DYNAMICS, THERMODYNAMIC FORMALISM AND PERTURBATIONS OF TRANSCENDENTAL ENTIRE FUNCTIONS OF FINITE SINGULAR TYPE

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In this dissertation, we study the dynamics, fractal geometry and the topology of the Julia set of functions in the family H which is a set in the class S, the Speiser class of entire transcendental functions which have only finitely many singular values.

One can think of a function from H as a generalized expanding function from the cosh family.

We shall build a version of thermodynamic formalism for functions in H and we shall show among others, the existence and uniqueness of a conformal measure. Then we prove a Bowen's type formula, i.e. we show that the Hausdorff dimension of the set of returning points, is the unique zero of the pressure function. We shall also study conjugacies in the family H, perturbation of functions in the family and related dynamical properties.

We define Perron-Frobenius operators for some functions naturally associated with functions in the family H and then, using fundamental properties of these operators, we shall prove the important result that the Hausdorff dimension of the subset of returning points depends analytically on the parameter taken from a small open subset of the n-dimensional parameter space.

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CHAPTER 1

INTRODUCTION

1.1. Discussion of the Problems

In this dissertation we investigate a family of transcendental entire functions. We study the dynamics of these maps, the topology and the geometry of their Julia sets.

The dynamics in the class of transcendental entire functions, denoted usually by *Ent* are different from the dynamics in the class of rational maps, for example, mainly because of the essential singularity at infinity and since the phase space is not compact. By the Picard theorem any neighborhood of infinity is mapped with infinite multiplicity over the entire plane missing at most one point. This particular situation makes the topology of the Julia set look very different from that of rational maps.

There is a class of entire maps whose dynamics is relatively well understood. This is the class $\mathcal{S}$ of entire functions that have finitely many asymptotic and critical values (maps of finite type) i.e. maps which have only finitely many singular values. More precisely, for every element $f \in Ent$, $\omega \in \hat{\mathbb{C}}$ is a singular value if f is not a regular covering map over any neighborhood of ω. The set of singular values is denoted by $Sing(f^{-1})$. Observe that, if ω is a non-singular value of f, then there exists a neighborhood V of ω where every branch of f^{-1} in V is well defined and is a conformal map of V. This set $Sing(f^{-1})$ is very important in our investigation and the last observation is used extensively throughout this thesis.

That is, let $\mathcal{S}$ denotes the set $\{f \in Ent : Sing(f^{-1}) \text{is a finite set}\}$. $\mathcal{S}$ is usually called the class of finite singular type (transcendental) entire functions or the Speiser class. This class has been studied for many years and we refer the readear only to [16], [14] or [20]. Most of work in the field of iterates of transcendental entire functions has been centered around this class of maps and we also refer the reader for example to [16], [14], [23] [29], or [2]. This class $\mathcal{S}$ includes the 1-parameter *exponential* family $\{\lambda e^z\}_{\lambda\in\mathbb{C}^*}$ or the 1-parameter families $\{\lambda \sin z\}_{\lambda\in\mathbb{C}^*}$ and $\{\lambda \cos z\}_{\lambda\in\mathbb{C}^*}$. Maps from these last two families are conjugated to maps in the 2-parameter *cosh* family $\{ae^z + be^{-z}\}_{(a,b)\in\mathbb{C}^2}$. In Chapters 1 and 2 of this thesis we shall recall also some of the most interesting properties of maps from these families.

It is known that ([2], [20], [16]) the No Wandering Domains theorem holds for this type of maps and that the Julia set of these maps, satisfying additional conditions, contains Cantor Bouquets, as it was observed in [14]. Cantor bouquets are Cantor sets of curves called hairs. For more details on these topological objects the reader may consult for example [37], [14] or [1]. In section 1.4 we shall give the definition of a Cantor bouquet and we shall present some of the basic properties of this topological object. Another important observation (see for example [16]) is that maps in the class $\mathcal{S}$ have the property that their Fatou set contains no Baker domain.

We shall present these and other fundamental properties of transcendental entire functions in Chapters 1 and 2. In Chapter 1 we will also discuss the previous results obtained in [29] and [30] by M.Urbański and A.Zdunik who studied expanding mappings $f_\lambda(z) = \lambda e^z$ that have an attracting periodic orbit. They developed the thermodynamic formalism for some special *potentials* associated with these functions. The present work is an answer to the question about the possibility of developing a similar theory for the *cosh family*. The fundamental difference between the exponential family and the cosine family is that a map from the second one has no asymptotic values (only critical values) and a map from the other has only one singular value, the asymptotic value at 0.

In Chapter 1, sections 1.3 and 1.4, we recall some of the basic properties we need on the Dynamics of transcendental entire functions, we define the Speiser class and then we consider Cantor bouquets, following the work of Devaney, Krych, Tangerman, Aarts and Oversteegen, via the fundamental concept of *straight brush* introduced by Aarts and Oversteegen in [1] in 1991.

In Chapter 2 we define the family $\mathcal{H} \subset \mathcal{S}$ and we will establish its basic dynamical properties. We discuss the topology of the Julia set of these maps and we shall observe that the Julia set of these maps is a Cantor bouquet. In section 2.2 we prove the uniformly expanding property and then in section 2.4 we study conjugacies in the family $\mathcal{H}$. Next, in Chapter 3 we build a version of thermodynamic formalism for maps in $\mathcal{H}$ and we show, among others, the existence and uniqueness of a $(t, \alpha)-$conformal measure for maps in the family, so then in section 3.3 it can be proven a Bowen's type formula. In Chapter 4 it is shown that the Hausdorff dimension of the set of points in the Julia set of f_a having non-escaping orbits (denoted by $J^r_{f_a}$) depends analytically on the parameter $a \in \mathbb{C}^{n+1}$.

In what follows we discuss the problems we shall study in this dissertation. There are two basic problems in iteration theory. The first, classical one is to study the iterative behavior of a single function. The second one is to study families of functions, especially how the dynamical behavior of a member in the family changes if the function is perturbed. The simplest (but already sufficiently complicated) case being a family of functions depending on one parameter (see for example the cases of *exponential* or *sine* families). A good understanding of the dynamics of an individual function is of course necessary for the study of problems involving perturbation of functions.

Mathematical models for phenomena in the natural sciences often lead to iteration of functions. But in what follows we study iteration theory for its own. Iteration theory of functions in one complex variable (or Holomorphic Dynamics) essentially originated with the work of Fatou [18] and Julia [19] at the begining of the last century. At the same time, the iteration of rational functions was also investigated by Ritt [27]. In 1926 Fatou [18] extended some of the results to the case of transcendental entire functions. Julia did not consider the iteration of transcendental functions. In the last 30 years there was a renewed interest in the iteration theory of Holomorphic functions. Nowdays there exist many introductory books in the field of Complex Dynamics. We mention only [3], [25] and [41]. There are comparatively few expositions of the Dynamics of transcendental entire functions. We refer to [13] for the iteration of the exponential function and we refer again to [41] which has a chapter on the Dynamics of transcendental entire functions. But many papers were written on the Dynamics of transcendental entire maps and we will mention some of them whenever the research conducted there will interfer with our present work. In this thesis we answer mainly to the following six questions. First question is about the existence and uniqueness of a $(t, \alpha)-$ measure for maps in the family $\mathcal{H}$. Second we ask the question if it is possible to prove a Bowen's type formula for these maps. The third interesting question is about the behavior of maps in the family under conjugating maps. The fourth, most important problem, is about the possibility of developing a thermodynamic formalism for maps in the family. The fifth important problem is on perturbation theory for maps in $\mathcal{H}$. This question will direct us to this fundamental sixth question: if we perturb a little bit the parameters on which a map in the family depend on, how the Hausdorff dimension of the Julia set is changed?

1.2. Previous Results. Discussion of the Methods

The research conducted in [29], [30], [31] between 1999 and 2003 by Mariusz Urbański and Anna Zdunik motivated essentialy our present work. In these papers, Urbański and Zdunik investigated the fractal geometry, the dynamics and the theromodynamic formalism of maps in the exponential family $f_\lambda(z) = \lambda e^z$. They considered both hyperbolic and non-hyperbolic situations, and they proved first the existence and uniqueness of a probability conformal measure (with an exponent greater than 1) for some maps associated with maps in the exponential family (simply, these maps are projections of exponential maps on the perodicity strips of height $2\pi i$).

Then they proved various dynamical related properties, inluding a Bowen's type formula and the fact that the Hausdorff dimension of the complement (in the Julia set J_{f_λ}) of the set of points escaping to infinity under forward iterates of f_λ, is less than 2. This set was denoted by $J^r_{f_\lambda}$.

Next, for the hyperbolic situation, considering the parameters λ such that f_λ has an atracting perodic orbit (this family was denoted by *Hyp*) it is known that the Julia set of a map in *Hyp* is a Cantor Bouquet (see Chapter 2). In [30] Urbański and Zdunik studied perturbations in the exponential family and then, with the methods of thermodynamic formalism, they showed that the function $\lambda \to HD(J^r_{f_\lambda})$ is real-analytic. They proved also that maps from *Hyp* are uniformly expanding on their Julia set and they defined appropriately the topological pressure for some potentials associated to these maps, Perron-Frobenius operators with some more general potentials, and generalized (Gibbs) measures. It is important to observe that, for these maps (in contrast to the case of subshifts of finite type or distance expanding maps), among other difficulties, the phase space is not compact, the potentials are unbounded, and Perron-Frobenius operators are expressed as infinite series of other appropiate operators.

The special methods used by Urbański and Zdunik for the analysis of the dynamics entire transcendental functions are, as we already mentioned, the methods of Thermodynamic formalism. A question frequently asked is why the name "thermodynamic formalism"? Altough the analogies are formal rather than physical, many of the ideas, such as the existence of Gibbs measures, were originally developed in statistical mechanics, and translated to dynamical systems many years later (refer to [17] for more comments).

The pair mathematics-physics is historically inseparable, with mathematics serving to "model physical reality with the intent to rationally understand and clearly expose its laws" (refer to [32]). Thermodynamics is a science created by (among others) Bollzman, Carnot, Kelvin, Maxwell, and Gibbs. This physical science gave birth to the (mathematical) theory of dynamical systems. Important to mention that the thermodynamic formalism generalized to the case of rational maps whose Julia set contains no critical points is due to Denker and Urbański (see [12]). Thermodynamic formalism applied to transcendental entire functions was also initiated, as we already observed, by Mariusz Urbański and Anna Zdunik, with the sequence of papers [29], [30], [31] published by the two authors between 2001 and 2003.

1.3. The Speiser Class

In this section we follow the expositions from [4] and [41] on the Dynamics of transcendental entire functions. We are going to collect the very basic but fundamental facts which we need later in our exposition. We recall basic definitions and theorems emphasizing the class of transcendental entire functions of finite singular type (or the Speiser class). First we consider topological properties of the Fatou sets and the Julia sets of these maps. The point at infinity is essential singularity of a transcendental entire function. The Picard theorem shows that, in a neighborhood of the point at infinity, the action of such a function is strongly "explosive". Thus, in general, iteration of transcendental entire functions is much more complicated than that of polynomials. For an arbitrary polynomial, the point at infinity is always a superattracting fixed point. Hence the Julia set of a polynomial is compact in $\mathbb{C}$. On the other hand we have the following.

Proposition(Julia set is unbounded)

The Julia set of a transcendental entire function is unbounded in $\mathbb{C}$.

Proof. Let f be a transcendental entire function. Choose a point ω in Julia set J_f which is not exceptional value in the sense of Picard. Let U be any neighborhood of the point at infinity. The Picard theorem shows that there is a point θ in U such that $f(\theta) = \omega$. Since the Julia set is backward invariant, θ is in J_f. Thus J_f is unbounded. □

Again recall that, in the case of a polynomial, the immediate basin of attraction of the point at infinity is completely invariant. Hence its boundary is the Julia set of the polynomial. It follows that all Fatou components except for this basin are simply connected.

Proposition(No Herman rings)

Let $f \in Ent \cup Poly$, *where by Ent we denoted the set of transcendental entire functions and by Poly we denote the set of polynomial functions. Then its Fatou set contains no cycles of Herman rings.*

Proof. Suppose that there were a Herman ring H. Since $\{f^n\}_{n=1}^{\infty}$ is uniformly bounded on H, the maximum principle shows that $\{f^n\}_{n=1}^{\infty}$ is uniformly bounded also in the bounded component U of $\mathbb{C} - H$. This is a contradiction because U contains points of J_f. $\square$

Remark

The above Proposition does not imply non-existence of multiply connected components of the Fatou set for transcendental entire functions. Baker was the first to give an example of a multiply connected Fatou component (refer to [41]-Th 3.4.1). Also recall that (see for example Theorem 3.15 in [41]) every unbounded Fatou component of a map in *Ent* is simply connected and, as a corollary, observe that the Julia set of a map in *Ent* is never totally disconnected.

Definition(Singular values)

For every $f \in Ent$ we call $\alpha \in \hat{\mathbb{C}}$ a singular value if f is not a smooth covering map over any neighborhood of α. We denote the set of all singular values by $Sing(f^{-1})$.

If α is a non-singular value of f, then there exists a neighborhood V of α where every branch of f^{-1} in V is well defined and is conformal map of V.

Definition(Eremenko-Lyubich)

We call a transcendental entire function to be of finite singular type or to belong to the Speiser class if it belongs to $\mathcal{S}$ where

$$\mathcal{S} = \{f \in Ent : Sing(f^{-1}) \textit{is a finite set}\}.$$

Observe that

$$\mathcal{S} = \{f \in Ent : \text{the set of critical and asymptotic values is finite}\}.$$

Recall that the set of critical points of a function f is defined by:

$$Crit(f) = \{z : f'(z) = 0\}$$

and the set of critical values is $f(Crit(f))$. Also we make the following observation.

Remark

For f a polynomial or rational function the dynamics is determined in large measure by the behavior of orbits of the critical values. For $f \in Ent$, the set of singular orbits must be extended as we can see for $f_a(z) = ae^z$ with $0 < a < \frac{1}{e}$, f_a has no critical points but the essential role is played by 0 which is an omitted value of f_a. In fact 0 is an asymptotic value.

For a map $f \in Ent$ a point $w \in \overline{\mathbb{C}}$ is an asymptotic value for f if there is a continuous curve $\gamma(t)$ (called a path of determination) satisfying

$$\lim_{t\to\infty} \gamma(t) = \infty$$

and

$$\lim_{t\to\infty} f(\gamma(t)) = w.$$

Any curve which tends to ∞ such that $\mathrm{Re}\, z \to -\infty$ is such a curve γ for f_a (take for example $\gamma(t) = -t^2 + it$) so 0 is an asymptotic value for f_a.

A Picard exceptional value (omitted value) is an asymptotic value for an entire function(see 0 and ∞ for e^z which are both omitted values).

We observe also that there is a dichotomy in the Speisser class as there exists for quadratic polynomials, where there are basically two types of Julia sets, Cantor sets and Julia sets that are connected; for maps in the Speiser class there is a similar dichotomy, either Julia set is $\mathbb{C}$ or Julia set is a Cantor bouquet.

Fundamental Theorem

If $f \in \mathcal{S}$ then the Fatou set F_f contains no wandering domains, no Baker domains and every component of F_f is simply connected

For the proof we refer the reader to [41] or [4].

We also recall that the set of repelling periodic points is dense in the Julia set. Hence the set of the points whose orbits are bounded is dense in the Julia set. Next we discuss the relation between the Julia set J_f and the set of *escaping* points (see[4] p.26).

$$I_f = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}.$$

If $f \in Poly$ then I_f is the immediate attractive basin of the superattracting fixed point ∞. In this case it was proved that $J_f = \partial I_f$. Eremenko showed first that if $f \in Ent$ then I_f is not an empty set and next he showed that if $f \in \mathcal{S}$ then $J_f = \overline{I_f}$. Eremenko asked then two fundamental questions: Is every component of I_f unbounded? Can every point in I_f be joined with ∞ by a curve in I_f? Clearly a positive answer to the second question will imply that the answer to the first question is also positive. Eremenko proved that $\overline{I_f}$ does not have bounded components and he remarked that a positive answer to the second question for a restricted class of functions follows from the results of Devaney and Tangerman (see [14]).

We are going to present the results of Devaney and Tangerman and some others about Cantor bouquets in the next section.

Cantor Bouquets. Hairs. Examples.

In a paper from 1986, [14], Devaney and Tangerman showed that maps in the Speiser class satisfying some growth conditions admit "Cantor bouquets" in their Julia set. All of the curves(hairs) in the bouquet tend to ∞ in the same direction, and the map behaves like the shift automorphism on the Cantor set. Hence the dynamics near ∞ for these maps may be analyzed completely. Among the maps in Ent, for which the Devaney and Tangerman methods apply, are $exp(z), sin(z), cos(z), cosh(z), sinh(z)$ and we will see that maps in the more general family $\mathcal{H}$ (that we deal with in the research presented in this disseratation) also satisfy Devaney-Tangerman conditions and the Julia set of these maps is itself a Cantor bouquet.

Cantor bouquets arise very often in the dynamics of maps in $\mathcal{S}$. Examples include maps for the exponential family (see [13] or [37]) $f_\lambda = \lambda e^z$ for parameters λ satisfying $0 < \lambda < \frac{1}{e}$ or maps in the *sine* family $f_\lambda = \lambda \sin z$ with λ real satisfying $0 < \lambda < 1$. Also (refer to [1]) it was shown that Julia sets of maps in the one parameter families $\{\lambda \cosh z\}$ with $0 < \lambda < 0.67)$ and $\{\lambda \sinh z\}$ for $0 < \lambda < 0.85$ are also Cantor bouquets. We do not go into detailes in this thesis but in my paper [7], which is now in preparation, I shall treat these problems with an accent on the family $\mathcal{H} \subset \mathcal{S}$ which will be defined in the next section. Here we just want to give a rigourous definition of a Cantor bouquet following [1] and [37]. To describe the topological structure of a Cantor Bouquet, we need to introduce the notion of a *straight brush*.

Definition (Straight brush-Aarts/Oversteegen)

To each irrational number ζ , we assign (there are many ways to do this (see for example [37])) an infinite string of integers $n_0 n_1 n_2 \ldots$ as follows. We will break up the real line into open intervals $I_{n_0 n_1 \cdots n_k}$ wich have the following properties

(i) $I_{n_0 \cdots n_{k+1}} \subset I_{n_0 \cdots n_k}$.

(ii) The endpoints of $I_{n_0 \cdots n_k}$ are rational.

(iii) $\zeta = \cap_{k=1}^{\infty} I_{n_0 \cdots n_k}$.

A straight brush B is a subset of $[0, \infty) \times \mathcal{N}$, where $\mathcal{N}$ is a dense subset of irrationals, having the following three properties

1. *B is "hairy" in the following sense. If $(y, \alpha) \in B$, then there exists a $y_\alpha \leq y$ such that $(t, \alpha) \in B$ iff $t \geq y_\alpha$. That is the "hair" (t, α) is contained in B where $t \geq y_\alpha$. And y_α is called the endpoint of the hair corresponding to α.*
2. *Given an endpoint $(y_\alpha, \alpha) \in B$ there are sequences $\beta_n \uparrow \alpha$ and $\gamma_n \downarrow \alpha$ in $\mathcal{N}$ such that $(y_{\beta_n}, \beta_n) \to (y_\alpha, \alpha)$ and $(y_{\gamma_n}, \gamma_n) \to (y_\alpha, \alpha)$. That is, any endpoint of a hair in B is the limit of endpoints of other hairs from both above and below.*
3. *B is closed subset of $\mathbb{R}^2$.*

We observe that a straight brush is a remarkable topological object and we view it as a subset of the Riemann sphere. Aarts and Oversteegen have shown that any two straight brushes are ambiently homeomorphic, i.e. there is a homeomorphism of $\mathbb{R}^2$ taking one brush onto another. This important observation led to the formal definition of the Cantor bouquet.

Definition

A Cantor bouquet is a subset of $\overline{\mathbb{C}}$ that is homeomorphic to a straight brush with ∞ mapped to ∞.

We shall make the observation that the Julia set of a map in the family $\mathcal{H}$, which we are going to define in the next Chapter, is a Cantor bouquet. We shall see that its existence follows from [14] and then we shall conclude that with Aarts and Oversteegen's methods, it can be shown that the Julia set of a map in $\mathcal{H}$ is indeed homeomorphic to a straight brush. We mention that a more extensive approach is done in my paper [7] which is in preparation and it will be ready to be submitted for publication in the near future.

CHAPTER 2

THE FAMILY $\mathcal{H}$

2.1. Definition of $\mathcal{H}$, Basic Properties

In this section we define the family $\mathcal{H}$ and we establish basic dynamical properties of a map $f_a \in \mathcal{H}$. Then we we prove the important Lemma 2.1.

Definition of $\mathcal{H}$

We define the family $\mathcal{H}$ as a family of maps in the Speiser class of transcendental entire functions of finite singular type.

Let $a = (a_0, a_1, \cdots, a_n) \in \mathbb{C}^{n+1}$ be a vector such that $a_0 \neq 0$, $a_n \neq 0$,

$$P_a(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

and

$$g_a(z) = \frac{P_a(z)}{z^k}$$

where k is a positive integer strictly less than $n = \deg(P_a) \geq 2$. Define

$$f_a(z) = g_a \circ \exp(z) = \frac{a_n e^{nz} + a_{n-1} e^{(n-1)z} + \cdots + a_1 e^z + a_0}{e^{zk}} = \sum_{j=0}^{n} a_j e^{(j-k)z}$$

Observe that maps of this form do not have any finite asymptotic values. This is the reason why we restricted ourselves to integers k satisfying condition $0 < k < n$. As it was mentioned in Chapter 1, the most well known examples of this type of maps are maps from the *cosine* family.

We denote by $\mathrm{Crit}(f_a)$ the set $\{z : f_a'(z) = 0\}$. Observe that

$$f_a'(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}$$

and that $g_a'(z) = 0$ if and only if $zP_a'(z) - kP_a(z) = 0$, which is equivalent to

$$\sum_{j=0}^{n} a_j(j-k)z^j = 0.$$

Therefore, there exist n non-zero complex numbers (counting multiplicities) $s_1, s_2, \cdots, s_n$ such that $z \in \mathrm{Crit}(f_a)$ if and only if $e^z = s_k$ for some $k = 1, 2, \cdots, n$ i.e.

$$\{z_k = \log s_k + 2\pi i m : m \in \mathbb{Z}, k = 1, \cdots, n\}$$

is the set of critical points and observe that the set of critical values of a map f_a is finite.

Denote by $\mathcal{H}$ the family of functions

$$\mathcal{H} = \left\{ f_a(z) = \frac{P_a(e^z)}{e^{kz}} : \deg P_a > k > 0 \ and \ \delta_a > 0 \right\},$$

where by $\mathcal{P}_{f_a}$ we denote the post-critical set of f_a i.e. the set

$$\mathcal{P}_{f_a} = \overline{\bigcup_{n \geq 0} f_a^n(Crit(f_a))}$$

and

$$\delta_a = \frac{1}{2} \min \left\{ \frac{1}{2}, \mathrm{dist}(J_{f_a}, \mathcal{P}_{f_a}) \right\},$$

where

$$\mathrm{dist}(J_{f_a}, \mathcal{P}_{f_a}) = \inf\{|z_1 - z_2| : z_1 \in J_{f_a}, z_2 \in \mathcal{P}_{f_a}\}$$

is the Euclidean distance between the Julia set of f_a, J_{f_a}, and the post-critical set of f_a, $\mathcal{P}_{f_a}$.

The reason we define δ_a in such a way will be more visible later on, starting with Chapter 3, and is due to the application (we shall need) of the Koebe Distortion Theorem since one can observe that, for every $y \in J_{f_a}$ and for every $n \geq 1$, there exists a unique holomorphic inverse branch

$$(f_a^n)_y^{-1} : B(f_a^n(y), 2\delta_a) \to \mathbb{C}$$

such that $(f_a^n)_y^{-1} \circ (f_a^n)(y) = y$.

Then there exists a numerical constant K such that, for $z_1, z_2 \in J_{f_a}$ with $|z_1 - z_2| < \delta_a$ and for $y \in f_a^{-n}(z_1)$,

$$(1) \qquad \frac{1}{K} \leq \frac{|((f_a^n)_y^{-1})'(z_1)|}{|((f_a^n)_y^{-1})'(z_2)|} \leq K.$$

Observe that $Crit(f_a) \subset F_{f_a}$, where F_{f_a} is the Fatou set of f_a. Consequently, maps in the family $\mathcal{H}$ do not have neither parabolic domains nor Herman rings nor Siegel disks. Moreover, as was written in Chapter 1 they do not have neither wandering nor Baker domains. Also for every point z in the Fatou set there exists (super)attracting cycle such that the trajectory of z converges to this cycle.

The cylinder and the definition of $J_{F_a}^r$.

Since the map $f_a \in \mathcal{H}$ is periodic with period $2\pi i$, we consider it on the quotient space $P = \mathbb{C}/\sim$ (the cylinder) where

$$z_1 \sim z_2 \text{ iff } z_1 - z_2 = 2k\pi i \text{ for some } k \in \mathbb{Z}.$$

If $\pi : \mathbb{C} \to P$ is the natural projection, then, since the map $\pi \circ f_a : \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it induces a holomorphic map

$$F_a : P \to P.$$

The cylinder P is endowed with Euclidean metric which will be denoted in what follows by the same symbol $|w - z|$ for all $z, w \in P$. The Julia set of F_a is defined to be

$$J_{F_a} = \pi(J_{f_a})$$

and observe that

$$F_a(J_{F_a}) = J_{F_a} = F_a^{-1}(J_{F_a}).$$

We shall study the set $J^r_{f_a}$ consisting of those points of J_{f_a} that do not escape to infinity under positive iterates of f_a. In other words, if

$$I_\infty(f_a) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_a^n(z) = \infty\},$$

then

$$J^r_{f_a} = J_{f_a} \backslash I_\infty(f_a)$$

and, if

$$I_\infty(F_a) = \{z \in P : \lim_{n\to\infty} F^n(z) = \infty\},$$

then

$$J^r_{F_a} = J_{F_a} \backslash I_\infty(F_a).$$

In what follows we fix $a \in \mathbb{C}^{n+1}$ and we denote for simplicity $f_a \in \mathcal{H}$ by f. The following Lemma reveals some background information for a better understanding of the dynamical behavior of maps in our family $\mathcal{H}$. This lemma will be used several times and it will be a key technical ingredient for many proofs.

Observe first that, if we consider $a = (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}$, since

$$f_a(z) = \sum_{j=0}^{n} a_j e^{(j-k)z} \tag{2}$$

we have

$$f'_a(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}. \tag{3}$$

Lemma

Let f_a be a function of form (2). Then there exist $M_1, M_2, M_3 > 0$ such that, for every z with $|Re\ z| \geq M_3$, the following inequalities hold.

(i) $M_1 e^{q|Re\ z|} \leq |f_a(z)| \leq M_2 e^{q|Re\ z|}$

(ii) $M_1 e^{q|Re\ z|} \leq |f'_a(z)| \leq M_2 e^{q|Re\ z|}$

(iii) $\frac{M_1}{M_2}|f'_a(z)| \leq |f_a(z)| \leq \frac{M_2}{M_1}|f'_a(z)|$

where $q = \begin{cases} k & \textit{if Re } z < 0 \\ n-k & \textit{if Re } z > 0. \end{cases}$

Proof. Note that (iii) follows from (i) and (ii). The proof of (i) and (ii) follows from the fact that

$$|f_a(z)| = |a_n| e^{(n-k)\mathrm{Re}\ z} + o(e^{(n-k)\mathrm{Re}\ z}) \text{ as } \mathrm{Re}\ z \to \infty$$

$$|f_a(z)| = |a_0| e^{-k\mathrm{Re}\ z} + o(e^{-k\mathrm{Re}\ z}) \text{ as } \mathrm{Re}\ z \to -\infty$$

and from the observation that f_a' is a function of the same (algebraic) type as f_a (see (3)). $\square$

2.2. The Uniformly Expanding Property

In this section we shall prove, mainly, the very important result, Proposition 2.2, using McMullen's result from [23], that any map $f_a \in \mathcal{H}$ is uniformly expanding on its Julia set.

Proposition

For every $f \in \mathcal{H}$ *there exist* $c > 0$ *and* $\gamma > 1$ *such that*

$$|(f^n)'(z)| > c\gamma^n$$

for every $z \in J_f$.

Proof. By [23, Proposition 6.1], for all $z \in J_f$,

$$\lim_{n\to\infty} |(f^n)'(z)| = \infty. \tag{4}$$

Since f is periodic with period $2\pi i$ we consider

$$A = J_f \cap \{z : \mathrm{Im}\ z \in [0, 2\pi]\}.$$

and we let A_m denotes the open set

$$\{z \in A : |(f^m)'(z)| > 2\}.$$

Then by (4) $\{A_m\}_{m\geq 1}$ is an open covering of A. Moreover, it follows from Lemma 2.1 that there exists M such that, if $|\mathrm{Re}\ z| > M$, then $|f'(z)| > 2$. Therefore

$$\{z \in A : |\mathrm{Re}\ z| > M\} \subset A_1.$$

Since $A \cap \{z : |\mathrm{Re}\ z| \leq M\}$ is a compact subset of A, it follows that there exists $k \geq 1$ such that the family $\{A_1, A_2, \dots, A_k\}$ covers A. It implies that, for every $z \in A$, there exists $k(z) \leq k$ for which $|(f^{k(z)})'(z)| > 2$. Therefore, for every $n > 0$ and every $z \in A$ we can split the trajectory $z, f(z), \dots, f^n(z)$ into $l \leq \lfloor \frac{n}{k} \rfloor + 1$ pieces of the form

$$z_i, f(z_i), \dots, f^{k(z_i)-1}(z_i)$$

for $i = 1, \dots, l-1$, and, for $i = l$,

$$z_l, f(z_l), \dots f^j(z_l) = f^n(z),$$

where $z_1 = z$, $z_i = f^{k(z_{i-1})}(z_{i-1})$ and j is some integer smaller than k. Then

$$|(f^n)'(z)| \geq 2^{\lfloor \frac{n}{k} \rfloor} \Delta^{k-1},$$

where

$$\Delta = \inf_{z \in J_f} |f'(z)| \neq 0,$$

since J_f contains no critical points and because of Lemma 2.1 (ii). It follows that

$$|(f^n)'(z)| \geq 2^{\frac{n}{k}-1} \Delta^{k-1} = \frac{\Delta^{k-1}}{2} (2^{\frac{1}{k}})^n.$$

$\square$

More remarks on the family $\mathcal{H}$.

We observed that the family $\mathcal{H}$ is a family in the class of transcendental entire functions, which was denoted by Ent. Moreover, every function f_a from $\mathcal{H}$ has only finitely many singular values, in other words f_a has finitely many asymptotic and critical values so $\mathcal{H}$ is a family of functions which belong to the Speiser class $\mathcal{S}$ defined in Section 1.3. Moreover, for every map in $\mathcal{H}$ the Julia set is a Cantor bouquet as it was observed in [14] ,[41] or [37] for maps in the Speiser class with an attracting cycle.

Note also that the assumption $0 < k < \deg P_a$ implies that any map $f_a \in \mathcal{H}$ does not have a finite asymptotic value since $P_a(z)/z^k$ converges to infinity when z aproaches 0 or ∞. If this condition is not satisfied then one of the limits is finite and it would be a finite asymptotic value of f_a. Even in this case, the main result from section 4.3 may be established, using the proofs from this thesis with some minor changes. We additionally assume that maps from the family $\mathcal{H}$, which we consider, satisfy the following extra-condition.

If z is a periodic point of period m then $|(f^m)'(z)| \neq 0$.

Of course we rise the question if it is possible to develop a similar theory we shall present in Chapters 3 and 4 and to prove the main result of section 4.3, without this extra-condition. This problem remains open but we believe the answer is positive.

Applying Thermodynamic Formalism on the family $\mathcal{H}$ (the reader interested in Thermodynamic formalism and its connection with dynamics is refered to [26], [32],[29] or [30]) we shall prove that the Hausdorff dimension of the subset of the Julia set of such maps, consisting of the points for which the forward orbit does not escape to infinity i.e. the set

$$J^r_{f_a} = J_{f_a} \setminus I_\infty(f_a),$$

where $I_\infty(f_a) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\}$, depends real-analytically on the parameter $a \in \mathbb{C}^{n+1}$.

In order to do that we study first quasiconformal conjugacies in the family $\mathcal{H}$ and then we define Perron-Frobenius operators associated with some special potentials. The classical theorem of Hartogs will help us to prove the main tool of this thesis (see section 4.1) which will allow us to prove the main result in section 4.3 which shows that these Perron-Frobenius operators can be embedded into a family of operators which depend holomorphicaly on the parameter a chosen from a designed open set $G \subset \mathbb{C}^{n+1}$ and then, using perturbation theory (Kato-Rellich theorem) and the results from Chapter 3, where we prove mainly that $\mathrm{HD}(J^r_{F_a}) = h$ is the unique zero of the pressure function $t \to P(t)$ for $t > 1$ and $a \in \mathbb{C}^{n+1}$, we obtain that the function $a \mapsto HD(J^r_{f_a})$ is real-analytic.

It is also important to remark that the derivative $f_b^{(s)}(z)$ of a map $f_b \in \mathcal{H}$ has the expression:

$$f_b^{(s)}(z) = \sum_{j=0}^{n} b_j (j-k)^s e^{(j-k)z}$$

for every $b = (b_0, \cdots, b_n) \in C^{n+1}$ and every positive integer $s \geq 0$. Moreover observe that the derivative with respect to the variable parameter $b \in \mathbb{C}^{n+1}$ has the expression:

$$\frac{\partial F_b'}{\partial b}(z) = \frac{\partial f_b'}{\partial b}(z) = \begin{bmatrix} \frac{\partial f_b'}{\partial b_0}(z) \\ \frac{\partial f_b'}{\partial b_1}(z) \\ \cdots \\ \frac{\partial f_b'}{\partial b_j}(z) \\ \cdots \\ \frac{\partial f_b'}{\partial b_n}(z) \end{bmatrix} = \begin{bmatrix} -ke^{-kz} \\ (1-k)e^{(1-k)z} \\ \cdots \\ (j-k)e^{(j-k)z} \\ \cdots \\ (n-k)e^{(n-k)z} \end{bmatrix}. \tag{5}$$

Hence

$$\left|\left|\frac{\partial f_b'}{\partial b}(z)\right|\right|^2 = \left|\left|\frac{\partial F_b'}{\partial b}(z)\right|\right|^2 = \sum_{j=0}^{n} (j-k)^2 e^{2(j-k)\mathrm{Re}\, z}, \tag{6}$$

where $||\cdot||$ means the norm on $\mathbb{C}^{n+1}$ definied by the formula

$$||(z_0, \ldots, z_n)|| = \sqrt{\sum_{j=0}^{n} |z_j|^2}$$

for $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. Similarly, it is clear that

$$\left\|\frac{\partial f_b}{\partial b}(z)\right\|^2 = \left\|\frac{\partial F_b}{\partial b}(z)\right\|^2 = \sum_{j=0}^{n} e^{2(j-k)\mathrm{Re}\, z}. \tag{7}$$

2.3. Bounded Orbits

We fix again $a \in \mathbb{C}^{n+1}$ and we denote f_a by f, F_a by F and the Julia set of F by J_F. Our goal in this section is to prove Proposition 2.3. In order to prove this proposition we apply the thermodynamic formalism for compact repellers.

Definition

Let f be a holomorphic function from an open subset V of $\mathbb{C}$ into $\mathbb{C}$ and J a compact subset of V.

The triplet (J, V, f) is a conformal repeller if

(i) there are $C > 0$ and $\alpha > 1$ such that $|(f^n)'(z)| \geq C\alpha^n$ for every $z \in J$ and $n \geq 1$.

(ii) $f^{-1}(V)$ is relatively compact in V with

$$J = \bigcap_{n\geq 1} f^{-n}(V).$$

(iii) for any open set U with $U \cap J$ not empty, there is $n > 0$ such that

$$J \subset f^n(U \cap J).$$

It is worth noting that there are no critical points of f in J.

Conformal repellers.

Let (J, V, g) be a (mixing) conformal expanding repeller(see for example [32] for more properties). In the proof of Proposition 2.3, $J = J_1(M)$ is a compact subset of $\mathbb{C}$, limit of a finite conformal iterated function system, $g = F$, is a holomorphic function for which J is invariant and for which there exist $\gamma > 1$ and $c > 0$ such that, for all $n \in \mathbb{N}$ and for all $z \in J$, $|(g^n)'(z)| \geq c\gamma^n$. For $t \in \mathbb{R}$ we consider the topological pressure defined by

$$P_z(t) = \lim_{n\to\infty} \frac{1}{n} \log P_z(n, t),$$

where

$$P_z(n,t) = \sum_{y\in g^{-n}(z)} |(g^n)'(y)|^{-t}.$$

The function $P(t) = P_z(t)$ as a function of t is independent of z, continuous, strictly decreasing, $\lim_{t\to-\infty} P(t) = +\infty$ and the following remarkable theorem holds.

Bowen's Formula *Hausdorff dimension of J is the unique zero of $P(t)$.*

For more details and definitions concerning the thermodynamic formalism of conformal expanding repellers (initiated by Bowen and Ruelle) we refer the reader to [32] or [26].

In order to prove Proposition 2.3, i.e. to show that $\mathrm{HD}(J) > 1$, we use Bowen's formula and we observe that, from the definition of $P_z(n,t)$, it is enough to find a constant $C > 1$ such that, for all $z \in J$,

$$P_z(1,1) \geq C. \tag{8}$$

Proposition

Let $f \in \mathcal{H}$. Then the Hausdorff dimension of the set of points in Julia set of f having bounded orbit is strictly greater than 1.

Proof. Let N be a large number, $H = \{z \in \mathbb{C} : \mathrm{Re}\, z > N\}$. Observe that there exists U such that $\overline{U} \subset \{z : s-\pi < \mathrm{Im}\, z < s+\pi\}$ for some $s \in (-\pi,\pi]$, $\mathrm{Re}\, U > 0$, $f|_U$ is univalent and $f(U) = H$. Note that, since N is large, by Lemma 2.1 there exists $\gamma_N > 1$ such that, if $\mathrm{Re}\, z \geq N$, then

$$|F'(z)| = |f'(z)| > \gamma_N. \tag{9}$$

For every $M > N$ define

$$P(M) = \{z \in \overline{U} : N \leq \mathrm{Re}\, z \leq M\}.$$

Then, for $j \in \mathbb{Z}$, let $L_j : H \to U$ be defined by the formula

$$L_j(z) = (f|_U)^{-1}(z + 2\pi i j),$$

and let

$$Q_j(M) = L_j(P(M)). \tag{10}$$

The set P(M) and the family of functions

$$\{L_j\}_{j \in \mathcal{K}_M}$$

with

$$\mathcal{K}_M = \{j \in \mathbb{Z} : Q_j(M) \subset \mathrm{Int} P(M)\},$$

define a finite conformal iterated function system . By $J_1(M)$ we denote its limit set. The set $J_1(M)$ is forward $F-$invariant. From (9) and from the fact that the Julia set is the closure of the set of repelling periodic points it follows that

$$J_1(M) \subset J_F. \tag{11}$$

Next we need a condition for j which guarantees that $Q_j(M) \subset \mathrm{Int} P(M)$ (equivalently $j \in \mathcal{K}_M$) for all M large enough. Observe that

$$\mathcal{K}_M \subset \mathcal{K}_{M+1} \tag{12}$$

for all M large enough. To prove (12), let $j \in \mathcal{K}_M$ and let $z \in Q_j(M+1) \setminus Q_j(M)$. Note that, if we assume that $M > M_2 e^{(n-k)(N+1)}$, then we can be sure that $\mathrm{Re}\, z > N+1$ (n and k are defined in section 2.1). Therefore, to get (12), it is enough to prove that $\mathrm{Re}\, z < M+1$. Since

$$F(Q_j(M+1) \setminus Q_j(M)) = P(M+1) \setminus P(M),$$

it follows from Lemma 2.1 that $|F'(z)| \geq \frac{M_1}{M_2}|f(z)| \geq M$ and, then,

$$Q_j(M+1) \setminus Q_j(M) \subset B\Big(z, \frac{M_2 2\pi}{M_1 M}\Big) \subset B(z, 1).$$

But we know, that, for $y \in Q_j(M)$, $\mathrm{Re}\, y \leq M$. This proves (12).

The next step is to prove that there exists $j_0 \in \mathbb{N}$ such that, for all $M \in \mathbb{N}$ large enough,

$$j_0, j_0 + 1, \dots, e^{\lfloor M/2 \rfloor} \in \mathcal{K}_M. \tag{13}$$

Note that we can find j_0 such that, for every $j \geq j_0$, $\mathrm{Re}\, Q_j(M) > N$. By Lemma 2.1 it is enough to take

$$j_0 = \left\lceil \frac{M_2 e^{(n-k)N} + 2\pi}{\pi} \right\rceil .$$

So, to prove (13) it remains to show that $j < e^{\lfloor M/2 \rfloor}$ implies

$$\mathrm{Re}\, Q_j(M) \leq M.$$

Striving for a contradiction, suppose that $j < e^{\lfloor M/2 \rfloor}$ and there exists $z \in Q_j(M)$ such that $\mathrm{Re}\, z > M$. Then by Lemma 2.1 we have

$$|f(z)| > M_1 e^{(n-k)M}. \tag{14}$$

Since $z \in Q_j(M)$, $f(z) \in P(M) + 2\pi ij$. Then the square of the distance from zero to the upper-right corner of $P(M) + 2\pi ij$ is greater than $|f(z)|^2$, i.e.

$$M^2 + (s + \pi + 2\pi j)^2 > |f(z)|^2.$$

By (14) and the assumption $j < e^{\lfloor M/2 \rfloor}$, it follows that

$$(M_1 e^{(n-k)M})^2 < M^2 + (s + \pi + 2\pi)^2 e^M.$$

Hence we have the required contradiction since for large M the inequality is false.

Finally observe that by Lemma 2.1, for $j \in \mathcal{K}_M$ and $z \in Q_j(M)$, the following is true

$$|F'(L_j(z + 2j\pi i))| \leq \frac{M_2}{M_1} |f(L_j(z + 2\pi ij))| \leq \frac{M_2}{M_1}(2j\pi + 2\pi + M).$$

Then

$$P_z(1,1) = \sum_{y \in F^{-1}(z) \cap J_1(M)} \tfrac{1}{|F'(y)|} = \sum_{j \in \mathcal{K}_M} |L_j'(z + 2j\pi i)|$$

$$\geq \sum_{j=j_0}^{e^{\lfloor M/2 \rfloor}} \frac{1}{\frac{M_2}{M_1}(2j\pi+2\pi+M)}.$$

Since , if M is large enough, the right side of this inequality can be as large as we want, (8) and the proposition are proved. $\square$

2.4. Quasiconformal Conjugacy

In this section we present some analytic and geometric properties of the family $\mathcal{H}$. We follow the analysis from [30], which in turn follows the more elaborated descriptions from [16] and [42]. As in [16] every $f \in \mathcal{H} \subset S$ is viewed as an element of a finite dimensional complex analytic manifold $M_f = \mathcal{H} \subset S$. In the refered paper [16] various analytical and geometrical results are proved on M_f.

For the theory of quasiconformal maps in the plane we refer the reader to the books written by Lehto and Virtanen [40], Ahlfors [34], the paper of Astala [35] and the first chapter of the book by F.Gardiner and N.Lakic [38].

A sense-preserving homeomorphism f of a domain G is called *quasiconformal* if its maximal dilatation $K(G)$ is finite. If $K(G) \leq K < \infty$ then f will be called $K-$*quasiconformal* (see [40, p.16]). Following the terminology used in the conformal case we also call a quasiconformal homeomorphism a *quasiconformal mapping*.

The topology of $\mathcal{H}$

The domain of all functions from $\mathcal{H}$ is the non-compact complex plane, and the most natural topology of $\mathcal{H}$ is the topology of uniform convergence on compact subset of $\mathbb{C}$. Observe that this topology is equivalent to the Euclidean topology on $\mathbb{C}^{n+1}$ when we identify a parameter a with the function f_a. Therefore, throughout this paper we sometimes write $a \in \mathcal{H}$ with the meaning that $f_a \in \mathcal{H}$. Moreover, whenever we say b is close to a we mean that f_b is close to f_a as well. We also say b is sufficiently close to a whenever we need b to be chosen from a small open neighborhood of $a \in \mathbb{C}^{n+1}$. (compare [16]).

After this short introduction on the topological structure of $\mathcal{H}$ we can formulate a lemma which follows from the results of Eremneko and Lyubich ([16], Proposition 5, p.1016) on structural stability of maps in the Speiser class (see also [42]).

Lemma

For $a \in \mathcal{H}$, f_a is structurally stable *i.e. if b is sufficiently close to a, then there exists a conjugating quasiconformal homeomorphism $h_b : \mathbb{C} \to \mathbb{C}$ such that*

$$f_b \circ h_b = h_b \circ f_a.$$

Moreover the map $b \mapsto h_b(z)$ is holomorphic for every $z \in \mathbb{C}$ and the mapping $(b, z) \mapsto h_b(z)$ is continuous. The quasiconformal constant converges to 1 *as b approaches a.*

This is the moment when we need our extra-condition, since, if f_a has a superatracting periodic point, then f_a is not structurally stable. This property of stability of the family $\mathcal{H}$ stated in the previous Lemma is a crucial fact. But we need to have some control over the changes resulted from the action of the quasiconformal homeomorphism in a neighborhood of a. This is stated in Proposition 2.4. To obtain this result we need to povide some information about quasiconformal maps and give some properties of functions from $\mathcal{H}$.

Let K, $\alpha > 0$. We say that a map $h : \mathbb{C} \to \mathbb{C}$ is (K, α)-Hölder continuous if

$$|h(z_1) - h(z_2)| \leq K|z_1 - z_2|^{\alpha}$$

for all $z_1, z_2 \in \mathbb{C}$ such that $|z_1 - z_2| < 1$.

But what we are really interested in is the distortion of Euclidean distances under normalized $K-$quasiconformal maps. Let us first recall the classical theorems of Koebe and Mori. For the proof of Koebe's theorems the reader can see [15] and for the proof of Mori's theorem see for example [40, p.66].

Koebe's One-Quarter Theorem & Koebe's Distortion theorem

Let $f : B(z_0, \rho) \to \mathbb{C}$ be a univalent map. Then

$$B(f(z_0), \frac{1}{4}|f'(z_0)|\rho) \subset f(B(z_0, \rho))$$

Moreover , for $0 < \eta < 1$ and for $z \in S(z_0, \eta\rho) = \{z \in \mathbb{C} : |z - z_0| = \eta\rho\}$

(i) $\frac{|f'(z_0)|\eta\rho}{(1+\eta)^2} < |f(z) - f(z_0)| < \frac{|f'(z_0)|\eta\rho}{(1-\eta)^2}$

(ii) $\frac{1-\eta}{(1+\eta)^3} < \frac{|f'(z)|}{|f'(z_0)|} < \frac{1+\eta}{(1-\eta)^3}$

(iii) $|\arg(\frac{f'(z)}{f'(z_0)})| \leq 2\ln(\frac{1+\eta}{1-\eta})$

Mori's theorem

Let f be a $K-$quasiconformal mapping of the unit disk onto itself normalized by $f(0) = 0$. Then for every pair of points z, w with $|z| < 1$ and $|w| < 1$ we have

$$|f(w) - f(z)| \leq 16|w - z|^{\frac{1}{K}}$$

The number 16 cannot be replaced by any smaller bound if the inequality is to be hold for all K.

Now we formulate three lemmas about functions from the family $\mathcal{H}$. The first is very similar to Lemma 2.1 but we bring it into attention, once again, for the sake of completness.

Lemma

For $a \in \mathcal{H}$ there exist positive numbers M_1, M_2, M_3 and r such that for all $b \in B(a, r)$ and for all $z \in \mathbb{C}$ with $|Re\ z| > M_3$ the following inequalities hold.

(i) $M_1 e^{|Re\ z| q(z)} \leq |f_b'(z)| \leq M_2 e^{|Re\ z| q(z)}$,

(ii) $M_1 e^{|Re\ z| q(z)} \leq |f_b''(z)| \leq M_2 e^{|Re\ z| q(z)}$,

(iii) $M_1 e^{|Re\ z| q(z)} \leq |\frac{\partial f_b'}{\partial b}(z)| = |\frac{\partial F_b'}{\partial b}(z)| \leq M_2 e^{|Re\ z| q(z)}$,

where

$$q(z) = \begin{cases} k & \text{if } Re\ z < 0 \\ n - k & \text{if } Re\ z > 0. \end{cases}$$

Another important observation is that we can maintain the bounds from Lemma 2.4 when we apply the quasiconformal homeomorphism h_b to the points of J_{f_a}. Note the parts (iii) and (iv) follow from the equalities (6) and (7).

Lemma

For $a \in \mathcal{H}$ there exists M_1, M_2, M_3 and r such that for all $b \in B(a,r)$ and for all $z \in \mathbb{C}$ with $|Re\ z| > M_3$ the following inequalities hold.

(i) $M_1 e^{|Re\ z| q(z)} \leq |f_b'(h_b(z))| \leq M_2 e^{|Re\ z| q(z)}$,

(ii) $M_1 e^{|Re\ z| q(z)} \leq |f_b''(h_b(z))| \leq M_2 e^{|Re\ z| q(z)}$,

(iii) $M_1 e^{|Re\ z| q(z)} \leq |\frac{\partial f_b'}{\partial b}(h_b(z))| = |\frac{\partial F_b'}{\partial b}(h_b(z))| \leq M_2 e^{|Re\ z| q(z)}$,

(iv) $M_1 e^{|Re\ z| q(z)} \leq |\frac{\partial f_b}{\partial b}(h_b(z))| = |\frac{\partial F_b}{\partial b}(h_b(z))| \leq M_2 e^{|Re\ z| q(z)}$,

where

$$q(z) = \begin{cases} k & \textit{if } Re\ z < 0 \\ n-k & \textit{if } Re\ z > 0. \end{cases}$$

Consequently, we can also generalize Proposition 2.2 from section 2.2 and we obtain that for a fixed parameter $a \in \mathcal{H}$ (i.e. $f_a \in \mathcal{H}$) , a map $f_b \in \mathcal{H}$ is expanding on its Julia set uniformly over a small neighborhood $B(a,r) \subset \mathcal{H}$.

Lemma

For every $a \in \mathcal{H}$ there exist $c > 0$, $\gamma > 1$, $r > 0$ such that, for all $b \in B(a,r)$,

$$|(f_b^n)'(z)| > c\gamma^n$$

for every $z \in J_{f_b}$.

We state now the main result of this section.

Proposition

Fix $a \in \mathcal{H}$. For b sufficiently close to a, we can choose $h_b : \mathbb{C} \to \mathbb{C}$, the quasiconformal conjugacy homeomorphism, such that the following three properties hold.

(i) $\sup_{z \in J_{f_a}} \left\{ |\frac{dh_b}{db}(z)| \right\}$ *is bounded.*

(ii) $h_b : \mathbb{C} \to \mathbb{C}$ *is* $(K(Q), 1/Q)$*-Hölder continuous, where* Q *is quasiconformal constant of* h_b *and* $K : [1, \infty) \to (0, \infty)$ *is increasing.*

(iii) *For every* $z \in \mathbb{C}$ *we have* $h_b(z + 2\pi i) = h_b(z) + 2\pi i$. *This shows that* h_b *is well defined on the cylinder* P.

Proof. First we will prove (i). Let f_a, f_b be as above. Also consider J_{f_b}, J_{f_a} and $h_b : \mathbb{C} \to \mathbb{C}$ with $|a - b| < \varepsilon$ for a small $\varepsilon > 0$. We need to show that

$$\sup_{z \in J_{f_a}, b \in B(a,\varepsilon)} \left| \frac{dh_b(z)}{db} \right| < \infty$$

By the conjugacy relation we get

$$h_b \circ f_a(z) = f_b \circ h_b(z) \text{ for every } z \in \mathbb{C}.$$

Therefore, for every $n \geq 0$, we have that

$$h_b(f_a^n(z)) = f_b^n(h_b(z)).$$

We consider first $z \in J_{f_a}$ a periodic point with period $n \geq 1$. Define the function $f : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ by the formula

$$f(b, z) = f_b(z)$$

Then by the conjugacy relation we obtain, for every $b \in B(a, \varepsilon)$,

$$f^n(b, h_b(z)) = h_b(z)$$

because $f_a^n(z) = z$. Differentiating the above relation with respect to the variable b, we get

$$D_1 f^n(b, h_b(z)) + D_2 f^n(b, h_b(z)) \cdot \frac{dh_b}{db}(z) = \frac{dh_b}{db}(z).$$

Since periodic points from the Julia set are not parabolic, this implies that

$$\frac{dh_b}{db}(z) = \frac{D_1 f^n(b, h_b(z))}{1 - D_2 f^n(b, h_b(z))}.$$

It follows then from Lemma 2.4, the expanding property of maps in the family $\mathcal{H}$ on the Julia set, that if the period n of z is large enough, then

$$\left|\frac{dh_b}{db}(z)\right| \leq \frac{|D_1 f^n(b, h_b(z))|}{|D_2 f^n(b, h_b(z))| - 1} \leq 2\frac{|D_1 f^n(b, h_b(z))|}{|D_2 f^n(b, h_b(z))|}. \tag{15}$$

Let w denotes $h_b(z)$. Then using the equality $f_b^n(w) = f_b(f_b^{n-1}(w))$ (which is equivalent to $f^n(b, w) = f(b, f_b^{n-1}(w)))$ we can estimate D_1 in terms of D_2 as follows. First write

$$\begin{aligned} D_1 f^n(b, w) = D_1\Big(f(b, f^{n-1}(b, w))\Big) \\ = D_1 f(b, f^{n-1}(b, w)) + D_2 f(b, f^{n-1}(b, w)) \cdot D_1 f^{n-1}(b, w). \end{aligned}$$

Therefore, repeating these computations for $n, n-1, \cdots, 1, 0$, by induction and using the chain rule we obtain

$$D_1 f^n(b, w) = \sum_{k=0}^{n-1} D_2 f^k(b, f^{n-k}(b, w)) \cdot D_1 f(b, f^{n-k-1}(w))$$

With ∂-notation, for $w = h_b(z)$, it looks like this.

$$\begin{aligned} D_1 f^n(b, h_b(z)) &= \tfrac{\partial f^n}{\partial b}(b, w) \\ &= \tfrac{\partial f^n}{\partial b}(b, f_b^{n-1}(w)) + f_b'(f_b^{n-1}(w))\tfrac{\partial f^{n-1}}{\partial b}(b, w) \\ &= \textstyle\sum_{k=0}^{n-1}(f_b^k)'(f_b^{n-k}(w))\tfrac{\partial F}{\partial b}(b, f_b^{n-k-1}(w)). \end{aligned}$$

Then

$$\begin{aligned} \frac{D_1 f^n(b, h_b(z))}{D_2 f^n(b, h_b(z))} = \frac{\frac{\partial f^n}{\partial b}(b, w)}{(f_b^n)'(w)} = \sum_{k=0}^{n-1} \frac{\frac{\partial f}{\partial b}(b, f_b^{n-k-1}(w)) \cdot (f_b^k)'(f_b^{n-k}(w))}{(f_b^n)'(w)} \\ = \sum_{k=0}^{n-1} \frac{\frac{\partial f}{\partial b}(b, f_b^{n-k-1}(w))}{(f_b^{n-k})'(w)} = \sum_{k=0}^{n-1} \frac{\frac{\partial f}{\partial b}(b, f_b^{n-k-1}(w))}{(f_b')(f_b^{n-k-1}(w))} \cdot \frac{1}{(f_b^{n-k-1})'(w)}. \end{aligned} \tag{16}$$

Next we would like to show that

$$\left|\frac{\frac{\partial f}{\partial b}(b, f_b^{n-k-1}(w))}{(f_b')(f_b^{n-k-1}(w))}\right|$$

is uniformly (with respect to b) bounded from above. It is worth reminding that $f_b^{n-k-1}(w) \in J_{f_b}$ and both function $\frac{\partial f}{\partial b}(b, \cdot)$ and f_b' are periodic with period $2\pi i$. Therefore, it is enough to prove that there exists a constant C such that

$$\frac{|\frac{\partial f}{\partial b}(b, z)|}{|(f_b')(z)|} \le C \tag{17}$$

for b sufficiently close to a and for $z \in J_{f_b} \cap \{z \in \mathbb{C} : \mathrm{Im}\, z \in [0, 2\pi]\}$.

To do this we split the set $J_{f_b} \cap \{z \in \mathbb{C} : \mathrm{Im}\, z \in [0, 2\pi]\}$ into two sets, a compact one $\{z \in J_{f_b} : x \in [-M_3, M_3] \times [0, 2\pi]\}$ and its complement. By Lemma 2.4

$$C' = \sup\left\{\frac{|\frac{\partial f}{\partial b}(b, x)|}{|(f_b')(x)|} : b \in B(a, \varepsilon), x \in J_{f_b}, x \in [-M_3, M_3] \times [0, 2\pi]\right\} < \infty$$

for ε small enough. Morover, by Lemma 2.4 (i) and (iii),

$$\frac{|\frac{\partial f}{\partial b}(b, x)|}{|(f_b')(z)|} \le \frac{M_2}{M_1}$$

if $|\mathrm{Re}\, x| \ge M_3$. Therefore, (17) is proved with $C = \max\{C', M_2/M_1\}$.

Note that, it follows from Lemma 2.4 that we can assume that $\varepsilon > 0$ satisfies the condition

$$\sum_{j=0}^{l} \frac{1}{|(f_b^j)'(w)|} \le \frac{1}{c(1 - (1/\gamma))}$$

for all n and $b \in B(a, \varepsilon)$. Then, putting (15), (16) and (17) together, we get

$$\sup_{z \in Per, b \in B(a,\varepsilon)} \left|\frac{dh_b}{db}(z)\right| < \infty.$$

Hence, since $\overline{Per} = J_{f_a}$ and since $b \mapsto h_b(z)$ is analytic, the part (i) follows. Next we will prove (ii). Obviously we want to use Mori's theorem and the result obtained before. The point (i) shows, in particular, that for small ε

$$\sup_{z \in J_{f_a}, b \in B(a,\varepsilon)} |z - h_b(z)| < 1. \tag{18}$$

Let $\varepsilon > 0$ be so small that for every $b \in B(a, \varepsilon)$ the maps f_b are sufficiently close to f_a. Fix $x \in J_{f_a}$ and consider the open disk $B(x, 1)$ of radius 1 with center at x. Then $G_b = h_b(B(x, 1))$ is an open simply connected set for every $b \in B(a, \varepsilon)$.

Let $R_b : D(0, 1) \to G_b$ be the conformal representation(Riemann map) of G_b such that $R(0) = h_b(x)$. Then the map

$$g_b = R_b^{-1} \circ h_b : B(x, 1) \to D(0, 1)$$

is a $Q-$quasiconformal homeomorphism between two disks of radius 1. Let now χ_x be a path in J_{F_a} which joins x and infinity. The existence of such a path is a consequence of the fact that all Fatou components are simply conected (see [41, p.90] and section 1.4). Let $\chi_x^\omega \subset \chi_x \cap \overline{B(x, 1)}$ be an arc inside $B(x, 1)$ joining x with a point on the boundary $\partial B(x, 1)$ call it ω. Then $h_b(\chi_x^\omega)$ is an arc joining $h_b(x)$ and $h_b(\omega) \in \partial G_b$.

Note that there exists $z \in D(0, 1)$ with $|z| = \frac{1}{2}$ and $y \in B(x, 1) \cap J_{F_a}$ such that $R_b(z) = h_b(y) \in J_{F_b}$ (or equvalently $g_b(y) = z$). From (18), for $|a - b| < \varepsilon$, it follows that

$$\begin{aligned}
|R_b(z) - R_b(0)| &= |h_b(y) - h_b(x)| \\
&\leq |h_b(y) - y| + |y - x| + |x - h_b(x)| \\
&= |h_b(y) - h_a(y)| + |y - x| + |x - h_b(x)| \\
&\leq 2\varepsilon \sup \left\{ \left| \frac{\partial h_b}{\partial b} \right| : z \in J_{f_a}, b \in B(a, \varepsilon) \right\} + 1 \\
&\leq 2\varepsilon + 1.
\end{aligned}$$

It follows that $R_b(B(0, 1/2))$ does not contain the ball $B(h_b(x), 2\varepsilon + 1)$ since $R_b\Big(B(0, \frac{1}{2})\Big)$ does not contain any ball centered at $h_b(x)$ with radius greater than $|R_b(z) - h_b(x)|$. Then, using Koebe's Distortion Theorem, we get $|R_b'(0)| \leq 4(2\varepsilon + 1)$.

Applying Mori's Theorem to the quasiconformal mapping g_b and to points $z_1, z_2 \in B(x, 1)$ we get

$$|g_b(z_1) - g_b(z_2)| < 16|z_1 - z_2|^{\frac{1}{Q}}.$$

If additionaly $z_1, z_2 \in B(x, 1/(32)^Q)$, then, using Koebe's Theorem with $K = K(1/2)$ for the function R_b, we get

$$|h_b(z_1) - h_b(z_2)| = |R_b(g_b(z_1) - R_b(g_b(z_2))| \leq$$

$$K|R'(0)||g_b(w) - g_b(z)| \leq 4K(2\varepsilon + 1)|w - z|^{\frac{1}{Q}}.$$

From the above computations it follows that h_b is $4K(2\varepsilon+1)$, $\frac{1}{Q}$-Hölder continuous on $1/(32)^Q$-neighborhood of J_{f_a}. But note, that there exists r, such that $r/(32)^Q$-neighborhood of J_{f_a} contains the whole plane $\mathbb{C}$. Therefore, considering the map $g_b^r(z) = \frac{1}{r} g_b$ insead of g_b (we have to increase the domain of g_b to $B(x, r)$), we can repeat the computations to prove that h_b is $4rK(2\varepsilon + 1)$, $\frac{1}{Q}$-Hölder continuous on $\mathbb{C}$.

Finally we will prove (iii). Consider the map $b \mapsto k_z(b) = h_b(z + 2\pi i) - h_b(z) \in \mathbb{C}$. Since $b \mapsto h_b(z)$ is continuous, the map k_z is continuous as well. If $b \in B(a, \varepsilon)$ for some small ε, as before, we get from the conjugacy relation that

$$f_b(h_b(z + 2\pi i)) = h_b(f_b(z + 2\pi i)) = h_b(f_b(z)) = f_b(h_b(z)). \tag{19}$$

Then, for every $b \in B(a, \varepsilon)$, the set of all possible values of k_z is a discret subset of $\mathbb{C}$ (in particular has a finite intersection with the stripe $\{z : \mathrm{Im}\, z \in [0, 2\pi)\}$). If $h_b(z + 2\pi i)$ and $h_b(z)$ are regular points of f_b, then, for c suficiently close to b, $h_c(z + 2\pi i)$ and $h_c(z)$ are regular points of f_c and, if $k_z(b) \neq 2\pi i$, then $k_z(c) \neq 2\pi i$, and if $k_z(b) = 2\pi i$, then $k_z(c) = 2\pi i$. Since $k_z(a) = 2\pi i$ for $z \in \mathbb{C}$ and since the set of critical points of f_a is discrete, k_z is the constant function $2\pi i$. This finishes the proof. $\square$

CHAPTER 3

PERRON-FROBENIUS OPERATORS

3.1. Conformal Measures and Semi-Conformal Measures

The main goal of this section is to find a relation between the Hausdorff dimension of J_F^r and the exponent of the unique conformal measure on J_F. This relation (actually equality) is stated in Corollary 3.1.

We give required definitions and then we prove this result going first through semiconformal measures. An observation of a general character is that all (semi)conformal measures in our paper are probabilistic measures. Let, as before, $a \in \mathbb{C}^{n+1}$, $f = f_a$, $F = F_a = \pi \circ f_a \circ \pi^{-1}$, and $\delta = \delta_a$.

Define

$$J(M) = \bigcap_{n \in \mathbb{N}} F^{-n}(D(M)),$$

where

$$D(M) = \{z \in J_F : |\mathrm{Re}\ z| \le M\}.$$

Similarly as in [11, Lemma 5.3] (see also [26]) it follows that for $t > 0$ there exists a real number $\alpha_M(t) \ge 0$ (we will often denote $\alpha_M(t)$ just by α_M) and (t, α_M)-*semiconformal measure* $m_{t,M}$ ($\alpha_M = \alpha_M(t)$) supported on $J(M)$, i.e. there exists a Borel probability measure $m_{t,M}$ on $J(M)$ such that for all Borel sets $A \subset J(M)$ for which $F|_A$ is 1-1,

$$m_{t,M}(F(A)) \ge \alpha_M \int_A |F'|^t dm_{t,M} \tag{20}$$

and, if additionally $A \cap \partial D(M) = \emptyset$, then

$$m_{t,M}(F(A)) = \alpha_M \int_A |F'|^t dm_{t,M}.$$

The number $\alpha_M(t)$ can be defined by the formula $\alpha_M(t) = e^{c_M(t)}$, where

$$c_M(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E^M} \sum_{w\in F|_{J_M}^{-n}(x)} |(F^n)'(w)|^{-t},$$

E^M is a finite collection of points such that the set

$$\mathcal{Z}_M = \{B(x, \delta_1/2) : x \in E^M\}$$

is an open cover of $J(M)$, and $\delta_1 \leq \delta$ is chosen so that

$$J_1(M) = \bigcap_{j\in\mathcal{K}_M} L_j(R(M)),$$

where

$$R(M) = \{z \in \mathbb{C} : \min_{z_0\in P(M)} |z - z_0| < \delta_1\}.$$

Recall that $J_1(M)$ is a compact set and F-forward invariant as a limit set of a finite conformal iterated function system (see the proof of Proposition 2.3) and, by (9), δ_1 can chosen such that, for $z \in R(M)$ and for $j \in \mathcal{K}_M$,

$$|L_j'(z)| < \frac{1}{a}$$

for some $a > 1$. Moreover, functions L_j are inverse branches of F^{-1} and

$$L_j(R(M)) \subset R(M). \tag{21}$$

Our objective is to prove that we can find $t_M > 1$ such that $\alpha_M(t_M) = \alpha_M = 1$ ($c_M(t) = 0$).

Lemma

For every large M there exists $t_M > 1$ with $\alpha_M(t_M) = 1$. Moreover, for all $p \geq \delta_1$, $1 < HD(J_1(M)) \leq t_{M+p}$.

Proof. Let

$$\mathcal{B}^n := \{L_{j_1} \circ \ldots \circ L_{j_n} : j_l \in \mathcal{K}_M, l = 1, \ldots, n\}.$$

Let $p \geq \delta_1$ and let $z \in E^{M+p}$ be such that $z \in R(M)$. We can always find such a point since δ_1 can be as small as we want. Then, for $B = B(z, \delta_1/2)$ and for $\phi \in \mathcal{B}^n$, we have that

$$\phi(B) \subset R(M).$$

Therefore

$$\sum_{\phi \in \mathcal{B}^n} |\phi'(z)|^t \leq \sum_{x \in E^M} \sum_{w \in F|_{J_{M+p}}^{-n}(x)} |(F^n)'(w)|^{-t}.$$

Hence

$$P_z(t) \leq c_{M+p}(t)$$

where $P_z(t)$ is the pressure function for the finite conformal iterated function system defined in the proof of Proposition 2.3. Since the zero of P_z is equal to $\mathrm{HD}(J_1(M)) > 1$, the zero point of c_{M+p} exists and is greater or equal to $\mathrm{HD}(J_1(M))$ if the function c_{M+p} is a continuous, decreasing and

$$\lim_{t \to \infty} c_{M+p}(t) = -\infty.$$

But this can be done by a direct computation using the Hölder inequality and the expanding property. $\square$

Therefore, we are able to find a *t_M-semiconformal measure* on $J(M)$ (i.e. $(t_M, 1)$-semiconformal) $m_{t_M,M}$ which we denote by m_M. Note that for the measure m_M it is true that

$$m_{t,M}(F^n(A)) \geq \int_A |(F^n)'|^t dm_{t,M} \tag{22}$$

for every Borel set $A \subset J(M)$ for which $F^n|_A$ is 1-1, and, if additionally $A \cap \partial D(M) = \emptyset$, then

$$m_{t,M}(F^n(A)) = \int_A |(F^n)'|^t dm_{t,M}. \tag{23}$$

Next we will use the following proposition and for a proof we refer the reader to [17] or [26].

Proposition

Let $E \subset \mathbb{R}^n$ be Borel set, let μ be a finite Borel measure on $\mathbb{R}^n$.

(i) *If* $\limsup_{r\to 0} \mu(B(x,r))/r^t < \infty$ *for all* $x \in E$ *then* $HD(E) \geq t$.

(ii) *If* $\limsup_{r\to 0} \mu(B(x,r))/r^t > 0$ *for all* $x \in E$ *then* $HD(E) \leq t$.

Lemma

Let m_M be the t_M-semiconformal measure supported on $J(M)$. Then

$$HD(J(M)) \geq t_M.$$

Proof. By Proposition 3.1 it is enough to prove that there exists a constant C such that for all $z \in J(M)$ and for all small $r > 0$

$$m_M(B(z,r)) \leq Cr^{t_M}.$$

So let $z \in J(M)$ (M is large) and let r be a positive number such that

$$\frac{4r}{\delta} < (\sup_{x\in J(M)} |F'(x)|)^{-1} < 1$$

(see (9)) where $\delta = \delta_a$. Since $|(F^{n+1})'(z)| = |F'(F^n(z))||(F^n)'(z)|$, we can find $n \geq 1$ such that

$$|(F^n)'(z)|^{-1} \geq \frac{4r}{\delta} > (|(F^n)'(z)| \sup_{x\in J(M)} |F'(x)|)^{-1}. \tag{24}$$

Since $P_{f_a} \cap B(z, 2\delta) = \emptyset$, we have a univalent function

$$F_z^{-n} : B(F^n(z), \delta) \to P$$

which is the inverse branch of F^n sending $F^n(z)$ to z. It follows from $\frac{1}{4}$–Koebe Theorem that

$$B(z,r) \subset B(z, \frac{\delta}{4}|(F^n)'(z)|^{-1}) \subset F_z^{-n}(B(F^n(z),\delta)).$$

Then from (22) and (1) we obtain the inequalities

$$
\begin{aligned}
1 &\geq m_M(B(F^n(z),\delta)) = m_M(F^n(F_z^{-n}(B(F^n(z),\delta)))) \\
&\geq \int_{F_z^{-n}(B(F^n(z),\delta))} |(F^n)'|^{t_M} dm_M \\
&\geq (\inf_{x\in B(F^n(z),\delta)} |(F^n)'(x)|^{t_M}) m_M(F_z^{-n}(B(F^n(z),\delta))) \\
&\geq \frac{1}{K^{t_M}} |(F^n)'(z)|^{t_M} m_M(F_z^{-n}(B(F^n(z),\delta))) \\
&\geq \frac{1}{K^{t_M}} |(F^n)'(z)|^{t_M} m_M(B(z,r)).
\end{aligned}
$$

Hence, by (24),

$$
\begin{aligned}
m_M(B(z,r)) &\leq K^{t_M} |(F^n)'(z)|^{-t_M} \\
&\leq ((4K/\delta) \sup_{x\in J(M)} |F'(x)|)^{t_M} r^{t_M}.
\end{aligned}
$$

□

So we know that the sequence $(t_M)_{M\in\mathbb{N}}$ is bounded from above by 2. Therefore we can find a convergent subsequence (t_{M_k}) to a finite point h(see Theorem 3.1).

It is also worth noting that, if we fix $t > 1$, then the sequence $(\alpha_M(t))_{M\in\mathbb{N}}$ is bounded. This follows from the fact that $\overline{P}_z(t)$ is finite, where $\overline{P}_z(t)$ is defined in the section 3.3 (see Lemma 3.3). Then it follows from Lemma 3.1 that, for $t > 1$, we can always find a convergent subsequence $(\alpha_{M_k}(t))$ to a finite point $\alpha(t)$ different from zero (see Remark 3.1).

More on cylinder P

Semiconformal measures introduced before will help us to construct conformal measures. But before we do this, we would like to show how strong property is the existence of such measures. So now we describe some subsets of the cylinder P, introduce some notation for these special subsets and we make some remarks on the behavior of a map $f \in \mathcal{H}$, far from its critical values, in a simply connected neighborhood of infinity (outside of a ball of a sufficiently big radius). This allows us to prove the uniqueness property of conformal measures. Therefore, we will be well prepared for the construction of such measures.

Consider for the beginning the following sets:

$$A_1(M) = \{z \in J_F : \mathrm{Re}\, z > M\},$$

$$A_2(M) = \{z \in J_F : \mathrm{Re}\, z < -M\},$$

$$A(M) = A_1(M) \cup A_2(M).$$

The function $f \circ \pi^{-1} : P \to \mathbb{C}$ is well defined by the relation $f \circ \pi^{-1}(\hat{z}) = f(z)$, where $\hat{z} \in P$ is the class of $z \in \mathbb{C}$ via the equivalence relation defined in 2.2, and, if it does not lead to misunderstanding, we will denote it sometimes by f and the class $\hat{z}$ by z. Because $f(z) = f_a(z) \in \mathcal{H}$ has the analytic expression

$$f_a(z) = \left(\frac{a_0}{e^{kz}} + \frac{a_1}{e^{(k-1)z}} + \cdots + \frac{a_{k-1}}{e^z}\right) + a_k + \left(a_{k+1}e^z + \cdots + a_n e^{(n-k)z}\right)$$

for $0 < k < n$, we observe that, for M large enough (see Lemma 2.1), the set $\pi(f^{-1}(A_1(M)))$ has $n-k$ connected components which intersect $A_1(M)$ and k components which intersect $A_2(M)$. We denote them respectively by $B(M,j)$ ($j = 1, \ldots, n-k$) and by $B(M,j)$ ($j = n-k+1, \ldots, n$). Analogously we can enumerate n connected components of $\pi(f^{-1}(A_2(M)))$ by $B(M,j)$ ($j = n+1, \ldots, 2n-k$) and $B(M,j)$ ($j = 2n-k+1, \ldots, 2n$). Let

$$B(M,j,l) = \{z \in B(M,j) : \mathrm{Im}\ f \circ \pi^{-1}(z) \in [(2l-1)\pi, (2l+1)\pi)\},$$

where $l \in \mathbb{Z}$. Note that

$$\bigcup_{j=1}^{2n} F(B(M,j)) \subset A(M). \tag{25}$$

Next, let

$$A^1(M) = \{z \in \mathbb{C} : |z| > M_1 e^M, \mathrm{Im}\ z > 0, \mathrm{Re}\ z \in [-M, M]\},$$

$$A^2(M) = \{z \in \mathbb{C} : |z| > M_1 e^M, \mathrm{Im}\ z < 0, \mathrm{Re}\ z \in [-M, M]\}.$$

Then, similarly, we can enumerate $2n = ((n-k)+k)+((n-k)+k)$ connected components of $\pi(f^{-1}(A^1(M) \cup A^2(M)))$ and denote them by $C(M,j)$ $(j = 1, \ldots, 2n)$. And, finally,

$$C(M,j,l) = \{z \in C(M,j) : \mathrm{Im}\ f \circ \pi^{-1}(z) \in [(2l-1)\pi, (2l+1)\pi)\},$$

where $l \in \mathbb{Z}$. But note that, if $C(M,j)$ is a preimage of $A^1(M)$ $(A^2(M))$, then $C(M,j,l)$ is empty for negative (res. positive) l. Moreover, using Lemma 2.1 (i) we can get that, for $z \in C(M,j)$,

$$|\mathrm{Im}\ f(z)| \geq \sqrt{M_1^2 e^{2M} - M^2}.$$

Therefore, for large M, if $|l| \leq e^M/2$, then $C(M,j,l) = \emptyset$ for every j.

Observe that $B(M,j,l)$ and $C(M,j,l)$ are subsets of P on which the function F is univalent. Moreover, it follows from Lemma 2.1 (i) that, for $z \in A(M)$, $|f(z)| \geq M_1 e^M$. Then

$$A(M) \subset \bigcup_{j=1}^{2n} (B(M,j) \cup C(M,j)). \tag{26}$$

So, the sets $\bigcup_{j=1}^{2n} B(M,j) \cap A(M)$ and $\bigcup_{j=1}^{2n} C(M,j) \cap A(M)$ give us a partitions of $A(M)$ into two sets. The image of the first one (under F) is contained in $A(M)$, while the image of the second one in contained in the complement of $A(M)$.

A conformal measure on $A(M)$

Let μ be (t,α)-conformal measure on J_F with $t > 1$. Above we have described a partition of $A(M)$ into two sets. So there is a natural question, which one is bigger with the respect to the measure μ. First, we will give an answer to this question and then we will show that there exists a constant c_μ such that for all M large enough

$$\mu(A(M)) \leq \frac{c_\mu}{e^{M(t-1)}}. \tag{27}$$

If follows from (25) and from Lemma 2.1 (iii) that

$$\begin{aligned} \mu(A(M)) &\geq \mu(F(B(M,j,l))) = \alpha \int_{B(M,j,l)} |F'|^t d\mu \\ &\geq \alpha \tfrac{M_1}{M_2}(\sqrt{(\pi l)^2 + M^2})^t \mu(B(M,j,l)). \end{aligned} \tag{28}$$

Therefore,

$$\frac{M_2}{\alpha M_1}\left(2n \sum_{l=-\infty}^{\infty} \frac{1}{(\sqrt{(\pi|l|)^2 + M^2})^t}\right)\mu(A(M)) \geq \mu(\bigcup_{j=1}^{2n} B(M,j)). \tag{29}$$

Then, since $t > 1$, there exists c'_μ such that

$$\mu(\bigcup_{j=1}^{2n} B(M,j)) < \frac{c'_\mu}{M^{t-1}}\mu(A(M))$$

for large enough M. In particular, we can get that that more than half of $A(M)$ (with respect to the measure μ) belongs to $\mu(\bigcup_{j=1}^{2n} C(M,j))$ i.e.

$$2\mu(\bigcup_{j=1}^{2n} C(M,j)) > \mu(A(M)).$$

So, to prove (27), first observe that, from Lemma 2.1 we have:

$$1 \geq \mu(F(C(M,j,l))) = \alpha \int_{C(M,j,l)} |F'|^t d\mu \geq \alpha \frac{M_1}{M_2} |l|^t \mu(C(M,j,l)). \tag{30}$$

Therefore, since $|l| \leq e^M/2$ implies $C(C(M,j,l)) = \emptyset$,

$$\mu(A(M)) \leq 2\frac{M_2}{\alpha M_1} 2n \sum_{|l| \geq \lfloor e^M/2 \rfloor} \frac{1}{|l|^t} \leq \frac{c_\mu}{e^{M(t-1)}}, \tag{31}$$

where c_μ is a constant which is, for large M, independent of M.

Uniqueness

The next lemma will be used especially for the proofs of Lemma 3.1 and Proposition 3.1. Let again

$$J_F^r(M) = \{z \in J_F : \liminf_{n\to\infty} |\mathrm{Re}\, F^n(z)| \leq M\}.$$

Lemma

Let μ be (t, α)-conformal measure on J_F with $t > 1$. Then there exists $M > 0$ such that for μ-a.e. x

$$\liminf_{n\to\infty} |Re\ F^n(x)| \leq M.$$

Proof. Fix $M > 0$. Let

$$X(M) = \{z \in J_F : \liminf_{n\to\infty} |\mathrm{Re}\ F^n(z)| > M\}.$$

Then

$$X(M) = \bigcup_{n\in\mathbb{N}} \{z \in J_F : F^k(z) \in A(M) \text{ for all } k \geq n\}.$$

Suppose $\mu(X(M)) > 0$. Then there exists $n \in \mathbb{N}$ such that

$$\{z \in J_F : F^k(z) \in A(M) \text{ for all } k \geq n\}$$

has a positive measure μ. Moreover, since μ is t-conformal, the measure μ of the set

$$Y(M) = F^n(\{z \in J_F : F^k(z) \in A(M) \text{ for all } k \geq n\})$$

is also positive. Using the fact that $Y(M)$ is F-forward invariant, similarly as in (28) and (30), we get

$$\begin{aligned} \mu(Y(M)) &\geq \mu(F(B(M, j, k) \cap Y(M))) \\ &\geq \alpha \frac{M_1}{M_2} (\sqrt{(\pi k)^2 + M^2})^t \mu(B(M, j, k) \cap Y(M)). \end{aligned}$$

Since $Y(M)$ and $F(Y(M))$ are subsets of $A(M)$, we have

$$\frac{2nM_2}{\alpha M_1} \sum_{k\in\mathbb{Z}} \frac{1}{(\sqrt{(\pi k)^2 + M^2})^t} \mu(Y(M)) \geq \mu(Y(M)).$$

But, for M large, the left side of this inequality is smaller then the right. Hence, for M large, the measure μ of $X(M)$ cannot be positive. $\square$

Next we are going to prove a lemma which is needed mainly in the proof of Proposition 3.1. This proposition together with its corollaries will be used in the proof Theorem 3.1, the main result of this section.

Lemma

Let μ be a (t,α)-conformal measure on J_F, where $t>1$. Then there exists $M>0$ and $b_\mu>0$ such that $\mu(J_F^r(M))=1$ and for all $x\in J_F^r(M)$ there exist sequences $r_k>0$ and n_k such that

$$\lim_{k\to\infty} r_k = 0, \lim_{k\to\infty} n_k = \infty$$

and

$$\frac{1}{b_\mu}\alpha^{-n_k} r_k^t \le \mu(B(x,r_k)) \le \alpha^{-n_k} b_\mu r_k^t.$$

Proof. By Lemma 3.1 there exists $M>0$ such that for $\mu-$a.e. $x\in J_F$

$$\liminf_{n\to\infty} |\mathrm{Re}\ F^n(x)| \le M. \tag{32}$$

This is equivalent to the equality $\mu(J_F^r(M))=1$. Then choose any finite cover $\mathcal{V}$ of $\{z\in J_F : |\mathrm{Re}\ z|\le M\}$ consisting of disks of radius $\delta/32$. Define

$$a_1 = \min_{B\in\mathcal{V}} \mu(B) > 0.$$

By (32) there exist $y\in J_F$ and a sequence n_k such that $B(y,\delta/32)\in\mathcal{V}$ and $F^{n_k}(x)\in B(y,\delta/32)$ for $k\in\mathbb{N}$. Consider the disk $B(x,\frac{\delta}{4|(F^{n_k})'(x)|})$. Then using Koebe's Theorem we get

$$B(z,\frac{\delta}{32}) \subset B\left(F^{n_k}(x),\frac{\delta}{16}\right) \subset$$
$$F^{n_k}\left(B\left(x,\frac{\delta}{4|(F^{n_k})'(x)|}\right)\right) \subset B(F^{n_k}(x),\delta) \subset B(y,2\delta),$$

where z is such a point that $B(z,\frac{\delta}{32})\in\mathcal{V}$. Since the measure μ is conformal, it follows that

$$a_1 \le \mu(F^{n_k}(B(x,\frac{\delta}{4|(F^{n_k})'(x)|})))$$

$$\leq \mu(B(x, \frac{\delta}{4|(F^{n_k})'(x)|}))\alpha^{n_k} K^t |(F^{n_k})'(x)|^t,$$

and

$$1 \geq \mu(F^{n_k}(B(x, \frac{\delta}{4|(F^{n_k})'(x)|})))$$

$$\geq \mu(B(x, \frac{\delta}{4|(F^{n_k})'(x)|}))\alpha^{n_k} \frac{1}{K^t} |(F^{n_k})'(y)|^t.$$

Hence

$$\frac{a_1}{K^t}\alpha^{-n_k}(\frac{1}{|(F^{n_k})'(z)|})^t \leq \mu(B(x, \frac{\delta}{4|(F^{n_k})'(x)|})),$$
$$K^t\alpha^{-n_k}(\frac{1}{|(F^{n_k})'(z)|})^t \geq \mu(B(x, \frac{\delta}{4|(F^{n_k})'(x)|})).$$

Since $\lim_{k\to\infty} \frac{\delta}{4|(F^{n_k})'(x)|} = 0$, we can take $r_k = \frac{\delta}{4|(F^{n_k})'(x)|}$ and the lemma is proved. $\square$

Proposition

Let μ_1 be (t_1, α)-conformal measure on J_F and μ_2 be (t_2, α)-conformal measure on J_F, where $t_1, t_2 > 1$. Then $t_1 = t_2$ and there exist a constant b_{μ_1,μ_2} such that

$$\frac{1}{b_{\mu_1,\mu_2}}\mu_1(E) \leq \mu_2(E) \leq b_{\mu_1,\mu_2}\mu_1(E).$$

for every E a bounded Borel subset of J_F.

Proof. Let μ_1 be (t_1, α)-conformal measure and μ_2 be (t_2, α)-conformal measure. It follows from Lemma 3.1 that there exists $M \in \mathbb{N}$ such that $\mu_1(J_F^r(M)) = \mu_2(J_F^r(M)) = 1$. Let $x \in J_F^r(M)$. Then

$$\liminf_{n\to\infty} \mathrm{Re}\, |F^n(x)| \leq M \tag{33}$$

Taking

$$r_k(x) = \frac{\delta}{4|(F^{n_k})'(x)|},$$

where n_k is chosen as in Lemma 3.1, we have that, for $j = 1, 2$,

$$\frac{1}{b}\alpha^{-n_k} r_k(x)^{t_j} \leq \mu_j(B(x, r_k(x))) \leq b\alpha^{-n_k} r_k(x)^{t_j},$$

where $b := \max\{b_{\mu_1}, b_{\mu_2}\}$ and is independent of the choice of x. Suppose that $t_2 < t_1$ and fix $\varepsilon > 0$. Then, for $r_k(x) \leq \varepsilon$,

$$\begin{aligned} \mu_1(B(x, r_k(x))) &\leq b\alpha^{-n_k} r_k(x)^{t_1 - t_2} r_k(x)^{t_2} \\ &\leq b^2 r_k(x)^{t_1-t_2} \mu_2(B(x, r_k(x))) \\ &\leq b^2 \varepsilon^{t_1-t_2} \mu_2(B(x, r_k(x))). \end{aligned} \tag{34}$$

Let E be bounded Borel subset of J_F and let E' be a subset of E which contains this points of E satisfying the property (33). Since the measures are regular, for every $x \in E'$ there exists $r(x) = r_k(x)$ (k depends on x) such that $\{B(x, r(x)) : x \in E'\}$ is a cover of E and

$$\mu_2(\bigcup_{x \in E'} B(x, r(x)) \setminus E) \leq \varepsilon.$$

By Besicovic theorem we can choose a countable subcover

$$\{B(x_j, r(x_j)) : j \in \mathbb{N}\}$$

of bounded multiplicity C which is independent of ε. Therefore

$$\begin{aligned} \mu_1(E) &\leq \textstyle\sum_{j \in \mathbb{N}} \mu_1(B(x_j, r(x_j))) \\ &\leq b^2 \varepsilon^{t_1-t_2} \textstyle\sum_{j \in \mathbb{N}} \mu_2(B(x, r_k(x))) \\ &\leq b^2 C \varepsilon^{t_1-t_2} \mu_2(\textstyle\bigcup_{k \in \mathbb{N}} B(x_k, r(x_k))) \\ &\leq b^2 C \varepsilon^{t_1-t_2} (\varepsilon + \mu_2(E)). \end{aligned} \tag{35}$$

Since $t_1 > t_2$, it follows that the measure μ_1 of the set E is as small as we want and, then, $\mu_1(J_F) = 0$. This is a contradiction. With the same argument, exchanging μ_1 with μ_2 we get that the inequality $t_2 > t_1$ cannot holds. Hence $t_1 = t_2$ and then the inequality (35) finishes the proof. $\square$

Remark

Similarly one can show that if μ_1 is a (t, α_1)-conformal measure on J_F and μ_2 is another (t, α_2)-conformal measure on J_F, where $t > 1$ then $\alpha_1 = \alpha_2$ and there exist a constant b_{μ_1, μ_2}

such that

$$\frac{1}{b_{\mu_1,\mu_2}}\mu_1(E) \le \mu_2(E) \le b_{\mu_1,\mu_2}\mu_1(E).$$

for every bounded Borel subset E of J_F.

Corollary

Every (t, α)-conformal measure μ on J_F with $t > 1$ is ergodic.

Proof. Suppose that μ is not ergodic. Then we can find an invariant set A (i.e. $F^{-1}(A) = A$) such that $0 < \mu(A) < 1$. Then two conditional measures μ_A and $\mu_{J_F\setminus A}$, where

$$\mu_X(Y) = \frac{\mu(Y \cap X)}{\mu(Y)},$$

are (t, α)-conformal. Moreover, they are mutually singular. This contradicts Proposition 3.1.

□

We are ready to state now the following important uniqueness property of a $(t, \alpha)-$conformal measure supported on J_F.

Proposition(Uniqueness)

(i) *Let μ_1 be (t_1, α)-conformal measure on J_F and μ_2 be (t_2, α)-conformal measure on J_F with $t_1, t_2 > 1$. Then $\mu_1 = \mu_2$.*

(ii) *Let μ_1 be (t, α_1)-conformal measure on J_F and μ_2 be (t, α_2)-conformal measure on J_F with $t > 1$. Then $\mu_1 = \mu_2$*

Proof. (i) It follows from Proposition 3.1 that there exists $h \in L^1(\mu_1)$ such that

$$\mu_1 = h \cdot \mu_2,$$

and, for $z \in J_F$, $h(z) \in [1/b_{\mu_1,\mu_2}, b_{\mu_1,\mu_2}]$. Consider the sets $H_1 = \{z \in J_F : h(z) \ge 1\}$ and $H_2 = \{z \in J_F : h(z) \le 1\}$. At least one of them has a positive measure μ_1. Say H_1. Let us show first that

$$\mu_1(F^{-1}(H_1) \setminus H_1) = 0.$$

To prove this, let B be a Borel subset of $F^{-1}(H_1)$ on which F is 1-1. Then

$$
\begin{aligned}
\int_B |F'|^t d\mu_1 &= \mu_1(F(B)) = \int_{F(B)} h d\mu_2 \\
&\geq \mu_2(F(B)) = \int_B |F'|^t d\mu_2.
\end{aligned} \tag{36}
$$

Hence, $h(z) \geq 1$ μ_1-almost everywhere on $F^{-1}(H_1)$ and, therefore,

$$\mu_1(\bigcup_{n\in\mathbb{N}} F^{-n}(H_1) \setminus H_1) = 0.$$

Since μ_1 is ergodic and since $\mu_1(H_1) > 0$, $\mu_1(H_1) = 1$. From the fact that $\mu_2(J_F) = 1$ we get that $h(z) = 1$ for μ_1-almost every z. Hence, $\mu_1 = \mu_2$.

The point (ii) can be proved similarly as the proof of (i).

□

Existence

For the proof of the next result we use Prochorov's Theorem (see [28, ChapterIII]) about convergence of measures on a non-compact space. For the sake of completness we include this and the definition of tightness in the last section 3.4 of this Chapter.

Theorem

The sequence $(m_M)_{M>M_0}$ is tight *on J_F for some positive M_0, which in particular means that for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $M > M_0$,*

$$m_M(\{z \in J_F : |\mathrm{Re}\, z| > N\}) < \varepsilon.$$

Moreover, there exists a subsequence (m_{M_k}) such that its weak limit m is an h-conformal measure, where

$$h = \lim_{k\to\infty} t_{M_k} = \lim_{M\to\infty} HD(J(M)).$$

Note that $h > 1$ since $HD(J(M)) > 1$ for M large enough.

Proof. The proof of tightness is similar to the proof of (27). In particular, we can get similar formulas like (28) and (30), but two equalities in the computations we have to replace by inequalities. Therefore,

$$(2n\sum_{k\in\mathbb{Z}}\frac{1}{(\sqrt{(\pi k)^2+M^2})^t}+2n\sum_{|k|\geq\lfloor M_1e^M\rfloor-1}\frac{1}{k^t})\geq m_M(A(M)).$$

Since $t_M > 1$,, the left hand side can be as small as we want for large M. Hence we obtain that the sequence is tight. The existence of weak limit follows from Prochorov's Theorem.

So we have to prove that any week limit of the sequence is h-conformal measure. Let A be such a set that $F|_A$ is one-to-one. We can assume that there exists $N \in \mathbb{N}$ such that

$$A \subset \{z : |\mathrm{Re}\ z| \leq N\}. \tag{37}$$

If the set is unbounded, we can divide it into countable many subsets. For $i = 1, 2, \ldots$ denote $N + i$ by N_i. Since m_{N_i} is t_{N_i}-semi-conformal,

$$m_{N_i}(F(A\cap J(N_i))) = \int_{A\cap J(N_i)}|F'|^{t_{N_i}}dm_{N_i}.$$

Then, if

$$F(A)\cap J(N_i) = F(A\cap J(N_i)) \tag{38}$$

and if $m(\partial(A)) = 0$

$$\begin{aligned}m(F(A)) = \lim_{i\to\infty} m_{N_i}(F(A)) = \lim_{i\to\infty}\int_A|F'|^{t_{N_i}}dm_{N_i}\\ = \lim_{i\to\infty}\int_A|F'|^h dm_{N_i} + \lim_{i\to\infty}\int_A(|F'|^{t_{N_i}}-|F'|^h)dm_{N_i} = \int_A|F'|^h dm.\end{aligned}$$

The last equality follows from (37), because from this we have that $|F'|$ is bounded on A and then $\lim_{i\to\infty}\int_A(|F'|^{t_{N_i}}-|F'|^h)dm_{N_i} = 0$. For an arbitrary Borel set A, such that $F|_A$ is injective, since by Lemma 2.4 in [10] the boundary condition is resolved, it suffices for us to to prove (38). Observe that

$$F(A\cap J(N_i)) \subset F(A)\cap J(N_i).$$

To get the opposite inclusion suppose, by contradiction, that there exists $z \in F(A) \cap J(N_i)$ but z is not an element of $F(A \cap J(N_i))$. Then we can find $y \in A$ such that $F(y) \in J(N_i)$ and $y \notin J(N_i)$. It follows from the definition of $J(N_i)$ (Section 3.1) that $F^n(y) \in D(N_i)$ for all natural n but zero. This is a contradiction because $y \in A \subset D(N_i)$. $\square$

Remark

It can be shown as before that, if $t > 1$ and $m_{t,M}$ is a (t, α_M)-semiconformal measure, then $(m_{t,M})$ is tight. Moreover, there exists a subsequence (m_{t,M_k}) such that its weak limit m_t is (t, α_t)-conformal measure on J_F where $\alpha_t = \lim_{k\to\infty} \alpha_{M_k}$.

Now we are ready to state the theorem which is the main result of this section. The proof of this theorem is obvious from the previous Remark 3.1, Proposition 3.1 and Theorem 3.1.

Theorem

For every $t > 1$ there exist a unique $\alpha_t > 0$ and a unique (t, α_t)-conformal measure m_t on J_F. Moreover, there exists a unique $h > 1$ such that $m = m_h$ is an h-conformal measure.

The set of returning points

Observe once again that for $a \in \mathbb{C}^{n+1}$ and $F = F_a$ as in 2.2 we have

$$J_F^r = \{z \in J_F : \liminf_{n\to\infty} |\mathrm{Re}\, F^n(z)| < \infty\}.$$

Recall that (see 2.3)

$$J_F^r = \pi\big(J_f \setminus \{z \in J_f : \lim_{n\to\infty} |f^n(z)| = \infty\}\big).$$

Corollary

If F and J_F^r are as before then

$$\mathrm{HD}(J_F^r) = h.$$

Proof. Since $J(M) \subset J_F^r$ and because $\mathrm{HD}(J(M)) \geq t_M$ (see Lemma 3.1 and Theorem 3.1) we have

$$\mathrm{HD}(J_F^r) \geq h.$$

Let

$$J_F^r(M) = \{z \in J_F : \liminf_{n\to\infty} |\mathrm{Re}\, F^n(z)| \leq M\}.$$

It follows from the proof of Lemma 3.1, that, for h-conformal measure m and for all M large enough, if $z \in J_F^r(M)$, then there exists a constant b_M and a sequence $r_k \to 0$ such that

$$\frac{1}{b_M} r_k^h \leq m(B(z, r_k)).$$

Then, by Proposition 3.1 (ii), $\mathrm{HD}(J_F^r(M)) \leq h$. Therefore, since $J_F^r = \bigcup_M J_F^r(M)$,

$$\mathrm{HD}(J_F^r) \leq h.$$

$\square$

3.2. Ionescu-Tulcea and Marinescu Theorem

In this section we fix $a \in \mathcal{H}$ and we denote $f = f_a$, $F = F_a$ and $\delta = \delta_a$. Let $\mathrm{CB}(J_F, \mathbb{C})$ be the Banach space of all bounded continuous functions $g : J_F \to \mathbb{C}$ with the norm $||\cdot||_\infty$. For $\alpha \in (0, 1]$ and for $g \in \mathrm{CB}(J_F, \mathbb{C})$ we denote by $v_\alpha(g)$ the α-variation of the function g which is

$$\inf\{L \geq 0 : |g(x) - g(y)| \leq L|x - y|^\alpha \text{ for all } x, y \in J_F \text{ with } |x - y| \leq \delta\}.$$

Let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty,$$

and define

$$H_\alpha = H_\alpha(J_F) = \{g \in \mathrm{CB}(J_F, \mathbb{C}) : ||g||_\alpha < \infty\}.$$

Then the set H_α with the norm $||\cdot||_\alpha$ is a Banach space and H_α is a dense subset of $\mathrm{CB}(J_F, \mathbb{C})$.

Remark

Observe that it follows immediately from Proposition 2.2 that there exist $L > 0$ and $0 < \beta < 1$ such that for every $n \geq 0$, every $v \in J_F$ and every $z \in B(F^n(v), \delta)$

$$|(F_v^{-n})'(z)| \leq L\beta^n.$$

Definitions

We say a continuous function $\phi : J_F \to \mathbb{C}$ is dynamically Hölder with an exponent $\alpha > 0$ if there exists $c_\phi > 0$ such that

$$|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))| \leq c_\phi |\phi_n(F_v^{-n}(x))||y - x|^\alpha$$

for all $n \geq 1$, all $x, y \in J_F$ with $|x - y| \leq \delta$ and all $v \in F^{-n}(x)$, where

$$\phi_n(z) = \phi(z)\phi(F(z)) \cdots \phi(F^{n-1}(z)). \tag{39}$$

We say that a continuous function $\phi : J_F \to \mathbb{C}$ is summable if

$$\sup_{z\in J_F}\Big\{\sum_{v\in F^{-1}(z)}||\phi\circ F_v^{-1}||_\infty\Big\}< \infty.$$

Next define

$$H_\alpha^s=\{\phi : J_F\to\mathbb{C} : \phi \text{ is a Hölder continuous summable function}\}.$$

If the function $\phi\in H_\alpha^s$ then the formula

$$\mathcal{L}_\phi g(z)=\sum_{x\in F^{-1}(z)}\phi(x)g(x)$$

defines a bounded operator $\mathcal{L}_\phi : CB(J_F,\mathbb{C})\to CB(J_F,\mathbb{C})$ called the Perron-Frobenius operator associated with the function(potential) ϕ.

Lemma (Ionescu-Tulcea and Marinescu inequality)

Let $\phi : J_F \to \mathbb{C}$ be a summable dynamically Hölder potential with an exponent $\alpha > 0$. Then

$$\mathcal{L}_\phi(H_\alpha)\subset H_\alpha.$$

Moreover, if $\phi(J_F))\subset[0,\infty)$ and $\sup_{n\geq 1}\{||\mathcal{L}_\phi^n(\mathbb{1})||_\infty\}<\infty$ then there exists a constant $c_1>0$ such that

$$||\mathcal{L}_\phi^n g||_\alpha\leq\frac{1}{2}||g||_\alpha+c_1||g||_\infty$$

for all n large enough and every $g\in H_\alpha$.

Proof. Fix $n \geq 1$, $g \in \mathcal{H}_\alpha$ and $x, y \in J_F$ with $|x-y| \leq \delta$. Then we can write

$$
\begin{aligned}
&|\mathcal{L}_\phi^n g(y) - \mathcal{L}_\phi^n g(x)| \\
&= |\textstyle\sum_{v \in F^{-n}(x)} \phi_n(F_v^{-n}(y)) g(F_v^{-n}(y)) - \sum_{v \in F^{-n}(x)} \phi_n(F_v^{-n}(x)) g(F_v^{-n}(x))| \\
&= |\textstyle\sum_{v \in F^{-n}(x)} \phi_n(F_v^{-n}(x))(g(F_v^{-n}(y)) - g(F_v^{-n}(x))) \\
&\quad + \textstyle\sum_{v \in F^{-n}(x)} g(F_v^{-n}(y))(\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n})(x)))| \\
&\leq \textstyle\sum_{v \in F^{-n}(x)} |\phi_n(F_v^{-n}(x))||g(F_v^{-n}(y)) - g(F_v^{-n}(x))| \\
&\quad + \textstyle\sum_{v \in F^{-n}(x)} |g(F_v^{-n})(y)||\phi_n(F_v^{-n})(y) - \phi_n(F_v^{-n}(x))| \\
&\leq \textstyle\sum_{v \in F^{-n}(x)} |\phi_n(F_v^{-n}(x))||v_\alpha(g)||F_v^{-n}(y) - F_v^{-n}(x)|^\alpha \\
&\quad + \textstyle\sum_{v \in F^{-n}(x)} \|g\|_\infty c_\phi \sum_{v \in F^{-n}(x)} |\phi_n(F_v^{-n}(x))||x-y|^\alpha \\
&\leq c_\phi \|g\|_\infty \mathcal{L}_{|\phi|}^n(1\!\!1)(x)|y-x|^\alpha + v_\alpha(g)(L\beta^n)^\alpha |y-x|^\alpha \textstyle\sum_{v \in F^{-n}(x)} |\phi_n(F_v^{-n})(x)| \\
&\leq (\mathcal{L}_{|\phi|}^n(1\!\!1))(c_\phi \|g\|_\infty + L^\alpha \beta^{n\alpha} v_\alpha(g))|y-x|^\alpha.
\end{aligned}
$$

This shows that

$$v_\alpha(\mathcal{L}_\phi^n g) \leq (\mathcal{L}_{|\phi|}^n(1\!\!1))(c_\phi \|g\|_\infty + L^\alpha \beta^{\alpha n} \|g\|_\infty) < \infty$$

i.e. $\mathcal{L}_\phi^n g \in H_\alpha$ or equivalently $\mathcal{L}_\phi(H_\alpha) \subset H_\alpha$.

If $\phi(J_F) \subset [0, \infty)$ and $Q_\phi = \sup_{n \geq 1}\{\|\mathcal{L}_\phi^n(1\!\!1)\|_\infty\} < \infty$ then

$$\|\mathcal{L}_\phi^n g\|_\alpha = v_\alpha(\mathcal{L}_\phi^n g) + \|\mathcal{L}_\phi^n g\|_\infty \leq Q_\phi L^\alpha \beta^{\alpha n} \|g\|_\alpha + (Q_\phi c_\phi + Q_\phi)\|g\|_\infty.$$

Taking now $n \geq 1$ large enough so that $Q_\phi L^\alpha \beta^{\alpha n} \leq \frac{1}{2}$, with $c_1 = Q_\phi c_\phi + Q_\phi$ we are done. $\square$

Definition(Urbański, Zdunik)

We say (see also [30]) that a summable dynamically Hölder potential $\phi : J_F \to (0, \infty)$ satisfies the $Q-$ condition if

$$Q_\phi = \sup_{n \geq 1}\{||\mathcal{L}_\phi^n(1\!\!1)||_\infty\} < \infty.$$

We say that ϕ is rapidly decreasing if

$$\lim_{|\mathrm{Re}\ z| \to \infty} \mathcal{L}(1\!\!1)(z) = 0.$$

Lemma

Let $\phi : J_F \to (0, \infty)$ be a rapidly decreasing summable dynamically Hölder potential satisfying the $Q-$condition. If B is a bounded subset of $(H_\alpha, ||\cdot||_\alpha)$ then $\mathcal{L}_\phi(B)$ is a pre-compact subset of the space

$$(\mathrm{CB}(J_F, \mathbb{C}), ||\cdot||_\infty).$$

Proof. Fix an arbitrary sequence $\{g_n\}_{n=1}^{\infty} \subset B$. By Ionescu-Tulcea and Marinescu inequality the family $\mathcal{L}_\phi(B)$ is equicontinuous. Since the operator $\mathcal{L}_\phi$ is bounded, the family $\mathcal{L}_\phi(B)$ is bounded. So, from Ascoli's theorem it follows that we can choose from $\{\mathcal{L}_\phi(g_n)\}_{n\geq 1}$ an infinite subsequence $\{\mathcal{L}_\phi(g_{n_k})\}_{k\geq 1}$ converging uniformly on compact subsets of J_F to a function $\psi \in \mathrm{CB}(J_F, \mathbb{C})$. Fix $\varepsilon > 0$. Since B is a bounded subset of $\mathrm{CB}(J_F, \mathbb{C})$ it follows (because ϕ is rapidly decreasing) that there exists $M > 0$ such that

$$|\mathcal{L}_\phi g(z)| \leq \frac{\varepsilon}{2}$$

for all $g \in B$ and $z \in J_F$ with $|\mathrm{Re}\, z| \geq M$. Hence $|\psi(z)| \leq \frac{\varepsilon}{2}$ for every $z \in J_F$ with $|\mathrm{Re}\, z| \geq M$. As a consequence we get

$$|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \leq \varepsilon$$

for every $j \geq 1$ and every $z \in J_F$ with $|\mathrm{Re}\, z| \geq M$. In addition, by uniform convergence on compact sets, there exists $p \geq 1$ such that

$$|\mathcal{L}_\phi(g_{n_j})(z) - \psi(z)| \leq \varepsilon$$

for every $j \geq p$ and every $z \in J_F$ such that $|\mathrm{Re}\, z| \leq M$. Therefore

$$|\mathcal{L}_\phi(g_{n_j}) - \psi| \leq \varepsilon$$

for every $j \geq p$ and every $z \in J_F$ i.e. $||\mathcal{L}_\phi(g_{n_j}) - \psi||_\infty \leq \varepsilon$ for every $j \geq p$. Letting now $\varepsilon \searrow 0$ we conclude that $\mathcal{L}_\phi(g_{n_j})$ converges uniformly on J_F to $\psi \in \mathrm{CB}(J_F, \mathbb{C})$. $\square$

Ionescu-Tulcea and Marinescu theorem

If the assumptions of the previous lemma are satisfied then there exist finite numbers $\gamma_1, \gamma_2, \cdots, \gamma_p \in S_1 = \{z \in \mathbb{C} : |z| = 1\}$, *finitely many bounded finitely dimensional operators* $Q_1, Q_2, \cdots, Q_p : H_\alpha \to H_\alpha$ *and an operator* $S : H_\alpha \to H_\alpha$ *such that*

$$\mathcal{L}_\phi^n = \sum_{i=1}^{p} \gamma_i^n Q_i + S^n$$

for all $n \geq 1$,

$$Q_i^2 = Q_i, Q_i \circ Q_j = 0, (i = j), Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \leq C\xi^n$$

for some constant $C > 0$, *some constant* $\xi \in (0, 1)$ *and all* $n \geq 1$. *In particular all numbers* $\gamma_1, \cdots, \gamma_p$ *are isolated eigenvalues of the operator* $\mathcal{L}_\phi : H_\alpha \to H_\alpha$ *and this operator is quasicompact.*

Proof. Combining Ionescu-Tulcea and Marinescu inequality with the results from the previous lemma we observe that the assumptions of [21, Theorem 1.5], are satisfied with Banach spaces H_α and the bounded operator $\mathcal{L}_\phi : CB \to CB$. Also compare [30, Theorem 4.3]. □

3.3. Bowen's Formula for the Family $\mathcal{H}$

In this last section we define the topological pressure and we study Perron-Frobenius operators for our type of potentials. We establish more properties and we prove Corollary 3.3, the main result of this chapter.

Topological Pressure

Fix $a \in \mathcal{H}$ and denote $f = f_a$, $F = F_a$. For every $t \geq 0$ and for every $z \in J_F$ define the lower and the upper topological pressure respectively by

$$\underline{P}_z(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} = \liminf_{n\to\infty} \frac{1}{n} \log P_z(n, t),$$

$$\overline{P}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} = \limsup_{n\to\infty} \frac{1}{n} \log P_z(n,t),$$

where $P_z(n,t) = \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t}$.

Lemma

$\underline{P}_z(t)$ and $\overline{P}_z(t)$ do not depend on the choice of $z \in J_F$.

Proof. Let $r > 0$ and $z \in J_F$ such that

$$B(z, 2r) \cap \pi(P_f) \neq 0. \tag{40}$$

First we will prove that for $y \in B(z, r)$

$$\underline{P}_z(t) = \underline{P}_y(t) \text{ and } \overline{P}_z(t) = \overline{P}_y(t). \tag{41}$$

For every $x \in F^{-n}(z)$ denote by $F^{-n}_{z,x}$ the branch of F^{-n} such that $F^{-n}_{z,x}(z) = x$. Note (compare (1)) that

$$\frac{1}{K^t} \sum_{x\in F^{-n}(z)} |(F^{-n}_{z,x})'(z)|^t \leq \sum_{x\in F^{-n}(z)} |(F^{-n}_{z,x})'(y)|^t$$
$$\leq K^t \sum_{x\in F^{-n}(z)} |(F^{-n}_{z,x})'(z)|^t.$$

Hence

$$\frac{1}{K^t} P_z(n,t) \leq P_y(n,t) \leq K^t P_z(n,t).$$

Thus (41) follows. Since the set $\bigcup_{n\in\mathbb{N}} f^n(Critf)$ has only finite number of accumulation points, any two points from J_F can be joined by a chain of balls satisfying (40). Therefore, the lemma follows. $\square$

From the previous lemma it follows that we can denote $\overline{P}(t) = \overline{P}_z(t)$ and $\underline{P}(t) = \underline{P}_z(t)$.

Lemma

Let as before $a \in \mathbb{C}^{n+1}$, $f = f_a \in \mathcal{H}$, $F = F_a$ and let $t > 1$. Then :

(i) $||P_z(1,t)||_\infty = \sup_{z \in J_F} |P_z(1,t)| < \infty$

(ii) $|P_z(1,t)| \to 0$ *as* $|Re\ z| \to \infty$

(iii) $|P_z(n,t)| \leq |P_z(1,t)|^n$

(iv) $\overline{P}(t) \leq \log ||P(1,t)||_\infty$.

Proof. Observe first that there exists $M_4 > nM_2$ such that, for $M \geq M_4$, if $|f(z)| > M$, then $|\mathrm{Re}\ z| > N_M > M_3$. Moreover,

$$N_M \geq \log \frac{M}{nM_2}.$$

It follows that

$$N_M \to \infty \text{ as } M \to \infty.$$

Then from Lemma 2.1 if $|\mathrm{Re}\ z| > M \geq M_4$,

$$P_z(1,t) \leq n\frac{M_2}{M_1}\sum_{j\in\mathbb{Z}}\left(\frac{1}{\sqrt{M^2 + (\mathrm{Im}\ (z) + 2\pi j)^2}}\right)^t. \tag{42}$$

Since $t > 1$, there exists a constant $C_1(F) > 0$ such that

$$P_z(1,t) \leq C_1(F)\frac{1}{M^{t-1}}.$$

In particular (ii) follows from this inequality.

So, in order to finish the proof of (i) it is enough to consider points $z \in J_F$ with $|\mathrm{Re}\ z| \leq M_4$. First note that there exists a constant $C(F) > 0$ such that for all $z \in J_F$

$$\sum_{x \in F^{-1}(z)\cap A'(M)} |F'(x)|^{-t} \leq C(F)$$

where $A'(M)$ is the set of points z such that $f(z)$ belongs to the square with the center at 0 and the length of the side equal to $2M_4$. Hence using also the argument leading to (42) we can write

$$P_z(1,t) \leq C(F) + C_2(F)\frac{1}{M_4^{t-1}}.$$

with a constant $C_2(F)$ independent of z and this finishes the proof of (i).

By a straight forward inductive argument the proof of (iii) is a consequence of the following:

$$\begin{aligned} P_z(n,t) &= \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t} \\ &= \sum_{y\in F^{-(n-1)}(z)} |(F^{n-1})'(y)| \sum_{x\in F^{-1}(y)} |F'(x)|^{-t} \\ &\leq ||P(1,t)||_\infty P_z(n-1,t). \end{aligned}$$

Now (iv) follows directly from (iii). □

Lemma

For $t>1$ both functions $t\mapsto \underline{P}(t), \overline{P}(t)$ are convex, continuous, strictly decreasing and $\lim_{t\to\infty}\overline{P}(t)=-\infty$.

Proof. We prove convexity. Let $\lambda\in(0,1)$. From Hölder inequality it follows that

$$\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-\lambda t_1}|(F^n)'(x)|^{-(1-\lambda)t_2}$$
$$\leq (\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t_1})^{\lambda}(\sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t_2})^{1-\lambda}.$$

Therefore,

$$P_z(n,\lambda t_1+(1-\lambda)t_2)\leq (P_z(n,t_1))^{\lambda}(P_z(n,t_2))^{1-\lambda},$$

$$\log P_z(n,\lambda t_1+(1-\lambda)t_2)\leq \lambda\log P_z(n,t_1)+(1-\lambda)\log P_z(n,t_2).$$

Hence, $\underline{P}(t), \overline{P}(t)$ are convex. Since continuity is a consequence of convexity, the other properties follow from the definition of the pressure and from the property that f is expanding on its Julia set, fact proved in Proposition 2.2. The proof is finished.

□

As in section 3.2 we let $CB=CB(J_F,\mathbb{C})$ be the Banach space of all bounded continuous complex-valued functions on J_F. We observed that the Perron-Frobenius operator $\mathcal{L}:CB\to$

CB, given by the formula

$$\mathcal{L}_t g(z) = \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} g(x)$$

is well defined. Its dual operator $\mathcal{L}_t^* : CB^* \to CB^*$ is given by the formula

$$\mathcal{L}_t^* \mu(g) = \mu(\mathcal{L}_t g).$$

Note that

$$\mathcal{L}_t^n(\mathbb{1})(z) = P_z(n, t),$$

where $\mathbb{1}(z) = 1$ for all z. Define also

$$\hat{\mathcal{L}}_t = \alpha_t^{-1} \mathcal{L}_t$$

and denote its dual operator by $\hat{\mathcal{L}}_t^*$. In the thermodynamic formalism of compact repellers, the conformal measure is a fixed point of $\hat{\mathcal{L}}_t^*$. This is also true for our operator (see also [10]).

Proposition

For every $t > 1$,

$$\mathcal{L}_t^* m_t = \alpha_t m_t,$$

or equivalently

$$\hat{\mathcal{L}}_t^* m_t = m_t.$$

Proof. First let $\{X_i\}_{i \in I}$ be a countable measurable partition of J_F such that $F|_{X_i}$ is measurable homeomorphism. Then, for measurable $B \subset X_i$,

$$m(F(B)) = \int_B \alpha_t |F'|^t dm_t.$$

Therefore, for measurable $B \subset F(X_i)$,

$$\int_{F(X_i)} \mathbb{1}_B dm_t = m_t(F((F|_{X_i})^{-1}(B))) = \int_{(F|_{X_i})^{-1}(B)} \alpha_t |F'|^t dm_t =$$

$$\int_{X_i} \mathbb{1}_{(F|_{X_i})^{-1}(B)} \alpha_t |F'|^t dm_t = \alpha_t \int_{X_i} (\mathbb{1}_B \circ F) \cdot |F'|^t dm_t.$$

Hence, for any $g \in L^1(m_t)$,

$$\int_{F(X_i)} g dm_t = \alpha_t \int_{X_i} (g \circ F) |F'|^t dm_t.$$

Since $|F'|$ is bounded away from zero, it follows that, for $h \in L^1(m_t)$,

$$(h \circ ((F|_{X_i})^{-1})) \cdot |F' \circ (F|_{X_i})^{-1}|^{-t} \in L^1(m_t).$$

Then

$$\int_{F(X_i)} (h \circ ((F|_{X_i})^{-1})) \cdot |F' \circ (F|_{X_i})^{-1}|^{-t} dm_t = \alpha_t \int_{X_i} h dm_t.$$

Therehore,

$$\int_{J_F} \mathcal{L}_t(h) dm_t = \int_{J_F} \sum_{x \in F^{-1}(z)} |F'(x)|^{-t} h(x) dm_t(z) =$$

$$\int_{J_F} \sum_{i \in I} \mathbb{1}_{F(X_i)}(z) |F'((F|_{X_i})^{-1}(z))|^{-t} h((F|_{X_i})^{-1}(z)) dm_t(z) =$$

$$\sum_{i \in I} \int_{F(X_i)} |F'((F|_{X_i})^{-1})|^{-t} h((F|_{X_i})^{-1}) dm_t =$$

$$\alpha_t \sum_{i \in I} \int_{X_i} h dm_t = \alpha_t \int_{J_F} h dm_t.$$

Hence,

$$\mathcal{L}_t^* m_t = \alpha_t m_t.$$

□

Observe now that if we fix any two points $x, y \in J_F$ then there exists a chain (simply connected) of balls of radius less than δ, joining x and y in $P \backslash \pi(\mathcal{P}_f)$. Moreover, there exists a constant such that, if $\mathrm{Re}\, x, \mathrm{Re}\, y \leq M$, then the numbers of balls are less than the constant. Then there exists a constant K_M such that for $x, y \in J_F$ with the property that

$\mathrm{Re}\ x, \mathrm{Re}\ y \le M$ and for a branch F_*^{-n} of F^{-n} defined on the union of the balls in the chain joining x and y

$$(43) \qquad \frac{|(F_*^{-n})'(x)|}{|(F_*^{-n})'(y)|} \le K_M.$$

Then, using the fact that

$$\mathcal{L}_t^n(\mathbb{1})(x) = P_x(n,t) = \sum_{u \in F^{-n}(x)} |(F^n)'(u)|^{-t},$$

we get

$$(44) \qquad K_M^{-t} \le \frac{\hat{\mathcal{L}}_t^n(\mathbb{1})(x)}{\hat{\mathcal{L}}_t^n(\mathbb{1})(y)} \le K_M^t.$$

Lemma

$$\sup_{n \ge 0}\{||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty\} < \infty.$$

Proof. We proceed by induction. From Lemma 3.3 (ii) it follows that, there exists $M > 0$ such that

$$(45) \qquad \sup\{\hat{\mathcal{L}}_t(\mathbb{1})(z) : |\mathrm{Re}\ z| > M\} < 1.$$

Let $D(M) = \{z \in J_F : |\mathrm{Re}\ z| \le M\}$. Then, by (5.5), for $z \in D(M)$,

$$\hat{\mathcal{L}}_t(\mathbb{1})(z) \le K_M^t \le \frac{K_M^t}{m_t(D(M))}.$$

Since $K_M^t \ge 1$, it follows from (45) that this inequality is also true for every $z \in J_F$. Now, suppose that

$$(46) \qquad ||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty \le \frac{K_M^t}{m_t(D(M))}.$$

We will now prove this inequality for $n+1$. Using Lemma 3.3 (iii) we get that

$$\hat{\mathcal{L}}_t^n(\mathbb{1})(z) \le \alpha_t^{-n}|P_z(1,t)|^n,$$

and, then, we can find, for $n \in \mathbb{N}$, such a point $z \in J_F$ that

$$||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty = \hat{\mathcal{L}}_t^{n+1}(\mathbb{1})(z).$$

If $z \notin D(M)$, then by (46) and (45)

$$\begin{aligned}
||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty &= \hat{\mathcal{L}}_t^{n+1}(\mathbb{1})(z) \\
&= \hat{\mathcal{L}}_t(\hat{\mathcal{L}}_t^{n}(\mathbb{1}))(z) \\
&= \alpha_t^{-1} \textstyle\sum_{x \in F^{-1}(z)} |F'(x)|^{-t} \hat{\mathcal{L}}_t^{n}(\mathbb{1})(x) \\
&\leq \alpha_t^{-1} \textstyle\sum_{x \in F^{-1}(z)} |F'(x)|^{-t} ||\hat{\mathcal{L}}_t^{n}(\mathbb{1})||_\infty \\
&\leq \hat{\mathcal{L}}_t(\mathbb{1})(z) \tfrac{K_M^t}{m_t(D(M))} \\
&\leq \tfrac{K_M^t}{m_t(D(M))}.
\end{aligned}$$

Otherwise, if $z \in D(M)$,

$$1 = \int \hat{\mathcal{L}}_t^{n+1}(\mathbb{1}) dm_t \geq \int_{D(M)} \hat{\mathcal{L}}_t^{n+1}(\mathbb{1}) dm_t \geq \frac{m_t(D(M))}{K_M^t} ||\hat{\mathcal{L}}_t^{n+1}(\mathbb{1})||_\infty.$$

Hence, by induction, the inequality (46) is true for $n \geq 1$ and the lemma follows. $\square$

Lemma

For every $\varepsilon > 0$ there exists $M > 0$ such that

$$\inf_{n \geq 0} \sup\{\hat{\mathcal{L}}_t^{n}(\mathbb{1})(z) : |Re\ z| \leq M\} \geq 1 - \varepsilon.$$

Therefore, there exists M such that

$$\inf_{n \geq 0} \inf\{\hat{\mathcal{L}}_t^{n}(\mathbb{1})(z) : |Re\ z| \leq M\} \geq \frac{1}{4K_M^t}.$$

Proof. Suppose that there exists $\varepsilon > 0$ such that for every $M > 0$,

$$\inf_{n \geq 0} \sup\{\hat{\mathcal{L}}_t^{n}(\mathbb{1})(n) : |\mathrm{Re}\ z| \leq M\} < 1 - \varepsilon.$$

Let $Q = \sup_{n\geq 1} ||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty$. From Lemma 3.3 $Q < \infty$. Let M be so large that

$$m_t(J_F \setminus D(M)) \leq \frac{\varepsilon}{2Q}$$

Then

$$\begin{aligned} 1 &= \int \hat{\mathcal{L}}_t^n(\mathbb{1}) dm_t \\ &= \int_{D(M)} \hat{\mathcal{L}}_t^n(\mathbb{1}) dm_t + \int_{J_F \setminus D(M)} \hat{\mathcal{L}}_t^n(\mathbb{1}) dm_t \\ &\leq (1-\varepsilon) m_t(D(M)) + Q m_t(J_F \setminus D(M)) \\ &\leq 1 - \varepsilon + Q\tfrac{\varepsilon}{2Q} = 1 - \tfrac{\varepsilon}{2}. \end{aligned}$$

This is a contradiction. The second inequality follows directly from (44). □

Proposition

For every $t > 1$,

$$P(t) = \overline{P}(t) = \underline{P}(t) = \log \alpha_t.$$

Proof. Let $Q = \sup_{n\geq 0}\{||\hat{\mathcal{L}}_t^n(\mathbb{1})||_\infty\}$. By Lemma 3.3 we have that Q is finite and consequently

$$P_z(n,t) = \mathcal{L}_t^n(\mathbb{1})(z) \leq Q\alpha_t^n.$$

Therefore,

$$\overline{P}(t) = \limsup_{n\to\infty} \frac{1}{n} \log P_z(n,t) \leq \log \alpha_t.$$

It follows from Lemma 3.3 that, if $z \in D(M)$, then

$$P_z(n,t) = \mathcal{L}_t^n(\mathbb{1})(z) \geq \frac{\alpha_t^n}{4K_M^t}.$$

Hence,

$$\underline{P}(t) = \liminf_{n\to\infty} \frac{1}{n} \log P_z(n,t) \geq \log \alpha_t.$$

□

Corollary

$HD(J_f^r) = h < 2$ is the unique zero of the function $t \mapsto P(t)$, $t > 1$.

Proof. The fact that $HD(J_f^r) = h$ is the unique zero of the function $t \mapsto P(t)$ $(t > 1)$ follows from the equality

$$\mathrm{HD}(J_f^r) = \mathrm{HD}(J_F^r)$$

and from Theorem 3.1, Corollary 3.1 and Proposition 3.3. So, it remains to prove that $h < 2$. By way of contradiction assume that there exists a $(2,1)$-conformal measure m. Then, as in the proof of Proposition 3.1, we obtain that m is absolutely continuous with respect to the Lebesgue measure leb. Therefore, by Lemma 3.1, $\mathrm{leb}(J_F^r(M)) > 0$. But this is impossible since almost every point in J_F escape to infinity (see for example [6]). $\square$

For the last part of this section our objective is to show Theorem 3.3. We establish first two lemmas and then we state and prove this theorem.

Lemma

(i) *The function $\phi = -t \log |F_a'|$ is 1-Hölder (Lipshitz).*

(ii) *The function $\phi_t(z) = |F_a'(z)|^{-t}$ is 1-Hölder.*

Proof. Observe that $\phi_t(z) = e^{\phi}$. To prove (i) we use Koebe's distortion theorem. Let $|z - w| < \delta$ and $|z - w| = \eta 2\delta$. Then

$$\frac{(1-\eta)^3}{(1+\eta)^3} \le \frac{|F_a'(z)|}{|F_a'(w)|} \le \frac{(1+\eta)^3}{(1-\eta)^3}.$$

Therefore

$$\big| -t\log|F_a'(z)| + t\log|F_a'(w)|\big| = |t| \cdot |\log \frac{|F_a'(z)|}{|F_a'(w)|}| \le$$
$$|t|(3\log(1+\eta) - 3\log(1-\eta)) \le |t|\frac{9}{2\delta}|z-w|$$

since, for $\eta \in (0, 1/2)$, $\log(1+\eta) \le \eta$ and $\log(1-\eta) < 2\eta$. This finishes the proof of part (i).

To get (ii), first observe that if $\mathrm{Re}\ t \geq 0$

$$\left||F_a'(z)|^{-t}\right| \leq \frac{1}{\Delta_a^{\mathrm{Re}\ t}},$$

where $\Delta_a = \min(1/2, \inf\{|f_a'(z)| : \mathrm{dist}(z, Crit(f_a)) > \delta\})$. There exists $M_t > 0$ such that, if $|x| \leq \log \frac{1}{\Delta_a^{\mathrm{Re}\ t}}$, then

$$|e^x - 1| \leq M_t |x|.$$

Therefore

$$\left||F_a'(z)|^{-t} - |F_a'(w)|^{-t}\right| = |e^{-t\log|F_a'(z)|} - e^{-t\log|F_a'(w)|}| =$$

$$\left|e^{-t\log|F_a'(w)|}\right| \cdot \left|e^{-t\log|F_a'(z)|+t\log|F_a'(w)|} - 1\right| \leq \frac{1}{\Delta_a^{\mathrm{Re}\ t}} M_t \frac{9}{2\delta}|z-w|.$$

□

Next, consider again the operator

$$\mathcal{L}_{\phi_t} g(z) = \sum_{x \in F^{-1}(z)} |F'(z)|^{-t} g(z)$$

and recall that by $\hat{\mathcal{L}}_t$ we denoted the operator $\hat{\mathcal{L}}_t = \alpha_t^{-1}\mathcal{L}_{\phi_t}$, where ϕ_t is the function from the previous Lemma. We proved before that $\alpha_t = e^{P(t)}$.

Lemma

If $\mathrm{Re}\ t > 1$ then $\phi := e^{-P(t)}\phi_t(z) = e^{-P(t)}|(F_a)'(z)|^{-t}$ is a rapidly decreasing summable dynamically Hölder function satisfying the $Q-$condition. Observe that $\mathcal{L}_\phi = \hat{\mathcal{L}}_t$.

Proof. Since $|(F_a^n)'(z)| = |F_a'(z)| \cdot |F_a'(F_a(z))| \cdots |(F_a^{n-1})'(z))|$ then $\phi_n(z)$ for the potential $|(F_a)'(z)|^{-t}$ is equal to $|(F_a^n)'(z)|^{-t}$. Therefore

$$\phi_n(F_v^{-n}(y)) = |(F_a^{-n})'(F_v^{-n}(y))|^{-t} = |(F_v^{-n})'(y)|^t.$$

Using the same argument as in the proof of Lemma 3.3 we have

$$|-t\log|(F_v^{-n})'(x)| + t\log|(F_v^{-n})'(y)|| \leq |t|\frac{9}{2\delta}|y-x|.$$

Then

$$\begin{aligned}
&||(F_v^{-n})'(x)|^t - |(F_v^{-n})'(y)|^t| \\
&\quad = |e^{t\log|(F_v^{-n})'(y)|}| \cdot |e^{t\log|(F_v^{-n})'(x)| - t\log|(F_v^{-n})'(y)|} - 1| \\
&\quad \leq |(F_v^{-n})'(y)|^{\mathrm{Re}\ t} M_t' |x-y|
\end{aligned}$$

for some constant M_t'. Observe that from the estimation above and the proof of the Lemma 3.3, $\phi_t(z)$ is rapidly decreasing summable dynamically Hölder potential satisfying the $Q-$condition. This means that all the assumptions of Ionescu-Tulcea and Marinescu theorem are satisfied. □

We obtain then the following important result.

Theorem

If $t > 1$ then 1 is an isolated simpe eigenvalue of $\hat{\mathcal{L}}_t : H_\alpha \to H_\alpha$ and the eigenspace of the eigenvalue 1 is generated by the nowhere vanishing function $\psi_t \in H_\alpha$ such that

$$\int \psi_t dm_t = 1$$

and

$$\lim_{|Re\ z| \to \infty} \psi_t(z) = 0.$$

Moreover, if $t > 1$ then the measure $\mu = \mu_t = \psi_t m_t$ is F-invariant, ergodic and equivalent to m_t. In particular $\mu(J_F^r) = 1$.

Proof. We have first that

$$\|\hat{\mathcal{L}}_t^n(1\!\!1)\|_\alpha \leq C_1$$

for some $C_1 > 0$ and every $n \geq 0$. Therefore

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \hat{\mathcal{L}}_t^k(1\!\!1) \right\|_\alpha = \left\| \hat{\mathcal{L}}_t \left(\frac{1}{n} \sum_{k=1}^{n-1} \hat{\mathcal{L}}_t^k(1\!\!1) \right) \right\|_\alpha \leq C_1 \tag{47}$$

for every $n \geq 1$. Then it follows from section 3.3 that there exists a strictly increasing sequence of positive integers $\{n_j\}_{j\geq 1}$ such that the sequence

$$\left\{\frac{1}{n_j}\sum_{k=1}^{n_j}\hat{\mathcal{L}}_t^k(\mathbb{1})\right\}_{j\geq 1}$$

converges in $\mathrm{CB}(J_F, \mathbb{C})$ to a function $\psi_t : J_F \to \mathbb{R}$. Then, since by (47) $\|\psi_t\|_\alpha \leq C_1$, we get that $\psi_t \in H_\alpha$. Moreover, by Remark 3.3 m_t is a fixed point of the dual operator $\hat{\mathcal{L}}_t^*$. Therefore, for every $j \geq 0$,

$$\int \hat{\mathcal{L}}_t^j(\mathbb{1})dm_t = 1,$$

and consequently

$$\int \frac{1}{n}\sum_{j=0}^{n-1}\hat{\mathcal{L}}_t^j(\mathbb{1})dm_t = 1.$$

Applying Lebesgue's dominated convergence theorem together with the fact that the function ϕ from the previous lemma has the $Q-$ property, we obtain the equality

$$\int \psi_t dm_t = 1.$$

Moreover, it follows then from Lemma 3.3 that $\psi_t > 0$ throughout J_F. Since $\psi_t = \hat{\mathcal{L}}_t\psi_t$ and since $\mathcal{L}_t g(z) \leq \|\mathcal{L}_t\mathbb{1}\|_\infty\|g\|_\infty$ and $\lim_{|\mathrm{Re}\ z|\to\infty}\mathcal{L}_t\mathbb{1}(z) = 0$ it follows that

$$\lim_{|\mathrm{Re}\ z|\to\infty}\psi_t(z) = 0.$$

The fact that 1 is isolated eigenvalue of $\hat{\mathcal{L}}_t$ follows from Ionescu-Tulcea and Marinescu theorem and the last statement of the Theorem follows immediately from above.

Therefore it remains to prove that the isolated eigenvalue 1 is simple i.e. the eigenspace of 1 is one dimensional. Let $g = g_1 + ig_2 \in H_\alpha$, where $g_1, g_2 \in H_\alpha$ are real-valued be such that

$$\hat{\mathcal{L}}_t(g) = g.$$

Since

$$g_1 + ig_2 = g = \hat{\mathcal{L}}_t(g_1 + ig_2) = \hat{\mathcal{L}}_t(g_1) + i\hat{\mathcal{L}}_t(g_2),$$

$$\hat{\mathcal{L}}_t(g_l) = g_l$$

for $l = 1, 2$. We shall prove that both g_l are equal to $\lambda_l \psi_t$ for some $\lambda_l \in \mathbb{R}$. So let us assume that $g_l \neq \lambda \psi_t$ for all $\lambda \in \mathbb{R}$. Since

$$0 \leq \mathbb{1} - \frac{g_l}{||g_l||_\infty},$$

$$0 \leq \frac{1}{n_j} \sum_{k=1}^{n_j} \hat{\mathcal{L}}_t^k \left(\mathbb{1} - \frac{g_l}{||g_l||_\infty} \right) = \frac{1}{n_j} \sum_{k=1}^{n_j} \hat{\mathcal{L}}_t^k(\mathbb{1}) - \frac{g_l}{||g_l||_\infty}.$$

Therefore the function

$$h = \frac{\psi_t - \frac{g}{||g||_\infty}}{\int (\psi_t - \frac{g}{||g||_\infty}) d m_t}$$

is a well-defined non-vanishing non-negative function which is a fixed point of $\hat{\mathcal{L}}_t$ and

$$\int h d m_t = 1.$$

But this gives us that $h = \psi_t$ since we know that m_t is ergodic. Then the required contradiction follows.

$\square$

3.4. Prochorov's Theorem and the Concept of Tightness

In this section we recall some basic properties regarding the concept of tightness and Prochrov's theorem and we refer the reader to the book of Billingsley, [5], for more detalies.

We recall that if S is an arbitrary metric space, B the class of Borel sets (i.e. the $\sigma-$field generated by the open sets) and if we consider probability measures P_n and P defined on B, then we say that P_n converge weakly to P if

$$P_n(A) \to P(A)$$

for every $A \in B$ such that $P(\partial A) = 0$.

Theorem

P_n converges weakly to P if and only if

$$\int_B f\,dP_n \to \int_B f\,dP$$

for every function f bounded, continuous real-valued defined on S.

Now, observe that every probability measure P on (S, B) is regular i.e. has the property that for every $A \in B$ and for every $\epsilon > 0$ there exists $F = \overline{F}$ and G open set such that $F \subset A \subset G$ and $P(G - F) < \epsilon$.

This implies that P is completely determined by the values of $P(F)$ for closed sets F.

Also observe that two probability measures P and Q on B coincide if

$$\int f\,dP = \int f\,dQ$$

for all $f : S \to \mathbb{R}$ bounded and uniformly continuous.

This last observation shows that P is also determined by the values of $\int f\,dP$ for bounded, continuous functions f.

Definition

A probability measure P on (S, B) is tight if for every $\epsilon > 0$ there exists a compact set K in S such that $P(K) > 1 - \epsilon$.

An observation with a more general character says that if S is a separable and complete metric space then each probability measure on (S, B) is tight.

Definition

Let π be a family of probability measures on (S, B). We call π relatively compact if every sequence of elements of π contains a weakly convergent subsequence.

We are concerned with the relative compactness of sequences but we formulate bellow the Prochrov's theorem in general.

Definition

The family π of probability measures on (S, B) is tight if for every $\epsilon > 0$ there exists K compact set in S such that

$$P(K) > 1 - \epsilon$$

for every $P \in \pi$.

Prochorov's Theorem

If π is tight then it is relatively compact.

The proof of this important theorem can be found in [5]. We mention that this is only the direct part of Prochorov's theorem but it is exactly what we needed for our sequence of semi-conformal measures. Both the direct and the converse theorems have hard proofs.

CHAPTER 4

HOLOMORPHIC PROPERTIES

4.1. Hartogs's Theorem and its Applications

In this section we prove Theorem 4.1, our main tool that we need in section 4.3 to prove the main result of this chapter. We show that (under some special conditions on the family $\{\Phi_b\}_{b\in G}$ of potentials) the function $b \mapsto \mathcal{L}_{\Phi_b}$ is holomorphic. In the proof we use the very well known result of Hartogs(see [39]). But first let us give several lemmas and definitions to make the proof more readable.

Lemma

Let G be a domain in $\mathbb{C}$ and $\{\Phi_b : J_{F_a} \to \mathbb{C}\}_{b\in G}$ be a family of continous summable potentials such that for every $z \in J_{F_a}$ the function $G \ni b \mapsto \Phi_b(z) \in \mathbb{C}$ is holomorphic and such that the map $G \ni b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$ is continous on G. Then the map $b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$ is holomorphic on G.

Proof. Let $\gamma \subset G$ be a simple closed curve homotopic to zero. Fix $g \in H_\alpha$ and $z \in J_F$. Let $W \subset G$ be a bounded open set such that $\gamma \subset W \subset \overline{W} \subset G$. Since for each $x \in F^{-1}(z)$ the function $b \mapsto g(x)\phi_b(x)$ is holomorphic on G and since for each $b \in W$ we have

$$\Big|\sum_{x\in F^{-1}(z)} g(x)\Phi_b(x)\Big| \le \|\mathcal{L}_{\Phi_b} g\|_\infty.$$

Then, since $\|\mathcal{L}_{\Phi_b} g\|_\infty \le \|\mathcal{L}_{\Phi_b} g\|_\alpha \le \|g\|_\alpha \sup\{\|\mathcal{L}_{\Phi_\theta}\|_\alpha : \theta \in \overline{W}\} < \infty$,

$$\Big|\sum_{x\in F^{-1}(z)} g(x)\Phi_b(x)\Big| < \infty.$$

and then by the compactness of $\overline{W}$ and continuity of the mapping $b \mapsto \mathcal{L}_{\Phi_b}$ we conclude that the function

$$W \ni b \mapsto \mathcal{L}_{\Phi_b} g(z) = \sum_{x \in F^{-1}(z)} \Phi_b(x) g(x) \in \mathbb{C}$$

is holomorphic. Hence, by Cauchy's theorem, $\int_\gamma \mathcal{L}_{\Phi_b} g(z) db = 0$. Since the function $b \mapsto \mathcal{L}_{\Phi_b} g \in H_\alpha$ is continous, the integral $\int_\gamma \mathcal{L}_{\Phi_b} g db$ exists, and for every $z \in J_F$ we have

$$\int_\gamma \mathcal{L}_{\Phi_b} g db(z) = \int_\gamma \mathcal{L}_{\Phi_b} g(z) db = 0.$$

Hence $\int_\gamma \mathcal{L}_{\Phi_b} g db = 0$.Now, since the function $b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$ is continous, the integral $\int_\gamma \mathcal{L}_{\Phi_b} db$ exists and for every $g \in H_\alpha$, we have

$$\int_\gamma \mathcal{L}_{\Phi_b} db(g) = \int_\gamma \mathcal{L}_{\Phi_b} g db = 0.$$

Thus, $\int_\gamma \mathcal{L}_{\Phi_b} db = 0$ and then by Morera's theorem, the function

$$b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$$

is holomorphic in G. □

Definition

Given $w \in J_F$ we define :

$$H_{\alpha,w} = \{g : B(w,\delta) \to \mathbb{C} : \text{g bounded for which there exists } C \geq 0$$

$$\text{such that if } x, y \in B(w,\delta) \text{ and } |y-x| \leq \delta \text{ then } |g(y) - g(x)| \leq C|y-x|^\alpha\}.$$

The $\alpha - \text{variation}\ v_\alpha(g)$ is the least C with the property above and we define $\|g\|_\alpha = v_\alpha(g) + \|g\|_\infty$. Observe that $(H_{\alpha,w}, \|.\|_\alpha)$ is a Banach space.

Lemma

(i) *If $v \in J_F$ and $\Phi \in H_\alpha$ then the operator $A_{v,\Phi} : H_\alpha \to H_{\alpha,F(v)}$ given by the formula*

$$A_{v,\Phi}g(z) = \Phi(F_v^{-1}(z))g(F_v^{-1}(z)), z \in B(F(v), \delta)$$

is continuous and

$$\|A_{v,\Phi}\|_\alpha \leq (2 + (L\beta)^\alpha)\|\Phi \circ F_v^{-1}\|_\alpha.$$

(ii) *If $\Phi : J_F \to \mathbb{C}$ is dynamically Hölder then for every $v \in J_F$ we have*

$$\|\Phi \circ F_v^{-1}\|_\alpha \leq (c_\Phi + 1)\|\Phi \circ F_v^{-1}\|_\infty.$$

(iii) *If $\Phi \in H_\alpha$ then for every $n \geq 1$ and every $v \in J_F$*

$$\|\Phi \circ F_v^{-n}\|_\alpha \leq (1 + L^\alpha \beta^{\alpha n})\|\Phi\|_\alpha$$

where $g \mapsto g \circ F_v^{-n} : B(F^n(v), \delta) \to \mathbb{C}$ is an operator from H_α to $H_{\alpha,F^n(v)}$.

(iv) *Let X be a metric space. If $\rho : X \to H_\alpha$ is a continuous mapping, then for every $v \in J_F$ the function*

$$X \ni x \mapsto A_{v,\rho(x)} \in L(H_\alpha, H_{\alpha,F^n(v)})$$

is continuous.

Proof. We start with (i). For every $g \in H_\alpha$ and $z \in B(F(v), \delta)$ we have

$$|A_{v,\Phi}g(z)| = |\Phi(F_v^{-1}(z))|.|g(F_v^{-1}(z))| \leq \|\phi \circ F_v^{-1}\|_\alpha.\|g\|_\alpha. \tag{48}$$

If in addition $w \in B(F(v), \delta)$ and $|w-z| \leq \delta$, then similarly as in the proof of Ionescu-Tulcea and Marinescu inequality we get

$$
\begin{aligned}
&|A_{v,\Phi}g(w) - A_{v,\Phi}g(z)| \leq |g(F_v^{-1}(w))||\Phi \circ F_v^{-1}(w) - \Phi \circ F_v^{-1}(z)| \\
&\ +|\Phi \circ F_v^{-1}(z)||g(F_v^{-1}(w)) - g(F_v^{-1}(z))| \\
&\leq \|g\|_\infty \|\Phi \circ F_v^{-1}\|_\alpha |w-z|^\alpha + \|\Phi \circ F_v^{-1}\|_\infty ||g||_\alpha L^\alpha \beta^\alpha |w-z|^\alpha \\
&\leq \|g\|_\alpha (1 + (L\beta)^\alpha) \|\Phi \circ F_v^{-1}\|_\alpha |w-z|^\alpha.
\end{aligned}
$$

Hence $v_\alpha(A_{v,\Phi}g) \leq (1 + (L\beta)^\alpha)\|\phi \circ F_v^{-1}\|_\alpha \|g\|_\alpha$ and combining this with (48) we obtain

$$
\|A_{v,\Phi}g\|_\alpha \leq (2 + (L\beta)^\alpha)\|\Phi \circ F_v^{-1}\|_\alpha \|g\|_\alpha.
$$

Thus $A_{v,\Phi}(H_\alpha) \subset H_{\alpha,F(v)}$, the operator $A_{v,\Phi} : H_\alpha \to H_{\alpha,F(v)}$ is continuous and $\|A_{v,\Phi}\|_\alpha \leq (2 + (L\beta)^\alpha)\|\Phi \circ F_v^{-1}\|_\alpha$.

Next (ii). We have that for $x, y \in B(F(v), \delta)$ with $|x - y| \leq \delta$

$$
|\Phi \circ F_v^{-1}(y) - \Phi \circ F_v^{-1}(x)| \leq c_\Phi |\Phi(F_v^{-1}(x))|.|y-x|^\alpha \leq c_\Phi \|\Phi \circ F_v^{-1}\|_\infty |y-x|^\alpha
$$

and therefore $v_\alpha(\Phi \circ F_v^{-1}) \leq c_\Phi \|\Phi \circ F_v^{-1}\|_\infty$. Thus we are done with (ii) and (iii) follows immediately.

To prove (iv) fix $x_0 \in X$, $\varepsilon > 0$ and take $\theta > 0$ small such that for every $x \in B(x_0, \theta)$and every $v \in J_F$

$$
\|\rho(x) - \rho(x_0)\|_\alpha \leq (2 + (L\beta)^\alpha)^{-2}\varepsilon
$$

Then applying Ionescu-Tulcea and Marinescu inequality and (iii) above we see that for every $x \in B(x_0, \theta)$ and every $v \in J_F$ we have

$$
\begin{aligned}
\|A_{v,\rho(x)} - A_{v,\rho(x_0)}\|_\alpha &= \|A_{v,\rho(x)-\rho(x_0)}\|_\alpha \\
&\leq (2 + (L\beta)^\alpha)\|(\rho(x) - \rho(x_0)) \circ F_v^{-1}\|_\alpha \\
&\leq (2 + (L\beta)^\alpha)(1 + (L\beta)^\alpha)\|\rho(x) - \rho(x_0)\|_\alpha \leq \varepsilon.
\end{aligned}
$$

The proof is complete. □

Theorem

Suppose that G is an open connected subset of $\mathbb{C}^n$, $n \geq 1$, and $\Phi_b : J_{F_a} \to \mathbb{C}$, $b \in G$, is a family of mappings such that

(i) *for every $b \in G$, $\Phi_b \in H_\alpha^s$,*

(ii) *for every $b \in G$, Φ_b is dynamically Hölder,*

(iii) *$G \ni b \mapsto \Phi_b \in H_\alpha$ is continuous,*

(iv) *family $\{c_{\Phi_b}\}_{b \in G}$ is bounded,*

(v) *the function $b \mapsto \Phi_b(z) \in \mathbb{C}$, $b \in G$ is holomorphic for every $z \in J_{F_a}$,*

(vi) *for every $d \in G$ there exists $r > 0$ and there exists $c \in G$ such that*

$$\sup\left\{\left|\frac{\Phi_b}{\Phi_c}(z)\right| : b \in \overline{B(d,r)}, z \in \mathbb{C}\right\} < \infty.$$

Then the function $b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$, $b \in G$, is holomorphic.

Proof. Due to Hartogs's theorem it is enough to prove the theorem for 1-dimensional case. So let's consider $n = 1$ and $G \subset \mathbb{C}$. Due to Lemma 4.1 it suffices to prove that the function $G \ni b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$ is continuous. Observe that in view of Lemma 4.1(i),(ii) and our assumption (iv) we have for every $v \in J_F$ and every $b \in \overline{B(d,r)}$ that

$$\|A_{v,\Phi_b}\|_\alpha \leq (2 + (L\beta)^\alpha)\|\Phi_b \circ F_v^{-1}\|_\alpha \leq M\|\Phi_b \circ F_v^{-1}\|_\infty$$

where $M = (2 + (L\beta)^\alpha)\sup\{c_{\Phi_b}, d \in G\} < \infty$. It follows that

$$\begin{aligned}
\|A_{v,\Phi_b}\|_\alpha \leq M\|\Phi_b \circ F_v^{-1}\|_\infty &= M\left\|\Phi_c \circ F_v^{-1}.\frac{\Phi_b \circ F_v^{-1}}{\Phi_c \circ F_v^{-1}}\right\|_\infty \\
&\leq M\|\Phi_c \circ F_v^{-1}\|_\infty.\left\|\frac{\Phi_b \circ F_v^{-1}}{\Phi_c \circ F_v^{-1}}\right\|_\infty \\
&\leq M\left\|\frac{\Phi_b}{\Phi_c}\right\|_\infty.\|\Phi_c \circ F_v^{-1}\|_\infty \leq MN\|\Phi_c \circ F_v^{-1}\|_\infty,
\end{aligned}$$

where N is the supremum taken from (vi). Then for every $z \in J_F$ we can define the operator

$$\mathcal{L}_{\Phi_b,z} : H_\alpha \to H_{\alpha,z}$$

by the formula

$$\mathcal{L}_{\Phi_b,z} = \sum_{v\in F^{-1}(z)} (\Phi_b \circ F_v^{-1}).(g\circ F_v^{-1}) = \sum_{v\in F^{-1}(z)} A_{v,\Phi_b}.$$

Observe that, $\mathcal{L}_{\Phi_b,z}(g) = \mathcal{L}_{\Phi_b}(g)|_{B(z,\delta)}$ for every $g \in H_\alpha$. Fix now $\varepsilon > 0$ and two elements $b, t \in B(d,r)$. Then there exist $g_\varepsilon \in B_{H_\alpha}(0,1)$ and two points $x, y \in J_F$ such that

$$\begin{aligned}
&\|\mathcal{L}_{\Phi_b} - \mathcal{L}_{\Phi_t}\|_\alpha = \sup\Big\{\|\mathcal{L}_{\Phi_b}(g) - \mathcal{L}_{\Phi_t}(g)\|_\alpha : g\in B_{H_\alpha}(0,1)\Big\}\\
&\le \|\mathcal{L}_{\Phi_b}(g_\varepsilon) - \mathcal{L}_{\Phi_t}(g_\varepsilon)\|_\alpha + \tfrac{\varepsilon}{5}\\
&= v_\alpha\Big(\mathcal{L}_{\Phi_b}(g_\varepsilon) - \mathcal{L}_{\Phi_t}(g_\varepsilon)\Big) + \tfrac{\varepsilon}{5} + \|\mathcal{L}_{\Phi_b}(g_\varepsilon) - \mathcal{L}_{\Phi_t}(g_\varepsilon)\|_\infty\\
&\le v_\alpha\Big(\mathcal{L}_{\Phi_b,x}(g_\varepsilon) - \mathcal{L}_{\Phi_t,x}(g_\varepsilon)\Big) + \tfrac{\varepsilon}{5} + \|\mathcal{L}_{\Phi_b,y}(g_\varepsilon) - \mathcal{L}_{\Phi_t,y}(g_\varepsilon)\|_\infty + \tfrac{\varepsilon}{5} + \tfrac{\varepsilon}{5}\\
&\le \|\mathcal{L}_{\Phi_b,x}(g_\varepsilon) - \mathcal{L}_{\Phi_t,x}(g_\varepsilon)\|_\alpha + \|\mathcal{L}_{\Phi_b,y}(g_\varepsilon) - \mathcal{L}_{\Phi_t,y}(g_\varepsilon)\|_\alpha + \tfrac{3\varepsilon}{5}\\
&\le 2\max_{w\in\{x,y\}} \|\mathcal{L}_{\Phi_b,w}(g_\varepsilon) - \mathcal{L}_{\Phi_t,w}(g_\varepsilon)\|_\alpha + \tfrac{3\varepsilon}{5} \le 2\|\mathcal{L}_{\Phi_b,w} - \mathcal{L}_{\Phi_b,w}\|_\alpha + \tfrac{3\varepsilon}{5},
\end{aligned}$$

where $w = x$ or $w = y$ as before. Since Φ_c is a summable function, there exists a finite set $V \subset F^{-1}(w)$ such that

$$\sum_{v\in F^{-1}(w)\setminus V} \|\Phi_c \circ F_v^{-1}\|\infty \le \frac{\varepsilon}{10MN}.$$

But the function $a \mapsto \Phi_a \in H_\alpha$ is continuous and therefore the function

$$a \mapsto A_{v,\Phi_a} \in L(H_\alpha, H_{\alpha,F(v)})$$

is also continuous. Thus there exists $r_1 \le r$ such that $\|A_{v,\Phi_b} - A_{v,\Phi_t}\|_\alpha \le \frac{\varepsilon}{10card(V)}$ for all $b, t \in B(d, r_1)$ and all $v \in V$. Splitting now the summation over the two sets V and $F^{-1}(w)\setminus V$ we get from the above computations that

$$\|\mathcal{L}_{\Phi_b} - \mathcal{L}_{\Phi_t}\|_\alpha \le \frac{3\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.$$

$\square$

4.2. Continuity of the Topological Pressure

In section 4.3 we shall prove the main result of this thesis. For a parameter $b \in \mathbb{C}^{n+1}$ and a map $f = f_b \in \mathcal{H}$ we shall show that the function $b \mapsto \mathrm{HD}(J^r_{f_b})$ is real-analytic. This ultimate goal is established in Theorem 4.3. But first we need to prove that, for $t > 1$, the function $a \mapsto P_a(t)$ is continuous on $\mathcal{H}$ and to obtain this result we need the following lemma.

Lemma

For every $a \in \mathcal{H}$ and for every $\epsilon > 1$, there exists $r > 0$ such that, for $b \in B(a, r)$ and for $z \in J_{f_a}$,

$$\left|\frac{f_b'(h_b(z))}{f_a'(z)} - 1\right| < \epsilon.$$

Proof. Write

$$\begin{aligned} \left|\frac{f_b'(h_b(z))}{f_a'(z)} - 1\right| &= \left|\frac{f_b'(h_b(z)) - f_a'(z)}{f_a'(z)}\right| \\ &\leq \left|\frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)}\right| + \left|\frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)}\right|. \end{aligned} \tag{49}$$

We split the proof in two cases.

Case 1 Assume that $|\mathrm{Re}\, z| \leq M_3 + 1$, where M_3 is the constant from Lemmas 2.4 and 2.4. Observe that there exists $M_5 < \infty$ such that

$$\sup\left\{\sum_{j=0}^{n} (j-k)^2 e^{2(j-k)\mathrm{Re}\, z} : |\mathrm{Re}\, z| \leq M_3 + 2\right\} \leq M_5^2.$$

If b is close to a, then $|\mathrm{Re}\, h_b(z)| \leq M_3 + 2$. Therefore, by (6), we get

$$\begin{aligned} \left|\frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)}\right| &\leq \frac{1}{\delta_a} \sup\{||\tfrac{\partial f_b'}{\partial b}(z)|| : |a-b| < r\}|b-a| \\ &\leq \frac{M_5}{\delta_a}|b-a|. \end{aligned}$$

Observe also that there exists $M_6 < \infty$ such that

$$\sup\{|f_a''(z)| : |\mathrm{Re}\, z| \leq M_3 + 2\} \leq M_6.$$

Then

$$\left|\frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)}\right| \leq \frac{M_6}{\delta_a}|h_b(z) - z| \leq \frac{M_6}{\delta_a}\left|\frac{\partial h_b}{\partial b}(z)\right||b-a|.$$

Since, by Proposition 2.4, $|\frac{\partial h_b}{\partial b}(z)|$ is bounded in a small neighborhood of a, we have that (49) can be as small as we want for b sufficiently close to a.

Case 2 Assume that $|\mathrm{Re}\, z| > M_3 + 1$. Then, similary like in Case 1 but using Lemma 2.4 instead of estimations by M_5, M_6 and δ_a, we get

$$\left|\frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)}\right| \le \frac{M_2}{M_1}|b-a|, \text{ and}$$
$$\left|\frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)}\right| \le \frac{M_2}{M_1}\left|\frac{\partial h_b}{\partial b}(z)\right||b-a|.$$

And again, if b is close to a, then (49) can be as small as we want for b sufficiently close to a. □

Lemma

For all $t > 1$ the function $a \mapsto P_a(t)$, $a \in \mathcal{H}$, is continuous.

Proof. Fix $a \in \mathcal{H}$ and, for this a, $r > 0$ from Lemma 4.2. Next, take any $z \in J_{F_a}$ and $n \ge 1$ and $x \in F_a^{-n}(z)$. Note first, by Proposition 2.4 (iii), h_b, which conjugates f_b and f_a, conjugates also F_b and F_a. Moreover, we have $h_b(F_a^{-n}(z)) = F_b^{-n}(z)$ and for every $i \in \{0, 1, \cdots, n\}$ and every $x \in F_a^{-n}(z)$ we have $h_b \circ f_a^i(x) = f_b^i \circ h_b(x)$. Now we can write

$$\frac{|(F_b^n)'(h_b(x))|}{|(F_a^n)'(x)|} = \frac{|(f_b^n)'(h_b(x))|}{|(f_a^n)'(x)|} = \prod_{i=0}^{n-1}\frac{|f_b'(f_b^i(h_b(x)))|}{|f_a'(f_a^i(x))|} = \prod_{i=0}^{n-1}\frac{|f_b'(h_b(f_a^i(x)))|}{|f_a'(f_a^i(x))|}.$$

Hence, by Lemma 4.2, for every $\gamma > 1$, there exists $0 < r_1 < r$ such that

$$\frac{1}{\gamma^n} < \frac{|(F_b^n)'(h_b(x))|}{|(F_a^n)'(x)|} < \gamma^n.$$

Since $h_b : F_a^{-n}(z) \to F_b^{-n}(h_b(z))$ is a bijection we obtain

$$\frac{1}{\gamma^{tn}} < \frac{\sum_{x \in F_b^{-n}(h_b(z))}|(F_b^n)'(x)|^{-t}}{\sum_{x \in F_a^{-n}(z)}|(F_a^n)'(x)|^{-t}} < \gamma^{tn}.$$

From this last relation, taking log and dividing by n we get

$$-t\log\gamma < P_b(t) - P_a(t) < t\log\gamma$$

for all $b \in B(a, r_1)$. We are done. □

4.3. Real Analyticity of the Hausdorff Dimension

Let $a \in \mathcal{H}$. Let $r_1 > 0$ be such a real number that for every $b \in B(a, r_1)$ there exists quasiconformal map h_b conjugating f_a and f_b and let

$$\alpha = \inf\left\{\frac{1}{Q(b)} : b \in B(a, r_1)\right\} > 0. \tag{50}$$

The existence of such r_1 follows from Lemma 2.4 and Proposition 2.4. Moreover, from now on by h_b we dentote the quasiconforomal map from Proposition 2.4. Therefore, for $b \in B(a, r_1)$ and $t > 1$, we can define a potential $\phi_{(\cdot)}(b, t) : J_{F_a} \to \mathbb{R}$ by the formula

$$\phi_z(b, t) = |F_b'(h_b(z))|^{-t}.$$

Lemma

If $Re\ t > 1$ then the functions

$$\phi = -t \log |F_b'(h_b(z))| \ \mathit{and}\ \phi_z(b, t) = |F_b'(h_b(z))|^{-t}$$

are α Hölder continuous, where α is the constant from (5.1).

Proof. Since h_b is as described in Lemma 2.4 the proof follows immediately from Lemma 3.3 because from Proposition 3.8(ii) we know that h_b is $(K(Q), \frac{1}{Q})$-Hölder and α is given by the formula (50). $\square$

Perron-Frobenius operator

Then by $\mathcal{L}^0_{b,t}$ we denote the Perron-Frobenius operator associated with $\phi_z(b, t)$, i.e.

$$\mathcal{L}^0_{b,t} g(z) = \sum_{x \in F_a^{-1}(z)} \phi_z(b, t) g(x)$$

for $g \in CB(J_{F_a})$.

Embedding of $\mathbb{C}^{n+1}$ into $\mathbb{C}^{2n+2}$

We would like to apply Theorem 4.1, but the function ϕ_z is not holomorphic as a function of b and t. For this reason we embed $\mathbb{C}^{n+1}$ into $\mathbb{C}^{2n+2}$, we extend ϕ_z and then we are able to use the results of Theorem 4.1.

Let $b = (b_0, b_1, \dots b_n) \in \mathbb{C}^{n+1}$. Write $b_j = b_j^1 + i b_j^2$ for $j = 0, 1, \dots n$ where i is the imaginary unit. Then $e_1 : \mathbb{C}^{n+1} \hookrightarrow \mathbb{C}^{2n+2}$ is the embedding defined by the following formula

$$e_1(b) = (b_0^1, b_0^2, b_1^1, b_1^2, \dots b_n^1, b_n^2),$$

and $e : \mathbb{C}^{n+1} \times \mathbb{R} \hookrightarrow \mathbb{C}^{2n+3}$ is the embedding defined as $e(b, t) = (e_1(b), t)$.

An extension of $\phi_z \circ e^{-1} : e(B(a, r_1) \times (1, \infty)) \to \mathbb{R}$

Fix $z \in J_{F_a}$(here e^{-1} is the left inverse of e). Observe that

$$\phi_z(b, t) = |F_b'(h_b(z))|^{-t} = \exp\left\{-t\left(\log|\psi_z(b)| + \log|F_a'(z)|\right)\right\},$$

where

$$\psi_z(b) = \frac{F_b'(h_b(z))}{F_a'(z)}.$$

We want to extend the function $\phi_z \circ e^{-1}$ in a neighborhood of $(e_1(a), t) \in \mathbb{C}^{2n+3}$ determined by the fact that b is sufficiently close to a so that $|b - a| < r_1$. Observe that, if we can well define Log that is a branch of $\exp^{-1}$ for $\psi_z(b)$, then

$$\mathrm{Re}\ \mathrm{Log}\psi_z(b) = \log|\psi_z(b)|.$$

But it follows from Lemma 4.2 that there exists $r_2 > 0$ such that, for $b \in B(a, r_2)$ and every $z \in J_{F_a}$,

$$|\psi_z(b) - 1| < \frac{1}{2}. \tag{51}$$

Therefore, the holomorphic function

$$\mathrm{Log}\psi_z : B(a, r_2) \to \mathbb{C}$$

is well defined, where Log is that branch of $\exp^{-1}$ satisfying the condition $\mathrm{Log}1 = 0$. Another consequence of (51) is the existence of $M_7 < \infty$ such that

$$|\mathrm{Log}\psi_z(b)| \leq M_7$$

for all $b \in B(a, r_2)$ and $z \in J_{F_a}$.

Extension of $\mathrm{Re}\ \mathrm{Log}\psi_z \circ e_1^{-1}$

Fix $z \in J_{F_a}$. The function $\mathrm{Log}\psi_z$ is an analytic function defined on $B_{\mathbb{C}^{n+1}}(a, r_2)$. Then

$$\mathrm{Log}\psi_z(b) = \sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} c_{i_0,\ldots,i_n}(z)(a_0 - b_0)^{i_0} \ldots (a_n - b_n)^{i_n}. \tag{52}$$

From the Cauchy's estimates it follows that

$$|c_{i_0,\ldots,i_n}(z)| \leq \frac{M_7}{r_2^{i_0+\ldots+i_n}}.$$

Recall that $b_j = b_j^1 + ib_j^2$, $a_j = a_j^1 + ia_j^2$ where $j = 0, \ldots, n$ and i is the imaginary unit. Note that

$$e_1(b) = (b_0^1, b_0^2, b_1^1, b_1^2, \ldots, b_n^1, b_n^2).$$

Since (52) can be written as

$$\sum_{(i_0,\ldots,i_n)\in\mathbb{N}^{n+1}} c_{i_0,\ldots,i_n}(z) \prod_{j=0}^{n} ((a_j^1 - b_j^1) + i(a_j^2 - b_j^2))^{i_j},$$

it follows that

$$\mathrm{Re}\ \mathrm{Log}\psi_z(b') = \sum_{(k_0,\ldots,k_{2n+1})\in\mathbb{N}^{2n+1}} c'_{k_0,\ldots,k_{2n+1}}(z) \prod_{l=0}^{2n+1} (a'_l - b'_l)^{i_j},$$

where

$$\begin{aligned} c'_{k_0,\ldots,k_{2n+1}}(z) &= \mathrm{Re}\ (c_{k_0+k_1,\ldots,k_{2n}+k_{2n+1}}(z) i^{k_1+k_3+\ldots+k_{2n+1}}) \\ &\quad \cdot \tbinom{k_0+k_1}{k_0}\tbinom{k_2+k_3}{k_2}\ldots\tbinom{k_{2n}+k_{2n+1}}{k_{2n}}, \end{aligned} \tag{53}$$

and $a'_l = a_j^m$, $b'_l = a_j^m$, where $l = 2j + (m-1)$. Note that

$$\begin{aligned} |c'_{k_0,\ldots,k_{2n+1}}(z)| &\leq |c_{k_0+k_1,\ldots,2n+2n+1}(z)| 2^{k_0+k_1+\ldots+k_{2n+1}} \\ &\leq M_7 \left(\tfrac{2}{r_2}\right)^{k_0+k_1+\ldots+k_{2n+1}}. \end{aligned}$$

Take $r_3 = r_2/4$. Then

$$|c'_{k_0,\ldots,k_{2n+1}}(z)| \leq M_7 \left(\frac{1}{2}\right)^{k_0+k_1+\ldots+k_{2n+1}}. \tag{54}$$

Hence

$$|\mathrm{Re}\ \mathrm{Log}\psi_z(b')| \leq M_7 \sum_{(k_0,\ldots,k_{2n+1}) \in \mathbb{N}^{2n+2}} \left(\frac{1}{2}\right)^{k_0+k_1+\ldots+k_{2n+1}} \leq M_7 2^{2n+2}. \tag{55}$$

Finally we define the extension of $\Phi_z(b,t)$ by the formula

$$\tilde{\Phi}_z(b,t) = \exp\{-t(\mathrm{Re}\ \mathrm{Log}\psi_z(b)) + \log |F'_a(z)|\} \tag{56}$$

where $(b,t) \in \mathbb{D}_{\mathbb{C}^{2n+2}}(a, r_3) \times B_{\mathbb{C}}(t_0, \rho)$ and $\rho = t_0 - 1$.

Proposition

Fix $a \in \mathcal{H}$ and $t_0 > 1$. Then there exist r_3 and ϱ such that, for

$$(b,t) \in G_3 = \mathbb{D}_{\mathbb{C}^{2n+2}}(e_1(a), r_3) \times B(t_0, \varrho),$$

Perron-Frobenius operator $\mathcal{L}_{\Phi(b,t)}$ for the potential $\Phi_{(\cdot)}(b,t)$ is well defined. Moreover, denote by $\mathcal{L}$ the function $G_3 \ni (b,t) \mapsto \mathcal{L}_{\Phi(b,t)} \in L(H_\alpha)$. Then $\mathcal{L}$ is holomorphic (α comes from (50) with r_1 replaced by r_3.)

To prove the proposition it is enough to check the conditions from Theorem 4.1

Condition (v)

It is satisfied from the construction 4.3-4.3.

Condition (i)

First, we prove that the function $\varphi_{(b,t)}(z)$ is summable. From (55) if follows that

$$
\begin{aligned}
|\varphi_{(b,t)}(z)| &= \exp\{\mathrm{Re}\ (-t\mathrm{Re}\ \mathrm{Log}\psi_z(b))\}|F_a'(z)|^{-\mathrm{Re}\ t} \\
&\leq e^{M_7 2^{2n+2}|t|}|F_a'(z)|^{-\mathrm{Re}\ t}.
\end{aligned}
$$

In Lemma 3.3 we proved that $|F_a'(z)|^{-\mathrm{Re}\ t}$ is summable. Therefore $\varphi_{(b,t)}$ is summable. Next, we show that the potential $\varphi_{(b,t)}(z)$ is Hölder. Note that, by Lemma 3.3(i) and Proposition 3.8(ii)

$$|\mathrm{Re}\ \mathrm{Log}\psi_z(b) - \mathrm{Re}\ \mathrm{Log}\psi_w(b)| \leq C|z-w|^\alpha.$$

From Koebe's estimation of the Arg it follows that

$$
\begin{aligned}
|\mathrm{Arg}\psi_z(b) - \mathrm{Arg}\psi_w(b)| &= \left|\mathrm{Arg}\frac{\psi_z(b)}{\psi_w(b)}\right| \\
&\leq \left|\mathrm{Arg}\frac{F_b'(h_b(z))}{F_b'(h_b(w))}\right| + \left|\mathrm{Arg}\frac{F_a'(z)}{F_a'(w)}\right| \\
&\leq \frac{6}{2\delta}|z-w| + \frac{6}{2\delta}|h_b(z)-h_b(w)| \\
&\leq C_{\mathrm{Arg}}|z-w|^\alpha
\end{aligned}
$$

Then we have that

$$
\begin{aligned}
&|\mathrm{Log}\psi_z(b) - \mathrm{Log}\psi_w(b)| \\
&\qquad \leq |\mathrm{Re}\ \mathrm{Log}\psi_z(b) - \mathrm{Re}\ \mathrm{Log}\psi_w(b)| + |\mathrm{Arg}\psi_z(b) - \mathrm{Arg}\psi_w(b)| \\
&\qquad \leq (C + C_{\mathrm{Arg}})|z-w|^\alpha
\end{aligned}
$$

To prove that the extension is Hölder first we show that the coefficients of the extension of $\mathrm{Re}\ \mathrm{Log}\psi_z$ are Hölder. This follows from Cauchy's estimates and the fact that the function is Hölder.

$$|c_{i_0,\dots,i_n}(z) - c_{i_0,\dots,i_n}(w)| \leq \frac{M_7}{r_2^{i_0+\dots+i_n}}|z-w|^\alpha.$$

Therefore, by (52),

$$|c'_{k_0,\dots,k_{2n+1}}(z) - c'_{k_0,\dots,k_{2n+1}}(w)| \le M_7 \left(\frac{2}{r_2}\right)^{i_0+\cdots+i_n} |z-w|^\alpha. \tag{57}$$

Recall that $r_3 = \frac{r_2}{4}$. Then

$$|\mathrm{Re}\ \psi_z(b) - \mathrm{Re}\ \psi_w(b)| \le M_8|z-w|^\alpha.$$

Next observe that there exists C such that, if

$$|x| \le (2^{n+1}M_7 + C)|t|,$$

then

$$|e^x - 1| \le M_9^{|t|}|x|.$$

Therefore

$$\begin{aligned}
&|\Phi_z(b,t) - \Phi_w(b,t)| \\
&\qquad = |e^{-t(\mathrm{Re\ Log}\Psi_z(b))+\log|F'_a(z)|}|M_9(M_8|z-w|^\alpha + M_{10}|z-w|) \\
&\qquad\qquad \le e^{|t|M_6+C}M_9(M_8|z-w|^\alpha + M_{10}|z-w|) = M_{11}\cdot|z-w|^\alpha.
\end{aligned}$$

Conditions (ii) and (iv)

We check now conditions (ii) and (iv) i.e we show that Φ_b is dynamically Hölder , with exponent α, for b in some region G (as before)in $\mathbb{C}^{2n+3}$ and with uniformly bounded constants $c_{\phi_{b,t}}$. So let

$$\phi = \Phi_z(b,t) = e^{-t(\mathrm{Re\ Log}\psi_z(b)+\log|F'_a(z)|)} = e^{\varphi_z(b,t)}$$

where $(b,t) \in G = \mathbb{D}_{\mathbb{C}^{2n+2}}(a, r_3) \times \{t : \mathrm{Re}\ t > 1\}$.

We first show that

$$|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))| \le c_\phi|\phi_n(F_v^{-n}(x))||y-x|^\alpha$$

for all $n \geq 1$, for any $x, y \in J_{F_a}$, $|x-y| \leq \delta$, $v \in F^{-n}(x)$ and

$$\phi_n(z) = \phi(z)\phi(F(z))\cdots\phi(F^{n-1}(z)$$

i.e.

$$\phi_n(z) = \prod_{k=0}^{n-1} \phi(F^k(F_v^{-n}(z)) = e^{\sum_{k=0}^{n-1} \varphi_{F^k(F_v^{-n}(z))}(b,t)}$$

where here for convenience recall that we denoted $\phi(z) = \Phi_z(b,t)$ and

$$\varphi(z) := \varphi_z(b,t) = -t(\mathrm{Re}\ \mathrm{Log}\psi_z(b) + \log|F_a'(z)|)$$

for fixed $(b,t) \in G$. Fix now t_0 with $\mathrm{Re}\ t_0 > 1$ and let $t \in B_{\mathbb{C}}(t_0, \rho)$ where $\rho := t_0 - 1$. Then we have

$$\begin{aligned}(58)\quad & |\varphi(z)-\varphi(w)| \\ & \leq (|t_0|+\rho)(|\mathrm{Re}\ \mathrm{Log}\psi_z(b) - \mathrm{Re}\ \mathrm{Log}\psi_w(b)| + |\log|F_a'(z)| - \log|F_a'(w)||) \\ & \qquad \leq (|t_0|+\rho)(M_{12}|z-w|^\alpha + \frac{9}{2\delta}|z-w|) \leq M_{15}(t_0+\rho)|z-w|^\alpha.\end{aligned}$$

From Remark 3.2 and (58) we easily obtain, by the use of triangle inequality and the mean value theorem that

$$\left|\sum_{k=0}^{n-1} \varphi_{F^k(F_v^{-n}(z))}(b,t) - \sum_{k=0}^{n-1} \varphi_{F^k(F_v^{-n}(w))}(b,t)\right| \leq \frac{M_{12}(|t_0|+\rho)L^\alpha}{1-\beta^\alpha}|z-w|^\alpha = M_{13}|z-w|^\alpha$$

Therefore putting

$$M_{14} = \sup\left\{\left|\frac{e_z-1}{z}\right| : |z| \leq M_{13}\right\} < \infty, \tag{59}$$

we get

$$\left|\phi_n(F_v^{-n})(z) - \phi_n(F_v^{-n}(w))\right| \leq M_{13}M_{14}|\phi_n(F_v^{-n}(z))||z-w|^\alpha$$

Consequently the potential ϕ is dynamically Hölder and we can see that c_{ϕ_b} is uniformly bounded. So the assumptions(ii) and (iv) of the Main Tool are verified.

Condition (iii)

Let $G = \mathbb{D}_{\mathbb{C}^{2n+2}}(a, r_3) \times B(t_0, \rho) \subset \mathbb{C}^{2n+3}$ We show that

$$G \ni (b, t) \mapsto \tilde{\Phi}_z(b, t) \in H_\alpha(J_{F_a})$$

is a continuous function.

Now observe that

$$\tilde{\Phi}_z(b, t) = e^{-t\text{Re Log}\psi_z(b)} |F_a'(z)|^{-t}.$$

We show continuity by proving that the following two functions are continuous in each of the variables b and t

$$(b, t) \mapsto e^{-t\text{Re Log}\psi_z(b)},$$

$$(b, t) \mapsto |F_a'(z)|^{-t}.$$

We already know that both functions

$$z \mapsto e^{-t\text{Re Log}\psi_z(b)}$$

and

$$z \mapsto |F_a'(z)|^{-t}$$

are in H_α.The function $(b, t) \mapsto e^{-t\text{Re Log}\psi_z(b)}$ is continuous in the variable t as a function $\mathbb{R} \mapsto H_\alpha$, for a fixed b as we can see

$$|| - t_1\text{Re Log}\psi_z(b) + t_2\text{Re Log}\psi - z(b)||_\alpha = |t_1 - t_2|||\text{Re Log}\psi_z(b)||_\alpha < M|t_1 - t_2|$$

for some constant M because $||tg||_\alpha = |t|||g_\alpha||$. For continuity with respect with the variable b we recall that

$$\begin{aligned} ||\text{Re Log}\psi_z(b) - \text{Re Log}\psi_z(c)||_\alpha &= v_\alpha(\text{Re Log}\psi_z(b) - \text{Re Log}\psi_z(c)) \\ &\quad +||\text{Re Log}\psi_z(b) - \text{Re Log}\psi_z(c)||_\infty \end{aligned}$$

Then we evaluate by using (55)

$$
\begin{aligned}
&|(\mathrm{Re\ Log}\psi_w(b) - \mathrm{Re\ Log}\psi_w(c)) - (\mathrm{Re\ Log}\psi_z(b) - \mathrm{Re\ Log}\psi_z(c))| \\
&= |\textstyle\sum_{k_0,k_1,\cdots,k_{2n+1}}(c'_{k_0\cdots k_{2n+1}}(w) - c'_{k_0,\cdots k_{2n+1}}(z))\prod_{l=0}^{2n+1}(b'_l - a'_l)^{k_l} \\
&\quad - \textstyle\sum_{k_0,\cdots,k_{2n+1}}(c'_{k_0,\cdots,k_{2n+1}}(w) - c'_{k_0,\cdots,k_{2n+1}}(z))\prod_{l=0}^{2n+1}(c'_l - a'_l)^{k_l}| \\
&= |\textstyle\sum_{k_0,\cdots,k_{2n+1}}(c'_{k_0,\cdots,k_{2n+1}}(w) - c'_{k_0,\cdots,k_{2n+1}}(z)) \\
&\quad (\textstyle\prod_{l=0}^{2n+1}(b'_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1}(c'_l - a'_l)^{k_l})| \\
&\leq M_6\left(\frac{2}{r_2}\right)^{k_0,\cdots,k_{2n+1}}|z-w|^{\alpha}\left|\textstyle\prod_{l=0}^{2n+1}(b'_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1}(c'_l - a'_l)^{k_l}\right|.
\end{aligned}
$$

But a simple computation will give us that

$$
\begin{aligned}
&|\textstyle\prod_{l=0}^{2n+1}(b'_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1}(c'_l - a'_l)^{k_l}| \\
&\quad \leq \frac{4}{r_2}(k_0 + \cdots + k_{2n+1})\left(\frac{r_2}{4}\right)^{k_0+\cdots+k_{2n+1}}||b-c||
\end{aligned}
$$

(we know that $||b-c|| \leq \frac{r_2}{4}$). So,this will show that

$$v_\alpha(\mathrm{Re\ Log}\psi_z(b) - \mathrm{Re\ Log}\psi_z(c)) \leq M_{15}||b-c||$$

for some constant $M_{15} > 0$ which is obvious from the above computations, and similarly

$$||\mathrm{Re\ Log}\psi_z(b) - \mathrm{Re\ Log}\psi_z(c)||_\infty \leq M_{16}||a-b||$$

for some $M_{16} > 0$. As a consequence

$$||\mathrm{Re\ Log}\psi_z(b) - \mathrm{Re\ Log}\psi_z(c)||_\alpha \leq M_{17}||a-b||$$

and the continuity is proven because now we can write

$$
\begin{aligned}
&|e^{-t\mathrm{Re\ Log}\psi_z(b)} - e^{-t\mathrm{Re\ Log}\psi_z(c)}| \\
&\qquad = |e^{-t\mathrm{Re\ Log}\psi_z(b)+t\mathrm{Re\ Log}\psi_z(c)} - 1||e^{-t\mathrm{Re\ Log}\psi_z(c)}| \\
&\qquad\qquad \leq e^{2^{n+1}M_6(t_0+\rho)}(t_0+\rho)M_{17}||b-c||
\end{aligned}
$$

Using the same arguments as before, when we checked conditions (ii) and (iv), we are done.

To prove continuity of

$$(b, t) \mapsto |F_a'(z)|^{-t} \in H_\alpha$$

we first observe that continuity in the variable b is clear simply because the function is constant as a function of b. For continuity in the variable t we first write

$$|||F_a'(z)|^{-t_2} - |F_a'(z)|^{-t_1}||_\alpha = v_\alpha(.) + ||.||_\infty$$

and repeating the computations we did before together with the argument used when we checked condition (iii) we are done.

Note that the quality of H_α of being a Banach algebra, in particular having the property that if $f, g \in H_\alpha$ then $||fg||_\alpha \leq ||f||_\alpha ||g||_\alpha$ played an essential role in our above computations.

Condition (vi)

Let $(d, t_d) \in B_{\mathbb{C}^{2n+3}}((a, t_0), \min\{r_3/4, \rho\})$. Let γ be such a real number that

$$B_{\mathbb{C}^{2n+3}}((d, t_d), 2\gamma) \subset B_{\mathbb{C}^{2n+3}}((a, t_0), \min\{r_3/4, \rho\})$$

This real number is the number r in Condition (vi)from Theorem 4.1 and for the point c of the same condition we choose (a, t_c) where t_c is an arbitrary point from the real interval $(1, \mathrm{Re}\ (t_b - \gamma))$. Now, let $(b, t_b) \in B_{\mathbb{C}^{2n+3}}((d, t_d), \gamma)$. Then we get that

$$|e^{t_c(\mathrm{Re\ Log}\psi_z(a) - \mathrm{Re\ Log}\psi_z(b))}| \leq e^{t_c 2^{2n+3} M_7},$$

$$|e^{(t_c - t_b)\mathrm{Re\ Log}\psi_z(b)}| \leq e^{\rho M_7 2^{2n+2}}.$$

Therefore

$$\left|\frac{\Phi_{b,t_b}(z)}{\Phi_{c,t_c}(z)}\right| \leq \frac{e^{-t_b \mathrm{Re\ Log}\psi_z(b)}}{e^{-t_c \mathrm{Re\ Log}\psi_z(c)}} |F_c'(z)|^{-(t_b - t_c)}$$

$$\leq |e^{t_c(\mathrm{Re\ Log}\psi_z(a) - \mathrm{Re\ Log}\psi_z(d))}||e^{(t_c - t_d)\mathrm{Re\ Log}\psi_z(d)}||F_a'(z)|^{\mathrm{Re\ }(t_c - t_d)}$$

$$\leq e^{t_c 2^{2n+3}(M_7 + \rho) M_7 2^{2n+2}} \Delta_a^{-\rho}.$$

Important Remark

For $b = (b_0, \ldots, b_n) \in B_{\mathbb{C}^{n+1}}(a, r_3)$ and $t \in (t_0 - \varrho, t_0 + \varrho)$,

$$\mathcal{L}^0_{b,t} = \mathcal{L}(e_1(b), t).$$

We will use now, directly, the following important perturbation theorem and we refer to [43] for a proof and to [32] for other applications.

Kato, Rellich Theorem

Let H be a complex Banach space and $L(H)$ the Banach space of bounded linear operators on H. If $\mathcal{L}_0 \in L(H)$ has a simple eigenvalue α_0 which is an isolated point of the spectrum of $\mathcal{L}_0$ with the associated eigenvector g_0 then for every $\varepsilon > 0$ small enough there exists $\delta > 0$ such that, if $||\mathcal{L} - \mathcal{L}_0|| < \delta$, then the operator $\mathcal{L}$ has a simple eigenvalue $\alpha(\mathcal{L})$ and eigenvector $g(\mathcal{L})$ with the properties

(i) the functions $\mathcal{L} \mapsto \alpha(\mathcal{L})$ and $\mathcal{L} \mapsto g(\mathcal{L})$ are holomorphic.

(ii) if $||\mathcal{L} - \mathcal{L}_0|| < \delta$, then $\mathrm{spectrum}(\mathcal{L}) \cap B(\alpha_0, \varepsilon) = \{\alpha(\mathcal{L})\}$.

Kato-Rellich Theorem works

From Theorem 3.3 and the fact that

$$\mathcal{L}_t = \mathcal{L}^0_{a,t_0} = \mathcal{L}(a, t)$$

for $t_0 > 1$ it follows that $e^{P_a(t_0)}$ is simple isolated eigenvalue. From Theorem 3.3 and Theorem 4.3 it follows that, for every $\varepsilon > 0$ small enough, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that, if $|b - e_1(a)| < \delta_1$ and $|t - t_0| < \delta_2$, then $\mathcal{L}(b, t)$ has a simple eigenvalue $\alpha_{(b,t)}$ with the properties

(i) the function $(b, t) \mapsto \alpha_{(b,t)}$ is holomorphic on

$B_{\mathbb{C}^{2n+2}}(e_1(a), \delta_1) \times B_{\mathbb{C}}(t_0, \delta_2)$,

(ii) if $|b - e_1(a)| < \delta_1$ and $|t - t_0| < \delta_2$, then

$$\mathrm{spectrum}(\mathcal{L}(b,t)) \cap B(e^{P_a(t_0)}, \varepsilon) = \{\alpha_{(b,t)}\}.$$

Diagram

We claim that the following diagram is commutative

$$\begin{array}{ccc} H_1(J_{F_b}) & \xrightarrow{\mathcal{L}_{b,t}} & H_1(J_{F_b}) \\ T_b \downarrow & & \downarrow T_b \\ H_\alpha(J_{F_a}) & \xrightarrow{\mathcal{L}(e_1(b),t)} & H_\alpha(J_{F_a}). \end{array}$$

where $T_b = g \circ h_b$. Since T_b is linear and continuous, T_b is bounded. Moreover, since h_b is Hölder, $T_b(H_1(J_{F_b})) \subset H_\alpha(J_{F_a})$. To prove the claim observe

$$(\mathcal{L}(e_1(b),t) \circ T_b)(z) = \sum_{x \in F_a^{-1}(z)} |(F_b' \circ h_b)(x)|^{-t} g(h_b(z)),$$

$$(T_b \circ \mathcal{L}_{b,t})(z) = \mathcal{L}_{b,t}(h_b(z)) = \sum_{x \in F_b^{-1}(h_b(z))} |(F_b)'(x)|^{-t} g(x).$$

Since F_a and F_b are conjugated by the homeomorphism h_b,

$$F_b^{-1}(h_b(z)) = \{h_b(y) : y \in F_a^{-1}(z)\},$$

It finishes the proof of the claim.

We prove that $\alpha(b,t) = e^{P_b(t)}$

Let $g_{b,t} \in H_1(J_{F_b})$ be an eigenvector that is associated to the eigenvalue $e^{P_b(t)}$ of the operator $\mathcal{L}_{b,t}$. (see Theorem 3.3). From the commutativity of the previous diagram it follows that

$$\mathcal{L}(e_1(b),t)(g_{b,t} \circ h_b) = e^{P_b(t)}(g_{b,t} \circ h_b).$$

Therefore, $\alpha_{(b,t)}$ and $e^{P_b(t)}$ are eigenvalues $\mathcal{L}(e_1(b),t)$. Since we have
$\alpha(e_1(a), t_0) = e^{P_a(t_0)}$ and since $e^{p_b(t)}$ is continuous, for (b,t) close to (a,t_0)we have that

$$e^{P_b(t)} \in B(e^{P_a(t_0)}, \varepsilon).$$

Then we are done, since we get $\alpha_{(b,t)} = e^{P_b(t)}$.

Corollary

The function $(b, t) \mapsto P_b(t)$ is real-analytic in some neighborhood of (a, t_0) in $\mathbb{C}^{n+1} \times (1, \infty)$.

Theorem(Real analiticity)

The Hausdorff dimension of $HD(J^r_{f_b})$ is real-analytitic.

Proof. To proof real-analyticity of the Hausdorff dimension observe that it is enough to prove that the function

$$b \mapsto HD(J^r_{F_b})$$

with $b \in \mathbb{C}^{n+1}$ is real analytic. Therefore we need to show that the solution of the equation

$$P_b(t) = 0,$$

(which as we know from Chapter 3 is the function $b \mapsto \mathrm{HD}(J^r_{F_b})$) exists and is real-analytic in a neighborhood of a.

This follows from the Implicit Function Theorem because

$$\frac{\partial P_b(t)}{\partial t} \neq 0.$$

The last relation is true since the function $P_b : (1, \infty) \to \mathbb{R}$ is real-analytic, convex and strictly decreasing(as we proved in Chapter 3). $\square$

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