QUANTIZATION OF SPIN DIRECTION FOR SOLITARY WAVES IN A UNIFORM MAGNETIC FIELD.

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It is known that there are nonlinear wave equations with localized solitary wave solutions. Some of these solitary waves are stable (with respect to a small perturbation of initial data)and have nonzero spin (nonzero intrinsic angular momentum in the centre of momentum frame). In this paper we consider vector-valued solitary wave solutions to a nonlinear Klein-Gordon equation and investigate the behavior of these spinning solitary waves under the influence of an externally imposed uniform magnetic field. We find that the only stationary spinning solitary wave solutions have spin parallel or antiparallel to the magnetic field direction.

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CONTENTS

CHAPTER 1

Introduction

Stable localized solitary wave solutions of nonlinear wave equations are known to exist (see [1], [3], [4], [5], [6], and [9]). In particular, the existence of stable solitary wave solutions has been proven for certain nonlinear Klein-Gordon and Schroedinger equations (see [3], [4], [6], and [9]) and in the case of a class of nonlinear Schroedinger equations, have been shown to have nonzero spin (intrinsic angular momentum) (see [6], [9]).

Consider the nonlinear Klein-Gordon equation (NLKG)

$$u_{tt} - \Delta u = \overrightarrow{g}(u) \tag{1.1}$$

where $u : \mathbb{R}^{3+1} \longrightarrow \mathbb{R}^N$ with N even and the nonlinearity $\overrightarrow{g} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is defined by $\overrightarrow{g}(y) = h(|y|^2)y$, $h : [0, \infty) \longrightarrow \mathbb{R}$ being a continuous function. We will examine the Noether conserved quantity $\overrightarrow{S}[\cdot]$, called spin, which results from the rotational invariance of NLKG. This functional gives the angular momentum about the origin of a solution u. The goal will be to find the spin of particular solitary waves (stationary spinning solitary waves) when exposed to an external uniform magnetic field, $\overrightarrow{B}$. Equation (1.1) represents the case with no magnetic field, i.e. $|\overrightarrow{B}| = 0$. It will be seen that for an arbitrary direction, $\hat{d}$, NLKG will have rotational invariance about $\hat{d}$ and the solitary waves under study can have nonzero spin components in that direction. However, when an external uniform magnetic field $\overrightarrow{B}$ is applied, the spin of these stationary solitary waves is confined to a direction either parallel or antiparallel to $\overrightarrow{B}$.

NLKG can be written compactly using relativistic index notation as

$$\partial^\alpha \partial_\alpha u = \overrightarrow{g}(u). \tag{1.2}$$

where for $X = (x, y, z, t)$ we have $X^\alpha = (x^1, x^2, x^3, t) = (x, y, z, t)$, $X_\alpha = (-x, -y, -z, t)$, $\partial_\alpha = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial t}) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})$, and $\partial^\alpha = (-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}, \frac{\partial}{\partial t})$.

We model the imposition of an external uniform magnetic field of strength $B = |\overrightarrow{B}|$ parallel to the z-axis by making the minimal-coupling substitutions $\partial^\alpha \longmapsto \partial^\alpha - \sigma A^\alpha$ and $\partial_\alpha \longmapsto \partial_\alpha - \sigma A_\alpha$ into NLKG giving us

$$(\partial^\alpha - \sigma A^\alpha)(\partial_\alpha - \sigma A_\alpha)u = \overrightarrow{g}(u) \tag{1.3}$$

where σ is a fixed $N \times N$ real skew-symmetric matrix with $\sigma^2 = -I$ and

$$\overrightarrow{A} = \frac{B}{2} \begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix}$$

It will be assume throughout that B is small. This will allow us to simplify matters by ignoring terms in (1.3) that involve B^2.

It is important to note that the discussion here is not a quantum mechanical one. Although many of the constructions have analogues in quantum mechanics, the interpretations are different.

A word about notation. There are several places, mostly in chapter 2, where there are two kinds of dot products. One kind is that between vectors in $\mathbb{R}^N$ and the other between vectors in $\mathbb{R}^3$. They will both be denoted by $\cdot$.

CHAPTER 2

The Situation With No Magnetic Field

In this chapter the spin functional $\overrightarrow{S}[\cdot]$ is found by applying Noether's principle (see [10]), and then it is shown that stationary spinning solitary wave solutions to NLKG can have a nonzero spin component in any direction. We begin by finding the Lagrange functional and then show this to be invariant under our definition of rotation.

Define: $H(s) = \int_0^s h(r)dr$ and $G(y) = \frac{1}{2}H(|y|^2)$, $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ (recall that h is the continuos function such that $\overrightarrow{g}(y) = h(|y|^2)y$ and $h : [0, \infty) \longrightarrow \mathbb{R}$). Then note the following:

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{G(u+\epsilon v) - G(u)}{\epsilon} &= \left.\frac{\partial G(u+\epsilon v)}{\partial \epsilon}\right|_{\epsilon=0} \\
&= \left.\frac{1}{2}h(|u+\epsilon v|^2)\frac{\partial}{\partial \epsilon}|u+\epsilon v|^2\right|_{\epsilon=0} \\
&= \left.\frac{1}{2}h(|u+\epsilon v|^2)\frac{\partial}{\partial \epsilon}(u+\epsilon v)\cdot(u+\epsilon v)\right|_{\epsilon=0} \\
&= \left.h(|u+\epsilon v|^2)(v)\cdot(u+\epsilon v)\right|_{\epsilon=0} \\
&= h(|u|^2)v \cdot u \\
&= \overrightarrow{g}(u)\cdot v
\end{aligned}$$

This result will be used when computing the Lagrangian.

Let

$$L(u) = \int_{\mathbb{R}^4} \left(\sum_{\alpha=1}^{4} \sum_{\beta=1}^{N} \frac{1}{2} (\partial^\alpha u^\beta)(\partial_\alpha u^\beta) + G(u) \right) d^3\overrightarrow{X} dt$$

$$= \int_{\mathbb{R}^4} \left(\frac{1}{2}|u_t|^2 - \frac{1}{2}|\overrightarrow{\nabla u}|^2 + G(u) \right) d^3\overrightarrow{X} dt$$

where

$$\overrightarrow{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Claim:L(u) is the Lagrange functional of $u_{tt} - \Delta u = \overrightarrow{g}(u)$.

Taking the Frechet derivative, we get

$$\begin{aligned}
& DL(u)(v) \\
= \;& \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}^4} (\frac{1}{2}|u_t + \epsilon v_t|^2 - \frac{1}{2}|\overrightarrow{\nabla}(u+\epsilon v)|^2 + G(u+\epsilon v) \\
- \;& \frac{1}{2}|u_t|^2 + \frac{1}{2}|\overrightarrow{\nabla u}|^2 - G(u)) d^3\overrightarrow{X} dt \\
= \;& \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\mathbb{R}^4} (|u_t|^2 + \epsilon^2 |v_t|^2 + 2\epsilon u_t \cdot v_t - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{N} (\partial_\alpha u^\beta)^2 - \epsilon^2 \sum_{\alpha=1}^{3}\sum_{\beta=1}^{N} (\partial_\alpha v^\beta)^2 \\
- \;& 2\epsilon \sum_{\alpha=1}^{3}\sum_{\beta=1}^{N} (\partial_\alpha u^\beta)(\partial_\alpha v^\beta) + 2G(u+\epsilon v) - |u_t|^2 + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{N} (\partial_\alpha u^\beta)^2 - 2G(u)) d^3\overrightarrow{X} dt \\
= \;& \frac{1}{2} \int_{\mathbb{R}^4} (2u_t \cdot v_t - 2 \sum_{\alpha=1}^{3}\sum_{\beta=1}^{N} (\partial_\alpha u^\beta)(\partial_\alpha v^\beta) + 2 \lim_{\epsilon \to 0} \frac{G(u+\epsilon v) - G(u)}{\epsilon}) d^3\overrightarrow{X} dt \\
= \;& \int_{\mathbb{R}^4} (u_t \cdot v_t - \overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} v + \overrightarrow{g}(u) \cdot v) d^3\overrightarrow{X} dt
\end{aligned}$$

Integrating $u_t \cdot v_t$ and $\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} v$ by parts and assuming $v \longrightarrow 0$ as $\overrightarrow{X}^2 + t^2 \longrightarrow \infty$, gives

$$
\begin{aligned}
DL(u)(v) &= -\int_{\mathbb{R}^4} (u_{tt} \cdot v - \Delta u \cdot v - \overrightarrow{g}(u) \cdot v) d^3\overrightarrow{X} dt \\
&= -\int_{\mathbb{R}^4} (u_{tt} - \Delta u - \overrightarrow{g}(u)) \cdot v d^3\overrightarrow{X} dt
\end{aligned}
$$

So $DL(u) = 0 \Longleftrightarrow u_{tt} - \Delta u = \overrightarrow{g}(u)$.

For a function $u : \mathbb{R}^{3+1} \longrightarrow \mathbb{R}^N$, denote a counterclockwise rotation of u about an axis through the origin in $\mathbb{R}^3$ through an angle θ by $T_\theta u(\overrightarrow{X}, t)$ and define it to be

$$
T_\theta u(\overrightarrow{X}, t) \equiv u(R_\theta^{-1} \overrightarrow{X}, t)
$$

where R_θ^{-1} is a 3×3 (counterclockwise) rotation matrix. A matrix R is considered to be a rotation iff $R^T R = I$ and $\det R = 1$, and it can be shown that R is a rotation iff R can be written as the exponential of a skew-symmetric matrix. Let

$$
\beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

Then $\beta_1, \beta_2, \beta_3$ form a basis for 3×3 real skew-symmetric matrices and $e^{a\beta_1}, e^{a\beta_2}$,

$e^{a\beta_3}$ are counterclockwise rotations about respectively the x-axis, y-axis and z-axis by an angle a. In general, a rotation about a vector $ai + bj + ck$ is given by $e^{a\beta_1+b\beta_2+c\beta_3}$ where the angle of rotation is $\theta = \sqrt{a^2 + b^2 + c^3}$. Note that if $\hat{d}$ is the unit vector in the direction $ai + bj + ck$ and $\overrightarrow{\beta} = (\beta_1, \beta_2, \beta_3)$, then $e^{a\beta_1+b\beta_2+c\beta_2} = e^{\theta\hat{d}\cdot\overrightarrow{\beta}}$.

It is now shown that L is invariant under these rotations, i.e. $L(T_\theta u(\overrightarrow{X}, t)) = L(u(\overrightarrow{X}, t))\ \forall\theta\ \forall u$.

$$\begin{aligned} L(T_\theta u(\overrightarrow{X}, t)) &= L(u(R_\theta^{-1}\overrightarrow{X}, t)) \\ &= \int_{\mathbb{R}^4} (\frac{1}{2}|u_t(R_\theta^{-1}\overrightarrow{X}, t)|^2 - \frac{1}{2}|\overrightarrow{\nabla} u(R_\theta^{-1}\overrightarrow{X}, t)|^2 + G(u(R_\theta^{-1}\overrightarrow{X}, t)))d^3\overrightarrow{X}dt \end{aligned}$$

Let $\overrightarrow{Z} = R_\theta^{-1}\overrightarrow{X}$.Then

$$\begin{aligned} L(u(R_\theta^{-1}\overrightarrow{X}, t)) &= \int_{\mathbb{R}^4} (\frac{1}{2}|u_t(\overrightarrow{Z}, t)|^2 - \frac{1}{2}|\overrightarrow{\nabla} u(\overrightarrow{Z}, t)|^2 + G(u(\overrightarrow{Z}, t)))(\frac{1}{detR_\theta^{-1}})d^3\overrightarrow{Z}dt \\ &= \int_{\mathbb{R}^4} (\frac{1}{2}|u_t(\overrightarrow{X}, t)|^2 - \frac{1}{2}|\overrightarrow{\nabla} u(\overrightarrow{X}, t)|^2 + G(u(\overrightarrow{X}, t)))d^3\overrightarrow{X}dt \\ &= L(u(\overrightarrow{X}, t)) \end{aligned}$$

$\Rightarrow L(T_\theta u(\overrightarrow{X}, t)) = L(u(\overrightarrow{X}, t))$.

It will turn out that the conserved quantity $\overrightarrow{S}[u]$ involves in its integrand the quantity $Mu \equiv \frac{\partial}{\partial\theta}u(R_\theta^{-1}\overrightarrow{X}, t)\Big|_{\theta=0}$ so we now find this. If $\hat{d}$, $\overrightarrow{\beta}$ and$\overrightarrow{X}$ are as mentioned before, with $\hat{d} = (l, m, n), l, m, n \epsilon \mathbb{R}$, then

$$\begin{aligned}
\hat{d} \cdot \overrightarrow{\beta} &= (l, m, n) \cdot (\beta_1, \beta_2, \beta_3) \\
&= l \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + m \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + n \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow (\hat{d} \cdot \overrightarrow{\beta})\overrightarrow{X} = \begin{pmatrix} 0 & -n & m \\ n & 0 & -l \\ -m & l & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -ny + mz \\ nx - lz \\ -mx + ly \end{pmatrix}$$

$$= \hat{d} \times \overrightarrow{X}$$

Define

$$\begin{aligned}
Mu &\equiv \frac{\partial}{\partial \theta} u(R_\theta^{-1} \overrightarrow{X}, t) \Big|_{\theta=0} \\
&= \overrightarrow{\nabla} u(R_\theta^{-1} \overrightarrow{X}, t) \cdot \frac{\partial}{\partial \theta} (R_\theta^{-1} \overrightarrow{X}) \Big|_{\theta=0} \\
&= \overrightarrow{\nabla} u(e^{-\theta \hat{d} \cdot \overrightarrow{\beta}} \overrightarrow{X}, t) \cdot \frac{\partial}{\partial \theta} (e^{-\theta \hat{d} \cdot \overrightarrow{\beta}} \overrightarrow{X}) \Big|_{\theta=0}
\end{aligned}$$

$$
\begin{aligned}
&= -\overrightarrow{\nabla} u(e^{-\theta \hat{d}\cdot \overrightarrow{\beta}}\overrightarrow{X}, t) \cdot e^{-\theta \hat{d}\cdot \overrightarrow{\beta}}(\hat{d}\cdot \overrightarrow{\beta})\overrightarrow{X}\Big|_{\theta=0} \\
&= -\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d}\cdot \overrightarrow{\beta})\overrightarrow{X} \\
&= -\left(\frac{\partial u}{\partial x}(-ny+mz) + \frac{\partial u}{\partial y}(nx-lz) + \frac{\partial u}{\partial z}(-mx+ly)\right) \\
&= -\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d}\times \overrightarrow{X})
\end{aligned}
$$

So

$$
\begin{aligned}
& L(T_\theta u) &&= L(u) \quad \forall u \forall \theta \\
\implies \quad & \tfrac{\partial}{\partial \theta} L(T_\theta u) &&= 0 \quad \forall u \forall \theta \\
\implies \quad & DL(T_\theta u)(\tfrac{\partial}{\partial \theta} T_\theta u) &&= 0 \quad \forall u \forall \theta \\
\implies \quad & DL(u)(Mu) &&= 0 \quad \forall u
\end{aligned}
$$

$$
\iff \int_{\mathbb{R}^4} (u_{tt} - \Delta u - \overrightarrow{g}(u)) \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t)(\hat{d}\times \overrightarrow{X}))d\overrightarrow{X}^3 dt = 0 \quad \forall u,
$$

Note that all of the above is true regardless of whether or not u is a solution of NLKG. The fact that the integral vanishes independent of u suggests that the integrand has the form of a divergence. We will show that this integrand can be written as a 4-divergence, that is, $(u_{tt} - \Delta u - \overrightarrow{g}(u)) \cdot (u(\overrightarrow{X}, t)(\hat{d}\times \overrightarrow{X})) = \partial_t Q(u) + \overrightarrow{\nabla} \cdot \overrightarrow{P}(u)$ with Q and $\overrightarrow{P}$ to be determined. If there exists such a Q and $\overrightarrow{P}$ and if u is a solution to NLKG, then $\partial_t Q(u) + \overrightarrow{\nabla} \cdot \overrightarrow{P}(u) = 0$. Integrating over $\mathbb{R}^3$ and assuming that $\overrightarrow{P}(u)$ vanishes at spatial infinity gives $\partial_t \int_{\mathbb{R}^3} Q d\overrightarrow{X}^3 = 0$ which implies that $\int_{\mathbb{R}^3} Q d\overrightarrow{X}^3$ is a conserved quantity. It is the Noether invariant corresponding to the symmetry of L with respect to T_θ.

The integrand can be written as this divergence by working through it term by term. The term $u_{tt} \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \times \overrightarrow{X}))$ gives $\frac{\partial Q}{\partial t}$ and a part of the divergence.

$$\begin{aligned} u_{tt} \cdot u_x(-ny + mz) &= \frac{\partial}{\partial t}(u_t \cdot u_x(-ny + mz)) + u_t \cdot u_{xt} ny - u_t \cdot u_{xt} mz \\ &= \frac{\partial}{\partial t}(u_t \cdot u_x(-ny + mz)) + \frac{\partial}{\partial x}(\frac{1}{2} ny|u_t|^2) - \frac{\partial}{\partial x}(\frac{1}{2} mz|u_t|^2) \\ &= \frac{\partial}{\partial t}(u_t \cdot u_x(-ny + mz)) + \frac{\partial}{\partial x}(\frac{1}{2}(ny - mz)|u_t|^2) \end{aligned}$$

Similarly

$$u_{tt} \cdot u_y(nx - lz) = \frac{\partial}{\partial t}(u_t \cdot u_y(nx - lz)) + \frac{\partial}{\partial y}(\frac{1}{2}(-nx + lz)|u_t|^2)$$

and

$$u_{tt} \cdot u_z(-mx + ly) = \frac{\partial}{\partial t}(u_t \cdot u_z(-mx + ly)) + \frac{\partial}{\partial z}(\frac{1}{2}(mx - ly)|u_t|^2)$$

$$
\begin{aligned}
& \therefore u_{tt} \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \times \overrightarrow{X})) \\
= \; & \frac{\partial}{\partial t}(u_t \cdot (u_x(-ny + mz) + u_y(nx - lz) + u_z(-mx + ly))) \\
& + \frac{\partial}{\partial x}(\frac{1}{2}(ny - mz)|u_t|^2) + \frac{\partial}{\partial y}(\frac{1}{2}(-nx + lz)|u_t|^2) + \frac{\partial}{\partial z}(\frac{1}{2}(mx - ly)|u_t|^2) \\
= \; & \frac{\partial}{\partial t}(u_t \cdot (u_x(-ny + mz) + u_y(nx - lz) + u_z(-mx + ly))) \\
& + \overrightarrow{\nabla} \cdot \left(\frac{1}{2}|u_t|^2 \begin{pmatrix} -ny + mz \\ nx - lz \\ -mx + ly \end{pmatrix} \right) \\
= \; & \frac{\partial}{\partial t}(u_t \cdot (\overrightarrow{\nabla} u \cdot (\hat{d} \times \overrightarrow{X}))) + \overrightarrow{\nabla} \cdot (\frac{1}{2}|u_t|^2 (\hat{d} \times \overrightarrow{X})) \\
= \; & \frac{\partial}{\partial t}(u_t \cdot (\hat{d} \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u))) + \overrightarrow{\nabla} \cdot (\frac{1}{2}|u_t|^2 (\hat{d} \times \overrightarrow{X}))
\end{aligned}
$$

The remainder of the integrand, $(-\Delta u - \overrightarrow{g}(u)) \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \times \overrightarrow{X}))$, gives the rest of the divergence. So first we find $(-\Delta u) \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \times \overrightarrow{X}))$. Note that

$$\overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_x(-ny + mz))) = -\Delta u \cdot (u_x(-ny + mz)) - \overrightarrow{\nabla} u \cdot \overrightarrow{\nabla}(u_x(-ny + mz))$$

Which implies that

$$-\Delta u \cdot (u_x(-ny + mz))$$

$$
\begin{aligned}
&= \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_x(-ny+mz))) + \overrightarrow{\nabla} u \cdot \overrightarrow{\nabla}(u_x(-ny+mz)) \\
&= \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_x(-ny+mz))) + \frac{\partial}{\partial x}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(-ny+mz)) \\
&\quad +\overrightarrow{\nabla} u \cdot (u\overrightarrow{\nabla}(-ny+mz)) \\
&= \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_x(-ny+mz))) + \frac{\partial}{\partial x}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(-ny+mz))
\end{aligned}
$$

since $\overrightarrow{\nabla} u \cdot (u\overrightarrow{\nabla}(-ny+mz)) = 0$. Similarly

$$
-\Delta u \cdot (u_y(nx-lz)) = \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_y(nx-lz))) + \frac{\partial}{\partial y}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(nx-lz))
$$

and

$$
-\Delta u \cdot (u_z(-mx+ly)) = \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_z(-mx+ly))) + \frac{\partial}{\partial z}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(-mx+ly))
$$

So,

$$
\begin{aligned}
&-\Delta u \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \cdot \overrightarrow{\beta})\overrightarrow{X}) \\
= \;& \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (u_x(-ny+mz) + u_y(nx-ly) + u_z(-mx+ly)) \\
&+\frac{\partial}{\partial x}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(-ny+mz)) + \frac{\partial}{\partial y}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(nx-lz)) \\
&+\frac{\partial}{\partial z}(\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(-mx+ly))
\end{aligned}
$$

$$
\begin{aligned}
&= \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (\overrightarrow{\nabla} u \cdot (\hat{d} \cdot \overrightarrow{\beta})\overrightarrow{X}) + \overrightarrow{\nabla} \cdot (\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u \begin{pmatrix} -ny + mz \\ \\ nx - lz \\ \\ -mx + ly \end{pmatrix}) \\
&= \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (\overrightarrow{\nabla} u \cdot (d \times \overrightarrow{X})) + \overrightarrow{\nabla} \cdot (\frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(\hat{d} \times \overrightarrow{X}))
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& \overrightarrow{\nabla} \cdot (G(u)(\hat{d} \times \overrightarrow{X})) \\
= \; & \frac{\partial}{\partial x} G(u)(-ny + mz) + \frac{\partial}{\partial y} G(u)(nx - lz) + \frac{\partial}{\partial z} G(u)(-mx + ly) \\
= \; & \frac{1}{2}\frac{\partial}{\partial x} H(|u|^2)(-ny + mz) + \frac{1}{2}\frac{\partial}{\partial y} H(|u|^2)(nx - lz) + \frac{1}{2}\frac{\partial}{\partial z} H(|u|^2)(-mx + ly) \\
= \; & \frac{1}{2}(-ny + mz)h(|u|^2)\frac{\partial}{\partial x}|u|^2 + \frac{1}{2}(nx - lz)h(|u|^2)\frac{\partial}{\partial y}|u|^2 \\
& + \frac{1}{2}(-mx + ly)h(|u|^2)\frac{\partial}{\partial z}|u|^2 \\
= \; & \frac{1}{2}(-ny + mz)h(|u|^2)2u_x \cdot u + \frac{1}{2}(nx - lz)h(|u|^2)2u_y \cdot u \\
& + \frac{1}{2}(-mx + ly)h(|u|^2)2u_z \cdot u \\
= \; & (-ny + mz)\overrightarrow{g}(u) \cdot u_x + (nx - lz)\overrightarrow{g}(u) \cdot u_y + (-mx + ly)\overrightarrow{g}(u) \cdot u_z \\
= \; & \overrightarrow{g}(u) \cdot (\overrightarrow{\nabla} u \cdot (\hat{d} \times \overrightarrow{X}))
\end{aligned}
$$

Adding all of these results gives

$$
(u_{tt} - \Delta u - \overrightarrow{g}(u)) \cdot (\overrightarrow{\nabla} u(\overrightarrow{X}, t) \cdot (\hat{d} \times \overrightarrow{X}))\overrightarrow{X}
$$

$$
\begin{aligned}
= \; & \frac{\partial}{\partial t}u_t \cdot (\hat{d} \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u)) - \overrightarrow{\nabla} \cdot (\frac{1}{2}|u_t|^2(\hat{d} \times \overrightarrow{X})) + \overrightarrow{\nabla} \cdot (-\overrightarrow{\nabla} u \cdot (\overrightarrow{\nabla} u \cdot (\hat{d} \times \overrightarrow{X})) \\
& + \frac{1}{2}(\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u \cdot (\hat{d} \times \overrightarrow{X})) + \overrightarrow{\nabla} \cdot (G(u)(\hat{d} \times \overrightarrow{X})) \\
= \; & \frac{\partial}{\partial t}u_t \cdot (\hat{d} \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u)) + \overrightarrow{\nabla} \cdot (-\frac{1}{2}|u_t|(\hat{d} \times \overrightarrow{X}) - \overrightarrow{\nabla} u \cdot (\overrightarrow{\nabla} u \cdot (\hat{d} \times \overrightarrow{X})) \\
& + \frac{1}{2}\overrightarrow{\nabla} u \cdot \overrightarrow{\nabla} u(\hat{d} \times \overrightarrow{X}) + G(u)(\hat{d} \times \overrightarrow{X}))
\end{aligned}
$$

$$
\therefore Q = u_t \cdot (\hat{d} \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u))
$$

So the conserved quantity is

$$
\begin{aligned}
S[u] &= \int_{\mathbb{R}^3} u_t \cdot (\hat{d} \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u)) d^3\overrightarrow{X} \\
&= \hat{d} \cdot \int_{\mathbb{R}^3} u_t \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u)) d^3\overrightarrow{X}
\end{aligned}
$$

and we define the spin functional

$$
\overrightarrow{S}[u] \equiv \int_{\mathbb{R}^3} u_t \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u) d^3\overrightarrow{X}
$$

So if $\hat{e}$ is a unit vector in $\mathbb{R}^3$, we define the spin in the direction $\hat{e}$ as the conserved quantity $\hat{e} \cdot \overrightarrow{S}[u]$. Note that this functional provides the spin for *any* solution to NLKG. If it is assumed that all boundary terms vanish whenever an integration by parts is performed, it is relatively easy to verify by direct computation that $\frac{d\overrightarrow{S}}{dt} = 0$.

We will now look for standing-wave solutions of the form $u(\overrightarrow{X},t) = e^{t\Omega}v(\overrightarrow{X})$ where Ω is the $N \times N$ skew-symmetric matrix with the 2×2 blocks

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

along the main diagonal with $\omega \in \mathbb{R}$, and all other entries being zero. So $\Omega^2 = -\omega^2 I$, I being the $N \times N$ identity matrix. The function $v(\overrightarrow{X}) = \hat{\Psi}(\hat{X})w(r)$, where $w[0,\infty) \longrightarrow \mathbb{R}$, with $r = (x^2+y^2+z^2)^{\frac{1}{2}}$, and $\hat{\Psi} : \mathbb{R}^{3+1} \longrightarrow \mathbb{R}^N$ is a unit-vector-valued eigenfunction of the spherical Laplacian. The Laplacian Δ can be decomposed into radial and angular components:$\Delta = \Delta_R + \frac{1}{r^2}\Delta_S$ where $\Delta_R = \partial_r^2 + \frac{2}{r}\partial_r$. The spherical component, Δ_S, is a second-order derivative operator with only angular derivatives, that acts on real-valued functions defined on the unit sphere. It also acts on real-valued functions defined on $\mathbb{R}^3$, leaving the radial dependence unchanged. As shown in [4], there exist unit-vector-valued eigenfunctions of Δ_S in 3-space with any of the eigenvalues $\mu_l = -l(l+1)$ where $l = 0, 1, 2, , ...$ Given a nonnegative integer l, such an eigenfunction $\hat{\Psi} : S^2 \longrightarrow S^{N-1}$ with eigenvalue μ_l exists provided that $N \geq 2l+1$. So $\Delta_S\hat{\Psi} = -l(l+1)\hat{\Psi}$, where we extend the action of Δ_S to vector $\mathbb{R}^N$-valued functions by allowing the operator to act componentwise. Thus the coordinate functions of $\hat{\Psi}$ are eigenfunctions of Δ_S (see [2]). It is also shown in [4] that the variable $w : [0,\infty) \longrightarrow \mathbb{R}$

satifies the ordinary differential equation

$$\Delta_R w(r) - \frac{l(l+1)}{r^2} w(r) + g(w(r)) - \omega^2 w(r) = 0.$$

For nonlinearities g of appropriate type (see [4]), there exist solutions w that are exponentially decreasing far from the origin. Hence forth we assume g satisfies these conditions.

For $u(\overrightarrow{X}, t) = e^{t\Omega}\hat{\Psi}(\hat{X})w(r)$

$$\begin{aligned}
\overrightarrow{S}[u] &= \int_{\mathbb{R}^3} u_t \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} \Omega e^{t\Omega} w(r)\hat{\Psi}(\hat{X}) \cdot (\overrightarrow{X} \times \overrightarrow{\nabla})(e^{t\Omega} w(r)\hat{\Psi}(\hat{X})) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} \Omega e^{t\Omega} w(r)\hat{\Psi}(\hat{X}) \cdot e^{t\Omega}(\overrightarrow{X} \times \overrightarrow{\nabla})(w(r)\hat{\Psi}(\hat{X})) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} \Omega e^{t\Omega} e^{-t\Omega} w(r)\hat{\Psi}(\hat{X}) \cdot (\overrightarrow{X} \times \overrightarrow{\nabla})(w(r)\hat{\Psi}(\hat{X})) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} w(r)\Omega\hat{\Psi}(\hat{X}) \cdot [w(r)(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X}) + \hat{\Psi}(\hat{X})(\overrightarrow{X} \times \overrightarrow{\nabla})w(r)] d^3\overrightarrow{X}
\end{aligned}$$

Let $\hat{e}_x, \hat{e}_y, \hat{e}_z$, be the unit vectors along the x, y, and z axes respectively. Then since

$$\begin{aligned}
(\overrightarrow{X} \times \overrightarrow{\nabla})w(r) &= [(y\partial_z - z\partial_y)\hat{e}_x - (x\partial_z - z\partial_x)\hat{e}_y + (x\partial_y - y\partial_x)\hat{e}_z]w(r) \\
&= [yw'(r)\partial_z r - zw'(r)\partial_y r]\hat{e}_x - [xw'(r)\partial_z r - zw'(r)\partial_x r]\hat{e}_y
\end{aligned}$$

$$
\begin{aligned}
&+ \quad [xw'(r)\partial_y r - yw'(r)\partial_x r]\hat{e}_z \\
&= \quad w'(r)[(y\frac{z}{r} - z\frac{y}{r})\hat{e}_x - (x\frac{z}{r} - z\frac{x}{r}) + \hat{e}_y(x\frac{y}{r} - y\frac{x}{r})\hat{e}_z] \\
&= \quad 0
\end{aligned}
$$

we get

$$
\begin{aligned}
\overrightarrow{S}[u] &= \int_{\mathbb{R}^3} w(r)\Omega\hat{\Psi}(\hat{X}) \cdot [w(r)(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X})]d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} w^2(r)\Omega\hat{\Psi}(\hat{X}) \cdot [(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X})]d^3\overrightarrow{X}
\end{aligned}
$$

To illustrate, let us take $l = 1$. Then a corresponding eigenfunction of Δ_S is

$$
\hat{\Psi}(\hat{X}) \equiv \frac{1}{r}\begin{pmatrix} x \\ y \\ z \\ 0 \\ \vdots \end{pmatrix}
$$

where the dots represent the other $N - 4$ entries which are all zero. To evaluate the spin with this choice of $\hat{\Psi}$ we first need to find $\Omega\hat{\Psi}(\hat{X})$ and $(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X})$.

$$\Omega\hat{\Psi}(\hat{X}) = \frac{1}{r}\begin{pmatrix} 0 & -\omega & 0 & 0 & \cdots \\ \omega & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\omega & \cdots \\ 0 & 0 & \omega & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 0 \\ \vdots \end{pmatrix} = \frac{\omega}{r}\begin{pmatrix} -y \\ x \\ 0 \\ z \\ \vdots \end{pmatrix}$$

$$\begin{aligned} (\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X}) &= \frac{1}{r}[(y\partial_z - z\partial_y)\hat{e}_x - (x\partial_z - z\partial_x)\hat{e}_y - (x\partial_y - y\partial_x)\hat{e}_z]\begin{pmatrix} x \\ y \\ z \\ 0 \\ \vdots \end{pmatrix} \\ &= \frac{1}{r}\begin{pmatrix} z\hat{e}_y - y\hat{e}_z \\ -z\hat{e}_x + x\hat{e}_z \\ y\hat{e}_x - x\hat{e}_y \\ 0 \\ \vdots \end{pmatrix} \end{aligned}$$

And so

$$\Omega\hat{\Psi}(\hat{X})\cdot[(\overrightarrow{X}\times\overrightarrow{\nabla})\hat{\Psi}(\hat{X})] = \frac{\omega}{r^2}\begin{pmatrix} -y \\ x \\ 0 \\ z \\ \vdots \end{pmatrix}\cdot\begin{pmatrix} z\hat{e}_y - y\hat{e}_z \\ -z\hat{e}_x + x\hat{e}_z \\ y\hat{e}_x - x\hat{e}_y \\ 0 \\ \vdots \end{pmatrix}$$

$$\begin{aligned} &= \frac{\omega}{r^2}(-yz\hat{e}_y + y^2\hat{e}_z - xz\hat{e}_x + x^2\hat{e}_z) \\ &= \frac{\omega}{r^2}((x^2+y^2)\hat{e}_z - xz\hat{e}_x - yz\hat{e}_y) \end{aligned}$$

Note that since the functions $-xz$ and $-yz$ are odd, their integrals over $\mathbb{R}^3$ vanish, which implies

$$\begin{aligned} \overrightarrow{S}[u] &= \omega\int_{\mathbb{R}^3}\frac{w^2(r)}{r^2}((x^2+y^2)\hat{e}_z - xz\hat{e}_x - yz\hat{e}_y)d^3\overrightarrow{X} \\ &= \hat{e}_z\omega\int_{\mathbb{R}^3}\frac{w^2(r)}{r^2}(x^2+y^2)d^3\overrightarrow{X} \\ &> 0 \end{aligned}$$

if $\omega > 0$. That there is a nonzero spin in the $\hat{e}_z$ direction. If the solution $u(\overrightarrow{X},t) = e^{t\Omega}\hat{\Psi}(\hat{X})w(r)$ is rotated counter clockwise by 90^o about the y-axis, the solution

$u(\overrightarrow{X}, t) = e^{t\Omega}\hat{\Psi}(C\hat{X})w(r)$ results where

$$C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Since NLKG is invariant under rotations, this is also a solution. To find the spin we find $\Omega\hat{\Psi}(C\hat{X})$ and $(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(C\hat{X})$ where

$$C\hat{X} = \frac{1}{r}\begin{pmatrix} -z \\ y \\ x \end{pmatrix}$$

and

$$\hat{\Psi}(C\hat{X}) = \frac{1}{r}\begin{pmatrix} -z \\ y \\ x \\ 0 \\ \vdots \end{pmatrix}$$

So

$$\Omega\hat{\Psi}(C\hat{X}) = \frac{\omega}{r}\begin{pmatrix} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} -z \\ y \\ x \\ 0 \\ \vdots \end{pmatrix} = \frac{\omega}{r}\begin{pmatrix} -y \\ -z \\ 0 \\ x \\ \vdots \end{pmatrix}$$

$$\begin{aligned}(\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(C\hat{X}) &= \frac{1}{r}[(y\partial_z - z\partial_y)\hat{e}_x - (x\partial_z - z\partial_x)\hat{e}_y + (x\partial_y - y\partial_x)\hat{e}_z]\begin{pmatrix} -z \\ y \\ x \\ 0 \\ \vdots \end{pmatrix} \\ &= \frac{1}{r}\begin{pmatrix} -y\hat{e}_x + x\hat{e}_y \\ -z\hat{e}_x + x\hat{e}_z \\ z\hat{e}_y - y\hat{e}_z \\ 0 \\ \vdots \end{pmatrix}\end{aligned}$$

And so

$$\begin{aligned}
\Omega\hat{\Psi}(C\hat{X}) \cdot [(\overrightarrow{X} \times \overrightarrow{\nabla})C\hat{\Psi}(\hat{X})] &= \frac{\omega}{r^2} \begin{pmatrix} -y \\ -z \\ 0 \\ x \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} -y\hat{e}_x + x\hat{e}_2 \\ -z\hat{e}_x + x\hat{e}_z \\ z\hat{e}_y - y\hat{e}_z \\ 0 \\ \vdots \end{pmatrix} \\
&= \frac{\omega}{r^2}(y^2\hat{e}_x - xy\hat{e}_y + z^2\hat{e}_x - xz\hat{e}_z) \\
&= \frac{\omega}{r^2}((y^2 + z^2)\hat{e}_x - xy\hat{e}_y - xz\hat{e}_z)
\end{aligned}$$

Note that since the functions $-xz$ and $-yz$ are odd, their integrals over $\mathbb{R}^3$ vanish. Thus, if $\omega > 0$,

$$\begin{aligned}
\overrightarrow{S}[u] &= \omega \int_{\mathbb{R}^3} \frac{w^2(r)}{r^2}((y^2 + z^2)\hat{e}_x - xy\hat{e}_y - xz\hat{e}_z)d^3\overrightarrow{X} \\
&= \hat{e}_x\omega \int_{\mathbb{R}^3} \frac{w^2(r)}{r^2}(y^2 + z^2)d^3\overrightarrow{X} \\
&> 0
\end{aligned}$$

giving nonzero spin in the $\hat{e}_x$ direction. Similarly by taking $u(\overrightarrow{X}, t) = e^{t\Omega}\hat{\Psi}(D\hat{X})w(r)$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

we get a solution with nonzero spin in the $\hat{e}_y$ direction.

By appropriate rotations, a solution to NLKG can be manufactured with a nonzero spin in *any* direction.

CHAPTER 3

The Spin Direction Of Stationary Solutions In A Uniform Magnetic Field

We now introduce an external uniform magnetic field. This will change the equation of interest. However, it will be demonstrated that if this field is assumed to be weak, some of the same stationary waves can be solutions to an equation which is an approximation to the new one. The spin direction will be found to be either parallel or antiparallel to the field. It should be noted here that in [6] the existence of complex-valued solutions have been shown for a class of nonlinear Schrodinger equations (with an external magnetic field) with the B^2 term present (which we drop).

By making the minimal-coupling substitutions $\partial^\alpha \longmapsto \partial^\alpha - \sigma A^\alpha$ and $\partial_\alpha \longmapsto \partial_\alpha - \sigma A_\alpha$ into NLKG we bring about

$$(\partial^\alpha - \sigma A^\alpha)(\partial_\alpha - \sigma A_\alpha)u = \overrightarrow{g}(u) \tag{3.1}$$

which introduces the external uniform magnetic field, $\overrightarrow{B}$. The vector potential

$$\overrightarrow{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix}$$

is chosen so that $\overrightarrow{B} = B\hat{z}$, the function $\overrightarrow{g}(u)$ is defined as before and σ is the $N \times N$ real skew-symmetric matrix with the 2×2 blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

along the main diagonal and zeros everywhere else. Expansion of (3.1) gives

$$\partial^\alpha \partial_\alpha u - 2\sigma A^\alpha \partial_\alpha u - A^\alpha A_\alpha u = \overrightarrow{g}(u)$$

since the term $\partial^\alpha A_\alpha = \frac{B}{2}(-\partial_x y + \partial_y x) = 0$. It will be assumed that $B \ll 1$ and since the solutions of interest have exponential decay far from the origin, the term $A^\alpha A_\alpha = -\frac{1}{4}B^2(x^2 + y^2)$ can be ignored. So the equation under study becomes

$$\partial^\alpha \partial_\alpha u - 2\sigma A^\alpha \partial_\alpha u = \overrightarrow{g}(u)$$

or

$$u_{tt} - \Delta u - B\sigma(x\partial_y - y\partial_x)u = \overrightarrow{g}(u) \tag{3.2}$$

We now look for stationary solutions, $u(\overrightarrow{X}, t) = e^{t\Omega}v(\overrightarrow{X})$, to (3.2) where Ω is an $N \times N$ skew-symmetric matrix and we will require that Ω commutes with σ. Substituting this form into (3.2) yields

$$\begin{aligned}
\Omega^2 e^{t\Omega} v(\overrightarrow{X}) - e^{t\Omega}\Delta v(\overrightarrow{X}) - 2e^{t\Omega}\sigma A^{\alpha}\partial_{\alpha} v(\overrightarrow{X}) &= e^{t\Omega}\overrightarrow{g}(v(\overrightarrow{X})) \\
\Rightarrow \Omega^2 v(\overrightarrow{X}) - \Delta v(\overrightarrow{X}) - 2\sigma A^{\alpha}\partial_{\alpha} v(\overrightarrow{X}) &= \overrightarrow{g}(v(\overrightarrow{X}))
\end{aligned}$$

Next, we look for solutions where $v(\overrightarrow{X}) = \hat{\Psi}(\hat{X})w(r)$. Then

$$\begin{aligned}
\Delta(\hat{\Psi}(\hat{X})w(r)) &= (\Delta_R + \frac{1}{r^2}\Delta_S)(\hat{\Psi}(\hat{X})w(r)) \\
&= \Delta_R((\hat{\Psi}(\hat{X})w(r)) + \frac{1}{r^2}\Delta_S(\hat{\Psi}(\hat{X})w(r)) \\
&= \hat{\Psi}(\hat{X})\Delta_R w(r) + \frac{1}{r^2}\Delta_S(\hat{\Psi}(\hat{X}))w(r) \\
&= \hat{\Psi}(\hat{X})\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}w(r)\left(-l(l+1)\hat{\Psi}(\hat{X})\right) \\
&= \left(w''(r) + \frac{2}{r}w'(r) - \frac{l(l+1)}{r^2}w(r)\right)\hat{\Psi}(\hat{X})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & \Omega^2\hat{\Psi}(\hat{X})w(r) - \left(\Delta_R w(r) - \frac{l(l+1)}{r^2}w(r) + g(w(r))\right)\hat{\Psi}(\hat{X}) \\
& -B\sigma(x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})w(r) = 0 \\
\Rightarrow\ & [\Omega^2\hat{\Psi}(\hat{X}) - B\sigma(x\partial_y - y\partial_x)\hat{\Psi}(\hat{X})]w(r) \\
& -\left(\Delta_R w(r) - \frac{l(l+1)}{r^2}w(r) + g(w(r))\right)\hat{\Psi}(\hat{X}) = 0
\end{aligned}$$

It will be shown that under certain conditions the equation

$$\Omega^2\hat{\Psi}(\hat{X}) - B\sigma L_z\hat{\Psi}(\hat{X}) = \eta\hat{\Psi}(\hat{X}) \tag{3.3}$$

results where η is some constant. This makes the ansatz result in a solution by allowing $\hat{\Psi}$ to factor out giving

$$\begin{aligned}
\left(\Delta_R w(r) - \frac{l(l+1)}{r^2}w(r) + g(w(r)) - \eta w(r)\right)\hat{\Psi}(\hat{X}) &= 0 \\
\Rightarrow w''(r) + \frac{2}{r}w'(r) - \frac{l(l+1)}{r^2}w(r) + g(w(r)) - \eta w(r) &= 0
\end{aligned}$$

The solutions of this radial equation are studied in [3] and [4]. Equation (3.3) will be achieved by expanding $\hat{\Psi}$ into terms involving an orthonormal basis of $\mathbb{C}^N$ and eigenfunctions of the spherical Laplacian.

The spin components of a solution u along the direction of the positive x and y

axes are:

$$\begin{aligned}\overrightarrow{S}_x[u] = \hat{e}_x \cdot \overrightarrow{S}[u] &= \int_{\mathbb{R}^3} w^2(r)\Omega\hat{\Psi}(\hat{X}) \cdot [\hat{e}_x \cdot (\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X})] d^3\overrightarrow{X} \\ &= \int_{\mathbb{R}^3} w^2(r)\Omega\hat{\Psi}(\hat{X}) \cdot [(y\partial_z - z\partial_y)\hat{\Psi}(\hat{X})] d^3\overrightarrow{X}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{S}_y[u] = \hat{e}_y \cdot \overrightarrow{S}[u] &= \int_{\mathbb{R}^3} w^2(r)\Omega\hat{\Psi}(\hat{X}) \cdot [\hat{e}_y \cdot (\overrightarrow{X} \times \overrightarrow{\nabla})\hat{\Psi}(\hat{X})] d^3\overrightarrow{X} \\ &= \int_{\mathbb{R}^3} w^2(r)\Omega\hat{\Psi}(\hat{X}) \cdot [(z\partial_x - x\partial_z)\hat{\Psi}(\hat{X})] d^3\overrightarrow{X}\end{aligned}$$

In order to evaluate these spin components, the action of $y\partial_z - z\partial_y$ and $z\partial_x - x\partial_z$ on $\hat{\Psi}$ must be found. As mentioned earlier, we will expand $\hat{\Psi}$ into terms involving an orthonormal basis of $\mathbb{C}^N$ and eigenfunctions of the spherical Laplacian, which will also allow us to compute these actions. It is then shown that $S_x + iS_y$ may be nonzero only when $\overrightarrow{B} = 0$, that is S_x or S_y may be nonzero only when $\overrightarrow{B} = 0$

As pointed earlier, each component of $\hat{\Psi}$ is an eigenfunction of Δ_S with the same eigenvalue, and the possible eigenvalues of Δ_S in 3-space are $\mu_l = -l(l+1)$ where $l = 0, 1, 2, ...$ (see [2]). For each value of l, there are $2l + 1$ linearly independent eigenfunctions of Δ_S each with the common eigenvalue μ_l (see [2]). Let

$$\overrightarrow{L} \equiv \begin{pmatrix} y\partial_z - z\partial_y \\ -x\partial_z + z\partial_x \\ x\partial_y - y\partial_x \end{pmatrix} = \overrightarrow{X} \times \overrightarrow{\nabla}$$

Then $\Delta_S = \overrightarrow{L}^2 = \overrightarrow{L} \cdot \overrightarrow{L}$. Let $L_x = y\partial_z - z\partial_y, L_y = -x\partial_z + z\partial_x$ and $L_z = x\partial_y - y\partial_x$

Then

$$\begin{aligned} L_x L_y &= -xy\partial_z^2 + xz\partial_y\partial_z + y\partial_x + yz\partial_z\partial_x - z^2\partial_y\partial_x \\ L_y L_x &= -xy\partial_z^2 + yz\partial_x\partial_z + x\partial_y + xz\partial_z\partial_y - z^2\partial_y\partial_x \end{aligned}$$

$$\begin{aligned} \Rightarrow [L_x, L_y] &= L_x L_y - L_y L_x \\ &= xz\partial_y\partial_z + y\partial_x - x\partial_y - xz\partial_z\partial_y \\ &= L_z \end{aligned}$$

Similarly $[L_y, L_z] = L_x$ and $[L_z, L_x] = L_y$. If A, B and C are linear operators then $[A, BC] = [A, B]C + B[A, C]$. Using this property of commutators gives

$$
\begin{aligned}
[L_z, L_x^2] &= [L_z, L_x]L_x + L_x[L_z, L_x] \\
&= L_y L_x + L_x L_y
\end{aligned}
$$

$$
\begin{aligned}
[L_z, L_y^2] &= [L_z, L_y]L_y + L_y[L_z, L_y] \\
&= -L_x L_y - L_y L_x
\end{aligned}
$$

and

$$
\begin{aligned}
[L_z, L_z^2] &= [L_z, L_z]L_z + L_z[L_z, L_z] \\
&= 0
\end{aligned}
$$

$\Rightarrow [L_z, \overrightarrow{L}^2] = 0$. So $L^2 = \Delta_S$ and L_z have a common set of orthonormal eigenfunctions

$$\xi_{-l}, ..., \xi_{-1}, \xi_0, \xi_1, ..., \xi_l$$

and so we can write $\Delta_S \xi_m = -l(l+1)\xi_m$. The corresponding $2l+1$ eigenvalues of L_z

are

$$-il, -i(l+1), ..., -i, 0, i, ..., i(l-1), il$$

as shown in [7]. Note that since $L_z\xi_m = im\xi_m$ implies $L_z\bar{\xi}_m = -im\bar{\xi}_m$, we can take $\xi_{-m} = \bar{\xi}_m$. This collection of eigenfunctions span the space of all eigenfunctions of Δ_S with eigenvalue $-l(l+1)$. So there exists $\overrightarrow{\alpha}_m \in \mathbb{C}^N$, $-l \leq m \leq l$, such that $\hat{\Psi}(\hat{X}) = \sum_{m=-l}^{l} \overrightarrow{\alpha}_m \xi_m(\hat{X})$.

Since Ω and σ are both real skew-symmetric matrices, then they are both normal, i.e., $\Omega^*\Omega = \Omega\Omega^*$ and $\sigma^*\sigma = \sigma\sigma^*$, where $*$ is the conjugate-transpose operation. By hypothesis, Ω and σ commute. So there exists an orthonormal basis of $\mathbb{C}^N$ consisting of vectors which are eigenvectors for both Ω and σ (see [8]). Let $\phi_1, \phi_2, ..., \phi_n$ be this collection of eigenvectors. The eigenvalues of σ are $\pm i$ and so $\sigma\phi_j = (-1)^j i\phi_j$, and $\Omega\phi_j = \lambda_j\phi_j$, $1 \leq j \leq N$, where λ_j is the (complex) eigenvalue corresponding to ϕ_j. It follows that the λ_j are pure imaginary since

$$\begin{aligned} 0 \leq \|\Omega\phi_j\|^2 &= \langle\Omega\phi_j|\Omega\phi_j\rangle \\ &= \langle\phi_j|\Omega^*\Omega\phi_j\rangle \\ &= -\langle\phi_j|\Omega^2\phi_j\rangle \\ &= -\langle\phi_j|\lambda_j^2\phi_j\rangle \\ &= -\lambda_j^2\|\phi\|^2 \end{aligned}$$

$$\Rightarrow \lambda_j^2 \leq 0$$

i.e. λ_j is pure imaginary, and so let $\lambda_j = i\nu_j$ where $\nu_j \in \mathbb{R}$. So if $i\nu_n, i\nu_m$ are distinct eigenvalues of Ω, then $\langle \phi_n, \phi_m \rangle = 0$, as the following proof shows:

$$\begin{aligned} \Omega\phi_n &= i\nu_n\phi_n \\ \Rightarrow \langle \Omega\phi_n, \phi_m \rangle &= \langle i\nu_n\phi_n, \phi_m \rangle \\ \Rightarrow -\langle \phi_n, \Omega\phi_m \rangle &= i\nu_n \langle \phi_n, \phi_m \rangle \end{aligned}$$

On the other hand $\Omega\phi_m = i\nu_m\phi_m$ implies

$$\begin{aligned} \langle \phi_n, \Omega\phi_m \rangle &= \langle \phi_n, i\nu_m\phi_m \rangle \\ \Rightarrow \langle \phi_n, \Omega\phi_m \rangle &= \overline{i\nu}_m \langle \phi_n, \phi_m \rangle \\ \Rightarrow \langle \phi_n, \Omega\phi_m \rangle &= -i\nu_m \langle \phi_n, \phi_m \rangle \end{aligned}$$

Adding the last two results

$$\begin{aligned} i\nu_n \langle \phi_n, \phi_m \rangle - i\nu_m \langle \phi_n, \phi_m \rangle &= 0 \\ \Rightarrow i(\nu_n - \nu_m) \langle \phi_n, \phi_m \rangle &= 0 \end{aligned}$$

$$\Rightarrow \langle \phi_n, \phi_m \rangle = 0$$

Since $\Omega\phi_j = i\nu_j\phi_j \Rightarrow \Omega\overline{\phi}_j = -i\nu_j\overline{\phi}_j$, we can take $\phi_2 = \overline{\phi}_1, \phi_4 = \overline{\phi}_3, ..., \phi_N = \overline{\phi}_{N-1}$. Let us call $\hat{j}$ the other index of the pair, that is let $\hat{n} = n+1$ if n is odd, and $\hat{n} = n-1$ if n is even. Hence $\{\phi_1, \phi_2, ..., \phi_N\}$ can be chosen to be an orthonormal basis of $\mathbb{C}^N$ so that $\overline{\phi}_n = \phi_{\hat{n}}$.

We can now write $\hat{\Psi}(\hat{X}) = \sum_{m=-l}^{l} \alpha_m \xi_m(\hat{X}) = \sum_{m=-l}^{l} \sum_{j=1}^{N} \alpha_{jm}\phi_j\xi_m(\hat{X}), \;\; \alpha_{jm} \in \mathbb{C}$. Also

$$
\begin{aligned}
\overline{\hat{\Psi}}(\hat{X}) &= \sum_{m=-l}^{l}\sum_{j=1}^{N} \overline{\alpha}_{jm}\overline{\phi}_j\overline{\xi}_m(\hat{X}) \\
&= \sum_{m=-l}^{l}\sum_{j=1}^{N} \overline{\alpha}_{\hat{j}(-m)}\phi_j\xi_m(\hat{X})
\end{aligned}
$$

Since $\hat{\Psi}$ is real, we require $\overline{\alpha}_{\hat{j}(-m)} = \alpha_{jm}$. Putting this expansion of $\hat{\Psi}(\hat{X})$ into (3.3) gives

$$
\sum_{m=-l}^{l}\sum_{j=1}^{N}[\alpha_{jm}(-\nu_j^2\phi_j\xi_m - B(-1)^j i\phi_j im\xi_m - \eta\phi_j\xi_m)] = 0
$$

Thus for every j and m, either $\alpha_{jm} = 0$ or $-\nu_j^2 + (-1)^j mB - \eta = 0$

Define $L_+ = L_x + iL_y$. We know $L_+\xi_m = i\sqrt{l(l+1) - m(m+1)}\;\xi_{m+1}$ with $L_+\xi_m = 0$ if $m = l$ (see [7]). Using this definition, the direction of spin is found by making the necessary substitutions into the quantity $S_x + iS_y$:

$S_x + iS_y$

$$
\begin{aligned}
&= \hat{e}_x \int_{\mathbb{R}^3} u_t \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u) d^3\overrightarrow{X} + i\hat{e}_y \int_{\mathbb{R}^3} u_t \cdot (\overrightarrow{X} \times \overrightarrow{\nabla} u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} u_t \cdot (L_x u) d^3\overrightarrow{X} + i \int_{\mathbb{R}^3} u_t \cdot (L_y u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} u_t \cdot (L_x + iL_y) u d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} u_t \cdot (L_+ u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} w^2 \Omega \overline{\hat{\Psi}} \cdot (L_+ \hat{\Psi}) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} w^2 \sum_{n=-l}^{l} \sum_{k=1}^{N} \overline{i\nu_k}\,\overline{\alpha}_{kn} \overline{\phi}_k \overline{\xi}_n \cdot \sum_{m=-l}^{l} \sum_{j=1}^{N} \alpha_{jm} \xi_{m+1} \phi_j i \sqrt{l(l+1) - m(m+1)}\ d^3\overrightarrow{X} \\
&= \sum_{n=-l}^{l} \sum_{k=1}^{N} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \int_{\mathbb{R}^3} w^2 (-i\nu_k) \overline{\alpha}_{kn} \phi_{\hat{k}} \xi_{-n} \cdot \alpha_{jm} \xi_{m+1} \phi_j i \sqrt{l(l+1) - m(m+1)}\ d^3\overrightarrow{X} \\
&= \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \int_{\mathbb{R}^3} w^2 \nu_{\hat{j}} \overline{\alpha}_{\hat{j}n} \phi_j \xi_{-n} \cdot \alpha_{jm} \xi_{m+1} \phi_j \sqrt{l(l+1) - m(m+1)}\ d^3\overrightarrow{X} \\
&= \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \nu_{\hat{j}} \overline{\alpha}_{\hat{j}n} \alpha_{jm} \sqrt{l(l+1) - m(m+1)} \int_{\mathbb{R}^3} w^2 \overline{\xi}_n \xi_{m+1} d^3\overrightarrow{X} \\
&= \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \nu_{\hat{j}} \alpha_{j(-n)} \alpha_{jm} \sqrt{l(l+1) - m(m+1)} \int_{\mathbb{R}^3} w^2 \xi_{-n} \xi_{m+1} d^3\overrightarrow{X} \\
&= \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \nu_{\hat{j}} \alpha_{jn} \alpha_{jm} \sqrt{l(l+1) - m(m+1)}\ \delta_{n(m+1)} \int_0^\infty r^2 w^2(r) dr \\
&= \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \nu_{\hat{j}} \alpha_{j(m+1)} \alpha_{jm} \sqrt{l(l+1) - m(m+1)} \int_0^\infty r^2 w^2(r) dr
\end{aligned}
$$

Suppose that for particular values of m and j, $\alpha_{j(m+1)} \neq 0$ and $\alpha_{jm} \neq 0$. Then $-\nu_j^2 = \eta - (-1)^j mB$ and $-\nu_j^2 = \eta - (-1)^j (m+1)B$ which implies $B = 0$. Thus if $B \neq 0$ then $S_x + iS_y = 0$. So when $B \neq 0$, the only allowable directions for nonzero spin are along the magnetic field lines.

We now provide a simple example showing nonzero spin in a direction parallel to the $\overrightarrow{B}$ field, i.e. parallel to the z-axis. Let $u(\overrightarrow{X}, t) = e^{t\Omega}\hat{\Psi}(\hat{X})w(r)$ with $u : \mathbb{R}^{3+1} \longrightarrow \mathbb{R}^4$ and where Ω is a 4×4 skew-symmetric matrix of the form

$$\begin{pmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\hat{\Psi} = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

We will subtitute the above form of Ω in (3.3) and it will be seen that we will be able to find ω_2 in terms of ω_1. So first find each of the terms in (3.3).

$$\Omega^2 = \begin{pmatrix} -\omega_1^2 & 0 & 0 & 0 \\ 0 & -\omega_1^2 & 0 & 0 \\ 0 & 0 & -\omega_2^2 & 0 \\ 0 & 0 & 0 & -\omega_2^2 \end{pmatrix}$$

So

$$\begin{aligned} \Omega^2 \hat{\Psi}(\hat{X}) &= \frac{1}{r} \begin{pmatrix} -\omega_1^2 & 0 & 0 & 0 \\ 0 & -\omega_1^2 & 0 & 0 \\ 0 & 0 & -\omega_2^2 & 0 \\ 0 & 0 & 0 & -\omega_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\omega_1^2 x \\ -\omega_1^2 y \\ -\omega_2^2 z \\ 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
L_z\hat{\Psi}(\hat{X}) &= \frac{1}{r}(x\partial_y - y\partial_x)\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \\
&= \frac{1}{r}\begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

And so

$$
\begin{aligned}
\sigma L_z\hat{\Psi}(\hat{X}) &= \frac{1}{r}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{r}\begin{pmatrix} -x \\ -y \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Now, substituting into (3.3)

$$\begin{pmatrix} -\omega_1^2 x \\ -\omega_1^2 y \\ -\omega_2^2 z \\ 0 \end{pmatrix} + \begin{pmatrix} Bx \\ By \\ 0 \\ 0 \end{pmatrix} = \eta \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} (-\omega_1^2 + B)x \\ (-\omega_1^2 + B)y \\ -\omega_2^2 z \\ 0 \end{pmatrix} = \eta \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

This gives us that $\omega_2^2 = \omega_1^2 - B$ which implies $\omega_2 = \pm\sqrt{\omega_1 - B}$ and therefore

$$\Omega = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\omega^2 - B} \\ 0 & 0 & \sqrt{\omega^2 - B} & 0 \end{pmatrix}$$

Using this form of Ω it is now shown that a solution of the form $u(\overrightarrow{X}, t) = e^{t\Omega}\hat{\Psi}(\hat{X})w(r)$ has nonzero spin parallel to the direction of the magnetic field. The component of spin parallel to $\overrightarrow{B}$ is

$$
\begin{aligned}
S_z[u] &= \overrightarrow{S}[u] \cdot \hat{e}_3 \\
&= \int_{\mathbb{R}^3} w^2 \Omega \hat{\Psi}(\hat{X}) \cdot L_z \hat{\Psi}(\hat{X}) d^3 \overrightarrow{X}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\Omega \hat{\Psi}(\hat{X}) \cdot L_z \hat{\Psi}(\hat{X}) &= \frac{1}{r^2} \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\omega^2 - B} \\ 0 & 0 & \sqrt{\omega^2 - B} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \cdot L_z \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \\
&= \frac{1}{r^2} \begin{pmatrix} -\omega y \\ \omega x \\ 0 \\ (\sqrt{\omega^2 - B}) z \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{\omega}{r^2}(y^2 + x^2) \\
&\geq 0
\end{aligned}
$$

if $\omega > 0$. So

$$S_z[u] \;\; = \;\; \int_{\mathbb{R}^3} \frac{w^2}{r^2}\omega(x^2+y^2)d^3\overrightarrow{X}$$

which is nonzero if $\omega \neq 0$.

Note that since $S_z[u]$ does not involve B, we could have set $B = 0$ at the begining of this example and we would then have a solution with a nonzero spin parallel to the z-axis in the absence of a magnetic field.

CHAPTER 4

Precessing Solutions

The solutions examined so far have their axis of rotation parallel to the uniform magnetic field. In order to generalize the results, we now investigate possible precessing solutions in the uniform magnetic field. It will be seen that precessing solutions to 3.2 of the same ansatz as before will have spin parallel or antiparallel to the magnetic field direction. This implies that there are no precessing solutions, since the spin does not lie along the axis of rotation.

Let $u(\overrightarrow{X},t) = e^{\Omega t}\hat{\Psi}(\hat{X})w(r)$be a solution to 3.2. A precessing solitary wave is obtained by first tilting one of the form $u(\overrightarrow{X},t) = e^{\Omega t}\hat{\Psi}(\hat{X})w(r)$. It can be tilted by rotating about say the y-axis, through some angle θ. This is achieved by multiplying the $\hat{X}$ in the argument of $\hat{\Psi}$ by the rotation matrix

$$T_\theta^{-1} = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}$$

The argument of $\hat{\Psi}$ then becomes

$$T_\theta^{-1}\hat{X} = \frac{1}{r}\begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{r}\begin{pmatrix} xcos\theta + zsin\theta \\ y \\ -xsin\theta + zcos\theta \end{pmatrix}$$

Since NLKG is invariant under rotations, if $e^{\Omega t}\hat{\Psi}(\hat{X})w(r)$ is a solution to 3.2, then so is $e^{\Omega t}\hat{\Psi}(T_\theta^{-1}\hat{X})w(r)$, that is, tilting a solitary wave solution of this form only results in another of the same form. Thus to examine possible precessing solutions, we need only set a solitary wave of the type $u(\overrightarrow{X}, t) = e^{\Omega t}\hat{\Psi}(\hat{X})w(r)$ precessing about the z-axis. This is done by multiplying the argument of $\hat{\Psi}$ by the 3×3 rotation matrix

$$\begin{aligned} R_{\mu t}^{-1} &= \begin{pmatrix} cos\mu t & sin\mu t & 0 \\ -sin\mu t & cos\mu t & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{-\mu t M} \end{aligned}$$

where

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and μ is the rate of precession. So precessing solitary waves take on the form $u(\overrightarrow{X}, t) = e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)$. We now proceed as in chapter 3. We wish to see if there are any precessing solutions to 3.2 with the above ansatz and then to find their direction of spin. Many of the things that will be need were discussed in chapter 3 and so they are stated here again.

We are looking for solutions to $u_{tt} - \Delta u - B\sigma(x\partial_y - y\partial_x)u = \overrightarrow{g}(u)$ (i.e. we are assuming $B \ll 1$) which have the form $u(\overrightarrow{X}, t) = e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)$. The matrix $R_{\mu t}^{-1}$ is as mentioned above, Ω is an $N \times N$ skew-symmetric matrix and as before, we require it to commute with σ. $\hat{\Psi}$ is the unit-vector-valued eigenfunction of the spherical Laplacian with eigenvalues $\mu_l = -l(l+1)$ where $l = 0, 1, 2, \ldots.$ The variable $w[0, \infty) \longrightarrow \mathbb{R}$ is exponentially decreasing far from the origin and satifies the ordinary differential equation

$$\Delta_R w(r) - \frac{l(l+1)}{r^2}w(r) + g(w(r)) - \eta w(r) = 0 \tag{4.1}$$

Recall that Δ_S and L_z have a common orthonormal set of eigenfunctions $\xi_{-l}, ..., \xi_{-1}, \xi_0, \xi_1, ...\xi_l$ where $\Delta_S \xi_m = -l(l+1)\xi_m$ and $L_z \xi_m = im\xi_m$ and we can take $\xi_{-m} = \overline{\xi}_m$. So there exists $\overrightarrow{\alpha}_m \in \mathbb{C}^N$, $-l \leq m \leq l$, such that $\hat{\Psi}(R_{\mu t}^{-1}\hat{X}) = \sum_{m=-l}^{l} \overrightarrow{\alpha}_m \xi_m(\hat{X})$. Also, Ω and σ have a common set of eigenvectors which form an orthonormal basis of $\mathbb{C}^N$ which we call $\phi_1, \phi_2, ..., \phi_n$. The corresponding eigenvalues of σ are $\pm i$, and so $\sigma\phi_j = (-1)^j i\phi_j$ and the corresponding eigenvalues of Ω are $i\nu_j$ where $\nu_j \in \mathbb{R}$, and so $\Omega\phi_j = i\nu_j\phi_j$, $1 \leq j \leq N$. The ϕ_j can be chosen so that $\phi_2 = \overline{\phi}_1, \phi_4 = \overline{\phi}_3, ..., \phi_N = \overline{\phi}_{N-1}$ and the notation $\overline{\phi}_n = \phi_{\hat{n}}$ is used where $\hat{n} = n+1$ if n is odd, and $\hat{n} = n-1$ if n is even. Thus $\hat{\Psi}$ may now be written as the expansion $\hat{\Psi}(R_{\mu t}^{-1}\hat{X}) = \sum_{m=-l}^{l} \overrightarrow{\alpha}_m \xi_m(\hat{X}) = \sum_{m=-l}^{l} \sum_{j=1}^{N} \alpha_{jm}\phi_j \xi_m(\hat{X})$, $\alpha_{jm} \in \mathbb{C}$. We proceed as in chapter 3 by substituting into $u_{tt} - \Delta u - B\sigma(x\partial_y - y\partial_x)u = \overrightarrow{g}(u)$. Taking $u(\overrightarrow{X}, t) = e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)$ we get

$$\begin{aligned}
& u_t(\overrightarrow{X}, t) \\
= \; & \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + e^{\Omega t}\overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot (-\mu M)e^{-\mu t M}\hat{X}w(r) \\
= \; & \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + (-\mu M^T)e^{\Omega t}\overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot e^{-\mu t M}\hat{X}w(r) \\
= \; & \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) - \mu e^{\Omega t} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\nabla}_1\hat{\Psi}(e^{-\mu t M}\hat{X}) \\ \overrightarrow{\nabla}_2\hat{\Psi}(e^{-\mu t M}\hat{X}) \\ \overrightarrow{\nabla}_3\hat{\Psi}(e^{-\mu t M}\hat{X}) \end{pmatrix}
\end{aligned}$$

$$\cdot \begin{pmatrix} cos\mu t & sin\mu t & 0 \\ -sin\mu t & cos\mu t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} w(r)$$

$$= \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) - \mu e^{\Omega t} \begin{pmatrix} -\overrightarrow{\nabla}_2\hat{\Psi}(e^{-\mu t M}\hat{X}) \\ \overrightarrow{\nabla}_1\hat{\Psi}(e^{-\mu t M}\hat{X}) \\ 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} xcos\mu t + ysin\mu t \\ -xsin\mu t + ycos\mu t \\ z \end{pmatrix} w(r)$$

$$= \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + \mu e^{\Omega t}((xcos\mu t + ysin\mu t)\overrightarrow{\nabla}_2\hat{\Psi}(e^{-\mu t M}\hat{X})$$

$$-(-xsin\mu t + ycos\mu t)\overrightarrow{\nabla}_2\hat{\Psi}(e^{-\mu t M}\hat{X}))w(r)$$

$$= \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + \mu e^{\Omega t}(L_z\hat{\Psi}(\hat{X}))\Big|_{e^{-\mu t M}\hat{X}} w(r)$$

$$= \Omega e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + \mu e^{\Omega t}(L_z\hat{\Psi})(e^{-\mu t M}\hat{X})w(r)$$

$$= \Omega u(\hat{X}, t) + \mu L_z u(\hat{X}, t)$$

Using this result gives

$$u_{tt}(\overrightarrow{X}, t)$$

$$= \Omega^2 e^{\Omega t}\hat{\Psi}(e^{-\mu t M}\hat{X})w(r) + 2\mu\Omega e^{\Omega t}(L_z\hat{\Psi})(e^{-\mu t M}\hat{X})w(r) + \mu^2 e^{\Omega t}(L_z\hat{\Psi})(e^{-\mu t M}\hat{X})w(r)$$

$$= \quad \Omega^2 u(\hat{X},t) + 2\Omega\mu L_z u(\hat{X},t) + \mu^2 L_z^2 u(\hat{X},t) w(r)$$

If both $e^{\Omega t}$ and $\hat{\Psi}$ are both time dependent, then (3.2) takes on the form

$$\Omega^2 u_{tt} - 2\mu\Omega L_z u + \mu^2 L_z^2 u - \Delta u - B\sigma L_z u - \overrightarrow{g}(u) = 0 \tag{4.2}$$

With $u(\overrightarrow{X},t) = e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)$ this is

$$\Omega^2 e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r) - 2\mu\Omega e^{\Omega t} L_z(\hat{\Psi}(R_{\mu t}^{-1}\hat{X}))w(r) + \mu^2 e^{\Omega t} L_z^2(\hat{\Psi}(R_{\mu t}^{-1}\hat{X}))w(r) -$$

$$\Delta(e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)) - B\sigma e^{\Omega t} L_z(\hat{\Psi}(R_{\mu t}^{-1}\hat{X}))w(r) - \overrightarrow{g}(e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)) = 0$$

Now,

$$\begin{aligned}
\Delta u &= \Delta(e^{\Omega t}\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)) \\
&= e^{\Omega t}(\Delta_R + \frac{1}{r^2}\Delta_S)(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)) \\
&= e^{\Omega t}(\Delta_R(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r)) + \frac{1}{r^2}\Delta_S(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})w(r))) \\
&= e^{\Omega t}(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})\Delta_R w(r) + \frac{1}{r^2}\Delta_S(\hat{\Psi}(R_{\mu t}^{-1}\hat{X}))w(r)) \\
&= e^{\Omega t}\left(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})\left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}w(r)\left(-l(l+1)\hat{\Psi}(R_{\mu t}^{-1}\hat{X})\right)\right) \\
&= e^{\Omega t}\left(w''(r) + \frac{2}{r}w'(r) - \frac{l(l+1)}{r^2}w(r)\right)\hat{\Psi}(R_{\mu t}^{-1}\hat{X})
\end{aligned}$$

and

$$
\begin{aligned}
L_+(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})) &= (L_x + iL_y)(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})) \\
&= (L_x + iL_y)(\hat{\Psi}(e^{-\mu t M}\hat{X})) \\
&= L_x(\hat{\Psi}(e^{-\mu t M}\hat{X})) + iL_y(\hat{\Psi}(e^{-\mu t M}\hat{X})) \\
&= (y\partial_z - z\partial_y)(\hat{\Psi}(e^{-\mu t M}\hat{X})) + i(-x\partial_z + z\partial_x)(\hat{\Psi}(e^{-\mu t M}\hat{X})) \\
&= \overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot (y\partial_z - z\partial_y)(e^{-\mu t M}\hat{X}) \\
&\quad + i\overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot (-x\partial_z + z\partial_x)(e^{-\mu t M}\hat{X}) \\
&= \overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot \begin{pmatrix} -zsin\mu t \\ zcos\mu t \\ y \end{pmatrix} + i\overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot \begin{pmatrix} zcos\mu t \\ -zsin\mu t \\ -x \end{pmatrix} \\
&= \overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot \begin{pmatrix} iz(cos\mu t + isin\mu t) \\ -z(cos\mu t + isin\mu t) \\ y - ix \end{pmatrix} \\
&= \overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X})e^{i\mu t} \cdot \begin{pmatrix} iz \\ -z \\ e^{-i\mu t}(y - ix) \end{pmatrix}
\end{aligned}
$$

$$= e^{i\mu t}\overrightarrow{\nabla}\hat{\Psi}(e^{-\mu t M}\hat{X}) \cdot \begin{pmatrix} iz \\ -z \\ e^{i\mu t}(y - ix) \end{pmatrix}$$

Since

$$\begin{aligned}
L_+\hat{\Psi}(\hat{X}) &= (L_x + iL_y)\hat{\Psi}(\hat{X}) \\
&= L_x\hat{\Psi}(\hat{X}) + iL_y\hat{\Psi}(\hat{X}) \\
&= (y\partial_z - z\partial_y)\hat{\Psi}(\hat{X}) + i(-x\partial_z + z\partial_x)\hat{\Psi}(\hat{X}) \\
&= \overrightarrow{\nabla}\hat{\Psi}(\hat{X}) \cdot (y\partial_z - z\partial_y)\hat{X} + i\overrightarrow{\nabla}\hat{\Psi}(\hat{X}) \cdot (-x\partial_z + z\partial_x)\hat{X} \\
&= \overrightarrow{\nabla}\hat{\Psi}(\hat{X}) \cdot \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} + i\overrightarrow{\nabla}\hat{\Psi}(\hat{X}) \cdot \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix} \\
&= \overrightarrow{\nabla}\hat{\Psi}(\hat{X}) \cdot \begin{pmatrix} iz \\ -z \\ y - ix \end{pmatrix}
\end{aligned}$$

then

$$L_+(\hat{\Psi}(R_{\mu t}^{-1}\hat{X})) = e^{i\mu t}L_+(\hat{\Psi})(R_{\mu t}^{-1}\hat{X})$$

Substituting into (4.2) gives us

$$\sum_{m=-l}^{l}\sum_{j=1}^{N} e^{\Omega t}\Big(\alpha_{jm}(\Omega^2\phi_j)\xi_m(\hat{X})w(r) + 2\mu\alpha_{jm}(\Omega\phi_j)(L_z\xi_m)(\hat{X})w(r)$$
$$+\mu^2\alpha_{jm}\phi_j(L_z^2\xi_m)(\hat{X})w(r) - \alpha_{jm}\phi_j\xi_m(\hat{X})\Delta_R w(r) + \frac{l(l+1)}{r^2}\alpha_{jm}\phi_j\xi_m(\hat{X})w(r)$$
$$-B\sigma\alpha_{jm}\phi_j(L_z\xi_m)(\hat{X})w(r) - g(w(r))\alpha_{jm}\phi_j\xi_m(\hat{X})\Big) = 0$$

$$\Rightarrow \sum_{m=-l}^{l}\sum_{j=1}^{N}\alpha_{jm}\phi_j\xi_m(\hat{X})\Big((-\nu_j^2 + 2\mu m\nu_j - \mu^2 m^2 + B(-1)^j m)w(r) - \Delta_R w +$$
$$\frac{l(l+1)}{r^2}w(r) - g(w(r))\Big) = 0 \quad (4.3)$$

For each j, m we require either $\alpha_{jm} = 0$ or $(\nu_j - \mu m)^2 - B(-1)^j m) = \eta$, where η is a positive constant independent of j and m. We want to know if there are any precessing solutions that have nonzero spin components not parallel to the $\overrightarrow{B}$ field. This is done by examining the vector $S_x + iS_y$ and using the result just found. First substitute the expansion of $\hat{\Psi}$ into this expression and use the results found earlier for $u_t(\overrightarrow{X}, t)$ and the actions of L_z and L_+ on $\hat{\Psi}$.

$$S_x + iS_y$$
$$= \hat{e}_x\int_{\mathbb{R}^3} u_t\cdot(\overrightarrow{X}\times\overrightarrow{\nabla}u)d^3\overrightarrow{X} + i\hat{e}_y\int_{\mathbb{R}^3} u_t\cdot(\overrightarrow{X}\times\overrightarrow{\nabla}u)d^3\overrightarrow{X}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} u_t \cdot (L_x u) d^3\overrightarrow{X} + i \int_{\mathbb{R}^3} u_t \cdot (L_y u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} u_t \cdot (L_x + iL_y) u d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} u_t \cdot (L_+ u) d^3\overrightarrow{X} \\
&= \int_{\mathbb{R}^3} w^2 (\Omega\overline{\hat{\Psi}} + \mu L_z \overline{\hat{\Psi}}) \cdot (L_+ \hat{\Psi}) d^3\overrightarrow{X} \\
&= e^{i\mu t} \int_{\mathbb{R}^3} w^2 \left(\sum_{n=-l}^{l} \sum_{k=1}^{N} \overline{\alpha}_{kn} \overline{\phi}_k (\overline{i\nu_k \xi}_n + \mu L_z \overline{\xi}_n) \right) \\
&\quad \cdot \left(\sum_{m=-l}^{l} \sum_{j=1}^{N} \alpha_{jm} \xi_{m+1} \phi_j i \sqrt{l(l+1) - m(m+1)} \right) d^3\overrightarrow{X} \\
&= e^{i\mu t} \int_{\mathbb{R}^3} w^2 \left(\sum_{n=-l}^{l} \sum_{k=1}^{N} \overline{\alpha}_{kn} \overline{\phi}_k \xi_{-n} (\nu_k - \mu n) \right) \\
&\quad \cdot \left(\sum_{m=-l}^{l} \sum_{j=1}^{N} \alpha_{jm} \xi_{m+1} \phi_j \sqrt{l(l+1) - m(m+1)} \right) \, d^3\overrightarrow{X} \\
&= e^{i\mu t} \sum_{n=-l}^{l} \sum_{k=1}^{N} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \int_{\mathbb{R}^3} w^2 \overline{\alpha}_{kn} \phi_{\hat{k}} \xi_{-n} (\nu_k - \mu n) \\
&\quad \cdot \alpha_{jm} \xi_{m+1} \phi_j i \sqrt{l(l+1) - m(m+1)} \, d^3\overrightarrow{X} \\
&= e^{i\mu t} \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \int_{\mathbb{R}^3} w^2 \overline{\alpha}_{\hat{j}n} \phi_j \xi_{-n} (\nu_{\hat{j}} - \mu n) \\
&\quad \cdot \alpha_{jm} \xi_{m+1} \phi_j \sqrt{l(l+1) - m(m+1)} \, d^3\overrightarrow{X} \\
&= e^{i\mu t} \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \alpha_{j(m+1)} \alpha_{jm} (\nu_{\hat{j}} - \mu(m+1)) \\
&\quad \sqrt{l(l+1) - m(m+1)} \int_{\mathbb{R}^3} w^2 \overline{\xi}_n \xi_{m+1} d^3\overrightarrow{X} \\
&= e^{i\mu t} \sum_{n=-l}^{l} \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \alpha_{j(m+1)} \alpha_{jm} (\nu_{\hat{j}} - \mu(m+1)) \\
&\quad \sqrt{l(l+1) - m(m+1)} \, \delta_{n(m+1)} \int_0^\infty r^2 w^2(r) dr \\
&= e^{i\mu t} \int_0^\infty r^2 w^2(r) dr \sum_{m=-l}^{l-1} \sum_{j=1}^{N} \alpha_{j(m+1)} \alpha_{jm} (\nu_{\hat{j}} - \mu(m+1)) \sqrt{l(l+1) - m(m+1)}
\end{aligned}
$$

If $\alpha_{j(m+1)}\alpha_{jm} \neq 0$, then $\alpha_{j(m+1)} \neq 0$ and $\alpha_{jm} \neq 0$.

Now if $\alpha_{j(m+1)} \neq 0$, then (4.1) and (4.2) imply

$$(\nu_j - \mu(m+1))^2 - B(-1)^j(m+1)) = \eta$$

$$\Rightarrow \nu_j^2 - 2\nu_j\mu(m+1) + \mu^2(m+1)^2 - B(-1)^j(m+1) = \eta \tag{4.4}$$

and $\alpha_{jm} \neq 0$ implies

$$(\nu_j - \mu m)^2 - B(-1)^j m = \eta \tag{4.5}$$

$$\Rightarrow \nu_j^2 - 2\nu_j\mu m + \mu^2 m^2 - B(-1)^j m = \eta \tag{4.6}$$

Subtracting (4.6) from (4.4) yields

$$2\nu_j\mu - 2m\mu^2 - \mu^2 + B(-1)^j = 0$$

Note that if $\mu = 0$, then this problem reduces to what was done in chapter 3, and $S_x + iS_y = 0$. So assume from here on that $\mu \neq 0$. Solving the above equation for ν_j

gives

$$\nu_j = \frac{1}{2\mu}(\mu^2(2m+1) - B(-1)^j)$$

Substituting this into (4.5) gives η.

$$\begin{aligned}
\eta &= \left(\frac{1}{2\mu}(\mu^2(2m+1) - B(-1)^j - \mu m\right)^2 - B(-1)^j m \\
&= \left(\frac{1}{2\mu}(\mu^2 - B(-1)^j\right)^2 - B(-1)^j m \\
&= \frac{1}{4\mu^2}(\mu^4 - 2\mu^2 B(-1)^j + B^2 - 4\mu^2 B(-1)^j m) \\
&= \frac{1}{4\mu^2}(\mu^4 - 2B(2m+1)(-1)^j\mu^2 + B^2)
\end{aligned}$$

This result will needed shortly.

We wish to have $|\hat{\Psi}|^2 = 1$ independent of $\hat{X}$. From [2] it is known that the ξ's can be chosen so that $\sum_n |\xi(\hat{X})|^2 = 1$ independent of of $\hat{X}$. However in order for for $|\hat{\Psi}|^2 = \sum_{j,m,n} \overline{\alpha}_{jn}\alpha_{jm}\overline{\xi}_n(\hat{X})\xi_m(\hat{X})$ to be independent of $\hat{X}$, we insist that the coefficient of $\overline{\xi}_n(\hat{X})\xi_m(\hat{X})$ vanishes for $n \neq m$. That is we require $\sum_j \overline{\alpha}_{jn}\alpha_{jm} = \delta_{mn}$. Then $|\hat{\Psi}|^2 = \sum_{j,m,n} \overline{\alpha}_{jn}\alpha_{jm}\overline{\xi}_n(\hat{X})\xi_m(\hat{X}) = \sum_{m,n} \delta_{mn}\overline{\xi}_n(\hat{X})\xi_m(\hat{X}) = \sum_n |\xi(\hat{X})|^2 = 1$ and this is independent of $\hat{X}$.

Let

$$[\alpha]_m \begin{pmatrix} \alpha_{1m} \\ \vdots \\ \alpha_{Nm} \end{pmatrix}$$

If $S_x + iS_y$ is to be nonzero, we need for some $j = j_0$ and some $m = m_0 \leq l - 1$, that $\alpha_{j_0 m_0} \neq 0$ and $\alpha_{j_0(m_0+1)} \neq 0$. Since $\sum_{j=1}^{N} \overline{\alpha}_{jn}\alpha_{jm} = \delta_{mn}$, then $[\alpha]_{m_0}$ is orthogonal to $[\alpha]_{m_0+1}$. So there must be another value of j, say $j = j_1$, ($j_1 \neq j_0$) for which $\alpha_{j_1 m_0} \neq 0$ and $\alpha_{j_1(m_0+1)} \neq 0$. It is now shown that this forces $S_x + iS_y = 0$, giving us a contradiction. So suppose $\alpha_{j_0 m_0} \neq 0$, $\alpha_{j_0(m_0+1)} \neq 0$, and $\alpha_{j_1 m_0} \neq 0$, $\alpha_{j_1(m_0+1)} \neq 0$ and the ansatz for the form of u produces a solution. Then this results in the four equations

$$(\nu_{j_0} - \mu m_0)^2 - B(-1)^{j_0} m_0 = \eta \tag{4.7}$$

$$(\nu_{j_0} - \mu(m_0+1))^2 - B(-1)^{j_0}(m_0+1)) = \eta \tag{4.8}$$

$$(\nu_{j_1} - \mu m_0)^2 - B(-1)^{j_1} m_0 = \eta \tag{4.9}$$

$$(\nu_{j_1} - \mu(m_0+1))^2 - B(-1)^{j_1}(m_0+1)) = \eta \tag{4.10}$$

(where, recall, η is a constant independent of j and m). As already shown, from (4.7) and (4.8)

$$\eta = \frac{1}{4\mu^2}(\mu^4 - 2B(2m_0 + 1)(-1)^{j_0}\mu^2 + B^2)$$

and (4.9) and (4.10) give

$$\eta = \frac{1}{4\mu^2}(\mu^4 - 2B(2m_0 + 1)(-1)^{j_1}\mu^2 + B^2)$$

If $(-1)^{j_0} \neq (-1)^{j_1}$, then two different values of η result, and there is no solution of the ansatz form. Thus it must be true that $(-1)^{j_0} = (-1)^{j_1}$, from which it follows that $\nu_{j_0} = \nu_{j_1}$ and thus $\nu_j = \frac{1}{2\mu}(\mu^2(2m+1) - B(-1)^j)$. Thus every j_1 for which $\alpha_{j_1 m} \neq 0$ and $\alpha_{j_1(m+1)} \neq 0$ has $\nu_{j_0} = \nu_{j_1}$. So in the formula for $S_x + iS_y$ the common factor of $\nu_j - \mu(m+1)$ can be taken out of the sum over j. The sum over j then reduces to the inner product of $[\alpha]_m$ and $[\alpha]_{m+1}$. Since these are orthogonal, $S_x + iS_y = 0$. So there are no solutions with the ansatz for the form of u which have nonzero spin components in a direction not parallel to the magnetic field. This implies that there are no precessing solutions since any nonzero spin components would not lie along the axis of rotation.

We have examined certain vector-valued solitary wave solutions to a nonlinear Klein-Gordon equation. Since the equation was seen to be invariant under rotations, we were able to find a functional which gave the Noether conserved quantity (which we call spin) for solutions to NLKG. The existence of solitary wave solutions with nonzero spin in any prescribed direction was then shown. When an external uniform magnetic field is applied, the only solutions to NLKG had spin parallel or antiparallel to the magnetic field. It should be noted that the solutions examined were in space-time $\mathbb{R}^{3+1}$ and were restricted to positive integral values of values of l. Futher work needs to be done to examine the existence of solitary wave solutions in general space-time $\mathbb{R}^{N+1}$ and for half-integral values of l.

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