THE STUDY OF TRANSLATION EQUIVALENCE ON INTEGER LATTICES

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This paper is a contribution to the study of countable Borel equivalence relations on standard Borel spaces. We concentrate here on the study of the nature of translation equivalence on $2^{\mathbb{Z}^n}$. We study these known hyperfinite spaces in order to gain insight into the approach necessary to classify $2^{\mathbb{Z}^{<\omega}}$ as either being hyperfinite or not. In Chapter 1, we will give the basic definitions and examples of spaces used in this work. The general construction of marker sets is developed in this work. These marker sets are used to develop several invariant tilings of the equivalence classes of $2^{\mathbb{Z}^n}$. Some properties that are equivalent to hyperfiniteness in the space $2^{\mathbb{Z}^{<\omega}}$ are also developed. Lastly, we will give the new result that there is a continuous injective embedding from $2^{\mathbb{Z}}$ into 2^{ω}.

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CHAPTER 1

INTRODUCTION

This paper is a contribution to the study of countable Borel equivalence relations on standard Borel spaces. We concentrate here on the study of the nature of translation equivalence on $2^{\mathbb{Z}^n}$. We study these known hyperfinite spaces in order to gain insight into the approach necessary to classify $2^{\mathbb{Z}^{<\omega}}$as either being hyperfinite or not. In Chapter 1, we will give the basic definitions and examples of spaces used in this work. The general construction of marker sets is developed in this work. These marker sets are used to develop several invariant tilings of the equivalence classes of $2^{\mathbb{Z}^n}$. Some properties that are equivalent to hyperfiniteness in the space $2^{\mathbb{Z}^{<\omega}}$ are also developed. Lastly, we will give the new result that there is a continuous injective embedding from $2^{\mathbb{Z}}$ into 2^{ω}.

1.1 Basic Definitions

Definition 1.1. An **equivalence relation** on a set X is a collection R of ordered pairs $(x, y) \in X^2$ such that the following hold:

1. Reflexive Property: For all $x \in X, (x, x) \in R$ [or for all $x \in X, xRx$].

2. Symmetric Property: If $(x, y) \in R$ then $(y, x) \in R$ [or if xRy then yRx].

3. Transitive Property: If $(x, y), (y, z) \in R$ then $(x, z) \in R$. [or if xRy and yRz then xRz].

Definition 1.2. A **Borel equivalence relation** on a topological space X is an equivalence relation R on X such that R is a Borel subset of X^2 equipped with the product topology.

Definition 1.3. For $x \in X$ the **equivalence class** of x for R is

$$[x]_R = \{y \in X \mid yRx\}.$$

Definition 1.4. An equivalence relation R on X is a **finite equivalence relation** if every $x \in X$ the equivalence class $[x]_R$ is finite.

Definition 1.5. An equivalence relation R on X is a **countable equivalence relation** if every $x \in X$ the equivalence class $[x]_R$ is countable.

Definition 1.6. A **Polish space** is a separable completely metrizable topological space.

Definition 1.7. X is a **standard Borel space** means X is a set equipped with a σ-algebra which is Borel isomorphic to the σ-algebra of the Borel sets in a Polish space.

Definition 1.8. A Borel equivalence relation E on a standard Borel space X is a **hyperfinite equivalence relation** if $E = \cup_n E_n$, where (E_n) is an increasing sequence of finite Borel equivalence relations(i.e., $xEy \Leftrightarrow \exists n(\forall m \geq n(xE_m y)))$. Alternatively, a Borel equivalence relation E on a standard Borel space X is a **hyperfinite equivalence relation** if $E = E_{\mathbb{Z}}$

i.e. there is a Borel automorphism T of X with $xEy \Leftrightarrow \exists n \in \mathbb{Z}(T^n(x) = y)$. [This is a result of T. Slaman and J. Steel [6].]

We will now give some examples of hyperfinite equivalence relations. Let X be a Polish space, $U : X \to X$ a Borel countable-to-one function. Define

$$xE_0(U)y \Leftrightarrow \exists n(U^n x = U^n y).$$

$$xE_t(U)y \Leftrightarrow \exists n \exists m(U^n x = U^m y).$$

Then $E_0(U)$ and $E_t(U)$ are both hyperfinite (see Dougherty-Jackson-Kechris [2]).

1.2 Translation Equivalence

In this section, we will introduce the main equivalence relations that will be studied throughout this paper. Let $2^{\mathbb{Z}} = \{f \mid f : \mathbb{Z} \to \{0,1\}\}$ and $n \in \mathbb{Z}$. We will define $\pi_n : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ as follows: for each $x \in 2^{\mathbb{Z}}, (\pi_n(x))(m) = x(m+n)$. Translation equivalence on $2^{\mathbb{Z}}$ is defined as follows:

$$xE_T y \Leftrightarrow \exists n \in \mathbb{Z}, \pi_n(x) = y.$$

Let $2^{\mathbb{Z}^n} = \{f \mid f : \mathbb{Z}^n \to \{0,1\}\}$ and $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. We will define $\pi_{(m_1,m_2,\ldots,m_n)} : 2^{\mathbb{Z}^n} \to 2^{\mathbb{Z}^n}$ as follows: for each $x \in 2^{\mathbb{Z}^n}$,

$$(\pi_{(m_1,m_2,\ldots,m_n)}(x))(k_1, k_2, \ldots, k_n) = x(m_1 + k_1, m_2 + k_2, \ldots, m_n + k_n).$$

Translation equivalence on $2^{\mathbb{Z}^n}$ is defined as follows:

$$xE_Ty \Leftrightarrow \exists(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n, \pi_{(m_1,m_2,\ldots,m_n)}(x) = y.$$

Definition 1.9. A translation equivalence class B of $2^{\mathbb{Z}^n}$ is called a **periodic class** when there exists $(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n$ such that for every element $x \in B$, $\pi_{(s_1,s_2,\ldots,s_n)}x = x$. Any equivalence class of $2^{\mathbb{Z}^n}$ which is not periodic will be called an **aperiodic class**.

Definition 1.10. For $x, y \in 2^{\mathbb{Z}^n}$ such that xE_Ty we will define a **distance** between x and y as follows:

$$dist(x, y) = \min\{\sqrt{m_1^2 + m_2^2 + \cdots + m_n^2} \mid \pi_{(m_1,m_2,\ldots,m_n)}(x) = y\}.$$

Let $2^{\mathbb{Z}^{<\omega}} = \{f \mid f : \mathbb{Z}^{<\omega} \to \{0, 1\}\}$ for each $(z_i)_{i\leq k} \in \mathbb{Z}^{<\omega}$, define $\pi_{(z_i)_{i\leq k}} : 2^{\mathbb{Z}^{<\omega}} \to 2^{\mathbb{Z}^{<\omega}}$ as follows: for $f \in 2^{\mathbb{Z}^{<\omega}}$ let

$$\pi_{(z_i)_{i\leq k}}(f)((x_i)_{i\leq k'}) = f((x_i + z_i)_{i\leq \max(k,k')})$$

where any undefined x_i or z_i on the right hand side is zero. Translation equivalence on $2^{\mathbb{Z}^{<\omega}}$is defined as follows:

$$xE_Ty \Leftrightarrow \exists(z_i)_{i\leq k} \in \mathbb{Z}^{<\omega}(\pi_{(z_i)_{i\leq k}}(x) = y).$$

CHAPTER 2

MARKER SETS

In this chapter, we will explain the general construction and purpose of marker sets in $2^{\mathbb{Z}^n}$. In the aperiodic equivalence classes of $2^{\mathbb{Z}^n}$, we choose a Borel set of points with certain geometric properties with respect to each other and the other points which are equivalent to them. This Borel set is called a marker set in $2^{\mathbb{Z}^n}$. These geometric properties usually involve a given distance, the equivalent marker points will stay apart and that every point in an aperiodic class has a marker point within a given distance of it. These marker sets serve as starting points for various constructions of collections of equivalent points in $2^{\mathbb{Z}^n}$.

We will first construct a marker set which is Borel and relatively clopen on the free part of $2^{\mathbb{Z}^n}$. It is well known that there exists Borel marker sets. We will show it is also possible to get the marker sets to be relatively clopen on the free part of $2^{\mathbb{Z}^n}$. These marker sets will be used in various constructions and tilings in the remainder of this work.

Theorem 2.1. *There is a Borel Marker Set S in $2^{\mathbb{Z}^n}$ and distance D where S is relatively clopen on the aperiodic part of $2^{\mathbb{Z}^n}$ with the following properties:*

1. *For each pair $x, y \in S, dist(x, y) > D$.*

2. *For each aperiodic $x \in 2^{\mathbb{Z}^n}$ $\exists y \in S,$ such that $dist(x, y) \leq D$.*

Proof. Fix $n \in \omega$. Let $\mathbb{W} = 2^{\mathbb{Z}^n} - \cup\{\mathbf{B} \mid \mathbf{B}\ \textit{is a periodic class}\}$ the aperiodic part of $2^{\mathbb{Z}^n}$.

Lemma 2.2. *For each distinct pair $x, y \in \mathbb{W}$ with xE_Ty there exists a large enough m such that $x\restriction[-m,m]^n \neq y\restriction[-m,m]^n$.*

Proof. Suppose not. Then the $(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n$ such that $\pi_{(s_1,s_2,\ldots,s_n)}(x) = y$ also translates the cube centered at $x_{(0,0,\ldots,0)}$ of width $2m$ onto the cube centered at $y_{(0,0,\ldots,0)}$ of width $2m$ since $x\restriction[-2m,2m]^n = y\restriction[-2m,2m]^n$. If this was true for all $m,\ x$ would be contained in a periodic class which is a contradiction. □

Lemma 2.3. *For each $x \in \mathbb{W}$ and distance D, there is a large enough N such that if $x\restriction[-N,N]^n = y\restriction[-N,N]^n, x \neq y, yE_Tx$ then $dist(x,y) > D$.*

Proof. Fix $x \in \mathbb{W}$ and distance D. For each yE_Tx with $dist(x,y) \leq D$, let N_y be such that

$$x\restriction[-N_y,N_y]^n \neq y\restriction[-N_y,N_y]^n.$$

Let $N' = \max\{N_y \mid yE_Tx\ \textit{and}\ dist(x,y) \leq D\}$.

Let $N = N' + D$. □

For each $x \in \mathbb{W}$, let $N(x)$ be the least positive integer such that

$$N_{x\restriction[-N(x),N(x)]^n} \bigcap \{y \in 2^{\mathbb{Z}^n} \mid yE_Tx\ \textit{and}\ 0 < dist(x,y) \leq D\} = \emptyset.$$

For each $x \in \mathbb{W}$, define $\varphi(x)$ to be the binary string of ones and zeros defined by $x\restriction[-N(x), N(x)]^n$. Let $N_1 = \min\{N(x) \mid x \in \mathbb{W}\}$ and

$$\varphi_1 = \min\{\varphi(x) \mid x \in \mathbb{W}, N(x) = N_1\}.$$

Note if $x, y \in \mathbb{W}$ such that $\varphi(x) = \varphi(y)$ then $x\restriction[-N_1, N_1]^n = y\restriction[-N_1, N_1]^n$ and $dist(x, y) > D$. Let $\hat{x}_1 \in \mathbb{W}$ be such that $N(\hat{x}) = N_1$ and $\varphi(\hat{x}) = \varphi_1$. Let $\mathbf{M}_1 = N_{\hat{x}\restriction[-N_1,N_1]^n}$. Now $\mathbf{M}_1$ has the property that for any pair $x, y \in \mathbf{M}_1$ and xE_Ty then $dist(x, y) > D$.

Now, for the second pass consider $N(y)$ for every $y \in \mathbb{W}_2 = \mathbb{W} - B(M_1, D)$. Let $N_2 = \min\{N(y) \mid y \in \mathbb{W}_2\}$ and $\varphi_2 = \min\{\varphi(y) \mid y \in \mathbb{W}_2, N(y) = N_2\}$. Choose $\hat{x}_2 \in \mathbb{W}_2$ such that $N(\hat{x}_2) = N_2, \varphi(\hat{x}_2) = \varphi_2$. Let $\mathbf{M}_2 = (N_{\hat{x}_2\restriction[-N_2,N_2]^n} \cap \mathbb{W}_2) \cup \mathbf{M}_1$. Now, $\mathbf{M}_2$ has the property that for any pair $x, y \in \mathbf{M}_2$ and xE_Ty then $dist(x, y) > D$.

Suppose $M_k \supset M_{k-1} \supset \cdots \supset M_1$ have been selected with the property that for any pair $x, y \in \mathbf{M}_k$ and xE_Ty then $dist(x, y) > D$, consider $N(y)$ for every $y \in \mathbb{W}_{k+1} = \mathbb{W} - B(M_k, D)$. Let $N_{k+1} = \min\{N(y) \mid y \in \mathbb{W}_{k+1}\}$ and $\varphi_{k+1} = \min\{\varphi(y) \mid y \in \mathbb{W}_{k+1}, N(y) = N_{k+1}\}$. choose $\hat{x}_{k+1} \in \mathbb{W}_{k+1}$ such that $N(\hat{x}_{k+1}) = N_{k+1}, \varphi(\hat{x}_{k+1}) = \varphi_{k+1}$. Let $\mathbf{M}_{k+1} = (N_{\hat{x}_{k+1}\restriction[-N_{k+1},N_{k+1}]^n} \cap \mathbb{W}_{k+1}) \cup \mathbf{M}_k$. Now $\mathbf{M}_{k+1}$ has the property that for any pair $x, y \in \mathbf{M}_{k+1}$ and xE_Ty then $dist(x, y) > D$. This process can go on only countably many times since for each N_k there are only finitely many φ_k that are possible and $N_k \le N_{k+1}$. Let $S = \cup S_i$, S is an open set.

Now, for each $y \in (\mathbb{W} - S)$, $N_{y\restriction[-N(y),N(y)]^n}$ was not a basic open set used in defining S. Thus, when S_i is determined by a neighborhood of width greater than $N(y)$, the procedure must not be considering $N(y)$ when minimizing over the neighborhood width. This implies that there exists $z \in S$ such that $dist(y, z) \leq D$. For some $i,\ z \in S_i$. Let $k = 2(dist(y, z)+N_i)$ now $N_{y\restriction[-k,k]^n} \subseteq (\mathbb{W} - S)$. Thus, S is relatively clopen in $\mathbb{W}$. $\square$

CHAPTER 3

A NEW PROOF THAT $2^{\mathbb{Z}\times\mathbb{Z}}$ IS HYPERFINITE

In this section, we will give an original proof that $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ is hyperfinite. This proof will yield a countable increasing union of finite equivalence relations which equals $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$. It was already known that $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ is hyperfinite by an unpublished result of S. Jackson.

For $x, y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ with xE_Ty, remember $dist(x,y) = \min\{\sqrt{s^2+t^2} \mid \pi_{(s,t)}(x) = y\}$ and for $x, y \in \mathbb{R}^2$, $\rho(x,y)$ is the normal Euclidean distance between x and y.

Theorem 3.1. *Translation equivalence on* $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ *is hyperfinite.*

Proof. Assuming the marker sets and distances (S_n, D_n) have been chosen with the following properties:

1. For each pair $x, y \in S_n, dist(x,y) > D_n$.
2. For each $x \in 2^{\mathbb{Z}\times\mathbb{Z}} \exists y \in S_n, such\ that\ dist(x,y) \leq D_n$.
3. $\frac{(\sum_{i\leq n} D_i)^2}{D_{n+1}} < \frac{1}{2^{n+1}}$.

Definition 3.2. A **marker sequence** is a sequence of equivalent points $(x_n)_{n\geq 0}$ in $2^{\mathbb{Z}\times\mathbb{Z}}$ with the following properties:

1. $\forall n \geq 1,\ x_n \in S_n$.

2. $\forall n \geq 1,\ dist(x_n, x_{n-1}) \leq 10D_n$.

Definition 3.3. Marker sequences $(x_n), (y_n)$ are said to be **equivalent marker sequences** when $x_0 E_T y_0$.

Fix an equivalence class $[x]$ in $2^{\mathbb{Z}\times\mathbb{Z}}$. Let Ψ be the set of all sequences $(x_n)_{n\geq 0}$ of elements in $[x]$ such that

$$\forall n \geq 1, dist(x_n, x_{n-1}) \leq 10D_n.$$

For $x, y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ with xE_Ty. Let $x \diamond y = (s,t) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ such that $\pi_{(s,t)}(x) = y$.

Let $B_k^{(x_n)_{n\geq 0}} = \{\frac{x_k \diamond y}{10D_{k+1}} \mid y \in S_{k+1}\} \subseteq \mathbb{R}^2$.

Claim 3.4. *For each sequence $(x_n)_{n\geq 0} \in \Psi$ there is a point $\alpha \in \mathbb{R}^2$ in the open unit ball and $(n_k)_{k\geq 0}$ such that for each $\epsilon > 0, \exists m \geq 0$ such that $\forall k \geq m, B_{n_k}^{(x_n)_{n\geq 0}} \cap B(\alpha, \epsilon) \neq \emptyset$.*

Proof. $\forall h \geq 1, B_h^{(x_n)_{n\geq 0}} \cap \overline{B((0,0), \frac{1}{5})} \neq \emptyset$. Thus, by the compactness of the closed ball there exists such an α and $(n_k)_{k\geq o}$. $\square$

Let $B_{(x_n)}$ be the collection of all points α satisfying the above claim for $(x_n)_{n\geq 0}$.

Claim 3.5. *(Drift) If $(x_n), (y_n) \in \Psi$ then $B_{(x_n)} = B_{(y_n)}$.*

Proof. Fix $\epsilon > 0$ and $\alpha \in B_{(x_n)}$. Choose N_1 so large such that

$$(dist(x_0, y_0) + 20\sum_{i\leq N_1} D_i) < (\frac{\epsilon}{4})10D_{N_1+1}.$$

Let $N_2 > N_1$ such that $B^{(x_n)}_{N_2} \cap B(\alpha, \epsilon/4) \neq \emptyset$. Let $N = N_2$. Thus $\exists z \in S_{N+1}$ such that

$$\rho(\frac{x_N \diamond z}{10D_{N+1}}, \alpha) < \frac{\epsilon}{4}$$

by the choice of $N_2 = N$. And since $N > N_1$ it follows that the vector translating x_N onto y_N is less than $(\frac{\epsilon}{4})10D_{N+1}$. Thus $\rho(\frac{x_N \diamond z}{10D_{N+1}}, \frac{y_N \diamond z}{10D_{N+1}}) < \frac{\epsilon}{4}$ and $\rho(\frac{y_N \diamond z}{10D_{N+1}}, \alpha) < \frac{\epsilon}{2}$. Therefore $\alpha \in B_{(y_n)}$ proving that $B_{(x_n)} = B_{(y_n)}$.

□

Fix $x \in [x]$, let $(x_n)_{n\geq 0}$ be such that $\forall n, x_n = x$. Let $B([x]) = B_{(x_n)}$. Note claim 3.5 implies $B([x])$ is well defined.

Claim 3.6. $B([x])$ *is the intersection of an open and closed set.*

Proof. $B([x]) = \overline{B} \bigcap B((0,0),1)$ since $B([x])$ is relatively closed in $B((0,0),1)$ using a similar argument as in Claim 3.5. □

For each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ in a Borel manner choose $\alpha(x) \in B([x])$ such that $\alpha(x)$ is invariant. Now for each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ define a canonical marker sequence $CMS(x) = (x_n)$ in the following manner. Let $x_0 = x$. For $n \geq 1$ choose $x_n \in S_n$ in a Borel manner satisfying the following:

1. $dist(x_{n-1}, x_n) < 10D_n$.

2. $\rho(\alpha(x), \frac{x_{n-1} \diamond x_n}{10D_n})$ is a minimum for the points satisfing (1).

Claim 3.7. *This definition of markers $CMS(x)$ produces an embedding from translation equivalence on $2^{\mathbb{Z}\times\mathbb{Z}}$ into the equivalence relation $((2^{\mathbb{Z}\times\mathbb{Z}})^{\mathbb{N}}, E_0)$.*

Proof. Fix $x, y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ such that $CSM(x) = (x_n), CSM(y) = (y_n)$ are equivalent marker sequences. Now two equivalent markers in S_n must be no closer than D_n unless they are equal, thus for $s_1, s_2 \in S_n \cap [x]$ and for any $z \in [x], \rho(\frac{z\diamond s_1}{10D_n}, \frac{z\diamond s_2}{10D_n})$ must be greater than $\frac{1}{10}$. Now choose N such that the following hold:

1. $\rho(\alpha, \frac{x_N \diamond x_{N+1}}{10D_{N+1}}) < \frac{1}{40}$.

2. $(dist(x_0, y_0) + 20\sum_{i\leq N} D_i) < \frac{D_{N+1}}{4}$.

By (2)

$$\rho(\frac{x_N \diamond x_{N+1}}{10D_{N+1}}, \frac{y_N \diamond x_{N+1}}{10D_{N+1}}) < \frac{1}{40}.$$

Thus $\rho(\alpha, \frac{y_N \diamond x_{N+1}}{10D_{N+1}}) < \frac{1}{20}$ and there is no other $z \in S_{N+1}$ which could satisfy

$$\rho(\alpha, \frac{y_N \diamond z}{10D_{N+1}}) < \frac{1}{20}.$$

So $y_{N+1} = x_{N+1}$. Now by the way the canonical marker sequences are chosen using z_n to choose z_{n+1}, it follows that $\forall k \geq N+1, y_k = x_k$. Therefore $CSM(x) = (x_n)$ and $CSM(y) = (y_n)$ are E_0 equivalent. Now suppose that $CSM(x) = (x_n)$ and $CSM(y) = (y_n)$ are E_0 equivalent. Then for any pair $m, n \in \omega$ it follows that $x_n E_T x_m$ and $y_n E_T y_m$ and there exists some $k \in \omega$ such that $x_k = y_k$. Thus we have that xE_Ty.

□

Now since $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ embeds into $((2^{\mathbb{Z}\times\mathbb{Z}})^\omega, E_0)$ it follows that translation equivalence on $2^{\mathbb{Z}\times\mathbb{Z}}$ is hyperfinite. For a proof of this see [2]. However, one can argue the hyperfiniteness directly as follows: for each $x \in S_1$ there is at most $(10D_1)^2$ equivalent points within $10D_1$ of x. And for each $x \in S_n$ there is at most $(10D_n)^2$ points in S_{n-1} within $10D_n$ of x. Therefore there are at most $\sum_{i=1}^{n}(10D_i)^2$ points that choose $x \in S_n$ as the n^{th} term in their canonical marker sequence. We could define $xE_{T_n}y$ when their canonical marker sequences agree at the n^{th} term. This shows $E_T = \bigcup_{i=1}^{\infty} E_{T_i}$ where each E_{T_i} is a finite equivalence relation, thus showing E_T is hyperfinite. □

CHAPTER 4

BOREL BOUNDEDNESS AND LATTICE ROUNDING PROPERTIES

In this section we introduce the property Borel Boundedness which holds for all hyperfinite equivalence relation. Moreover, if $(2^{\mathbb{Z}^{<\omega}}, E_T)$ were Borel Bounded then it would be hyperfinite. Also some rounding properties that holds for all countable equivalence relations.

Definition 4.1. An equivalence relation (X, E) is **Borel Bounded** means for all Borel functions $F : X \to \omega^\omega$ there exists a Borel function $G : X \to \omega^\omega$ such that

1. for all $x \in X$, $F(x) \leq_\star G(x)$,
2. for all $x, y \in X$ if xEy then $G(x) =_\star G(y)$.

Note for $x, y \in \omega^\omega$, $x \leq_\star y$ means for all but finitely many n it follows that $(x(n) \leq y(n))$ and likewise for other relations.

Claim 4.2. *Any Hyperfinite (X, E) is Borel bounded.*

Proof. Suppose $E = \cup E_n$ where the E_n are finite and increasing equivalence relations, $F : X \to \omega^\omega$ is a Borel function.

For each $n \in \omega$, define $G_n(x) = max\{F(y)(n) \mid yE_nx\}$.

Define $G : X \to \omega^\omega$ by $G(x) = (G_n(x))$.

For $x \in X$, $G(x) \geq F(x)$ and if xEy there exists an n such that for all $m \geq n$, $yE_m x$ so $G(x) =_\star G(y)$. $\square$

Claim 4.3. *If (X, E) is a Borel bounded countable equivalence relation such that we can represent E as an increasing union $E = \cup E_n$ and each E_n is hyperfinite then (X, E) is hyperfinite.*

Proof. Suppose $E = \cup_n E_n$, $E_n = \cup_m E_n^m$ such that for all m, n E_n^m is finite and $E_n \subseteq E_{n+1}$, $E_n^m \subseteq E_n^{m+1}$. Also let $G = \{g_1, g_2, \ldots\}$ be a countable group which generates E. Define $F(x)(n)$ as follows: consider $B_n(x) = \{g_1 x, g_2 x, \ldots, g_n x\} \subseteq [x]_E$ let $F(x)(n)$ be the least integer m such that

$$\forall i \leq n \ (g_i x E_n x \implies g_i x E_n^m x).$$

Let $G : X \to \omega^\omega$ be defined by the Borel bounded property of (X, E). Define $\pi : X \to X^\omega$ as follows:

$$\pi(x)(n) = \hat{x} \in [x]_{E_n^{G(x)(n)}}$$

where $\hat{x}$ is chosen in an invariant Borel manner. If xEy then there exists an n_0 such that $y = g_i x$ for some $i \leq n_0$ and $xE_{n_0}y$. Then for all $n \geq n_0$ where $G(x)(n) = G(y)(n)$ it follows that

$$[x]_{E_n^{G(x)(n)}} = [y]_{E_n^{G(x)(n)}}.$$

Now π embeds (X, E) into the equivalence relation (X^ω, E_0) thus (X, E) is hyperfinite. This is a result of Dougherty, Jackson and Kechris which appears in [2]. ☐

Corollary 4.4. *Let (X, E) be an increasing union of hyperfinite equivalence relations. Then (X, E) is hyperfinite if and only if (X, E) is Borel bounded.*

Corollary 4.5. *Let $X = 2^{\mathbb{Z}^{<\omega}}$ and E_T be translation equivalence. If (X, E_T) is Borel Bounded then (X, E_T) is Hyperfinite.*

Proof. Suppose (X, E_T) is Borel bounded. Define (X, E_T^n) by $xE_T^n y$ if $y = \pi_{(s_1,\ldots,s_n)}(x)$ for some $(s_1, \ldots, s_n) \in \mathbb{Z}^n$. Now each E_T^n is hyperfinite and $E_T = \cup_n E_T^n$. Thus if (X, E_T) is Borel bounded then it is also hyperfinite. ☐

Definition 4.6. Given Borel equivalence relations $(X, E), (Y, F)$ we say E is **Borel reducible** to F, means there is a Borel map $\phi : X \to Y$ such that $xEy \Leftrightarrow \phi(x)F\phi(y)$. This is denoted by $E \leq_B F$.

Lemma 4.7. *If $E \leq_B F$ and F is Borel bounded then E is Borel bounded.*

Proof. Suppose $\phi : X \to Y$ is a Borel reduction of (X, E) to (Y, F) and F is Borel bounded. Also suppose $K : X \to \omega^\omega$ is a Borel function. Let $B_y = \{(x, y) \mid x \in \phi^{-1}(y)\}$ and $B = \cup\{B_y \mid y \in image(\phi)\}$ there is a uniformization $u : image(\phi) \to X$ of B since each section B_y of B is countable and B is Borel. Also let $\{g_0, g_1, g_2, \ldots\}$ be a countable group with g_0 being the identity of the group which generates E. Define $\hat{K} : Y \to \omega^\omega$ by $\hat{K}(y)(n) = \max\{K(g_i u(y))(n) \mid i \leq n\}$ if $y \in image(\phi)$, else let $\hat{K}(y)$ be all ones. Let $\hat{G} : Y \to \omega^\omega$ bound $\hat{K}$ be defined by the Borel boundedness of F. We will now show

$G : X \to \omega^\omega$ defined by $G(x) = (\hat{G} \circ \phi)(x)$ will Borel bound K. Fix $x \in X$ there exists some $n \in \omega$ such that $g_n \cdot (u(\phi(x))) = x$ thus for all $m \geq n$ it follows that $K(x)(m) \leq G(x)(m)$ satisfying property (1) of Borel boundedness. Now if xEy then $\phi(x)F\phi(y)$. Thus, by the Borel boundedness of F there exists some $n \in \omega$ such that if $m \geq n$ then $G(x)(m) = \hat{G}(\phi(x))(m) = \hat{G}(\phi(y))(m) = G(y)(n)$ which shows that property (2) of Borel boundedness is also satisfied. Thus E is Borel bounded. □

Lemma 4.8. *If $E \subseteq F$ and F is Borel bounded then E is Borel bounded.*

Proof. Suppose $K : X \to \omega^\omega$ is a Borel function. The bounding function $G : X \to \omega^\omega$ by the Borel boundedness of F will also work for E. □

Lemma 4.9. *If $|F/E| < \infty$ and E is Borel bounded then F is Borel bounded.*

Proof. Suppose $(X, E), (X, F)$ are countable Borel equivalence relations such that $|F/E| < \infty$ and E is Borel bounded. Also suppose $K : X \to \omega^\omega$ is a Borel function. Let $\hat{G} : X \to \omega^\omega$ be a Borel function that Borel bounds K with respect to E. Let $\{g_1, g_2, \ldots\}$ be a countable group which generates F and g_0 be the identity of the group. Fix $x \in X$ then $[x]_F = \cup_{i=0}^{k} [g_{n_i} x]_E$ for some Borel choice of $(n_0, n_1, n_2, \ldots, n_k)$ where $n_0 = 0$ such that for $i \neq j$, $\neg(g_{n_i} x E g_{n_j} x)$. Now we will let $G(x)(n) = \max\{\hat{G}(g_{n_i} x)(n) \mid 0 \leq i \leq k\}$. $\forall x \in X, n \in \omega, (G(x)(n) \geq \hat{G}(x)(n))$ so G satisfies property (1) of Borel boundedness since $\hat{G}$ satisfies property (1). Now suppose xFy thus

$$[x]_F = \cup_{i=0}^{k} [g_{n_i} x]_E = \cup_{i=0}^{k} [g_{m_i} y]_E = [y]_F.$$

These unions are the same so there exists k distinct pairs $(i,j) \in \{0,1,\ldots,k\}^2$ such that $[g_{n_i}x]_E = [g_{m_j}y]_E$. Now we will choose $q_{(i,j)}$ large enough such that for the pair (i,j) it follows that $\forall p \geq q_{(i,j)}$, $(\hat{G}(g_{n_i}x)(p) = \hat{G}(g_{m_i}y)(p))$, let $q = \max\{q_{(i,j)}\}$ then $\forall p \geq q(G(x)(p) = G(y)(p))$ since G is taking maximum for x and y over the same set of integers when $p \geq q$. Thus G Borel bounds K with respect to F showing that F is also Borel bounded. $\square$

We will now define a weaker property called Weakly Borel Bounded and will eventually show all countable Borel equivalence relations are Weakly Borel bounded.

Definition 4.10. Let $h_1, h_2 : \omega \to \omega$ such that $\lim_{n\to\infty} \frac{h_1(n)}{h_2(n)} = 0$. (X,E) is **Weakly Borel Bounded** with respect to h_1, h_2 if for all Borel $F : X \to \omega^\omega$ satisfying

$$(A)\ \forall x E y \forall^{\star}_n \mid F(x)(n) - F(y)(n) \mid < h_1(n).$$

Then there exists a Borel function $G : X \to \omega^\omega$ such that

1. $\forall x \forall^{\star}_n (F(x)(n) \leq G(x)(n))$,
2. $\forall x \forall^{\star}_n (\mid F(x)(n) - G(x)(n) \mid < 3h_2(n))$,
3. $\forall x E y \exists^{\infty}_n (G(x)(n) = G(y)(n))$.

The following are some interesting open questions concerning Borel boundedness:

1. Is Borel boundedness equivalent to Hyperfiniteness?

2. If (X, E) and (Y, F) are both Borel bounded then is $(X \times Y, E \times F)$ Borel bounded? If not, what if E is the equality relation?

We will now introduce some rounding properties used to map points with real coordinates to points that have integer coordinates.

Definition 4.11. (X, E) a countable Borel equivalence relation has the **1-Lattice Rounding Property** $(1 - LRP)$ if for every Borel function $F : X \to \mathbb{R}^\omega$ satisfying

$$\forall x E y \lim_{n\to\infty} |F(x)(n) - F(y)(n)| = 0.$$

There is a Borel function $G : X \to \mathbb{Z}^\omega$ such that

1. $\forall x \forall n (|F(x)(n) - G(x)(n)| \leq 2)$,
2. $\forall x \forall n (F(x)(n) \leq G(x)(n))$,
3. $\forall x E y (\exists^\infty_n (G(x)(n) = G(y)(n)))$.

Lemma 4.12. *If (X, E) has the $1 - LRP$ then (X, E) is Weakly Borel Bounded.*

Proof. Suppose (X, E) has $1 - LRP$ and h_1, h_2 and $F : X \to \omega^\omega$ satisfies property (A) in the definition of weakly Borel bounded. Define $\hat{F} : X \to \mathbb{R}^\omega$ by $\hat{F}(x)(n) = \frac{F(x)(n)}{h_2(n)}$. Thus if xEy then

$$\lim_{n\to\infty} |\hat{F}(x)(n) - \hat{F}(y)(n)| = 0.$$

Now by the $1 - LRP$ there exists a Borel function $\hat{G} : X \to \mathbb{Z}^\omega$ such that

1. $\forall x \forall n(|\hat{F}(x)(n) - \hat{G}(x)(n)| \leq 2)$,

2. $\forall x \forall n(\hat{F}(x)(n) \leq \hat{G}(x)(n))$,

3. $\forall x E y(\exists_n^\infty(\hat{G}(x)(n) = \hat{G}(y)(n)))$.

Let $G(x)(n) = \hat{G}(x)(n) \cdot h_2(n)$. Thus we have

$$G(x)(n) = \hat{G}(x)(n) \cdot h_2(n) \geq \hat{F}(x)(n) \cdot h_2(n) = F(x)(n)$$

showing property (1) of Weakly Borel Boundedness is satisfied. Property (3) of Weakly Borel Boundedness follows from property (3) of $1-LRP$. Now by property (1) of $1-LRP$ we have that $\forall x \forall n |\hat{F}(x)(n) - \hat{G}(x)(n)| \leq 2$ thus we have $\forall x \forall n |F(x)(n) - G(x)(n)| \leq 2h_2(n)$ which shows we satisfy property (2) of Weakly Borel Boundedness. Therefore (X, E) is Weakly Borel Bounded. □

Lemma 4.13. *Any countable Borel equivalence relation (X, E) has the $1 - LRP$.*

Proof. Fix $F : X \to \mathbb{R}^\omega$ satisfying the hypothesis of $1 - LRP$ and let

$$X_1 = \{x \in X \mid \lim_{n \to \infty} \rho(F(x)(n), \mathbb{Z}) = 0\}$$

and let $X_2 = X - X_1$. Note X_1 is invariant because F satisfies the hypothesis of $1 - LRP$. Define $G(x)(n)$ by the following:

1. If $x \in X_1$ let $G(x)(n) =$ the nearest integer to $F(x)(n) + 1$.

2. If $x \in X_2$ let $G(x)(n) = \lceil F(x)(n) \rceil$ where $\lceil z \rceil$ is the smallest integer larger than or equal to z.

Clearly G is a Borel function that satisfies (1) and (2) of $1 - LRP$. To show property (3) of $1 - LRP$, first suppose $x \in X_1$ and yEx. Then $y \in X_1$ by the invariance of X_1 and since $\lim_{n\to\infty} |F(x)(n) - F(y)(n)| = 0$ we have for all but finitely many n, $(G(x)(n) = G(y)(n))$. Now suppose $x \in X_2$ and yEx, $y \in X_2$ as well. Since $x, y \in X_2$ there is some $\epsilon > 0$ and infinitely many n such that $|F(x)(n) - F(y)(n)| < \frac{\epsilon}{4}$ and $\rho(F(x)(n), \mathbb{Z}) > \epsilon$. For these n it follows that $G(x)(n) = G(y)(n)$. Thus we have shown that property (3) of $1 - LRP$ is satisfied for all $x \in X$. Therefore (X, E) has the $1 - LRP$. $\square$

Corollary 4.14. *Any countable Borel equivalence relation (X, E) is Weakly Borel Bounded.*

Proof. This follows directly from the previous 2 lemmas. $\square$

Definition 4.15. Let (X, E) be a countable Borel equivalence relation then (X, E) has the **2-Lattice Rounding Property** $(2 - LRP)$ means for all Borel functions $F : X \to (\mathbb{R}^2)^\omega$ satisfying

$$(A)\ \forall x E y \lim_{n\to\infty} d(F(x)(n), F(y)(n)) = 0$$

there exists a Borel $G : X \to (\mathbb{Z}^2)^\omega$ such that

1. $\forall x \forall n (d(F(x)(n), G(x)(n)) \leq 2)$

2. $\forall x E y(\exists^\infty n(G(x)(n) = G(y)(n)))$.

Definition 4.16. (X, E) a countable Borel equivalence relation has the **$2'$-Lattice Rounding Property** $(2' - LRP)$ if for all positive integers b and for all Borel functions $F : X \to ((\mathbb{R}^2)^{\leq b})^\omega$ satisfying:

$$(A)\ \forall x E y\ \forall \epsilon > 0\ \exists k\ \forall l > k$$

$$\forall p \in (F(x)(l) \cap B((0,0),3))\ \exists q \in F(y)(l)[d(p,q) < \epsilon] \wedge$$

$$\forall p \in (F(y)(l) \cap B((0,0),3))\ \exists q \in F(x)(l)[d(p,q) < \epsilon]$$

there exists a Borel function $G : X \to ((\mathbb{Z}^2)^{\leq b})^\omega$ such that

1. $\forall x \forall n,\ d^{Haus}(F(x)(n), G(x)(n)) \leq 2$,

2. $\forall x E y(\exists^\infty n(\forall p \in B((0,0),3)[p \in G(x)(n) \leftrightarrow p \in G(y)(n)]))$.

Lemma 4.17. $2' - LRP \to 2 - LRP$

Proof. Note for any F satisfying (A) of $2 - LRP$ let $(p_x, p_y) = F(n)(z)$. We could then define $\hat{F}(n)(z) \doteq (\{p_x\}, \{p_y\})$ where $\{d\} = d - \lceil d \rceil$ is the fractional part of d. Now suppose the $2' - LRP$ with $b = 1$ holds. Thus there exists a $\hat{G}$ for $\hat{F}$ from the $2' - LRP$. Let $G = F - \hat{F} + \hat{G}$ notice this would satisfy the $2 - LRP$. Thus $2' - LRP$ implies the $2 - LRP$. $\square$

Definition 4.18. (X, E) a countable Borel equivalence relation has the **$2''$-Lattice Rounding Property** $(2'' - LRP)$ if for all positive integers b and for each Borel function $F : X \to$

$((\mathbb{R}^2)^{\leq b})^\omega$ satisfying:

$$(A)\ \forall x E y\ \forall \epsilon > 0\ \exists k\ \forall l > k$$

$$\forall p \in (F(x)(l) \cap B((0,0),3))\ \exists q \in F(y)(l)[d(p,q) < \epsilon] \wedge$$

$$\forall p \in (F(y)(l) \cap B((0,0),3))\ \exists q \in F(x)(l)[d(p,q) < \epsilon]$$

There exists an invariant Borel function which assigns to each $x \in X$ a function r_x : $(\mathbb{R}^2)^{<\leq b} \to (\mathbb{Z}^2)^{\leq b}$ with the following property. For each pair of Borel functions F', F'' : $X \to ((\mathbb{R}^2)^{<b})^\omega$ with the properties:

1. $\forall x \in X,\ \ d^{Haus}(F'(x)(n), F(x)(n)) \to 0$

2. $\forall x \in X,\ \ d^{Haus}(F''(x)(n), F(x)(n)) \to 0$

we have for every $x \in X$, there exists infinitely many n such that $r_x(F'(x)(n)) = r_x(F''(x)(n))$.

Lemma 4.19. *Any countable Borel equivalence relation* (X, E) *has the* $2' - LRP$*.*

Proof. Fix F satisfying (A) for the $2' - LRP$ for some $b \in \omega$. Now we will define for each equivalence class $[x]_E$ a cut point $(z_1, z_2) \in \mathbb{R}^2$ such that there is a fixed $\epsilon > 0$ such that

$$\forall x \in [x]_E\ \exists^\infty n\ \forall p \in (F(x)(n) \cap B((0,0),3)) \text{ where } (p = (p_x, p_y))$$

$$|\{p_x\} - z_1| > \epsilon \wedge |\{p_y\} - z_2| > \epsilon$$

where again $\{p_x\}$ is the fractional part of p_x. To define z_1, z_2 in a Borel manner we proceed as follows. Let $x \in X$. Let

$$p_n^x = \{(\{p_x\}, \{p_y\}) \mid (p_x, p_y) \in F(x)(n)\} \in [0,1]^{\leq b}.$$

By compactness, we can get a subsequence $p_{i_k}^x \to F$, $F \subseteq [0,1]^2$. Let $\epsilon_0 = \frac{1}{2(b+1)}$ then pick in a Borel manner as a function of x, $(z_1, z_2) \in [0,1]^2$ with the following properties:

(a) $(\rho(z_1, \mathbb{Z}) \geq \epsilon_0) \wedge (\rho(z_2, \mathbb{Z}) \geq \epsilon_0)$

(b) $\forall (p_x, p_y) \in F$, $(|\{p_x\} - z_1| \geq \epsilon_0) \wedge (|\{p_y\} - z_2| \geq \epsilon_0)$

We have defined an invariant Borel function assigning for each $x \in X$ a $(z_1(x), z_2(x))$ satisfying

$$(\dagger)\ \forall y E x \exists^\infty i \forall p \in (F(y)(i) \cap B((0,0),3))$$

$$(|\{p_x\} - z_1| \geq \frac{\epsilon_0}{2} \wedge |\{p_y\} - z_2| \geq \frac{\epsilon_0}{2}).$$

Now to define $G(x)(i)$ use (z_1, z_2) to round each $p = (p_1, p_2) \in F(x)(i)$ as follows: For $j = 1, 2$ if $\{p_j\} \geq z_j(x)$ round p_j up to the next biggest integer, else round p_j down to the next smallest integer. Now property (1) of the $2' - LRP$ is satisfied since we did not adjust any point by more than 1 unit in either coordinate. It suffices to show that G satisfies property (2) of the $2' - LRP$. Suppose $x \in X$ and $x E_T y$, let $\epsilon_1 = \frac{\epsilon_0}{4}$. By property (A) there exists some k such that for all $l > k$

$$\forall p \in (F(x)(l) \cap B((0,0),3)) \exists q \in F(y)(l)[d(p,q) < \epsilon_1] \wedge$$

$$\forall p \in (F(y)(l) \cap B((0,0),3)) \exists q \in F(x)(l)[d(p,q) < \epsilon_1].$$

Therefore we have satisfied property (2) of the $2' - LRP$ for the infinitely many $i > k$ which satisfy the condition (†). $\square$

Lemma 4.20. *Any countable Borel equivalence relation* (X, E) *has the* $2'' - LRP$.

The proof would be very similar to the proof for the $2' - LRP$. So we will omit the proof.

The previous proof can be generalized to show that any countable Borel equivalence relation (X, E) has the following $n - LRP$, $n' - LRP$, and $n'' - LRP$. Where for example we will define the $n - LRP$. The $n' - LRP$, and $n'' - LRP$ will be left to the reader.

Definition 4.21. (X, E) a countable Borel equivalence relation has the **n-Lattice Rounding Property** $(n - LRP)$. If for every Borel function $F : X \to (\mathbb{R}^n)^\omega$ such that

$$\forall x E y, \lim_{n\to\infty} \xi(F(x)(n), F(y)(n)) = 0$$

where $\xi((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)) = \max(|a_1 - a_2|, \ldots, |a_n - b_n|)$. There is a Borel function $G : X \to (\mathbb{Z}^n)^\omega$ which satisfies:

1. $\forall x \forall m(\xi(F(x)(m), G(x)(m)) \leq 2)$,

2. $\forall x \forall m \forall i \leq n(F(x)(m)_i \leq G(x)(m)_i)$,

3. $\forall x E y(\exists^\infty_m (G(x)(m) = G(y)(m)))$.

One significant use of the $n''{-}LRP$ is that it could be used to give another that translation equivalence of $2^{\mathbb{Z}^n}$ is hyperfinite. The spirit of the proof would be similar to the proof from Chapter 3 that $2^{\mathbb{Z}\times\mathbb{Z}}$ is hyperfinite. However instead of looking for a limit point here we would look for an anti-limit point.

CHAPTER 5

TILINGS OF THE EQUIVALENCE CLASSES OF $2^{\mathbb{Z}\times\mathbb{Z}}$

In this section we will investigate different possibilities for dividing each aperiodic equivalence class of $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ into finite subclasses. These subclasses are determined by invariant tilings of the equivalence classes. First, it will be shown that there is an invariant rectangular tiling of the aperiodic classes of $2^{\mathbb{Z}\times\mathbb{Z}}$ by nearly uniform-sized rectangles. Assume the Marker Sets and distances (S_n, D_n) have been chosen with the following properties:

1. For each pair $x, y \in S_n, dist(x, y) > D_n$.

2. For each $x \in 2^{\mathbb{Z}\times\mathbb{Z}} \exists y \in S_n, such\ that\ dist(x, y) \leq D_n$.

3. $\frac{(\sum_{i \leq n} D_i)^2}{D_{n+1}} < \frac{1}{2^{n+1}}$.

For any unit distance of at least 10 we have the following theorem:

Theorem 5.1. *Each equivalence class of $2^{\mathbb{Z}\times\mathbb{Z}}$ can be divided up into rectangles that contain a ball of diameter one half unit and are contained in a ball of diameter four units.*

Proof. Fix an equivalence class $[x]$. By the above properties of the Marker Sets, a sufficiently large n can be chosen such that squares of unit size (these squares will be called the marker squares) can be placed at each origin of each $y \in S_n \cap [x]$ with those squares having the following properties:

1. The squares are all the unit size.

2. The pairs of these squares are all at least a distance of 20 apart.

We will now give an algorithm to produce vertical columns. To create columns of rectangles, stack squares of the same size on top of these marker squares until one of the stacked squares overlaps a marker square. Remove the overlapping square and then make the last non-overlapping square a rectangle, which fills the space to the above marker square. This procedure is illustrated in figure 5.1.

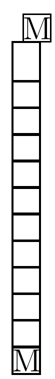

Figure 5.1: Illustrates building a vertical column in the upward direction between two marker blocks which are the blocks with an M inside them.

Now if any marker square is not sharing a bottom edge with a stacking rectangle, do the above procedure in the downward direction until overlapping a marker block or a block produced in the upward direction. Then remove the overlapping square and make the last non-overlapping square a rectangle. This fills the space to the marker square or block produced in the upward construction which was overlapped by the downward column construction.

This procedure is illustrated in figure 5.2. This produces columns of stacked rectangles.

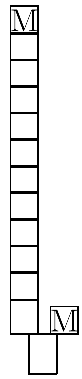

Figure 5.2: Illustrates building a vertical column in the downward direction between a marker block and a rectangle built in the upward direction.

Lemma 5.2. *Let $\mathbb{G}$ be the set of rectangles minus their horizontal edges produced in the above column algorithm. Then $\mathbb{G}$ is a pair wise disjoint set in $\mathbb{R}^2$.*

Proof. Suppose not. There would have to be at least two distinct blocks that intersect each other. Thus we must have one of the following:

1. A marker block intersecting a column block.

2. A column block intersecting another column block.

Case (1) could not happen since the column block would be stopped by the marker block in either direction of the column algorithm. For case (2) to happen it would occur in either the upward direction or the downward direction of the column construction. Suppose two column blocks produced in the upward direction of the column algorithm $B_1, B_2 \in \mathbb{G}$ are such that $B_1 \cap B_2 \neq \emptyset$. Then there exists M_1, M_2 marker blocks that produced B_1, B_2 respectively. If $M_1 \cap M_2 \neq \emptyset$, then $M_1 = M_2$ implying that B_1 and B_2 are not distinct. If $M_1 \cap M_2 = \emptyset$ then M_1 and M_2 are at different horizontal levels and whichever of M_1, M_2 is higher would stop the column blocks produced by the lower of the two marker blocks M_1, M_2, thus could not produce intersecting blocks. Now suppose two column blocks produced in the downward direction of the column algorithm $B_1, B_2 \in \mathbb{G}$ are such that $B_1 \cap B_2 \neq \emptyset$. If B_1, B_2 are both produced in the downward direction. Then, an argument similar to the upward case would work. Now if B_1 was produced in a downward direction and B_2 was produced in an upward direction then either B_2 is directly below a marker block or there is a column block directly above B_2. In either case, the marker block or column block or B_2 itself would certainly prevent any column block produced in the downward direction from intersecting B_2. $\square$

Now fix a connected component D of $K = \mathbb{R}^2 - \textit{the columns}$.

Definition 5.3. A **left (right) column block** with respect to D is the column block B intersecting $\overline{D}$ such that a horizontal line through a point $x \in D$ intersects the interior of the column block B before intersecting any other column block, to the left (right) of the point x.

Note if B is a left column block with respect to D then $sup\{y \mid (x, y) \in D\} \geq sup\{z \mid (x, z) \in B\}$.

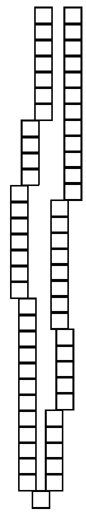

Figure 5.3: Illustration of part of a possible connected component D of $K = \mathbb{R}^2 -$ *the columns*.

Lemma 5.4. *No column block B is both a right and left column block of D.*

Proof. Suppose not. Let B be a left and right column block of D. For this to be true there would have to exist distinct $x, y \in D$ such that x is to the left of B and y is to the right of B. However, there is some connected path of column blocks touching B from above that intersects every horizontal line in $\mathbb{R}^2$ above B. Likewise below B. This would place x and y in different connected components of K which is A contradiction. $\square$

Corollary 5.5. *If two distinct points $(x_1, y), (x_2, y) \in D$, then the line segment from (x_1, y) to (x_2, y) is in D.*

Corollary 5.6. *There is at most one left column block with respect to D for each horizontal level.*

Lemma 5.7. *If B is a left column block with respect to D then one and only one of the following is true:*

1. $sup\{y \mid (x,y) \in D\} = sup\{z \mid (x,z) \in B\}$.

2. *The right most column block sitting on top of B is also a Left column block with respect to D.*

Proof. Suppose B is a left column block with respect to D. Let B' be the right most column block sitting on top of B. Now suppose $sup\{y \mid (x,y) \in D\} > sup\{z \mid (x,z) \in B\}$. Thus there exists a point $(x,y) \in D$ such that the horizontal line through (x,y) intersects B'. If B' is not a left column with respect to D then there exists a left column block C with respect to D determined by (x,y). Note $C \bigcap B' = \emptyset$ so there exists $(x_1, y) \in D$ between C and B' implying C contains part of the horizontal line segment between two distinct points in D. This is a contradiction. Next suppose $sup\{y \mid (x,y) \in D\} < sup\{z \mid (x,z) \in B\}$. This can not happen because column blocks do not share vertical edges and B is a left column block with respect to D. $\square$

Corollary 5.8. *If B_1 and B_2 are left column blocks of D then there is a path of left column blocks of D from B_1 to B_2.*

Lemma 5.9. *For any left column block B_l of the region D there exists a horizontal line which passes through block B_l no less than $\frac{1}{3}$ unit from either vertical edge of B_l. Furthermore, this* *line either passes through a vertical edge of a right column block of D or is no less than $\frac{1}{3}$ unit from any horizontal edge of a right column block of D.*

Proof. Let B_l be the left column block in the lemma. Consider the horizontal line L that is $\frac{1}{3}$ unit below the top edge of B_l. If this line satisfies the lemma, we are done. Suppose not, then there are two cases to consider.

Case 1 There is a vertical edge of a right column block within at least $\frac{1}{3}$ unit below L.

Case 2 L passes through a right column block within $\frac{1}{3}$ unit of this blocks top edge.

In either case, there is a horizontal line within $\frac{1}{3}$ unit of L which satisfies the lemma. This is due to the fact that each column block is at least one unit in height.

□

Let D be some connected component of $K = \mathbb{R}^2 -$ *the columns*.

Definition 5.10. The **left (right) column** of D is the set of left (right) column blocks with respect to D.

Note D is determined by a left and a right column. These columns can be viewed as rectangles R that are one unit width and at least twenty units in height. Each of these rectangles consists of a marker square with column blocks which are stacked above or below the marker square as shown in figures 5.1, 5.2.

Now consider one of the rectangles R in the left column of D. Using the previous lemma we can put a horizontal line through the marker block determining R and every fifth column block of R stacked above or below this marker block except the last one. If we do this for all the rectangles used in the left column of D, we will partition D into horizontal strips that are at least 4 units but no more than 11 units in height which are determined by these horizontal lines and left and right column blocks.

Now consider one of the above horizontal strips HS of D. We will now give an algorithm for filling these horizontal strips. We will either add rectangles that have edges that are at least $\frac{1}{3}$ unit but no more than 2 units in length or we will expand the portion of the column blocks in the horizontal strip HS of D by no more than $\frac{1}{3}$ unit in width.

Definition 5.11. A **corner** of a left (right) column of D is the right (left) most point of some horizontal line segment of length greater than one unit contained in the left (right) column of D. A corner of a left column is illustrated in figure 5.4.

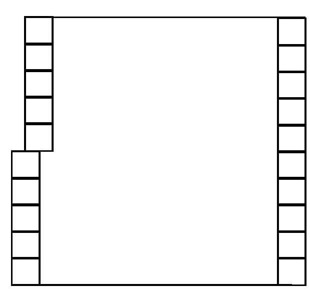

Figure 5.4: Illustration of a possible horizontal strip HS with a corner in the left column.

Claim 5.12. *There exists a vertical line segment contained in D which connects the two horizontal lines determining the horizontal strip HS.*

Proof.

Case 1 One of the columns of HS has no corner. Then since the right column of D does not intersect the left column, the claim is satisfied.

Case 2 Both of the columns of HS have a corner. Note: there is at most one corner to either vertical edge of HS. Then each of these corners must be a marker block or have a distinct marker block adjacent to them and these distinct marker blocks must be at least 20 units apart. Since these marker blocks both intersect the same horizontal strip which has a height of no more than 11 units they must be at least 10 units apart horizontally. Now all the column blocks were 1 unit in width, thus the right most point of the left column of D must be at least 6 units horizontally from the left most point of the right column of D. Therefore, the claim must be satisfied in this case.

□

Now since the horizontal lines determining HS either coincide with a column block's horizontal edge or are at least $\frac{1}{3}$ unit from a horizontal edge of a column block, we can use a similar construction to that used to produce the columns to fill from the column blocks (or portion of column block adjacent to HS) to the vertical line from the above claim. We proceed as follows:

1. If one of the horizontal lines determining HS passes through the middle of a column block of D, use this horizontal line to divide the column block into two blocks. Note these two blocks are at least $\frac{1}{3}$ unit in height.

2. If the distance from a column block (or portion) determining HS to the vertical line is less than $\frac{1}{3}$ unit, we will enlarge horizontally the column block (or portion) adjacent to HS to the vertical line.

3. If the distance from a column block (or portion) determining HS to the vertical line is at least $\frac{1}{3}$ unit, we will stack blocks the same height as the column block (or portion) beside the column block until we get to the vertical line. These blocks will all be one unit in width except the one intersecting the vertical line which will have a width of at least $\frac{1}{3}$ unit but less than two units.

Using the above algorithm for each horizontal strip of each connected component of K, we have produced an invariant tiling of the plane for $[x]$ which consists of rectangles with edge length at least $\frac{1}{3}$ units but less than 2 units.

$\square$

Thus we can divide each equivalence class of $2^{\mathbb{Z}\times\mathbb{Z}}$ into bounded-size rectangles. Now we will show that there are also square tilings, however the squares will not be the same size.

Lemma 5.13. *If p_1, q_1, p_2, q_2 are all pairwise relatively prime positive integers then any rectangle with sufficiently larger integer dimensions can be constructed using only squares of* *length* p_1, q_1, p_2, q_2.

Proof. Let p_1, q_1, p_2, q_2 be pairwise relatively prime positive integers. For any sufficiently large H, W we write:

1. $s_1 p_1 + t_1 q_1 = H$

2. $s_2 p_2 + t_2 q_2 = H$

3. $s_3(p_1 q_1) + t_3(p_2 q_2) = W$

are all positive integer combinations. There are solutions to these equations for sufficiently large values of H and W.

A Type 1 rectangle of width $p_1 q_1$ and height H can be constructed in the following manner using only squares of length p_1 or q_1 only. First, construct a rectangle of width $p_1 q_1$ and height $s_1 p_1$ with squares of width p_1. Then, on top of this place a rectangle of width $p_1 q_1$ and height $t_1 q_1$ constructed of squares of width q_1. This is illustrated in figure 5.5.

A Type 2 rectangle of width $p_2 q_2$ and height H can be constructed in a similar manner using only squares of length p_2 or q_2.

Now we can use s_3 Type 1 rectangles and t_3 Type 2 rectangles to construct a rectangle of width W and height H using only squares of length p_1, q_1, p_2, or q_2. This is illustrated in figure 5.6. $\square$

Corollary 5.14. *There are invariant square tilings of the aperiodic part of* $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$. *Furthermore the squares used in the tiling are nearly uniform (i.e. for* $\epsilon > 0$ *there exists*

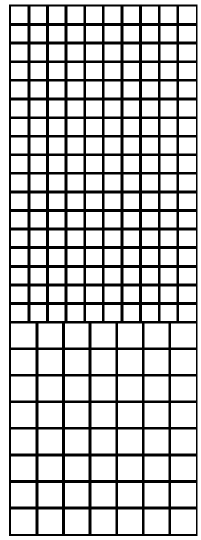

Figure 5.5: Shows a Type I rectangle of width p_1q_1 height H made using only squares of length p_1 and q_1.

square tilings such that if S_1, S_2 *are two squares in the tiling of widths* w_1, w_2 *respectively then* $|\frac{w_1}{w_2} - 1| < \epsilon$*).*

Proof. We can get sufficiently large rectangular tilings and then divide each rectangle into squares satisfying the desired width uniformity using the previous lemmas. ☐

Now we will give an alternate and more general way to produce a rectangular tiling of the aperiodic part of $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$, in particular this proof will generalize to higher dimensions. Let $d > 0$ and choose a Marker set M with a marker distance $D \geq 10,000d$ with the following properties:

1. For all $y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ there exists $m \in M$ such that $dist(y, m) \leq D$.

2. If $x, y \in M$ and xE_Ty then $dist(x, y) > D$.

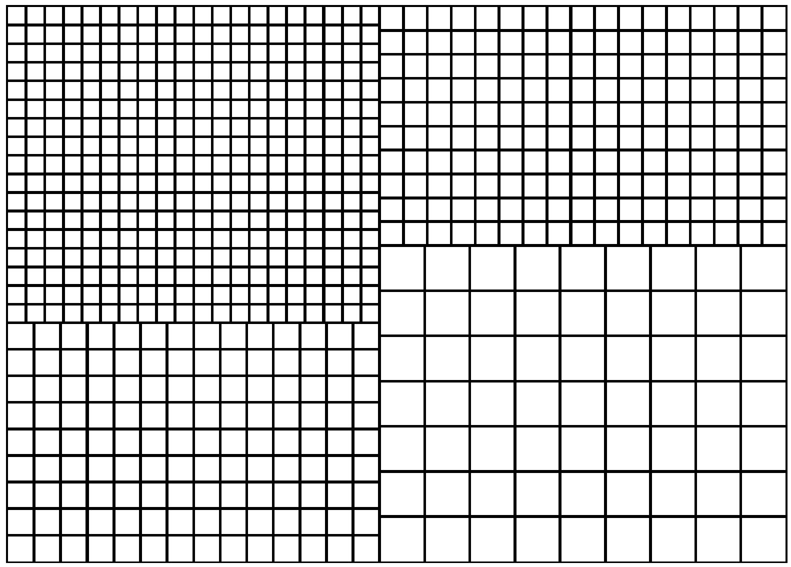

Figure 5.6: Shows a rectangle of dimensions $W \times H$ made using only squares of length p_1, q_1, p_2, or q_2.

Let $\hat{D} \geq 100D$. Partition M into finitely many subsets M_i such that if $x, y \in M_i$ are distinct and xE_Ty then $dist(x,y) > \hat{D}$. This can be done in the following manner. First use the marker algorithm from Theorem 2.1 to choose $M_1 \subseteq M$ such that:

1. for all $y \in M$ there exists $x \in M_1, xE_Ty$ with $dist(x,y) \leq \hat{D}$,

2. if $x, y \in M_1, xE_Ty$ then $dist(x,y) > \hat{D}$.

After $M_1, \ldots, M_k$ have been selected select M_{k+1} such that:

1. $M_{k+1} \subseteq M - \bigcup_{i=1}^{k} M_i$,

2. if $y \in M - \bigcup_{i=1}^{k} M_i$ there exists $x \in M_{k+1}$ such that $dist(x,y) \leq \hat{D}$,

3. if $x, y \in M_{k+1}, xE_Ty$ then $dist(x, y) > \hat{D}$.

Every $x \in M$ has a finite number of points of $2^{\mathbb{Z}\times\mathbb{Z}}$ within $\hat{D}$ of x. Thus this process must terminate within a finite number of steps by (2) above. This process will partition M into a finite pairwise disjoint collection of subsets $\{M_1, \ldots, M_n\}$ such that $M = \bigcup_{i=1}^{n} M_i$ with the above properties. For each $x \in M$ put a square S_x of diameter $2D$ centered at x. Note: for every $y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ there is some $x \in M$ with $y \in S_x$. Let $\mathbb{S} = \{S_x \mid x \in M\}$.

Definition 5.15. Two distinct rectangles A, B are said to have the **separation property** when one of the following hold:

1. $d^{Haus}(A, B) > 2.5D$.

2. If $d^{Haus}(A, B) \leq 2.5D$ then both of the following hold:

   (a) If $(a, b), (s, t)$ are points on a vertical edges of A, B respectively then $|a - s| > 5d$.

   (b) If $(a, b), (s, t)$ are points on a horizontal edges of A, B respectively then $|b - t| > 5d$.

Now we want to adjust the edges of each square S_x to get a rectangle $\hat{S}_x$ such that any distinct pair of rectangles $\hat{S}_x, \hat{S}_y$ with xE_Ty have the separation property. To do this, we will show for any $x \in M$ no edge of S_x will needed to be adjusted by more than $200d$ in order to achieve the separation property for the adjusted rectangles. Now we will proceed by inductively adjusting the M_i squares and avoiding the edges of the previously adjusted squares.

We will leave the S_x with $x \in M_1$ alone. Note, for $x, y \in M_1, xE_Ty$ it follows that S_x, S_y have the separation property since they are sufficiently far apart. Suppose $k > 1$ and the adjustments have been made for all squares in $\cup_{l<k} M_l$ and let $\hat{\mathbb{S}}_k = \{\hat{S}_x \mid x \in \cup_{l<k} M_l\}$ where $\hat{S}_x$ is the adjusted S_x square. Fix $x \in M_k$, we will first adjust the lower edge of S_x. Note, when adjusting the lower edge of S_x we will only adjust in order to avoid the horizontal edges of the the squares in $\hat{\mathbb{S}}_k$. This will suffice since S_x is at least $90D$ from any other M_k square and so if we adjust by no more than $200d$, the set of adjusted rectangles for points in M_k will have the separation property.

To adjust the lower horizontal edge of S_x, we will observe there is at most fifteen squares with a horizontal edge within $200d$ vertical distance of the lower horizontal edge of S_x and intersecting the vertical strip determined by vertical lines placed a distance of $2.5D$ horizontally to either side of S_x. This is illustrated in Figure 5.7.

This is true since for some $\hat{S}_y \in \hat{\mathbb{S}}_k$ to have a horizontal edge within $200d$ below the lower horizontal edge of S_x, $\hat{S}_y$ would have one of the following two properties:

1. $\hat{S}_y$ is at about the same horizontal level as S_x and has the property that the $(s,t) \in \mathbb{Z}^2$ such that$\pi_{(s,t)}(x) = y$ satisfies $|t| \leq 400d$.

2. S_y is below S_x horizontally and has the property that the $(s,t) \in \mathbb{Z}^2$ such that$\pi_{(s,t)}(x) = y$ satisfies $0 \geq t + 2D \geq -400d$.

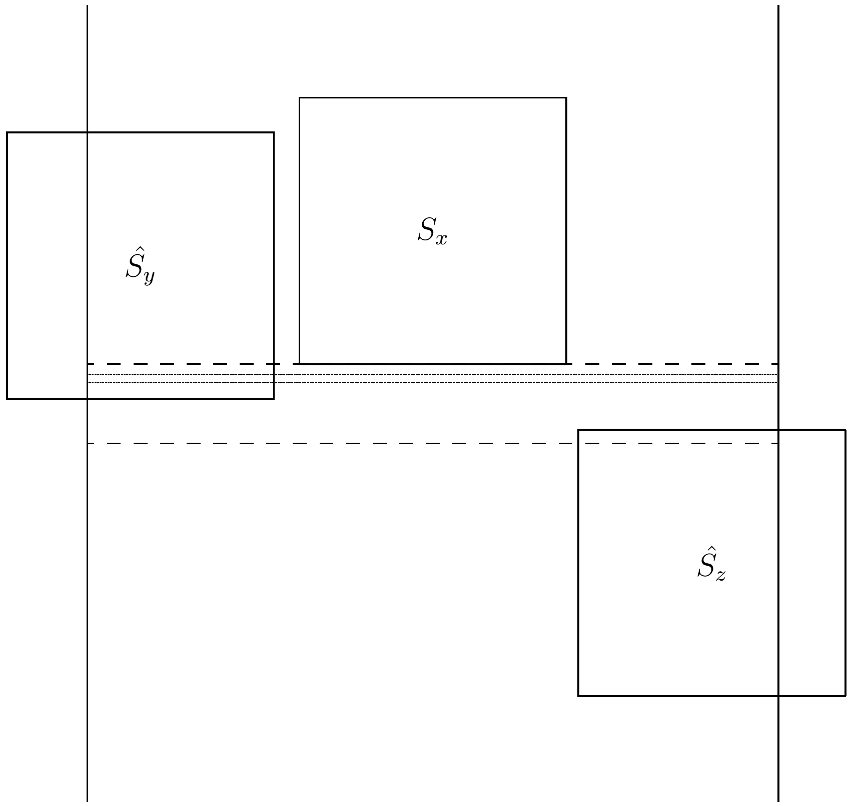

Figure 5.7: Note: picture of setup not to scale. This shows S_x square with the vertical strip and possible locations of nearby rectangles.

There would be at most seven distinct $\hat{S}_y \in \hat{\mathbb{S}}_k$ satisfying property (1) which intersect the vertical strip determined by S_x and at most eight distinct $\hat{S}_z \in \hat{\mathbb{S}}_k$ satisfying property (2), which intersect the vertical strip determined by S_x [see Figure 5.7]. This is at most a total of fifteen horizontal edges to avoid when adjusting the edge of S_x. Thus there must be a gap of at least $10d$ within $200d$ below the lower horizontal edge of S_x inside the vertical lines $2.5D$ to either side of S_x for which no $\hat{S}_y \in \hat{\mathbb{S}}_k$ has a horizontal edge intersecting this gap. Thus we can adjust the lower edge of S_x down to the middle of the first $10d$ vertical gap inside the vertical strip which contains no horizontal edges of any $\hat{S}_y \in \hat{\mathbb{S}}_k$. Note: this adjustment was by no more than $200d$. Now the other directions can be adjusted in a similar manner. This will produce a $\hat{S}_x$ which satisfies the separation property with any $\hat{S}_y \in \hat{\mathbb{S}}_k$. Furthermore, as mentioned before, since distinct S_x, S_y squares for $x, y \in M_k$ are at least $90D$ apart the adjusted rectangles $\hat{S}_x, \hat{S}_y$ must have the separation property thus, completing the inductive step.

Now since any distinct pair $\hat{S}_y, \hat{S}_z$ that intersect some other $\hat{S}_x$ have the separation property, we can extend the edges of $\hat{S}_y, \hat{S}_z$ in the interior of $\hat{S}_x$ to the boundary of $\hat{S}_x$. This will be illustrated in figure 5.8. This will partition $\hat{S}_x$ into rectangles that have length

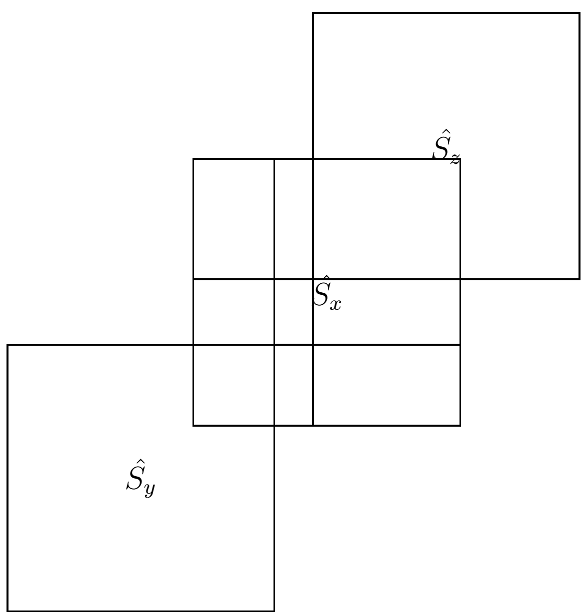

Figure 5.8: This illustrates extending the edges of the $\hat{S}_y, \hat{S}_z$ rectangles inside the $\hat{S}_x$ rectangle.

and width at least $5d$. Now we have given an invariant rectangular tiling of the equivalence classes of $2^{\mathbb{Z}\times\mathbb{Z}}$ which could be generalized to higher dimensions.

The following theorem will show we cannot do better than the nearly uniform square tilings in Corollary 5.14.

Theorem 5.16. *The equivalence classes of translation equivalence E_T on $2^{\mathbb{Z}\times\mathbb{Z}}$ cannot be divided up in an invariant Borel manner into a grid of squares.*

Proof. Suppose there is a Borel squaring of $2^{\mathbb{Z}\times\mathbb{Z}}$ of blocks of size $n_0 \times n_0$ with $n_0 > 1$. Let $x \mapsto n(x)$ by defining $n(x)$ to be the horizontal distance from x to the left-hand edge of the square containing x when x is not on a horizontal edge of a square. If x is on a horizontal edge of a square define $n(x)$ to be 0. This is a Borel map. Let $C_0 \subseteq 2^{\mathbb{Z}\times\mathbb{Z}}$ be a comeager, invariant set such that $n(x)$ is continuous on C_0 and C_0 is a subset of the free part of $2^{\mathbb{Z}\times\mathbb{Z}}$. For each $k \in \omega$, let $B_k \doteq \{x \in C_0 : n(x) = k\}$ which is a relatively clopen subset in C_0 and $C_0 = \cup_{0\leq k\leq n_0} B_k$. Let each $D_n \subseteq 2^{\mathbb{Z}\times\mathbb{Z}}$ be a dense and open subset such that $\cap D_n \subseteq C_0$. Now build an equivalence class as follows. Let $x_0 \in D_0 \cap B_0 \cap C_0$. Note: $D_0 \cap B_0$ is a relatively open set in C_0. Thus we can choose an $m \in \omega$ such that $(N_{x_0 \restriction [-m,m]^2} \cap C_0) \subseteq (D_0 \cap B_0 \cap C_0)$. Choose distance $a > 2m$ such that $a \equiv 1 \mod n_0$. Let $U_1 \subseteq N_{x\restriction[-m,m]^2}$ be the open set

$$U_1 = \{y \in 2^{\mathbb{Z}\times\mathbb{Z}} \mid (y \in N_{x\restriction[-m,m]^2}) \wedge (\pi_{(a,0)}(y) \in N_{x\restriction[-m,m]^2})\}.$$

Let $x_1 \in C_0 \bigcap U_1$. Then $x_1 \in B_0$ and $\pi_{(a,0)}(x_1) \in B_0$. This implies both origin and $(a, 0)$ are on the edge of a square. However this can not be true since $a \equiv 1 \mod n_0$. Therefore there is no invariant Borel tiling of $2^{\mathbb{Z}\times\mathbb{Z}}$ into squares of width n_0 with $n_0 > 1$.

$\square$

Suppose one of the previous rectangular tiling arguments could be generalized to produce a sequence of invariant rectangular partitions of the plane (T_n) for each equivalence class with the following properties:

1. If T_m, T_n are distinct tilings from the sequence and $m < n$ then T_m partitions T_n.

2. For each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ the sequence of tilings has the property that the Hausdorff distance of the origin of x to the set of edges of the rectangles in the tilings goes to infinity.

Then we would have a proof that $(2^{\mathbb{Z}\times\mathbb{Z}}, E_T)$ is hyperfinite by directly producing the increasing union of finite equivalence relations on $2^{\mathbb{Z}\times\mathbb{Z}}$. However, we will now show there is not even a sequence of Borel invariant polygonal tilings with a bounded geometry on the equivalence classes of $2^{\mathbb{Z}\times\mathbb{Z}}$ which satisfy partition property in (1).

Definition 5.17. Let $\alpha, \beta \in \mathbb{R}^2$, define $l(\alpha, \beta)$ to be the **line segment** connecting α to β which includes α but not β.

Definition 5.18. Let $(\alpha_0, \alpha_1, \cdots, \alpha_n)$ be finite length sequence of points in $\mathbb{R}^2$ satisfying the following properties:

1. $\alpha_0 = \alpha_n$.

2. $\alpha_0, \cdots, \alpha_{n-1}$ are distinct.

3. If $i \neq j$ then $l(\alpha_{i-1}, \alpha_i)) \bigcap l(\alpha_{j-1}, \alpha_j)) = \emptyset$.

Let

$$B(\alpha_0, \alpha_1, \cdots, \alpha_n) = \bigcup_{i=0}^{n} (l(\alpha_{n-1}, \alpha_n)).$$

Let $I(\alpha_0, \alpha_1, \cdots, \alpha_n)$ be the nonempty bounded connected component of

$$\mathbb{R}^2 - B(\alpha_0, \alpha_1, \cdots, \alpha_n).$$

We will define the **polygon** $P(\alpha_0, \alpha_1, \cdots, \alpha_n)$ by

$$P(\alpha_0, \alpha_1, \cdots, \alpha_n) = I(\alpha_0, \alpha_1, \cdots, \alpha_n) \cup B(\alpha_0, \alpha_1, \cdots, \alpha_n).$$

For $(a, b), (c, d) \in \mathbb{R}^2$, let

$$\pi_{(a,b)}(c, d) = (a + c, b + d).$$

For $(a, b) \in \mathbb{R}^2$, we define the (a, b) translation of the polygon $P(\alpha_0, \cdots, \alpha_n)$ by

$$\pi_{(a,b)}[P(\alpha_0, \cdots, \alpha_n)] = [P(\pi_{(a,b)}(\alpha_0), \cdots, \pi_{(a,b)}(\alpha_n))].$$

Recall we defined the π action of $\mathbb{Z}^2$ on $2^{\mathbb{Z}\times\mathbb{Z}}$ in the following manner, for $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$, $(a,b) \in \mathbb{Z}^2$ define

$$\pi_{(a,b)}(x) = y \Leftrightarrow y(i,j) = x(i+a, j+b).$$

Definition 5.19. A **Borel Polygonal Tiling** of $2^{\mathbb{Z}\times\mathbb{Z}}$ is a Borel function $F : 2^{\mathbb{Z}\times\mathbb{Z}} \to \mathbb{R}$ such that $F(x)$ codes polygons $(P_i(\alpha_0^i, \cdots, \alpha_{n_i}^i))_{i\geq 0}$ such that

1. $\mathbb{R}^2 = \bigcup_{i\geq 0} P_i(\alpha_0^i, \cdots, \alpha_{n_i}^i)$.

2. If $i \neq j$ then $int[P_i(\alpha_0^i, \cdots, \alpha_{n_i}^i)] \cap int[P_j(\alpha_0^j, \cdots, \alpha_{n_j}^j)] = \emptyset$.

The reader should note in the previous definition the function $F(x)$ used some reasonable mechanism to code the set of polygons

$$\{P_i(\alpha_0^i, \cdots, \alpha_{n_i}^i) \mid i \geq 0\}$$

into the reals. $P \subseteq \mathbb{R}^2$ is coded in $F(x)$ means P is a polygon which is coded by $F(x)$.

Definition 5.20. A Borel Polygonal Tiling of $2^{\mathbb{Z}\times\mathbb{Z}}$ is **invariant** means if $x, y \in 2^{\mathbb{Z}\times\mathbb{Z}}$ such that $\pi_{(s,t)}(x) = y$ and P is coded in $F(x)$ then $\pi_{(-s,-t)}(P)$ is coded in $F(y)$.

Definition 5.21. For two Borel Polygonal Tilings F_1, F_2 of $2^{\mathbb{Z}\times\mathbb{Z}}$, F_1 is said to **respect** F_2 if for each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ and pair of polygons P, Q such that P is coded in $F_1(x)$ and Q is coded in $F_2(x)$ where $P \cap Q \neq \emptyset$ then $Q \subseteq P$.

For a Borel invariant polygonal tiling $F : 2^{\mathbb{Z}\times\mathbb{Z}} \to \mathbb{R}$, define the Borel function $f : 2^{\mathbb{Z}\times\mathbb{Z}} \to 2^{\mathbb{Z}}$ as follows: Let $P = P(\alpha_0, \alpha_1, \cdots, \alpha_n)$ be the polygon coded in $F(x)$ such that $(0,0) \in I(P)$, if it exists.

1. If P exists then $f(x)$ codes the points $\{(a,b) \in \mathbb{Z}^2 \mid (a,b) \in P\}$
2. If P does not exist (i.e. $(0,0)$ intersects an edge of a polygon coded in $F(x)$) then let $f(x)$ be the code for the empty set.

Now since f is a Borel function there exists a comeager set S such that f is continuous on S. The saturation of a meager set is still meager, so without loss of generality we can assume S is a comeager invariant set. Suppose $x \in S$ such that $f(x)$ is not the code for the empty set. f is continuous at x, thus there exists a $z \in \omega$ such that for all $y \in N_{x\restriction[-z,z]^2} \cap S$, $f(y) = f(x)$.

Lemma 5.22. *Translation Lemma: Let U be the open set constructed by placing a copy of $x \restriction [-z,z]^2$ centered at the point $(s,t) \in \mathbb{Z}^2$ that is*

$$U = \{y \in 2^{\mathbb{Z}\times\mathbb{Z}} \mid \pi_{(-s,-t)}(y) \in N_{x\restriction[-z,z]^2}\}.$$

For any $y \in U \cap S$ it follows that $f(\pi_{(-s,-t)}(y)) = f(x)$.

Proof. $\pi_{(-s,-t)}(y) \in S$ since $y \in S$ and S is invariant. Also

$$(\pi_{(-s,-t)}(y)) \restriction [-z,z]^2 = x \restriction [-z,z]^2 \Leftrightarrow f(\pi_{(-s,-t)}(y)) = f(x).$$

$\square$

Note this implies that any $y \in U \cap S$ from above lemma has a polygon coded in $F(y)$ which has the same structure about the point (s,t) as the structure of the polygon about $(0,0)$ coded by $f(x)$. In other words, the polygon coded in $F(y)$ which contains (s,t) also contains $\pi_{(a,b)}(s,t)$ if and only if (a,b) is in the polygon containing $(0,0)$ which is coded in $f(x)$.

Definition 5.23. For $x \in 2^{\mathbb{Z}\times\mathbb{Z}}, (a,b),(c,d) \in \mathbb{R}^2$ which are vertices of polygons in $F(x)$ a **walk** from (a,b) to (c,d) is a sequence of distinct edges of the $F(x)$ polygons whose union connect (a,b) to (c,d).

Theorem 5.24. *There is no sequence (F_n) of Borel Invariant Polygonal Tilings of $2^{\mathbb{Z}\times\mathbb{Z}}$ that have the following properties:*

1. *For all integers $i,j \geq 0$ with $i < j$, F_j respects F_i.*

2. *Diameter of the F_n polygons are bounded by*

$$b_n = \sup\{diam(P) \mid P \text{ coded in } F_n(x),\, x \in 2^{\mathbb{Z}\times\mathbb{Z}}\} < \infty.$$

3. *For each n, there exists $(e_n, l_n) \in \mathbb{Z}^2$ with $\lim_{n\to\infty} l_n = \infty$ such that for any $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ and for any walk along edges of the $F_n(x)$ polygons there could be at most e_n edges in a row of length less than l_n.*

4. *The F_0 polygons are constructed using only finitely many slopes $m_1, m_2, \cdots, m_k$ and* $\max(m_1, m_2, \cdots, m_k) < \infty$.

5. *For each equivalence class γ of $2^{\mathbb{Z}\times\mathbb{Z}}$, there is some $x \in \gamma$ and polygon P coded in $F(x)$ and $(a,b) \in \mathbb{Z}^2$ such that*

$$\{(a+i, b+j) \,|, i, j \in \{0,1,2\}\} \subseteq P.$$

The motivations for the above properties are:

1. If we use the polygons to generate finite subequivalence classes of $2^{\mathbb{Z}\times\mathbb{Z}}$ the equivalence classes will be increasing.

2. Insures for each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}, n \in 2^{\mathbb{Z}}$, there is an edge of a $F_n(x)$ polygon within b_n of $(0,0)$.

3. Insures for each $x \in 2^{\mathbb{Z}\times\mathbb{Z}}, n \in 2^{\mathbb{Z}}$, there is a long edge of a $F_n(x)$ polygon fairly close to $(0,0)$.

4. There are no vertical edges to the polygons.

5. For each equivalence class γ of $2^{\mathbb{Z}\times\mathbb{Z}}$ and $x \in \gamma$ there is a polygon coded in $F_0(x)$ with a two-by-two square contained inside the polygon.

Proof. By way of contradiction, suppose there there exists $(F_n)_{n\geq 0}$ a sequence of Borel Invariant Polygonal Tilings of $2^{\mathbb{Z}\times\mathbb{Z}}$ with the above properties. For each F_n, let $f_n : 2^{\mathbb{Z}\times\mathbb{Z}} \to$

$2^{\mathbb{Z}}$ be defined as before the Translation Lemma and $S_n \subseteq 2^{\mathbb{Z}\times\mathbb{Z}}$ an invariant comeager set on which f_n is continuous. Let $S = \bigcap S_n$ which is an invariant comeager set. Fix $x \in S$ such that $f_0(x)$ codes a set P_0 and $\{(i,j) \mid 0 \leq i,j \leq 2\} \subseteq P_0$, we know such an $x \in S$ exists by the invariance of S and property (5) from the statement of Theorem 5.24. Therefore there is a polygon P_0 coded in $F_0(x)$ which contains a two-by-two square with lower left corner at $(0,0)$. $x \in f_0^{-1}(f_0(x))$ which is an open set in $2^{\mathbb{Z}\times\mathbb{Z}} \cap S$. So there is a neighborhood $N_0 = N_{x\restriction_{[-z,z]^2}}$ such that $z > 2b_0$ [b_0 is the bound on F_0 polygons from property (2)] and if $y \in N_0 \cap S$ then $f_0(y) = f_0(x)$. Note that $f_0(y)$ codes the set of integer points contained in the polygon P coded in $F_0(y)$ such that P contains $(0,0)$. Thus for $F_0(y)$, $(0,0)$ is the lower left corner of a two-by-two square contained in the $F_0(y)$ polygon used to define the code $f_0(y) = f_0(x)$.

Let $G_0 = x \restriction_{[-z,z]^2}$ be this grid of 1's and 0's. We will now describe an algorithm for building an open set U defined by a vertical strip of fixed width, with the property that no edge of slope m_1 of a polygon coded in $F_m, m > 0$ will pass through the entire vertical strip and respect the F_0 polygons.

1. Place G_0 centered at the origin. Now for any $y \in S \cap N_{G_0}$ there is a $F_0(y)$ polygon which contains a square of width two with lower left corner at the origin. Define s_0 to be the unit square with lower left corner at the origin then let (s_i) be unit squares stacked vertically above s_0.

2. Continue to place translated copies of G_0 centered at the points $(a_i, b_i) = (2zi +$

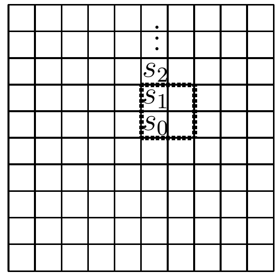

Figure 5.9: Illustration shows the vertical stack of unit squares (s_i) and the square of width two contained in a $F_0(y)$ polygon.

$i, [m_1(2zi + i) + i])$ for $i = 1$ to $2z + 1$ where $[\cdot]$ is the greatest integer function. A line with slope m_1 will not be able to pass through both s_i and the i^{th} translation of G_0 without separating points in the two-by-two square with lower left corner at $(2zi + i, [m_1(2zi + i) + i])$. The grid G_0 was translated by $(2zi + i, [m_1(2zi + i) + i])$ which has a 2^{nd} coordinate within one unit below $m_1(2zi + i) + i$. This implies that the two-by-two square determined by the i^{th} translate of G_0 has lower left corner at $(2zi+i, [m_1(2zi+i)+i])$ and upper left corner at $(2zi+i, [m_1(2zi+i)+i]+2)$. Note that any line of slope m_1 passing through s_i would also pass through $(2zi+i, y)$ where $[m_1(2zi+i)+i] \leq m_1(2zi+i)+i \leq y \leq m_1(2zi+i)+i+1 < [m_1(2zi+i)+i]+2$. In particular, no edge of slope m_1 from an F_n polygon could pass through s_i and the i^{th} translate of G_0 and respect the F_0 polygons. If an edge passes through s_i and the i^{th} translate of G_0 it would separate the top left corner points from the lower right corner point of the two-by-two square contained in the i^{th} translate of G_0. This is illustrated in Figure 5.10.

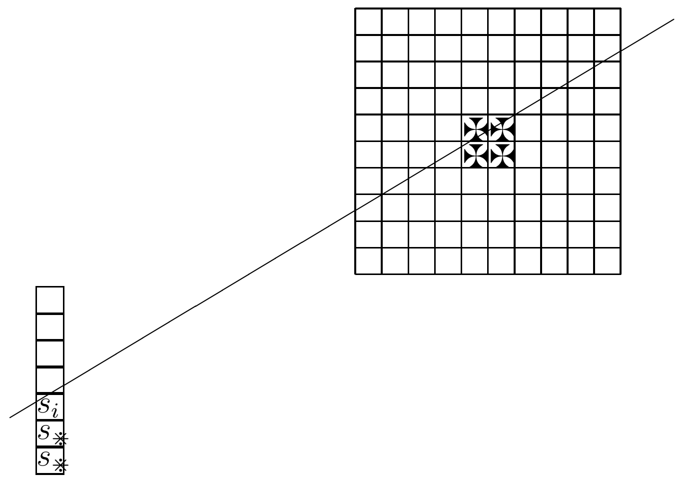

Figure 5.10: Illustrates an edge of a polygon of slope m_1 passing through s_i unit square and a square of width 2 contained inside a F_0 polygon determined by the i^{th} translate of G_0.

This will produce $2z+1$ copies of G_0 in a diagonal pattern with centers at the points (a_i, b_i) as illustrated in figure 5.11 .

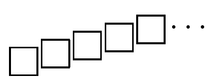

Figure 5.11: $2z+1$ copies of G_0.

3. Make a copy of the above neighborhood translated $2z+1$ vertically to produce a more restrictive neighborhood which will prevent a polygon edge with slope m_1 from passing through s_i and not separate corner points of the two-by-two square contained in the polygon determined by the translate of G_0 corresponding to s_i. We can continue to place vertically translated copies of the diagonal pattern to produce as tall of a vertical

strip as desired which prevents edges of polygons with slope m_1 from crossing the vertical strip. This is illustrated in figure 5.12.

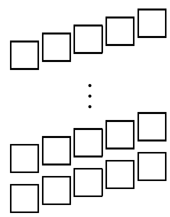

Figure 5.12: Vertical strip of translated copies of G_0 which will prevent a polygon edge from passing through this strip with slope m_1 which respects the F_0 polygons.

This process can be continued to make a vertical strip with the following properties:

(a) No polygon edge with slope m_1 will be able to pass completely through the vertical strip.

(b) The strip is no wider than $SW_1 = (2z + 1)^2$.

Now by modifying the algorithm to use m_2 instead of m_1 we can build a vertical strip of width SW_2 to the right of the strip for m_1 which will prevent an edge with slope m_2 from passing through this second vertical strip without separating points in the some polygon coded in F_0. Figure 5.13 will illustrate this process.

Continue to use this algorithm for each slope m_i to create a vertical strip of width SW_i which no edge of a polygon will cross without separating points of some polygon coded in F_0. This will create a collection of vertical strips we will call the master strip with the property

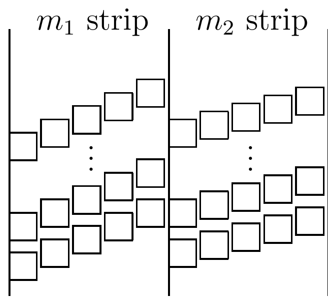

Figure 5.13: Vertical strips of translated copies of G_0 which will prevent a polygon edge from passing through these strips with slope m_1 or m_2 which respects the F_0 polygons.

that no edge of a polygon will be able to completely cross the master strip and respect the polygons coded in F_0. The sum of the widths of these strips $SW = \sum SW_i$ is no more than $k(2z+1)^2$. SW is the width of the master strip. Let n be large such that

$$l_n \cos\theta > 2SW, \ \theta = \max\{\theta_i \mid \theta_i = |\arctan(m_i)|\}$$

for the slopes (m_i) of the polygons coded in F_0 . Recall the following properties from the hypothesis of Theorem 5.24.

Property (2) b_n is the bound on how large the polygons are in F_n. If $x \in P$ where P is coded in F_n then the ball of radius b_n centered at x contains P.

Property (3) e_n is the maximum number of distinct edges of polygons in F_n which can be walked before there is an edge with length at least l_n.

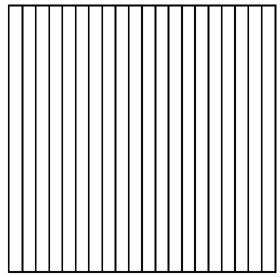

Figure 5.14: Illustration of A where each vertical strip is a copy of the master strip of height w.

Let r be the positive integer such that

$$r(SW) \leq [(e_n+1)l_n + b_n] < (r+1)(SW).$$

Now we can use the algorithm to build an $x \in 2^{\mathbb{Z}\times\mathbb{Z}}$ which consists of $\mathbb{Z}$ copies of the master strip extended vertically in both directions. Let $A = x \upharpoonright [-w,w]^2$. See Figure 5.14 for an illustration of A. Now for any point $x \in (N_A \cap S)$, if we go to the closest vertex of the F_n polygons and walk distinct edges, we will encounter an edge with length at least l_n within a distance of $(e_n+1)l_n + b_n$ from the origin of x. Note: $l_n \cos\theta > 2SW$, therefore this long edge will have to completely cross a master vertical strip used in constructing A. This long edge of a F_n polygon would separate points is some F_0 polygon since it crosses a master vertical strip. Thus the F_n polygons would not respect the F_0 polygons, thus proving the theorem. $\square$

CHAPTER 6

A CONTINUOUS INJECTIVE EMBEDDING FROM $2^{\mathbb{Z}}$ TO 2^{ω}.

In this section we give a continuous injective embedding from $2^{\mathbb{Z}}$ into 2^{ω} which is an extension of the well known fact that a Borel embedding exists from $2^{\mathbb{Z}}$ to 2^{ω}. Let $\mathbb{W}$ be the aperiodic part of $2^{\mathbb{Z}}$.

Lemma 6.1. *Let t be a positive integer. There are pairwise disjoint relatively clopen sets $(M_i)_{i=1}^{N}$ and marker set $M = \bigcup_{i=1}^{N} M_i$ in $\mathbb{W}$, for any marker distance D with the following properties:*

1. *For each pair $x, y \in M, dist(x, y) > D$.*
2. *For each $x \in \mathbb{W}, \exists y \in M, such\ that\ dist(x, y) \leq D$.*
3. *For each pair $x, y \in M_i, dist(x, y) > 10 \cdot t \cdot D + 1$.*

Proof. Using the standard marker algorithm from Theorem 2.1 we can generate a clopen marker set $\hat{M}$ with marker distance $10 \cdot t \cdot D + 1$ such that:

1. For each pair $x, y \in \hat{M}, dist(x, y) > 10 \cdot t \cdot D + 1$.
2. For each $x \in \mathbb{W}, \exists y \in \hat{M}, such\ that\ dist(x, y) \leq 10 \cdot t \cdot D + 1$.

For each $\hat{x} \in \hat{M}$, let $s(\hat{x})$ be the largest integer s such that

$$\{\phi^\alpha(\hat{x}) \mid 1 \leq \alpha < (s+1)(D+1)\} \cap \hat{M} = \emptyset.$$

Note that $s(\hat{x})$ is no larger than $22 \cdot t$. Let $M_0 = \hat{M}$, for $i \geq 1$ let

$$M_i = \{\phi^{i(D+1)}(\hat{x}) \mid i \leq s(\hat{x})\}.$$

Let N be the largest i such that $M_i \neq \emptyset$ and $M = \bigcup_{i=1}^{N} M_i$. Now to show each M_i is clopen we note the following:

(a) The map $\hat{x} \to s(\hat{x})$ is continuous.

(b) $\{i \mid i \leq s(\hat{x})\}$ is a clopen set.

(c) For each i, $\{\hat{x} \in M \mid s(\hat{x}) \geq i\}$ is clopen.

(d) ϕ is a homeomorphism.

So M_i is the homeomorphic image of a clopen set, thus M_i is clopen. (2) of Lemma 6.1 is easily verified by the definition of $s(\hat{x})$.

□

Lemma 6.2. *let t be a positive integer. Let $\epsilon > 0$ and $(D_i)_{i \geq 1}$ be a sequence of integers such* *that* $D_1 \cdot \epsilon > 2\frac{(2t+1)^2}{1-\epsilon}$ *and for all* $n \geq 2$,

$$\sum_{i=1}^{n-1}(\frac{D_n}{D_i(1-\epsilon)}+1) < \frac{D_n \cdot \epsilon}{(2t+1)^2}.$$

There are clopen Marker Sets $(M_i, D_i)_{i\in\omega}$ *in* $\mathbb{W}$ *with the following properties:*

1. *For each pair* $x, y \in M_i, dist(x, y) > D_i(1-\epsilon)$.

2. *For each* $x \in \mathbb{W}, \exists y \in M_i, such\ that\ dist(x, y) \leq D_i(1+\epsilon)$.

3. *The following collection of sets are all pairwise disjoint:*

$$\{\phi^{iD_j}(M_j) : j \geq 1, |i| \leq t\}$$

Proof. This will be done by generating marker sets $(\hat{M}_i, D_i)$ with the previous lemma and adjusting the points by no more than $(D_i \cdot \epsilon)$. Let $\hat{M}_1 = \bigcup_{i=1}^{N} \hat{T}_i$ be chosen as in the previous lemma for the distance D_1. Now let $T_1 = \hat{T}_1$. Note that $\{\phi^{iD_1}(T_1) : |i| \leq t\}$ are all pairwise disjoint since points in T_1 are at least $10 \cdot t \cdot D_1 + 1$ apart. Suppose that sets $T_1, T_2, \cdots, T_{l-1}$ have been chosen which satisfy the following properties:

1. $\{\phi^{iD_1}(T_j) : 0 < j \leq l-1, |i| \leq t\}$ are all pairwise disjoint. Let

$$B = \bigcup_{0<j\leq(l-1)} \bigcup_{|i|\leq t} \phi^{iD_1}(T_j).$$

2. Each $x \in T_i$ is a right shift by no more than $(D_1 \cdot \epsilon)$ of a point in $\hat{T}_i$.

Claim 6.3. *For each point $\hat{x} \in \hat{T}_l$ there exists a $\alpha < (D_1 \cdot \epsilon)$ such that*

$$\{\phi^{\alpha+iD_n}(\hat{x}) : |i| \leq t\} \cap B = \emptyset.$$

Proof. For each point $\hat{x} \in \hat{T}_l$,

$$|\{\phi^s(\hat{x}) \mid 0 \leq s \leq D_1\} \cap B| \leq 2(\frac{2t+1}{1-\epsilon})$$

This is true since all pairs of points in $T = \bigcup_{u=1}^{l-1} T_u$ are at least $D_1 \cdot (1-\epsilon)$ apart and we have $2t+1$ translations of this set T in B. Thus

$$|\{\phi^s(\hat{x}) | 0 \leq s \leq D_1 \cdot \epsilon\} \cap (\bigcup_{|i| \leq 2t+1} \phi^{iD_1}(B))| \leq 2\frac{(2t+1)^2}{1-\epsilon}.$$

Since $(D_1 \cdot \epsilon) > 2\frac{(2t+1)^2}{1-\epsilon}$ the above claim is true. $\square$

For each $\hat{x} \in \hat{T}_l$, let $s(\hat{x})$ be the least s satisfying the claim. Let $T_l = \{\phi^{s(\hat{x})}(\hat{x}) \mid \hat{x} \in \hat{T}_l\}$ and $M_1 = \bigcup_{i=1}^{N} T_i$. Note that $\{\phi^{iD_1}(M_1) : |i| \leq t\}$ are all pairwise disjoint. Now we can inductively adjust the rest of the $\hat{M}_n$ in a similar manner also avoiding $\{\phi^{iD_j}(M_j) : |i| \leq$

$t, 1 \leq j < n\}$ Suppose that $M_1, \cdots, M_{n-1}$ have been chosen satisfying the claim. Let

$$B = \bigcup_{0<j<n} \bigcup_{|i|\leq t} \phi^{iD_1}(M_j).$$

Notice for any set $C \in 2^{\mathbb{Z}\times\mathbb{Z}}$ and distance D with the property $x, y \in C$ implies

$$dist(x, y) > D \cdot (1 - \epsilon)$$

it follows that

$$|\{\phi^s(\hat{x})|0 \leq s \leq D_n \cdot \epsilon\} \cap (\bigcup_{|i|\leq t} \phi^{iD_n}(C))| \leq (\frac{D_n}{D(1-\epsilon)} + 1)(2t+1) < D_n\epsilon.$$

Hence for each $\hat{x} \in \hat{M}_n$

$$|\{\phi^s(\hat{x})|0 \leq s \leq D_n \cdot \epsilon\} \cap (\bigcup_{|i|\leq t} \phi^{iD_n}(B))\ |\leq 2(2t+1)^2(\sum_{i=1}^{n-1}(\frac{D_n}{D_i(1-\epsilon)} + 1))$$

For each point $\hat{x} \in \hat{M}_n$ let $k(\hat{x})$ be the least integer such that we can translate $\hat{x}$ to the right by $k(\hat{x}) < (D_n \cdot \epsilon)$ producing $M_n = \{\phi^{k(\hat{x})}(\hat{x}) \mid \hat{x} \in \hat{M}_n\}$ such that $\{\phi^{iD_n}(M_n) : |i| \leq t\}$ are pairwise disjoint and each is disjoint from B. Note that $\hat{x} \to k(\hat{x})$ is continuous thus it follows that M_n is clopen in $\mathbb{W}$.. $\hat{M}_n$ had the properties that:

1. For each pair $x, y \in \hat{M}_n, dist(x, y) > D_n$.

2. For each $x \in \mathbb{W}, \exists y \in \hat{M}_n, such\ that\ dist(x,y) \le D_n$.

We have translated each point of $\hat{M}_n$ to the right by no more than $D_n \cdot \epsilon$ therefore M_n has the following properties:

1. For each pair $x, y \in M_n, dist(x,y) > D_n(1-\epsilon)$.

2. For each $x \in \mathbb{W}, \exists y \in M_n$, such that $dist(x,y) \le D_n(1+\epsilon)$.

This will generate marker sets $(M_i, D_i)_{i\in\omega}$ with the properties:

1. For each pair $x, y \in M_i, dist(x,y) > D_i(1-\epsilon)$.

2. For each $x \in \mathbb{W}, \exists y \in M_i, such\ that\ dist(x,y) \le D_i(1+\epsilon)$.

3. The following are all pairwise disjoint:

$$\{\phi^{iD_j}(M_j) : |i| \le t, j \ge 1\}$$

$\square$

Now the above procedure producing clopen marker sets in $\mathbb{W}$ can be used with $t = 3$ and the further stipulation that $n!$ divides D_n. This means we can write each marker set as the union of basic open sets in $2^{\mathbb{Z}}$ intersected with $\mathbb{W}$. Thus the marker sets $(M_i, D_i)_{i\in\omega}$ can be extended to all of $2^{\mathbb{Z}}$ and remain open in $2^{\mathbb{Z}}$. The extended marker sets would have the following properties:

1. For each n,

   (a) For each pair $x, y \in M_n, dist(x,y) > D_n(1-\epsilon)$.

   (b) For each $x \in \mathbb{W}, \exists y \in M_n, such\ that\ dist(x,y) \leq D_n(1+\epsilon)$.

2. The following sets are all pairwise disjoint :

$$M_1, \phi^{D_1}(M_1), \phi^{-D_1}(M_1), \phi^{2D_1}(M_1), \phi^{-2D_1}(M_1), \phi^{3D_1}(M_1), \phi^{-3D_1}(M_1),$$

$$M_2, \phi^{D_2}(M_2), \phi^{-D_2}(M_2), \phi^{2D_2}(M_2), \phi^{-2D_2}(M_2), \phi^{3D_2}(M_2), \phi^{-3D_2}(M_2),$$

$$\ldots,$$

$$M_k, \phi^{D_k}(M_k), \phi^{-D_k}(M_k), \phi^{2D_k}(M_k), \phi^{-2D_k}(M_k), \phi^{3D_k}(M_k), \phi^{-3D_k}(M_k),$$

$$\ldots$$

   [This is true in the extension since each of these sets are open and $\mathbb{W}$ is dense.]

3. Each M_i is relatively clopen in $\mathbb{W}$.

4. $n!$ divides D_n.

For each $n \in \omega$ define the following intervals if they exist. For each $a \leq n$, we can define the triple (I_a, k_1^a, k_2^a) as follows, provided (I_a, k_1^a, k_2^a) have not been defined at an earlier stage:

1. Let $\hat{k}_1^a$ be the least integer k such that

$$0 \leq k < 3D_a,\ \phi^{-k}(x) \restriction_{[-10D_n,10D_n]} \subseteq M_a,$$

provided that such a k exists.

2. Let $\hat{k}_2^a$ be the least integer k such that

$$0 \leq k < 3D_a,\ \phi^{k}(x) \restriction_{[-10D_n,10D_n]} \subseteq M_a,$$

provided that such a k exists.

3. If both $\hat{k}_1^a$ and $\hat{k}_2^a$ exist and (I_a, k_1^a, k_2^a) has not been defined at an earlier stage, let $k_1^a = \hat{k}_1^a, k_2^a = \hat{k}_2^a, I_a = x \restriction_{[-k_1^a,k_2^a]}$.

We can define the following function $f : 2^{\mathbb{Z}} \to \omega^\omega$ by: $f(x)(n)$ is the integer that codes the following information:

Type I. For each interval I_a defined at the n^{th} stage code the following:z

(a) I_a.

(b) a.

(c) The translation of the right endpoint of I_{a-1} to the right endpoint of I_a, if I_{a-1} exists by the n^{th} stage.

(d) The translation of the right endpoint of I_a to the right endpoint of I_{a+1}, if I_{a+1} exists by the n^{th} stage.

Type II. The lexicographically least cyclic permutation of s where s is defined in the following manner: Let l_n be the maximum of 0 and the integers k_1^a determined by the n^{th} stage and r_n to be the maximum of 0 and the integers k_2^a determined by the n^{th} stage. Now define $s_n = (s_0, \ldots, s_{p_{n-1}})$ to be the shortest substring of $u_n(x) \doteq x \restriction_{[-(l_n+D_n), r_n+D_n]}$ so that s_n is a potential period of u_n. That is u_n occurs as a substring of s_n^*, where s_n^* is determined by repeating infinitely many copies of s_n in both directions.

Type III. The following pair of integers $\hat{a}_n, \hat{b}_n$ defined by the following:

(a) $a_{-1} = b_{-1} = 0$

(b) Let k_n be the integer $0 \leq k_n < p_n$ such that $x \restriction [-k_n, p_n - k_n - 1] = s_n$.

(c) $a_n = (l_n + D_n - k_n) \mod (p_n)$.

(d) $b_n = l_n + D_n$.

We will let $\hat{a}_n = a_n - a_{n-1}$ and $\hat{b}_n = b_n - b_{n-1}$. Notice that a_n is the distance from the left boundary of $u_n(x)$ to the first start of a 'period' s_n and b_n is the distance of x to the left boundary of $u_n(x)$.

To define $f(x)(n)$ we need to consider $x \restriction_{[-14D_n, 14D_n]}$ only. Thus, this process will produce a continuous function. Also for $x \in \mathbb{W}$, $n \in \omega$ $f(x)$ will eventually code the interval I_n by some stage.

Claim 6.4. *If $x \in \mathbb{W}$ and xE_Ty then $f(x)E_0f(y)$.*

Proof. The disjointness property of the marker sets and their translations implies that for any $z \in 2^{\mathbb{Z}}$, $k \in \mathbb{Z}$ there is at most one pair $(n, i) \in \omega \times \{0, 1, 2, 3\}$ such that $\phi^{\pm i \cdot D_n + k}(z) \in M_n$. Suppose $x \in \mathbb{W}$ and xE_Ty. Now, if xE_Ty there exists an integer k_0 such that $\phi^{k_0}(x) = y$ without loss of generality suppose $k_0 > 0$. Now for any $n \in \omega$,

$$\{\phi^t(x) \mid |t| \leq 3D_n\} - \{\phi^t(y) \mid |t| \leq 3D_n\} = \{\phi^{-3D_n+t}(x) \mid 0 \leq t < k_0\}.$$

Thus by the above property for some $N \in \omega$, for all $m \geq N$,

$$(\{\phi^k(x) : |k| \leq 3D_m\} \cap M_m) \subseteq (\{\phi^k(y) : |k| \leq 3D_m\} \cap M_m).$$

Similarly it follows that for some $M \in \omega$, for all $m \geq M$,

$$(\{\phi^k(y) : |k| \leq 3D_m\} \cap M_m) \subseteq (\{\phi^k(x) : |k| \leq 3D_m\} \cap M_m).$$

Thus there exists a N_0 such that for all $m \geq N_0$,

$$(\{\phi^k(x) : |k| \leq 3D_m\} \cap M_m) = (\{\phi^k(y) : |k| \leq 3D_m\} \cap M_m).$$

This implies for each $m \geq N_0$ that $f(x)$ and $f(y)$ are eventually at some stage coding the same interval for I_m . Let $N_1 \geq N_0$ be a stage so that for all $a \leq (N_0 + 1)$ the interval I_a has been defined by the N_1^{th} stage. Note: for all $m > N_1$ both $f(x)$ and $f(y)$ will be defining the same intervals I_a for the m^{th} stage with the same links to I_{a-1} and I_{a+1}. Thus for all $m > N_1$, f is coding the same Type I information for both x and y.

Now we need to show that $f(x)$ and $f(y)$ will also eventually code the same Type II information. Let L be the maximum length of $u_{N_1}(x), u_{N_1}(y)$. Choose $B \geq N_1$ such that for all $n \geq B, dist(x, M_n) > L$ and $dist(y, M_n) > L$. To see this can be done, note since the marker sets and their translates are disjoint there can be only finitely many within L of either x or y. Let $N_2 \geq N_1$ be a stage where I_B has been defined. Now for all stages $n \geq N_2$ it follows that $u_n(x) = u_n(y)$. This implies for all $n \geq N_2$, $f(x)(n)$ and $f(y)(n)$ are coding the same Type II information.

We will now show $f(x)$ and $f(y)$ will also eventually code the same Type III information. Now we know for all $n \geq N_2$ it follows that $u_n(x) = u_n(y)$. Thus for all $n \geq N_2$ it follows that $a_n(x) = a_n(y)$ and $b_n(x) = b_n(y) - v$ where $\pi_v(x) = y$. This implies that for all $n > N_2$ we have that $\hat{a}_n(x) = \hat{a}_n(y)$ and $\hat{b}_n(x) = \hat{b}_n(y)$. Thus for all $n > N_2$ we have that $f(n)(x) = f(n)(y)$. $\square$

Claim 6.5. *If $x \in \mathbb{W}$ and $f(x)E_0f(y)$ then xE_Ty.*

Proof. Suppose $x \in \mathbb{W}$ and $f(x)E_0f(y)$. This implies that there is an M such that for all $m > M, f_m(x) = f_m(y)$. Thus there is some integer N such that for $m \geq N$, $I_m(x) = I_m(y)$ and these intervals are linked to I_{m-1}, I_{m+1} in the same way for both x and y. Now since the marker sets are disjoint, and on the free part we do define all I_m, by the disjointness property of the marker sets it follows that the right (left) endpoints of these intervals are of unbounded distances from x. This means that the intervals $\{I_m\}_{m\geq 1}$ and their links determine the equivalence class of x. In fact, any tail of $\{I_m\}_{m\geq 1}$ and their links will determine the equivalence class of x. Thus xE_Ty. $\square$

Claim 6.6. *If $x \in 2^{\mathbb{Z}}$ is periodic with a period length c and $N \geq c$ then for any string s of length $d \leq c$ if $u \doteq x \upharpoonright [-N, N]$ occurs as a substring of s^* then the length of s must be c, and s is a period of x.*

Proof. First note the following simple fact. If $t, u, v \in 2^{<\omega}$ such that $t = u^\wedge v = v^\wedge u$ then there exists $w \in 2^{<\omega}$ and positive integers l, k such that u is k copies of w and v is l copies of w. This follows from an induction argument on the length of the longer of u and v.

Suppose $d < c$ and $u \doteq x \upharpoonright [-N, N)$ occurs as a substring of t^* where t has length $d < c$. Note$u = s^\wedge s$ where s is a period of x. Without loss of generality we can assume $t = x \upharpoonright [-N, -N + d - 1]$. Let $m = c \mod d$ and $u = (t_0, \ldots, t_{m-1})$ and $v = (t_m, \ldots, t_{d-1})$. Note $t = u^\wedge v$. Since $s^\wedge s$ is a substring of t^* it follows by looking at the second copy of s that s extends $v^\wedge u$ and thus $v^\wedge u = u^\wedge v$. By the above simple fact it follows that u and v are both some number of copies of a string w, hence s is also. This contradicts the period length of x being c since $s^* E_T w^*$. $\square$

Claim 6.7. *If $x \in 2^{\mathbb{Z}}$ is periodic and xE_Ty then $f(x)E_0f(y)$.*

Proof. Suppose $x \in 2^{\mathbb{Z}}$ is periodic and xE_Ty. Choose N_0 such that D_{N_0} is greater than three times the period of x. There could be no points from the M_{N_0} marker set that are E_T equivalent to x or y since this would put points within D_{N_0} of each other in M_{N_0}. Thus for both x and y there is no Type I information coded by f from stages N_0 and beyond. So it suffices to show the Type II information agrees for all stages past some point. This follows directly from Claim 6.6. Now we need to show the Type III information agrees for all stages past some point. The fact that x and y are periodic means there exists some N greater than the period length of x such that for all $n \geq N$ there are no M_n markers. Thus there exists some $N_1 > N$ such that for all $n \geq N_1$ it follows that $\hat{b}_n(x) = \hat{b}_n(y) = D_n - D_{n-1}$. Note also since the period of x divides D_n that $\hat{a}_n(x) = 0$, likewise $\hat{a}_n(y) = 0$ since y has the same period as x. Therefore for all $n \geq N_1$ we have that $f(n)(x)$ codes the same Type III information as $f(n)(y)$. Thus $f(x)E_0f(y)$. □

Claim 6.8. *If $x \in 2^{\mathbb{Z}}$ is periodic and $f(x)E_0f(y)$ then xE_Ty.*

Proof. Now both x and y are periodic with the same period (i.e. there exists some finite length string s such that xE_Ts^* and yE_Ts^*) thus xE_Ty. □

Claim 6.9. *The function f is one-to-one.*

Proof. We will first show that in the case x is periodic $f(x) = f(y) \Leftrightarrow x = y$. Suppose $f(x) = f(y)$ and x is periodic. Note this implies that yE_Tx, thus y is also periodic. And x and y have the same length period c. The Type III information codes that for all n,

$\hat{a}_n(x) = \hat{a}_n(y)$ and $\hat{b}_n(x) = \hat{b}_n(y)$ which implies that $a_n(x) = a_n(y)$ and $b_n(x) = b_n(y)$. Thus from Type III-(c),(d) it follows that $k_n(x) \equiv k_n(y) mod p_n$ hence $k_n(x) = k_n(y)$. Let N_0 be sufficiently large so that for all $n \geq N_0$ we have that s_n is of length $p_n = c$ thus $xE_T s_n^*$ and $x \restriction [-k_n, p_n - k_n - 1] = y \restriction [-k_n, p_n - k_n - 1]$. Thus $x = y$

Now suppose x is aperiodic and $f(x) = f(y)$. This implies that $xE_T y$. It has already been shown that for some N_0 for all $n \geq N_0$ we have that $u_n(x) = u_n(y)$. Furthermore $f(x) = f(y)$ implies $b_n(x) = b_n(y)$. Thus for each $n \geq N_0$ we have that

$$y \in N_{x \restriction_{[-(l_n+D_n), r_n+D_n]}} (\text{also } x \in N_{y \restriction_{[-(l_n+D_n), r_n+D_n]}}).$$

This implies that $x = y$ since $D_n \to \infty$. Thus f is one-to-one. $\square$

BIBLIOGRAPHY

[1] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Series 232, Cambridge University Press, 1989.

[2] R. Dougherty, S. Jackson, and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations.* Trans. Amer. Math. Soc., 341 (1994), 193-225.

[3] G. Hjorth, *Classification and Orbit Equivalence Relations.* Math. Surveys and Monographs, vol 75, Amer. Math. Soc., 2000.

[4] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations.* Jor. of Math. Logic, 2 (2002), 1-80.

[5] A. S. Kechris, *Classical Descriptive Set Theory.* Springer-Verlag, 1995.

[6] T. Slaman and J. Steel, *Definable functions on degrees.*, Cabal Seminar 81-85, Lecture Notes in Math., vol. 1333, Springer-Verlag, 1988, 37-55.