LYAPUNOV EXPONENTS, ENTROPY AND DIMENSION

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We consider diffeomorphisms of a compact Riemann Surface. A development of Oseledec's Multiplicative Ergodic Theorem is given, along with a development of measure theoretic entropy and dimension. The main result, due to L.S. Young, is that for certain diffeomorphisms of a surface, there is a beautiful relationship between these three concepts; namely that the entropy equals dimension times expansion.

PREFACE

This work was part of an interdisciplinary project supported by the National Science Foundation (NSF) Biocomplexity in the Environment grant CNH BCS-0216722. This project studies coupled human-natural systems for an understanding of ways that human behavior could impact the environment of the future. Human system models are based on multiagent methods and environmental models include land use change in forest landscapes and hydrological responses. The contents of this paper are related to the mathematical models we are using for these systems.

The models we encountered were non-homogenous semi-Markov processes. We were quickly able to adapt these to form non-homogeneous Markov processes. While very little is currently known about this non-homogenous variety, we were able to embed our specific models into $\mathbb{R}^n$, and consider a continuous transformation representing the evolution of the systems.

One of the goals of our group was to be able to detect the presence of chaos in these models, in particular to find chaotic attractors. The simplest method was to test the systems for a positive lyapunov exponent. We adapted existing software to test our models, but found no indication of chaos.

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CHAPTER 1

INTRODUCTION

1.1 History

In 1932 Birkhoff proved the Individual Ergodic Theorem, which related the time averages of individual orbits to the space average for certain types of dynamical systems. While this theorem is a remarkable result, it is constrained by the types of averages that are used. In 1963 Kingman extended Birkhoff's theorem by proving the Subadditive Ergodic Theorem, which allowed more general types of averages for subadditive sequences. Shortly after this, in 1968, Oseledec proved the Multiplicative Ergodic Theorem, which allowed for the computation of geometric means (rather than arithmetic means) for similar ergodic processes. A direct result of Olsedec's Multiplicative Ergodic theorem is the existence of Lyapunov exponents, which opened a door to practical analysis of dynamical systems in a context outside of pure mathematics.

These ideas from chaos theory quickly overtook the scientific community. In the 1970's and 80's researchers were using Lyapunov exponents to indicate whether "chaos" was present in a slew of systems, arising in fields from physics to biology. Non-linear studies had become the wave of the future. This was a result of Oseledec's theorem, and newfound availability of computational equipment brought by computers.

But the implications of Oseledec's theorem were not just confined to the areas of applied mathematics. In 1981 Anthony Manning confirmed a relationship between the Lyapunov exponents, entropy, and dimension for Axiom A diffeomorphisms of a surface [Ma]. In the next year Lai-Sang Young proved a more general result: that for a $C^{1+\alpha}$ diffeomorphism of a surface, there is a similar relationship[Y1]. This work was extended by Ledrappier and Young to diffeomorphisms of higher dimensional manifolds [LY1] [LY2]. In short their results state that dimension is equal to the entropy divided by the exponential rate of expansion. This line of research continues today, in try to expand this relationship to more and more general types of dynamical systems, such as conformal systems. [MU]

1.2 What Will Be Covered

In this paper, Oseledec's Multiplicative Ergodic Theorem for surfaces will be proven, using Kingman's Theorem. We will then examine some of the implications of this theorem on the structure of invariant sets arising from dynamical systems.

Next we will build some of the tools needed to prove Young's 1982 result. This will involve an introduction to measure theoretic entropy, and the theory of dimensions. Much of the work here will be stated without proof, due to the large amount of background needed.

The main result will be the proof of Young's Formula in Chapter 9. Though the proof is somewhat technical, requiring Lyapunov charts, we will take an almost naive approach to the charting process, allowing the reader to clearly see the ideas in the proof rather than a lump of technical lemmas. The formula is a result for diffeomorphisms of a surface, and as of yet there are no results generalizing this formula to higher dimensions.

We will then conclude with a short discussion of the work in the 1985 paper of Ledrappier and Young, which gives great insight in to the beautiful link between the exponents, entropy, and dimension. We also mention the Kaplan-Yorke conjecture which is partially verified by these works.

Let us will conclude this introduction with two examples showing the link between exponents, entropy, and dimension.

1.3 Examples

In this paper we will prove Young's Formula, that the dimension varies directly with entropy and inversely with the rate of expansion; hence the formula $d = \frac{h}{\lambda}$. Here are a few concrete examples.

Example 1.3.1. *(The Cantor Middle Thirds Set)*

The cantor set, $C := \{x \in [0,1) : x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$ *where* $x_i = 0, 2\}$ *is invariant under* $T(x) = 3x \mod 1$. *The Hausdorff dimension of C, HD(C), can be shown to be* $\frac{log\ 2}{log\ 3}$. *Also* T *has Lyapunov exponent* $log\ 3$ *and entropy* $h = log\ 2$. *Thus the formula* $HD = \frac{h}{\lambda}$ *holds.*

Example 1.3.2. *(Hyperbolic Toral Automorphisms)*

Consider the torus $\mathbb{T} = \mathbb{R}^2 \ / \mathbb{Z}^2$ *Let* $A \in SL_2(\mathbb{Z})$. *Let* $f : \mathbb{T} \to \mathbb{T}$ *be the action induced by* A *on the torus. The attractor for this function is all of* $\mathbb{T}$, *and* $HD(\mathbb{T})$*=2. By the Perron-Frobenius theorem* $h = log(\gamma_1)$ *where* γ_1 *is the largest eigenvalue of* A *[PY]. The Lyapunov exponents of* f *are* $\lambda_1 = -\lambda_2 = log(\gamma_1)$. *Thus, we have the formula* $HD = h(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})$.

More examples of this equality can be found in [Y1] [Ma].

CHAPTER 2

PRELIMINARIES AND CONTEXT

2.1 Manifolds

The setting for the main results discussed in this paper will be a diffeomorphism of a Riemannian manifold. A thorough treatment of the development of manifolds is beyond the scope of this paper, therefore we will introduce this topic very naively. A reader who would like a more thorough introduction is referred to [L1] , and [S].

A n-dimensional Riemannian manifold is a connected, second countable, Hausdorff space endowed with a metric which makes the manifold locally homeomorphic to $\mathbb{R}^n$. The homeomorphisms are called local coordinates or charts, and we will develop a particular type of charting systems in Chapter 8. Intuitively a n-dimensional Riemannian manifold is a space which "looks like" $\mathbb{R}^n$ locally but has no particular orientation. Some examples of 2-manifolds, or surfaces, are the sphere, the torus, and the plane.

The fact that a Riemannian manifold locally looks like $\mathbb{R}^n$ also gives rise to the tangent space. One can think of the tangent space at x, T_xM, as the set the set of all vectors tangent to the manifold, M, at x. This resembles looking at the vector space created when you lay $\mathbb{R}^n$ across the manifold at x via charts.

Definition 2.1.1. *Let M be a n-dimensional Riemannian manifold. Let $f : M \to M$. The derivative of f at a point $x \in M$, denoted $D_x f$ is a linear transformation going between $T_x M$ and $T_{f(x)} M$ that has the following property:*

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - D_x f \cdot h|}{h} = 0$$

A function which has a derivative at every point is called differentiable. Furthermore if derivative is also continuous, the function is bijective, and the inverse is differentiable the function is called a diffeomorphism. Diffeomorphisms are classified by their degree of differentiability. A function which is twice differentiable is called a C^2-diffeomorphism and so on. A $C^{1+\alpha}$-diffeomorphism is a diffeomorphism with α-Holder continuous first derivative.

In this paper we will consider a compact Riemannian manifold X with an associated $C^{1+\alpha}$-diffeomorphism, $f : X \to X$, representing the evolution of the system.

2.2 Measure Theory

Much of this paper is built on the back of measure theory. For the purposes of this paper, we will only develop a few theorems and definitions which are commonly used in ergodic theory. For a more thorough treatment see [Roy] or [Ro]. As a note: Measure spaces are a collection of a set, X, a measure, μ, and a sigma algebra Σ. In this paper we do not reference the sigma algebra, as it will always be assumed to be the standard Borel sigma algebra generated by the open sets on the manifold.

The next few definitions lay out some of the properties which the measures encountered in this paper will have.

Definition 2.2.1. *Let X, μ be a measure space. The measure μ is said to be a probability measure if $\mu(X) = 1$.*

Definition 2.2.2. *Let (X, μ) be a measure space, and let $f : X \to X$ be measurable. A measure μ is called f-invariant if for every $A \subset X$, $\mu(f^{-1}(A)) = \mu(A)$. When the function is unambiguous, we may simply call μ an invariant measure*

Peterson's derivation of ergodicity. Another property we would like our measures to have is ergodicity. There are many different equivalent definitions of the word ergodic. For this paper the following definition will be used.

Definition 2.2.3. *Let (X, μ) be a measure space, and let $f : X \to X$ be measurable. An invariant probability measure μ is called ergodic if for every $A \subset X$ such that $f^{-1}(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$.*

We will denote the set of probability measures on X by $M(X)$ and the set of f-invariant probability measures $M(X, f)$. We will frequently rely on the existence of a ergodic measure (with respect to f), μ, on X. This is given by the following theorem:

Theorem 2.2.4. *(Krylov and Bogolioubov)*

If (X, d) is a compact metric space with $f : X \to X$ continuous, then $M(X, f)$ is a non-empty convex subset M(X) which is compact in the weak* topology. Additionally the extreme points of $M(X, f)$ are the ergodic f-invariant measures.

We will also use frequently the following ergodic theorem due to Birkhoff stating that the time averages along most orbits are the same as the space averages.

Theorem 2.2.5. *(Birkhoff's Ergodic Theorem)*

Let (X, B, μ) be a probability space and $T : X \to X$ be ergodic with respect to μ.

Then $\forall\ f \in L^1(\mu)$

$$\lim_{n\to\infty} \quad \tfrac{1}{n}\sum_{i=0}^{n-1} f \circ (T^i(x)) = \int_X f d\mu \qquad \text{for } \mu\text{-a.e. } x \in X$$

Theorem 2.2.6. *Poincaré Recurrence Theorem*

Let (X, μ) be a probability space, and let $f : X \to X$ be a measure preserving transformation. Let $A \subset X$ be measurable such that $\mu(A) > 0$. Then for μ- almost every $x \in A$. The orbit of x, $\{f^n(x)\}_{n=0}^{\infty} \bigcap A$ is infinite.

Definition 2.2.7. *Absolute Continuity*

A measure μ is said to be absolutely continuous with respect to the measure ν, written $\mu << \nu$, if $\nu(A) = 0 \Rightarrow \mu(A) = 0$.

2.3 Linear Algebra and Matrix Theory

To prove the special case of the Multiplicative Ergodic Theorem, we will need a few theorems from linear algebra relating to symmetric matrices, their eigenvalues and eigenvectors. To prove the general version of this theorem requires more advanced tools such involving matrix decompositions and representations (these will be excluded here). In this paper the norm of the matrix A, denoted $\|A\|$, will always be defined as $\|A\| = \sup_{v \in V} \|Av\|$ where the second norm is the usual vector norm on V.

CHAPTER 3

THE SUBADDITIVE ERGODIC THEOREM

Kingman's Subadditive Ergodic theorem extends the Individual Ergodic theorem to subadditive sequences. It will allow us to obtain ergodic type results for functions involving the logarithm of the norm of a matrix. In particular, we will use a direct corollary of the subadditive theorem to prove the Multiplicative Ergodic Theorem. First a statement of Kingman's Subadditive Ergodic Theorem.

Theorem 3.0.1. *The Subadditive Ergodic Theorem (Kingman 1968) Let (X, B) be a measure space, $T : X \to X$ be a measurable transformation, and μ be an ergodic measure (with respect to T). Let $(F_n)_{n=1}^{\infty} \subset L^1(\mu)$ be such that for every $n, k \geq 1$ the following condition holds:*

$$F_{n+k}(x) \leq F_n(x) + F_k(T^n(x)) \qquad for \quad \mu - a.e.\ x \in X. \tag{3.1}$$

Then there exists $\lambda \in \mathbb{R} \cup \{-\infty\}$ such that for μ- a.e. $x \in X$, $\lim\limits_{n\to\infty} \frac{1}{n} F_n(x) = \lambda$. Furthermore:

$\lambda = \underset{n>0}{inf} \ \frac{1}{n} \ \int F_n d\mu.$

Proof of this theorem follows in a fashion similar to the proof of Birkhoff's Ergodic Theorem. In this paper we will use this theorem without proof. For more details and a complete proof, the reader is referred to [P]. Also note that by taking $F_n(x) = \sum_{i=0}^{n-1} f \circ T^i(x)$, Kingman's theorem implies Birkhoff's theorem. Of particular interest to us is the following corollary, which is what is actually needed to prove the Multiplicative Ergodic Theorem.

Corollary 3.0.2. *Let $f : M \to M$ be a C^1- diffeomorphism of a compact Riemann manifold, M, and let μ be ergodic with respect to f. Then there exists $\lambda \in \mathbb{R}$ such that $\underset{n\to\infty}{lim} \ \frac{1}{n} log\|D_x f^n\| = \lambda$ for μ-a.e. $x \in M$.*

Proof.

We employ the Subadditive Ergodic Theorem. For each $n \in \mathbb{N}$, take $F_n(x) = log\|D_x f^n\|$. Since f is a C^1-diffeomorphism, for each n, there exists positive constants, A_n, B_n, such that $0 < A_n \leq \|D_x f^n\| < B_n < \infty$. Thus $log\|D_x f^n\|$ is bounded for every n, and $F_n \in L_1(\mu)$. To establish subadditivity consider the following inequality using the chain rule:

$$\|D_x f^{(n+k)}\| = \|D_{f^{n+k-1}(x)} f \circ D_x f^{n+k-1}\| = ... = \|D_{f^n(x)} f^k \circ D_x f^n\| \leq \|D_{f^n(x)} f^k\| \ \|D_x f^n\|.$$

Thus taking the log of both sides one gets:

$$F_{n+k}(x) = log\|D_x f^{n+k}\| \leq log\|D_{f^n(x)} F^k\| + log\|D_x f^n\| = F_k(f^n(x)) + F_n(x).$$

Therefore there exists $\lambda \in \mathbb{R} \cup \{-\infty\}$ such that $\lim\limits_{n\to\infty} \frac{1}{n} log\|D_x f^n\| = \lambda$ for μ-a.e. $x \in M$. To rule out the possibility that $\lambda = -\infty$, let $\|Df^{-1}\| = \sup\limits_{x\in M}\{\|D_x f^{-1}\|\}$. The following inequality uniformly bounds $\frac{1}{n} log\|D_x f^n\|$ from below:

$\frac{1}{\|Df^{-1}\|^n} \leq \|D_x f^n\|$ and thus, $-log\|Df^{-1}\| \leq \frac{1}{n} F_n$ for every $n \in \mathbb{N}$.

This gives us the desired λ.

$\square$

CHAPTER 4

LYAPUNOV EXPONENTS AND OSELEDEC'S MULTIPLICATIVE ERGODIC THEOREM

Throughout this section we will consider a compact Riemannian manifold M with an associated C^1-diffeomorphism, $f : M \to M$, representing the evolution of the system. We would like to find a quantitative measurement of how chaotic the system is. In particular we want to measure the degree of "sensitive dependence on initial conditions" inherent in the system. Intuitively if some point $x \in M$ is perturbed slightly how will the trajectories of the original point be related to those of the perturbed point. At first glance, one would think that this could depend on a variety of factors, such as the initial point, the direction of the perturbation etc. We will first examine the general one dimensional case, then a specific example, standard continued fractions on [0,1] with the shift map, $f(x) = \frac{1}{x} \bmod 1$. In these scenarios, one finds via Birkhoff's Ergodic Theorem a global measure of the sensitive dependence which is constant for almost every $x \in [0,1]$ with respect to Lebesgue measure.

4.1 The One Dimensional Case: A Heuristic Derivation

Let M be a 1-dimensional Riemann Manifold. Pick a point $x \in M$. Now perturb x by an infinitesimal amount dx. We wish examine the (exponential) rate at which the trajectories of x and $x + dx$ diverge. Let $x_0 = x$ and $y_0 = x + dx$. Continue inductively to define $x_n = f^n(x_0)$ and $y_n = f^n(y_0)$. Using the derivative of f we can also inductively approximate the distance between x_n and y_n:

$d(x_0, y_0) = dx$

$d(x_1, y_1) = d(f(x_0), f(y_0)) \approx d(f(x_0), f(x_0) + f'(x_0)dx) = |f'(x_0)|dx$

$d(x_2, y_2) = d(f(x_1), f(y_1)) \approx d(f(x_1), f(x_1) + f'(x_1)d(x_1, y_1)) = |f'(x_1)|d(x_1, y_1) = |f'(x_0)f'(x_1)|dx$

...

$$d(x_n, y_n) \approx dx \prod_{i=0}^{n-1} |f'(x_i)|$$

Now the ratio of expansion(contraction) after n iterations of f can be found by taking:

$$\frac{d(x_n,y_n)}{dx} = \prod_{i=0}^{n-1} f'(x_i).$$

Taking the absolute value and then the log of both sides, we have:

$$log(|\frac{d(x_n,y_n)}{dx}|) = log(|\prod_{i=0}^{n-1} f'(x_i)|) = \sum_{i=0}^{n-1} log(|f'(x_i)|).$$

Now averaging and taking the limit gives us the average rate of expansion:

$$\lim_{n\to\infty}\tfrac{1}{n}\sum_{i=0}^{n-1} log(|f'(x_i)|) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} log(|f'(f^i(x))|.$$

Birkhoff's Ergodic Theorem says that the above average exists and if μ is ergodic this average is constant for μ-a.e. x, and even gives us a specific method of calculating it:

$$\lambda = \lim_{n\to\infty}\tfrac{1}{n}\sum_{i=0}^{n-1} log(|f'(x_i)|) = \int log|f'|d\mu \qquad \text{(provided } log|f'| \in L^1(\mu) \text{).}$$

The number λ is called the Lyapunov exponent of the dynamical system. Note: In the case of a one dimensional manifold, there is only one Lyapunov exponent, though as we will see in the next section higher dimensional systems can have multiple Lyapunov exponents, each corresponding to a particular direction. A similar heuristic derivation for higher dimensional manifolds can be found in [O].

Example 4.1.1. *Let* $M = (0,1]$ *and let* $f : M \to M$ *be define by* $f(x) = \frac{1}{x}$ *mod 1. The Gauss measure,* μ *given by* $\mu(A) = \frac{1}{\ln\ 2}\int_A \frac{1}{x+1}dx$*, is ergodic with respect to f. Thus the Individual Ergodic theorem again yields the Lyapunov exponent:*

$$\lambda = \int_0^1 ln\tfrac{1}{x^2}d\mu = \tfrac{1}{ln\ 2}\int_0^1 ln\left(\tfrac{1}{x^2}\right)\tfrac{1}{x+1}dx = \tfrac{\pi^2}{6ln\ 2} \approx 2.37 \qquad \textit{for } \mu\textit{-a.e. } x \in (0,1]$$

Since μ *is equivalent (has same sets of measure zero) to Lebesgue measure, we know that*

λ is constant for l-a.e. $x \in (0, 1]$.

4.2 Oseledec's Multiplicative Ergodic Theorem

In the general case, the existence of the Lyapanov exponent(s) is given by Oseledec's Multiplicative Ergodic Theorem. There are many incarnations of this theorem for various types of dynamical systems. For the case of diffeomorphisms of Riemannian manifold, its general form is:

Theorem 4.2.1. *(Oseledec's Multiplicative Ergodic Theorem)*

Let $f : M \to M$ be a C^1- diffeomorphism of a compact manifold of dimension n, and let μ be ergodic with respect to f. Then one can find:

i. real numbers $\lambda_1 > \lambda_2 > ... > \lambda_k$ where $k \leq n$

such that there exists

ii. positive integers $n_1, n_2, ..., n_k$ such that $n_1 + n_2 + ...n_k = n$,

that invoke

iii. a measurable splitting $T_xM = E_x^1 \bigoplus E_x^2 \bigoplus ... \bigoplus E_x^k$ with $dim(E_x^i) = n_i$

and $D_xf(E_x^i) = E_{f(x)}^i$;

such that for $\mu - a.e.x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} log\|D_x f^n(v)\| = \lambda_l \text{ for each } v \in T_xM$$

where l is the unique integer satisfying $v \in E_x^1 \bigoplus E_x^1 \bigoplus ... \bigoplus E_x^l$ but $v \notin E_x^1 \bigoplus E_x^2 \bigoplus ... \bigoplus E_x^{l-1}$

We will call the numbers λ_i the Lyapunov Exponents for μ. Furthermore we will call the set of points for which parts i,ii, and iii of Oseledec's Multiplicative Theorem hold for a particular measure, μ, μ-regular or just regular when there is no ambiguity. In some papers, these points may be called "regular in the sense of Pesin".

In this paper we will prove a simplified version of this theorem for 2-dimensional manifold. The proof has all the features of the general theorem, but requires less linear algebra. This proof can be found in [P] and is due to Ruelle.

Theorem 4.2.2. *Oseldec's Multiplicative Ergodic Theorem for Surfaces*

Let $f : M \to M$ be a C^1-diffeomorphism of a compact surface, and let μ be ergodic with respect to f. Then either:

i.) There exists $\lambda \in \mathbb{R}$ such that for every $v \in T_xM$, $\lim_{n\to\infty} \frac{1}{n} log\|D_x f(v)\| = \lambda$ for μ-a.e. $x \in M$,

or

ii.) There exists $\lambda_1 > \lambda_2$ *and a measurable splitting* $T_xM = E_x^1 \bigoplus E_x^2$ *such that*

$$\text{For } v \in E_x^1, \quad \lim_{n\to\infty} \frac{1}{n} log\|D_x f(v)\| = \lambda_1 \qquad , \text{ and}$$

$$\text{For } v \in E_x^2 \quad \lim_{n\to\infty} \frac{1}{n} log\|D_x f(v)\| = \lambda_2$$

for μ*-a.e.* $x \in M$

Proof. The proof of this theorem will basically involve three parts. First we will find the exponents λ_1 and λ_2. Then we will find the necessary splitting. Finally we will show that our exponents work correctly on our splitting.

Finding the Exponents:

Fix $x \in M$ and let $B_n = D_x f^n$. Since these matrices B_n are not necessarily symmetric we define the following matrices, A_n. See [Ru]. For each n, there exists A_n, a symmetric matrix, such that $A_n^2 = BB^T$ and $\|A_n v\| = \|B_n v\|$ for every $v \in \mathbb{R}^2$. Since the A_n's are symmetric, for each n, there exists real eigenvalues λ_1^n, λ_2^n and orthonormal eigenvectors v_1^n, v_2^n such that:

1.) $A_n v_1^n = \lambda_1^n v_1^n$

2.) $A_n v_2^n = \lambda_2^n v_2^n$

3.) $\|v_1\| = \|v_2\| = 1$

4.) $\lambda_1^n \geq \lambda_2^n$

By Corollary 4.2 there exists $\lambda_1 \in \mathbb{R}$ such that:

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} log \|D_x f^n\| = \lim_{n\to\infty} \frac{1}{n} log \|A_n\| = \lim_{n\to\infty} \frac{1}{n} log \lambda_1^n \text{ for } \mu \text{ a.e. } x \in M.$$

Using f^{-1} in the place of f, there exists:

$$\lambda_2 = \lim_{n\to\infty} \frac{1}{n} log \|D_x f^{-n}\| = \lim_{n\to\infty} \frac{1}{n} log \|A_n^{-1}\| = \lim_{n\to\infty} \frac{1}{n} log \frac{1}{\lambda_2^n} \text{ for } \mu \text{ a.e. } x \in M.$$

The numbers λ_1, λ_2 will be the Lyapunov exponents for f.

Construction of the Splitting

To construct the splitting we will look at the sequence of eigenvectors and show that it is Cauchy. We will in fact find a rate of convergence which will be used in part three of this proof.

Suppose $\lambda_1 > \lambda_2$. For each $x \in M$ consider $E_1^n = span(v_1^n)$. We will show that the sequence v_1^n is Cauchy, thus the subspaces, E_1^n, "converge" to a space E_1.We will prove this inductively. Let e_1^n be a unit basis vector for E_1^n, and e_2^n be unit basis vector for E_2^n. With $x \in M$ again fixed, first look at $\|e_1^{n+1} - e_1^n\|$. If this sequence converges this norm will either go to 0 or to 2 (the directions may be reversed).

$\|e_1^{n+1} - e_1^n\|^2 = | < e_1^{n+1} - e_1^n, e_1^{n+1} - e_1^n > | = | < e_1^{n+1}, e_1^{n+1} > -2 < e_1^{n+1}, e_1^n > + < e_1^n, e_1^n > |$

$$= |2(1- < e_1^{n+1}, e_1^n >)| = |2(1 \pm \sqrt{1 - | < e_2^n, e_1^{n+1} > |^2}| \qquad (*)$$

With the last equality coming from the orthonormal decomposition of $< \cdot, \cdot >$. We now consider $| < e_2^n, e_1^{n+1} > |$. For each $\epsilon > 0$ there is an n such that the following inequality holds. (This is just using the properties of the inner product, and the convergence of the exponents found in part one of this proof.)

$$| < e_2^n, e_1^{n+1} > | = | < \frac{A_{n+1}e_1^{n+1}}{\lambda_1^{n+1}}, e_2^n > | \leq \frac{|<e_1^{n+1}, A_{n+1}e_2^n>|}{e^{(n+1)(\lambda_1-\epsilon)}} \leq \frac{\|A_{n+1}e_2^n\|}{e^{(n+1)(\lambda_1-\epsilon)}} = \frac{\|D_x f^{n+1} e_2^n\|}{e^{(n+1)(\lambda_1-\epsilon)}}$$

$$= \frac{\|(D_{f^n x} f) D_x f^n e_2^n\|}{e^{(n+1)(\lambda_1-\epsilon)}} \leq \frac{\|D_{f^n x} f\| \quad \|A_n e_2^n\|}{e^{(n+1)(\lambda_1-\epsilon)}}$$

Now since M is compact, there exists $W \in \mathbb{R}$ such that $\|D_x f\| \leq W$ for every $x \in M$. Using this and the definition of λ_2, for sufficiently large n:

$$| < e_1^{n+1}, e_2^n > | \leq \frac{We^{(\lambda_2+\epsilon)n}}{e^{(n+1)(\lambda_1-\epsilon)}}.$$

Since $\lambda_1 > \lambda_2$, and we can take ϵ as small as we like, there exists real numbers $C', \delta > 0$ such that:

$$| < e_1^{n+1}, e_2^n > | \leq C' e^{-\delta n}.$$

Now considering (*): $\|e_1^{n+1} - e_1^n\|$ either goes to 0 or to 2. Without loss of generality suppose it goes to 0. Then triangle inequality gives the following bound:

$$\|e_1^{n+1} - e_1^n\| = \sqrt{|2(1 - \sqrt{1 - | < e_2^n, e_1^{n+1} > |^2})|} \leq \sqrt{|2(1 - \sqrt{1 - C' e^{-2\delta n}})|}.$$

Now adjusting the constants:

$$\leq C\sqrt{1 - \sqrt{1} + \sqrt{e^{-2\delta n}}} \leq Ce^{-\frac{\delta n}{2}}, \qquad \text{and by induction}$$

$$\|e_1^{n+k} - e_1^n\| \leq \sum_{i=n}^{n+k-1} \|e_1^{i+1} - e_1^i\| \leq \frac{Ce^{-\frac{\delta n}{2}}}{1 - e^{-\frac{\delta}{2}}}. \tag{4.1}$$

Therefore the sequence is Cauchy, and must converge. A similar argument for E_2 can be made using f^{-1}. This shows that the exponents and splitting exist, now all that remains is to show that the function as advertised.

Putting it Together

To put the previous two parts together, fix $x \in M$ let $u \in E_1$ such that $\|u\| = 1$. We will show $\frac{1}{n} \log \|B_n u\| \to \lambda_1$ using the following inequality arising from the triangle inequality.

$| \ \|B_n u\| \ - \ \|B_n e_1^n\| \ | \ \leq \ \|B_n(u - e_1^n)\|$

or

$$\|B_n e_1^n\| \ - \ \|B_n(u - e_1^n)\| \ \leq \ \|B_n u\| \ \leq \ \|B_n e_1^n\| \ + \ \|B_n(u - e_1^n)\| \tag{4.2}$$

We will establish bounds for each piece of the above inequality separately. First consider the $\|B_n(u - e_1^n)\|$. It is clearly less than or equal to $\|B_n\| \ \|u - e_1^n\|$. Take Inequality 4.1, and let $k \to \infty$. Since the sequence converges, and for each n, $\|e_1^n\| = 1$, meaning $e_1^{n+k} \to u$. Thus for sufficiently large n, the following inequality holds:

$\|u - e_1^n\| \leq \frac{Ce^{-\delta n}}{1-e^{-\delta}}$.

Looking at the way λ_1 was found, for sufficiently large n:

$\|B_n\| \leq e^{(\lambda_1+\delta/2)n}$

Thus $\|B_n(u - e_1^n)\| \ \leq \ \frac{Ce^{(\lambda_1-\delta/2)n}}{1-e^{-\delta}}$.

Now we bound $\|B_n e_1^n\|$ by considering the following $\|B_n e_1^n\| = \|A_n e_1^n\| = \lambda_1^n$. Now for $\epsilon < \delta/2$, the following inequality arises again from the way λ_1 was found.

$e^{(\lambda_1-\epsilon)n} \leq \lambda_1^n = \|B_n e_1^n\| \leq e^{(\lambda_1+\epsilon)n}$

Now these can be put together with Inequality 4.2 to finish the problem. We will determine that $\lambda_1 - \epsilon \leq \underline{lim}_{n\to\infty} \ \frac{1}{n} log\|B_n u\|$, a similar inequality can be formed for the limit superior, to finish the proof.

$\|B_n e_1^n\| \ - \ \|B_n(u - e_1^n)\| \ \leq \ \|B_n u\|$

$e^{(\lambda_1-\epsilon)n} - \frac{Ce^{(\lambda_1-\delta/2)n}}{1-e^{-\delta}} \ \leq \ \|B_n u\|$

$e^{(\lambda_1-\epsilon)n}\left[1 - \frac{e^{(\epsilon-\delta/2)n}}{1-e^{-\delta}}\right] \ \leq \ \|B_n u\|$

$\lambda_1 - \epsilon + \frac{1}{n}\log\left(1 - \frac{e^{(\epsilon-\delta/2)n}}{1-e^{-\delta}}\right) \leq \frac{1}{n}\log\|B_n u\|$

Now letting $n \to \infty$ yields $\underline{lim}_{n\to\infty} \ \frac{1}{n}\log\|B_n u\| \geq \lambda_1 - \epsilon$ since $\epsilon < \delta/2$. The other direction follows similarly, as does $u \in E_2$. Note: The case of $\lambda_1 = \lambda_2$ is relatively trivial, as the above computation is all that is necessary.

$\square$

CHAPTER 5

ENTROPY

The Lyapunov exponents give one way of quantifying chaotic behavior. Another is entropy. There are two different versions of the entropy of a transformation: a topological version and a measure theoretical version. While these differ for most maps the variational principle gives a relationship for the two.

In this section the standard definitions of measure theoretical entropy, and topological entropy are given. Several equivalent definitions are also given, along with some pointwise formulas and approximations of measure theoretical entropy.

5.1 Measure-theoretic Entropy

Let (X, B, μ) be a probability space, and let $T : X \to X$ be a measure preserving transformation. To define the measure theoretic entropy of T, we will first need to develop some specified partitions. Let $\{A_n\}$ be a finite partition of X such that:

1.) For each i, A_i is measurable and $\mu(A_i) > 0$

2.) $\mu(\bigcup_i A_i) = 1$

3.) For each $i \neq j$, $\mu(A_i \bigcap A_j) = 0$

Definition 5.1.1. *Entropy of a Partition*

Let α be a partition satisfying 1,2,and 3. We define the entropy of α to be:

$$H(\alpha) = -\sum_i \mu(A_i) log \mu(A_i)$$

A axiomatic derivation which shows why this quantity agrees with the concept of entropy can be found in Khinchin's beautiful book.

For two partitions α, β of this type define their join by:

$$\alpha \vee \beta = \{A_i \bigcap B_j : A_i \in \alpha, B_j \in \beta\}$$

We are interested in how T acts on members of a partition. In particular, we would like to know the behavior of $\alpha \vee T^{-1}(\alpha)$. Inductively define:

$$\alpha^n = \vee_{i=0}^{n-1} T^{-i}(\alpha) = \alpha^{n-1} \vee T^{-1}(\alpha^{n-1})$$

Definition 5.1.2. *Entropy of a partition relative to T.*

$$h(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H(\alpha^n)$$

The previous limit can be shown to exist due to the fact that $\{H(\alpha^i)\}_{i=0}^{\infty}$ is subadditive. ($H(\alpha^{i+j}) \leq H(\alpha^i) + H(\alpha^j)$) Finally we are ready to give the definition of the measure theoretical entropy of a transformation.

Definition 5.1.3. *Measure Theoretical Entropy*

$$h_\mu(T) = \sup_\alpha \; h(T,\alpha) = \sup_\alpha \lim_{n\to\infty} \; -\frac{1}{n} \sum_{A_j \in \; \bigvee_{i=0}^{n-1} T^{-i}(\alpha)} \mu(A_j) log \mu(A_j)$$

In certain cases a suitable partition, called a generator, can be found so that it is not necessary to take the sup over all partions [W].

Now that the measure theoretic entropy has been defined, an intuitive description will be given. We can consider the act of partitioning the space to be equivalent to performing an experiment. In this experiment we only determine which member of the partition the result is in. By taking the joins of the inverse images, we replicate past experiments; the more experiments the more we know aboout the orbit of the particle. The entropy of the partition relative to T examines the limiting behavior of predicting the orbit of the particle based on n-past experiments. Finally the entropy of a transformation, h_μ, examines all possible ways of constructing an experiment(partition) for a given system. For more information on entropy in the sense of Information Theory, the reader is referred to [Bi].

5.2 Entropy on Manifolds

The previous section required only a measure space, and a transformation. When considering the measure theoretic entropy on a manifold, you can find many formulas which use the metric properties of the manifold rather than the partitions found in the previous definitions. Of particular importance are the following two theorems due to Mane and Katok and Brin respectively. [M] [BK] [Y1] Mane's theorem gives an inequality relating the metric entropy to the measure of a special set. Katok and Brin's theorem gives improves on this, giving an equality under some circumstances. The beauty of both of these theorems is that they provide global estimates based on pointwise calculations, removing the need for difficult partitions.

Theorem 5.2.1. *Mane*

Let $f : M \to M$ be a diffeomorphism, $\phi : M \to R^+$ and μ be an ergodic borel probability measure. Furthermore suppose $\int -log\ \phi\ d\mu < \infty$, then for μ-a.e. $x \in M$:

$$\overline{lim}_{n_1,n_2\to\infty} \tfrac{-1}{n_1+n_2} log\ \mu(V(x,\phi,n_1,n_2)) \leq h_\mu$$

Where $V(x,\phi,n_1,n_2) := \{y \in M : d(f^k(x), f^k(y)) \leq \phi(f^k(x))\ \ for\ each\ -n_2 \leq k \leq n_1\}$.

The following theorem improves this estimate if you take $\phi(x) = \epsilon > 0$.

Theorem 5.2.2. *Brin - Katok*

For μ - a. e. $x \in M$:

$$\lim_{\epsilon\to 0} \underline{lim}_{n_1,n_2\to\infty} \tfrac{-1}{n_1+n_2} log\ \mu(V(x,\epsilon,n_1,n_2)) = h_\mu$$

Due to Theorem 6.4, the result for 6.5 also holds for the lim sup. We will make strong use of these theorems in Chapter 9.

5.3 Topological Entropy

Let M be a compact metric space. Consider $f : M \to M$. To define the topological entropy the following metric is needed:

Define $d_n(x, y) = \max_{i \leq n} \quad d(f^i(x), f^i(y))$.

Definition 5.3.1. *A set A is called a (n, ϵ)- separated set if for every $a, b \in A$, $d_n(a, b) > \epsilon$. To say that A is a maximal (n, ϵ)- separated set means that A is a (n, ϵ)- separated set, and if B is also an (n, ϵ) separated set then $card(A) \geq card(B)$.*

Definition 5.3.2. *The topological entropy of a transformation is given by:*

$$h_{top}(f) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \quad \sup_{A_n} \frac{1}{n} \log \, card(A_n)$$

where A_n is a maximal $(n, \epsilon) -$ separated set.

5.4 Variational Principle

The variational principle relates the topological entropy to the measure theoretical entropy.

Theorem 5.4.1. *Variational Principle*

$h_{top} \geq h_\mu$ for every μ. Furthermore $h_{top} = \sup_{\mu} \; h_\mu$.

The Variational Principle is a very powerful tool. To see some of it uses in dynamical systems see [P]. Due to this strength, researchers have found other incarnations of variational principle for different dynamical quantities. See [PY] and [W].

CHAPTER 6

DIMENSION

The notion of dimension is of great interest to mathematicians studying invariant sets of transformations. Like entropy, there are multiple ways of quantifying the dimension of a set. In this paper we will deal with two different versions of dimension: metric dimension and measure theoretic dimension.

6.1 Metric Dimension

Consider a compact subset, A, of a metric space. For each $\epsilon > 0$, A can be covered by finitely many balls of radius ϵ. The box counting dimension looks at the asymptotic growth of the logarithm of the number of balls needed, divided by the negative logarithm of ϵ.

Definition 6.1.1. *The upper (or lower) box counting dimension or Capacity,* $\overline{C}$ *(or* $\underline{C}$*) , of a set* X *is given by:*

$$\overline{C}(X) = \overline{lim}_{\epsilon \to 0} \frac{logN(\epsilon)}{-log(\epsilon)} \quad or \quad \underline{C}(X) = \underline{lim}_{\epsilon \to 0} \frac{logN(\epsilon)}{-log(\epsilon)} \qquad respectively$$

Where $N(\epsilon)$ is the number of cubes of diameter ϵ needed to cover X. If $\overline{C}(X) = \underline{C}(X)$ then this common called the box counting dimension denoted $C(X)$.

One of the big advantages of using the box counting dimension is that it is quite easy to calculate, both theoretically and experimentally. On the downside though it does not always behave in the way one would like. There are countable sets for which the box counting dimension is positive. A better notion is the Hausdorff dimension. It is constructed using the Hausdorff measure, and fixes many of the problems found in capacity.

Definition 6.1.2. *Let A be a subset of a compact metric space. The δ-mesh α-dimensional Hausdorff measure is defined as follows:*

$$H_\delta^\alpha(A) := inf\{\sum_{G\in F}(\mathrm{diam}(G))^\alpha : F \ is \ a \ \delta - mesh \ cover \ of \ A\}$$

For each α, the δ-mesh Hausdorff measure is a metric outer measure. To define the α-dimensional Hausdorff measure, take the limit as $\delta \to 0$.

$$H^\alpha(A) = \lim_{\delta\to 0}\big(H_\delta^\alpha(A)\big)$$

Definition 6.1.3. *The Hausdorff dimension, HD, of a set is given by:*

$$HD(X) = \inf_{\alpha}\{\alpha : H^\alpha = 0\}$$

Note: While the definition of the Hausdorff dimension uses a measure, this is still a metric definition because this measure is based entirely on the metric properties of X.

For further information on capacity and Hausdorff dimension see [Fal].

6.2 Measure-Theoretic Dimension

The disadvantage of a metric definition of dimension, is that it looks at all points in the space equally. Frequently one would like to know about the probable points rather than the whole set. One way to fix this shortcoming is to look at the metric dimension of a set of full measure. Our first two definitions of measure theoretic dimension produced using this method.

There are measure theoretic versions of Hausdorff dimension and Capacity. Given a measure μ, define the following:

Definition 6.2.1. $HD(\mu) = \underset{\substack{A \subset X \\ \mu(A)=1}}{inf} HD(A)$

$$\overline{C}(\mu) = \underset{\delta \to 0}{sup} \underset{\substack{A \subset X \\ \mu(a) \geq 1-\delta}}{inf} \overline{C}(A)$$

$$\underline{C}(\mu) = \underset{\delta \to 0}{sup} \underset{\substack{A \subset X \\ \mu(a) \geq 1-\delta}}{inf} \underline{C}(A)$$

Renyi modified the capacity in a different way. He picked the important points based on partitions and information theory, rather than covering with balls.

Definition 6.2.2. *The Renyi or Information Dimension of a set X. Let $\{\alpha_i\}_{i\in I}$ be the set of all partitions satisfying 1,2,3 in 5.1. Define $H(\alpha)$ as in Definition 5.1.1. Define:*

$$H(\epsilon) = \inf_{\substack{\alpha_i \\ diam(\alpha_i)\leq\epsilon}} H(\alpha_i).$$

The upper and lower Reyni dimensions of μ denoted $\overline{R}(\mu)$ and $\underline{R}(\mu)$respectively are defined by:

$$\overline{R}(\mu)= \overline{lim}_{\epsilon\to 0} \frac{H(\epsilon)}{-log(\epsilon)}$$

$$\underline{R}(\mu)= \underline{lim}_{\epsilon\to 0} \frac{H(\epsilon)}{-log(\epsilon)}$$

If $\overline{R}(\mu)= \underline{R}(\mu)$, then this common number is called the Reyni dimension of μ and is denoted $R(\mu)$.

The last version of measure theoretic dimension we will examine is the pointwise dimension. Rather than being a global property of the whole set, the pointwise dimension looks at a neighborhood of a given point, then determines how "thick" the set is around that point. Hence this quantity can vary from point to point. We will exploit this local property in the proof of Young's formula.

Definition 6.2.3. *Let μ be a Borel probability measure on a compact metric space. The upper or lower Pointwise dimension at x with respect to μ, denoted $\overline{P}_\mu(x)$ or $\underline{P}_\mu(x)$ respectively, is given by*

$$\overline{P}_\mu(x) = \overline{lim}_{\epsilon \to 0} \frac{log \ \mu B_\epsilon(x)}{log \ \epsilon}$$

$$\underline{P}_\mu(x) = \underline{lim}_{\epsilon \to 0} \frac{log \ \mu B_\epsilon(x)}{log \ \epsilon}$$

If $\overline{P}_\mu(x) = \underline{P}_\mu(x)$, this common value is called the pointwise dimension of μ at x, and is denoted $P_\mu(x)$

There are many additional notions of dimension. In particular the upper and lower Ledrappier dimensions, $\overline{C_L}(\mu)$, and $\underline{C_L}(\mu)$ respectively are very similar to the upper and lower measure theoretic Capacity, but instead uses a slightly different limiting procedure. When these two agree, their common value is called simply the Ledrappier Dimension,(or Ledrappier Capacity). See [Y1].

6.3 Relationships Between Various Notions of Dimension

A great amount of study has attempted to determine when various notions of dimension agree. It is conjectured that for certain types attractors each of the topological notions of the dimension agree. This common value is sometimes called the fractal dimension. d_F of the set. It is also conjectured that under the right conditions the values of the measure theoretical dimensions will agree. This dimension is referred to as the '"dimension of the natural measure" ,d_μ. There are cases where the topological and measure theoretical dimensions agree, though this is thought to be the exception rather than the rule, but in general $d_\mu \leq d_F$. For a more thorough explanation see [Far]. The following inequalities give some relationships between the various dimensions.

$HD(\mu) \leq HD(X)$

$\overline{C}_L(\mu) \leq \overline{C}(\mu) \leq \overline{C}(\mathrm{X})$

$\underline{C}_L(\mu) \leq \underline{C}(\mu) \leq \underline{C}(\mathrm{X})$.

The following four theorems can be found in [Y1]. Their amalgamation results in the final theorem of this chapter: that all the measure theoretic notions of dimension are the same on a manifold where $P_\mu(x)$ is constant for μ- a.e. $x \in M$.

Theorem 6.3.1. *Let μ be a probability measure on a compact metric space X. Then $HD(\mu) \leq \underline{C}_L(\mu)$.*

Theorem 6.3.2. *Let μ be a continuous Borel probability measure on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be measurable and bounded such that $\mu(A) > 0$ and for every $x \in A$:*

$a \leq \underline{P}_\mu(x) \leq \overline{P}_\mu(x) \leq b.$

Then $\quad a \leq \underline{C}(\mu) \leq \overline{C}(\mu) \leq b.$

Theorem 6.3.3. *Let μ be a Borel probability measure on a compact metric space X with $\overline{C}(x) < \infty$. Then :*

1.) $\overline{R}(\mu) \leq \overline{C}_L(\mu)$

2.) If for μ-a.e. x, $P_\mu(x) \geq a$, then $\underline{R}(\mu) \geq a$.

Theorem 6.3.4. *The Big Equality*

Let μ be a Borel Probability measure on a compact Riemannian manifold, and suppose that for μ-a.e. x, $P_\mu(x) = a$. Then:

$HD(\mu) = \underline{C}(\mu) = \overline{C}(\mu) = \underline{C}_L(\mu) = \overline{C}_L(\mu) = \underline{R}(\mu) = \overline{R}(\mu) = a$

For a very good primer on dimension of an attractor, especially from an applied standpoint see [Far].

CHAPTER 7

LYAPUNOV EXPONENTS AND ENTROPY

The Lyapunov exponents provide a substantial amount of information about the structure of sets which are invariant under the transformation. The first part of this section deals with the Pesin Formula and Ruelle Inequality which relate the exponents to the entropy. Next we will focus on the information the exponents give as to the structure of invariant sets. We finally give a brief introduction to the charting process needed to prove Young's Formula.

7.1 Results of Pesin and Ruelle

Theorem 7.1.1. *Ruelle Inequality*

Let $f : M \to M$ be a C^1- diffeomorphism of a compact Riemannian Manifold. Let μ be ergodic with respect to f, with associated Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d$. Then:

$$h_\mu \leq \sum_{\lambda_i > 0} \lambda_i$$

Theorem 7.1.2. *Pesin Formula*

Let $f : M \to M$ be a C^2-diffeomorphism of a compact Riemannian manifold. Let μ be a ergodic measure with respect to f that is equivalent to lebesgue measure, with associated Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d$. Then:

$$h_\mu = \sum_{\lambda_i > 0} \lambda_i$$

Young reformulated the inequality in a pointwise manner, and used this to prove the following theorem [LY1][LY2].

Theorem 7.1.3. *Let $f : M \to M$ be a C^2-diffeomorphism of a compact Riemannian manifold preserving μ, a Borel probability measure. Then μ has absolutely continuous conditional measures on unstable manifolds if and only if:*

$$h_\mu = \int \sum_i \lambda_i^+(x) dim E_i(x) d\mu(x) \quad \textit{where } a^+ = max(a, 0)$$

and in particular if μ is also ergodic: $h_\mu = \sum_i \lambda_i^+ \; dim \; E_i$.

The decompositions necessary to get these conditional measures are quite complicated, and can be found in [LY1],[LY2], [Ro].

7.2 The Structure of Invariant Sets

Lyapunov exponents are frequently used to detect "chaos" in dynamical systems. The main reason for this is because they are much easier to calculate than entropy. With entropy one must make partitions and find the join of the inverse images, while the exponents only require the iteration of a single point. With less than 200 lines of code, a researcher can simulate the dynamical system under various parameters and initial conditions, and calculate the exponents. The exponents can then be used to determine the general structure of the invariant set. They can be used to differentiate between periodic attractors, strange attractors, divergence to infinity, attracting fixed points,etc [O]. We now will examine how the Lyapunov exponents are used to make this distinction. We will break the problem into four cases: all exponents are positive, all exponents are negative, at least one positive and one negative exponent with no zero exponents, and zero exponents.

<u>All Positive Exponents:</u> If each of the exponents are positive there is expansion in all directions. If the orbits are also bounded, then this indicates a chaotic condition in the whole space. If the orbits are unbounded, this will signify divergence to infinity.

<u>All Negative Exponents:</u> If each of the exponents are negative, there is an attractor. If additionally the sequence of iterates of a point converge to some z_0, then z_0 is a point attractor. If not there is a periodic attractor.

<u>No zero exponents:</u> The situtation in which there are both positive and negative exponents with no zero exponents is known as <u>hyperbolicity</u>. This frequently produces what is know nas a "strange attractor", and is a strong indicator of chaos.

<u>Zero Exponents:</u> If some of the exponents are zero, then there are indifferent directions.

This makes classification difficult. The presence of zero exponents is also very difficult from a theoretical standpoint, and as of yet, have not had a thorough treatment. However see [MU] for recent work on these systems in a conformal setting.

Due to the fact that zero exponents make things difficult, we will need the following lemma, which removes them from consideration in the context of diffeomorphisms of a surface.

Lemma 7.2.1. *Let $T : M \to M$ be a C^1-diffeomorphism of a compact surface such that $h_\mu > 0$, then $\lambda_1 \ > \ 0 \ > \ \lambda_2$.*

Proof. Since $\mathrm{h}_\mu > 0$, Ruelle's inequality says that $\lambda_1 > 0$ also. Since $\mathrm{h}_\mu(T^{-1}) = \mathrm{h}_\mu(T)$, $-\lambda_2 > 0$ similarly. Thus $\lambda_1 \ > \ 0 \ > \ \lambda_2$

$\square$

7.3 Lyapanov Charts

Let M be a Surface, $f : M \to M$ a $C^{1+\alpha}$ diffeomorphism, and μ an ergodic borel probability measure on M with Lyapunov exponents $\lambda_1 \geq \lambda_2$. Fix $\epsilon > 0$. Let $A : M \to \mathbb{R}$ be a function such that for $\mu - a.e.x \in M$ with the following properties:

1.)$A(f(x)) \geq (1-\epsilon)A(x)$

2.)$A(f^{-1}(x)) \geq (1-\epsilon)A(x)$

Define $R(r) = [-r, r] \times [-r, r] \subset \mathbb{R}^2$.

Here we assume the existance of the function, $A(x)$, but it can be shown to exist given f is at least $C^{1+\alpha}$. This proof is quite technical and hinges on finding a uniformly hyperbolic metric, hence the need for $C^{1+\alpha}$. A thorough treatment can be found in [LY1].

Definition 7.3.1. *Let $\Phi_x : R(A(x)) \to M$ have the following properties:*

1.) $\Phi_x(0) = x$

2.) $\|\tilde{f}_x(x) - \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \cdot x\| < \epsilon\|x\|$ *for each $x \in M$ where* $\tilde{f}_x := \Phi^{-1}_{f(x)} \circ f \circ \Phi_x$

The function Φ_x is called a $(\epsilon, A, x) -$ chart, or just an x-chart when it is clear. The function $\tilde{f}_x$ is called the connecting map. A point x is called a chart point if x satisfies the conditions above.

To distinguish between metrics on M and $\mathbb{R}^2$, we will always use $d(\cdot,\cdot)$ for the Riemannian metric on M, and $\|\cdot - \cdot\|_x$ for Euclidean distance in the x-chart. The following lemma gives a relationship between $d(\cdot,\cdot)$ and $\|\cdot - \cdot\|_x$.

Lemma 7.3.2. *There exists $C \in \mathbb{R}$ and $K(x) : M \to \mathbb{R}$ such that for every $z, z' \in R(A(x))$:*

1.) $d(\Phi_x(z), \Phi_x(z')) \leq C\|z - z'\|_x$

2.) $\|z - z'\|_x \leq K(x) d(\Phi_x(z), \Phi_x(z'))$

The next general lemma will be used to describe the relationship between an x-chart, and its iterate an f(x)-chart.

Lemma 7.3.3. *Suppose* $2\epsilon < \chi_1 - 1$ *and* $2\epsilon < 1 - \chi_2$*, and for* $k = 1,2,\ldots$ $g_k : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfy* $g_k(0) = 0$ *and* $\|g_k(x) - \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \cdot x\| < \epsilon\|x\|$*. Then:*

1.) For any $r > 0$ *,* $g_k \circ \ldots \circ g_2 \circ g_1(R(r)) \subset R((\chi_1 + 2\epsilon)^k r)$

2.) if $y = (y_1, y_2) \in \mathbb{R}^2$ *such that* $|y_1| \geq |y_2|$ *then:*

$|(g_k \circ \ldots \circ g_1(y))_1| \geq |(g_k \circ \ldots \circ g_1(y))_2|$

$|(g_k \circ \ldots \circ g_1(y))_1| \geq (\chi_1 - 2\epsilon)^k |y_1|$

Proof. 1.) Let ϵ and g_k be defined as above.

$\|g_i(x) - \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} (x)\| < \epsilon\|x\|$

Since $x \in R(r)$, $\|x\| \leq 2r$, thus:

$|(g_i(x))_1 - \chi_1 x_1| \leq \|g_i(x) - \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} (x)\| < \epsilon\|x\| \leq 2\epsilon r$

This allows us to bound $(g_i(x))_1$ as follows:

$\chi_1 x_1 - 2\epsilon r < (g_i(x))_1 < \chi_1 x_1 + 2\epsilon r$

Again since $x \in R(r)$, $|x_1| \leq r$, which gives the following inequalities:

$$-\chi_1 r - 2\epsilon r < (g_i(x))_1 < \chi_1 r + 2\epsilon r$$

$$|(g_i(x))_1| < (\chi_1 + 2\epsilon)r.$$

A similar argument holds for $(g_i(x))_2$ giving:

$$|(g_i(x))_2| < (\chi_2 + 2\epsilon)r \leq (\chi_1 + 2\epsilon)r.$$

Thus for each i:

$$g_i(R(r)) \subset R((\chi_1 + 2\epsilon)r),$$

and by induction we obtain the desired result:

$$g_k \circ ... \circ g_2 \circ g_1(R(r)) \subset R((\chi_1 + 2\epsilon)^k r)$$

The proof of part 2 follows similarly.

$\square$

CHAPTER 8

YOUNG'S FORMULA

Theorem 8.0.4. *Let $f : M \to M$ be a $C^{1+\alpha}$-diffeomorphism of a compact surface, and let μ be ergodic with respect to f, with Lyapunov exponents $\lambda_1 \geq \lambda_2$. Then*

$HD(\mu) = h_\mu(\frac{1}{\lambda_1} - \frac{1}{\lambda_2})$ *whenever the right side is not $\frac{0}{0}$.*

Proof. If λ_1, λ_2 are both positive or both negative, μ is supported on a finite set. Thus $HD(\mu)$ and h_μ are both zero. Thus the remaining case is when $\lambda_1 > 0$ and $\lambda_2 < 0$. Fix $\epsilon > 0$. We will construct a set Λ_1 such that $\mu(\Lambda_1) = 1$ and for μ-a.e. $x \in \Lambda_1$:

$$\underline{lim}_{p \to 0} \frac{log\ \ \mu(B_p(x))}{log\ p} \quad \geq \quad (\mathrm{h}_\mu - \epsilon) \left[\frac{1}{log \frac{\chi_1 + 2\epsilon}{1-\epsilon}} + \frac{1}{log \frac{\chi_2^{-1} + 2\epsilon}{1-\epsilon}} \right] \tag{8.1}$$

where $\chi_1 = e^{\lambda_1}$ and $\chi_2 = e^{\lambda_2}$.

We then fix $\delta > 0$ and construct a set Λ_δ, such that $\mu(\Lambda_\delta) > 1 - \delta$ and for $\mu - a.e\ x \in \Lambda_\delta$:

$$\overline{lim}_{p \to 0} \frac{log\ \mu(B_p(x)}{log\ p} \ \leq \ \mathrm{h}_\mu \ \left[\frac{1}{log\ (\chi_1 - 2\epsilon)} + \frac{1}{log(\chi_2^{-1} - 2\epsilon)} \right] \tag{8.2}$$

Thus the intersection $\Lambda = \Lambda_1 \bigcap \Lambda_\delta$ has measure larger than $1-\delta$ and has the desired property.

Proof of equation 1.

Take $\Lambda_1 = \{x \in M : x$ is Pesin regular, and Katok's Entropy formula holds for $x.\}$. Let $\chi_i = e^{\lambda_i}$ and fix $x \in \Lambda_1$. Choose $p_0 > 0$ such that :

$$\underline{lim}_{n_1,n_2 \to \infty} \tfrac{-1}{n_1+n_2} \; log \; \mu(V(x, p_0, n_1, n_2)) \geq \mathrm{h}_\mu - \epsilon$$

Let $p_1 = min\{A(x), \frac{p_0}{C}\}$ and let $p << p_1$. Now define $n_1(p)$ and $n_2(p)$ to be the unique integers satisfying:

$$(\chi_1 + 2\epsilon)^{n_1(p)} p \leq \tfrac{(1-\epsilon)^{n_1(p)} p_1}{2} < (\chi_1 + 2\epsilon)^{n_1(p)+1} p$$

$$(\chi_2^{-1} + 2\epsilon)^{n_2(p)} p \leq \tfrac{(1-\epsilon)^{n_2(p)} p_1}{2} < (\chi_2^{-1} + 2\epsilon)^{n_2(p)+1} p$$

This implies:

$$n_1(p) \leq \frac{log \; \frac{P_1}{2} - log \; p}{log \frac{\chi_1 + 2\epsilon}{1-\epsilon}} \leq n_1(p) + 1 \qquad \text{and}$$

$$n_2(p) \leq \frac{log \; \frac{P_1}{2} - log \; p}{log \frac{\chi_2^{-1} + 2\epsilon}{1-\epsilon}} \leq n_2(p) + 1 \qquad \text{and in particular}$$

$$\frac{log \; \frac{p_1}{2} - log \; p}{n_1(p)+n_2(p)+2} \left[\frac{1}{log \; \frac{\chi_1+2\epsilon}{1-\epsilon}} \; + \; \frac{1}{log \frac{\chi_2^{-1}+2\epsilon}{1-\epsilon}}\right] \;\; \leq \;\; 1$$

For p small enough we will show $B_{K^{-1}(x)p}(x) \subset V(x, p_0, n_1(p), n_2(p))$.

Thus $log \; \mu(B_{K^{-1}(x)p}(x)) \leq log \; \mu(V(x, p_0, n_1(p), n_2(p)))$. And for very small p:

$$\frac{log\ \mu(B_{K^{-1}(x)p}(x))}{log\ p + log\ K(x)^{-1}} \geq \frac{log\ \mu(V(x,p_0,n_1(p),n_2(p)))}{log\ p - log\ K(x)}$$

$$\frac{log\ \mu(B_{K^{-1}(x)p}(x))}{log\ p + log\ K(x)^{-1}} \geq \frac{log\ \mu(V(x,p_0,n_1(p),n_2(p)))}{log\ p - log\ K(x)} \frac{log\ \frac{p_1}{2} - log\ p}{n_1(p)+n_2(p)+2} \left[\frac{1}{log\ \frac{\chi_1+2\epsilon}{1-\epsilon}} + \frac{1}{log \frac{\chi_2^{-1}+2\epsilon}{1-\epsilon}} \right]$$

$$\frac{log\ \mu(B_{K^{-1}(x)p}(x))}{log\ p + log\ K(x)^{-1}} \geq \frac{-log\ \mu(V(x,p_0,n_1(p),n_2(p)))}{n_1(p)+n_2(p)+2} \frac{log\ \frac{p_1}{2} - log\ p}{log\ K(x) - log\ p} \left[\frac{1}{log\ \frac{\chi_1+2\epsilon}{1-\epsilon}} + \frac{1}{log \frac{\chi_2^{-1}+2\epsilon}{1-\epsilon}} \right]$$

$$\underline{lim}_{p\to 0} \ \frac{log\ \ \mu(B_p(x))}{log\ p} \ \ \geq \ \ (\mathrm{h}_\mu - \epsilon) \left[\frac{1}{log \frac{\chi_1+2\epsilon}{1-\epsilon}} + \frac{1}{log \frac{\chi_2^{-1}+2\epsilon}{1-\epsilon}} \right] \tag{8.3}$$

Now for the first direction all that is left is to prove $B_{K(x)^{-1}p} \subset V(x, p_0, n_1(p), n_2(p))$. For this we use Lyapunov charts and Lemma 8.7.

Let $y \in B_{K(x)^{-1}p}$. Thus $d(x, y) \leq K(x)^{-1}p$. For small enough p, y can be found in the x-chart, and by the Lipschitz condition on the metrics $\tilde{y} \in R(p)$ in the x- chart. By lemma 8.7, $\tilde{f}_x(\tilde{y}) \in R((\chi_1 + 2\epsilon)^k p)$ in the $\tilde{f}^k_x$- chart, for each $0 \leq k \leq n_1(p)$. Thus

$d(f^k(x), f^k(y)) \leq C\|\tilde{f}^k(\tilde{x}) - \tilde{f}^k(\tilde{y})\|_x = C\|\tilde{f}^k(\tilde{y})\|_x \leq 2C(\chi_1 + 2\epsilon)^k p \leq C(1 - \epsilon)^k p_1 \leq Cp_1 \leq p_0$

A similar argument can be made for $-n_0(p) \leq k < 0$ using f^{-1}. Thus $y \in V(x, p_0, n_1(p), n_2(p))$, and $B_{K(x)^{-1}p} \subset V(x, p_0, n_1(p), n_2(p))$.

Proof of Part 2.

Fix $\delta > 0$. For the reverse inequality we will construct a set Λ_δ with measure greater than $1 - \delta$, and a function $\phi : M \to \mathbb{R}$ with the property $\int -log\ \phi\ d\mu < \infty$. This will allow us to use Mane's estimate, and arrive at the following inequality:

$$\overline{lim}_{p\to\infty} \frac{log\ \mu(B_p(x))}{\log\ p} \leq \mathrm{h}_\mu[\frac{1}{log(\chi_1 - 2\epsilon)} + \frac{1}{log(\chi_1^{-1} - 2\epsilon)}]. \tag{8.4}$$

From there we will complete the proof in a similar fashion as the first part.

Construction of Λ_δ:

Let $\Lambda_2 = \{x \in M : x$ is a chart point $\}$. By Definition $\mu(\Lambda_2) = 1$. We now want to remove a few bad points. Let $B_n = \{x \in \Lambda_2 : K(x) \geq n\}$. Clearly the B_n are a nested set of subsets, and $\Lambda_2 = \bigcup_{i=0}^{\infty} B_i$. Thus there exists B_n such that $\mu(B_n) > 1 - \frac{\delta}{2}$. Likewise construct $E_n = \{x \in \Lambda_2 : A(x) \geq \frac{1}{n}\}$. These sets are again nested, and $\Lambda_2 = \bigcup_{i=1}^{\infty} E_i$, and there exists E_m such that $\mu(E_m) \geq 1 - \frac{\delta}{2}$. Now define $\Lambda_\delta = B_n \bigcap E_m$. This set has measure at least $1-\delta$, and both $K(x)$ and $A(x)$ restricted to Λ_δ are bounded (above and below respectively). Define $K_1 := \underset{x\in\Lambda_\delta}{sup}\ K(x)$, and $A_1 := \underset{x\in\Lambda_\delta}{inf}\, A(x)$. Since $A(x) = \frac{A(x)}{2}$ also satisfies the conditions for Lyapunov charts, we can assume without loss of generality that $A_1 C \leq 1$.

Construction of ϕ:

To produce the function ϕ we use Poincare's Recurrence theorem. Again without loss of generality we may assume that every $x \in \Lambda_\delta$ returns to Λ_δ under both f and f^{-1}. Define:

$r_1(x) = \{$ the smallest positive integer such that $f^k(x) \in \Lambda_\delta$, and

$r_2(x) = \{$ the smallest positive integer such that $f^{-k}(x) \in \Lambda_\delta$.

Define $\phi : M \to \mathbb{R}$ by :

$$\phi(x) := \left\{ \begin{matrix} min \left\{ \begin{matrix} A_1 K_1^{-1}(\chi_1+2\epsilon)^{-r_1(x)} \\ A_1 K_1^{-1}(\chi_2^{-1}+2\epsilon)^{-r_2(x)} \end{matrix} \right\} & if\ x \in \Lambda_\delta \\ 1 & if\ x \notin \Lambda_\delta \end{matrix} \right\}$$

By Kac's Formula both r_1, r_2 are integrable ([PY], pg.92). Thus $\int -log\ \phi\ d\mu < \infty$, and Mane's inequality holds for $\mu - a.e.\ x \in M$.

Now will we finish the proof in the same way as Part 1. Let $p > 0$, define $n_1(p)$ and $n_2(p)$ to be the smallest integers such that:

$$(\chi_1 - 2\epsilon)^{n_1(p)} p \geq A_1 \qquad \text{and} \qquad (\chi_2^{-1} - 2\epsilon)^{n_2(p)} p \geq A_1$$

The Subset Argument:

We now show that $V(x, \phi, n_1(p), n_2(p)) \subset B_{2Cp}(x)$ for all $x \in \Lambda_\delta$. Let $x \in \Lambda_\delta$ and $y \in V(x, \phi, n_1(p), n_2(p))$. Since $d(x,y) \leq \phi(x) \leq A_1 K_1^{-1}(\chi_1 + 2\epsilon)^{-r_1(x)} \leq A_1 \leq A(x)$, y is in the x-chart. y can be written (y_1, y_2). Without loss of generality suppose $|y_1| \geq |y_2|$. We will show that $|y_1| \leq p$.

Let $\{s_0, s_1, ...\}$ be the set of all points such that $f^{s_i} \in \Lambda_\delta$ written in ascending order. There exists and $n \in \mathbb{N}$ such that $s_n \leq n_1(p) < s_{n+1}$. Since $y \in V(x, \phi, n_1(p), n_2(p))$, $d(f^{s_i}(x), f^{s_i}(y) \leq \phi(f^{s_i}(x)) \leq A_1 K_1^{-1}(\chi_1 + 2\epsilon)^{-r_1(f^{s_i}(x))} \leq A_1 \leq A(f^{s_i}(x)$ for each $0 \leq i \leq n$, $f^k(\ y)$ is in the $f^k(x)$-chart. Now by Lemma 8.7:

$|(f^{s_n}(y))_1| \geq (\chi_1 - 2\epsilon)^{s_n}|y_1| \geq \frac{A_1|y_1|}{(\chi_1-2\epsilon)^{n_1(p)-s_n}p}$.

Now by the way n was picked,

$|(f^{s_n}(y))_1| \geq \frac{A_1|y_1|}{(\chi_1-2\epsilon)^{r_1(f^{s_n}(x))}p} \geq \frac{\phi(f^{s_n}(x))K_1|y_1|}{p}$.

Using the fact that $d(f^{s_n}(x), f^{s_n}(y)) \leq \phi(f^{s_n}(x))$ and working on the other side yields:

$K_1\phi(f^{s_n}(x)) \geq |(f^{s_n}(x))_1| \geq \frac{\phi(f^{s_n}(x))K_1|y_1|}{p}$.

and finally $\frac{|y_1|}{p} \leq 1$. Thus $\|x - y\| \leq 2p$ and by the lipschitz condition on the charts, $d(x,y) \leq 2Cp$, which puts $y \in B_{2Cp}(x)$.

Putting It Together:

$V(x,\phi,n_1(p),n_2(p)) \subset B_{2Cp}(x) \quad \Rightarrow log\ \mu(V(x,\phi,n_1(p),n_2(p))) \leq log\ \mu(B_{2Cp}(x))$, and:

$$-log\ \mu(V(x,\phi,n_1(p),n_2(p))) \geq -log\ \mu(B_{2Cp}(x)) \tag{8.5}$$

Now looking at the way $n_1(p)$ and $n_2(p)$ were picked yields:

$n_1(p) \geq \frac{log\ A_1 - log\ p}{log(\chi_1-2\epsilon)} \geq n_1(p) - 1$ and

$n_2(p) \geq \frac{log\ A_1 - log\ p}{log(\chi_2^{-1}-2\epsilon)} \geq n_2(p) - 1$ Thus

$\frac{1}{n_1(p)+n_2(p)-2} \geq \frac{1}{log\ A_1 - log\ p}\left(\frac{1}{log(\chi_1-2\epsilon)} + \frac{1}{log(\chi_2^{-1}-2\epsilon)}\right)^{-1}$

Putting this together yields:

$$\frac{-log\ \mu(V(x,\phi,n_1(p),n_2(p)))}{n_1(p)+n_2(p)-2}\left(\frac{1}{log(\chi_1-2\epsilon)}\ +\ \frac{1}{log(\chi_2^{-1}-2\epsilon)}\right)\geq\frac{log\ \mu(B_{2Cp}(x))}{log\ p-log\ A_1}.$$

Now taking the lim sup as $p\to 0$, then putting parts one and two together gives:

$$(\mathrm{h}_\mu-\epsilon)\left[\frac{1}{log\frac{\chi_1+2\epsilon}{1-\epsilon}}+\frac{1}{log\frac{\chi_2^{-1}+2\epsilon}{1-\epsilon}}\right]\ \leq\ \underline{P}_\mu(x)\ \leq\ \overline{P}_\mu(x)\ \leq \mathrm{h}_\mu\left[\frac{1}{\chi_1-2\epsilon}+\frac{1}{\chi_2^{-1}-2\epsilon}\right].\quad(8.6)$$

Now taking the $\epsilon\to 0$ gives the desired result.

$\square$

CHAPTER 9

CONCLUSION

In this paper, we showed a strong relationship between the dimension, entropy, and Lyapunov exponents of a diffeomorphism on a surface. There has been further work trying to relate this to other dynamical systems. In [LY1] [LY2] Ledrappier and Young prove:

Let $f : M \to M$ be a C^2-diffeomorphism of a compact Riemannian manifold preserving μ, a Borel probability measure. Then μ has absolutely continuous conditional measures on unstable manifolds if and only if:

$$\mathrm{h}_\mu = \int \sum_i \lambda_i^+(x) dim E_i(x) d\mu(x) \quad \text{where } a^+ = max(a, 0)$$

and in particular, if μ is also ergodic:

$$\mathrm{h}_\mu = \sum_i \lambda_i^+ dim E_i$$

A *b*rief sketch of the proof:

Many of the ideas in Young and Ledrappier's proof can be found in the proof of Theorem 9.1. They define a function h_i in a similar manner as Mane, Katok, and Brin, but the sets $V(\cdot)$ are restricted to tangent subspace E^i, so there is only one Lyapunov exponent at work.

The idea is to run through the unstable foliations one exponent at a time, and use a similar method as in Theorem 9.1. First prove $h_1 = \lambda_1\delta_1$. Next show $h_n - h_{n-1} = \lambda_n(\delta_n - \delta_{n-1})$ by looking a 'quotient' space W^n/W^{n-1}. Finally show $h_u = \mathrm{h}_\mu$. Each piece of the inductive proof requires a more thorough version of Lyapunov charts, and many complex estimates, but still analogous to the proof of Young's formula.

There has also been more work done on the applied side of dynamical systems, notably Taken's Embedding Theorem. [T] It states that attractors arising from a diffeomorphism of a manifold, can be reconstructed by taking certain measurements of the system then embedding them in $\mathbb{R}^n$ through time-delay series. This allows researchers to take field data from a dynamical system, and recreate the attractor, determine its exponents, and so on. All of this can be found without knowing anything about the equations for its evolution.

Finally, an interesting aspect of this problem is the fact that it occurs where manifolds and measure theory cross. Questions arise on whether there is a purely measure theoretic relationship between the dimension, entropy, and expansion. Or possibly a purely metric relationship, or even a purely topological relationship.

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