THE CONCEPT OF COLLISION STRENGTH AND ITS APPLICATIONS

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Collision strength, the measure of strength for a binary collision, hasn't been defined clearly. In practice, many physical arguments have been employed for the purpose and taken for granted. A scattering angle has been widely and intensively used as a measure of collision strength in plasma physics for years. The result of this is complication and unnecessary approximation in deriving some of the basic kinetic equations and in calculating some of the basic physical terms. The Boltzmann equation has a five-fold integral collision term that is complicated. Chandrasekhar and Spitzer's approaches to the linear Fokker-Planck coefficients have several approximations. An effective variable-change technique has been developed in this dissertation as an alternative to scattering angle as the measure of collision strength. By introducing the square of the reduced impulse or its equivalencies as a collision strength variable, many plasma calculations have been simplified. The five-fold linear Boltzmann collision integral and linearized Boltzmann collision integral are simplified to three-fold integrals. The arbitrary order linear Fokker-Planck coefficients are calculated and expressed in a uniform expression. The new theory provides a simple and exact method for describing the equilibrium plasma collision rate, and a precise calculation of the equilibrium relaxation time. It generalizes bimolecular collision reaction rate theory to a reaction rate theory for plasmas. A simple formula of high precision with wide temperature range has been developed for electron impact ionization rates for carbon atoms and ions. The universality of the concept of collision strength is emphasized. This dissertation will show how Arrhenius' chemical reaction rate theory and Thomson's ionization theory can be unified as one single theory under the concept of collision strength, and how many
important physical terms in different disciplines, such as activation energy in chemical reaction theory, ionization energy in Thomson's ionization theory, and the Coulomb logarithm in plasma physics, can be unified into a single one -- the threshold value of collision strength. The collision strength, which is a measure of a transfer of momentum in units of energy, can be used to reconcile the differences between Descartes' opinion and Leibnitz's opinion about the "true" measure of a force. Like Newton's second law, which provides an instantaneous measure of a force, collision strength, as a cumulative measure of a force, can be regarded as part of a law of force in general.
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CHAPTER 1

OUTLINE OF THE DISSERTATION

When we say that one body is warmer or colder than another one, it means that the "temperature" of one body is higher or lower than that of another. When we say that one collision is "stronger" than another one, what does "stronger" really mean? Is there any scientific or physical term, which acts like the role of temperature, to measure the strength of a collision so that we can determine which one is stronger? Surprisingly, there has been no simple answer to this basic problem[1]. It is the purpose of this dissertation to define a physical term, collision strength, and show some applications about collision strength.

While an absolute collision strength is not defined clearly, in practice, people usually use a relative measure of collision strength for a special set of binary collision events. For all head-on binary hard sphere collisions of molecules $A$ and $B$, the relative kinetic energy at the moment of contact has been used as the measure of collision strength for those kinds of collisions[2]. Large kinetic energy collisions are strong collisions while small kinetic energy collisions are weak collisions. In another case, both scattering angle and impact parameter can be used as the measure of collision strength for collisions in Rutherford alpha particle scattering experiments. In this case, small impact parameter or large scattering angle collisions are strong collisions while large impact parameter or small scattering angle collisions are weak collisions. We call the kinetic energy in the first example, scattering angle
and impact parameter in the second example as collision strength variables because these variables provide a good relative measure of collision strength for special sets of collision events.

One principle of the application of the collision strength concept is that using collision strength as an independent variable can simplify most of calculations that are associated with collisions. In practice, people usually select adequate variables in their calculations. However, there are some cases in which a better selection is possible. The penalty of not using the best collision strength variable is added complication and unnecessary approximation. Switching to a better collision strength variable can simplify many calculations. The main content of this dissertation are examples of switching to a better collision strength variable by employing a variable change technique. The order of the chapters are carefully arranged to illustrate one important concept, collision strength.

Due to the long range property of the Coulomb force, the total collision rate is divergent. The practically used plasma collision rate is an approximate description, where many weak collisions are equivalent to one strong collision [3, 4]. A new approach to obtain a plasma collision rate is derived in Chapter 2. In order to provide a simple and exact description of the plasma collision rate, the square of the reduced impulse is introduced to represent the measure of collision strength. A simple differential collision rate distributed on the collision strength is obtained[5].

It is shown in Chapter 3 that the use of the square of the reduced impulse as a general
collision strength variable has long been implicated in chemical reaction rate theory[2, 6].

While appearing different and being exclusively defined for hard sphere interactions, the definition of the measure of collision strength in chemical reaction rate theory is consistent with our new more general definition of collision strength. The new and more general definition naturally generalizes bimolecular collision reaction rate theory to a general reaction rate theory that can include non-hard sphere interactions. A new reaction rate formula, much like Arrhenius’ formula, is obtained for Coulomb interactions. The Arrhenius-like formula is successfully applied to the electron impact ionization rate for carbon atoms and carbon ions [6].

A careful selection of a collision strength variable can simplify many problems. In Chapter 4, the pre-collision velocity of a representative test particle is selected as the relative collision strength variable. Then, the five-fold linear Boltzmann collision integral is simplified to a three-fold integral [7, 8].

In Chapter 5, we prove that the singularity point of the Boltzmann collision integral for an inverse-square force law is at $H = 0$, where $H$ is the square of the reduced impulse. When the relative speed has varied values instead of being constant, the customarily used scattering angle cutoff is incapable of solving the divergence difficulty of the integral. Making a cutoff on $H$, we can further simplify the linearized Boltzmann collision operator by integrating over the scattering angle.

In the case of the calculation of linear Fokker-Planck coefficients, the velocity change
of the test particle is taken as the collision strength variable. The probability function for a Maxwellian distribution is first calculated. With the probability function, the arbitrary higher order Fokker-Planck coefficients are obtained [9, 10] and expressed in a uniform formula in Chapter 6. It is shown that a cutoff must be taken on the collision strength variable. A cutoff on the scattering angle cannot take into account the effect caused by the variation of relative speed.

As an example of the application of the new expression of the linear Fokker-Planck coefficients, a new expression for an equilibration time is obtained in Chapter 7. Comparing with the standard equilibration time obtained by Spitzer[4], based on Chandrasekhar’s formulas for the velocity space friction and diffusion coefficients, the difference between the standard expression and our new expression for equilibration time is characterized by a replacement of the Coulomb logarithm[11]. A zeroth order incomplete gamma function with the minimum non-dimensional collision strength as its argument replaces the Coulomb logarithm.

Chapter 8 is a summary and generalization of the new results from the previous chapters. A concept of relative reduced temperature $T'$ is first introduced as an average of the collision strength for a two-temperature system of hard sphere interactions. With the relative reduced temperature, a non-dimensional collision strength variable is defined as $y = H/(k_B T')$. Then, all the new results can be expressed uniformly in terms of $y$ and $y_{\text{min}} = H_{\text{min}}/(k_B T')$. The differential collision rate in Chapter 2 and reaction rate coefficient in Chapter 3 developed for a one-temperature system are now generalized to be for a two-temperature system in
Chapter 8. We show that an Arrhenius-like formula[6] is in fact a reaction rate formula based on Thomson’s ionization theory[12]. Therefore Arrhenius’ bimolecular reaction rate theory[2, 13] and Thomson’s ionization theory[12] are unified as one theory. The exponential part of the Arrhenius formula and the Coulomb logarithm in plasma physics are also unified as incomplete gamma functions with different orders $\Gamma(\alpha, y_{\text{min}})$ as the factor that appeared in the Arrhenius-like formula. A special function $\Upsilon_j(\alpha, x)$, which can be used to express all of the averages of physical terms for a test particle in Maxwellian field particles, is obtained. All of the physical terms including arbitrary orders of Fokker-Planck coefficients, collision frequencies, energy lose rates, and the average relative speed are well summarized by the special set of functions.

The concept of collision strength can unify many theories from different disciplines. The universality of collision strength is associated with another more basic concept—force in general. A general discussion of the law of force and measurements of a force is provided in Chapter 9. The usefulness of reduced impulse as a collision strength variable can be arrived at by analyzing the basic property of a force. To be consistent with Newton’s third law, momentum transfer must be used as a cumulative effect of a force. To remove the mass dependence, a modification on momentum transfer is necessary to make it consistent with our experiences. It is shown that such a definition of collision strength can reconcile Descartes’ view about the true measure of force with that of Leibnitz’s. The square of reduced impulse can be regarded as a cumulative effect of a force, just like the product of
mass and acceleration as an instantaneous effect of a force. The dissertation is finished with the final conclusion that collision strength might be the right choice for the true measure of force.
2.1 Introduction

The concept of collision rates and frequencies plays an import role in the description of many-particle systems. The same concept has been widely used in kinetic gas theory and plasma theory. Because of the complexity of many particle interactions, a variety of different collision rates and collision frequencies are needed for an adequate description of kinetic phenomena. The varieties are due to different considerations. Different species and different types of collisions, e.g. elastic and inelastic ones, have different forms for collision frequencies. The variations also come about from the purpose of kinetic calculations. For a theory of viscosity or thermal conductivity, an infinite set of collision frequencies are generally required to give a complete description of elastic collisions between even one single species. Each transport coefficient can then be expressed approximately by these transport collision frequencies. A nice discussion of these kinds of collision frequencies and their relations to the phenomenological transport coefficients has been given by Suchy[15]. Pure collision rates refer only to the rate of collisions. In this chapter, our interest is limited to the pure elastic collision rates in statistical equilibrium.

For pure collision rates, there are still several kinds of descriptions dependent on the
number and type of independent variables. The more independent variables, the more detailed the description and the more complex. The simplest one is the total collision rate that has no independent variable and gives only the number of collision events per unit time and volume. Other kinds of collision rates have differential forms that give more details of collision rates distributed on one or several independent variables. The most complex classical differential collision rates have up to eight independent variables. While the most complex collision rates give the most details of collision rates, one would prefer to have the simple total collision rates or differential collision rates with fewer independent variables, from which significant physical information is much easier to extract. The most complex differential collision rates are usually obtained through an analysis of microscopic collision processes. All other simple and useful collision rates with fewer independent variables are obtained from the complex ones by integration.

Knowledge about collision rates for hard sphere interaction gases is more complete than that for plasmas. The total collision rate for hard sphere interaction is exact. Each differential collision rate with fewer variables has been derived from the differential collision rates that have more variables. For hard sphere interactions, there is a set of exact expressions of differential collision rates with the number of independent variables reduced from eight down to one. A complete calculation and expressions of collision rates for hard sphere interactions can be found in Ref. [16].

In plasma physics, the situation is quite different. The practical total plasma collision
rate that appears in the literature is a reduced approximation. In calculating a “total” collision rate, many small angle collision events are made equivalent to a single large angle collision event[3, 4]. Because of a divergence, an exact total collision rate for a plasma cannot be obtained when binary Coulomb collisions are considered. However, it is still possible to derive other simple differential forms of collision rates with fewer independent variables[5]. The simplest exact expression of a collision rate should be one with only one independent variable. It is the purpose of this chapter to derive a new set of simple and exact expressions of differential collision rates for plasmas. The simplest differential collision rate has only one independent variable for equilibrium plasmas. That special variable is found to be the collision strength variable, \( H = (\Delta p)^2/(8\mu) \) where \( \Delta p \) is the magnitude of momentum transfer for a binary Coulomb collision and \( \mu \) is the reduced mass for the binary collision.

The concept of collision cross-section and the definition of collision rates are introduced in Section 2.2. Starting from the exact complex differential collision rate for a plasma, various simple differential collision rates are derived in Section 2.3. In Section 2.4, a definition of the absolute collision strength is provided and conditions for several other relative collision strength variables are discussed based on the absolute one. The concept of collision strength and the method of calculation can equally apply to the case for hard sphere interactions. In Section 2.5, brief results for hard sphere interaction are given. Comparing with that for the hard sphere interaction, a reduce total collision rate for a plasma is proposed in Section 2.6.
2.2 The Definition Of Collision Frequency and Collision Rate

The situation would be quite simple if we have always the same relative velocity between colliding particles. Under this (unrealistic) hypothetical situation, let us consider elastic hard sphere collisions. Assume the interaction distance is

$$\rho_H = (d + d_F)/2,$$  \hspace{1cm} (2.1)

where \(d\) and \(d_F\) are diameters of the test and field particles. With a given impact parameter \(b\), a collision would occur for an impact parameter smaller than the interaction radius \(\rho_H\) and would not occur for an impact parameter greater than that limit. This situation can thus be described by indicating as collision cross-section the disc with surface

$$S = \pi \rho_H^2.$$  \hspace{1cm} (2.2)

Once the collision cross-section is known, the number of collisions is easily calculated for a given relative speed \(V\). Let us consider one test particle colliding with field particles of density \(n_F\). Then the number of collision events per unit time is simply the number of possible field particles in a cylinder with length \(V\) and cross-section \(S\).

$$\nu = n_F V S.$$  \hspace{1cm} (2.3)
Here $\nu$ represents the collision frequency of one test particle with that of field particles $F$. The collision frequency, $\nu$, of a test particle with field particles has the dimension of a reciprocal time. Assuming the density of the test particles is $n$, the total collision rate between test particles and field particles is

$$\omega = nn_F V S. \quad (2.4)$$

In Eq.(2.4), the collision rate $\omega$ represents the total number of collisions per unit time and unit volume.

In reality, not all collision events have the same relative speed $V$. Only a small infinitesimal fraction of the collision events have the relative speed $V$. The number of that small fraction of collisions can be expressed by a differential collision rate. A differential expression with six independent variables is obtained from Eq.(2.4) by replacing each density of particles by differentials obtained from their velocity distribution functions

$$d\omega = f(v)f_F(v_F)V S dv dv_F. \quad (2.5)$$

Here the relative speed $V = |v - v_F|$ is a variable depending on the velocities of the test and field particles. When the gas is in thermal equilibrium, the statistical distribution $f$ and $f_F$ are the well-known Maxwellian distribution. The exact expression of the total collision rate
can be obtained by integration of Eq.(2.5) as [16]

$$\omega = n n_F S \sqrt{8k_B T/ (\pi \mu)},$$  \hspace{1cm} (2.6)

where $k_B$ is Boltzmann’s constant, $T$ is the equilibrium temperature, and

$$\mu = m m_F / (m + m_F)$$  \hspace{1cm} (2.7)

is the reduced mass with $m$ the mass of a test particle and $m_F$ the mass of a field particle. Equation (2.6) can be written as

$$\omega = n n_F v_{av} \pi \rho_H^2,$$  \hspace{1cm} (2.8)

in which

$$v_{av} = \sqrt{8k_B T/ (\pi \mu)}$$  \hspace{1cm} (2.9)

is the average of the relative speed. The process of integration and other expressions of differential collision rates for hard sphere interactions can be found in Chapman’s book [16].

A plasma is a collection of charged particles. The collision between two charged particles cannot be stated by a “yes or no” answer. For plasmas, it is impossible to obtain equivalent expressions for Eqs. (2.5-2.6), because $S$ can be infinitely large due to long range Coulomb interactions. However, it is possible to obtain expressions for the differential collision rates.

If one wants to know the change of the direction of the relative velocity produced by a
collision, one needs to find the most detailed differential collision rate. In the case of hard sphere interactions, one can replace the total cross-sections $S$ in Eq. (2.5) by the differential form $\frac{1}{4}\rho_H^2 d\Omega$ to get the most detailed differential collision rate with eight independent variables as

$$d\omega = f(v)f_F(v_F)V\frac{\rho_H^2}{4}d\Omega d\nu d\nu_F,$$

where $d\Omega = \sin \theta d\theta d\varphi$ is the differential of solid scattering angle, and $\theta$ and $\varphi$ are respectively the scattering angle and azimuthal angle around the relative velocity $V = v - v_F$. Equation (2.10) is the most detailed differential collision rate which has eight independent variables. Six of the eight independent variables are the velocities of two kinds of colliding particles, the other two variables are solid scattering angles which are determined by the directions of collisions. It is notable that Eq. (2.5) can be recovered from Eq. (2.10) by integration over the solid scattering angles, because

$$\int \frac{\rho_H^2}{4} d\Omega = \pi \rho_H^2 = S.$$

Therefore, the most detailed differential form of collision rate, such as Eq. (2.10), is the most comprehensive one. All the other differential collision rates can be derived from the most comprehensive one by integration.
2.3 Simplified Differential Collision Rates for a Plasma

In the case of a plasma, while it impossible to get collision rates that are equivalent to Eqs.(2.5-2.6), it is possible to obtain a detailed differential collision rate like Eq. (2.10). For a plasma in equilibrium, the differential collision rate with eight independent variables is expressed as

\[ d\omega(v, v_F, \theta, \varphi) = f_0(v)f_0(v_F)V\sigma_R\sin\theta d\theta d\varphi dv d\nu_F, \tag{2.12} \]

where the Maxwellian distribution function for the test particles is

\[ f_0(v) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right), \tag{2.13} \]

and for the field particles is

\[ f_0(v_F) = n_F \left( \frac{m_F}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m_Fv_F^2}{2k_B T} \right). \tag{2.14} \]

The only difference between Eq.(2.10) and Eq. (2.12) is the differential scattering cross-section. Here, in Eq. (2.12),

\[ \sigma_R = \left( \frac{Z_{p}e^2}{4\pi \varepsilon_0 \mu} \right)^2 \frac{1}{4V^4 \sin^4(\theta/2)} \tag{2.15} \]
is the well-know Rutherford differential collision cross-section in which \( Z \) and \( Z_F \) are the charge states for a test and field particle, \( e \) is the unit charge, \( \varepsilon_0 \) is the permittivity of free space, and \( \theta = \arccos(\mathbf{V}' \cdot \mathbf{V}/V^2) \) is the scattering angle in which \( \mathbf{V}' \) and \( \mathbf{V} \) are relative velocities before and after a collision.

The traditional approach[16] to obtaining a collision rate for hard sphere interactions cannot be applied to obtain a the collision rate for plasmas. A divergence occurs when integrating over the scattering angle \( \theta \). The denominator, \( 4V^4 \sin^4(\theta/2) = 0 \), causes the divergence, which is in the Rutherford’s differential collision cross-section. When the scattering angle \( \theta = 0 \) is used as an integration limit, the integral is divergent, which is the customary divergence difficulty of the scattering angle integral. However, another divergence associated with the relative speed \( V \) is usually over looked or covered up by a so-called thermal velocity approximation[9]. In fact, the six-variable velocity integral is also divergent for the present set of independent variables.

For the standard selection of independent variables, more than one integration may diverge. However, a different set of independent variables with only one integration that diverges can be found. We notice that the denominator of Rutherford’s differential collision cross-section is proportional to the magnitude of the velocity change of the test particle \( \Delta v \), because

\[
\Delta V = |\mathbf{V}' - \mathbf{V}| = 2V \sin(\theta/2), \tag{2.16}
\]
and
\[ m\Delta v = \mu \Delta V, \]  
(2.17)

where \( \Delta V \) is the magnitude of the change of relative velocity. Therefore, the magnitude of the velocity change of the test particle \( \Delta v \) can be selected as the only integration variable that causes a divergence.

In order to get other simplified differential collision rates with fewer independent variables, we change the old set of independent variables \((v, v_F, \theta, \phi)\) to a new set \((v, \Delta v, \theta, \phi)\) where \( \phi \) is the azimuthal angle around the test particle velocity change of a collision \( \Delta v = v' - v \).

The advantage of the new set of variables is that it includes \( \Delta v = |\Delta v| \), which is the only variable that causes a divergence. The differential collision rate under the new set of variables can be obtained from Eq. (2.12) by variable change as

\[ d\omega(v, \Delta v, \theta, \phi) = f_M(v) f_M(v_F) V \sigma_R \sin \theta \, |J| \, d\theta d\phi d\Delta v, \]  
(2.18)

where \( |J| \) is the Jacobian for the transformation. The relations between the new variables and the old variables have been calculated in Appendix A. The expressions for the square speed of a field particle \( v_F^2 \), the relative speed \( V \), and the Jacobian of the transformation \( |J| \) in terms of the new variables \((v, \Delta v, \theta, \phi)\) are derived in Appendix A as

\[ v_F^2 = v^2 + 2v \cdot \frac{\Delta v}{a} + \left(\frac{\Delta v}{a}\right)^2 \csc^2 \left(\frac{\theta}{2}\right) - 2 \left| v \times \frac{\Delta v}{a} \right| \frac{\sin(\phi + \phi_0)}{\tan(\theta/2)}, \]  
(2.19)
\[ J = (a \sin(\theta/2))^{-3}, \quad (2.20) \]

\[ V = a^{-1} \Delta v \csc(\theta/2), \quad (2.21) \]

where the mass factor \( a \) is defined as

\[ a = 2\mu/m. \quad (2.22) \]

In Eq. (2.19), \( \phi_0 \) is a function of velocity \( v \) and velocity change \( \Delta v \) of the test particle. Since \( \phi_0 \) will disappear after the integral over the azimuthal angle \( \phi \), we do not give the exact expression of \( \phi_0 \) here.

Before substituting Eqs. (2.19-2.21) into Eq. (2.18), we can go one step further. It is convenient to scale the independent variables to non-dimensional variables. With the field particle thermal velocity

\[ v_{\text{th}} = \sqrt{2k_B T/m_F}, \quad (2.23) \]

we define the non-dimensional velocity \( u \) and velocity change \( u_\delta \) of the test particle as

\[ u = v/v_{\text{th}}, \quad (2.24) \]

\[ u_\delta = \Delta v/(av_{\text{th}}). \quad (2.25) \]
Then we have

\[ d\mathbf{v} = v_{\text{th}}^3 d\mathbf{u}, \quad (2.26) \]

\[ d\Delta \mathbf{v} = (av_{\text{th}})^3 d\mathbf{u}_\delta. \quad (2.27) \]

Substituting Eqs. (2.24) and (2.25) into Eqs. (2.19-2.21) first, then substituting Eqs. (2.26-2.27) and Eqs. (2.19-2.21) with all non-dimensional variables into Eq. (2.18), the eight-variable differential collision rate can be expressed as

\[ d\omega(u, u_\delta, \theta, \phi) = \omega_0 \exp\left(-\frac{2a^{-1}u^2 - 2u \cdot u_\delta}{(2a)^{3/2} \pi^{7/2} u_\delta^3} \right) \exp\left(-u_\delta^2 \csc^2(\theta/2) \right) \]

\[ \times \exp\left(\frac{2 |u \times u_\delta|}{\tan(\theta/2)} \right) \frac{\sin \theta d\theta d\phi}{\sin^4(\theta/2)} d\mathbf{u} d\mathbf{u}_\delta, \quad (2.28) \]

where \( \omega_0 \), which has the dimensions of reaction rate and can be considered a close collision rate (see below), is

\[ \omega_0 = \left( \frac{1}{m} + \frac{1}{m_F} \right)^{1/2} \frac{nn_F Z_F^2 Z_e^2 e^4}{2^{9/2} \pi^{3/2} \varepsilon_0^2 (k_B T)^{3/2}}. \quad (2.29) \]

Equations. (2.28) and (2.29) are the new form of a differential collision rate with eight non-dimensional variables. Other simplified differential collision rates with fewer independent variables can be found by integration of Eq. (2.28). The only singularity point of Eq. (2.28) is at an integration limit \( u_\delta = 0 \). The rest of this section is a series of integrations over the other seven independent variables.

Using the integral formula Eq.(B.1) proved in Appendix B to integrate over the azimuthal
angles $\phi$ and following a variable change $x = \cot^2(\theta/2)$, we have

$$d\omega(u, u_\delta, x) = 4\omega_0 \frac{\exp(-2a^{-1}u^2 - 2u \cdot u_\delta - u_\delta^2)}{(2a)^{3/2} \pi^{5/2} u_\delta^3} \times \exp(-u_\delta^2 x) I_0(2 |u \times u_\delta| \sqrt{x}) dx du du_\delta,$$  \hspace{1cm} (2.30)

where $I_0$ is the first order modified Bessel function[17].

In Eq.(2.30), the integral over $x$ can be done by applying the integral formula Eq.(B.4) in Appendix B. Then, we have a differential collision rate with six velocity variables as

$$d\omega(u, u_\delta) = 4\omega_0 \frac{\exp(-2a^{-1}u^2 - 2u \cdot u_\delta - u_\delta^2 + |u \times u_\delta|^2 u_\delta^{-2})}{(2a)^{3/2} \pi^{5/2} u_\delta^3} dudu_\delta.$$

Equation (2.31) gives a description of the plasma collision rate depending on the velocity and velocity change of the test particles.

Using spherical coordinates, there is no difficulty to integrate Eq.(2.31) over all of the angles except for the angle $\chi$ that is formed by $u$ and $u_\delta$:

$$d\omega(u, u_\delta, \chi) = \frac{\exp(-2a^{-1}u^2 - 2uu_\delta \cos \chi - u_\delta^2 + u^2 \sin^2 \chi)}{(32\omega_0)^{-1} (2a)^{3/2} \sqrt{\pi} u^{-2} u_\delta^3} \sin \chi d\chi du du_\delta.$$ \hspace{1cm} (2.32)

Equation (2.32) can be rewritten as

$$d\omega(u, u_\delta, \chi) = 32\omega_0 \frac{e^{-\xi u^2} e^{-(u_\delta + u \cos \chi)^2}}{(2a)^{3/2} \sqrt{\pi} u^{-2} u_\delta^3} \sin \chi d\chi du du_\delta,$$ \hspace{1cm} (2.33)
where a second mass coefficient is defined as

\[ \zeta = 2a^{-1} - 1 = m/m_F. \tag{2.34} \]

Integrating over the angle \( \chi \) by applying the integral formula Eq.(B.6) proved in Appendix B, we have

\[ d\omega(u, u_\delta) = 16\omega_0 \frac{\text{erf}(u_\delta + u) + \text{erf}(u_\delta - u)}{(2a)^{3/2}u_\delta^3} e^{-\zeta u^2 |u|} u du u_\delta, \tag{2.35} \]

where \( \text{erf} \) is the standard error function [18].

In Eq.(2.35), the integral over the non-dimensional speed of the test particles can be done. Using the integral formulas Eqs.(B.18) and (B.19), we get a differential collision rate with only one variable as

\[ d\omega(u_\delta) = \frac{16\omega_0}{(2a)^{3/2}u_\delta^3} \frac{1}{\zeta \sqrt{1 + \zeta}} \exp \left( -\frac{\zeta}{1 + \zeta} u_\delta^2 \right) du_\delta. \tag{2.36} \]

Equation (2.36) can be further simplified by making a variable change. Substituting

\[ y = \frac{\zeta}{1 + \zeta} u_\delta^2, \tag{2.37} \]

into Eq. (2.36), we obtain

\[ d\omega(y) = \omega_0 e^{-y} y^{-2} dy. \tag{2.38} \]
Because further integration over $y$ is divergent for a lower limit at $y = 0$, Eq. (2.38) can be regarded as the simplest exact differential collision rate for plasma in equilibrium.

2.4 The Definition of Collision Strength

In the previous section, a simple and exact differential expression for the collision rate in plasmas in thermal equilibrium is obtained as

$$d\omega(y) = \omega_0 e^{-y^2} dy,$$  \hspace{1cm} (2.39)

where $y$ is defined in Eq. (2.37), and $\omega_0$ is defined in Eq. (2.29). For a better understanding of $\omega_0$, we rewrite Eq. (2.29) in a more significant form as

$$\omega_0 = n n_F v_{av} \pi \rho_0^2,$$ \hspace{1cm} (2.40)

where the average relative speed is $v_{av} = \sqrt{8k_B T / (\pi \mu)}$ which is the same definition as Eq. (2.9), and the interaction radius $\rho_0$ for plasmas is defined as

$$\rho_0 = Z Z_F e^2 / (8\pi \varepsilon_0 k_B T).$$ \hspace{1cm} (2.41)

In this case, the interaction radius happens to be half of the Landau length $\rho_0 = \lambda_L / 2$. The usual Landau length is defined as $\lambda_L = Z Z_F e^2 / (4\pi \varepsilon_0 k_B T)$ [3]. We would like to point out
here that the definition of interaction radius Eq. (2.41) is by no means a redundancy because a more generalized version of the definition of the interaction radius, which is much different from the Landau length will be provided in Chapter 8. Clearly, $\omega_0$ has the unit of reaction rate because $y$ is dimensionless and Eq. (2.40) is much like Eq. (2.8).

From Eqs. (2.37), (2.34), (2.25) and (2.22), we find that $y$ can be written as

$$y = \frac{H}{(k_B T)},$$  

(2.42)

where the square of the “reduced” impulse of a collision event $H$ is defined as

$$H = \frac{\Delta p^2}{8\mu},$$  

(2.43)

in which $\Delta p = m\Delta v$ is the magnitude of momentum transfer of a collision. Eqs. (2.39-2.43) provide a simple and exact expression for the collision rate of plasmas in thermal equilibrium.

There exist some arbitrary choices in the selection of independent variables if a differential collision rate has two or more independent variables. The simplest differential collision rate for a plasma in equilibrium has only one independent variable. The variable $H$ is chosen here to reflect the collision strength of collision events. While it is possible to make a variable change for a single variable differential collision rate, all of the possible new variables have to be monotonically increasing or decreasing functions of $H$ to keep the collision rate distribution function as a monotonically decreasing or increasing function. The square of the
reduced impulse $H$ apparently provides the simplest form of the collision rate distribution function for a plasma in equilibrium. We call $H$ the measure of collision strength, or simply collision strength. Large $H$ collisions means strong collision events while small $H$ collisions are weak collision events. It is clear from Eq.(2.39) that the collision rate decreases when collision strength $H$ increases. It is weak collisions (small $H$ collisions) that cause the divergence of the plasma collision rate.

Many physical terms, such as scattering angle $\theta$, impact parameter $b$, momentum transfer $\Delta p$, and velocity change $\Delta v$, have been used as a measure of collision strength in plasma physics. However, there has been no in depth analysis about the consistency of these terms. If we assume that $H$ is the best choice as the basic measure of collision strength, we can analyze other physical terms to find out under what conditions they are consistent with the reduced impulse $H$.

Because the masses of both field and test particles never change, the reduced mass $\mu$ is a constant. It is found from Eq. (2.43) that the magnitude of the momentum transfer $\Delta p$ is a monotonically increasing function of $H$. So momentum transfer is consistent with $H$ as a measure of collision strength. For the same reason, it is easy to find that the magnitude of velocity change $\Delta v$ is also consistent with $H$. In fact, Eq. (2.36) is a form of collision frequency distribution, which depends on a non-dimensional velocity change $u_\delta = \Delta v/(av_{th})$. 
The relation between scattering angle $\theta$ and $H$ can be found through Eqs. (2.16-2.17),

$$H = \frac{1}{2} \mu V^2 \sin^2 \left( \frac{\theta}{2} \right).$$  \hfill (2.44)

It is clear that scattering angle $\theta$ is a monotonically increasing function only under the condition that the relative speed $V$ is a constant. However this is a condition that cannot hold for most of the problems in plasma physics. In Rutherford alpha particle scattering experiments, because the velocity of the heavy target nucleus is almost zero and alpha particles have almost the same incident velocity, the relative velocity can be regarded as a constant. So the scattering angle in Rutherford alpha particle scattering experiments is a monotonically increasing function with $H$. In other words, the scattering angle is the right choice as a measure of collision strength in Rutherford alpha particle scattering experiments because it is consistent with $H$ and is much more convenient for measurement.

To make the impact parameter consistent with our definition of collision strength $H$, it needs the condition that the scattering angle $\theta$ be constant. The relation between impact parameter $b$ and collision strength $H$ is

$$H = \frac{Z Z_F e^2 \sin \theta}{16 \pi \varepsilon_0 b}.$$  \hfill (2.45)

Because $H$ depends weakly on scattering angle $\theta$ in Eq. (2.45), the impact parameter can sometimes be *approximately* regarded as a monotonically decreasing function of $H$. 

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The hypothesis proposed for the application of collision strength $H$ is that the use of an independent variable that is a consistent with $H$ can simplify to the extent possible many problems associated with collisions. The reason that many traditional approaches fail to simplify differential collision rates for plasmas is that none of the independent variables in Eq.(2.12) are a consistent measure of collision strength. This point of view will be supported throughout the rest of this dissertation.

2.5 Collision Rates For Hard Sphere Interaction

Initially the variable change was made to deal with the divergence of collision integrals for plasmas. In the process, we came up with the definition of the collision strength variable $H$. We would like to emphasize that the collision strength variable $H$ is not just a convenient variable to solve mathematical divergence difficulties for plasmas alone. It is a convenient physical variable in many other areas. In this section, the collision rate for hard sphere interactions is explored to show the universality of the concept hypothesized for collision strength. The concept applies not only for long-range binary Coulomb interactions but also for the shortest possible interaction range.

Differential collision rates for hard sphere interactions can be calculated the same way as that in Section 2.3. The simplest differential collision rate for hard sphere interactions can be expressed as

$$d\omega(y) = \omega_0 e^{-y} dy, \quad (2.46)$$
where the non-dimension collision strength variable $y = H/(k_B T)$ is the same as in Eq. (2.42), and $\omega_0$ is defined as

$$\omega_0 = n n_F v_{av} \pi \rho_H^2. \quad (2.47)$$

As before, $v_{av} = \sqrt{8k_B T/(\pi \mu)}$ is the average relative speed, and $\rho_H = (d + d_F)/2$ is the interaction radius for hard sphere interactions, with $d$ and $d_F$ the diameters of the test and field particles. Unlike Eq. (2.39), Eq. (2.46) has no singularity. The total collision frequency for hard sphere interactions can be obtained by integrating Eq. (2.46) over $y$. We find

$$\omega = \omega_0 = n n_F v_{av} \pi \rho_H^2, \quad (2.48)$$

which is result we would expect [16].

2.6 Reduced Total Collision Rate for a Plasma

Since the total collision rate is divergent due to long range Coulomb interactions, the practical total plasma collision rate that appears in the literature is a reduced approximation. There are several *ad hoc* procedures in the calculation of the reduced collision rate[3, 4]. For example, many small angle collision events are equivalent to a single $\pi/2$ angle collision event. This $\pi/2$ is an *ad hoc* selection. In this section, a definition for a reduced total collision frequency for a plasma with less of an *ad hoc* treatment is proposed. The new definition is
made by comparing and referring to a collision frequency for hard sphere interactions.

The differential collision rate for hard sphere interactions is described by Eq. (2.46). For a hard sphere collision event, the collision strength \( H \) can be as small as 0 and as large as infinite. The average collision strength for hard sphere interaction can be found as

\[
\bar{H} = \frac{1}{\omega} \{ H \} = \frac{1}{\omega} \int_0^\infty H \omega_0 e^{-y} dy = k_B T. \tag{2.49}
\]

Therefore, \( \bar{H} = k_B T \) is the average collision strength for hard sphere interactions in equilibrium.

It is reasonable to selecte the average collision strength for a plasma to also be \( \bar{H} = k_B T \). With the differential collision rate distributed on collision strength Eq. (2.39), the reduced total collision rate for a plasma in equilibrium is naturally defined as

\[
\omega_p = \frac{\{ H \}}{\bar{H}} = \frac{1}{k_B T} \int_{y_{\text{min}}}^\infty H \omega_0 e^{-y} y^{-2} dy = \omega_0 \Gamma(0, y_{\text{min}}), \tag{2.50}
\]

where \( \Gamma(0, y_{\text{min}}) \) is the zeroth order incomplete gamma function defined as Eq. (6.5.3) in Ref.[19], and \( y_{\text{min}} = H_{\text{min}}/(k_B T) \) is the minimum non-dimensional collision strength. The value of \( y_{\text{min}} \) can be selected based on Debye shielding theory [10]. If \( y_{\text{min}} \) is selected so that \( \Gamma(0, y_{\text{min}}) = 2\lambda \) with \( \lambda \) the Coulomb logarithm, the reduced total collision frequency for the plasma \( \omega_p \) is the same as the standard definition[3, 4].
CHAPTER 3

GENERALIZATION OF CHEMICAL REACTION RATE THEORY

In the previous chapter, a square of the reduced impulse, $H$ in Eq. (2.43), is defined as an absolute measure of collision strength in order to get a simple differential collision rate for plasmas in equilibrium. It is the purpose of this chapter to show how the general concept of collision strength is used to generalize a successful theory in one discipline into other disciplines. We find that the idea of using $H$ as a measure of collision strength has been implicated in collision theory of bimolecular gas reaction rates. Then, the collision theory of bimolecular gas reaction rates is generalized to be independent of the type of force in action. For hard sphere interactions, Arrhenius’ formula is recovered. For Coulomb interactions, a new Arrhenius-like formula is obtained. The new formula is verified[6] by satisfactory agreement with empirical data of electron-impact ionization rates for carbon atoms and ions [20].

3.1 Bimolecular Gas Reaction Rate Theories

Bimolecular gas reaction theories are concerned with the rates of chemical reactions. For many reactions and particularly elementary reactions, the rate expression can be written as a product of a temperature-dependent term and a composition-dependent term. Because the
reaction rate is directly proportional to the composition term, the main concern of reaction rate theories is the temperature-dependent term, the reaction rate constant (or coefficient) $\alpha(T)$. The first formula for the reaction rate constant was proposed in 1889 by Arrhenius[13] who suggested that only those molecules having energy equal to or greater than a given level had the potential for reaction. Later, the Arrhenius formula was supported by many experiments and the other two more precise theories: the bimolecular collision theory and the transition state theory. The reaction rate constant $\alpha(T)$ for elementary reactions is almost always expressed as

$$\alpha(T) = (k_B T)^m D \exp \left( -\frac{E_A}{k_B T} \right), \quad (3.1)$$

where $D$ is a constant, which is associated with a steric or probability factor[21]. The quantity $E_A$ is usually called the activation energy, and $m$ has a different value depending on the specific theory and specific reactions. The case of $m = 0$ corresponds to classical Arrhenius theory[13]; $m = 0.5$ is derived from the collision theory of bimolecular gas phase reactions[22, 23]; $m = 1$ corresponds to transition state theory[2, 24]. For fitting experimental data, $m$ has a value that ranges from $-1.5$ to $4$ [25, 26]. The relatively small variation in the rate constant due to the pre-exponential temperature dependence $T^m$ from different theories is effectively masked by the much more temperature-sensitive exponential term $\exp(-E_A/(k_B T))$. Due to this reason, these theories are regarded to be consistent.
None of these theories is sufficiently well developed to predict reaction rates from first principles. It is important to understand the basic assumption of each theory as it is explained in Ref.[25]: “... it is interesting to note the difference in approach between the collision and transition state theories. Consider $A$ and $B$ colliding and forming an unstable intermediate which then decomposes into product, or

$$A + B \longrightarrow A^* \longrightarrow \text{products}$$ \hspace{1cm} (3.2)

Collision theory views the rate to be governed by the number of energetic collisions between reactants. What happens to the unstable intermediate is of no concern. The theory simply assumes that this intermediate breaks down rapidly enough into products so as not to influence the rate of over-all process. Transition state theory, on the other hand, views the reaction rate to be governed by the rate of decomposition of intermediate. The rate of formation of intermediate is assumed to be so rapid that it is present in equilibrium concentrations at all times. How it is formed is of no concern. Thus collision theory views the first step of [Eq. (3.2)] to be slow and rate-controlling, whereas transition state theory views the second step of [Eq. (3.2)] combined with the determination of complex concentration to be the rate-controlling factors. In a sense, then, these two theories complement each other. ” In summary, the pre-exponential temperature dependence $T^m$ can not be determined precisely by theory. In practice, $m$ is used as a fitting parameter determined by sets of experimental
The chemical reaction rate theory is very successful. The Arrhenius formula has been verified by many chemical reactions[26]. In Section 3.2, we will show that the idea of using $H$ as a measure of collision strength has been implicated in collision theory of bimolecular gas reaction rates. Then, the collision theory of bimolecular gas reaction rates is generalized to be independent of the type of force in action. Based on the general collision reaction rate theory, an Arrhenius-like reaction rate formula is derived for Coulomb interactions. In Section 3.3, the Arrhenius-like formula is used to develop semi-empirical electron-impact ionization rates for carbon atoms and ions. It is verified by satisfactory agreement between the formula and empirical data.

3.2 An Arrhenius-like Formula for Coulomb Interactions

The Arrhenius equation is highly effective. For many chemical reactions the reaction constant $\alpha(T)$ has been found in practically all cases to be well represented by Eq. (3.1). First, let us review one of the theories: the collision theory of bimolecular gas phase reactions.

The postulates for the collision theory of bimolecular reactions are as follows [2]:

1. A collision between the two reacting molecules $A$ and $B$ must occur before they can react.

2. Of the total number of collisions, only those molecules react that have a kinetic energy equal to or greater than $E_A$ (the activation energy) along the line of centers at the
moment of contact, provided that the molecules are properly orientated.

In the second postulate, the definition for strong collisions is exclusively defined for hard sphere interactions. For any other kind of interaction, such as Coulomb interactions, there is no such moment of contact as stated. It is necessary to compare the definition for strong collisions here in the second postulate with the one proposed in Chapter 2 based on the square of the reduced impulse $H$ defined in Eq.(2.43). In a collision, assume the kinetic energy is $E$ along the line of centers at the moment of contact, the momentum along the same direction can be calculated as

$$p = \sqrt{2\mu E},$$

where $\mu$ is the reduce mass of the two molecules. For an elastic collision of hard spheres, the momentum change $\Delta p$ is 2 times the momentum

$$\Delta p = 2p = 2\sqrt{2\mu E}.$$  \hfill (3.4)

The reduced square of impulse $H$ for the collision can be calculated according Eq. (2.43)

$$H = \frac{(\Delta p)^2}{8\mu} = E.$$  \hfill (3.5)

In other words, using $H$ as the definition of collision strength is consistent with collision
theory of bimolecular reactions.

Based on the above analysis, new generalized postulates for the collision theory can be rewritten as follows.

1. *A collision between the two reacting particles* $A$ and $B$ *must occur before they can react.*

2. *Of the total number of collisions, only those particles react that have a collision strength equal to or greater than* $E_A = H_{\text{min}}$ (*the minimum collision strength*), *provided that the particles are properly orientated.*

These new postulates are equivalent to the previous ones when applied for hard sphere interactions. The advantage of the new postulates is that they make the binary collision theory independent of the *type of force* in action.

Under the new generalized postulates, the general temperature-dependent reaction rate constant $\alpha(T)$ can be formulated as

$$\alpha(T) = \frac{q}{nn_F} \int_{H \geq H_{\text{min}}} d\omega,$$

where $q$ is a steric factor that represents the fraction of collisions having the proper orientation for a reaction to take place. A steric factor $q$ is expected to be a constant that is less than one, $n$ and $n_F$ are the densities of the two component particles, and $d\omega$ is the differential collision rate.
If we substitute the differential collision rate for hard sphere interactions Eq. (2.46) into Eq. (3.6), it is not surprising that we recover the formula Eq. (3.1) for the case of $m = 0.5$

$$\alpha(T) = D(k_B T)^{0.5} \exp\left(-\frac{H_{\text{min}}}{k_B T}\right),$$

(3.7)

where $D = q\sqrt{\pi/(2\mu)(d + d_F)^2}$, in which $d$ and $d_F$ are diameters of the two colliding particles. If we substitute the differential collision rate for Coulomb interactions Eq. (2.39) into Eq. (3.6), an Arrhenius-like formula of the reaction rate constant is obtained for Coulomb interactions

$$\alpha(T) = D(k_B T)^{-1.5} \Gamma\left(-1, \frac{H_{\text{min}}}{k_B T}\right),$$

(3.8)

where $D = q\sqrt{8\pi/\mu(ZZ_F e^2/(8\pi\varepsilon_0))^2}$, in which $Z$, $Z_F$ are charge states of the colliding particles, $e$ is the unit charge, and $\varepsilon_0$ is the permittivity of free space.

3.3 Electron-Impacted Ionization Rate Constant for Carbon Atom and Ions

The Arrhenius formula Eq.(3.1) has been verified by many chemical reaction experiments [26]. For example, the experimental results[27] for the reaction

$$2\text{HI} \rightarrow \text{H}_2 + \text{I}_2,$$

(3.9)
over the temperature range 670 K to 700 K are fit within experimental accuracy [23].

The purpose of this section is to show an example of satisfactory agreement between the Arrhenius-like formula and experimental data.

Since the collision theory for reaction rates is a highly simplified theory, the exponent $m$ is commonly left as a fitting parameter in Eq.(3.1), although specific values, namely $m = 0.5$, are given by the theory as in Eq.(3.7). Following this convention, the reaction rate coefficient for Coulomb interactions Eq.(3.8) can be written as

$$
\alpha(T) = (k_BT)^m D \Gamma \left( -1, \frac{H_{\text{min}}}{k_BT} \right),
$$

(3.10)

where $D$, $m$, and $H_{\text{min}}$ are three adjustable parameters which depend on the particular reaction considered. These parameters have the same physical meanings as those in the Arrhenius formula Eq.(3.1). Like the Arrhenius formula, the Arrhenius-like formula Eq.(3.10) may potentially be useful for describing a variety of reaction rates.

Because of the wide use of graphite and carbonized plasma-facing surfaces, simple expressions for ionization reaction rates of carbon atoms and ions in plasmas are needed for computer modeling of atomic processes in plasmas. The reaction of electron-impacted single ionization of carbon atoms and carbon ions can be expressed as

$$
e + C^k \longrightarrow 2e + C^{k+1},
$$

(3.11)
where the integer $k$ represents charge state and has a value from 0 to 5. In the present work, tabulated values from Ref.[20] for the electron-impact single-ionization of carbon atoms and ions are fit with Eq.(3.10). The fitting parameters, $D$, $m$, and $H_{\text{min}}$, associated with the single-ionization reaction rate for each ionizable charge state of carbon are given in Table (3.1). The fits are shown in Fig. 3.1. Although there are only three fitting parameters, Eq.(3.10) provides excellent qualitative agreement with the reaction rate data which spans nine orders of magnitude over a temperature range which spans four orders of magnitude. High precision fitting is possible over a smaller temperature range as needed. In Table (3.1), the fitted values for $D$ are all relatively close, within about a factor of ten, although the reaction rate data changes by about nine orders of magnitude. The parameter $m$ has almost the same value, $-1.3$, which is near the value of the exact simplified collision theory from Eq.(3.8), $-1.5$. The minimum collision strength or ‘activation’ energy, $H_{\text{min}} = E_A$, has values close to the carbon ionization potentials as expected. It can be concluded that the parameters in Eq.(3.10) have definite correspondence with the expected physical parameters although the theory does not include all of the complex phenomena which govern reaction rates.

In summary, a generalized approach has been presented for describing reaction rates associated with collisions of an arbitrary interaction force. In the case of hard sphere interactions, Arrhenius’ formula is recovered. For Coulomb interactions, an Arrhenius-like formula, Eq.(3.10), has been developed for describing electron-impact single-ionization of
carbon atoms and ions. Eq.(3.10), much like the Arrhenius formula, Eq.(3.1), provides a semi-empirical reaction rate. However, Eq.(3.10) is for long range (e.g., Coulomb) interactions while the Arrhenius formula, Eq.(3.1), is for short-range (e.g. hard sphere) interactions. In Ref.[28], 13 more sets of satisfactory agreements between Eq.(3.10) and the experimental data of rates for electron-impact ionization and excitation of hydrogen and helium[29], and for electron-impact single-ionization of oxygen atoms and ions[20] are found. As a semi-empirical formula, Eq.(3.10), is found to be suitable not only as an ionization rate coefficient but also as an excitation rate coefficient. This appears to be possible because these reactions result predominantly from Coulomb interactions. It can be concluded that, like the widely used Arrhenius formula for chemical reaction rates, the semi-empirical formula, Eq.(3.10), may have widespread applications in describing reaction rates when the dominate interaction of reactants is of the Coulomb type. More data are needed to further extend the applicability of the new Arrhenius-like formula Eq.(3.10).

Table 3.1: Equation(3.10) fitting parameters for single-ionization of carbon atoms and ions.

<table>
<thead>
<tr>
<th>Reactions (k)</th>
<th>-m</th>
<th>$H_{\text{min}}$(eV)</th>
<th>$D$(cm$^3$/s)</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>18.61</td>
<td>$13.14 \times 10^{-6}$</td>
</tr>
<tr>
<td>1</td>
<td>1.280</td>
<td>32.14</td>
<td>$7.09 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>1.280</td>
<td>56.24</td>
<td>$4.04 \times 10^{-6}$</td>
</tr>
<tr>
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<td>67.25</td>
<td>$1.17 \times 10^{-6}$</td>
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<td>4</td>
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<td>446.5</td>
<td>$3.29 \times 10^{-6}$</td>
</tr>
<tr>
<td>5</td>
<td>1.280</td>
<td>459.2</td>
<td>$1.08 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
Figure 3.1: Single-ionization reaction rate coefficients for carbon atoms and ions from Ref.[20], and the 3-parameter semi-empirical fitting function Eq.(3.10).
CHAPTER 4

SIMPLIFICATION OF LINEAR BOLTZMANN COLLISION INTEGRAL

In order to provide a simple expression of the differential collision rates for plasmas, a square of the reduced impulse $H$ is defined to represent collision strength in Chapter 2. It is the purpose of this chapter to show that a pre-collision velocity $v'$ can be used as a convenient collision strength variable for simplifying a linear Boltzmann integral. By employing the pre-collision velocity $v'$ as an independent variable, the five-fold linear Boltzmann integral can be simplified to a three-fold integral[7, 8].

4.1 Introduction

In dealing with the Boltzmann collision integral for Coulomb interactions, it is customary to assume that small-angle scattering dominates the integral. This weak coupling assumption provides justification of the practice, followed by Landau[30] and Rosenbluth et al.[31], in evaluating a distribution function at pre-collision velocities using a two-term Taylor series expansion about their post-collision values. The integral over scattering angles is divergent, however. The usual way to deal with this divergence is by using an *ad hoc* cutoff at a minimum scattering angle, $\theta_{\text{min}}$, which is related to the Debye shielding distance[32]. Under such an assumption, the Boltzmann collision integral is simplified to the Landau-RMJ collision...
term, which is widely used in solving the Fokker-Planck equation[33]. However, a small angle cutoff neglects some large-relative-speed strong collision events, which occur within the Debye sphere, and includes some small-relative-speed weak collision events, which occur outside the Debye sphere[7]. It is desirable to find an approach to the collision operator without using a minimum scattering angle, \( \theta_{\text{min}} \), as a Debye cutoff parameter.

The variable change technique developed in Chapter 2 can be used to simplify the linear Boltzmann collision integral[7]. An improved result for the simplified Boltzmann collision term was presented in our recent paper[8]. It is notable that a different approach with a similar result was also developed by Shoub [34]. Here in Section 4.2, an improved version of the variable change method and a more organized result is presented. The choice of using the pre-collision velocity of the test particle, \( v' \), as a new independent variable is no longer a mathematical strategy. It is introduced for two reasons. First, it is a convenient variable to represent the distribution function. Second, it is a monotonically increasing function of the collision strength \( H \) defined in Chapter 2. The property of the kernel function of the linear Boltzmann collision term is considered in Section 4.3. We also explain why, when using the new approach, the scattering angle does not cause the divergence of Boltzmann collision term for Coulomb interactions. We conclude that the Debye cutoff can be changed from the usual minimum scattering angle, \( \theta_{\text{min}} \), to the minimum collision strength \( H_{\text{min}} \).
4.2 Simplification of the Linear Boltzmann Collision Integral for Plasmas

Consider the linear Boltzmann collision integral

\[
\left( \frac{\partial f}{\partial t} \right)_e = \int (f(v')f_0(v_F') - f(v)f_0(v_F)) V\sigma_R \sin \theta d\theta d\varphi dv_F, \tag{4.1}
\]

where \(f(v)\) is the test particle velocity distribution function and \(f_0(v_F)\) is the field particle distribution function, which is assumed to be Maxwellian

\[
f_0(v_F) = n_F \left( \frac{m_F}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m_F v_F^2}{2k_B T} \right). \tag{4.2}
\]

The relative velocity is \(V = v - v_F\). The pre-collision velocities \(v'\) and \(v'_F\) are related to the post-collision velocities of the test particle \(v\) and the field particle \(v_F\) through the expressions

\[
v' = v - \frac{2\mu}{m}(V \cdot k)k, \tag{4.3}
\]
\[
v'_F = v_F + \frac{2\mu}{m_F}(V \cdot k)k, \tag{4.4}
\]

where the unit vector \(k\) is the external bisector of the angle between \(V\) and \(V'\), with \(\mu = mm_F/(m+m_F)\) the reduced mass. The unit vector \(k\) represents a direction that can uniquely determine the solid scattering angle \(\Omega(\theta, \varphi)\). The collision cross section is taken to be

\[
\sigma_R = \left( \frac{Z Z_F e^2}{4\pi\varepsilon_0 \mu} \right)^2 \frac{1}{4V^4 \sin^4(\theta/2)}, \tag{4.5}
\]
for Coulomb interactions. $\sigma_R$ is the well-known Rutherford differential collision cross-section, where $Z$ and $Z_F$ are charge states, $e$ is unit charge, and $\epsilon_0$ is the permittivity of free space.

The scattering angle cannot be integrated directly since $f(v')$ is an unknown function, and the pre-collision velocity, $v' = v'(v, v_F, \theta, \phi)$, is a function of the scattering angle. To integrate over the scattering angle, the customary approach is to approximate the distribution function at the pre-collision velocity by a Taylor series expansion that leads to the Landau collision terms[30]. Here we provide another approach. By taking the pre-collision velocity $v'$ as an independent variable, the scattering angle can be integrated exactly.

The introduction of collision strength $H$ in Chapter 2 greatly simplified the derivation of collision rates for both Coulomb interactions and hard sphere interactions. The results also suggest that the problem is simplified if there is an independent variable which is a monotonically increasing function of $H$. The collision strength $H$ is defined as the square of the reduced impulse

$$H = \frac{\Delta p^2}{8\mu},$$

(4.6)

where the magnitude of the momentum transfer is $\Delta p = m\Delta v$, with $\Delta v = |v' - v|$ the velocity change of a collision event. It is obvious that none of the five independent variables $(v_F, \theta, \varphi)$ in Eq.(4.1) is a monotonically increasing function of the collision strength $H$. It is not a good idea to use the collision strength $H$ directly as one independent variable because the collision strength $H$ is not a convenient argument to represent in a distribution function. The velocity change $\Delta v$ is a monotonically increasing function of $H$. However, the velocity
change is not as convenient as the pre-collision velocity $v'$ to represent in a distribution function. That the pre-collision velocity $v'$ can be used as a collision strength variable for this special case is due to the following observation. The velocity of the test particle $v$ is just a parameter that can be treated as a constant vector. Therefore, the pre-collision velocity $v'$ is chosen as the favored independent variable. To simplify Eq.(4.1), a variable change to make the pre-collision velocity $v'$ independent is necessary.

Before doing any variable changes, Eq.(4.1) can be rearranged to exclude the pre-collision velocity of the field velocity $v'_F$. Rewrite Eq.(4.1) as

$$
\left( \frac{\partial f}{\partial t} \right)_c = \int \left( \frac{f(v')}{f_0(v')} \frac{f_0(v'_F)}{f_0(v)} \right) f_0(v) f_0(v'_F) V \sigma R \exp(\frac{-mv^2}{2k_B T}) d\theta d\phi dv_F,
$$

(4.7)

where $f_0(v)$ is the Maxwellian velocity distribution function with the same temperature as that of field particles

$$
f_0(v) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right).
$$

(4.8)

Now, define the relative distribution function of the test particle as

$$
h(v) = f(v)/f_0(v).
$$

(4.9)
Owing to energy conservation during a collision, we have

\[ f_0(v')f_0(v_F) = f_0(v)f_0(v_F). \]  \tag{4.10}

Substituting Eqs\(4.9, 4.10\) into Eq\.(4.7), a collision integral without the pre-collision velocity of the field particle is obtained as

\[
\left( \frac{\partial f}{\partial t} \right)_c = \int (h(v') - h(v)) f_0(v)f_0(v_F)V \sigma_R \sin \theta d\theta d\phi dv_F. \]  \tag{4.11}

In Eq\.(4.11), the only unknown function is \(h\). If the pre-collision velocity \(v'\) is chosen as an independent variable of the integral, the other independent variables only appear in known functions. Therefore, the integral about the scattering angle can be separated.

Changing the independent variables of the integral in Eq\.(4.11) from \((v_F, \theta, \phi)\) to \((v', \theta, \phi)\) with \(\phi\) the azimuthal angle around \(\Delta v = v' - v\), Eq\.(4.11) becomes

\[
\left( \frac{\partial f}{\partial t} \right)_c = \int (h(v') - h(v)) C(v, v') dv'. \]  \tag{4.12}

where

\[ C(v, v') = f_0(v)P(v, v'), \]  \tag{4.13}

and

\[ P(v, v') = \int f_0(v_F)V \sigma_R |J| \sin \theta d\theta d\phi. \]  \tag{4.14}
Here $J$ is the Jacobian of the integral transform defined as

$$J = \frac{\partial(v_F, \theta, \varphi)}{\partial(v', \theta, \phi)}. \tag{4.15}$$

In order to calculate Eq. (4.14), any terms such as $v_F^2$, $V$, and $J$ must be expressed in terms of the new variables $(v', \theta, \phi)$. The calculation has been done in Appendix A as Eqs. (A.3), (A.10), and (A.20) which are.

$$v_F^2 = v^2 + 2v \cdot \Delta v \frac{\Delta v}{a} + \left( \frac{\Delta v}{a} \right)^2 \csc^2 \left( \frac{\theta}{2} \right) - 2 \left| v \times \frac{\Delta v}{a} \right| \frac{\sin(\phi + \phi_0)}{\tan(\theta/2)}, \tag{4.16}$$

$$J = (a \sin(\theta/2))^{-3}, \tag{4.17}$$

$$V = a^{-1} \Delta v \csc(\theta/2), \tag{4.18}$$

where

$$\phi_0 = \arctan \left( \begin{vmatrix} \Delta v & v_x & \Delta v_x \\ v_y & \Delta v_y \\ v_z & \Delta v_z \end{vmatrix} \right), \tag{4.19}$$

and the mass factor $a$ is defined as

$$a = 2 \mu/m. \tag{4.20}$$
We notice that the new independent variable $v'$ always appears in the form of $\Delta v = v' - v$ in the transform relations Eqs.(4.16-4.18). So $\mathcal{P}(v, v')$ can be denoted as $\mathcal{P}(v, \Delta v)$. Substituting Eqs.(4.2, 4.16-4.18) into Eq.(4.14) we obtain

$$\mathcal{P}(v, \Delta v) = \frac{\nu_0 e^{-v_{\text{th}}^2 (v^2 + 2v \cdot \Delta v)}}{4\pi^{5/2}(\Delta v)^3} \int_0^\pi \exp \left( - \left( \frac{\Delta v}{av_{\text{th}}} \right)^2 \csc^2 \left( \frac{\theta}{2} \right) \right) \frac{\sin \theta}{\sin^4(\theta/2)} d\theta$$

$$\times \int_0^{2\pi} \exp \left( 2 \left| \frac{v}{v_{\text{th}}} \times \frac{\Delta v}{av_{\text{th}}} \right| \frac{\sin(\phi + \phi_0)}{\tan(\theta/2)} \right) d\phi,$$

where the close collision frequency is defined

$$\nu_0 = n_F v_{\text{th}} \pi \rho_1^2,$$

in which

$$\rho_1 = \frac{ZZ_F e^2}{8\pi\varepsilon_0 k_B T (\mu/m_F)},$$

Here, the thermal velocity of the field particle is $v_{\text{th}} = \sqrt{2k_B T/m_F}$, and the interaction radius $\rho_1$ is different from the usual Landau length $\lambda_L = ZZ_F e^2/(4\pi\varepsilon_0 k_B T)$. Using the integral formula Eq.(B.1) derived in Appendix B to integrate over the azimuthal angle $\phi$, and substituting $x = \cot^2(\theta/2)$ into Eq.(4.21) yields

$$\mathcal{P}(v, \Delta v) = \frac{\nu_0 e^{-v_{\text{th}}^2 (v^2 + 2v \cdot \Delta v) + (\Delta v)^2}}{\pi^{3/2}(\Delta v)^3} \int_0^\infty e^{-\frac{(\Delta v)^2}{(av_{\text{th}})^2}x^2} I_0 \left( 2 \left| \frac{v}{v_{\text{th}}} \times \frac{\Delta v}{av_{\text{th}}} \right| \sqrt{x} \right) dx,$$
where $I_0$ is the zeroth order of the first kind modified Bessel function defined as Eq.(9.6.10) in Ref.[17]. Applying the integral formula Eq.(B.4) derived in Appendix B to Eq.(4.24), we obtain

$$
P(v, \Delta v) = \frac{\nu_0}{(\sqrt{\pi} a v_{th})^3} \left( \frac{\Delta v}{a v_{th}} \right)^{-5} e^{-v_{th}^{-2} \left( v^2 + 2v \cdot \frac{\Delta v}{a} + \frac{(\Delta v)^2}{a^2} - |v \times \Delta v|^2 / (\Delta v)^2 \right)}.
$$

(4.25)

Equation (4.25) can be further simplified as

$$
P(v, \Delta v) = \frac{\nu_0}{(\sqrt{\pi} a v_{th})^3} \left( \frac{\Delta v}{a v_{th}} \right)^{-5} e^{-\left( \frac{\Delta v}{a v_{th}} + \frac{v}{v_{th}} \cdot \frac{\Delta v}{\Delta v} \right)^2}.
$$

(4.26)

Considering Eq.(4.13), the kernel function $C(v, \nu')$ is obtained as

$$
C(v, \nu') = \frac{\nu_0 f_0(v)}{(\sqrt{\pi} a v_{th})^3} \left( \frac{\Delta v}{a v_{th}} \right)^{-5} e^{-\left( \frac{\Delta v}{a v_{th}} + \frac{v}{v_{th}} \cdot \frac{\Delta v}{\Delta v} \right)^2}.
$$

(4.27)

The linear Boltzmann integral for Coulomb interaction is simplified to a three-fold integral as given by Eqs.(4.12, 4.27).

4.3 Discussion

The scattering angle of the linear Boltzmann collision term for Coulomb interactions has been integrated without encountering a divergence difficulty. Unlike most traditional theories, it is not necessary to introduce a scattering angle cutoff when using the new variable
change technique. With the mathematical approach employed, the scattering angle $\theta = 0$ was not found to be a singularity point for Coulomb interactions. Therefore, the usual scattering angle cutoff is not necessary, strictly speaking. Consider Eq. (4.27), the divergence problem remains, but at $\Delta v = 0$, or equivalently at $H = 0$. It is known that the impact parameter

$$ b = \frac{Z Z_F e^2 \cot(\theta/2)}{4\pi\varepsilon_0\mu} \frac{1}{V^2} \tag{4.28} $$
approaches infinity if either the relative velocity, $V$, or scattering angle, $\theta$, goes to zero. As it can be shown in Fig. 4.1, a sole scattering angle cutoff $\theta \geq \theta_{\text{min}}$ can not efficiently limit the impact parameter $b$ to be finite if the relative velocity, $V$, goes to zero. However, Eq.(4.28) can be written as

$$b = \frac{\rho k_B T (\mu/\mu_F)}{2H} \sin \theta. \quad (4.29)$$

where the collision strength $H = (\Delta p)^2/(8\mu)$. Fig. 4.2 is a plot of Eq. (4.29). From

Figure 4.2: The relation between the impact parameter $b$ with scattering angle $\theta$ and collision strength $H$. 

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Fig. 4.2, we find that a cutoff on collision strength $H \geq H_{\text{min}}$ efficiently limits the impact parameter to be a finite value. We would like to point out that the impact parameter $b$ is by no means a good Debye cutoff parameter. For example, even if the impact parameter $b = 0$, two ions cannot get closer than the Debye length if their relative speed $V$ is close to zero. Such distant collisions with small $b$, which cannot be excluded by an impact parameter cutoff $b \leq b_{\text{max}}$, should be excluded according to the Debye shielding theory. To avoid the divergence of Boltzmann collision term for Coulomb interaction, we feel a cutoff should be made on the velocity change of the test particle $\Delta v_{\text{min}}$, or equivalently the collision strength variable $H_{\text{min}}$.

The method for simplifying the linear Boltzmann collision integral for Coulomb interactions can also be applied for hard sphere interactions. For hard sphere interactions, the linear Boltzmann collision term can be simplified to be the same as Eq. (4.12) with a kernel function

$$\mathcal{C}(\mathbf{v}, \mathbf{v}') = \frac{\nu_0 f_0(v)}{(\sqrt{\pi} a v_{\text{th}})^{\frac{3}{2}}} \left( \frac{\Delta v}{a v_{\text{th}}} \right)^{-1} e^{- \left( \frac{\Delta v}{av_{\text{th}}} + \frac{\mathbf{v} \cdot \Delta \mathbf{v}}{av_{\text{th}}} \right)^2},$$

where $\nu_0 = n_F v_{\text{th}} \pi \rho_H^2$ with the interaction radius for hard sphere interaction defined as

$$\rho_H = \frac{d + d_F}{2},$$

where $d, d_F$ are the diameters of test and field particles.
It can be shown that the arguments of the exponential function in $C(v, v')$ are

$$\frac{-mv^2}{2k_BT} - \left(\frac{\Delta v}{av_{th}} + \frac{v}{v_{th}} \cdot \frac{\Delta v}{\Delta v}\right)^2 = -v_{th}^{-2} \left(\frac{|v' - v|^2}{a^2} + 2 \frac{v' \cdot v}{a} - \frac{|v' \times v|^2}{|v' - v|^2}\right). \tag{4.32}$$

The left side of Eq.(4.32) is the form used in Ref.[7, 8]. From Eq.(4.32), it is easy to show the symmetric relation of the kernel function

$$C(v, v') = C(v', v). \tag{4.33}$$

With the symmetric property of the kernel function, particle conservation can now be verified by integrating the Boltzmann collision term Eq. (4.12) over the velocity space of the test particle, $v$.

$$\int \left(\frac{\partial f}{\partial t}\right)_{c} d\mathbf{v} = \int \int (h(v') - h(v))C(v, v')d\mathbf{v}'d\mathbf{v} = 0. \tag{4.34}$$

From the kernel function $C(v, v')$, we can calculate $W(v', v) = C(v, v')/f_0(v')$ which is

$$W(v', v) = \frac{\nu_0(a v_{th})^2}{\pi^{3/2} (\Delta v)^5} \exp \left(-\left(\frac{(m_F - m)v' + (m_F + m)v}{2m_Fv_{th}} \cdot \frac{v - v'}{|v - v'|}\right)^2\right). \tag{4.35}$$

Equation (4.35) is the expression used by Shoub as Eq.(19) in Ref.[34].

Assuming the mass of the field particles is the same as the mass of the test particles
$m_F = m$, we have $a = 1$. Then Eq.(4.32) can be expressed as

$$\frac{-mv^2}{2k_BT} - \left( \frac{\Delta v}{av_{th}} + \frac{\mathbf{v} \cdot \Delta \mathbf{v}}{v_{th}} \right)^2 = -v_{th}^{-2} \left( v^2 + \frac{(v' \cdot \Delta \mathbf{v})^2}{\Delta v^2} \right).$$

(4.36)

The right side of Eq.(4.36) is the form used by Hilbert as Eq.(23) in Ref.[35].

Among the several equivalent expressions of the kernel functions, Eq.(4.26) for $P(v, \Delta v)$ is more convenient for discussing the physical significance. In Fokker-Planck theory, $P(v, \Delta v)$ is used as the form of the probability function. Here, we have obtained an explicit probability function, Eq.(4.26), for a Maxwellian background. The probability function, $P(v, \Delta v)$ in Eq.(4.26), is dominated by the term $\Delta v/(av_{th})$. The larger the term $\Delta v/(av_{th})$ is, the smaller the probability function $P(v, \Delta v)$. We also find that the probability for collisions with $\mathbf{v} \cdot \Delta \mathbf{v} < 0$ is larger than with $\mathbf{v} \cdot \Delta \mathbf{v} > 0$. A test particle of velocity $\mathbf{v}$ feels more collisions that slow it down than the collisions that accelerate it. That is where the friction comes from.

In the foregoing, the scattering angle of the Boltzmann collision term for Coulomb interactions has been integrated without encountering a divergence difficulty. Unlike most traditional theories, it is not necessary to introduce a scattering angle cutoff when using the variable change technique. The linear Boltzmann integral is reduced to a three-fold integral as Eqs.(4.12) and (4.27). Compared to our previous works[7, 8], the probability function in this dissertation has been written in a more significant form as Eq.(4.26), which can be used
to explain the friction property in velocity space.
CHAPTER 5

LINEARIZED BOLTZMANN COLLISION OPERATOR FOR INVERSE-SQUARE FORCE LAW

In this chapter, we simplify the linearized Boltzmann collision operator for the inverse-square force law by implementing the cutoff on the collision strength variable $H$. Unlike the linear Boltzmann collision operator, which assumes a Maxwellian background, the linearized Boltzmann collision operator is derived from a general non-linear Boltzmann collision integral by a perturbation procedure. The new collision operator may have applications in plasma physics and astrophysics where the inverse-square force law dominates the behavior of the collection of particles.

5.1 Introduction

The Boltzmann transport equation can be used to find the distribution function associated with a collection of particles which are undergoing binary collisions. The equation contains a non-linear quadratic integral operator which sets up the main barrier for obtaining a solution. The perturbation procedure[36] developed by Hilbert and Chapman-Enskog is a most powerful method for solving the Boltzmann equation for non-equilibrium situations. In the perturbation procedure, collision operators for any arbitrary higher order perturbation can
be split into a known source term and one having the same form as the linearized Boltzmann collision operator. Therefore, the linearized collision operator is of basic importance in dealing with perturbation procedures for solving the Boltzmann equation.

Simplification of the linearized Boltzmann collision operator plays an important role in solving the Boltzmann transport equation. The simplifications are largely dependent on the particular models of the interactions. In 1912, Hilbert [35] succeeded in simplifying the collision operator for hard sphere interactions. The linearized Boltzmann collision operator was simplified to an orthogonal integral operator with a symmetrical kernel that has only velocities as its arguments. In 1917, Enskog [37] developed the general mathematical theory of the linearized collision operator for any elastic interactions. The linearized collision operator for inverse-power potentials \((U = \kappa r^{1-k})\) was derived by Grad [38]. Compared to the results for hard sphere interactions, the later derived operators are less simplified because the scattering angle is not integrated. To our knowledge, Hilbert’s linearized collision operator for hard sphere interactions is the only collision operator that has the scattering angle integrated. In the present work, the linearized collision operator for the inverse-square force law is simplified by integrating over the scattering angle.

The linearized Boltzmann collision operator has four terms. Two of them have the same form as that of the linear Boltzmann collision operator. In other words, the linear Boltzmann collision integral is part of the linearized Boltzmann collision operator. The present work can also be regarded as an extension of our previous work [7, 8], where the linear
Boltzmann collision integral for the inverse-square force law has been integrated over the scattering angle. With a different approach, a similar result for the linear Boltzmann collision operator was also obtained by Shoub[34]. Because of the complexity of the problem, all of the previous authors[7, 8, 34, 35, 37, 38] have independently developed specialized mathematical approaches to the results. In fact, going from Enskog’s general result to Grad’s result for the inverse-power potentials requires integrating over the azimuthal angle[39]. From Grad’s result to our result and Shoub’s result for the inverse-square force law requires integrating over the scattering angle. Thus, we will start from Grad’s result and simplify the linearized Boltzmann collision operator for the inverse-square force law.

The simplification of the linearized Boltzmann collision operator for the inverse-square force law is largely dependent on a proper treatment for a singularity point. The singularity is commonly considered to occur when the scattering angle equals zero. With the mathematical treatment of using the thermal velocity to replace the relative speed, which is a variable, the scattering angle cutoff has been widely employed in astrophysics [40, 41, 42] and in plasma physics [4, 31]. Even in the rarefied gas theory, the scattering angle has also been employed for the purpose of a cutoff [43]. In this work, we will show that a scattering angle cutoff is incapable of eliminating the divergence of the integral. After having taken the scattering angle cutoff, the divergence still exists in the following integral over velocity space. In the conventional theory [4, 31, 40, 41, 42], the singularity in the velocity space is dealt with using the thermal velocity to replace the relative speed. Several authors[44, 45] have noticed the
problem of justifying such an “approximation”. The divergence is caused by non-realistic weak collision events. To characterize the singularity point requires finding a variable that represents the collision strength—a variable that can be used to separate weak collisions and strong collisions. Previous works [5, 7, 8, 9, 10, 46] have shown that the singularity of the collision operator for the inverse-square force law is associated with collision strength, $H$, instead of scattering angle. In this Chapter, we will use the collision strength cutoff, $H_{\text{min}}$, to derive the linearized Boltzmann collision operator for the inverse-square force law.

To show the basic importance of the linearized Boltzmann integral, the perturbation theory is briefly reviewed in Section 5.2. In Section 5.3, we show that the singularity is not associated with scattering angle but with collision strength $H$. With a cutoff on the collision strength variable, the linearized Boltzmann collision integral for the inverse-square force law is derived in Section 5.4. A discussion is presented in Section 5.5.

5.2 Perturbation Methods for the Boltzmann Collision Operator

In this chapter our primary concern is with a system containing particles of one kind only. In order to find the distribution function $f(x, \tilde{\xi}, t)$ associated with a collection of particles, which are acted upon by external forces and which are undergoing collisions, it is necessary to solve the Boltzmann transport equation. In its most general form, the equation may be written

$$\frac{\partial f}{\partial t} + \tilde{\xi} \cdot \frac{\partial f}{\partial x} + \frac{F}{m} \cdot \frac{\partial f}{\partial \tilde{\xi}} = Q(f, f), \quad (5.1)$$
where $F$ is the external force, and $Q(f, f)$ is the collision integral which describes the change in $f$ due to collisions. The later is

$$Q(f, f) = \int_{\Xi} \int_0^{2\pi} \int_0^{\pi/2} \left( f' f_1' - f f_1 \right) V \sigma(V, \theta) \sin \theta d\phi d\theta d\xi_1,$$

(5.2)

where $f = f(\xi)$ is the unknown velocity distribution function normalized to number density, the space $x$ and time $t$ arguments are omitted here and later for simplicity in notation, and $f_1 = f(\xi_1), f' = f(\xi'), f_1' = f(\xi_1'), \sigma(V, \theta)$ the differential cross-section of scattering angle $\theta$ and relative speed $V = |\xi - \xi_1|$, the integral limits for azimuthal angle $\varphi$ are from $0$ to $2\pi$, the limits for scattering angle $\theta$ are from $0$ to $\pi$, and the integration with respect to $\xi_1$ covers all velocity space $\Xi$.

In the perturbation theory, $Q$ is defined as a slightly more general operator, the bilinear quantity

$$Q(f, g) = \frac{1}{2} \int \left( f' g_1' + f_1 g' - f g_1 - f_1 g \right) V \sigma(V, \theta) \sin \theta d\varphi d\theta d\xi_1.$$

(5.3)

The limits of integration are the same as in Eq.(5.2). For simplicity in notation, they have been omitted here and later. When $g = f$, it is found that Eq.(5.3) reduces to Eq.(5.2). Also, the operator $Q$ has a symmetric property,

$$Q(f, g) = Q(g, f).$$

(5.4)
To solve the Boltzmann equation for realistic non-equilibrium situations, we rely upon approximation methods, in particular, perturbation procedures. Assume $\epsilon$ is a parameter which can be small in some situations and can be used to expand $f$ in a series of powers of $\epsilon$,

$$f = \sum_{k=0}^{\infty} \epsilon^k f_k. \tag{5.5}$$

By substituting this expansion into, Eq.(5.3), the collision operator is changed to

$$Q(f, f) = \sum_{k=0}^{\infty} \epsilon^k \sum_{j=0}^{k} Q(f_j, f_{k-j}). \tag{5.6}$$

The terms with $j = j_0$ and $j = k - j_0$ (for any $j_0$ between 0 and $k$) are combined based on Eq.(5.5), the symmetric property. This equation shows that expanding $f$ in a power series in parameter $\epsilon$ is equivalent to an expansion for the collision term $Q$ as

$$Q_k = \sum_{j=0}^{k} Q(f_j, f_{k-j}), \quad (k \geq 0). \tag{5.7}$$

In most of the perturbation expansions, a Maxwellian distribution is usually assumed to be the solution for the zeroth order collision operator. The parameters appearing in the Maxwellian (density, mass velocity, temperature) are determined as macroscopic variables
of the time and space. The zeroth order collision operator can be written as

\[ Q(f_0, f_0) = 0, \quad \text{i.e.,} \quad Q_0 = 0, \quad (5.8) \]

where \( f_0 \) is known to be Maxwellian

\[ f_0(\xi) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right). \quad (5.9) \]

Here \( n \) is the number density, \( v = |\mathbf{v}| \) is a speed. Here \( \mathbf{v} = \mathbf{\xi} - \mathbf{\xi}_0 \) is the velocity with respect to the mass velocity, equals to \( \mathbf{\xi} \) if the Maxwellian is taken to have zero mass velocity \( \mathbf{\xi}_0 = 0 \). For any higher order collision operator \( Q_k \) \((k \geq 1)\) as given in Eq.(5.7), only the first and the last terms in the sum contain \( f_0 \), which is known to be a Maxwellian. \( f_k \) is the \( k \)th-order coefficient in the expansion of \( f \). The remaining terms \((1 \leq j \leq k - 1)\) contain \( f_j \) of order less than \( k \). These lower order terms \( j \leq k - 1 \) have been found in previous steps. At the \( k \)th step, only \( f_k \) is unknown. Thus, \( Q_k \) can be split into the sum

\[ Q_k = 2Q(f_0, f_k) + \sum_{j=1}^{k-1} Q(f_j, f_{k-j}), \quad (k \geq 1). \quad (5.10) \]

The second term can be written as a source term \( S_k \) since it is known at the \( k \)th step. Therefore, the relevant operator to be considered at each step is the linear operator \( 2Q(f_0, f_k) \) acting upon the unknown \( f_k \). For convenience, a relative distribution function, \( h_k = f_k/f_0 \),
is usually introduced instead of $f_k$. Then, the $k$th order collision operator can be written as follows:

$$Q_k = f_0 L h_k + S_k, \quad (k \geq 1),$$

(5.11)

where the linearized Boltzmann operator $L$ is defined as

$$L h = 2 f_0^{-1} Q(f_0, f_0 h).$$

(5.12)

Because the linearized collision operator, $L$, appears in each step of the perturbation procedure, it is of basic importance in solving the Boltzmann equation.

Considering the definition of the linearized collision operator $L$, Eq.(5.12), as well as the definition of the bilinear operator $Q$, Eq.(5.3), we can write the linearized collision operator $L$ in an explicit way,

$$L h = f_0^{-1} \int \left( f_0 f_0'(h_1' + h') - f_0 f_0(h_1 + h) \right) V \sigma(V, \theta) \sin \theta d\varphi d\theta d\vec{\xi}_1.$$  

(5.13)

For a Maxwellian distribution, it has the detailed equilibrium property,

$$f_0 f_0' = f_0 f_0.$$ 

(5.14)

By substituting Eq.(5.14) into Eq.(5.13), the standard expression of the linearized collision
operator is obtained,

\[ \mathcal{L} h = \int f_0 (h' + h - h) V \sigma (V, \theta) \sin \theta d\varphi d\theta d\xi_1. \] \hspace{1cm} (5.15)\

Because \( f_0 \) does not depend upon the integration variables, it is taken out of the integral and canceled with \( f_0^{-1} \).

Further simplification of Eq.(5.15) requires \( \sigma (V, \theta) \), which depends on a particular mode of interaction. For hard sphere interactions, the differential cross-section has a simple form

\[ \sigma (V, \theta) = \rho_H^2 / 4, \] \hspace{1cm} (5.16)\

where \( \rho_H \) is the diameter of a particle. With Eq.(5.16) and series manipulations, Hilbert could simplify the linearized Boltzmann collision operator for hard sphere interactions as[35]

\[ \mathcal{L} h = \int K (\vec{\xi}, \vec{\xi}_1) h(\vec{\xi}_1) d\vec{\xi}_1 - \nu (\xi) h(\vec{\xi}), \] \hspace{1cm} (5.17)\

where the kernel function of the linearized collision operator for hard sphere interaction is

\[ K (\vec{\xi}, \vec{\xi}_1) = \frac{\nu_0}{n} f_0 (v_1) \left( \frac{2v_{th}}{|v_2 - v|} \exp \left( \frac{|v_1 \times v|^2}{v_{th}^2 |v_1 - v|^2} \right) - \frac{|v_1 - v|}{v_{th}} \right), \] \hspace{1cm} (5.18)
\[ \nu(\xi) = \nu_0 \Phi_H(v/v_{th}). \]  
(5.19)

Here the close collision frequency is \( \nu_0 = n v_{th} \pi \rho_H^2 \), the thermal velocity is \( v_{th} = \sqrt{2k_B T/m} \), and the function \( \Phi_H(x) \) is

\[ \Phi_H(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} + \left( x + \frac{1}{2x} \right) \text{erf}(x), \]  
(5.20)

in which the error function is defined as

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2)dy. \]  
(5.21)

To our knowledge, Eq.(5.17) remains the only linearized Boltzmann integral that has the scattering angle integrated.

In the next two sections, we are going to simplify the linearized Boltzmann integral for the inverse-square force by integrating over the scattering angle. The main difference between the hard sphere interaction and the inverse-square force is that the hard sphere interaction has a finite range while the inverse-square force has an infinite range of interaction. The infinite range of interaction causes the collision integral divergence which is the main difficulty in its simplification.
5.3 The Singularity Point of The Boltzmann Collision Operator and Collision Strength Variables

When the interaction between two particles is governed by the inverse-square force law, or the potential is $U = \kappa/r$, the differential scattering cross-section $\sigma(V, \theta)$ is Rutherford’s formula

$$\sigma_R(V, \theta) = (\kappa/m)^2 (V \sin(\theta/2))^{-4}, \quad (5.22)$$

If we substitute Eq.(5.22) into Eq.(5.15), the linearized Boltzmann collision integral obtained is divergent. Before we try to simplify Eq.(5.15) to get a form like Eq.(5.17), we consider what causes the singularity.

A physically meaningful two-body interaction should be based on a potential with a finite range. However, using a finite-range potential, such as a Debye screened Coulomb potential, causes the theory to become analytically intractable[47, 48]. Consequently, an infinite range potential plus a cutoff procedure is used[4, 30, 31, 40, 41, 42, 43].

There are two basic purposes for the cutoff procedure. From a physical point of view, the purpose is to cut off weak collision events that physically do not exist. From a mathematical point of view, the purpose is to cut off the singularity point in the collision integral. However, a usual cutoff variable, the scattering angle $\theta$, can achieve neither of these purposes. First, we would like to show that the scattering angle cutoff does not remove the divergence. If the linearized Boltzmann collision operator for the inverse-square force can be simplified as
Eq.(5.17) with scattering angle cutoff, \( \nu (\xi) \) should be calculated as

\[
\nu (\xi; \theta_{\text{min}}) = \int_{\theta \geq \theta_{\text{min}}} f_{01} V (\kappa/m)^2 (V \sin(\theta/2))^{-4} \sin \theta \varphi d\theta d\xi_1.
\] (5.23)

When \( \theta \to 0 \), the integral Eq.(5.23) diverges. Assume the traditional scattering angle cutoff is introduced and let

\[
A_0 = \int_0^{2\pi} \int_{\theta_{\text{min}}}^{\pi} (\kappa/m)^2 \sin^{-4}(\theta/2) \sin \theta d\phi d\theta.
\] (5.24)

Substituting Eq.(5.24) into Eq.(5.23) we have

\[
\nu (\xi; \theta_{\text{min}}) = A_0 \int f_{01} |\mathbf{v} - \mathbf{v}_1|^{-3} d\mathbf{v}_1.
\] (5.25)

It is not difficult to find out that Eq.(5.25) is still divergent. Thus, introducing a scattering angle cutoff, as we did in Eq.(5.24), is incapable of keeping the integral from diverging.

5.4 Simplification of The Linearized Boltzmann Collision Operator For Inverse-Square Force

In Chapter 4, we found the singularity point to be at \( H = 0 \). If we make a cutoff on collision strength \( H \geq H_{\text{min}} \), the linearized Boltzmann collision operator for the inverse-square force
law can be written as

\[ \mathcal{L} h = \int_{H \geq H_{\text{min}}} f_{01}(h_1' + h_1' - h_1 - h) V \sigma_R(V, \theta) \sin \theta d\phi d\theta d\xi_1, \quad (5.26) \]

where \( \sigma_R \) is defined as Eq.(5.22). The main purpose of this section is to simplify Eq.(5.26) by integrating over the scattering angle. Simplifications of the terms associated with \( h_1 \) and \( h \) can be done in a straightforward way. The terms associated with \( h_1' \) and \( h' \) require some manipulation. The purpose of the manipulation is to separate the scattering angle from the unknown perturbation distribution function \( h \). Due to the mathematical complexity, there appear several different methods developed independently by Hibert[35], Enskog[37], Grad[38], Chang[7, 8], and Shoub[34]. The consistency of these different approaches has established this theory on a firm ground. To show the consistency and to avoid repeated derivation, here we use the result by Grad as expressed by Cercignani[36]. We can obtain the result for the inverse-square force law from the general result for a power-law potential \( U = \kappa r^{1-k} \) of Eqs.(3.20-3.24) in [36] by taking \( k = 2 \). However, care must be taken regarding the following changes: For making the cutoff, we do not unify the dummy integration variables \( \xi_1', \xi', \xi_1 \) in order to keep the exact physical meaning of each integral term. The angle used in Grad’s general result equals \( (\pi - \theta)/2 \), while here we use the scattering angle \( \theta \). Boltzmann’s constant \( k_B \) is used here, while the universal gas constant \( R \) was used in the general formula. With a cutoff on collision strength variable \( H \geq H_{\text{min}} \), we get the linearized Boltzmann
collision operator for inverse-square force

$$\mathcal{L} h = \int_{H \geq H_{\text{min}}} h(\xi') \mathcal{K}_3(\xi, \xi', \theta) d\theta d\xi' + \int_{H \geq H_{\text{min}}} h(\xi') \mathcal{K}_2(\xi, \xi', \theta) d\theta d\xi'$$

$$- \int_{H \geq H_{\text{min}}} h(\xi) \mathcal{K}_1(\xi, \xi_1, \theta) d\theta d\xi_1 - h(\xi) \nu(\xi; H_{\text{min}}), \quad (5.27)$$

where

$$\mathcal{K}_3(\xi, \xi', \theta) = \frac{2\pi f_0(v_1') I_0 \left(2v_{\text{th}}^{-2} |v_1| \times v \tan(\theta/2)\right)}{|v_1 - v|^3 \exp \left(v_{\text{th}}^{-2} |v_1| \times v |\cot(\theta/2)\right)} \beta(\theta), \quad (5.28)$$

$$\mathcal{K}_2(\xi, \xi', \theta) = \frac{2\pi f_0(v') I_0 \left(2v_{\text{th}}^{-2} |v'| \times v \cot(\theta/2)\right)}{|v' - v|^3 \exp \left(v_{\text{th}}^{-2} |v'| \times v |\cot(\theta/2)\right)} \beta(\theta), \quad (5.29)$$

$$\mathcal{K}_1(\xi, \xi_1, \theta) = 2\pi f_0(v_1)|v_1 - v|^{-3} \beta(\theta), \quad (5.30)$$

$$\nu(\xi; H_{\text{min}}) = \int_{H \geq H_{\text{min}}} 2\pi f_0(v_1)|v_1 - v|^{-3} \beta(\theta) d\theta d\xi_1, \quad (5.31)$$

in which $I_0$ is the modified Bessel function, and

$$\beta(\theta) = 2(\kappa/m)^2 \csc^3(\theta/2) \cos(\theta/2). \quad (5.32)$$

To implement the cutoff on collision strength variable $H \geq H_{\text{min}}$, we must find the
relationship between the collision strength variable and the independent variables of each integrand.

The collision strength is defined as $H = (\Delta p)^2/(8\mu)$ where $\Delta p = m|\vec{\xi} - \vec{\xi}'|$ is the momentum transfer and $\mu = m/2$ is the reduced mass. For integrating the scattering angle $\theta$ in $K_2(\vec{\xi}, \vec{\xi}', \theta)$, we have

$$|\vec{\xi} - \vec{\xi}'| \geq 2\sqrt{H_{\text{min}}/m}. \quad (5.33)$$

It is interesting that the $H_{\text{min}}$ cutoff makes no restriction (cutoff) on scattering angle when $\vec{\xi}'$ is an independent variable. Therefore, the integration range of the scattering angle for $K_2(\vec{\xi}, \vec{\xi}', \theta)$ is from 0 to $\pi$. Making a variable change by letting $x = \cot^2(\theta/2)$, we have

$$K_2(\vec{\xi}, \vec{\xi}') = \int_0^\pi K_2(\xi, \xi', \theta) d\theta = 4\pi \frac{\kappa^2}{m^2} f_0(v') \times \int_0^\infty \exp \left( -v_{\text{th}}^{-2} |\vec{v}' - \vec{v}|^2 x \right) \frac{n}{n_{\text{th}} \pi (\kappa/\k_B T)^2} dx. \quad (5.34)$$

The integral over $x$ can be evaluated as

$$K_2(\vec{\xi}, \vec{\xi}') = \frac{\nu_0 v_{\text{th}} f_0(v')}{n |\vec{v}' - \vec{v}|^2} \exp \left( \frac{|\vec{v}' \times \vec{v}|^2}{v_{\text{th}}^2 |\vec{v}' - \vec{v}|^2} \right), \quad (5.35)$$

where $\nu_0 = n v_{\text{th}} \pi (\kappa/\k_B T)^2$ is the close collision frequency. Many applications have been developed from the term $K_2(\vec{v}, \vec{v}')$ as in the expression Eq.(5.35). The function $K_2(\vec{v}, \vec{v}')$ has the same form as the probability function for Maxwellian field particles in Fokker-Planck
theory. A proper mathematical treatment for calculating the arbitrary higher-order Fokker-Planck coefficients[9] has been developed from the probability function. If the distribution function is isotropic, Shoub[34] has further simplified the probability function to depend only on the speeds as $K_2(v, v')$ which is the result that was originally obtained by Agekyan[49]. With the simple form $K_2(v, v')$, Retterer[50] developed a numerical approach for solving the linear Boltzmann equation and calculated the escape rate.

We do not integrate Eq.(5.31) directly for the collision frequency term $\nu(\xi; H_{\min})$ since there is a better approach. In fact, $\nu(\xi; H_{\min})$ can be calculated by integrating $K_2(\vec{\xi}, \vec{\xi}')$ over $\vec{\xi}'$ as

$$\nu(\xi; H_{\min}) = \int_{|\vec{\xi}'\cdot \vec{\xi}| \geq 2\sqrt{H_{\min}/m}} K_2(\vec{\xi}, \vec{\xi}') d\vec{\xi}'. \quad (5.36)$$

Using the variable change $\Delta v = \vec{\xi} - \vec{\xi}'$, Eq.(5.36) can be evaluated in polar coordinates to give

$$\nu(v; H_{\min}) = \nu_0 \Phi(\frac{v^2}{v_{th}^2}; H_{\min}), \quad (5.37)$$

where the function $\Phi(x; H_{\min})$ is defined as

$$\Phi(x; H_{\min}) = \frac{1}{2x^{1/2}} \int_{\frac{2H_{\min}}{xH_{\min}}}^{\infty} \frac{\text{erf}(\sqrt{x} + \sqrt{y}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y^2} dy, \quad (5.38)$$

and we have written $\nu(v; H_{\min})$ in place of $\nu(\vec{\xi}; H_{\min})$ in order to emphasize that the dependence is only upon the magnitude of $v$. 

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Figure 5.1: The relation of velocities before and after collisions. The cutoff on collision strength $H \geq H_{\text{min}}$ requires that $|\vec{\xi} - \vec{\xi}'| \geq |\vec{\xi} - \vec{\xi}_1'|_{\text{min}}$. Other cutoffs for different independent variables can be derived from the relation.

The scattering angle integral for $K_1(\vec{\xi}, \vec{\xi}_1)$ is a simple case. However, the cutoff is a two-step procedure. With reference to Fig. 5.1, we have

\[
\sin \frac{\theta}{2} \geq 2\sqrt{H_{\text{min}}/m}/|\vec{\xi} - \vec{\xi}_1|,
\]

\[
|\vec{\xi} - \vec{\xi}_1| \geq 2\sqrt{H_{\text{min}}/m}.
\]

Using the variable change $x = \sin(\theta/2)$, we obtain $K_1(\vec{\xi}, \vec{\xi}_1)$ as

\[
K_1(\vec{\xi}, \vec{\xi}_1) = \int_{\theta_{\text{min}}}^{\pi} K_1(\vec{\xi}, \vec{\xi}_1, \theta) d\theta = 4\pi \frac{\kappa^2}{m^2} \frac{f_0(v')}{|v' - \vec{v}|^3} \int_{2\sqrt{H_{\text{min}}/m}/|\vec{\xi} - \vec{\xi}_1|}^{1} \frac{2dx}{x^3}
\]

\[
= \nu_0 \frac{v_0^3 f_0(v_1)}{n |v_1 - \vec{v}|^3} \left( \frac{m|v - v_1|^2}{4H_{\text{min}}} - 1 \right).
\]

From Fig. 5.1, we find $\tan(\theta/2) = |\vec{\xi} - \vec{\xi}'|/|\vec{\xi} - \vec{\xi}_1'|$. The relation between collision strength
and the independent variables for $\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1, \theta)$ is

$$\tan \frac{\theta}{2} \geq \frac{2\sqrt{H_{\text{min}}/m}}{|\vec{\xi} - \vec{\xi}_1^2|}. \quad (5.41)$$

Let $x = \tan^2(\theta/2)$. Then $\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1)$ can be calculated as

$$\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1) = \int_{\theta_{\text{min}}}^{\pi} \mathcal{K}_3(\vec{\xi}, \vec{\xi}_1, \theta)d\theta = 4\pi \frac{\kappa^2}{m^2} \frac{f_0(v')}{|\vec{v} - \vec{v}'|^3}$$

$$\times \int_{4H_{\text{min}}}^{\infty} \frac{m|\vec{\xi} - \vec{\xi}_1|^2}{|\vec{v}'_1 - \vec{v}|^2} \exp \left( -v_{\text{th}}^{-2}|\vec{v}'_1 - \vec{v}|^2 x \right) I_0(2v_{\text{th}}^{-2}|\vec{v}'_1 \times \vec{v}|\sqrt{x}) x^{-2} dx. \quad (5.42)$$

Using the series form of the modified Bessel function Eq.(9.6.10) in Ref.[17], Eq.(5.42) can be integrated and expressed in terms of the incomplete gamma function, defined as Eq.(6.5.3) in Ref.[19].

$$\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1) = \frac{\nu_0}{n} \frac{v_{\text{th}}f_0(v')}{|\vec{v}'_1 - \vec{v}|} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left( \frac{|\vec{v}'_1 \times \vec{v}|^2}{v_{\text{th}}^2|\vec{v}'_1 - \vec{v}|^2} \right)^j \Gamma \left( j + 1, \frac{2H_{\text{min}}}{k_B T} \right). \quad (5.43)$$

The scattering angle has been integrated for all four terms as Eqs.(5.35,5.40,5.43). Because the domain of the integral for Eqs.(5.35,5.40,5.43) is different, the dummy integral variable can not be unified directly. Using the Heaviside step function $\Theta$, we can unify the dummy integral variable. The linearized Boltzmann can be expressed as

$$\mathcal{L} h = \int h(\vec{\xi}_1) \mathcal{K}(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) d\vec{\xi}_1 - h(\vec{\xi})\nu(\vec{\xi}; H_{\text{min}}), \quad (5.44)$$
Where

\[
\nu(v; H_{\text{min}}) = \nu_0 \Phi(v^2/v_{\text{th}}^2; H_{\text{min}}),
\]

\[\text{(5.45)}\]

\[
\mathcal{K}(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \mathcal{K}_3(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) + \mathcal{K}_2(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) - \mathcal{K}_1(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}),
\]

\[\text{(5.46)}\]

in which

\[
\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{\nu_0 v_{\text{th}} f_0(v_1)}{n |\mathbf{v}_1 - \mathbf{v}|} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left( \frac{|\mathbf{v}_1 \times \mathbf{v}|^2}{v_{\text{th}}^2 |\mathbf{v}_1 - \mathbf{v}|^2} \right)^j \Gamma \left( j - 1, \frac{2H_{\text{min}}}{k_B T} \right),
\]

\[\text{(5.47)}\]

\[
\mathcal{K}_2(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{\nu_0 v_{\text{th}} f_0(v_1)}{n |\mathbf{v}_1 - \mathbf{v}|^5} \exp \left( \frac{|\mathbf{v}_1 \times \mathbf{v}|^2}{v_{\text{th}}^2 |\mathbf{v}_1 - \mathbf{v}|^2} \right) \Theta \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right),
\]

\[\text{(5.48)}\]

\[
\mathcal{K}_1(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{\nu_0 v_{\text{th}} f_0(v_1)}{n |\mathbf{v}_1 - \mathbf{v}|^3} \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right) \Theta \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right),
\]

\[\text{(5.49)}\]

\[
\Phi(x; H_{\text{min}}) = \frac{1}{2x^{1/2}} \int_{2H_{\text{min}}/k_B T}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y^2} dy,
\]

\[\text{(5.50)}\]

\[
\nu_0 = n v_{\text{th}} \pi \left( \frac{\kappa}{k_B T} \right)^2.
\]

\[\text{(5.51)}\]

Eqs.(5.44-5.51) are the main result of this chapter. It is interesting if we compare
Hilbert’s results Eqs.(5.17-5.21) with Eqs.(5.44-5.51). Considering the difference between the cross-sections for hard sphere and inverse-square force laws, we can recover Hilbert’s results Eqs.(5.17-5.21) from Eqs.(5.44-5.51) by proper modification. For each different independent variables \( v_1, v', \) and \( v'_1, \) the cross-section for the inverse-square force law has three different forms

\[
\sigma_R = \frac{\sigma_H v_{th}^4}{|v - v_1|^4 \sin^4(\theta/2)} = \frac{\sigma_H v_{th}^4}{|v - v'|^4} = \frac{\sigma_H v_{th}^4}{|v - v'_1|^4 \tan^4(\theta/2)},
\]

(5.52)

where \( \sigma_H = \kappa^2/(4k_B^2T^2) \) is an equivalent hard-sphere cross-section. Considering the relation between \( \sigma_R \) and \( \sigma_H, \) we can get \( \mathcal{K}_2(\vec{\xi}, \vec{\xi}_1) \) for hard-sphere interaction by multiplying \( |v - v_1|^4/v_{th}^4 \) with Eq.(5.48) and letting \( H_{\text{min}} = 0. \) We obtain \( \mathcal{K}_3(\vec{\xi}, \vec{\xi}_1) \) for hard-sphere interactions by changing the \( \Gamma(j - 1, 2H_{\text{min}}/(k_B T)) \) in Eq.(5.47) to \( \Gamma(j + 1, 0) = j! \). The summation in Eq.(5.47) turns out to be an exponential function, and we find out that \( \mathcal{K}_3(\vec{\xi}, \vec{\xi}_1) = \mathcal{K}_2(\vec{\xi}, \vec{\xi}_1) \) for hard-sphere interactions. The relation of \( \mathcal{K}_1(\vec{\xi}, \vec{\xi}_1) \) between hard sphere and inverse-square force laws is the well-known factor of the integrand \( 1/\sin^4(\theta/2). \) And the function \( \Phi_H \) defined in Eq.(5.20) can be obtained from the function \( \Phi \) by multiplying \( y^2 \) into Eq.(5.50) and letting \( H_{\text{min}} = 0. \)

5.5 Discussion and Summary

The traditional misconception of scattering angle cutoff is clarified. A cutoff on scattering is incapable of solving the divergence in collision operators for inverse-square force law. The
proper cutoff should be made on the collision strength variable $H$.

With the proper cutoff on $H$, the linearized Boltzmann collision operator is simplified by integrating over the scattering angle. The simplified collision operator might have applications in solving the Boltzmann transport equation in plasma physics or astrophysics. To solve the linearized Boltzmann equation, we must rely upon perturbation procedures. Both Chapman\cite{51, 52} and Enskog\cite{37, 53} have solved the linearized Boltzmann equation for hard-sphere interactions. Their method for hard-sphere interactions can be applied for the inverse-square force law because there is no basic difference between Eqs.(5.17-5.21) and Eqs(5.44-5.51).

Both Hilbert’s results Eqs.(5.17-5.21) and the present results Eqs(5.44-5.51) are limited for one kind of particle. The next step is to generalize the theory for different kinds of particles. Except for the term $\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1)$, there is no difficulty to generalize the results for one kind of particle to the results for more than one kind of particles. For different kinds of particles, Enskog\cite{37} once developed a mathematical theory for the calculation of $\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1)$. However, the difference in mass of the particles brings tough mathematical difficulty that keeps $\mathcal{K}_3(\vec{\xi}, \vec{\xi}_1)$ unsolved.
CHAPTER 6
CALCULATION OF LINEAR FOKKER-PLANCK COEFFICIENTS OF ARBITRARILY HIGH ORDER

In Chapter 4 we showed how the pre-collision velocity $v'$ can be used as the equivalent collision strength variable to simplify the linear Boltzmann integral. In this Chapter, the velocity change of the test particles $\Delta v$ is employed as the collision strength variable to derive linear Fokker-Planck coefficients of arbitrarily high order.

6.1 Introduction

The original Fokker-Planck equation was derived from the random motion model, in which a stochastic process is described by a transition probability $P(v, \Delta v)$, where the velocity $v$ and its transfer $\Delta v$ are independent variables[54, 55]. In dealing with non-equilibrium systems whose particles obey an inverse-square force law, people usually deviate from the original Fokker-Planck theory because of the lack of an explicit expression for the transition probability function. Starting from the Boltzmann collision integral, Landau[30] developed the first Fokker-Planck equation for plasmas. In stellar dynamics, Chandrasekhar first calculated the friction coefficient[41] and diffusion coefficient[42], which are equivalent to the first two linear Fokker-Planck coefficients for gravitational interactions. Later, these coefficients
were applied to plasmas by Spitzer[4]. Rosenbluth et al. expressed the Fokker-Planck coefficients using, as they are now called, the Rosenbluth potentials[32]. In essence, a scattering angle cutoff and an approximation of using thermal velocity to replace variable relative speed has been employed by all of the previous authors[31, 41, 42]. An impact parameter cutoff was used by Landau[30]. However, other approximations were introduced in Landau’s approach. With the explicit expression for the transition probability $P(v, \Delta v)$ obtained in Chapter 4, we calculate linear Fokker-Planck coefficients based on the original Fokker-Planck theory. The lower order Fokker-Planck coefficients are derived by making a cutoff on collision strength, $H \geq H_{\text{min}}$.

The lower order Fokker-Planck collision operator is primarily used for plasmas with a large Coulomb logarithm $\lambda > 10$. The definition of the Coulomb logarithm and more precise calculations of the Coulomb logarithm based on screened Coulomb interactions can be found in Ref.[47]. Higher order Fokker-Planck collision operators[33] are needed when the Coulomb logarithm is not large. By including the third order Fokker-Planck collision term, a collision operator for moderately coupled plasmas ($2 < \lambda < 10$) has been developed[56]. The fourth order Fokker-Planck collision operator was derived by Shoub[33]. Even higher order Fokker-Planck collision terms are necessary for more precise calculations of collision operators when Coulomb logarithm is very small. However, the previous standard approach is inefficient for calculating the higher order Fokker-Planck coefficients because the integrals over the scattering angles for every order of the coefficients have to be done differently and
repeatedly. The calculation complexity from lower order to higher order increases quickly. To our knowledge, Fokker-Planck coefficients over fifth order have not been calculated. Using the explicit expression for the transition probability $\mathcal{P}(v, \Delta v)$, we can calculate arbitrarily high order Fokker-Planck coefficients. For a Maxwellian background, a complete set of the Fokker-Planck coefficients as well as the collision frequency are expressed in a uniform expression in this chapter.

After a brief introduction of the Fokker-Planck equation, we calculate higher order Fokker-Planck coefficients in Section 6.2. The exact expressions for the friction and diffusion coefficients, the first two order Fokker-Planck coefficients, will be derived in Section 6.3. In Section 6.4, we recover the standard results of the friction and diffusion coefficients by approximation under the assumption of a large Coulomb logarithm. Similar coefficients for hard-sphere interactions as well as the collision frequency are expressed uniformly in Section 6.5.

6.2 Calculation of the Higher Order Linear Fokker-Planck Coefficients

Let $f(v)$ denote the velocity distribution function of the test particles. Then the complete Fokker-Planck collision operator giving the rate of change of $f(v)$ due to collisions is expressed as the following infinite series\cite{57, 58, 59}

$$
\left( \frac{\partial f}{\partial t} \right)_c = \sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \frac{\partial^N}{\partial v^N} \cdot \left( \langle \Delta v^N \rangle f \right), \quad (6.1)
$$
where $\Delta v^N$ is the $N$th order dyad of velocity change of the test particle. The Fokker-Planck coefficients $\langle \Delta v^N \rangle$ should be calculated directly from the probability function $P(v, \Delta v)$ as follows\cite{54, 55}

$$\langle \Delta v^N \rangle = \int \Delta v^N P(v, \Delta v) d\Delta v. \quad (6.2)$$

However, the explicit form of the probability function $P(v, \Delta v)$, even for the Maxwellian field particles $f_0(v_F)$, was not known when the standard Fokker-Planck coefficients were calculated. All previous authors employed instead the following five-fold integral for the linear Fokker-Planck coefficients

$$\langle \Delta v^N \rangle = \int \Delta v^N f_0(v_F) V \sigma R \sin \theta d\theta d\phi dv_F, \quad (6.3)$$

where $\sigma_R = (Z Z_F e^2 / (4 \pi \varepsilon_0 \mu))^2 (4V^4 \sin^4(\theta/2))^{-1}$ is the Rutherford cross section, in which $\mu = mm_F / (m + m_F)$ is the reduced mass, $\theta$ is the scattering angle in the center-of-mass system, and $\varphi$ is the azimuthal angle around the relative velocity $V = v - v_F$, in which $v$ and $v_F$, respectively, are the velocity of the test particle and field particle; $Z$ and $Z_F$, respectively, are the charge states of the test and field particle. In the customary approach of directly integrating Eq.(6.3), the velocity change $\Delta v$ had to be expressed in terms of field particle velocity and solid scattering angle, that is, $\Delta v(v_F, \theta, \varphi)$. For the calculation of the lower order coefficients ($N \leq 2$), there is a divergence difficulty when integrating Eq.(6.3) over scattering angle. Therefore, the traditional scattering angle cutoff, $\theta \geq \theta_{\text{min}}$, has to be
introduced in the first integration over scattering angle. In order to facilitate the remaining integral over the field particle velocity $v_F$, all previous authors[31, 41, 42] replaced the varied relative speed, $V$, by a thermal velocity, $v_{th} = \sqrt{2k_BT/m_F}$. Rosenbluth et al. pointed out that such a treatment is made mainly because no better way could be found[31].

Instead of the customary approach of direct integration of Eq.(6.3), we use the variable change technique developed in Chapter 4. By choosing $(\Delta v, \phi)$ as independent variables, Eq.(6.3) can be written as

$$\langle \Delta v^N \rangle = \int \Delta v^N f_M(v_F)V \sigma_R \sin \theta \left| \frac{\partial(v_F, \varphi)}{\partial(\Delta v, \phi)} \right| d\theta d\phi d\Delta v, \quad (6.4)$$

where $\phi$ is the azimuthal angle around the velocity change $\Delta v$. Comparing Eq.(6.2) and Eq.(6.4), the probability function can be expressed as

$$\mathcal{P}(v, \Delta v) = \int f_0(v_F)V \sigma_R \sin \theta \left| \frac{\partial(v_F, \varphi)}{\partial(\Delta v, \phi)} \right| d\theta d\phi. \quad (6.5)$$

Consider that the velocity of the test particle $v$ is a parameter that can be treated as a constant vector. We then have $d\Delta v = d(v' - v) = d\mathbf{v}'$, and

$$\left| \frac{\partial(v_F, \varphi)}{\partial(\Delta v, \phi)} \right| = \left| \frac{\partial(v_F, \varphi)}{\partial(\mathbf{v}', \phi)} \right|. \quad (6.6)$$

The probability function $\mathcal{P}(v, \Delta v)$ defined in Eq.(6.5) is therefore the same as Eq.(4.26)
obtained in Chapter 4:

\[ P(v, \Delta v) = \frac{\nu_0}{(\sqrt{\pi}a v_{th})^3} \left( \frac{\Delta v}{av_{th}} \right)^{-5} e^{-\left( \frac{\Delta v}{av_{th}} + \frac{v}{v_{th}} \frac{\Delta v}{\Delta v_{av}} \right)^2}, \]  

(6.7)

where \( a = 2\mu/m \) is the mass coefficient, and \( \nu_0 = n_F \pi \rho_1^2 \) is the close collision frequency, in which \( n_F \) is the density of the field particles, and \( \rho_1 = Z Z_F/(8\pi \varepsilon_0 k_B T(\mu/m_F)) \) is the interaction radius. With the explicit expression of the probability function Eq.(6.7), we can calculate the linear Fokker-Planck coefficients by the original definition Eq.(6.2).

For the convenience of the calculation, we introduce a non-dimensional velocity and velocity change

\[ u = v/v_{th}, \]  

(6.8)

\[ u_\delta = \Delta v/(av_{th}). \]  

(6.9)

Substituting Eqs.(6.8, 6.9) into Eq.(6.2), the \( N \)th order Fokker-Planck coefficient can be written as

\[ \langle \Delta v^N \rangle = (av_{th})^N \langle u_\delta^N \rangle, \]  

(6.10)

where

\[ \langle u_\delta^N \rangle = \int u_\delta^N P(u, u_\delta) du_\delta, \]  

(6.11)

in which

\[ P(u, u_\delta) = (av_{th})^3 P(v, \Delta v) = \frac{\nu_0}{\pi^{3/2}} u_\delta^{-5} e^{-\left( u_\delta + u \cos \chi \right)^2}. \]  

(6.12)
Introducing \((e_\parallel, e_{\perp 1}, e_{\perp 2})\) as an orthogonal triplet of unit vectors with \(e_\parallel = v/v\), then \(u_\delta\) can be expressed as

\[
\begin{align*}
\mathbf{u}_\delta &= u_\parallel e_\parallel + u_{\perp 1} e_{\perp 1} + u_{\perp 2} e_{\perp 2} \\
&= u_\delta (e_\parallel \cos \chi + e_{\perp 1} \sin \chi \cos \psi + e_{\perp 2} \sin \chi \sin \psi),
\end{align*}
\]

(6.13)

where \(\chi\) is the angle between \(u_\delta\) and \(u\), and \(\psi\) is the azimuthal angle of \(u_\delta\) around \(u\). Here, the unit vectors, \(e_\parallel\), \(e_{\perp 1}\) and \(e_{\perp 2}\) respectively denote the directions parallel and perpendicular to \(u\). Choosing a spherical coordinate system with \(u\) as the polar axis, we then have \(d\mathbf{u}_\delta = u_\delta^2 \sin \chi d\psi d\delta\). Thus, the factor \(\langle u_\delta^N \rangle\) in Eq.(6.11) can be written as

\[
\langle u_\delta^N \rangle = \frac{\nu_0}{\pi^{3/2}} \int u_\delta^N u_\delta^{-3} e^{-(u_\delta + u\cos \chi)^2} \sin \chi d\chi d\psi d\delta.
\]

(6.14)

Substituting Eq.(6.13) into Eq.(6.14) and using the integral formula Eq.(B.5) derived in Appendix B, the nonzero elements of the \(N\)th dyad of \(\langle u_\delta^N \rangle\) are as follows

\[
\langle u_\delta^{N-2(J+K)} u_{\parallel 1}^{2J} u_{\perp 1}^{2K} \rangle = \langle u_\delta^N \cos^{N-2(J+K)} \chi \sin^{2(J+K)} \chi \cos^{2J} \psi \sin^{2K} \psi \rangle
\]

\[
= 2\pi^{-3/2} \nu_0 B\left(J+\frac{1}{2}, K+\frac{1}{2}\right) \int e^{-(u_\delta + u\cos \chi)^2} u_\delta^{-3} \cos^{N-2(J+K)} \chi \sin^{2(J+K)+1} \chi d\chi d\delta,
\]

(6.15)

where \(B\) is the beta function defined as Eq.(6.2.2) in Ref. [60]. With the binomial theorem
Eq.(3.1.1) in Ref.[61], the sine function can be expressed in terms of a cosine function
\[
\cos^{N-2(J+K)} \chi \sin^{2(J+K)} \chi = (-1)^{J+K} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i \cos^{N-2i} \chi.
\] (6.16)

Substituting Eq.(6.16) into Eq.(6.15), we have
\[
\langle u_{N-2(J+K)}^{2J} u_{N-2(J+K)}^{2K} \rangle = \frac{\nu_0 B(J+(1/2), K+(1/2))}{(-1)^{J+K} \pi} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i \times \frac{2}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^\infty e^{-(u_j+u \cos \chi)^2} u^N_\delta \cos^{N-2i} \chi \sin \chi d\chi du_\delta.
\] (6.17)

For high order Fokker-Planck coefficients \( N \geq 3 \), Eq.(6.17) is a convergent integral. Applying the two-fold integral formula Eq.(C.1) derived in Appendix C, we have
\[
\langle u_{N-2(J+K)}^{2J} u_{N-2(J+K)}^{2K} \rangle = \frac{\nu_0 B(J+(1/2), K+(1/2))}{(-1)^{J+K} \pi} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i \times \sum_{j=0}^{N-1} C_{(N-1)/2}^{j} \frac{\Gamma ((3/2)+[(N-1)/2]-i)}{\Gamma(3/2)+j} \frac{\Upsilon_{j} ((3/2)+j-i, u^2)}{(-u)^{N-2}},
\] (6.18)

where \( \Gamma \) is the gamma function, the operator \([\ ]\) represents the greatest integer less than or equal to the value inside, and the function \( \Upsilon_{j}(\alpha, x) \) is defined as
\[
\Upsilon_{j}(\alpha, x) = x^j \exp(-x) \frac{d^j}{dx^j} \left( x^j \exp(x) \gamma^*(\alpha, x) \right),
\] (6.19)
in which the analytical incomplete gamma function [62] is

\[ \gamma^*(\alpha, x) = \frac{1}{x^\alpha \Gamma(\alpha)} \int_0^x e^{-t} t^{\alpha-1} dt. \] (6.20)

All of the high order, \( N \geq 3 \), linear Fokker-Planck coefficients are expressed as Eqs.(6.10) and (6.18-6.20).

### 6.3 Low Order Fokker-Planck Coefficients

In Section 6.2, we derived the probability function Eq.(6.7) and all the high order \( (N \geq 3) \) linear Fokker-Planck coefficients. Here, expressions for low order \( (N \leq 2) \) Fokker-Planck coefficients are derived. Since field particles are assumed to have an isotropic Maxwellian velocity distribution, the low order Fokker-Planck coefficients have the following symmetric relations

\[ \langle \Delta v_{\perp 1} \rangle = \langle \Delta v_{\perp 2} \rangle = 0, \] (6.21)

\[ \langle \Delta v_{\perp 1}^2 \rangle = \langle \Delta v_{\perp 2}^2 \rangle. \] (6.22)

With the symmetric relations, Eqs.(6.21) and (6.22), the friction coefficient \( \langle \Delta v_\parallel \rangle \) and two diffusion coefficients \( \langle \Delta v_\parallel^2 \rangle \) and \( \langle \Delta v^2 \rangle \) are considered the independent low order Fokker-Planck coefficients. Using Eqs.(6.10-6.13), the three low order Fokker-Planck coefficients are
expressed as

\[
\langle \Delta v \| \rangle = \frac{\nu_0 a v_{th}}{\pi^{3/2}} \int u_\delta^{-2} e^{-(u_\delta + u \cos \chi)^2} \cos \chi \sin \chi d\chi d\psi du_\delta; \quad (6.23)
\]

\[
\langle \Delta v^2 \| \rangle = \frac{\nu_0 (a v_{th})^2}{\pi^{3/2}} \int u_\delta^{-1} e^{-(u_\delta + u \cos \chi)^2} \cos^2 \chi \sin \chi d\chi d\psi du_\delta, \quad (6.24)
\]

\[
\langle \Delta v \rangle = \frac{\nu_0 (a v_{th})^2}{\pi^{3/2}} \int u_\delta^{-1} e^{-(u_\delta + u \cos \chi)^2} \sin \chi d\chi d\psi du_\delta, \quad (6.25)
\]

Integrating over \( \psi \), and applying integral formulas Eqs.(B.6-B.8) derived in Appendix B, we have

\[
\langle \Delta v \| \rangle = -\nu_0 a v_{th} \int \frac{\text{eef}(u_\delta + u, u^2_\delta) - \text{eef}(u_\delta - u, u^2_\delta)}{u^2 u_\delta} du_\delta; \quad (6.26)
\]

\[
\langle \Delta v^2 \| \rangle = \nu_0 (a v_{th})^2 \left( \int \frac{\text{eef}(u_\delta + u, u^2_\delta) - \text{eef}(u_\delta - u, u^2_\delta)}{2u^3 u_\delta} du_\delta + \frac{\mathcal{F}(u, u_\delta)}{u_\delta u^3} du_\delta \right); \quad (6.27)
\]

\[
\langle \Delta v \rangle = \nu_0 (a v_{th})^2 \int \frac{\text{erf}(u_\delta + u) - \text{erf}(u_\delta - u, u^2_\delta)}{u u_\delta} du_\delta; \quad (6.28)
\]

where \( \text{erf} \) is the error function [18], \( \text{eef} \) is a combination of exponential and error functions

\[
\text{eef}(x, y) = \text{erf}(x) + \frac{\exp(-x^2)}{\sqrt{\pi} y}. \quad (6.29)
\]

and \( \mathcal{F}(u, u_\delta) \) is defined as

\[
\mathcal{F}(u, u_\delta) = u_\delta^2 (\text{erf}(u_\delta + u) - \text{erf}(u_\delta - u)) + \frac{-(2u_\delta)^{-1} + u_\delta - u}{\sqrt{\pi} \exp((u_\delta + u)^2)} + \frac{(2u_\delta)^{-1} - u_\delta - u}{\sqrt{\pi} \exp((u_\delta - u)^2)}. \quad (6.30)
\]
The integrals with respect $u_\delta$ in Eqs.(6.26-6.28) are divergent, except for the second right side integral of Eqs.(6.27). Using the integral formula Eq.(B.11) in Appendix B, we have

$$\int_0^\infty \frac{\mathcal{F}(u, u_\delta)}{u^3 u_\delta} du_\delta = \Upsilon_1(5/2, u^2),$$

(6.31)

where $\Upsilon_1(5/2, x)$ is defined as Eq.(6.19). It is interesting that the general expression for high order Fokker-Planck coefficients Eq.(6.18) can also include all low order Fokker-Planck coefficients if we make supplementary definitions for $\Upsilon_0(1/2, x)$ and $\Upsilon_0(3/2, x)$ as

$$\Upsilon_0(1/2, x) = \frac{1}{2x^{1/2}} \int_{y_{\text{min}}}^\infty \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y} dy,$$

(6.32)

$$\Upsilon_0(3/2, x) = \frac{1}{2x^{3/2}} \int_{y_{\text{min}}}^\infty \frac{\text{erf}(\sqrt{y} + \sqrt{x}, y) - \text{erf}(\sqrt{y} - \sqrt{x}, y)}{y} dy,$$

(6.33)

where $y$ is the non-dimensional collision strength variable which is

$$y = u_\delta^2 = \frac{H}{k_B T (\mu/m_F)}.$$  

(6.34)

Here $H = (\Delta p)^2/(8\mu)$ is the collision strength, and $\Delta p$ is the momentum transfer of a collision event.

In summary, a complete set of linear Fokker-Planck coefficients is expressed uniformly as

$$\langle \Delta v_{\parallel}^{N-2(J+K)} \Delta v_{\perp 1}^{2J} \Delta v_{\perp 2}^{2K} \rangle = \nu_0 B(J+(1/2), K+(1/2)) \frac{J+K}{(-1)^{J+K} \pi (av_{\text{th}})^{-N}} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i$$

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\[
\times \sum_{j=[(N-1)/2]}^{N-i-1} C_{j-[(N-1)/2]}^{j-[(N-1)/2]} \frac{\Gamma((3/2)+[(N-1)/2]-i)}{\Gamma((3/2)-[N/2]+j)} \frac{\Upsilon_j((3/2)+j-i, u^2)}{(-u)^{N-2}}
\]

(6.35)

with the function \(\Upsilon_j(\alpha, x)\) defined as Eqs.(6.19,6.32,6.33).

We notice that a collision frequency \(\nu(v)\), can be regarded as the zeroth order Fokker-Planck coefficient. Therefore, Eq.(6.35) can be used to represent collision frequency when \(N = 0\). Thus,

\[
\nu(v) = \langle 1 \rangle = \nu_0 u^2 \Upsilon_{-1} \left(1/2, u^2\right),
\]

(6.36)

where \(\Upsilon_{-1}(1/2, u^2)\) is defined as

\[
\Upsilon_{-1}(1/2, x) = \frac{1}{2x^{3/2}} \int_{y_{\text{min}}}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y^2} dy.
\]

(6.37)

With the definition Eq.(6.37), Eq.(6.36) is consistent with the collision frequency Eq.(5.45) derived in Chapter 5.

In Eqs.(6.32, 6.33, 6.37), there is only one singularity point which is at the integral limit \(y = 0\) corresponding to distance weak collision events. Unlike some traditional approaches, no singularity point is encountered corresponding to close collision events. In Ref.[30], Landau points out “If the exact formulae are used then there would, of course, be no divergence at small \(b\)” (\(b\) is impact parameter; small \(b\) corresponding to close collision events). Here, we prove Landau’s assertion by exact mathematical treatment. In the theory of Lenard-Balescu[63, 64], a divergence does occur in the range of infinite \(k\) (\(k\) is wave vector; large \(k\)
corresponding to close collision events). It is wrong to regard the divergence in the Lenard-
Balescu theory as the same divergence that occurs in Landau’s work, as claimed in Ref.[63].
Basically, no divergence occurs for close collision events in an exact theory for the Fokker-
Planck collision operator.

6.4 Friction and Diffusion Coefficients for Plasmas with a Large Coulomb Logarithm

No approximation is introduced when arbitrarily high order Fokker-Planck coefficients are reduced to Eq.(6.35). However, the low order Fokker-Planck coefficients are expressed in term of one-fold integrals \( \Upsilon_0(3/2, x) \) as in Eq.(6.33), and \( \Upsilon_0(1/2, x) \) as in Eq.(6.32). In this section, we evaluate the first and second order Fokker-Planck coefficients under the large Coulomb logarithm assumption \( \lambda >> 1 \).

According Eq.(6.35), the three low order Fokker-Planck coefficients are

\[
\langle \Delta v_{||} \rangle = -\nu_0 a v_{th} u \Upsilon_0(3/2, u^2),
\]

(6.38)

\[
\langle \Delta v_{\perp}^2 \rangle = \nu_0(a v_{th})^2 \left( \frac{1}{2} \Upsilon_0(3/2, u^2) + \Upsilon_1(5/2, u^2) \right),
\]

(6.39)

\[
\langle \Delta v^2 \rangle = \langle \Delta v_{||}^2 \rangle + 2 \langle \Delta v_{\perp 1}^2 \rangle = \nu_0(a v_{th})^2 \Upsilon_0(1/2, u^2).
\]

(6.40)

It is believed that weak collision events dominate the low order Fokker-Planck coefficients when the Coulomb logarithm is large \( \lambda >> 1 \). Here, a large Coulomb logarithm corresponds
to a small collision strength $y_{\text{min}} \ll 1$. If $y_{\text{min}} \ll 1$, we can prove from Eqs.(6.32,6.33) by using a Taylor series expansion that

$$\Upsilon_0(1/2, x) \approx \ln(y^{-1}_{\text{min}}) \gamma^*(1/2, x),$$  \hspace{1cm} (6.41)

$$\Upsilon_0(3/2, x) \approx \ln(y^{-1}_{\text{min}}) \gamma^*(3/2, x).$$  \hspace{1cm} (6.42)

Because $\Upsilon_1(5/2, u^2)$ does not depend on the Coulomb logarithm or $\ln(y^{-1}_{\text{min}})$, we can drop the term from the expression for $\langle \Delta v_\parallel^2 \rangle$. Then, the friction and diffusion coefficients Eqs.(6.38-6.40) can be reduced to

$$\langle \Delta v_\parallel \rangle \approx -\nu_0 \ln(y^{-1}_{\text{min}}) (a v_{\text{th}}) u \gamma^*(3/2, u^2),$$  \hspace{1cm} (6.43)

$$\langle \Delta v_\parallel^2 \rangle \approx \frac{1}{2} \nu_0 \ln(y^{-1}_{\text{min}}) (a v_{\text{th}})^2 \gamma^*(3/2, u^2),$$  \hspace{1cm} (6.44)

$$\langle \Delta v^2 \rangle \approx \nu_0 \ln(y^{-1}_{\text{min}}) (a v_{\text{th}})^2 \gamma^*(1/2, u^2).$$  \hspace{1cm} (6.45)

The analytical incomplete gamma function $\gamma^*$ is related to the error function, erf, and Chandrasekhar’s function, $G(x) = -\frac{d}{dx} \frac{\text{erf}(x)}{x}$ [32], through

$$\gamma^*(1/2, u^2) = \frac{\text{erf}(u)}{u},$$  \hspace{1cm} (6.46)

$$\gamma^*(3/2, u^2) = \frac{2G(u)}{u}.$$  \hspace{1cm} (6.47)
If we assume $\lambda = \frac{1}{2} \ln(y_{\text{min}}^{-1})$, standard expressions for the friction and diffusion coefficients are recovered from Eqs.(6.43-6.45) as

\begin{align*}
\langle \Delta v_\parallel \rangle &\approx -4\nu_0\lambda a v_{\text{th}} G(u), \\
\langle \Delta v_\parallel^2 \rangle &\approx 2\nu_0\lambda (a v_{\text{th}})^2 \frac{G(u)}{u}, \\
\langle \Delta v^2 \rangle &\approx 2\nu_0\lambda (a v_{\text{th}})^2 \frac{\text{erf}(u)}{u}.
\end{align*}

(6.48) \quad (6.49) \quad (6.50)

These expressions agree to within 5% with Eqs.(6.38-6.40) for Coulomb logarithm values larger than 10 and test particle speeds less than the thermal speed of the field particles ($u < 1$). When the Coulomb logarithm is not large, the friction and diffusion coefficients can be obtained by numerical calculation of Eqs.(6.38-6.40).

6.5 Moments for Hard Sphere Interactions

The present approach is not limited to Coulomb interactions. It can also apply for hard sphere interactions. The $N$th order moments for hard sphere interactions are

\begin{align*}
\langle \Delta v_\parallel^{N-2(J+K)} \Delta v_{\perp 1}^{2J} \Delta v_{\perp 2}^{2K} \rangle &= \frac{\nu_0 \mathcal{B}(J + (1/2), K + (1/2))}{(-1)^{J+K} \pi (a v_{\text{th}})^{-N}} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i \\
&\times \sum_{j=[(N+3)/2]}^{N-i+1} C_{\lceil N/2 \rceil - i}^{j - \lfloor (N+3)/2 \rfloor} \frac{\Gamma((1/2) + [(N + 1)/2] - i) \Gamma_j(j - i - (1/2), u^2)}{\Gamma((1/2) - [(N + 2)/2] + j) (-u)^{N+2}},
\end{align*}

(6.51)
where the close collision frequency is \( \nu_0 = n_F \nu_{th} \pi \rho_H^2 \), in which the interaction radius is defined as

\[
\rho_H = (d + d_F)/2. \tag{6.52}
\]

Here \( d \) and \( d_F \) are the diameters of the test and field particles respectively.

The collision frequency for a test particle can be evaluated from Eq.(6.51) as

\[
\nu(v) = \langle 1 \rangle = \nu_0 u^{-2} \Upsilon_1 \left(1/2, u^2\right). \tag{6.53}
\]

It is found that Eq. (6.53) is the same as Eq.(5.19) in Chapter 5, because

\[
u^{-2} \Upsilon_1 \left(1/2, u^2\right) = \pi^{-1/2} e^{-u^2} + \left(u + \frac{1}{2u}\right) \text{erf}(u). \tag{6.54}
\]

In summary, Fokker-Planck coefficients as well as collision frequencies for both Coulomb interactions and hard sphere interactions are effectively expressed in terms of the function \( \Upsilon_j (\alpha, x) \) as Eqs.(6.35) and (6.51).
CHAPTER 7

THE EQUILIBRIUM TIME

As an application of the exact expression for the Fokker-Planck coefficients obtained in Chapter 6, we derive an exact expression for an equilibrium time in this chapter. Unlike Spitzer’s expression, the new equilibrium time is expressed in terms of the minimum collision strength $H_{\text{min}}$.

7.1 Introduction

An equilibration time scale characterizes the rate for a group of test particles with a Maxwellian velocity distribution but different temperature from that of Maxwellian field particles to relax to the same temperature of the field particles. The equilibration time scale for a two-component-plasma was first calculated by Landau[30]. In stellar dynamics, a more precise expression for equilibration time was obtained by Spitzer[40]. Later, Spitzer[4] reformulated the equilibration time based on Chandrasekhar’s formula for the velocity space friction and diffusion coefficients[41, 42]. The derivation going from the friction and diffusion coefficients to the equilibration time is exact. However, the equilibration time is limited to plasmas with large Coulomb logarithm values due to the approximations employed in the calculation of the friction and diffusion coefficients. Recently, exact expressions for the
friction and diffusion coefficients have obtained by a variable change technique[10]. With the new coefficients, we can develop an equilibration time that has no limit on applicable Coulomb logarithm values. The difference between the new equilibration time and the old one is characterized by a factor associate with the definition of Coulomb logarithm[11].

The definition for equilibration time is provided in Section 7.2. We will also show how the equilibration time relates to the velocity space friction and diffusion coefficients. Sipitzer’s approach to the equilibration time is reviewed in Section 7.3. The new approach to the equilibration time is presented in Section 7.4. We also provided the equilibration time for hard sphere interactions.

7.2 Equilibration Time

Consider a single test particle moving in plasma of Maxwellian field particles. The test particle will exchange its energy with the field particles due to collisions. In a single encounter of a test particle with a field particle, the exchange of energy, $\Delta E$, is given by

$$\Delta E = \frac{1}{2}m \left( \Delta v^2 + 2v\Delta v_\parallel \right), \quad (7.1)$$

where $m$ is the mass of the test particle, $v$ is the velocity of the test particle, $\Delta v$ is the velocity change due to the collision, and $\Delta v_\parallel$ is the magnitude of its velocity change along its original direction. Averaging Eq.(7.1) over a Maxwellian velocity distribution for the field
particles, the average time rate of change of the test particle energy is written as

\[ \langle \Delta E \rangle = \frac{1}{2} m \left( \langle \Delta v^2 \rangle + 2v \langle \Delta v \parallel \rangle \right), \]  

(7.2)

where \( \langle \Delta v \parallel \rangle \) and \( \langle \Delta v^2 \rangle \) are the usual Fokker-Planck velocity-space friction and diffusion coefficients. There are various applications for \( \langle \Delta E \rangle \). Stopping power \([48], dE/dl\), characterizes the energy loss per unit distance for a high speed test particle. For Maxwellian field particles,

\[ \frac{dE}{dl} = \frac{1}{v} \langle \Delta E \rangle = \frac{1}{2v} m \left( \langle \Delta v^2 \rangle + 2v \langle \Delta v \parallel \rangle \right). \]  

(7.3)

For obtaining an equilibration time, let us suppose that a group of test particles have a Maxwellian velocity distribution with temperature \( T \), which is different from that of the field particles of temperature \( T_F \). The equilibration time scale can be obtained simply by averaging Eq.(7.2) over a Maxwellian velocity distribution for the test particles. The rate of change of the test particle temperature is defined through the expression,

\[ \frac{3}{2} n k_B \frac{dT}{dt} = \int \langle \Delta E \rangle f_0 (v, T) \, dv, \]  

(7.4)

where \( n \) is the density of the test particle species, \( k_B \) is Boltzmann’s constant, and \( f_0 (v, T) \)
is the Maxwellian velocity distribution with temperature $T$,

$$f_0(v, T) = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{m}{2k_B T} v^2 \right). \quad (7.5)$$

Because the test particle velocity distribution is isotropic, the integral over solid angle can be carried out. After integrating over the solid angle, Eq.(7.4) can be written as

$$\frac{dT}{dt} = \frac{8\pi}{3nk_B} \int_0^\infty \langle \Delta E \rangle f_0(v, T) v^2 dv. \quad (7.6)$$

Substituting Eq.(7.5) into Eq.(7.6) and using a variable change, $u = v/v_{th}$ with the thermal velocity of the field particle defined as $v_{th} = \sqrt{2k_B T_F/m_F}$, a non-dimensional integral form of Eq. (7.6) can be written as

$$\frac{dT}{dt} = \frac{8\zeta^{3/2}}{3\sqrt{\pi}k_B} \int_0^\infty \langle \Delta E \rangle \exp \left( -\zeta u^2 \right) u^2 du, \quad (7.7)$$

where $\zeta = T_F m/(T m_F)$ with $m_F$ the mass of the field particle. After integrating over $u$, Eq.(7.7) can be reduced to

$$\frac{dT}{dt} = \frac{T_F - T}{\tau_{eq}}. \quad (7.8)$$

Here, $\tau_{eq}$ in Eq.(7.8) is defined as the equilibration time or time scale. The next step for the calculation of the equilibration time scale is to integrate Eq. (7.7) over $u$. In order to integrate Eq. (7.7), one must substitute in Eq. (7.2) with expressions for friction and diffusion.
coefficients.

7.3 Spitzer’s Approach To The Equilibration Time Scale

In his second version of the derivation, Spitzer employed Chandrasekhar’s expressions for the friction and diffusion coefficients[4],

\[
\langle \Delta v_\parallel \rangle = -4\nu_0 av_{th} \lambda G(u), \quad (7.9)
\]

and

\[
\langle \Delta v^2 \rangle = 2\nu_0 (av_{th})^2 \lambda \text{erf}(u)/u, \quad (7.10)
\]

where \( a = 2\mu/m \), the reduced mass is \( \mu = mm_F/(m + m_F) \), \( \lambda \) is the Coulomb logarithm, the close collision frequency \( \nu_0 = n_F v_{th} \pi \rho_1^2 \), with \( n_F \) is the density of the field particles, the interaction radius for one test particle is defined as \( \rho_1 = ZZ_F e^2/(8\pi\varepsilon_0 k_B(\mu T_F/m_F)) \), \( Z \) and \( Z_F \) are the charge state of a test particle and field particle, \( e \) is the unit charge, \( \varepsilon_0 \) is the permittivity of free space, \( \text{erf} \) is the error function, and Chandrasekhar’s function[32] is defined as \( G(u) = -\frac{1}{2} \frac{d}{du} (\text{erf}(u)/u) \). Substituting Eqs. (7.9) and (7.10) into Eq. (7.2), the integral in Eq. (7.7) can be obtained as

\[
\frac{dT}{dt} = \frac{4m (av_{th})^2 \lambda \zeta^{1/2} (1 + \zeta - 2a^{-1})}{3\sqrt{\pi} \tau_0 k (1 + \zeta)^{3/2}}, \quad (7.11)
\]
where $\zeta = T_F m/(T m_F)$. Equation (7.11) can be rearranged into the standard form as

$$\frac{dT}{dt} = \frac{T_F - T}{\tau_{eq}^{\text{Spitzer}}}, \quad (7.12)$$

where $\tau_{eq}^{\text{Spitzer}}$ is the equilibration time scale obtained by Spitzer

$$\tau_{eq}^{\text{Spitzer}} = \frac{3mm_F \tau_0}{16 \mu^2 \lambda} = \frac{3mm_F}{8\sqrt{2\pi n_F} \lambda} \left( \frac{4\pi \varepsilon_0}{ZZ_F e^2} \right)^2 \left( \frac{k_B T}{m} + \frac{k_B T_F}{m_F} \right)^{3/2}, \quad (7.13)$$

where the close collision time is $\tau_0 = (n_F v_{av} \pi \rho_2^2)^{-1}$, the average relative speed between test and field particles is $v_{av} = \sqrt{8k_B T'/(\pi \mu)}$, and the interact radius for a group of test particles is defined as

$$\rho_2 = ZZ_F e^2 / (8\pi \varepsilon_0 k_B T'), \quad (7.14)$$

in which

$$T' = \mu \left( \frac{T}{m} + \frac{T_F}{m_F} \right). \quad (7.15)$$

Spitzer’s equilibration time scale, Eq.(7.13), applies only for plasmas with large Coulomb logarithm values because of an approximation employed in the process of obtaining Chandrasekhar’s expressions for the friction and diffusion coefficients.
7.4 New Approach To The Equilibration Time Scale

Recently, a variable change technique[10] has been developed for the calculation of the Fokker-Planck coefficients. Exact one-fold integral expressions for Fokker-Planck velocity-space friction and diffusion coefficients have been obtained from the new technique. These exact expressions make it possible to re-calculate the equilibration time scale without placing a restriction on the value of the Coulomb logarithm. The exact expressions for the friction and diffusion coefficients are Eqs.(6.38,6.40) as derived in Chapter 6.

\[
\langle \Delta v \rangle = -\nu_0 a v_{th} u \Upsilon_0(3/2, u^2), \tag{7.16}
\]

\[
\langle \Delta v^2 \rangle = \nu_0 (a v_{th})^2 \Upsilon_0(1/2, u^2), \tag{7.17}
\]

where the functions \( \Upsilon_0(1/2, x) \) and \( \Upsilon_0(3/2, x) \) are defined as one-fold integrals as Eqs.(6.32,6.33), which are

\[
\Upsilon_0(1/2, x) = \frac{1}{2x^{1/2}} \int_{y_{min}}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y} dy, \tag{7.18}
\]

\[
\Upsilon_0(3/2, x) = \frac{1}{2x^{3/2}} \int_{y_{min}}^{\infty} \frac{\text{eef}(\sqrt{y} + \sqrt{x}, y) - \text{eef}(\sqrt{y} - \sqrt{x}, y)}{y} dy, \tag{7.19}
\]

in which \( \text{eef}(x, y) = \text{erf}(x) + (\pi y)^{-1/2}\exp(-x^2) \) is a combination of error and exponential functions, and \( y \) is the non-dimensional collision strength variable, as

\[
y = \frac{H}{k_B T_F (\mu/m_F)}. \tag{7.20}
\]
Here $H = (\Delta p)^2/(8\mu)$ is collision strength, and $\Delta p$ is momentum transfer for a collision event. Substituting Eqs.(7.16, 7.17) into Eq.(7.2), we have the expression for the energy exchange rate as

$$\langle \Delta E \rangle = \frac{1}{2} m v_0 (av_{th})^2 \left( Y_0(1/2, u^2) - \frac{2}{a} u^2 Y_0(3/2, u^2) \right).$$  \hspace{1cm} (7.21)$$

Substituting Eq.(7.21) into Eq.(7.7), we have

$$\frac{dT}{dt} = \frac{4 m v_0 (av_{th})^2}{3 \sqrt{\pi k_B}} \frac{\zeta^{3/2}}{3 \sqrt{\pi k_B}} \int_0^{\infty} \left( \frac{1}{2} Y_0(1/2, u^2) - \frac{2}{a} u^2 Y_0(3/2, u^2) \right) e^{-\zeta u^2} du.$$ \hspace{1cm} (7.22)$$

If we use the one-fold integral forms, Eqs.(7.18) and (7.19), for functions $Y_0(1/2, u^2)$ and $Y_0(3/2, u^2)$ in Eq.(7.22). The integrand is a combination of error and exponential functions. Applying the integral formulas Eqs.(B.16-B.19) derived in Appendix B, we can integrate Eq.(7.22) over $u$ as

$$\frac{dT}{dt} = \frac{2 m v_0 (av_{th})^2}{3 \sqrt{\pi k_B}} \frac{\zeta^{1/2}}{(1 + \zeta)^{3/2}} \int_{y_{\min}}^{\infty} \exp \left( - \frac{\zeta y}{1 + \zeta} \right) y^{-1} dy.$$ \hspace{1cm} (7.23)$$

Equation (7.23) can be further reduced as

$$\frac{dT}{dt} = \frac{2 m v_0 (av_{th})^2}{3 \sqrt{\pi k_B}} \frac{\zeta^{1/2}}{(1 + \zeta)^{3/2}} \frac{\Gamma \left( 0, \frac{H_{\min}}{k_B T'} \right)}{\Gamma \left( 0, H_{\min} / k_B T' \right)}.$$ \hspace{1cm} (7.24)$$
where $\Gamma$ is zeroth order incomplete gamma function, $H_{\text{min}}$ is the minimum collision strength, and $T'$ is defined as Eq.(7.15). Let us rewrite Eq. (7.24) as

$$\frac{dT}{dt} = \frac{T_F - T}{\tau_{\text{new}}},$$

(7.25)

We obtain the new equilibration time scale as

$$\tau_{\text{new}}^{\text{eq}} = \frac{3m m_{F} \tau_0}{8 \mu^2 \Gamma(0, H_{\text{min}}/(k_b T'))}.$$  

(7.26)

The difference of the equilibrium time between Spitzer’s approach and the present approach is characterized by the ratio

$$\frac{\tau_{\text{eq}}^{\text{Spitzer}}}{\tau_{\text{eq}}^{\text{new}}} = \frac{\Gamma(0, H_{\text{min}}/(k_b T'))}{2\lambda}.$$  

(7.27)

In other word, the new expression is the same as Spitzer’s expression if the Coulomb logarithm is replaced by $\frac{1}{2} \Gamma(0, H_{\text{min}}/(k_b T'))$. No approximation is introduced in obtaining the new expression of the equilibrium time Eq.(7.26). Therefore, Eq.(7.26) is not restricted in applicability by the value of the Coulomb logarithm.

It is notable that the approach can also applied for hard sphere interactions. In the case
of hard-sphere interactions, the equilibrium time is obtained as

\[ \tau_{eq}^H = \frac{3m_m v_0}{8\mu^2 \Gamma(2, H_{\text{min}}/(k_B T'))}, \]  

(7.28)

where \( \tau_0 = (n_F v_{av} \pi \rho_H^2)^{-1} \), and \( \rho_H = (d + d_F)/2 \) is the interaction radius for hard-sphere interactions, where \( d \) and \( d_F \) are diameters of the test and field particles. If we compare Eq.(7.26) and Eq.(7.28), we find that the main difference between Coulomb and hard-sphere interactions is to increase the order of the incomplete Gamma function by 2. For hard-sphere interactions, the minimum collision strength can be set to zero \( H_{\text{min}} = 0 \). Then, we have

\[ \tau_{eq}^H = \frac{3m_m v_0}{8\mu^2}. \]  

(7.29)

In the foregoing, the equilibrium times for both Coulomb interactions and hard sphere interactions are derived based on exact expressions of the friction and diffusion coefficients. The difference between Coulomb interactions and hard-sphere interactions is of the order of the incomplete gamma function. For the equilibrium time for Coulomb interactions, the difference of the new result with Spitzer’s result is characterized by a replacement of the Coulomb logarithm.
CHAPTER 8

REDUCED TEMPERATURE AND A UNIFIED THEORY

In this chapter, we review all the newly obtained results in the previous chapters. A concept of reduced temperature, \( T' \), is introduced as an average collision strength. With the reduced temperature, many results for a one-temperature system can be generalized to results for a two-temperature system. By introducing an interaction parameter, \( \eta \), which equals 1 for hard-sphere interactions and \(-1\) for Coulomb interactions, these two kinds of interactions can be unified. A non-dimensional collision strength variable formed by combining collision strength and reduced temperature uniformly appears in many final expressions. And the cutoff collision strength \( H_{\text{min}} \) can unify activation energy in chemical reaction theory, ionization energy in Thomson’s classical ionization theory, and the Coulomb logarithm in plasma physics.

8.1 Average Collision Strength and Reduced Temperature

In Chapter 2, the average collision strength \( \overline{H} \) was calculated as \( k_B T \) for hard-sphere interactions when both field and test particles are in thermal equilibrium [see Eq. (2.49)]. Here, we consider a more general case for the average collision strength \( \overline{\eta} \). When the temperatures of field and test particles are different, the average collision strength \( \overline{H} \) is defined as
\[ \overline{H} = \frac{1}{\omega} \{ H \} = \frac{1}{\omega} \int H f_0(v; T) f_0(v_F; T_F) \sigma_H V d\Omega dvdv_F, \]  

(8.1)

where the integration limit extends over the whole six-dimensional velocity space and two-dimensional solid scattering angle. Using the same variable change technique developed in Chapter 3, we get, for hard-sphere interactions,

\[ \overline{H} = k_B T', \]  

(8.2)

where

\[ T' = \mu \left( \frac{T}{m} + \frac{T_F}{m_F} \right), \]  

(8.3)

and \( \mu = mm_F/(m + m_F) \) is reduced mass. We emphasize that the average collision strength, Eq.(8.2), is a calculation just for the case of hard-sphere interactions. We can not exactly calculate the average collision strength \( \overline{H} \) for Coulomb interaction systems because of the divergence problem. However, the temperature \( T' \) defined in Eq.(8.3) can be defined that way for any kind of interaction. For whatever system, we can always use \( k_B T' \) to characterize the typical collision strength. In this dissertation, the temperature is so important that \( T' \) deserves a name of its own. We call \( T' \) the reduced temperature for the system. In this dissertation, we have defined three basic physical terms: collision strength, \( H = (\Delta p)^2/(8\mu) \), cutoff collision strength \( H_{\text{min}} \), and reduced temperature \( T' \). These three physical terms form the basic elements to build all the new results derived in previous chapters.
If the temperature of the test particles equals the temperature of the field particles \( T = T_F \), then we have \( T' = T \). Many previous results of one temperature system can be generalized for a two-temperature system simply by replacing \( T \) by \( T' \). The purpose of this chapter is to review all of the main results of the previous chapters in terms of collision strength, cutoff collision strength, and the reduced temperature \( T' \).

The reduced temperature \( T' \) as a basic element can provide some useful formulas. The average relative speed \( v_{av} \), for example, is one of them. If the temperature of the test particles is different from that of the field particles, the average relative speed is

\[
v_{av} = \sqrt{\frac{8k_BT'}{(\pi\mu)}}, \tag{8.4}
\]

where \( \mu \) is the reduce mass. If we compare Eq.(8.4) with Eq.(2.9) which is the average speed for a one-temperature system, they have the same form. Therefore, we hypothesize that Eq.(8.4) may be considered a more general form of the average relative speed for a two-temperature system.

The main purpose of this chapter is to provide a uniform and generalized theory. The reduced temperature helps to generalize one-temperature problems into two temperature problems. To unify hard sphere interactions and Coulomb interactions, we introduce a parameter \( \eta \) for interaction type. The parameter \( \eta \) has only two values. It represents hard sphere interactions when \( \eta = +1 \), and represents Coulomb interactions when \( \eta = -1 \). With the reduced temperature \( T' \) and the parameter \( \eta \), a uniform interaction radius for both hard
sphere interactions and Coulomb interactions can be defined as

\[
\rho_\eta = \begin{cases} 
\rho_+ = (d + d_F)/2, & \text{for hard sphere interactions} \\
\rho_- = \kappa/(2k_B T'), & \text{for the inverse-power potential, } U(r) = \kappa/r \end{cases}
\] (8.5)

where \(k_B\) is Boltzmann’s constant, and \(T'\) is the reduced temperature Eq.(8.3). For hard sphere interactions, \(d\) and \(d_F\) are the diameters of the two kinds of particles. For inverse-square force law, \(\kappa\) is a force constant. For example, \(\kappa\) equals \(ZZ_F e^2/(4\pi\varepsilon_0)\) for Coulomb interactions.

With the definition of the interaction radius \(\rho_\eta\) as in Eq.(8.5), we can define a typical collision rate \(\omega_0\), a typical collision time \(\tau_0\), and a typical collision frequency \(\nu_0\) in a uniform expression:

\[
\omega_0 = nn_F v_{av} \pi \rho_\eta^2, \quad \text{typical collision rate} \quad (8.6)
\]

\[
\tau_0 = n/\omega_0 = (n_F v_{av} \pi \rho_\eta^2)^{-1}, \quad \text{typical collision time} \quad (8.7)
\]

\[
\nu_0 = n_F v_{th} \pi \rho_\eta^2, \quad \text{typical collision frequency} \quad (8.8)
\]

where \(n, n_F\) are number densities of test and field particles, \(v_{av}\) is the average relative speed between test and field particles Eq.(8.4), and \(v_{th} = \sqrt{2k_B T_F/m_F}\) is the thermal velocity of the field particles. It is notable that a typical collision rate \(\omega_0\) and a typical collision time \(\tau_0\)
are for a group of test particles that interact with all field particles, while a typical collision frequency $\nu_0$ is for a single test particle that interacts with all field particles. A single test particle can be regarded as a Maxwellian distribution of zero temperature $T = 0$. Therefore, the reduced temperature, according to Eq.(8.3), is always expressed as $T' = \mu T_F / m_F$ for single test particle interactions.

A parameter $\eta$ is introduced to indicate hard sphere interactions or inverse-square force law. We have also defined three physical terms: typical collision rate $\omega_0$, typical collision time $\tau_0$, and single test particle collision frequency $\nu_0$. With the three physical terms, and the three basic elements: collision strength, $H$, cutoff collision strength $H_{\text{min}}$, and reduced temperature $T'$ we are ready to summarize and generalize the main results of the previous chapters in a uniform way.

8.2 Collision Rate for a Two-Temperature System

In Chapter 2, we have developed a collision rate theory for equilibrium, namely when the temperatures of the test particles and field particles are the same. In this section, we generalize the theory for a two-temperature system. For both hard sphere interactions and Coulomb interactions, the differential collision rates can be unified as

$$d\omega(y) = \omega_0 e^{-y^\eta(y-1)} dy,$$

(8.9)
where
\[ y = \frac{H}{(k_B T')} \]  \hspace{1cm} (8.10)

is the non-dimensional collision strength.

For the Coulomb interaction, the total collision rate is divergent. However, we can assume the average collision strength equals the typical collision strength \( \overline{H} = k_B T' \) as that for hard sphere interaction Eq.(8.2). With a cutoff on collision strength \( H \geq H_{\text{min}} \), the total average collision rate for a two-temperature system is

\[ \omega = \frac{1}{k_B T'} \{ H \} = \omega_0 \Gamma \left( 1 + \eta, \frac{H_{\text{min}}}{k_B T'} \right), \quad (8.11) \]

where \( \Gamma \) is the incomplete gamma function. In the case of hard sphere interactions, the cutoff collision strength can be taken as zero. Then \( \Gamma \left( 1 + \eta, \frac{H_{\text{min}}}{k_B T'} \right) = \Gamma(2, 0) = 1 \), we have \( \omega = \omega_0 \) as expected.

In the case of Coulomb interaction, we have \( \omega = \omega_0 \Gamma \left( 0, \frac{H_{\text{min}}}{k_B T'} \right) \). In the conventional theory, the total collision rate of a plasma is \( \omega = 2 \lambda \omega_0 \) where \( \lambda \) is the Coulomb logarithm. So, \( \frac{1}{2} \Gamma \left( 0, \frac{H_{\text{min}}}{k_B T'} \right) \) can be regarded as a replacement of the Coulomb logarithm.
8.3 Reaction Rate Coefficients for a Two-Temperature System

From the differential collision rate to a derivation for the reaction rate coefficient is straightforward by using the following formula

\[ \alpha(T') = \frac{q\omega_0}{nn_F} \int_{H \geq H_{\text{min}}} d\omega = \frac{q\omega_0}{nn_F} \Gamma \left( \eta, \frac{H_{\text{min}}}{k_B T'} \right), \]  

(8.12)

where \( q \) is the steric factor. Using the results of the differential collision rate in Section 8.2, we have that the two-temperature reaction rate coefficient for hard sphere interactions is

\[ \alpha_H(T, T_F) = \frac{q\omega_0}{nn_F} \Gamma \left( 1, \frac{H_{\text{min}}}{k_B T_F} \right) = \frac{q\omega_0}{nn_F} e^{\frac{H_{\text{min}}}{k_B T_F}}, \]  

(8.13)

and the reaction rate coefficient for Coulomb interactions is

\[ \alpha_p(T, T_F) = \frac{q\omega_0}{nn_F} \Gamma \left( -1, \frac{H_{\text{min}}}{k_B T_F} \right). \]  

(8.14)

If \( T = T_F \), Eq.(8.13) is reduced to the Arrhenius formula. In chemical reaction theory, the minimum collision strength \( H_{\text{min}} \) in Eq.(8.13) is the activation energy.

If the electron temperature is different from the temperature of ions, Eq.(8.14) can be used to describe the ionization coefficients for electron impact ionization. An alternative approach is to derive Eq.(8.14) from Thomson's classical cross-section for electron impact
ionization [66],

\[ \sigma(E) = 4N \left( \frac{E_H}{E_i} \right) \frac{E_i}{E} \pi a_0^2, \]  

(8.15)

where \( N \) is the number of atomic electrons in the outer shell, \( E_H \) is the ionization potential of the hydrogen atom, \( a_0 \) is the Bohr radius, \( E_i \) is the ionization potential, and \( E \) is the incident energy. Using Thomson’s classical cross-section Eq.(8.15), and a Maxwellian distribution with different temperatures for both electrons and ions, Eq.(8.14) can be recovered by integration. In this case, we find that the minimum collision strength is the ionization potential \( H_{\text{min}} = E_i \). Table 8.1 is a comparison of the standard ionization potential data [67] for carbon atoms and ions with the fit value of the minimum collision strength in Chapter 3.

<table>
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<td>67.25</td>
<td>446.5</td>
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<td>47.89</td>
<td>64.49</td>
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<td>490.0</td>
</tr>
</tbody>
</table>

Table 8.1: Comparison between fitting data, Table 3.1, of minimum collision strength \( H_{\text{min}} \) with standard data of ionization potential in Ref.[67] for carbon atoms and ions

In terms of collision strength, the activation energy [2, 13] in chemical reaction rate theory and the ionization potential in Thomson’s ionization theory [6, 12] are unified as the minimum collision strength, \( H_{\text{min}} \). The collision theory for chemical reaction rate and Thomson’s ionization theory is unified as one theory by the general concept of collision strength.
8.4 Linear Boltzmann Collision Integral and Probability Function

When the field particles are Maxwellian, the scattering angle of the linear Boltzmann collision integral has been integrated in Chapter 4. We summarize the linear Boltzmann collision integral in terms of the probability function $P(v, \Delta v)$. In the astrophysical literature, the linear Boltzmann collision integral is known as the Kolmogorov-Feller equation[34, 68]. Here, we rewrite the collision integral Eq.(4.12) in the Kolmogorov-Feller form. For both hard sphere interactions and Coulomb interactions, we have

$$\left( \frac{\partial f(v)}{\partial t} \right)_{c} = \int f(v') f_{0}^{-1}(v') f_{0}(v) P(v, \Delta v) dv' - f(v) \int P(v, \Delta v) dv', \quad (8.16)$$

where $f_{0}(v)$ is Maxwellian distribution function

$$f_{0}(v) = n \left( \frac{m}{2\pi k_{B}T} \right)^{3} \exp \left( - \frac{mv^{2}}{2k_{B}T} \right), \quad (8.17)$$

and the probability function is

$$P(v, \Delta v) = \frac{\nu_{0}}{(\sqrt{\pi a}v_{th})^{3}} \left( \frac{\Delta v}{av_{th}} \right)^{2\eta-3} \exp \left( - \left( \frac{\Delta v}{av_{th}} + \frac{v}{v_{th}} \cdot \frac{\Delta v}{\Delta v} \right)^{2} \right), \quad (8.18)$$

where $a = 2\mu/m$ is the mass coefficient, and $\Delta v = v' - v$ is velocity change of a collision. The probability function strongly depends on the velocity $\Delta v$ which is a monotonically increasing
function of collision strength $H$:

$$\frac{\Delta v}{av_{th}} = \sqrt{\frac{H}{k_B T'}}.$$  \hspace{1cm} (8.19)

The dependence of the probability function $\mathcal{P}$ on $v$ appears in the argument of the exponential function as $v \cdot \Delta v/(v_{th}\Delta v)$. This term makes the probability of collisions that decrease the velocity of a test particle, $v$, larger than that of increasing the velocity of the test particle. This is exactly where the collisional friction comes from.

8.5 Linearized Boltzmann Collision Operator for Inverse-Square Force Law

For one component particle, the simple linearized Boltzmann collision operator for hard sphere interactions has been obtained by Hilbert[35]. We have obtained the simple linearized Boltzmann collision operator for the inverse-square force law $U = \kappa/r$ as

$$\mathcal{L}h = \int h(\tilde{\xi}_1)\mathcal{K}(\tilde{\xi}, \tilde{\xi}_1; H_{\text{min}})d\tilde{\xi}_1 - h(\tilde{\xi})\nu(v; H_{\text{min}}),$$  \hspace{1cm} (8.20)

where

$$\nu(v; H_{\text{min}}) = \nu_0(v/v_{th})^2\Upsilon_{-1}(1/2, (v/v_{th})^2),$$  \hspace{1cm} (8.21)

$$\mathcal{K}(\tilde{\xi}, \tilde{\xi}_1; H_{\text{min}}) = \mathcal{K}_3(\tilde{\xi}, \tilde{\xi}_1; H_{\text{min}}) + \mathcal{K}_2(\tilde{\xi}, \tilde{\xi}_1; H_{\text{min}}) - \mathcal{K}_1(\tilde{\xi}, \tilde{\xi}_1; H_{\text{min}}),$$  \hspace{1cm} (8.22)
in which

\[ K_3(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{v_0 v_{th} f_0(v_1)}{n} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left( \frac{|\mathbf{v}_1 \times \mathbf{v}|^2}{v_{th}^2 |\mathbf{v}_1 - \mathbf{v}|^2} \right)^j \Gamma \left( j - 1, \frac{H_{\text{min}}}{k_B T'} \right), \] (8.23)

\[ K_2(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{v_0^5 v_{th} f_0(v_1)}{n |\mathbf{v}_1 - \mathbf{v}|^5} \exp \left( \frac{|\mathbf{v}_1 \times \mathbf{v}|^2}{v_{th}^2 |\mathbf{v}_1 - \mathbf{v}|^2} \right) \Theta \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right), \] (8.24)

\[ K_1(\vec{\xi}, \vec{\xi}_1; H_{\text{min}}) = \frac{v_0^3 v_{th} f_0(v_1)}{n |\mathbf{v}_1 - \mathbf{v}|^3} \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right) \Theta \left( \frac{m|\mathbf{v} - \mathbf{v}_1|^2}{4H_{\text{min}}} - 1 \right), \] (8.25)

\[ \Upsilon_{-1}(1/2, x) = \frac{1}{2x^{3/2}} \int_{x_{\text{min}}}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y^2} dy, \] (8.26)

where \( \Theta \) is the Heaviside step function, \( \vec{\xi} \) represent the velocity, and \( \mathbf{v} = \vec{\xi} - \vec{\xi}_0 \) is the velocity relative to the mass velocity \( \vec{\xi}_0 \), equal to \( \vec{\xi} \) if a Maxwellian is taken to have zero mass velocity \( \vec{\xi}_0 = 0 \). In this case, the mass of field particles equals the mass of test particle. Therefore, the relative temperature for one test particle interaction according Eq.(8.3) is

\[ T' = \mu \left( \frac{0}{m} + \frac{T}{m} \right) = \frac{T}{2}. \] (8.27)

According to the relative temperature \( T' \), the cutoff non-dimensional collision strength is still of the form of \( H_{\text{min}}/(k_B T') = 2H_{\text{min}}/(k_B T) \). This is the argument in Eq.(8.23) and the lower limit in Eq.(8.26). The reduced temperature \( T' \), collision strength \( H \), and the minimum collision strength \( H_{\text{min}} \) are the most important elements in writing the linearized
Boltzmann collision operator.

8.6 Arbitrarily High Order Linear Fokker-Planck Coefficients

When the distribution of field particles is Maxwellian, Fokker-Planck Coefficients have been calculated in Chapter 6. The $N$th order of the linear Fokker-Planck coefficients is defined as

$$\langle \Delta v^N \rangle = \int \Delta v^N \mathcal{P}(v, \Delta v)d\Delta v,$$

where $\mathcal{P}$ is probability function. Then, the non-zero elements of the $N$th order Fokker-Planck coefficients for both hard-sphere interactions $\eta = 1$ and the Coulomb interactions $\eta = -1$ are unified as

$$\langle \Delta v^{N-2(J+K)} \Delta v_{11}^{2J} \Delta v_{12}^{2K} \rangle = \frac{\nu_0 \mathcal{B}(J+(1/2), K+(1/2))}{(-1)^{J+K} \pi (av_{th})^{-N}} \sum_{i=0}^{J+K} (-1)^i C_{J+K}^i \times \sum_{j=\lfloor(N+1)/2\rfloor}^{N-i} C_{\lfloor(N+1)/2\rfloor-i}^{j-\lfloor(N+1)/2\rfloor} \frac{\Gamma((1/2) + \lfloor(N + 1)/2\rfloor - i)}{\Gamma((1/2) - \lfloor(N/2\rfloor + j)} \frac{\Upsilon_{j+\eta}((1/2) + j - i, u^2)}{(-u)^{N+2\eta}},$$

where $\mathcal{B}$ is the beta function[60], $\Gamma$ is the gamma function, the operator $\lfloor \cdot \rfloor$ represents the greatest integer less than or equal to the value inside, $C$ is the binomial factor, $u = v/v_{th}$ is the non-dimensional speed of the test particle, and the special function $\Upsilon_j(\alpha, x)$ is defined as

$$\Upsilon_j(\alpha, x) = \frac{x^j e^{-x}}{j} \frac{d^j}{dx^j} \left(x^j e^{-x} \gamma^*(\alpha, x)\right),$$
in which the analytical incomplete gamma function \[62\] is

\[
\gamma^*(\alpha, x) = \frac{1}{x^\alpha \Gamma(\alpha)} \int_0^x \exp(-t)t^{\alpha-1}dt.
\] (8.31)

\(\Upsilon_{j+\eta}\) can not be calculated from Eq.(8.30) when \(j + \eta \leq 0\). For the case of \(j + \eta \leq 0\), we define

\[
\Upsilon_{-1}(1/2, x) = \frac{1}{2x^{3/2}} \int_{\frac{u_{\min}}{k_B T}}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y^2}dy,
\] (8.32)

\[
\Upsilon_{0}(1/2, x) = \frac{1}{2x^{1/2}} \int_{\frac{u_{\min}}{k_B T}}^{\infty} \frac{\text{erf}(\sqrt{y} + \sqrt{x}) - \text{erf}(\sqrt{y} - \sqrt{x})}{y}dy,
\] (8.33)

\[
\Upsilon_{0}(3/2, x) = \frac{1}{2x^{3/2}} \int_{\frac{u_{\min}}{k_B T}}^{\infty} \frac{\text{eef}(\sqrt{y} + \sqrt{x}, y) - \text{eef}(\sqrt{y} - \sqrt{x}, y)}{y}dy,
\] (8.34)

and the function eef is a combination of error and exponent functions

\[
\text{eef}(x, y) = \text{erf}(x) + \frac{\exp(-x^2)}{\sqrt{\pi y}}.
\] (8.35)

No approximation has been introduced to arrived at Eqs.(8.29-8.35). It is interesting that Eqs.(8.29-8.35) also include collision frequency as the zeroth order Fokker-Planck coefficient,

\[
\nu(v) = \langle \nu^0 \rangle = \langle 1 \rangle = \nu_0 u^{-2\eta} \Upsilon_{\eta}(1/2, u^2),
\] (8.36)
where \( u \) is non-dimensional speed of the test particle \( u = v/v_{th} \). If the non-dimensional cutoff collision strength \( \frac{H_{\text{min}}}{k_B T} \ll 1 \), the standard Chandrasekhar expression can be recovered by Taylor expansion of Eqs.(8.33,8.34),

\[
\Upsilon_0(1/2, x) \approx -\ln\left( \frac{H_{\text{min}}}{k_B T'} \right) \frac{\text{erf}(\sqrt{x})}{\sqrt{x}},
\]

\[
\Upsilon_0(3/2, x) \approx -\ln\left( \frac{H_{\text{min}}}{k_B T'} \right) \frac{2G(\sqrt{x})}{\sqrt{x}},
\]

where \( \text{erf} \) is the error function, and \( G \) is Chandrasekhar function. By comparison with the standard expression, we find that the Coulomb logarithm must be replaced by

\[
-\frac{1}{2} \ln \frac{H_{\text{min}}}{k_B T'}.
\]

8.7 Thermal Equilibration Time

If two-components of a plasma have different temperatures, but no relative drift velocity, the equilibration is described by

\[
\frac{dT}{dt} = \frac{T_F - T}{\tau_{eq}}.
\]

The rate of change of the test particle temperature over time \( \frac{dT}{dt} \) can be calculated as

\[
\frac{3}{2} n k_B \frac{dT}{dt} = \int \langle \Delta E \rangle f_0(v, T) dv,
\]
where the average energy change of a test particle $\langle \Delta E \rangle$ can be obtained in terms of friction coefficient $\langle \Delta v_\parallel \rangle$ and diffusion coefficient $\langle \Delta v^2 \rangle$, which are also called the first two orders of the linear Fokker-Planck coefficients. For Coulomb interactions, we have

$$\langle \Delta E \rangle = \frac{1}{2} m \left( \langle \Delta v^2 \rangle + 2 v \langle \Delta v_\parallel \rangle \right) = \frac{\nu_0}{2} m (a v_{th})^2 \left( \frac{1}{2} \Upsilon_0(1/2, u^2) - \frac{2}{a} u^2 \Upsilon_0(3/2, u^2) \right).$$

(8.42)

We can integrate Eq.(8.41) from which we obtain the equilibrium time $\tau_{eq}$ by comparing it with Eq.(8.40)

$$\tau_{eq} = \frac{3 m m_F \tau_0}{8 \mu^2 \Gamma \left( 0, \frac{H_{\min}}{k_B T^*} \right)}.$$  

(8.43)

We would like to point out that Eq.(8.43) is derived without any approximations. If we compare Eq.(8.43) with Spitzer’s results, the Coulomb logarithm should be replaced by

$$\frac{1}{2} \Gamma \left( 0, \frac{H_{\min}}{k_B T^*} \right).$$

(8.44)

In the case of hard sphere interactions, the equilibrium time can also be calculated. For hard sphere interactions, the equilibrium time is

$$\tau_{eq} = \frac{3 m m_F \tau_0}{8 \mu^2 \Gamma \left( 2, \frac{H_{\min}}{k_B T^*} \right)} = \frac{3 m m_F \tau_0}{8 \mu^2}.$$  

(8.45)
Here the minimum collision strength can go to zero, therefore \( \Gamma \left( 2, \frac{H_{\text{min}}}{k_B T} \right) = \Gamma(2, 0) = 1. \)

8.8 The Value of the Minimum Collision Strength

Most of the new results need the value of the cutoff collision strength \( H_{\text{min}} \). However, there is no uniform approach for calculating \( H_{\text{min}} \). The method for determining the cutoff collision strength \( H_{\text{min}} \) depends on each particular problem.

In the case of chemical reaction theory, the value of the cutoff collision strength \( H_{\text{min}} \) is the activation energy, which is a constant and determined by the structure of the reactant molecules. For electron impact ionization, \( H_{\text{min}} \) is the ionization energy, which is also a constant and determined by the ionization potential.

In plasma physics, \( H_{\text{min}} \) is not any discrete value. The value of the cutoff collision strength \( H_{\text{min}} \) is associated with the Debye-Hückel screening radius, which is determined by the temperature and density of the plasma. In Ref[10], a method for determining \( H_{\text{min}} \) by associating it with the Coulomb logarithm is proposed. An alternant approach is to determine \( H_{\text{min}} \) by experiments. By measuring the equilibrium time \( \tau_{\text{eq}} \), the \( H_{\text{min}} \) can be calculated from Eq.(8.43).

8.9 Summary

In this chapter, we have reviewed all the new results of the previous chapters. One purpose was to show how the seemingly different interactions, Coulomb and hard sphere
interactions, can be unified under the concept of collision strength. The average collision strength for a two-temperature hard sphere interaction system is found to be $k_B T'$, where the reduced temperature $T'$ is defined as Eq.(8.3). The reduced temperature has appeared almost everywhere in the unified formula system. With the reduced temperature, the interaction radius for a Coulomb interaction is defined as $\rho_\kappa = \kappa/(2k_B T')$. If we keep in mind the difference in the interaction radius between Coulomb interactions and hard sphere interactions as given in Eq. (8.5), all the coefficients that carry dimensions can be expressed as uniform expressions. The typical collision frequency $\nu_0 = n_F v_{th} \pi \rho_\eta^2$, the typical collision rate $\omega_0 = n n_F v \pi \rho_\eta^2$, and typical collision time $\tau_0 = (n_F v \pi \rho_\eta^2)^{-1}$ are uniformly defined for both Coulomb and hard sphere interactions.

The 8-variable differential collision rate, Eqs. (2.12), gives the details of the collision rate per unit volume. Because the change of any physical term for a single collision can be expressed generally as $\Psi(v, v')$, the differential changing rate of the physical term equals $\Psi(v, v')$ times the 8-variable differential collision rate

$$\Psi d\omega = \Psi(v, v') f_0(v) f_0(v_F) V \sigma \sin \theta d\theta d\varphi d\nu d\nu_F. \quad (8.46)$$

From a mathematical point of view, the main content of this dissertation is a systematic
integration of the following 8-fold integral:

$$\{\Psi\} = \int \Psi(v, v') f_0(v) f_0(v_F) V \sigma \sin \theta d\theta d\varphi dv d\nu_F. \quad (8.47)$$

The integration of Eq.(8.47) can be divided into two parts as

$$\{\Psi\} = \int \langle \Psi \rangle f_0(v) dv, \quad (8.48)$$

$$\langle \Psi \rangle = \int \Psi(v, v') f_0(v_F) V \sigma \sin \theta d\theta d\varphi dv_F, \quad (8.49)$$

where $\langle \Psi \rangle$ represents the average change of a physical term between one test particle of a particular velocity and Maxwellian field particles.

The first step is to integrate the scattering angle using a variable change. The change of variable using velocity change as an independent variable brings about two important benefits. It makes the integral over the scattering angle independent of the form of the physical term $\Psi(v, v')$, and it avoids the divergence difficulty in the integral over scattering angle. When the field particles have a Maxwellian distribution, Eq.(8.49) is reduced to

$$\langle \Psi \rangle = \int \Psi(v, v') P(v, \Delta v) d\Delta v, \quad (8.50)$$

where the probability function is expressed as Eq.(8.18).

The second step is to integrate over the velocity change. When $\Psi(v, v')$ are the moments
of any order of the velocity change of test particles, a complete set of Fokker-Planck coefficients are obtained as Eq.(8.29). All the results are expressed as a combination of a series function $\Upsilon_j(\alpha, x)$ defined as Eq.(8.30). Because any significant physical terms $\Psi(v, v')$ can be expressed in terms of the moments of velocity change, the result of Eq.(8.50) is also a combination of the series function $\Upsilon_j(\alpha, x)$. Therefore, Eq.(8.50) can be integrated with the form

$$\langle \Psi \rangle = \nu_0(\alpha v_{th})^N \sum_{i,j} C(i,j) \frac{\Upsilon_{j+i}(-i+(1/2), u^2)}{(-u)^{N+2\eta}},$$

(8.51)

where the coefficients $C(i,j)$ can be calculated by associating the physical term $\Psi(v, v')$ with the moments of velocity change (Fokker-Planck coefficients as Eq.(8.29)). It is notable that Eq.(8.51) has summarized many physical terms for both Coulomb interactions and hard sphere interactions. These physical terms include arbitrarily high order Fokker-Planck coefficients Eq.(8.29), collision frequencies Eq.(8.36), and energy loss rate of a test particle Eq.(8.42).

The final step is to integrate over the velocities of the test particles. Assuming the velocity distribution function for test particles is Maxwellian with a different temperature, the averages of the physical terms are calculated giving a uniform result

$$\{\Psi\} = \int \langle \Psi \rangle f_0(v) dv = \omega_0 CT(\alpha, H_{\text{min}}/(k_BT')),$$

(8.52)

where $C$ is a coefficient that carries the information of the dimension of $\Psi$ and can be
regarded as the macroscopic correspondence of $\Psi$. For $\Psi = 1$, $\{\Psi\}$ represents collision rate, and we have $C = 1$. For $\Psi = H$, we have $C = k_B T'$, which is the typical collision strength and proportion to the reduced temperature. For $\Psi = \frac{m}{2}(v'^2 - v^2)$ as in the calculation of equilibrium time, we have $C = \frac{4\mu^2}{m_{\text{eff}}} k_B (T_F - T)$, which proportions to the difference of the temperatures between field and test particles.

The order of the incomplete gamma function in Eq. (8.52) can be expressed as $\alpha = (N/2) + \eta$ where $N$ is the index of the velocity in the physical term $\Psi$. For $\alpha = 1$ as in Eq.(8.13), it is the exponential factor in the Arrhenius formula for chemical reaction rate theory. For $\alpha = -1$ as in Eq.(8.14), it is the factor in the ionization rate coefficient based on Thomson’s classical ionization theory. For $\alpha = 0$ as in Eq.(8.43), it is a replacement for the Coulomb logarithm in the exact expression for the equilibrium time. Although the incomplete gamma function $\Gamma(\alpha, H_{\text{min}}/(k_B T'))$ should appear in many of these final results, it has not appeared explicitly. In the calculation of the equilibrium time for hard-sphere interactions, the incomplete gamma function is collapsed to 1 because $\Gamma(\alpha, H_{\text{min}}/(k_B T')) = \Gamma(2, 0) = 1$. For the calculation of chemical reaction rate theory, the incomplete gamma function has transformed into the Arrhenius exponential factor because $\Gamma(1, H_{\text{min}}/(k_B T')) = \exp(-H_{\text{min}}/(k_B T'))$. In the calculation of the equilibrium time for Coulomb interactions, the approximations employed in the traditional approach have deformed the incomplete gamma function into the so called Coulomb logarithm as $\Gamma(0, H_{\text{min}}/(k_B T')) \to 2\lambda$. Another place that the incomplete gamma function should keep its form is in the reaction rate formula.
for Coulomb interactions. However, such a calculation is not commonly used even though the associate ionization cross-section formula was found as early as in 1912 by Thomson[12]. It is interesting to associate all these results with the incomplete gamma function. With this unification, we found that the cutoff collision strength $H_{\min}$ has unified many important physical terms. In the case of chemical reaction rate theory, the cutoff collision strength is called activation energy. In the case of electron impact ionization theory, it is called ionization potential. It also replaces the Coulomb logarithm, which is an important parameter in plasma physics. The present theory has unified all these theories of different disciplines under one uniform theory.

In conclusion, the integral for Eq.(8.47) has been integrated. All the results have been summarized in three formulas, Eqs.(8.18), (8.51), and (8.52). The integration over scattering angle provided the probability function, which has simplified the linear Boltzmann integral. All the results for the calculation of any physical term for one test particle with Maxwellian field particles have been done and are summarized by a series special function $\Upsilon_j(\alpha, x)$. When the test particles are also Maxwellian, an incomplete gamma function $\Gamma(\alpha, H_{\min}/(k_B T'))$ has summarized all the results for a complete 8-fold integral. In short, the main results of the previous chapters can be summarized by three special functions which are a probability function $\mathcal{P}(v, \Delta v)$, $\Upsilon_j(\alpha, u^2)$, and an incomplete gamma function $\Gamma(\alpha, H_{\min}/(k_B T'))$. The concept of minimum collision strength has unified activation energy in chemical reaction rate theory, ionization potential in ionization theory, and the Coulomb logarithm in plasma physics.
physics. We conclude that the universality of collision strength is associated with the concept of the measure of a force. In the last chapter, we will explain that collision strength is a measure of a force. Like Newton’s second law which is a measure of instantaneous force, collision strength is a measure of a force over a period of time.
CHAPTER 9

THE CONCEPT OF COLLISION STRENGTH

In the previous chapters, we have shown many applications of collision strength defined as $H = \frac{\Delta p^2}{(8\mu)}$, where $\Delta p$ is momentum change of a collision and $\mu$ is the reduced mass of two colliders. In this chapter, we would like to show that $H$ is a good definition for the measure of a force in a period of time. It might be a candidate for the “true” measure of a force proposed in [1].

9.1 Introduction

Newton’s second law can be regarded as the definition for the instantaneous measure of a force. Our knowledge about the measure of a force is incomplete because there are many situations in which a cumulative measure of a force is needed. For example, there is no difference for all binary hard sphere interactions based on the instantaneous measure of a force. Each hard sphere interaction will have the same measure—it is infinite at the moment of contact and zero otherwise. A measure of a cumulative effect of a force is required to reflect the differences of such collisions.

The cumulative effect, collision strength, was once called a “true” measure of a force [1]. The natural choice for the “true” measure of a force is to integrate the instantaneous force.
However, there are two ways to define the integral. One is to integrate instantaneous force over time, which is momentum transfer, and the other over space, which is the transfer of kinetic energy. Descartes proposed that the “true” measure of a force is momentum transfer while Leibnitz attacked this view and argued that it is the change of kinetic energy. Because of the two conflicting opinions, the “true” measure of a force, the absolute collision strength, hasn’t been defined clearly.

Although there is no precise definition of collision strength for a general case, terminologies such as “weak collisions” and “strong collisions” are used frequently in different disciplines. In practice, many variables have been introduced as a measure of collision strength. For all head-on binary hard sphere collisions of particles $A$ and $B$, the relative kinetic energy at the moment of contact has been used as a measure of collision strength\[2]. Large kinetic energy represents strong collisions while small kinetic energy represents weak collisions. In chemical reaction-rate theory, this kinetic energy is compared to activation energy to determine if a chemical reaction is going to take place. If the kinetic energy is below the activation energy, there is no reaction. Otherwise, there is some possibility that a chemical reaction occurs. In another case, both scattering angle and impact parameter can be used as measures of collision strength for collisions in Rutherford’s alpha particle scattering experiments. In this case, small impact parameter or large scattering angle collisions are strong collisions while large impact parameter or small scattering angle collisions are weak collisions. However, these collision strength variables are of limited applicability, and the
limit in each case is not clear. Consequently, it is desirable to clarify the ambiguity and to unify the collision strength definitions.

It is the purpose of this chapter to provide a definition of collision strength by analyzing the basic properties of a force. In Section 9.2, we analyze the conventional definition of a collision. Then we broaden the definition of collision as the simplest form of a force in general. Therefore, the concept of collision strength is the same as the measure of a force in a period of time. In Section 9.3, we will show how to appreciate Newton’s three laws of motion as three laws of force. When Newton’s second law is appreciated as a definition of an instantaneous measure of a force, it has four merits. It is consistent with the third law, is independent of inertial reference frame, is measurable by spring, considers the mass effect. With the four merits, we can compare the existing definitions for the measure of a force over a period of time. By comparison, we provide a definition that has all four merits in 9.4. Finally in Section 9.5, we show some direct applications of the concept of the “true” measure of a force.

9.2 General Existing One-Force System Is a Binary Collisions

What is a collision? According to [1], a collision is defined “a relatively large force acts on each colliding particle for a relatively short time.” How short a time period can be called short? How large a force can be called large? As it is impossible to set any critical value for a time period that is “short” and for a force that is “large,” we had better broaden the
definition of collision as *a force acts on each colliding particle for a finite interval* in the dissertation.

The general one-force system is a binary collision. If we isolate a system that has only one particle, it is impossible for the particle to experience a force. For a one-force system, there are at least two particles. The simplest one-force system is a binary interaction. If we assume a general force exists only in a finite interval, the force system is nothing but a binary collision. Here, the word “general” has excluded a special case of a one-force system where the interaction is forever. For example, inside a hydrogen atom, the binary interaction between an electron and a proton is forever with no finite interval. We will not consider such binary interaction that have no finite interval.

A force in a general one-force system has two kinds of effects that need two kinds of measures. The two kinds of effects are an instantaneous effect and an accumulative effect. The instantaneous effect of a force at some moment is accelerations of the particles. The measure of a force for the instantaneous effect is defined as Newton’s second law of motion. The cumulative effect of a force is velocity change or momentum transfer. However, there is no widely accepted definition of the measure of a force for the cumulative effect. Historically, the measure of a force for a cumulative effect is called the “true” measure of a force. Because a general one-force system is also a binary collision, the concept of collision strength is the same as the “true” measure of a force.
9.3 Measure of a Force for an Instantaneous Effect

Newton’s three laws of motion are also laws of force. The first law describes a phenomenon if there is no force. The third law is not directly a law of motion, rather a law of force. It is about force exerted and experienced by particles regardless of how they might be moving. The second law can be regarded as a definition of a measure of a force for an instantaneous effect.

Newton’s second law as a definition of a measure of a force for an instantaneous effect has many merits. We think the most important four merits are: consistent with the third law, measurable by a spring, independence of inertial reference system, and mass effect consideration. If someone has selected acceleration, for example, directly as a measure of a force for an instantaneous effect, then, it will violate the third law if the two colliding particles have different masses. At any moment, the acceleration of the larger mass is smaller than that of the smaller mass. However, if a force is defined as mass times acceleration as it does in Newton’s second law, the definition is always consistent with the third law as $m_1a_1 = -m_2a_2$ for any two colliding particles. In physics, a spring is special for the concept of a force. If two particles have a collision, at any moment we can image the force between the two particles are sourced from a spring. At any moment, $m_1|a_1| = k\Delta x$, where $k$ is spring constant and $\Delta x$ is the change of the spring from its natural length. The force of the two colliding particles has been indicated by $\Delta x$ of the spring. The third merit states that measures of a force from any inertial reference systems are the same. Based on Galileo’s transformation,
the statement is alright. However, it is not so right in relativistic transformations. That is why a better definition of a measure force for an instantaneous effect should be \( F = \frac{dp}{dt} \) when relativity is concerned, where \( p \) is momentum. The last merit of Newton’s second law is mass effect consideration. Newton’s second law always gives a reasonable value for an instantaneous measure of a force regardless of how big and small the mass of the particles might be. These merits may seem natural for an instantaneous measure of a force. For a cumulative measure of a force, however, not all the definitions have all four merits.

9.4 Definition of a Measure of a Force over a Period of Time

Collision strength must be momentum transfer in order to be consistent with Newton’s third law—the law of action and reaction. If we integrate the instantaneous force over space, we find that the cumulative force of particle one from particle two, \( \int F_{12} dx_1 \), is different from the cumulative force of particle two from particle one, \( \int F_{21} dx_2 \). According to the third law, \( F_{12} = -F_{21} \), but \( dx_2 \neq dx_1 \) in general. If we defined the collision strength as the integration of the force over time, then \( \int F_{12} dt = -\int F_{21} dt \), which reflects the third law. Considering the third law, the cumulative effect of a force in a period of time must be momentum transfer \( \Delta p = | \int F_{12} dt | \).

Momentum transfer must be modified to represent collision strength considering a mass effect. Assume there is a head-on elastic collision between two particles of equal mass \( m \). Before the collision, the two particles move with the same speed \( v \) but in opposite directions.
After the collision, the two particles exchange their velocity and move away. The magnitude of the momentum transfer of the collision is $\Delta p = 2mv$. To have a feeling about the force, we can image that there is a spring between the interfaces of the two particles. Before the collision, the spring is free. At the first period of the collision, the spring is compressed and the speed of the two particles decreases. After the spring is compressed to some point, the spring expands to push the two particles away. Regardless of the value of the spring constant, the maximum potential energy of the spring is $E = mv^2$. The momentum transfer $\Delta p$ and the maximum potential energy of the spring $E$ is related as

$$ \Delta p = 2\sqrt{mE}. \quad (9.1) $$

If we keep $E$ a constant and finite, and we change the collision by increasing the mass of each particle to be infinitely large, then, the momentum transfer, according to Eq.(9.1) can be infinitely large. It is unacceptable that such a collision, which causes only a small stress on a spring, is of infinity large collision strength. To get rid of the mass effect, momentum transfer must be modified to agree with our experience.

With a unit of energy, collision strength $H$ is defined as a modification of momentum transfer

$$ H = \frac{(\Delta p)^2}{8\mu}, \quad (9.2) $$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass. This definition of a measure of a force for
a cumulative effect has all four merits. Because the collision strength is based on momentum transfer, it is always consistent with Newton’s third law. Momentum transfer $\Delta p$ is the same from any inertial reference system. By the modification, the collision strength has a unit of energy. The value of the energy is the same as the maximum potential energy of the spring if we assume the force is sourced from the spring. Regardless of the magnitude of the mass of the colliding particles, Eq.(9.2) always provides a value consistent with our experience. Here, Descartes’ opinion and Leibnitz’s opinion about the “true” measure of a force have been reconciled.

9.5 Simple Direct Applications of The “True” Measure of Force

It is interesting to find the limits of other definitions of collision strength by comparing them with the definition of collision strength given by Eq.(9.2). Momentum transfer, as in Descartes’ definition, is a valid collision strength for all the collision events with the same reduced mass. It can be applied for collision theory to derive Arrhenius’ chemical reaction rate formula[6]. In the traditional chemical reaction theory, the definition of collision strength is kinetic energy. However, the kinetic energy has a direction which is in the direction of collision[2]. Only in the special inertial reference system that one of the particles has a zero velocity before a collision, kinetic energy change—as Leibnitz’s definition, is a valid collision strength. In the classical electron impact ionization theory, Thomson[12] has employed exactly the same definition for collision strength. When scattering angle
and impact parameter are used as collision strength variables, it requires that both the reduced mass and the relative speed of the collision are the same. However, constant relative speed is a condition that cannot hold for most of the problems in plasma physics. That is why scattering angle is not a collision strength variable in plasma physics. When we used $H$ as a collision strength variable, we have obtained useful results, such as the differential collision rate in Chapter 2, reaction rate for plasmas in Chapter 3, simplified linear Boltzmann collision integral in Chapter 4, simplified linearized Boltzmann collision operator in Chapter 5, arbitrarily high order linear Fokker-Planck coefficients in Chapter 6, and equilibrium time in Chapter 7.

A collision strength variable is an important physics concept as a measure of a force. This concept can be used to relate other problems. In 1994, Comet Shoemaker-Levy 9 had a collision with Jupiter. The collision was described with the force of tens of millions of megatons of TNT\[69\]. Here, the kinetic energy of the comet before it hit Jupiter is used to describe the collision strength. Though right, this definition of collision strength is not precise in physics. The collision is after all a binary collision. Instead of thinking of the collision as the comet hit Jupiter, we can regard the collision as Jupiter hit the comet. Why can’t we use the kinetic energy of Jupiter to describe collision strength for the same collision? If we did, the collision strength would be much larger and simply wrong. If the collision strength is regarded as momentum transfer with a unit of energy as in Eq.(9.2), we have a better definition of collision strength with no ambiguous physics.
The use of a collision strength variable as an independent variable can bring mathematical simplification. Since the second edition of the book *Classical Electrodynamics*, Jackson stresses momentum transfer instead of impact parameter with “more satisfactory results”[70]. Now, we can explain why momentum transfer brings more satisfactory results than impact parameter. The physics reason is that momentum transfer is a collision strength variable while the impact parameter is not.

The reduced temperature $T'$ Eq.(8.3), which is associated with the average of collision strength $\langle H \rangle = k_B T'$, and the threshold value of collision strength $H_{\text{min}}$ can summarize many important physical terms. The activation energy in chemical reaction theory[6, 2], the ionization energy in Thomson’s ionization theory[12], and a replacement for the Coulomb logarithm (Chapter 8) are the threshold values of collision strength. We expect more applications in the future.
CHAPTER 10

APPENDICES

MATHEMATICAL FORMULAE
APPENDIX A
TRANSORMATION RELATIONS FOR THE VARIABLE CHANGE

A.1 The Relative Velocity

Let a test particle have mass $m$ and velocity $v$ and $v'$ before and after an elastic collision with a scatter of mass $m_F$ and velocity $v_F$, $v'_F$ before and after the collision. Then with $V = v - v_F$ and $V' = v' - v'_F$ denoting the relative velocities, $\Delta V = V' - V$ and $\Delta v = v' - v$ denoting the velocity change of the relative velocity and test particle velocity, $\varphi$ is the azimuthal angle for $V'$ around $V$, and $\phi$ is the azimuthal angle for $V'$ around $\Delta v$, the relations of the variable transformation from $(v_F, \varphi)$ to $(\Delta v, \phi)$ are readily derived as follows.

With the definition of the reduced mass $\mu = mm_F/(m + m_F)$, it is shown from the conservation of momentum that

$$m\Delta v = \mu \Delta V.$$  \hspace{1cm} (A.1)

The angle between $V$ and $V'$ is the scattering angle $\theta$. The relative speed $V$ is unchanged by elastic collision, then

$$\Delta V = |V' - V| = 2V \sin(\theta/2).$$  \hspace{1cm} (A.2)
Substituting Eq.(A.2) into Eq.(A.1), we have

\[ V = \frac{\Delta v}{a} \csc(\theta/2), \]  
(A.3)

where \( a = 2\mu/m \). Eq.(A.3) is the transformation relation for relative speed.

A.2 The Speed of Field Particles

Let \((e_x, e_y, e_z)\) be an orthogonal triplet of unit vectors in a fixed Cartesian reference system. We chose \( \varphi = \phi = 0 \) when \( V' \) is on the plane established by the vectors \( e_z \) and \( V \), and let \((e_1, e_2, e_3)\) be a second orthogonal triplet with \( e_1 \) aligned along the velocity change \( \Delta v \) of the test particle. The two systems are related by

\[ e_1 = \frac{\Delta v}{\Delta v'}, \]  
(A.4)

\[ e_2 = \frac{e_z \times \Delta v}{\Delta v_{\perp}}, \]  
(A.5)

\[ e_3 = e_1 \times e_2, \]  
(A.6)

where \( \Delta v_{\perp} = \sqrt{(v'_{x} - v_{x})^2 + (v'_{y} - v_{y})^2} \).

As shown in Fig. (A.1), we have

\[ V = V (-e_1 \sin(\theta/2) + (e_2 \sin \phi + e_3 \cos \phi) \cos(\theta/2)) . \]  
(A.7)

Substituting Eq.(A.3) and Eqs.(A.4-A.6) into Eq.(A.7) gives

\[ V = \frac{a^{-1} \Delta v}{\tan(\theta/2)} \left( \left( \frac{\Delta v_x}{\Delta v_{\perp}} e_y - \frac{\Delta v_y}{\Delta v_{\perp}} e_x \right) \sin \phi + \left( \frac{\Delta v_x}{\Delta v_{\perp}} e_y - \frac{\Delta v_y}{\Delta v_{\perp}} e_z \right) \cos \phi \right) - \frac{\Delta v}{a} . \]  
(A.8)
Figure A.1: Vector relationship for the variable change from \((v_F, \varphi)\) to \((\Delta v, \phi)\).
From the relation $V = v - v_F$, we obtain

$$v_F^2 = v^2 + V^2 - 2v \cdot V. \quad (A.9)$$

Substituting Eq.(A.3) and Eq.(A.8) into Eq.(A.9) gives

$$v_F^2 = v^2 + 2v \cdot \Delta v / a + \left( \frac{\Delta v}{a} \right)^2 \csc^2 \left( \frac{\theta}{2} \right) - 2 \left| v \times \frac{\Delta v}{a} \right| \sin(\phi + \phi_0) \frac{\tan(\theta/2)}{\tan(\theta/2)}, \quad (A.10)$$

where

$$\phi_0 = \arctan \left( \frac{\Delta v \cdot \Delta v}{v \times \Delta v} \right). \quad (A.11)$$

### A.3 Jacobian of the Transformation

The relation between the element $dvFd\phi$ and $d\Delta v d\phi$ is given by $dvFd\phi = |J| d\Delta v d\phi$, the Jacobian of the transformation $J$ is defined by

$$J = \frac{\partial(v_F, \phi)}{\partial(\Delta v, \phi)} = \begin{vmatrix} \frac{\partial vF_x}{\partial \Delta v_x} & \frac{\partial vF_x}{\partial \Delta v_y} & \frac{\partial vF_x}{\partial \Delta v_z} & \frac{\partial vF_x}{\partial \phi} \\ \frac{\partial vF_y}{\partial \Delta v_x} & \frac{\partial vF_y}{\partial \Delta v_y} & \frac{\partial vF_y}{\partial \Delta v_z} & \frac{\partial vF_y}{\partial \phi} \\ \frac{\partial vF_z}{\partial \Delta v_x} & \frac{\partial vF_z}{\partial \Delta v_y} & \frac{\partial vF_z}{\partial \Delta v_z} & \frac{\partial vF_z}{\partial \phi} \\ \frac{\partial \phi}{\partial \Delta v_x} & \frac{\partial \phi}{\partial \Delta v_y} & \frac{\partial \phi}{\partial \Delta v_z} & \frac{\partial \phi}{\partial \phi} \end{vmatrix}. \quad (A.12)$$
Since $v_F = v - V$, using Eq.(A.8) the velocity of the field particle can be evaluated as

$$v_{Fx} = v_x + \frac{\Delta v_x}{a} + \left( \frac{\Delta v_x \Delta v_z}{a \Delta v_\perp} \cos \phi + \frac{\Delta v_y \Delta v_z}{a \Delta v_\perp} \sin \phi \right) \cot \frac{\theta}{2}, \quad (A.13)$$

$$v_{Fy} = v_y + \frac{\Delta v_y}{a} + \left( \frac{\Delta v_y \Delta v_z}{a \Delta v_\perp} \cos \phi - \frac{\Delta v_x \Delta v_z}{a \Delta v_\perp} \sin \phi \right) \cot \frac{\theta}{2}, \quad (A.14)$$

$$v_{Fz} = v_z + \frac{\Delta v_z}{a} - \frac{\Delta v_\perp}{a} \cos \phi \cot \frac{\theta}{2}. \quad (A.15)$$

In order to calculate $\varphi$, we defined two unit vectors $e_\zeta$ and $e_\eta$ as

$$e_\zeta = (e_z \times V) / |e_z \times V|, \quad (A.16)$$

$$e_\eta = (e_1 \times V) / |e_1 \times V|. \quad (A.17)$$

It is defined that the azimuthal angle $\varphi$ is the angle between the plane $\zeta$ established by the vectors $(e_z, V)$ and the plane $\eta$ established by the vectors $(e_1, V)$ as shown in Fig (A.1), therefore

$$\cos \varphi = (e_\zeta \cdot e_\eta). \quad (A.18)$$

It follows from Eqs.(A.16-A.18) and Eqs.(A.4-A.6) that $\varphi$ is calculated as

$$\varphi = \arccos \frac{\Delta v_z + \Delta v_\perp \cos \phi \tan(\theta/2)}{\sqrt{\Delta v^2 \sin^2 \phi + (\Delta v_\perp \tan(\theta/2) + \Delta v_z \cos \phi)^2}}. \quad (A.19)$$

Substituting Eqs.(A.13-A.15) and Eq.(A.19) into Eq.(A.12), the Jacobian can be calculated as

$$J = (a \sin(\theta/2))^{-3}. \quad (A.20)$$
Equations (A.3), (A.10), and (A.20) are the complete set of transformations from independent variables \((v_F, \varphi)\) to \((\Delta v, \phi)\).

It is notable that the calculation of Eq.(A.20) is very complicated. However, it can be evaluated easily by Stephen Wolfram’s software Mathematica[71]. The following is the Mathematica code for the calculation of the Jacobian.

(* THE CALCULATION OF TRANSFORM JACOBIAN *)

```mathematica
p[x_, y_] = Sqrt[x^2 + y^2]; (* p is velocity change in the perpendicular direction *)

v[x_, y_, z_] = Sqrt[x^2 + y^2 + z^2]; (* v is magnitude of the velocity change *)

(* (vFx, vFy, vFz) is the velocity of field particle *)
(* (vx, vy, vz) is the the velocity of test particle *)
(* (x, y, z) is the velocity change *)
(* w is the azimuthal angle around velocity change *)
(* pHi is the azimuthal angle around relative velocity *)
(* a is mass coefficient, b is scattering angle *)

vFx[x_, y_, z_, w_] = vx + x/a + ((x z/p[x, y]) Cos[w] + (y v[x, y, z]/p[x, y]) Sin[w])/(a Tan[b/2]);

vFy[x_, y_, z_, w_] = vy + y/a + ((y z/p[x, y]) Cos[w] - (x v[x, y, z]/p[x, y]) Sin[w])/(a Tan[b/2]);

vFz[x_, y_, z_, w_] = vz + z/a - p[x, y] Cos[w]/(a Tan[b/2]);

pHi[x_, y_, z_, w_] = ArcCos[(z + p[x, y] Cos[w] Tan[b/2])/
Sqrt[(v[x, y, z])^2 (Sin[w])^2 + (p[x, y] Tan[b/2] + z Cos[w])^2]]; (* Calculate the elements of the Jacobian *)

e11 = D[vFx[x, y, z, w], x]; e12 = D[vFx[x, y, z, w], y];
e13 = D[vFx[x, y, z, w], z]; e14 = D[vFx[x, y, z, w], w];
```
\[
\begin{align*}
e_{21} &= \frac{\partial v_{y}(x,y,z,w)}{\partial x}, \quad e_{22} = \frac{\partial v_{y}(x,y,z,w)}{\partial y}, \\
e_{23} &= \frac{\partial v_{y}(x,y,z,w)}{\partial z}, \quad e_{24} = \frac{\partial v_{y}(x,y,z,w)}{\partial w}, \\
e_{31} &= \frac{\partial v_{z}(x,y,z,w)}{\partial x}, \quad e_{32} = \frac{\partial v_{z}(x,y,z,w)}{\partial y}, \\
e_{33} &= \frac{\partial v_{z}(x,y,z,w)}{\partial z}, \quad e_{34} = \frac{\partial v_{z}(x,y,z,w)}{\partial w}, \\
e_{41} &= \frac{\partial p_{H_{i}}(x,y,z,w)}{\partial x}, \quad e_{42} = \frac{\partial p_{H_{i}}(x,y,z,w)}{\partial y}, \\
e_{43} &= \frac{\partial p_{H_{i}}(x,y,z,w)}{\partial z}, \quad e_{44} = \frac{\partial p_{H_{i}}(x,y,z,w)}{\partial w};
\end{align*}
\]

(* Calculate the Jacobian determinant *)

\[
m = \{e_{11}, e_{12}, e_{13}, e_{14}\}, \{e_{21}, e_{22}, e_{23}, e_{24}\}, \\
{e_{31}, e_{32}, e_{33}, e_{34}\}, \{e_{41}, e_{42}, e_{43}, e_{44}\};
\]

\[
j = \text{PowerExpand}[\left(\text{Simplify}\left[\text{Simplify}\left[\text{Apart}[\text{Det}[m]^2]\right]\right]\right)]^{1/2}
\]

(* The treatment of square the determinant followed *)

(* by taking the square root is just for simplification *)

\[
\text{Out}[23] = \frac{\csc\left(\frac{b}{2}\right)}{a^3}
\]

The Mathematica program proves that the transformation Jacobian is indeed Eq.(A.20).
APPENDIX B

INTEGRAL FORMULAE

B.1 Integral Over Azimuthal Angles

Integral Formula for Azimuthal Angle Around $\Delta \mathbf{v}$

$$
\int_0^{2\pi} \exp(\alpha \sin(\phi + \phi_0)) d\phi = 2\pi I_0(\alpha),
$$

(B.1)

where $I_0$ is the zeroth order of the first kind modified Bessel function[17]. Because the function of the integrand is a period function, we have

$$
\int_0^{2\pi} \exp(\alpha \sin(\phi + \phi_0)) d\phi = \int_0^{2\pi} \exp(\alpha \sin \phi) d\phi.
$$

(B.2)

Using Mathematica to integrate the right side of Eq.(B.2), we get

$$
\int_0^{2\pi} \exp(\alpha \sin \phi) d\phi = 2\pi I_0(\alpha).
$$

(B.3)

Eq.(B.1) is proved from Eqs.(B.2, B.3).
B.2 Integral Over Scattering Angle

To integrate the scattering, we encounter the integral

\[ \int_{0}^{\infty} \exp(\alpha x) I_0(2\beta \sqrt{x}) dx = \frac{\exp(\beta^2/\alpha)}{\alpha}. \]  

(B.4)

Eq. (B.4) can be directly obtained from Mathematica.

B.3 Integral Over Azimuthal Angle Around the Velocity of Test Particle

The integral about the azimuthal angle of velocity change around the velocity of the test particle is

\[ \int_{0}^{2\pi} \sin^m \psi \cos^n \psi d\psi = \begin{cases} 2B(J + (1/2), K + (1/2)), & \text{when } m = 2J, \ n = 2K \\ 0, & \text{otherwise} \end{cases} \]  

(B.5)

where \( B \) is the beta function, and \( m, n, J, K \) are all integers.

Consider (4.3.127) in Ref. [72] and (6.2.1) in Ref.[60] , there is no difficult to verify Eq.(B.5).
B.4 Integral Over the Angle $\chi$ Formed Between Velocity Change and Test Particle Velocity

There are three integrals about the angle formed by velocity change and velocity of the test particle

\[
\frac{2}{\sqrt{\pi}} \int_{0}^{\pi} e^{-(u_\delta+u \cos \chi)^2} \sin \chi d\chi = \frac{\text{erf}(u_\delta + u) - \text{erf}(u_\delta - u)}{u}, \tag{B.6}
\]

\[
\frac{2}{\sqrt{\pi}} \int_{0}^{\pi} e^{-(u_\delta+u \cos \chi)^2} \cos \chi \sin \chi d\chi = -\frac{\text{eef}(u_\delta + u, u_\delta^2) - \text{eef}(u_\delta - u, u_\delta^2)}{u^2u_\delta^{-1}}, \tag{B.7}
\]

\[
\frac{2}{\sqrt{\pi}} \int_{0}^{\pi} e^{-(u_\delta+u \cos \chi)^2} \cos^2 \chi \sin \chi d\chi = \frac{\text{eef}(u_\delta + u, u_\delta^2) - \text{eef}(u_\delta - u, u_\delta^2)}{2u^3} + \frac{\mathcal{F}(u, u_\delta)}{u^3}, \tag{B.8}
\]

where \(\text{erf}\) is error function \[18\], and \(\text{eef}\) is a function of combination of error and exponent functions

\[
\text{eef}(x, y) = \text{erf}(x) + \frac{\exp(-x^2)}{\sqrt{\pi}y}, \tag{B.9}
\]

and \(\mathcal{F}(u, u_\delta)\) is defined as

\[
\mathcal{F}(u, u_\delta) = u_\delta^2(\text{erf}(u_\delta+u) - \text{erf}(u_\delta-u)) + \frac{-(2u_\delta)^{-1} + u_\delta - u}{\sqrt{\pi} \exp((u_\delta+u)^2)} + \frac{(2u_\delta)^{-1} - u_\delta - u}{\sqrt{\pi} \exp((u_\delta-u)^2)}. \tag{B.10}
\]

Equations (B.6-B.8) can be calculated directly by Mathematica. However, the results are regrouped in terms of the new defined functions \(\text{eef}(x, y)\) and \(\mathcal{F}(u, u_\delta)\).
B.5 Integral Over the Velocity Change

The integral about velocity change is for the non-dominate term of the parallel diffusion terms

\[
\int_0^\infty \frac{F(u, u_\delta)}{u^3 u_\delta} du_\delta = \Upsilon_1(5/2, u^2), \tag{B.11}
\]

where the function \( \Upsilon_1(5/2, x) \) is defined as

\[
\Upsilon_1(5/2, x) = xe^{-x} \frac{d}{dx} \left( xe^{x^*}(5/2, x) \right), \tag{B.12}
\]

in which \( x^*(5/2, x) \) is an analytic incomplete gamma function[62].

Substituting Eq.(B.10) into Eq.(B.11), using partial integral formula, we can get ride of the error function in Eq.(B.11) to get

\[
\int_0^\infty \frac{F(u, u_\delta)}{u^3 u_\delta} du_\delta = \int_0^\infty \left( \frac{-u_\delta^3 - (2u_\delta)^{-1} + u_\delta - u}{\sqrt{\pi} \exp((u_\delta + u)^2)} + \frac{u_\delta^3 + (2u_\delta)^{-1} - u_\delta - u}{\sqrt{\pi} \exp((u_\delta - u)^2)} \right) \frac{du_\delta}{u_\delta u^3}. \tag{B.13}
\]

Four terms of the integrant in the right side of Eq. (B.13) are infinite. However, the combination of the terms is finite. Using Mathematica, we have the following results

\[
\int_0^\infty \frac{F(u, u_\delta)}{u^3 u_\delta} du_\delta = \frac{1}{2u^3} \left( \frac{6u \exp(-u^2)}{\sqrt{\pi}} + (2u^2 - 3) \text{erf}(u) \right). \tag{B.14}
\]
It can be proved that

\[ \mathcal{Y}_1(5/2, u^2) = \frac{1}{2u^3} \left( \frac{6u \exp(-u^2)}{\sqrt{\pi}} + (2u^2 - 3)\text{erf}(u) \right). \] (B.15)

Then Eq. (B.11) is proved from Eqs. (B.14-B.15).

B.6 Integral Over the Speed of Test Particle

Integral about the speed of test particles needs the following four integral formulae

\[ \int_0^\infty e^{-\zeta u^2 - (u+u_\delta)^2} \, u \, du = \frac{e^{-u_\delta^2}}{2(1+\zeta)} + \frac{\sqrt{\pi} u_\delta e^{-\frac{\zeta u_\delta^2}{1+\zeta}}}{2(1+\zeta)^{3/2}} \left( -1 + \text{erf} \left( \frac{u_\delta}{\sqrt{1+\zeta}} \right) \right), \] (B.16)

\[ \int_0^\infty e^{-\zeta u^2 - (u-u_\delta)^2} \, u \, du = \frac{e^{-u_\delta^2}}{2(1+\zeta)} + \frac{\sqrt{\pi} u_\delta e^{-\frac{\zeta u_\delta^2}{1+\zeta}}}{2(1+\zeta)^{3/2}} \left( 1 + \text{erf} \left( \frac{u_\delta}{\sqrt{1+\zeta}} \right) \right), \] (B.17)

\[ \int_0^\infty \text{erf} (u + u_\delta) e^{-\zeta u^2} \, u \, du = \frac{\text{erf} (-u_\delta)}{2\zeta} + \frac{e^{-\frac{\zeta u_\delta^2}{1+\zeta}}}{2\sqrt{1+\zeta}} \left( 1 - \text{erf} \left( \frac{u_\delta}{\sqrt{1+\zeta}} \right) \right), \] (B.18)

\[ \int_0^\infty \text{erf} (u - u_\delta) e^{-\zeta u^2} \, u \, du = -\frac{\text{erf} (-u_\delta)}{2\zeta} + \frac{e^{-\frac{\zeta u_\delta^2}{1+\zeta}}}{2\sqrt{1+\zeta}} \left( 1 + \text{erf} \left( \frac{u_\delta}{\sqrt{1+\zeta}} \right) \right). \] (B.19)

The integrals of Eqs. (B.16,B.17) could be directly obtained from Mathematica. To calculate Integrate Eqs. (B.18,B.19), we first do partial integrate the right side of Eqs. (B.18,B.19).
Consider \( \text{erf}(-u_\delta) = -\text{erf}(u_\delta) \), we have

\[
\int_0^\infty \text{erf} \left( u + u_\delta \right) e^{-\zeta u^2} du = \frac{\text{erf}(u_\delta)}{2\zeta} + \frac{1}{\sqrt{\pi\zeta}} \int_0^\infty e^{-\zeta u^2 - (u + u_\delta)^2} du, \tag{B.20}
\]

\[
\int_0^\infty \text{erf} \left( u - u_\delta \right) e^{-\zeta u^2} du = -\frac{\text{erf}(u_\delta)}{2\zeta} + \frac{1}{\sqrt{\pi\zeta}} \int_0^\infty e^{-\zeta u^2 - (u - u_\delta)^2} du. \tag{B.21}
\]

Then, using Mathematica to get Eq.(B.18) from Eq.(B.20), and Eq.(B.19) from Eq.(B.21).
APPENDIX C

A TWO-FOLD INTEGRAL FORMULA

C.1 The Two-fold Integral Formula

In the calculation of higher order Fokker-Planck coefficients, the following two-fold integral is encountered

\[ \int_0^\infty du_\delta \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} e^{-\left(u_\delta + u \cos \chi\right)^2} u_\delta^M \cos^{M-2L+1} \chi \sin \chi d\chi \]

where \( \Gamma \) is gamma function, the operator \([ ]\), represents the biggest integer less than or equal to the value inside, integers \( M \geq 0 \), \( L \leq [(M+1)/2] \), and the function \( \Upsilon_j(\alpha, x) \) is defined as

\[ \Upsilon_j(\alpha, x) = \frac{x^j e^{-x}}{j} \frac{d^j}{dx^j} \left( x^j e^x \gamma^*(\alpha, x) \right), \]
in which the analytical incomplete gamma function\cite{62} is

\[ \gamma^*(\alpha, x) = \frac{1}{x^\alpha \Gamma(\alpha)} \int_0^x e^{-t} t^{\alpha - 1} dt. \] (C.3)

The derivation of the two-fold integral formula is complicated. It could be done by steps. In Section C.2, we first prove two useful expansion formulas. In Section C.3, a special one-fold integral associate with the integral over the angle $\chi$, is calculated. The final calculation of the two-fold integral is completed in Section C.4.

C.2 Two Expansion Formulas

In order to derive the integral formula, we need two expansion formulas. The first one is

\[ \frac{\Gamma(K + L + \alpha)}{\Gamma(K + \alpha)} = \sum_{i=0}^{\min(L,K)} C_L^i \frac{\Gamma(L + \alpha)}{\Gamma(i + \alpha)} \frac{K!}{(K - i)!}, \] (C.4)

where $K$ and $L$ are integers, and $\alpha$ is a positive real number.

To prove Eq.(C.4), consider the $L$th derivative of the function $x^{K+L+\alpha-1}$

\[ \left(x^{K+L+\alpha-1}\right)^{(L)} = \frac{\Gamma(K + L + \alpha)}{\Gamma(K + L)} x^{K+\alpha-1}. \] (C.5)
On the other hand, the $L$th derivative of the function $x^{K+L+\alpha-1}$ can be calculated by Leibnitz’s theorem Eq.(3.3.8) in Ref.[73]

$$
\left(x^{K+L+\alpha-1}\right)^{(L)} = \left(x^{L+\alpha-1}x^K\right)^{(L)} = \sum_{i=0}^{\min(K,L)} C_L^i \left(x^{L+\alpha-1}\right)^{(L-i)} \left(x^K\right)^{(i)}
$$

$$
= \sum_{i=0}^{\min(K,L)} C_L^i \frac{\Gamma(L+\alpha)}{\Gamma(i+\alpha)} \frac{K!}{(K-i)!} x^{K+\alpha-1}. \quad (C.6)
$$

Eq.(C.4) is proved by comparing Eq.(C.5) with Eq.(C.6).

The second expansion formula is

$$
\exp\left(x^2y^{-1} - y\right) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1,y)}{n!y^n} J_n(2x)x^n, \quad (C.7)
$$

where $J_n$ is the $n$th order of the first kind Bessel function defined as Eq.(9.1.10) in Ref.[74].

Using the following expansion formula for incomplete gamma function

$$
\Gamma(n+1,y) = n!e^{-y} \sum_{m=0}^{n} \frac{y^m}{m!} \quad (C.8)
$$

to expand the right side of Eq.(C.7), we have

$$
\sum_{n=0}^{\infty} \frac{\Gamma(n+1,y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{n=0}^{\infty} J_n(2x) \left(\frac{x}{y}\right)^n \sum_{m=0}^{n} \frac{y^m}{m!}. \quad (C.9)
$$
Changing the order of the summation of the right side of Eq.(C.9)

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1, y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} J_n(2x) \left( \frac{x}{y} \right)^n \frac{y^m}{m!}.
\]  

(C.10)

Let \( n = j + m \), to shift the summation about the \( n \) of the right side of Eq.(C.10) so that the summation about the new index \( j \) can start from zero

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1, y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} J_{m+j}(2x) \left( \frac{x}{y} \right)^{m+j} \frac{y^m}{m!}.
\]  

(C.11)

Expanding the Bessel function in the right side of Eq.(C.11)

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1, y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m+j}}{k!(k+m+j)!} \left( \frac{x}{y} \right)^{m+j} \frac{y^m}{m!}.
\]  

(C.12)

Then, reorganize Eq.(C.12) as

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1, y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m+j}}{k!(k+m+j)!m!} \left( \frac{x}{y} \right)^{k+m+j} \frac{y^m}{m!}.
\]  

(C.13)

Reorder the summations in Eq.(C.13), let \( N = k + m + j \) and \( L = m + k \). Then the range of the new indexes are \( 0 \leq N \leq \infty \), \( 0 \leq L \leq N \), and \( 0 \leq k \leq L \)

\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+1, y)}{n!y^n} J_n(2x)x^n = e^{-y} \sum_{N=0}^{\infty} \left( \frac{x^2 y^{-1}}{N!} \right)^N \sum_{L=0}^{N} y^L \sum_{k=0}^{L} \frac{(-1)^k}{k!(L-k)!}.
\]  

(C.14)
Change the right side of Eq.(C.14) by times and divides the term $L!$

$$
\sum_{n=0}^{\infty} \frac{\Gamma(n + 1, y)}{n! y^n} J_n(2x) x^n = e^{-y} \sum_{N=0}^{\infty} \frac{(x^2 y^{-1})^N}{N!} \sum_{L=0}^{N} \frac{y^L}{L!} \sum_{k=0}^{L} \frac{L!(-1)^k 1^{L-k}}{k!(L-k)!}.
$$

(C.15)

Consider

$$(1 - 1)^L = \sum_{k=0}^{L} \frac{L!(-1)^k 1^{L-k}}{k!(L-k)!},
$$

(C.16)

and if $L \neq 0$, Eq.(C.16) is zero. Therefore, only the term of $L = 0$ makes contribution to the summation for right side of Eq.(C.15). Then, we have

$$
\sum_{n=0}^{\infty} \frac{\Gamma(n + 1, y)}{n! y^n} J_n(2x) x^n = e^{-y} \sum_{N=0}^{\infty} \frac{(x^2 y^{-1})^N}{N!}.
$$

(C.17)

The summation about $N$ in Eq.(C.17) is exactly an exponential function, that

$$
\sum_{N=0}^{\infty} \frac{(x^2 y^{-1})^N}{N!} = \exp(x^2 y^{-1}).
$$

(C.18)

Eq.(C.7) is proved by substituting Eq.(C.18) into Eq.(C.17).
C.3 Integral Formula Over the Angle

For deriving the two-fold integral formula, we encounter the following one-fold integral formula about \( \chi \)

\[
\int_0^\pi \cos^N \chi \exp(-z \cos \chi) J_n(z \sin \chi) \sin^{n+1} \chi d\chi
\]

\[
= \left(-\frac{z}{2}\right)^{2-N} \sum_{l=\lceil(N-1)/2\rceil}^{N-1} \frac{\sqrt{\pi} \Gamma((3/2)+\lfloor(N-1)/2\rfloor)(z/2)^{n+2l}}{\Gamma((3/2) - \lfloor N/2 \rfloor + l) \Gamma((5/2) + n + l)},
\]

(C.19)

where \( J_n \) is the first kind Bessel function, and \( \Gamma \) is the gamma function.

For convenient, the left side integral of Eq.(C.19) is denoted as \( S(N, n, z) \). First, we consider the case when \( N \) is an even number, namely \( N = 2L \). The magnitude of the integral does not change if the exponential function \( \exp(-z \cos \chi) \) is changed into a hyperbolic function \( 2 \cosh(z \cos \chi) \), meanwhile the upper limit \( \pi \) is changed into \( \pi/2 \). Thus one has

\[
S(2L, n, z) = 2 \int_0^{\pi/2} \cos^{2L} \chi \cosh(-z \cos \chi) J_n(z \sin \chi) \sin^{n+1} \chi d\chi.
\]

(C.20)

By expanding the hyperbolic cosine function \( \cosh(z \cos \chi) \), and using the integral formula (11.4.11) in Ref.[76], the integral of Eq.(C.20) is found to be

\[
S(2L, n, z) = \sum_{i=0}^{\infty} \frac{\Gamma(L + i + (1/2))2^{2i}}{(2i)! (z/2)^{L-i+(1/2)}} J_{n+L+i+(1/2)}(z).
\]

(C.21)
After expanding the Bessel function $J_{n+L+i+(1/2)}(z)$ with (9.1.10) in Ref.[74], we obtain

\[ S(2L, n, z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(L + i + (1/2)) 2^{2i}}{\Gamma(n + i + L + j + (3/2))(2i)!j!} \left( \frac{z}{2} \right)^{n+2(i+j)}. \] (C.22)

Reorder the summation of Eq.(C.22) by the transform $k = i + j$ as

\[ S(2L, n, z) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi} (z/2)^{n+2k}}{\Gamma(n + k + L + (3/2))} \sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + L + (1/2))}{j!(k - j)!\Gamma(k - j + (1/2))}. \] (C.23)

We now consider the second case when $N = 2L + 1$,

\[ S(2L + 1, n, z) = 2 \int_0^{\pi/2} \cos^{2L+1} \chi \sinh(z \cos \chi) J_n(z \sin \chi) \sin^{n+1} \chi d\chi. \] (C.24)

Expanding the hyperbolic sine function $\sinh(z \cos \chi)$ and repeating the procedure from Eq.(C.21) to Eq.(C.23), we get

\[ S(2L + 1, n, z) = -\frac{z}{2} \sum_{k=0}^{\infty} \frac{\sqrt{\pi} (z/2)^{n+2k}}{\Gamma(n + k + L + (5/2))} \sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + L + (3/2))}{j!(k - j)!\Gamma(k - j + (3/2))}. \] (C.25)

Combining Eq.(C.23) and Eq.(C.25) we get the integral

\[ S(N, n, z) = \left( -\frac{z}{2} \right)^{\sigma} \sum_{k=0}^{\infty} \frac{\sqrt{\pi} (z/2)^{n+2k}}{\Gamma(n + k + [N/2] + \sigma + (3/2))}. \]
\[
\sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + \lceil N/2 \rceil + \sigma + (1/2))}{j!(k - j)!\Gamma(k - j + \sigma + (1/2))},
\]

where \(\lceil N/2 \rceil\) is the largest integer equal to or smaller than \(N/2\), and \(\sigma\) is a parity of \(N\), \(\sigma = 1\) when \(N\) is odd, and \(\sigma = 0\) when \(N\) is even.

Applying the expansion relation Eq.(C.4) into the second summation of Eq.(C.26), we have

\[
\sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + \lceil N/2 \rceil + \sigma + (1/2))}{j!(k - j)!\Gamma(k - j + \sigma + (1/2))} = \sum_{j=0}^{k} \frac{(-1)^j}{j!(k - j)!} \sum_{i=0}^{\min(k-j, \lceil N/2 \rceil)} C_i^{\lceil N/2 \rceil} \frac{\Gamma([N/2] + \sigma + (1/2))}{\Gamma(\sigma + i + (1/2))} \frac{(k-j)!}{(k-j-i)!}. \tag{C.27}
\]

Changing the order of the summation in Eq.(C.27), get

\[
\sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + \lceil N/2 \rceil + \sigma + (1/2))}{j!(k - j)!\Gamma(k - j + \sigma + (1/2))} = \sum_{i=0}^{\lceil N/2 \rceil} C_i^{\lceil N/2 \rceil} \frac{\Gamma([N/2] + \sigma + (1/2))}{\Gamma(\sigma + i + (1/2))} \sum_{j=0}^{k-i} \frac{(-1)^j}{(k-j-i)!j!}. \tag{C.28}
\]

The summation about \(j\) in Eq.(C.28) can be calculated as

\[
\sum_{j=0}^{k-i} \frac{(-1)^j}{(k-j-i)!j!} = \frac{1}{(k-i)!} \sum_{j=0}^{k-i} \frac{(-1)^j 1^{k-i-j}(k-i)!}{(k-j-i)!j!} = \frac{(1-1)^{k-i}}{(k-i)!} = \delta_{i,k}, \tag{C.29}
\]
where \( \delta_{i,k} = 1 \) if \( k = i \), and \( \delta_{i,k} = 0 \) if \( k \neq i \). Substituting Eq.(C.29) into Eq.(C.28), get

\[
\sum_{j=0}^{k} \frac{(-1)^j \Gamma(k - j + [N/2] + \sigma + (1/2))}{j!(k - j)!\Gamma(k - j + \sigma + (1/2))} = \sum_{i=0}^{[N/2]} C_{[N/2]}^i \frac{\Gamma([N/2] + \sigma + (1/2))}{\Gamma(\sigma + i + (1/2))} \delta_{i,k}. \tag{C.30}
\]

Substituting Eq.(C.30) into Eq.(C.26), we have

\[
S(N, n, z) = \left(-\frac{z}{2}\right)^{\sigma} \frac{\sqrt{\pi}(z/2)^{n+2k}}{\Gamma(n + k + [N/2] + \sigma + (3/2))} \times \sum_{i=0}^{[N/2]} C_{[N/2]}^i \frac{\Gamma([N/2] + \sigma + (1/2))}{\Gamma(\sigma + i + (1/2))} \delta_{i,k}. \tag{C.31}
\]

With the relation \( \sigma = [(N - 1)/2] + 1 - [N/2] \), we can get rid of \( \sigma \) in Eq.(C.31). Let \( i = l - [(N - 1)/2] \), using the relation \( [(N - 1)/2] + [N/2] = N - 1 \), Eq.(C.31) can be simplified as

\[
S(N, n, z) = \left(-\frac{z}{2}\right)^{2-N} \sum_{l=([(N-1)/2])}^{N-1} C_{[N/2]}^{l-[(N-1)/2]} \frac{\sqrt{\pi}\Gamma((3/2) + [(N - 1)/2])(z/2)^{n+2l}}{\Gamma((3/2) - [N/2] + l)\Gamma((5/2) + n + l)}. \tag{C.32}
\]

Eq.(C.19) is proved from Eq.(C.32).
C.4 The Calculation of the Two-fold Integral

Applying the expansion formula Eq.(C.7) to the exponential function of the integrand in Eq.(C.1)

\[
e^{-\left(u_\delta + u \cos \chi \right)^2} = e^{-u^2 - 2uu_\delta \cos \chi} \exp \left( (uu_\delta \sin \chi)^2 \left(u_\delta^2\right)^{-1} - u_\delta^2 \right)
= e^{-u^2 - 2uu_\delta \cos \chi} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1, u_\delta^2)}{n!u_\delta^{2n}} J_n(2uu_\delta \sin \chi)(uu_\delta \sin \chi)^n.
\] (C.33)

Substituting Eq.(C.33) into the left side of Eq.(C.1), we have

\[
\int_0^{\infty} du_\delta \int_0^{\pi} \frac{2}{\sqrt{\pi}} e^{-\left(u_\delta + u \cos \chi \right)^2} (uu_\delta \cos M - 2L + 1) \sin \chi d\chi = \int_0^{\infty} e^{-2uu_\delta \cos \chi} J_n(2uu_\delta \sin \chi) \sin^{n+1} \chi \cos^{M-2L+1} \chi d\chi.
\] (C.34)

For simplicity, we using \( I \) to denote the two-fold integral of the left side of Eq.(C.34). Using the integral formula Eq.(C.19) to the right side of Eq.(C.34), we have

\[
I = \frac{2}{\sqrt{\pi}} \int_0^{\infty} du_\delta e^{-u^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1, u_\delta^2)}{n!u_\delta^{2n}} (uu_\delta)\cos^{M-2L+1} \chi \sin^{n+1} \chi \cos^{M-2L+1} \chi d\chi.
\] (C.35)
Shifting the summation about \( l \) by letting \( l = j - L - 1 \), we have

\[
I = 2 \int_0^\infty du_3 e^{-u_3^2} \sum_{n=0}^\infty \frac{\Gamma(n+1, u_3^2)}{n!(-u)^{M+1}} \sum_{j=[(M+1)/2]}^{M-L+1} C_j^{-(M+2)/2} \frac{\Gamma(1/2 + [(M+2)/2] - L)}{\Gamma((1/2) - [(M+1)/2] + j) \Gamma(3/2 + n + j - L)}.
\]

(C.36)

Change the order of the integral and summations in Eq.(C.36), using the variable change \( y = u_3^2 \), we have

\[
I = \sum_{j=[(M+2)/2]}^{M-L+1} C_j^{-(M+2)/2} \frac{\Gamma(1/2 + [(M+2)/2] - L)}{\Gamma((1/2) - [(M+1)/2] + j) \Gamma(3/2 + n + j - L)} \int_0^\infty \frac{\Gamma(n+1, y)}{n!} \frac{(u^2)^{n+j}}{y^{j-1}} dy.
\]

(C.37)

Because \( M \geq 0 \), it is clear that \( j \geq [(M+2)/2] \geq 1 \). For \( j \geq 1 \), we have the following integral formula as Eq.(6.5.37) in Ref.[77],

\[
\int_0^\infty \frac{\Gamma(n+1, x)x^{j-1}}{(n+j)\Gamma(3/2 + n + j - L)} dx = (n+j)!/j.
\]

(C.38)

Substituting Eq.(C.38) into Eq.(C.37), we obtain

\[
I = \sum_{j=[(M+2)/2]}^{M-L+1} C_j^{-(M+2)/2} \frac{\Gamma(1/2 + [(M+2)/2] - L)}{\Gamma((1/2) - [(M+1)/2] + j) \Gamma(3/2 + n + j - L)} \int_0^\infty \frac{\Gamma(n+1, y)}{n!} \frac{(u^2)^{n+j}}{y^{j-1}} dy.
\]

(C.39)
The summation with respect to the index \( n \) in Eq.(C.39) can be expressed in a finite form with the help of the property of analytical gamma function \( \gamma^*(\alpha, x) \),

\[
\left( x^j e^x \gamma^*(\alpha, x) \right)^{(j)} = \frac{d^j}{dx^j} \sum_{n=0}^{\infty} \frac{x^{n+j}}{\Gamma(n+\alpha+1)} = \sum_{n=0}^{\infty} \frac{x^n(n+j)!}{n!\Gamma(n+\alpha+1)}. \tag{C.40}
\]

Applying Eq.(C.40) into Eq.(C.39), we get

\[
I = \sum_{j=\lceil (M+2)/2 \rceil}^{M-L+1} C_j^{\lceil (M+1)/2 \rceil -\lceil (M+2)/2 \rceil - L} \Gamma((1/2) + \lceil (M + 2)/2 \rceil - L, u^2) \frac{\Upsilon_j((1/2) + j - L, u^2)}{(-u)^{M+1}}, \tag{C.41}
\]

where the function \( \Upsilon_j(\alpha, x) \) is defined as

\[
\Upsilon_j(\alpha, x) = j^{-1} x e^{-x} \frac{d^j}{dx^j} \left( x^j e^x \gamma^*(\alpha, x) \right). \tag{C.42}
\]

Then Eq.(C.1) has been proved as Equations (C.41-C.42).
BIBLIOGRAPHY


