $\rho(n)$ for a

random time τ drawn from the distribution $\psi_S(\tau)$ of Eq. (3.73): at the end of this interval the transition to another frozen state occurs. The physical meaning is the same as that of the theoretical results of Section 4.5, in the limiting case $\tau_m \to \infty$, based on the statistical density matrix representation. An alternative picture rests on this collision-like picture: the inphasing process is applied to the system for a very short time $\tau_u = 1$. The randomness responsible for the ensuing decoherence process is determined by the fact that the time distance between two consecutive, and very fast, collisions fluctuates. It is worth pointing out that conditions of this kind are adopted in the quantum chaos literature, a well known case being the quantum Arnol'd cat [47]. This paradigmatic case is an ordinary oscillator, kicked at regular times, the distance between one kick and the next being constant. The subordination procedure discussed in this paper, valid for coherent and partially coherent processes, as well as for incoherent natural processes, can be thought to be a kind of extension of the kicked quantum models to the case where the kicking process occurs at random times. It is remarkable that when the distribution of sojourn times between one kick and the next is of a non-Poissonian kind, the procedure proposed in this paper generates a non-Markovian master equation, with infinite memory, with no violation of the density matrix positivity.

4.6   Conclusions

As I have pointed out in the Introduction, the main purpose of this thesis has been to construct a Generalized Master Equation (GME), generating a survival probability with inverse power law decay. The physical motivation has been given by the Blinking Quantum Dots (BQD) physics. This goal has been realized by means of a subordination approach, resting on a quantum Liouville superoperator that corresponds to the joint action of an inphasing a dephasing term. This means that I have studied a two-state system, with

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the two states representing the "light on" and "light off" states of the BQD system. The inphasing term establishes a coherent linear superposition of the two states, with expansion coefficient depending harmonically on time. The dephasing term mimics the wave-function collapse into one of these two states. The sum of these two superoperators is the natural Liouville superoperator L_N.

I have imagined the GME , describing the physics of the BQD system, as resulting from the intermittent L_N action. On an intuitive ground, this aims at reproducing the fluorescent intermittency of the BQD system. From a formal point of view, this procedure corresponds to using the subordination function $\psi_S(t)$. This subordination function has either the exponential form, $\lambda \exp\left(-\lambda t\right)$, or an inverse power law form (Nutting law), with power index μ. I have shown that the Zeno effect is formally equivalent to using as a primary process the Pauli master equation in a condition equivalent to determining the choice of the "light on" or the "light off" state by tossing tossing a coin. Then I apply the subordination procedure using a the subordination function with $1/\lambda >> \tau_u = 1$. This means that the random walker spends a significant amount of time in either the "light on" or the "light off" state.

Let us consider now the three limiting conditions mentioned in the Introduction.

Condition (a). The dephasing term is predominant, and the resulting Pauli master equation corresponds to the quantum Zeno effect. The final result is equivalent to the consecutive use of two subordination functions, the former being exponential with damping $\lambda << 1$, and the latter being an inverse power law with index $\mu - 1$, where $\mu < 2$. In this case the survival probability decays as a Mittag-Leffler function. The Mittag-Leffler function is ubiquitous in the field of complex processes. This thesis affords a way to derive the Mittag-Leffler relaxation in a quantum mechanical context. There are no off-diagonal density matrix elements.

This means that no correlation exists between the "light on" and the "light off" state.

Condition (b). The inphasing term is predominant. In this case, the survival probability decays with the inverse power law, of index $\mu-1$, compatible with the BQD physics. The off-diagonal density elements relax to zero with the same inverse power law decay as the diagonal density matrix elements. This means that the "light on" states should be significantly correlated to the "light off" states. It is important to stress that this condition is, to some extent, a generalization of the celebrated quantum Arnold cat . The classical Arnold cat is a classical oscillator kicked at regular times. The quantum Arnold cat, studied by Ford et al. [47], is the quantum version of this kicked classical dynamics. The work of Ford establishes the breakdown of the correspondence principle, on the basis of the observation that the quantum case yields no chaos. The model of Ford becomes identical to the model of this thesis, in condition (b), by assuming that only the ground and first excited state of the quantum oscillator are occupied (low-temperature condition) and that the time distance between one kick and the next is distributed according to an inverse power law. Adopting this perspective, I conclude that the random distribution of kicking times has the effect of destroying quantum coherence, thereby producing randomness also in the quantum case.

Condition (c). The inphasing time scale is comparable to the dephasing time scale. In this case the subordination prescription generates inverse power law decay for the survival probability, with no correlation between the "light on" and the "light off" states.

The preliminary statistical analysis of real BQD data that has been recently carried out in the UNT Center for Nonlinear Science [7], and has established that the survival probability decays as an inverse power law, with no significant signs of a stretched exponential time window. This means that the Mittag-Leffler relaxation does not appear. Furthermore, this statistical analysis has shown that there is no correlation between the "light on " and the

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"light off" states. For these reasons, we conclude that this thesis shows that a plausible model for BQD dynamics is derived from the subordination approach applied to a natural time scale dynamics, where inphasing and dephasing term act on the same time scale. This time scale is very short, being very close to the unit time τ_u.

BIBLIOGRAPHY

[1] M. Kuno, D.P. Fromm, H.F. Hamann, A. Gallagher, and D.J. Nesbitt, J. Chem. Phys. 115, 1028 (2001).

[2] K. T. Shimizu, R. G. Neuhaser, C. A. Leatherdale, S. A. Empedocles, W. K. Woo and M. G. Bawendi, Phys. Rev. B. 63, 205316 (2001).

[3] R.G. Neuhaser, K. T. Shimizu, W.K. Woo, S.A. Empedocles, and M. G. Bawendi, Phys. Rev. Lett. 85, 3301 (2000).

[4] M. Kuno, D.P. Fromm, H.F. Hamann, A. Gallagher, J. Chem. Phys. 1122, 3117 (2000).

[5] Y. J. Jung, E. Barkai, R.J. Silbey, Chem. Phys. 284, 181 (2002).

[6] X. Brokmann, J. P. Hermier, G. Messin, P. Desbiolles, J.-P. Bouchaud, and M. Dahan, Phys. Rev. Lett. 90, 104101 (2003).

[7] S. Bianco, P. Grigolini, P. Paradisi, submitted to J. Chem. Phys.

[8] R. J. Cook, H. J. Kimble, Phys. Rev. Lett. 54, 1023 (1985).

[9] H. Dehmelt, Rev. Mod. Phys. 62, 525 (1990).

[10] C. H. Bennet, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. Wootters, Phys. Rev. Lett. **70**, 1895 (1993)

[11] C. H. Bennet, D. P. DiVincenzo, J. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996)

[12] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998)

[13] S. Abe and A. K. Rajagopal, Quantum entanglement inferred by the principle of maximum nonadditive entropy, Phys. Rev. A **60**, 3461 (1999)

[14] F. Giraldi and P. Grigolini, Phys. Rev. A **64**, (2001)

[15] R. Kubo, M. Toda, N. Hashitsume, Statistical Physics II, Nonequilibrium Statistical Mechanics, Springer Verlag, Berlin (1985)

[16] V.Scarani, M. Ziman, P. Stelmachovic, N. Gisin and V. Buzek, Termalizing quantum machines: Dissipation and Entanglement, Phys. Rev. Lett. **88**, 097905 (2002)

[17] V. B. Braginsky, F. Y. Khalili K. S. Thorne, *Quantum Measurement*, Cambridge University Press, UK (1995)

[18] D. A. Lindar, Z. Bihary, K. B. Whaley, Chem. Phys. **268**, 35 (2001)

[19] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[20] V. Gorini, A. Kossakowski and E. C. G. Sudarshan, J. Math Phys. **17**, 821 (1976)

[21] R. Kosloff, M. A. Ratner and W. B. Davis, J. Chem. Phys. **106** 7036 (1997)

[22] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).

[23] A. Suárez, R. Silbey, and I. Oppenheim, J. Chem. Phys. **97**, 5101 (1992).

[24] M. von Smoluchowsky, Ann. Phys. (Leipzig) **48**, 1103 (1915).

[25] P. Grigolini, Adv. Chem. Phys. **62** 1 (1985)

[26] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 178 (1965)

[27] I. M. Sokolov, Phys. Rev. E **66**, 041101 (2002)

[28] A. A. Budini, Phys. Rev. A **69**, 042107 (2004)

[29] G. H. Weiss, *Aspects and applications of the Random Walk*, North-Holland (1994)

[30] J. Korevaar, *Tauberian Theory. A century of developments.* Springer Verlag (2004)

[31] A. Erdéley, *Higher Transcendental Functions, Vol. 3*, McGraw-Hill, New York (1953).

[32] R. K. Saxena, A. M. Mathai and H. J. Haubold, Astrophys. Space Sci. **282**, 281 (2002).

[33] M. Bologna, P. Grigolini and B. J. West, Chem. Phys. **284** (2002) 115.

[34] B. Misra, E.C.G Sudarshan, J. Math. Phys. **18**, 756 (1977).

[35] B. J. West, M. Bologna and P. Grigolini, *Physics of Fractal Operators*, Springer, New York (2003).

[36] G. Aquino, P. Grigolini, N. Scafetta, Chaos, Solitons and Fractals, **12**, 2023 (2001).

[37] G. Zumofen, J. Klafter, Phys. Rev. E **47**, 851 (1993).

[38] P. Allegrini, G. Aquino, P. Grigolini, L. Palatella, and A. Rosa, Phys. Rev. A **68**, 056123 (2003).

[39] V. Capek, and I. Barvík, Physica A **294** , 388 (2001).

[40] P. Grigolini, M. G. Pala, L. Palatella, Phys. Lett. A **285**, 49 (2001).

[41] W. Neuhauser, M. Hohenstatt, P. Toscheck, and H. Dehmelt, Phys. Rev. Lett **41**, 233 (1978).

[42] W. Neuhauser, M. Hohenstatt, P. Toscheck, and H. Dehmelt, Phys. Rev. A **22**, 1137 (1980).

[43] W. Nagourney, G. Janik, and H. Dehmelt, Proc. Natl. Acad. Sci. U.S.A. **80**, 643 (1983).

[44] M. Kuno, D.P. Fromm, H.F. Hamann, A. Gallagher, and D.J. Nesbitt, J. Chem. Phys. **112**, 3117 (2000).

[45] J. L. Goldberg and A. J. Schwartz, *Systems of Ordinary Differential Equations: an introduction*, Harper & Row, New York (1972).

[46] S. O. Flomembon, J. Klafter, and A. Szabo, *What can one learn from two-state single molecule trajectories?*, submitted.

[47] J. Ford, G. Mantica and G. H. Ristow, Physica D **50**, 493 (1991).