QUANTUM SIZE OSCILLATIONS AND SIZE EFFECTS

THESIS

Presented to the Graduate Council of the
North Texas State University in Partial
Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

By

John R. Garlow, M. S.

Denton, Texas

May, 1976

Garlow, John R., Quantum Size Oscillations and Size Effects. Doctor of Philosophy (Physics), May, 1976, 72 pp., 2 illustrations, bibliography, 26 titles.

Early theories on the conduction of electrons in metals were based on the classical Boltzmann kinetic theory of gases. The classical model was unable to explain some of the bulk electronic properties. These features were explained by the introduction of Fermi-Dirac statistics which are quantum-mechanical. The next generation of electron theory for metals was characterized as semiclassical in nature. They employed the new Fermi-Dirac statistics but still used the classical Boltzmann transport equation. A quantum-mechanical treatment of the electronic properties is more complicated, requiring the use of a density matrix and the exact summation over the allowed states of the electron system. However, quantum-mechanical features of the system, such as the discreteness of the energy or momentum, are responsible for some of the features of the electron system and must therefore be taken into account.

The direct current size effect and the radio frequency size effect oscillations in the magnetoconductivity have been explained by a semiclassical theory which is based on the matching of the sample thickness and the classical spiral orbit of the electron about a magnetic field. The Shubnikov-de Haas effect is oscillation in the bulk magnetoconductivity

at large fields where the magnetic quantization is strong. The explanation is quantum-mechanical. Because the size effects occur at low magnetic fields where the quantization is not strong, they are referred to as classical, while the Shubnikov-de Haas effect is referred to as quantum-mechanical. However, the finiteness of the size of a thin sample quantizes the momentum of the electron system and the presence of the magnetic field quantizes the energy levels. Therefore it should be possible to derive both types of oscillatory behavior with a quantum-mechanical calculation of the magnetoconductivity.

The quantum-mechanical magnetoconductivity tensor for a thin sample is calculated with a linear response theory. In that theory the response of a free electron gas to an electromagnetic disturbance is assumed to be linear. In addition, it is assumed that each Fourier component of the disturbance gives rise to an independent response. The Hamiltonian for one electron is then written as the sum of an unperturbed part and a perturbation due to the electromagnetic disturbance. The equation of motion for the one particle density matrix is used to obtain the response of the system. The rate of change of the density matrix due to collisions is taken into account by the use of a constant relaxation time approximation. To apply this theory to a thin film, the stationary states of the system are assumed to be those of an electron in a magnetic field and an infinite

square well potential with Landau energy levels described by quantum numbers for the magnetic field and the z-component of momentum. The non-local magnetoconductivity in the long-wavelength limit is calculated from this. The diffusion current response, however, is neglected. The discreteness of the Landau levels gives rise to the usual Shubnikov-de Haas oscillations, but with a size modulation factor. Due to the discrete nature of the energy levels and the z-component of momentum, poles in the matrix elements give rise to oscillations in the conductivity corresponding to the radio frequency and direct current size effects. Physically, this corresponds to a matching of the energy between adjacent Landau levels and allowed momentum states at the intersection of the Fermi surface and the Fermi momentum as the magnetic field changes.

Bm

TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	iv
Chapter	
I. INTRODUCTION	1
II. CALCULATION	6
Derivation of the Wave Function	
Density Matrix Elements	
Current Density	
Matrix Elements of the Velocity Operator	
XY Element of Conductivity Tensor	
Summation over Magnetic Quantum Numbers	
Summation over Final Momentum States	
Summation over Initial Momentum States	
III. DISCUSSION	56
Mathematical Basis of Quantum Size	
Oscillations	
Physical Basis for Quantum Size	
Oscillations	
Scattering and the Uncertainty Principle	
Comparison with Experiments	
APPENDIX	68
BIBLIOGRAPHY	70

LIST OF ILLUSTRATIONS

Figure	Page
1. First Resonance.	60
2. Second Resonance.	61

CHAPTER I

INTRODUCTION

Many theories of the conduction of electrons in metals have been produced since the Drude theory based on the classical Boltzmann kinetic theory of gases, near the turn of the 20th century (5). Most of these theories represent the metal as a free-electron gas where the electron moves through a uniform distribution of positive charges of equal but opposite charge density. While the early classical model was able to explain some features of the conductivity of a bulk sample, the representation of the electrons by the Boltzmann statistics did not explain the temperature dependence of the conductivity (2, p. 148) or why the electron gas made no contribution to the specific heat of the metal (18, p. 11). The application of the Fermi-Dirac statistics by Sommerfeld (14) in 1928 indicated that the quantum-mechanical features of the electron gas must be taken into account in order to explain the temperature dependence in the conductivity and the lack of a specific heat contribution.

The full quantum-mechanical treatment of the conductivity involves the exact sum over the possible initial and final states of the electron, and the use of a density matrix which

represents an ensemble rather than an average state (1). However, the treatment is complicated and many of the same results can be obtained by a semiclassical treatment, such as the work by Cohen, Harrison, and Harrison (4), which uses the Fermi-Dirac statistics and the Boltzmann transport equation. Nevertheless, the information contained in the theory for the bulk direct current conductivity doesn't completely characterize the dynamical properties of a metal.

One way of characterizing the dynamical properties of a metal is to describe the properties of the Fermi surface, which is the locus of the momentum vectors in momentum space whose energy corresponds to the Fermi energy. The Fermi energy is the energy of the highest occupied states in the Fermi-Dirac distribution scheme. The properties of the Fermi surface can be studied by the introduction into the system of various modifications such as a magnetic field (9, 16), high frequency electromagnetic fields (12), limitations of finite size on the sample (15), sound waves (3), or a combination of the above (13, 6). The conditions placed on the electron gas by the introduction of a magnetic field quantize the energy levels of the system (10). The exact summation over states is then required to take into account the discreteness of the energy. Such a quantum-mechanical theory was needed to explain the Shubnikov-de Haas effect (13) which has been used to study parameters

of the Fermi surface. A limitation on the thickness of the metal will quantize the momentum states in the electron gas. To properly take this into account one should use a quantum-mechanical theory employing the exact summation over the quantized states.

In 1950 Sondheimer (15) studied what is now called the direct current size effect. This semiclassical theory was based on the classical spiraling motion of an electron in a magnetic field and the notion of diffuse scattering at the surfaces. Using a thin metal slab placed in a magnetic field oriented normal to the slab surface, he found the resistivity for electrons whose mean free paths were greater than the sample thickness to vary periodically with the magnetic field B . Since then workers have studied the oscillations experimentally and mapped out Fermi surface parameters (17). In 1962 Gantmahker (7) discovered oscillations in the surface impedance of a thin slab with radio frequency electromagnetic radiation incident upon it. This radio frequency size effect had the same periodicity as the direct current size effect and was also explained by a matching of the sample thickness and the spiral orbit about the magnetic field.

The size oscillations in the conductivity occur at low values of the magnetic field where the magnetic quantization is not pronounced. For this reason, the oscillations have not been ascribed to the effects of magnetic quantization

even though the presence of the field is necessary. One of the effects of magnetic quantization on the bulk conductivity was discovered in 1930 by Shubnikov and de Haas (16), for whom it was named. It was found that the oscillations due to the energy quantization by a high magnetic field B were periodic in $1/B$.

In the work done on gallium by Munarin, Marcus, and Bloomfield (11), the semiclassical oscillations periodic in B were observed simultaneously with the Shubnikov-de Haas quantum oscillations periodic in $1/B$. This suggests that to call one the quantum effect for a high magnetic field and the other a semiclassical effect for a low magnetic field tends to obscure any possible relationship between the two. A quantum-mechanical calculation based upon the quantization due to the magnetic field and the finite size of the sample should be able to predict both oscillatory effects. Such a quantum-mechanical calculation requires the use of boundary conditions in order to quantize the component of momentum in the direction of the finite dimension of the sample. Because the results for a finite system have already been observed, the calculation can also serve as a test on the validity of the simple boundary condition that the wave function for the electron vanish at the surfaces.

The quantum-mechanical magnetoconductivity tensor for a thin sample is calculated with the linear response theory of Green et. al. (8). In that theory the response of a free

electron gas to an electromagnetic disturbance is assumed to be linear. In addition, it is assumed that each Fourier component of the disturbance gives rise to an independent response. The Hamiltonian for one electron is then written as the sum of an unperturbed part and a perturbation due to the electromagnetic disturbance. The equation of motion for the one particle density matrix is used to obtain the response of the system. The rate of change of the density matrix due to collisions is taken into account by the use of a constant relaxation time approximation. To apply this theory to a thin film, the stationary states of the system are assumed to be those of an electron in a magnetic field and an infinite square well potential with Landau energy levels described by quantum numbers for the magnetic field and the z-component of momentum. The non-local magnetoconductivity in the long-wavelength limit is calculated from this. The diffusion current response, however, is neglected. The discreteness of the Landau levels gives rise to the usual Shubnikov-de Haas oscillations but with a size modulation factor. Due to the discrete nature of the energy levels and the z-component of momentum, poles in the matrix elements give rise to oscillations in the conductivity corresponding to the radio frequency and direct current size effects. Physically this corresponds to a matching of the energy between adjacent Landau levels and allowed momentum states at the intersection of the Fermi surface and the Fermi momentum as the magnetic field changes.

CHAPTER BIBLIOGRAPHY

1. Adams, E.N., and T.D. Holstein, "Quantum Theory of Transverse Galvanomagnetic Phenomena," Journal of Physics and Chemistry of Solids, 10(February, 1959), 254-276.
2. Blakemore, J.S., Solid State Physics, Philadelphia, W.B. Saunders Company, 1969.
3. Bommel, H.E., "Ultrasonic Attenuation in Superconducting and Normal-Conducting Tin at Low Temperature," Physical Review, 100(October, 1955), 758-759.
4. Cohen, M.H., M.J. Harrison and W.A. Harrison, "Magnetic Field Dependence of the Ultrasonic Attenuation in Metals," Physical Review, 117(February, 1960), 937-952.
5. Drude, P., "Zur Elektronen Theorie der Metalle," Annalen der Physik, 1(June, 1900), 566.
6. Gantmahker, V.F., "Investigation of the Fermi Surface of Tin by the Size Effect," Soviet Physics-JETP, 17(September, 1963), 549-560.
7. _____ "Method of Measuring the Momentum of Electrons in a Metal," Soviet Physics-JETP, 15 (November, 1962), 982-983.
8. Green, M.P., H.K. Lee, J.J. Quinn, and S. Rodriguez, "Linear Response Theory for a Degenerate Electron Gas in a Strong Magnetic Field," Physical Review, 177(January, 1969), 1019-1036.
9. Kapitza, P., "The Change of Electrical Conductivity in Strong Magnetic Fields," Proceedings of the Royal Society of London, A123(1929), 292.
10. Landau, L., "Diamagnetismus der Metalle," Zeitschrift fur Physik, 64(1930), 629.
11. Munarin, J.A., J.A. Marcus, and P.E. Bloomfield, "Size-Dependent Oscillatory Magnetoresistance Effect in Gallium," Physical Review, 172(August, 1968), 718.

12. Pippard, A.B., "Metallic Conduction at High Frequencies and Low Temperatures," Advances in Electronics-Electron Physics, 6(1954), 1.
13. _____ "Proposal for Determining the Fermi Surface by Magneto-Acoustic Resonance," Philosophical Magazine, Ser. 8, 2(September, 1957), 1147.
14. Sommerfeld, A., "Zur Elektronen Theorie der Metalle auf Grund der Fermischen Statistik," Zeitschrift fur Physik, 47(1928), 1.
15. Sondheimer, E.H., "The Influence of a Transverse Magnetic Field on the Conductivity of Thin Metallic Films," Physical Review, 80(November, 1950), 401-406.
16. Shubnikov L., and W.J. de Haas, "Neue Erscheinungen bei der Widerstandsanderung bei der Temperatur von Flussigen Wasserstoff. I," Leiden Commun, 207d (1930).
17. Zebouni, N.H., R.E. Hamburg, and H.J. Mackey, "Magneto-morphic Oscillations in the Hall Effect and Magnetoresistance in Cadmium," Physical Review Letters, 11(August, 1963), 260-263.
18. Ziman, J.M., Electrons in Metals, London, Taylor and Francis Ltd., 1963.

CHAPTER II

CALCULATION

The linear response theory of Green et. al. (4) for a free electron gas in a direct current magnetic field is used to calculate the non local wave number dependent quantum-mechanical magnetoconductivity $\sigma(q, q', \omega, B_0)$ for a thin metal slab in a magnetic field B_0 normal to the slab surface with a transverse electromagnetic wave of the form $\exp(iqz - i\omega t)$ incident normally upon the slab. In the small q and ω limit oscillations are found in the xy element of σ . The diffusion current response is neglected by picking a gauge in which the scalar potential is zero and by ignoring oscillations in the chemical potential.

Derivation of Wave Function

The surfaces of the sample are located at $\pm a/2$ on the z -axis and the direct current magnetic field $\vec{B}_0$ is oriented normal to the slab surfaces so that $\vec{B}_0 = (0, 0, B_0)$. The transverse electromagnetic wave is the form $\exp(iqz - i\omega t)$ such that the wave number $\vec{q} = (0, 0, q)$. In the gauge where the scalar potential is zero, the electric field $\vec{E}$ and the magnetic field $\vec{B}$ of the electromagnetic wave are related to the vector potential $\vec{A}(\vec{r}, t)$ by the relations

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \nabla \times \vec{A}.$$

The Hamiltonian for an electron of charge $-e$ and mass M subject to the above fields is

$$H = H_0 + H_1 = (\vec{p} + \vec{A}_0 e/c + \vec{A} e/c)/2M,$$

where

$$H_0 = \frac{1}{2} M v_0^2 = (\vec{p} + \vec{A}_0 e/c)^2/2,$$

$$H = (\vec{v}_0 \cdot \vec{A} + \vec{A} \cdot \vec{v}_0) e/2c,$$

$\vec{A}_0 = (0, B_0 x, 0)$, $\vec{p}$ is the linear momentum, and c is the speed of light. With the z dependent part written as a sum of sine and cosine functions, the eigenstates of the unperturbed Hamiltonian H_0 for the metal slab with N free conduction electrons and a volume $L_x L_y a$ are characterized by a set $\mathcal{V}$ of quantum numbers and written as

$$|\mathcal{V}\rangle = (1/L_y)^{\frac{1}{2}} A \cos k_z z + B \sin k_z z \exp(ik_y y) U_n(x + \hbar k_y / M \omega_c)$$

where U_n is a normalized off center harmonic oscillator wave function and $k_y = n_y \pi / L_y$ with $n_y = 0, \pm 1, \pm 2, \dots$.

The slab surfaces in the xy planes are represented by an infinite potential. The wave function must then vanish at $z = \pm a$. If the y and x dependent parts of the wave function are non-zero, then at $z = \pm a/2$,

$$A \cos k_z z + B \sin k_z z = 0.$$

At $z = -a/2$,

$$A \cos k_z(-a/2) + B \sin k_z(-a/2) = 0. \quad (1)$$

At $z = +a/2$,

$$A \cos k_z(a/2) + B \sin k_z(a/2) = 0. \quad (2)$$

The sum of equation (1) and equation (2) is $2A \cos k_z a/2 = 0$.

In general $A \neq 0$, so $\cos k_z a/2 = 0$. This is true if $k_z a/2 = (m - \frac{1}{2})\pi$ or $k_z = 2(m - \frac{1}{2})\pi/a$, where $m = 0, \pm 1, \pm 2, \dots$. The difference in equation (1) and equation (2) is $2B \sin k_z a/2 = 0$. In general $B \neq 0$, so $\sin k_z a/2 = 0$. This is true if $k_z a/2 = m\pi$ or $k_z = 2m\pi/a$, where $m = \pm 1, \pm 2, \pm 3, \dots$. The z -dependent part of the wave function becomes either an even function of z , $A \cos 2\pi(m - \frac{1}{2})z/a$, or an odd function of z , $B \sin 2\pi m z/a$.

Normalization of this part of the wave function requires that

$$1 = \int_{-a/2}^{a/2} A^2 \cos^2 2\pi(m - \frac{1}{2})z/a \, dz$$

Based on the standard integral

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin x \cos x}{2},$$

the normalization gives $A = (2/a)^{\frac{1}{2}}$. Similarly the normal-

ization for the odd function becomes $B = (2/a)^{\frac{1}{2}}$. The

stationary states of H_0 are thus given by

$$|\psi\rangle = |n, k_y, k_z\rangle = (2/aL_y)^{\frac{1}{2}} \exp(ik_y y) U_n(x + \hbar k_y / M\omega_c) \sin(2\pi m z/a) \quad (3)$$

or

$$|\psi\rangle = |n, k_y, k_z\rangle = (2/aL_y)^{\frac{1}{2}} \exp(ik_y y) U_n(x + \hbar k_y / M\omega_c) \cos 2\pi(m - \frac{1}{2})z/a, \quad (4)$$

where ω_c is the cyclotron frequency eB_0/MC . The eigenvalues of H_0 are given by

$$E_{\nu} = E_n(k_z) = (n + \frac{1}{2})h\omega_c + h^2 k_z^2 / 2M, \quad (5)$$

where $k_z = 2\pi m/a$ or $2\pi(m - \frac{1}{2})/a$.

Density Matrix Elements

The response of the system is found from the equation of motion for the single particle density matrix

$$\frac{\partial f}{\partial t} + i[H, f]/\hbar = -[f - f_0(H, \eta)]/\tau \quad (6)$$

where η is the equilibrium chemical potential, $f_0(H, \eta)$ is a local thermal equilibrium value of f , the single particle density matrix, and τ is a constant relaxation time. Oscillations in the chemical potential are ignored. The Fermi-Dirac function

$$f_0(H, \eta) = [1 + \exp \beta(H - \eta)]^{-1}$$

is used for the local thermal equilibrium value of the one particle density matrix, where $\beta = 1/kT$, k is the Boltzmann constant, and T is the absolute temperature. With H written as $H_0 + H_1$

$$f_0(H, \eta) = [1 + \exp \beta(H_0 + H_1 - \eta)]^{-1}$$

which can be rearranged to give

$$f_0(H, \eta) = \left\{ 1 + \exp \beta[(H_0 - \eta) + H_1] \right\}^{-1}.$$

The expansion theorem of Karplus and Schwinger (5) for a function $\exp \lambda(R + S) = G(\lambda) \exp \lambda R$, says $G(\lambda)$ can be written as $1 + \int_0^\lambda d\lambda' S(\lambda') + \int_0^\lambda d\lambda' \int_0^\lambda d\lambda'' S(\lambda'') S(\lambda') + \dots$, if $S(\lambda) = (\exp \lambda R) S[\exp(-\lambda R)]$. The expansion theorem can be used to write

$$\exp \beta [(H_0 - \eta) + H_1] = \left[1 + \int_0^\beta d\beta' \tilde{H}_1(\beta') + \dots \right] \left[\exp \beta (H_0 - \eta) \right]$$

where

$$\tilde{H}_1(\beta') = \exp \beta' (H_0 - \eta) H_1 \exp \left[-\beta' (H_0 - \eta) \right]$$

This can be used to write

$$f_0(H, \eta) = \left\{ 1 + \exp \beta (H_0 - \eta) + \left[\int_0^\beta d\beta' \tilde{H}_1(\beta') + \dots \right] \exp (H_0 - \eta) \right\}^{-1}$$

Factorization gives

$$f_0(H, \eta) = \left[1 + \exp \beta (H_0 - \eta) \right]^{-1} \left\{ 1 + \left[\int_0^\beta d\beta' \tilde{H}_1(\beta') + \dots \right] \exp \beta (H_0 - \eta) \right. \\ \left. / \left[1 + \exp \beta (H_0 - \eta) \right] \right\}^{-1}$$

The terms in the braces can be expanded to first order in H , by use of the form $1 + x^{-1} = 1 - x$, for $x \leq 1$. Therefore

$$f_0(H, \eta) \simeq f_0(H_0, \eta) \left\{ 1 - \left[\int_0^\beta d\beta' \tilde{H}_1(\beta') \right] \exp \beta (H_0 - \eta) f_0(H_0, \eta) \right. \\ \left. f_0(H, \eta) \simeq f_0(H_0, \eta) - f_0(H_0, \eta) \left[\int_0^\beta d\beta' \tilde{H}_1(\beta') \right] \exp (H_0 - \eta) f_0(H_0, \eta) \right\}$$

This expansion may be written as $f_0(H, \eta) = f_0(H_0, \eta) + f_2$

where

$$f_2 = -f_0(H_0, \eta) \left[\int_0^\beta d\beta' \tilde{H}_1(\beta') \right] \exp \beta (H_0 - \eta)$$

and is linear in A .

The matrix elements of f_2 between the eigenstates of H_0 are

$$\langle \nu | f_2 | \nu' \rangle = -\langle \nu | f_0(H_0, \eta) \int_0^\beta d\beta' \left[\exp \beta' (H_0 - \eta) \right] H_1 \\ \cdot \exp \left[-\beta (H_0 - \eta) \right] \exp (H_0 - \eta) f_0(H_0, \eta) | \nu' \rangle$$

Because $|\nu\rangle$ and $|\nu'\rangle$ are eigenstates of $f_0(H_0, \eta)$ with

eigenvalues of $f_0(E_\nu, \eta)$ and $f_0(E_{\nu'}, \eta)$ respectively

$$\begin{aligned} \langle \nu | f_2 | \nu' \rangle &= -f_0(E_\nu) \langle \nu | \int_0^\beta d\beta' \left[\exp \beta' (H_0 - \eta) \right] H_1 \\ &\quad \cdot \exp \left[-\beta' (H_0 - \eta) \right] \exp \beta (H_0 - \eta) | \nu' \rangle f_0(E_{\nu'}) \end{aligned}$$

where $f_0(E_\nu, \eta)$ is written as $f_0(E_\nu)$ for convenience. Successive operations from the right and left give

$$\begin{aligned} \langle \nu | f_2 | \nu' \rangle &= -f_0(E_\nu) \int_0^\beta d\beta' \langle \nu | \left[\exp \beta' (H_0 - \eta) \right] H_1 \\ &\quad \cdot \exp \left[-\beta' (H_0 - \eta) \right] | \nu' \rangle \exp \beta (E_{\nu'} - \eta) f_0(E_{\nu'}) \\ \langle \nu | f_2 | \nu' \rangle &= -f_0(E_\nu) \int_0^\beta d\beta' \exp \beta' (E - \eta) \langle \nu | H_1 | \nu' \rangle \\ &\quad \cdot \exp \left[-\beta' (E_{\nu'} - \eta) \right] \left[\exp \beta (E_{\nu'} - \eta) \right] f_0(E_{\nu'}) \end{aligned}$$

This may be factored and rearranged to give

$$\begin{aligned} \langle \nu | f_2 | \nu' \rangle &= -f_0(E_\nu) f_0(E_{\nu'}) \exp \beta (E_{\nu'} - \eta) \langle \nu | H_1 | \nu' \rangle \\ &\quad \cdot \int_0^\beta d\beta' \exp \beta' (E_\nu - E_{\nu'}) \end{aligned}$$

Integration gives

$$\begin{aligned} \langle \nu | f_2 | \nu' \rangle &= -f_0(E_\nu) f_0(E_{\nu'}) \langle \nu | H_1 | \nu' \rangle \exp \beta (E_{\nu'} - \eta) \\ &\quad \cdot \left[\exp \beta (E_\nu - E_{\nu'}) - 1 \right] / [E_\nu - E_{\nu'}] \end{aligned}$$

The factors

$$f_0(E_\nu) f_0(E_{\nu'}) \exp \beta (E_{\nu'} - \eta) \left[\exp \beta (E_\nu - E_{\nu'}) - 1 \right]$$

may be rearranged to give $f_0(E_\nu) f_0(E_{\nu'}) \left[\exp \beta (E_\nu - \eta) - \exp \beta (E_{\nu'} - \eta) \right]$.

If $\left[\exp \beta (E_\nu - \eta) - \exp \beta (E_{\nu'} - \eta) \right]$ is written as $\left[1 + \exp \beta (E_\nu - \eta) \right] - \left[1 + \exp \beta (E_{\nu'} - \eta) \right]$, then

$$\langle \nu | f_2 | \nu' \rangle = \langle \nu | H_1 | \nu' \rangle \left[f_0(E_{\nu'}) - f_0(E_\nu) \right] / [E_{\nu'} - E_\nu]. \quad (7)$$

Substitute $f - f_0(H, \eta) = f_1$, $H = H_0 + H_1$ and $f_0(H, \eta) = f_0(H_0, \eta) + F_2$ into equation (6) to get

$$\frac{\partial f_0(H_0, \eta)}{\partial t} + \frac{\partial f_2}{\partial t} + \frac{\partial f_1}{\partial t} + \frac{f_1}{\tau} = -i \left[H_0 + H_1, f_0(H_0, \eta) + f_2 + f_1 \right] / \hbar \quad (8)$$

The Poisson bracket on the right side of equation (8) can be to give $[H_0, f_0(H_0, \eta)] + [H_0, f_1] + [H_0, f_2] + [H_1, f_0(H_0, \eta)] + [H_1, f_2] + [H_1, f_1]$. The first term above is zero because H_0 and $f_0(H_0, \eta)$ commute. The fifth and sixth terms above will be disregarded because they are non-linear in A . The first term on the left hand side of equation (8) is zero because $f_0(H_0, \eta)$ is the equilibrium distribution. If the disturbance varies as $\exp(i\omega t)$, in the linear theory f_1 and f_2 will vary as $\exp(i\omega t)$. Then

$$\frac{\partial f_2}{\partial t} = i\omega f_2$$

and

$$\frac{\partial f_1}{\partial t} = i\omega f_1.$$

Equation (8) becomes

$$i\omega f_2 + i\omega f_1 + f_1/\tau = -i \left\{ [H_0, f_1] + [H_0, f_2] + [H_1, f_0(H_0, \eta)] \right\} / \hbar.$$

Multiplication by $i\hbar$ gives

$$-\hbar\omega f_2 - \hbar\omega f_1 + i\hbar f_1/\tau = [H_0, f_1] + [H_0, f_2] + [H_1, f_0(H_0, \eta)].$$

Rearrangement gives

$$-(\hbar\omega - i\hbar/\tau)f_1 - [H_0, f_1] - [H_0, f_2] - [H_1, f_0(H_0, \eta)] = \hbar\omega f_2. \quad (9)$$

The matrix elements of equation (9) are found from

$$\begin{aligned} \langle \nu | \hbar\omega f_2 | \nu \rangle &= -\langle \nu | (\hbar\omega - i\hbar/\tau) f_1 | \nu \rangle - \langle \nu | [H_0, f_1] | \nu \rangle \\ &\quad - \langle \nu | [H_0, f_2] | \nu \rangle - \langle \nu | [H_1, f_0(H_0, \eta)] | \nu \rangle. \end{aligned}$$

Expansion of the Poisson Brackets, $[A, B] = AB - BA$, gives

$$\begin{aligned} \hbar\omega \langle \nu | f_2 | \nu' \rangle = & - (\hbar\omega - i\hbar/\tau) \langle \nu | f_1 | \nu' \rangle - \langle \nu | H_0 f_1 | \nu' \rangle + \langle \nu | f_1 H_0 | \nu' \rangle \\ & - \langle \nu | H_0 f_2 | \nu' \rangle + \langle \nu | f_2 H_0 | \nu' \rangle - \langle \nu | H_1 f_0(H_0, \eta) | \nu' \rangle \\ & + \langle \nu | f_0(H_0, \eta) H_1 | \nu' \rangle. \end{aligned} \quad (10)$$

Because $|\nu\rangle$ and $|\nu'\rangle$ are eigenvectors of H_0 and $f_0(H_0, \eta)$, equation (10) becomes

$$\begin{aligned} \hbar\omega \langle \nu | f_2 | \nu' \rangle = & - (\hbar\omega - i\hbar/\tau) \langle \nu | f_1 | \nu' \rangle - E_{\nu'} \langle \nu | f_1 | \nu' \rangle + E_{\nu} \\ & \cdot \langle \nu | f_1 | \nu' \rangle - E_{\nu'} \langle \nu | f_2 | \nu' \rangle + E_{\nu} \cdot \langle \nu | f_2 | \nu' \rangle \\ & - f_0(E_{\nu'}) \langle \nu | H_1 | \nu' \rangle + f_0(E_{\nu}) \langle \nu | H_1 | \nu' \rangle. \end{aligned}$$

Factorization gives

$$\begin{aligned} [-(\hbar\omega - i\hbar/\tau) + E_{\nu'} - E_{\nu}] \langle \nu | f_1 | \nu' \rangle = & [\hbar\omega - (E_{\nu'} - E_{\nu})] \langle \nu | f_2 | \nu' \rangle \\ & + [f_0(E_{\nu'}) - f_0(E_{\nu})] \langle \nu | H_1 | \nu' \rangle. \end{aligned} \quad (11)$$

substitution of equation (7) into equation (11) gives

$$\begin{aligned} [-(\hbar\omega - i\hbar/\tau) + E_{\nu'} - E_{\nu}] \langle \nu | f_1 | \nu' \rangle = & \left\{ \hbar\omega [f_0(E_{\nu'}) - f_0(E_{\nu})] \right. \\ & \left. / [E_{\nu'} - E_{\nu}] \right\} \langle \nu | H_1 | \nu' \rangle - [f_0(E_{\nu'}) - f_0(E_{\nu})] \\ & \cdot \langle \nu | H_1 | \nu' \rangle + [f_0(E_{\nu'}) - f_0(E_{\nu})] \langle \nu | H_1 | \nu' \rangle. \end{aligned}$$

The above equation becomes

$$\begin{aligned} \langle \nu | f_1 | \nu' \rangle = & \langle \nu | H_1 | \nu' \rangle [f_0(E_{\nu'}) - f_0(E_{\nu})] \hbar\omega / \{ (E_{\nu'} - E_{\nu}) \\ & \cdot (E_{\nu'} - E_{\nu} - \hbar\omega') \}, \end{aligned} \quad (12)$$

where $\hbar\omega' = \hbar\omega - i\hbar/\tau$.

The sum of equation (7) and equation (12) is

$$\begin{aligned} \langle \nu | f_1 + f_2 | \nu' \rangle = & \left\{ [f_0(E_{\nu'}) - f_0(E_{\nu})] / [E_{\nu'} - E_{\nu}] \right\} \\ & \cdot [\hbar\omega / (E_{\nu'} - E_{\nu} - \hbar\omega') + 1] \langle \nu | H_1 | \nu' \rangle. \end{aligned} \quad (13)$$

The term $\hbar\omega/[(E_{\nu_1} - E_{\nu})(E_{\nu_1} - E_{\nu} - \hbar\omega')]$ can be rewritten as $(\hbar\omega/\hbar\omega') \hbar\omega'/[(E_{\nu_1} - E_{\nu})(E_{\nu_1} - E_{\nu} - \hbar\omega')]$. The factor $\hbar\omega/\hbar\omega'$ can be rearranged to give

$$\hbar\omega/\hbar\omega' = i\omega\tau/(1 + i\omega\tau).$$

If $\hbar\omega'$ is written as $\hbar\omega' = (E_{\nu_1} - E_{\nu}) + (E_{\nu_1} - E_{\nu} - \hbar\omega')$, then

$$\hbar\omega'/[(E_{\nu_1} - E_{\nu})(E_{\nu_1} - E_{\nu} - \hbar\omega')] = -1/(E_{\nu_1} - E_{\nu}) + 1/(E_{\nu_1} - E_{\nu} - \hbar\omega')$$

Equation (13) becomes

$$\begin{aligned} \langle \nu | f_1 + f_2 | \nu' \rangle &= [f_0(E_{\nu_1}) - f_0(E_{\nu})] \left\{ [i\omega\tau/(1 + i\omega\tau)] \right. \\ &\quad \cdot [1/(E_{\nu_1} - E_{\nu} - \hbar\omega') - 1/(E_{\nu_1} - E_{\nu})] \\ &\quad \left. + 1/(E_{\nu_1} - E_{\nu}) \right\} \langle \nu | H_1 | \nu' \rangle. \end{aligned}$$

$$\begin{aligned} \langle \nu | f_1 + f_2 | \nu' \rangle &= [f_0(E_{\nu_1}) - f_0(E_{\nu})] \left\{ [i\omega\tau/(1 + i\omega\tau)] \right. \\ &\quad \cdot [1/(E_{\nu_1} - E_{\nu} - \hbar\omega') - i\omega\tau/(1 + i\omega\tau)(E_{\nu_1} - E_{\nu})] \\ &\quad \left. + 1/(E_{\nu_1} - E_{\nu}) \right\} \langle \nu | H_1 | \nu' \rangle. \end{aligned}$$

Addition of the last two terms gives

$$\begin{aligned} \langle \nu | f_1 + f_2 | \nu' \rangle &= \left\{ [f_0(E_{\nu_1}) - f_0(E_{\nu})] / (1 + i\omega\tau) \right\} \\ &\quad \cdot [i\omega\tau/(E_{\nu_1} - E_{\nu} - \hbar\omega') + 1/(E_{\nu_1} - E_{\nu})] \\ &\quad \cdot \langle \nu | H_1 | \nu' \rangle. \end{aligned}$$

This may be written as

$$\langle \nu | f_1 + f_2 | \nu' \rangle = \Lambda_{\nu\nu'} \langle \nu | H_1 | \nu' \rangle \quad (14a)$$

where

$$\begin{aligned} \Lambda_{\nu\nu'} &= [i\omega\tau/(1 + i\omega\tau)] [f_0(E_{\nu_1}) - f_0(E_{\nu})] / (E_{\nu_1} - E_{\nu} - \hbar\omega') \\ &\quad + [1/(1 + i\omega\tau)] [f_0(E_{\nu_1}) - f_0(E_{\nu})] / (E_{\nu_1} - E_{\nu}). \end{aligned} \quad (14b)$$

Current Density

The current density is given as the trace of the product of the density matrix f and the current operator j_{op} (6, p. 108)

$$j(\vec{r}_0, t) = \text{Tr}\{f j_{op}\},$$

where

$$j_{op} = -\frac{1}{2}e\vec{v}\delta(\vec{r}_0 - \vec{r}) + (-\frac{1}{2}e)\delta^+(\vec{r}_0 - \vec{r})\vec{v}^+.$$

Substitution of $\vec{v} = \vec{v}_0 + e\vec{A}/Mc$ gives

$$j(\vec{r}_0, t) = \sum_{\nu} \left\{ \langle \nu | (-e/2)(\vec{v}_0 + e\vec{A}/Mc)\delta(\vec{r}_0 - \vec{r}) [f_0(H_0, \eta) + f_1 + f_2] | \nu \rangle \right. \\ \left. + \langle \nu | (-e/2)\delta^+(\vec{r}_0 - \vec{r})(\vec{v}_0^+ + e\vec{A}^+/Mc) [f_0(H_0, \eta) + f_1 + f_2] | \nu \rangle \right\}.$$

This becomes

$$j(\vec{r}_0, t) = \sum_{\nu} \left\{ \langle \nu | (-e/2)\vec{v}_0\delta(\vec{r}_0 - \vec{r})(f_0 + f_1 + f_2) | \nu \rangle \right. \\ + \langle \nu | (-e^2/2Mc)\vec{A}\delta(\vec{r}_0 - \vec{r})(f_0 + f_1 + f_2) | \nu \rangle \\ + \langle \nu | (-e/2)\delta^+(\vec{r}_0 - \vec{r})\vec{v}_0^+(f_0 + f_1 + f_2) | \nu \rangle \\ \left. + \langle \nu | (-e^2/2Mc)\delta^+(\vec{r}_0 - \vec{r})\vec{A}^+(f_0 + f_1 + f_2) | \nu \rangle \right\}$$

Separation of f_0 and $f_1 + f_2$, and collection of similar terms gives

$$j(\vec{r}_0, t) = \sum_{\nu} \left\{ \langle \nu | (-e/2)\vec{v}_0\delta(\vec{r}_0 - \vec{r})f_0 | \nu \rangle \right. \quad (15) \\ + \langle \nu | (-e/2)\delta^+(\vec{r}_0 - \vec{r})\vec{v}_0^+ f_0 | \nu \rangle \\ + \langle \nu | (-e/2)\vec{v}_0\delta(\vec{r}_0 - \vec{r})(f_1 + f_2) | \nu \rangle \\ + \langle \nu | (-e/2)\delta^+(\vec{r}_0 - \vec{r})\vec{v}_0^+(f_1 + f_2) | \nu \rangle \\ + \langle \nu | (-e^2/2Mc)\vec{A}\delta(\vec{r}_0 - \vec{r})f_0 | \nu \rangle \\ + \langle \nu | (-e^2/2Mc)\delta^+(\vec{r}_0 - \vec{r})\vec{A}^+ f_0 | \nu \rangle \\ \left. + \langle \nu | (-e^2/2Mc)\vec{A}\delta(\vec{r}_0 - \vec{r})(f_1 + f_2) | \nu \rangle \right\}$$

$$+ \langle \nu | (-e^2/2Mc) \delta^+(\vec{r}_0 - \vec{r}) \vec{A}^+(f_1 + f_2) | \nu \rangle .$$

Because they are hermitian $\vec{v}_0^+ = \vec{v}_0$ and $\vec{A}^+ = \vec{A}$. With the delta function written as

$$\delta(\vec{r} - \vec{r}_0) = 1/(2\pi)^3 \int_{-\infty}^{\infty} \exp[i\vec{q}' \cdot (\vec{r} - \vec{r}_0)] d\vec{q}' ,$$

then

$$\delta(\vec{r} - \vec{r}_0) = \delta(\vec{r}_0 - \vec{r})$$

because the integration is over positive and negative values of $\vec{q}'$. The first two terms in equation (15) represent the current in the absence of the perturbing fields. This current must be zero. The last two terms in equation (15) are non linear in A. Therefore, to the first order in A equation (15) becomes

$$j(\vec{r}_0, t) = \sum_{\nu} -e \langle \nu | [\frac{1}{2} \vec{v}_0 \delta(\vec{r} - \vec{r}_0) + \frac{1}{2} \delta(\vec{r} - \vec{r}_0) \vec{v}_0] (f_1 + f_2) | \nu \rangle \\ - (e^2/Mc) \langle \nu | \vec{A} \delta(\vec{r} - \vec{r}_0) f_0 | \nu \rangle ,$$

where the last term is the result of $\vec{A} \delta(\vec{r} - \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \vec{A}$ in the next to the last two terms of equation (15). If spins are taken into account with a factor of two,

$$j(\vec{r}_0, t) = \sum_{\nu} \left\{ -2e \langle \nu | [\frac{1}{2} \vec{v}_0 \delta(\vec{r} - \vec{r}_0) + \frac{1}{2} \delta(\vec{r} - \vec{r}_0) \vec{v}_0] (f_1 + f_2) | \nu \rangle \right. \\ \left. - (2e^2/Mc) \langle \nu | \vec{A} \delta(\vec{r} - \vec{r}_0) f_0 | \nu \rangle \right\} .$$

Consider the current density in two parts as j_{o_1} and j_{o_2}

where

$$j_{o_1}(\vec{r}_0, t) = \sum_{\nu} \left\{ -2e \langle \nu | [\frac{1}{2} \vec{v}_0 \delta(\vec{r} - \vec{r}_0) + \frac{1}{2} \delta(\vec{r} - \vec{r}_0) \vec{v}_0] (f_1 + f_2) | \nu \rangle \right\}$$

and

$$j_{o_2}(\vec{r}_0, t) = \sum_{\nu} \left\{ (-2e^2/Mc) \langle \nu | \vec{A} \delta(\vec{r} - \vec{r}_0) f_0 | \nu \rangle \right\}$$

To obtain $j_1(\vec{q}, \omega)$ for a finite sample, and arbitrary $\vec{A}$ the fourier transform of $j_1(\vec{r}_0, t)$ is written as

$$j_1(\vec{q}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j_1(\vec{r}_0, t) \exp i(\omega t - \vec{q} \cdot \vec{r}_0) d\vec{r}_0 dt. \quad (16)$$

This gives

$$\begin{aligned} j_1(\vec{q}, \omega) = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu} \{ (-2e/(2\pi)^3) \langle \nu' | [\frac{1}{2} \vec{v}_0 \exp[i\vec{q}' \cdot (\vec{r}_0 - \vec{r})] \\ & + \frac{1}{2} \exp[i\vec{q}' \cdot (\vec{r}_0 - \vec{r})] \vec{v}_0 (f_1 + f_2) | \nu' \rangle \\ & \cdot \exp[i(\omega t - \vec{q} \cdot \vec{r}_0)] d\vec{r}_0 dt d\vec{q}' \}. \end{aligned}$$

This can be expanded using the basis set of eigenvectors

$\sum_{\nu} |\nu\rangle \langle \nu|$ to give

$$\begin{aligned} j_1(\vec{q}, \omega) = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \{ (-2e/(2\pi)^3) \langle \nu' | [\frac{1}{2} \vec{v}_0 \exp i\vec{q}' \cdot (\vec{r}_0 - \vec{r}) \\ & + \frac{1}{2} \exp i\vec{q}' \cdot (\vec{r}_0 - \vec{r}) \vec{v}_0] | \nu \rangle \langle \nu' | (f_1 + f_2) | \nu' \rangle \\ & \cdot \exp[i(\omega t - \vec{q} \cdot \vec{r}_0)] d\vec{r}_0 dt d\vec{q}' \}. \end{aligned}$$

The matrix elements $\langle \nu' | f_1 + f_2 | \nu' \rangle$ can be moved outside the integral over $\vec{r}_0$, and $\exp(i\vec{q}' \cdot \vec{r}_0)$ commutes with $\vec{v}_0$ so it can be factored to give

$$\begin{aligned} j_1(\vec{q}, \omega) = & (-2e/(2\pi)^3) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | [\frac{1}{2} \vec{v}_0 \exp(-i\vec{q}' \cdot \vec{r}) \\ & + \frac{1}{2} \exp(-i\vec{q}' \cdot \vec{r}) \vec{v}_0] \int_{-\infty}^{\infty} \exp[i(\vec{q}' - \vec{q}) \cdot \vec{r}_0] d\vec{r}_0 | \nu \rangle \\ & \cdot \langle \nu' | f_1 + f_2 | \nu' \rangle \exp(i\omega t) dt d\vec{q}' \}. \end{aligned}$$

However,

$$1/(2\pi)^3 \int_{-\infty}^{\infty} \exp[i(\vec{q}' - \vec{q}) \cdot \vec{r}_0] d\vec{r}_0 = \delta(\vec{q}' - \vec{q}).$$

Substitution of equation (14) and use of $\delta(\vec{q}' - \vec{q})$ gives

$$j_1(q,) = -2e \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | \left[\frac{1}{2} \vec{v}_0 \exp(-i\vec{q} \cdot \vec{r}) + \frac{1}{2} \exp(-i\vec{q} \cdot \vec{r}) \vec{v}_0 \right] | \nu \rangle \cdot \Delta_{\nu, \nu'} \langle \nu' | H_1 | \nu \rangle \exp(i\omega t) dt.$$

Because $H_1 = (e/2c) [\vec{v}_0 \cdot \vec{A}(\vec{r}, t) + \vec{A}(\vec{r}, t) \cdot \vec{v}_0]$,

$$j_1(\vec{q}, \omega) = (-e^2/c) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | V(\vec{q}) | \nu \rangle \Delta_{\nu, \nu'} \langle \nu' | \vec{v}_0 \cdot \vec{A}(\vec{r}, t) + \vec{A}(\vec{r}, t) \cdot \vec{v}_0 | \nu \rangle \exp(i\omega t) dt$$

where

$$\vec{V}(\vec{q}) = \frac{1}{2} \vec{v}_0 \exp(-i\vec{q} \cdot \vec{r}) + \frac{1}{2} \exp(-i\vec{q} \cdot \vec{r}) \vec{v}_0.$$

The Fourier transform

$$\vec{A}(\vec{r}, t) = (1/2\pi) \int_{-\infty}^{\infty} A(\vec{q}', t) \exp(-i\vec{q}' \cdot \vec{r}) d\vec{q}'$$

can be used to give

$$j_1(\vec{q}, \omega) = (-e^2/c) \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | V(\vec{q}) | \nu \rangle \Delta_{\nu, \nu'} \langle \nu' | \vec{v}_0 \cdot \vec{A}(\vec{q}', t) \exp(-i\vec{q}' \cdot \vec{r}) + \vec{A}(\vec{q}', t) \exp(-i\vec{q}' \cdot \vec{r}) \cdot \vec{v}_0 | \nu \rangle \exp(i\omega t) dt d\vec{q}'.$$

Also

$$\int_{-\infty}^{\infty} \vec{A}(\vec{q}', t) \exp(i\omega t) dt = \vec{A}(\vec{q}', \omega),$$

so

$$j_1(\vec{q}, \omega) = (-e^2/2\pi c) \sum_{\nu, \nu'} \langle \nu' | V(\vec{q}) | \nu \rangle \Delta_{\nu, \nu'} \langle \nu' | \vec{v}_0 \cdot \vec{A}(\vec{q}', \omega) \exp(-i\vec{q}' \cdot \vec{r}) + \vec{A}(\vec{q}', \omega) \exp(-i\vec{q}' \cdot \vec{r}) \cdot \vec{v}_0 | \nu \rangle.$$

Because $\exp(-i\vec{q}' \cdot \vec{r})$ is not a vector it can be factored past the dot product. Also $\vec{A}(\vec{q}', \omega)$ is not an operator so it commutes with $\exp(-i\vec{q}' \cdot \vec{r})$ to give

$$j_1(\vec{q}, \omega) = (-e^2/2\pi c) \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | \vec{V}(\vec{q}) | \nu \rangle \Delta_{\nu, \nu'} \langle \nu' | \vec{v}_0 \exp(-i\vec{q}' \cdot \vec{r}) + \exp(-i\vec{q}' \cdot \vec{r}) \vec{v}_0 \cdot \vec{A}(\vec{q}', \omega) | \nu \rangle d\vec{q}'.$$

Expansion with the set of basis vectors $\sum_{\nu''} |\nu''\rangle \langle \nu'|$ gives

$$j_1(\vec{q}, \omega) = (-e^2/\pi c) \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | \vec{V}(\vec{q}) | \nu \rangle \langle \nu | \vec{V}(\vec{q}') | \nu'' \rangle \sum_{\nu''} | \nu'' \rangle \cdot \langle \nu'' | \vec{A}(\vec{q}', \omega) | \nu \rangle d\vec{q}' \Lambda_{\nu\nu'}.$$

Because $\vec{A}(\vec{q}', \omega)$ is diagonal

$$j_1(\vec{q}, \omega) = (-e^2/\pi c) \int_{-\infty}^{\infty} \sum_{\nu, \nu'} \langle \nu' | \vec{V}(\vec{q}) | \nu \rangle \langle \nu | \vec{V}(\vec{q}') | \nu' \rangle \cdot \vec{A}(\vec{q}', \omega) \Lambda_{\nu\nu'} d\vec{q}'. \quad (17)$$

$j_2(\vec{q}, \omega)$ for a finite sample is obtained by using the subscript two in equation (16) and writing $\delta(\vec{r} - \vec{r}_0)$ as before to give

$$j_2(\vec{q}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu'} [-2e^2/Mc(2\pi)^3] \langle \nu' | \vec{A}(\vec{r}, t) \exp[i\vec{q}' \cdot (\vec{r}_0 - \vec{r})] \cdot f_0 | \nu' \rangle \exp[i(\omega t - \vec{q} \cdot \vec{r}_0)] d\vec{r}_0 dt d\vec{q}'.$$

Expansion with the basis set of vectors $\sum_{\nu} | \nu \rangle \langle \nu |$ gives

$$j_2(\vec{q}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu'} [-2e^2/Mc(2\pi)^3] \langle \nu' | \vec{A}(\vec{r}, t) \exp[i\vec{q}' \cdot (\vec{r}_0 - \vec{r})] \sum_{\nu} | \nu \rangle \cdot \langle \nu | f_0 | \nu' \rangle \exp[i(\omega t - \vec{q} \cdot \vec{r}_0)] d\vec{r}_0 dt d\vec{q}'.$$

Factorization of $\exp(i\vec{q}' \cdot \vec{r}_0)$ and $\langle \nu | f_0 | \nu' \rangle$ gives

$$j_2(\vec{q}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{\nu, \nu'} [-2e^2/Mc(2\pi)^3] \langle \nu' | \vec{A}(\vec{r}, t) \exp(-i\vec{q}' \cdot \vec{r}) | \nu \rangle \int_{-\infty}^{\infty} \cdot \exp[i(\vec{q}' - \vec{q}) \cdot \vec{r}_0] d\vec{r}_0 d\vec{q}' \langle \nu | f_0 | \nu' \rangle \exp(i\omega t) dt.$$

But

$$j_2(\vec{q}, \omega) = (-2e^2/Mc) \sum_{\nu} \langle \nu | \int_{-\infty}^{\infty} \vec{A}(\vec{r}, t) \exp(-i\vec{q} \cdot \vec{r}) \exp(i\omega t) dt | \nu \rangle f_{0\nu}.$$

Because

$$\int_{-\infty}^{\infty} \vec{A}(\vec{r}, t) \exp(i\omega t) dt = \vec{A}(\vec{r}, \omega)$$

$$j_2(\vec{q}, \omega) = (-2e^2/Mc) \sum_{\nu} \langle \nu | \vec{A}(\vec{r}, \omega) \exp(-i\vec{q} \cdot \vec{r}) | \nu \rangle f_{0\nu}.$$

Substitution of

$$A(\vec{r}, \omega) = (1/2\pi) \int_{-\infty}^{\infty} d\vec{q}' \vec{A}(\vec{q}', \omega) \exp(i\vec{q}' \cdot \vec{r})$$

gives

$$j_2(\vec{q}, \omega) = (-e^2/Mc\pi) \sum_{\nu} \langle \nu | \int_{-\infty}^{\infty} d\vec{q}' \vec{A}(\vec{q}', \omega) \exp[-i(\vec{q} - \vec{q}') \cdot \vec{r}] | \nu \rangle f_{0\nu\nu}.$$

Expansion with the set of basis vectors $\sum_{\nu'} | \nu' \rangle \langle \nu' |$ gives

$$j_2(\vec{q}, \omega) = (-e^2/Mc\pi) \sum_{\nu'} \langle \nu | \int_{-\infty}^{\infty} d\vec{q}' \vec{A}(\vec{q}', \omega) | \nu' \rangle \cdot \langle \nu' | \exp[-i(\vec{q} - \vec{q}') \cdot \vec{r}] | \nu \rangle f_{0\nu\nu}.$$

Therefore

$$j_2(\vec{q}, \omega) = (-e^2/Mc) \sum_{\nu\nu'} \int_{-\infty}^{\infty} d\vec{q}' \langle \nu | \vec{A}(\vec{q}', \omega) | \nu' \rangle \cdot \langle \nu' | \exp[-i(\vec{q} - \vec{q}') \cdot \vec{r}] | \nu \rangle f_{0\nu\nu}.$$

Because $\vec{A}(\vec{q}', \omega)$ is diagonal

$$\begin{aligned} \langle \nu | \vec{A}(\vec{q}', \omega) | \nu' \rangle &= \vec{A}(\vec{q}', \omega) \delta_{\nu\nu'}, \\ j_2(\vec{q}, \omega) &= (-e^2/Mc\pi) \sum_{\nu} \int_{-\infty}^{\infty} d\vec{q}' \vec{A}(\vec{q}', \omega) \cdot \langle \nu | \exp[-i(\vec{q} - \vec{q}') \cdot \vec{r}] | \nu \rangle f_{0\nu\nu}. \end{aligned} \quad (18)$$

Because $j_1(\vec{q}, \omega)$ leads to size dependent oscillations in the conductivity and $j_2(\vec{q}, \omega)$ does not, only j_1 will be used from this point on. The calculation for j_2 will be done in the appendix. The coefficient can be rewritten as

$$-e^2/\pi c = (Ne^2/VMc)(-VM/\pi N) = (\omega_p^2/4\pi c)(-VM/\pi N)$$

where N is the number of particles in the system and ω_p is the plasma frequency. Equation (17) becomes

$$j(\vec{q}, \omega) = (\omega_p^2/4\pi c) \left\{ (-VM/\pi N) \sum_{\nu\nu'} \int_{-\infty}^{\infty} d\vec{q}' \Lambda_{\nu\nu'} \langle \nu' | \vec{v}(\vec{q}) | \nu \rangle \cdot \langle \nu' | \vec{v}(\vec{q}') | \nu \rangle^* \right\} \cdot \vec{A}(\vec{q}', \omega) d\vec{q}' \quad (19)$$

For low frequencies where the diffusion current is negligible,

$$j(\vec{q}, \omega) = \sigma(\vec{q}, \vec{q}', \omega) \cdot \vec{E}(\vec{q}', \omega) . \quad (20)$$

From Maxwell's equations

$$\vec{E} = - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

for the gauge where $\phi = 0$. If $\vec{A}$ varies as $\exp(i\omega t)$, then

$$\vec{E} = - \frac{1}{c} (i\omega) \vec{A}$$

and

$$\vec{A} = - \frac{c}{i\omega} \vec{E}.$$

Substitution of $-c\vec{E}/i\omega$ for $\vec{A}$ in equation (19) and comparison with equation (20) gives

$$\sigma(\vec{q}, \vec{q}', \omega) = (\omega_p^2 / 4\pi i \omega) \left[(VM/\pi N) \sum_{\nu, \nu'} \int_{-\infty}^{\infty} \Lambda_{\nu, \nu'} \langle \nu' | \vec{v}(\vec{q}) | \nu \rangle \cdot \langle \nu' | \vec{v}(\vec{q}') | \nu \rangle^* d\vec{q}' \right] \quad (21)$$

Matrix Elements of the Velocity Operator

The effects of the size on the conductivity show up in the xy element. From equation (21)

$$\sigma_{xy} = (\omega_p^2 / 4\pi i \omega) (VM/N) \int_{-\infty}^{\infty} d\vec{q}' \sum_{\nu, \nu'} \Lambda_{\nu, \nu'} \langle \nu' | v_x(\vec{q}) | \nu \rangle \cdot \langle \nu' | v_y(\vec{q}') | \nu \rangle^* , \quad (22)$$

where

$$v_x(\vec{q}) = \frac{1}{2} v_{o_x} \exp(-i\vec{q} \cdot \vec{r}) + \exp(-i\vec{q} \cdot \vec{r}) v_{o_x}$$

and

$$v_y(\vec{q}) = \frac{1}{2} v_{o_y} \exp(-i\vec{q} \cdot \vec{r}) + \frac{1}{2} \exp(-i\vec{q} \cdot \vec{r}) v_{o_y}$$

For the case where $\vec{q} = (0, 0, q_z)$, $\exp(-i\vec{q} \cdot \vec{r})$ becomes $\exp(-iq_z z)$

and may then be factored out past the operators V_{o_x} and V_{o_y} which operate only on x and y respectively. Because

$$\exp(-iq_z z) = \cos q_z z - i \sin q_z z,$$

$V_x(q)$ and $V_y(q)$ can be considered in two parts,

$$V_x(\vec{q}) = V_{x_1} - iV_{x_2}$$

and

$$V_y(\vec{q}) = V_{y_1} - iV_{y_2}$$

where

$$V_{i_1} = V_{o_i} \cos q_z z$$

and

$$V_{i_2} = V_{o_i} \sin q_z z$$

for $i = x, y$. Because the stationary states are either even or odd, there are four combinations for possible transitions for the matrix element evaluation of the velocity operator (q) . These four cases will be indicated as follows:

$$(1) \langle \nu'_o | v | \nu_o \rangle ,$$

$$(2) \langle \nu'_e | v | \nu_e \rangle ,$$

$$(3) \langle \nu'_o | v | \nu_e \rangle ,$$

$$(4) \langle \nu'_e | v | \nu_o \rangle .$$

For case number one

$$\begin{aligned} \langle \nu' | v_{x_1}(\vec{q}) | \nu \rangle &= \int_{-a/2}^{a/2} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx (2/L_y a) \exp(-ik'_y y) \\ &\cdot \sin k'_z z U_n^*(x + \hbar k'_y / M \omega_c) \cos q_z z v_{o_x} \end{aligned}$$

$$\begin{aligned}
& \cdot \exp(ik_y y) \sin k_z z U_n(x+hk_y/M\omega_c) \\
\langle v' | v_{x_1}(\vec{q}) | v \rangle &= (2/L_y a) \int_{-a/2}^{a/2} dz \sin k'_z z \cos q_z z \\
& \cdot \sin k_z z \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \left[\exp(-ik_y y) \right. \\
& \cdot U_n^*(x+hk_y/M\omega_c) v_{o_x} \exp(ik_y y) \\
& \cdot U_n(x+hk_y/M\omega_c) \left. \right].
\end{aligned}$$

If the integrals over x and y above are labelled as X_i where $i = x, y$ according to whether v_{o_i} is v_{o_x} or v_{o_y} , then:

$$\begin{aligned}
\langle v' | v_{x_1}(\vec{q}) | v \rangle &= \left[(2/L_y a) \int_{-a/2}^{a/2} dz \sin k'_z z \cos q_z z \sin k_z z \right] X_x, \\
\langle v' | v_{x_2}(\vec{q}) | v \rangle &= \left[(2/L_y a) \int_{-a/2}^{a/2} dz \sin k'_z z \sin q_z z \sin k_z z \right] X_x, \\
\langle v' | v_{y_1}(\vec{q}) | v \rangle &= \left[(2/L_y a) \int_{-a/2}^{a/2} dz \sin k'_z z \cos q_z z \sin k_z z \right] X_y, \\
\langle v' | v_{y_2}(\vec{q}) | v \rangle &= \left[(2/L_y a) \int_{-a/2}^{a/2} dz \sin k'_z z \sin q_z z \sin k_z z \right] X_y.
\end{aligned}$$

The integrals $\int_{-a/2}^{a/2} dz \sin k'_z z \sin q_z z \sin k_z z$ will be zero since the integrand is an odd function of z and the limits are symmetric about zero. Therefore the matrix elements become:

$$\langle v'_o | v_x(\vec{q}) | v_o \rangle = \langle v'_o | v_{x_1}(\vec{q}) | v_o \rangle, \quad (23a)$$

$$\langle v'_o | v_y(\vec{q}) | v_o \rangle = \langle v'_o | v_{y_1}(\vec{q}) | v_o \rangle. \quad (23b)$$

For case number two the integrals will be the same as for the case number one matrix elements with $\cos k_z z$

everywhere replaced by $\sin k_z z$. This gives the following:

$$\langle \nu' | v_{x_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \cos k_z z \right] X_x,$$

$$\langle \nu' | v_{x_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \cos k_z z \right] X_x,$$

$$\langle \nu' | v_{y_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \cos k_z z \right] X_y,$$

$$\langle \nu' | v_{y_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \cos k_z z \right] X_y.$$

The integrals $\int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \cos k_z z$ will be zero since the integrand is an odd function of z and the limits are symmetric about zero. The matrix elements become:

$$\langle \nu'_e | v_x(\vec{q}) | \nu_e \rangle = \langle \nu'_e | v_{x_1}(\vec{q}) | \nu_e \rangle, \quad (24a)$$

$$\langle \nu'_e | v_y(\vec{q}) | \nu_e \rangle = \langle \nu'_e | v_{y_1}(\vec{q}) | \nu_e \rangle. \quad (24b)$$

For case number three the integrals over x and y are the same as for case number one and case number two, but the integral over z will depend on $\sin k'_z z$ for $|\nu'\rangle$ odd and $\cos k_z z$ for $|\nu\rangle$ even. This gives the following:

$$\langle \nu' | v_{x_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \cos k_z z \right] X_x,$$

$$\langle \nu' | v_{x_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \cos k_z z \right] X_x,$$

$$\langle \nu' | v_{y_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \cos k_z z \right] X_y,$$

$$\langle \nu' | v_{y_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \cos k_z z \right] X_y.$$

The integrals $\int_{-a/2}^{a/2} dz \sin k'_z z \cos q_z z \cos k_z z$ will be zero

because the integrand is an odd function of z and the limits are symmetric about zero. Therefore the matrix elements become:

$$\langle \nu'_o | v_x(\vec{q}) | \nu_e \rangle = \langle \nu'_o | v_{x_2}(\vec{q}) | \nu_e \rangle (-i) , \quad (25a)$$

$$\langle \nu'_o | v_y(\vec{q}) | \nu_e \rangle = \langle \nu'_o | v_{y_2}(\vec{q}) | \nu_e \rangle (-i) . \quad (25b)$$

For case number four the integrals are the same as for case number three, but with $\cos k_z z$ and $\sin k_z z$ interchanged. This gives the following:

$$\langle \nu' | v_{x_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \sin k_z z \right] X_x ,$$

$$\langle \nu' | v_{x_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \sin k_z z \right] X_x ,$$

$$\langle \nu' | v_{y_1}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \sin k_z z \right] X_y ,$$

$$\langle \nu' | v_{y_2}(\vec{q}) | \nu \rangle = \left[(2/L_y a) \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \sin k_z z \right] X_y .$$

The integrals $\int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \sin k_z z$ will be zero because

the integrand is an odd function of z and the limits are symmetric about zero. Therefore the matrix elements become:

$$\langle \nu'_e | v_x(\vec{q}) | \nu_o \rangle = (-i) \langle \nu'_e | v_{x_2}(\vec{q}) | \nu_o \rangle , \quad (26a)$$

$$\langle \nu'_e | v_y(\vec{q}) | \nu_o \rangle = (-i) \langle \nu'_e | v_{y_2}(\vec{q}) | \nu_o \rangle . \quad (26b)$$

The evaluation of the matrix elements depends on the

evaluation of the four integrals:

$$\begin{aligned}
 I_1 &= \int_{-a/2}^{a/2} dz \sin k'_z z \cos q_z z \sin k_z z, \\
 I_2 &= \int_{-a/2}^{a/2} dz \cos k'_z z \cos q_z z \cos k_z z, \\
 I_3 &= \int_{-a/2}^{a/2} dz \sin k_z z \sin q_z z \cos k'_z z, \\
 I_4 &= \int_{-a/2}^{a/2} dz \cos k'_z z \sin q_z z \sin k_z z.
 \end{aligned}$$

The trigonometric identities

$$\cos(a-b) = \cos a \cos b + \sin a \sin b,$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b,$$

and

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

are used to evaluate the integrals.

For the odd state to odd state transition

$$I_1 = \int_{-a/2}^{a/2} dz \left[\frac{1}{2} \cos(k'_0 - k_0) z - \frac{1}{2} \cos(k'_0 + k_0) z \right] \cos q_z z,$$

where $k_0 = 2\pi m/a$ and $k'_0 = 2\pi m'/a$.

$$I_1 = \int_{-a/2}^{a/2} dz \left[\frac{1}{2} \cos(k'_0 - k_0) z \cos q_z z \right] - \int_{-a/2}^{a/2} dz \left[\frac{1}{2} \cos(k'_0 + k_0) z \cos q_z z \right]$$

Use of the standard integral,

$$\int \cos x \cos y = \frac{\sin(x-y)}{2(x-y)} + \frac{\sin(x+y)}{2(x+y)}$$

gives

$$I_1 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_0 - q)z}{2(k'_0 - k_0 - q)} + \frac{\sin(k'_0 - k_0 + q)z}{2(k'_0 - k_0 + q)} \right]_{-a/2}^{a/2} \\ - \frac{1}{2} \left[\frac{\sin(k'_0 + k_0 - q)z}{2(k'_0 + k_0 - q)} + \frac{\sin(k'_0 + k_0 + q)z}{2(k'_0 + k_0 + q)} \right]_{-a/2}^{a/2}.$$

Evaluated at the end points, this becomes

$$I_1 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_0 - q)a/2}{2(k'_0 - k_0 - q)} - \frac{\sin(k'_0 - k_0 - q)(-a/2)}{2(k'_0 - k_0 - q)} + \frac{\sin(k'_0 - k_0 + q)(a/2)}{2(k'_0 - k_0 + q)} \right. \\ \left. - \frac{\sin(k'_0 - k_0 + q)(-a/2)}{2(k'_0 - k_0 + q)} \right] - \frac{1}{2} \left[\frac{\sin(k'_0 + k_0 - q)a/2}{2(k'_0 + k_0 - q)} - \frac{\sin(k'_0 + k_0 - q)(-a/2)}{2(k'_0 + k_0 - q)} \right. \\ \left. + \frac{\sin(k'_0 + k_0 + q)a/2}{2(k'_0 + k_0 + q)} - \frac{\sin(k'_0 + k_0 + q)(-a/2)}{2(k'_0 + k_0 + q)} \right].$$

Because $\sin(-x) = -\sin x$

$$I_1 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_0 - q)a/2}{k'_0 - k_0 - q} + \frac{\sin(k'_0 - k_0 + q)a/2}{k'_0 - k_0 + q} - \frac{\sin(k'_0 + k_0 - q)a/2}{k'_0 + k_0 - q} \right. \\ \left. - \frac{\sin(k'_0 + k_0 + q)a/2}{k'_0 + k_0 + q} \right]$$

$$I_1 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_0)a/2 \cos qa/2 - \cos(k'_0 - k_0)a/2 \sin qa/2}{k'_0 - k_0 + q} \right. \\ + \frac{\sin(k'_0 - k_0)a/2 \cos qa/2 + \cos(k'_0 - k_0)a/2 \sin qa/2}{k'_0 - k_0 + q} \\ - \frac{\sin(k'_0 + k_0)a/2 \cos qa/2 - \cos(k'_0 + k_0)a/2 \sin qa/2}{k'_0 + k_0 - q} \\ \left. - \frac{\sin(k'_0 + k_0)a/2 \cos qa/2 + \cos(k'_0 + k_0)a/2 \sin qa/2}{k'_0 + k_0 + q} \right]$$

Because $k_0 = 2\pi m/a$

$$a/2(k'_0 - k_0) = (2m'\pi/a - 2m\pi/a)a/2 = (m' - m)\pi$$

and

$$a/2(k'_0 + k_0) = (m' + m)\pi .$$

Therefore $\sin(k'_0 - k_0)a/2 = 0$, $\sin(k'_0 + k_0)a/2 = 0$,

$$\cos(k'_0 - k_0)a/2 = (-1)^{m' - m} \text{ and } \cos(k'_0 + k_0)a/2 = (-1)^{m' + m} .$$

If $m' - m$ is an even or odd integer, $m' + m$ will correspondingly be even or odd so,

$$I_1 = \frac{1}{2}(-1)^{m' + m} \sin qa/2$$

$$\cdot \left[\frac{-1}{k'_0 - k_0 - q} + \frac{1}{k'_0 - k_0 + q} + \frac{1}{k'_0 + k_0 - q} + \frac{-1}{k'_0 + k_0 + q} \right] .$$

For the even state to even state transition

$$I_2 = \int_{-a/2}^{a/2} dz \frac{1}{2} \left[\cos(k'_e - k_e)z + \cos(k'_e + k_e)z \right] \cos q_z z .$$

This integral is like I_1 except k_0 is replaced by k_e and the last half of the terms are preceded by a (+) instead of a (-). Because $k_e = 2\pi(m - \frac{1}{2})/a$,

$$a/2(k'_e - k_e) = (2\pi/a)(a/2)(m' - \frac{1}{2}) - (2\pi/a)(a/2)(m - \frac{1}{2}) = (m' - m)\pi$$

and

$$a/2(k'_e + k_e) = (m' + m + 1)\pi .$$

Therefore $\sin(k'_e - k_e)a/2 = 0$, $\sin(k'_e + k_e)a/2 = 0$,

$$\cos(k'_e - k_e)a/2 = (-1)^{m' - m}, \text{ and } \cos(k'_e + k_e)a/2 = -(-1)^{m' + m}$$

Because $\cos(k'_0 + k_0)a/2 = -(-1)^{m' + m}$ places a negative sign in front of the last two terms I_2 is the same as I_1 except for odd replaced by even everywhere.

$$I_2 = \frac{1}{2}(-1)^{m' + m} \sin qa/2$$

$$\cdot \left[\frac{-1}{k'_e - k_e - q} + \frac{1}{k'_e - k_e + q} + \frac{1}{k'_e + k_e - q} + \frac{-1}{k'_e + k_e + q} \right]$$

For the even state to odd state transition

$$I_3 = \int_{-a/2}^{a/2} dz \frac{1}{2} \left[\sin(k'_0 - k_e)z + \sin(k'_0 + k_e)z \right] \cos q_z z . \text{ This}$$

becomes

$$I_3 = \frac{1}{2} \int_{-a/2}^{a/2} dz \sin(k'_0 - k_e)z \cos q_z z + \frac{1}{2} \int_{-a/2}^{a/2} dz \sin(k'_0 + k_e)z \cos q_z z$$

$$I_3 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_e - q)z}{2(k'_0 - k_e - q)} - \frac{\sin(k'_0 - k_e + q)z}{2(k'_0 - k_e + q)} \right]_{-a/2}^{a/2} \\ + \frac{1}{2} \left[\frac{\sin(k'_0 + k_e - q)z}{2(k'_0 + k_e - q)} - \frac{\sin(k'_0 + k_e + q)z}{2(k'_0 + k_e + q)} \right]_{-a/2}^{a/2}$$

Because $\sin(-x) = -\sin x$

$$I_3 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_e - q)a/2}{k'_0 - k_e - q} - \frac{\sin(k'_0 - k_e + q)a/2}{k'_0 - k_e + q} \right] \\ + \frac{1}{2} \left[\frac{\sin(k'_0 + k_e - q)a/2}{k'_0 + k_e - q} - \frac{\sin(k'_0 + k_e + q)a/2}{k'_0 + k_e + q} \right] .$$

$$I_3 = \frac{1}{2} \left[\frac{\sin(k'_0 - k_e)a/2 \cos qa/2 - \cos(k'_0 - k_e)a/2 \sin qa/2}{k'_0 - k_e - q} \right. \\ \left. - \frac{\sin(k'_0 - k_e)a/2 \cos qa/2 + \cos(k'_0 - k_e)a/2 \sin qa/2}{k'_0 - k_e + q} \right. \\ \left. + \frac{\sin(k'_0 + k_e)a/2 \cos qa/2 - \cos(k'_0 + k_e)a/2 \sin qa/2}{k'_0 + k_e - q} \right. \\ \left. - \frac{\sin(k'_0 + k_e)a/2 \cos qa/2 + \cos(k'_0 + k_e)a/2 \sin qa/2}{k'_0 + k_e + q} \right]$$

$$\begin{aligned}
& - \frac{\sin(k'_0 - k_e)a/2 \cos qa/2 + \cos(k'_0 - k_e)a/2 \sin qa/2}{k'_0 - k_e + q} \\
& + \frac{\sin(k'_0 + k_e)a/2 \cos qa/2 - \cos(k'_0 + k_e)a/2 \sin qa/2}{k'_0 + k_e - q} \\
& - \frac{\sin(k'_0 + k_e)a/2 \cos qa/2 + \cos(k'_0 + k_e)a/2 \sin qa/2}{k'_0 + k_e + q} \Big].
\end{aligned}$$

For this case, $(k'_0 \pm k_e)a/2$ becomes

$$(k'_0 - k_e)a/2 = (2\pi/a)(a/2)m' - (2\pi/a)(a/2)(m - \frac{1}{2}) = (m' - m) + \pi/2,$$

and

$$(k'_0 + k_e)a/2 = (2\pi/a)(a/2)m' + (2\pi/a)(a/2)(m - \frac{1}{2}) = (m' + m) - \pi/2.$$

Therefore $\cos(k'_0 - k_e)a/2 = 0$, $\cos(k'_0 + k_e)a/2 = 0$,

$$\sin(k'_0 - k_e)a/2 = (-1)^{m' - m}, \text{ and } \sin(k'_0 + k_e)a/2 = -(-1)^{m' + m}.$$

If $m' - m$ is an even or odd integer, $m' + m$ will correspondingly be an even or odd integer, so

$$\begin{aligned}
I_3 &= -\frac{1}{2}(-1)^{m' + m} \cos qa/2 \\
& \cdot \left[\frac{-1}{k'_e - k'_0 - q} + \frac{1}{k'_e - k'_0 + q} + \frac{1}{k'_e + k'_0 - q} + \frac{-1}{k'_e + k'_0 + q} \right].
\end{aligned}$$

For the odd state to even state transition

$$I_4 = \int_{-a/2}^{a/2} dz \frac{1}{2} \left[-\sin(k'_0 - k_e)z + \sin(k'_0 + k_e)z \right] \cos q_z z.$$

This has the same form as I_3 with even and odd interchanged except the first half of the terms have a negative sign in front. For this case $(k'_e \pm k'_0)a/2$ becomes

$$(k'_e - k'_0)a/2 = (2\pi/a)(a/2)(m' - \frac{1}{2}) - (2\pi/a)(a/2)m = (m' - m) - \pi/2$$

and

$$(k'_e + k_o)a/2 = (2\pi/a)(a/2)(m' - \frac{1}{2}) + (2\pi/a)(a/2)m = (m' + m) - \pi/2.$$

$$\text{Therefore } \cos(k'_e + k_o)a/2 = 0, \sin(k'_e - k_o)a/2 = -(-1)^{m' - m},$$

and $\sin(k'_e + k_o)a/2 = -(-1)^{m' - m}$. Because $\sin(k'_e - k_o)a/2$ differs from $\sin(k'_e + k_o)a/2$ by a negative sign, I_4 is the same as I_3 except for the interchange of even and odd so,

$$I_4 = -\frac{1}{2}(-1)^{m' + m} \cos qa/2$$

$$\cdot \left[\frac{-1}{k'_e - k_o - q} + \frac{1}{k'_e - k_o + q} + \frac{1}{k'_e + k_o - q} + \frac{-1}{k'_e + k_o + q} \right].$$

The integrals I_1 , I_2 , I_3 , and I_4 maybe rewritten by replacing $k' \pm k \pm q$ with $2m'(\pi/a) \pm 2m(\pi/a) \pm q = (m' \pm m)(2\pi/a) \cdot [1 \pm (qa/2\pi)/(m' \pm m)]$. Let

$$M_Z = \left[\frac{1}{k' + k - q} - \frac{1}{k' + k + q} + \frac{1}{k' - k + q} - \frac{1}{k' - k - q} \right]$$

then

$$M_Z = \left[\frac{1}{(m' + m)(\frac{2\pi}{a})(1 - \frac{qa/2\pi}{m' + m})} - \frac{1}{(m' + m)(\frac{2\pi}{a})(1 + \frac{qa/2\pi}{m' + m})} \right. \\ \left. + \frac{1}{(m' - m)(\frac{2\pi}{a})(1 + \frac{qa/2\pi}{m' - m})} - \frac{1}{(m' - m)(\frac{2\pi}{a})(1 - \frac{qa/2\pi}{m' - m})} \right].$$

This can be rearranged in a form suitable for approximation, such as

$$M_Z = \frac{a}{2\pi} \left\{ \frac{1}{(m' + m)} \left[(1 - \frac{qa/2\pi}{m' + m})^{-1} - (1 + \frac{qa/2\pi}{m' + m})^{-1} \right] \right. \\ \left. + \frac{1}{(m' - m)} \left[(1 + \frac{qa/2\pi}{m' - m})^{-1} - (1 - \frac{qa/2\pi}{m' - m})^{-1} \right] \right\}.$$

If

$$\frac{qa/2}{m' + m} < 1,$$

which means the thickness of the sample is less than the wavelength of the disturbance, the approximation of the form

$$(1 \pm x)^{-1} = 1 \mp x,$$

for $x^2 < 1$ can be used.

This gives

$$M_Z = \frac{a}{2\pi} \left\{ \frac{1}{m' + m} \left[\left(1 + \frac{qa/2\pi}{m' + m} \right) - \left(1 - \frac{qa/2\pi}{m' + m} \right) \right] + \frac{1}{m' - m} \left[\left(1 - \frac{qa/2\pi}{m' - m} \right) - \left(1 + \frac{qa/2\pi}{m' - m} \right) \right] \right\}.$$

Cancellation gives

$$M_Z = \frac{a}{2\pi} \left[\frac{1}{m' + m} \left(\frac{qa}{m' + m} \right) + \frac{1}{m' - m} \left(\frac{-qa}{m' - m} \right) \right].$$

Factorization leads to

$$M_Z = \frac{q}{2} \left(\frac{a}{\pi} \right)^2 \left[\frac{1}{(m' + m)^2} - \frac{1}{(m' - m)^2} \right].$$

Addition of the two terms gives

$$M_Z = \frac{q}{2} \left(\frac{a}{\pi} \right)^2 \left\{ \frac{-4m'm}{[(m' + m)(m' - m)]^2} \right\}.$$

This may be rewritten as

$$M_Z = \frac{q}{2} \left(\frac{a}{\pi} \right)^2 \left[\frac{-4m'm}{(m'^2 - m^2)^2} \right]. \quad (27)$$

If $\vec{q} = (0, 0, q)$,

$$\int_{-\infty}^{\infty} dy \exp(ik'_y y) \exp(ik_y y) = \delta(k'_y, k_y) L_y.$$

Schiff (8, p. 65) shows that if $U_n(x)$ is a harmonic oscillator wave function, then

$$\int U_n^*(x) U_n dx = \frac{1}{\alpha} \begin{cases} (n+1)^{\frac{1}{2}} \delta_{n',n+1} \\ n^{\frac{1}{2}} \delta_{n',n-1} \end{cases}$$

where $1/\alpha$ is a normalization factor. For the free electron case, with $\vec{q} = (0,0,q)$ (p. 2489)

$$X_x = i(\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \left[(n+1)^{\frac{1}{2}} \delta_{n',n+1} - n^{\frac{1}{2}} \delta_{n',n-1} \right] L_y.$$

and

$$X_y = (\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \left[(n+1)^{\frac{1}{2}} \delta_{n',n+1} + n^{\frac{1}{2}} \delta_{n',n-1} \right] L_y.$$

Substitution of the results of I_1, I_2, I_3, I_4, X_x , and X_y back into equation (23) through equation (26) give the following results if equation (27) is used. For case number one

$$\begin{aligned} \langle \nu' | V_x(\vec{q}) | \nu \rangle &= (1/a)(-1)^{m'+m} \sin(qa/2) 2q(a/\pi)^2 [-m'm/(m'^2 - m^2)^2] \\ &\cdot i(\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) (n+1)^{\frac{1}{2}} \delta_{n',n+1} - n^{\frac{1}{2}} \delta_{n',n-1} \end{aligned}$$

and

$$\begin{aligned} \langle \nu' | V_y(\vec{q}) | \nu \rangle &= (1/a)(-1)^{m'+m} \sin(qa/2) 2q(a/\pi)^2 [-m'm/(m'^2 - m^2)^2] \\ &\cdot (\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \left[(n+1)^{\frac{1}{2}} \delta_{n',n+1} + n^{\frac{1}{2}} \delta_{n',n-1} \right]. \end{aligned}$$

For case number two, where m' and m in equation (27) become $m' - \frac{1}{2}$ and $m - \frac{1}{2}$ respectively,

$$\begin{aligned} \langle \nu' | V_x(\vec{q}) | \nu \rangle &= (1/a)(-1)^{m'+m} \sin(qa/2) 2q(a/\pi)^2 \\ &\cdot \left\{ -(m' - \frac{1}{2})(m - \frac{1}{2}) / [(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2] \right\}^2 \\ &\cdot i(\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\cdot \left[(n+1)^{\frac{1}{2}} \delta_{n',n+1} - n^{\frac{1}{2}} \delta_{n',n-1} \right] \end{aligned}$$

and

$$\begin{aligned} \langle \nu' | v_y(\vec{q}) | \nu \rangle &= (1/a)(-1)^{m'+m} \sin(qa/2) 2q(a/\pi)^2 \left\{ -(m' - \frac{1}{2})(m - \frac{1}{2}) \right. \\ &\quad \left. / [(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2] \right\} (\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\quad \cdot [(n+1)^{\frac{1}{2}} \delta_{n', n+1} + n^{\frac{1}{2}} \delta_{n', n-1}]. \end{aligned}$$

For case number three, where m is replaced by $m - \frac{1}{2}$ in equation (27),

$$\begin{aligned} \langle \nu' | v_x(\vec{q}) | \nu \rangle &= i(1/a)(-1)^{m'+m} \cos(qa/2) 2q(a/\pi)^2 \left\{ -m'(m - \frac{1}{2}) \right. \\ &\quad \left. / [m'^2 - (m - \frac{1}{2})^2] \right\} i(\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\quad \cdot [(n+1)^{\frac{1}{2}} \delta_{n', n+1} - n^{\frac{1}{2}} \delta_{n', n-1}] \end{aligned}$$

and

$$\begin{aligned} \langle \nu' | v_y(\vec{q}) | \nu \rangle &= i(1/a)(-1)^{m'+m} \cos(qa/2) 2q(a/\pi)^2 \left\{ -m'(m - \frac{1}{2}) \right. \\ &\quad \left. / [m'^2 - (m - \frac{1}{2})^2] \right\} (\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\quad \cdot [(n+1)^{\frac{1}{2}} \delta_{n', n+1} + n^{\frac{1}{2}} \delta_{n', n-1}]. \end{aligned}$$

For case number four, where m' is replaced by $m' - \frac{1}{2}$ in equation (27),

$$\begin{aligned} \langle \nu' | v_x(\vec{q}) | \nu \rangle &= i(1/a)(-1)^{m'+m} \cos(qa/2) 2q(a/\pi)^2 \left\{ -(m' - \frac{1}{2})m \right. \\ &\quad \left. / [(m' - \frac{1}{2})^2 - m^2] \right\} i(\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\quad \cdot [(n+1)^{\frac{1}{2}} \delta_{n', n+1} - n^{\frac{1}{2}} \delta_{n', n-1}] \end{aligned}$$

and

$$\begin{aligned} \langle \nu' | v_y(\vec{q}) | \nu \rangle &= i(1/a)(-1)^{m'+m} \cos(qa/2) 2q(a/\pi)^2 \left\{ -(m' - \frac{1}{2})m \right. \\ &\quad \left. / [(m' - \frac{1}{2})^2 - m^2] \right\} (\hbar\omega_c/2M)^{\frac{1}{2}} \delta(k_y', k_y) \\ &\quad \cdot [(n+1)^{\frac{1}{2}} \delta_{n', n+1} + n^{\frac{1}{2}} \delta_{n', n-1}]. \end{aligned}$$

XY Element of Conductivity Tensor

σ_{xy} can be calculated for the four cases by substitution of the matrix elements into equation (22). For case number one,

$$\begin{aligned} \sigma_{xy} = & (\omega_p^2 / 4\pi i \omega) (VM/\pi N) \sum_{\nu\nu'} \int_{-\infty}^{\infty} dq' \sin(qa/2) \sin(q'a/2) qq' \\ & \cdot (i\hbar\omega_c^2 / 2M\pi^4) \{ m'^2 m^2 / [m'^2 - m^2]^4 \} \\ & \cdot [(n+1) \delta_{n', n+1} - n \delta_{n', n-1}] \Lambda_{\nu\nu'} \end{aligned}$$

$\Lambda_{\nu\nu'}$ can be rewritten if $f_0(E_{\nu'})$ is expanded in a Taylor series to get

$$f_0(E_{\nu'}) = f_0(E_{\nu}) + (E_{\nu'} - E_{\nu}) (\partial f_0 / \partial E) + \dots$$

To first order in $\partial f_0 / \partial E$

$$\frac{f_0(E_{\nu'}) - f_0(E_{\nu})}{E_{\nu'} - E_{\nu} - \hbar\omega + i\hbar/\tau} = \frac{(E_{\nu'} - E_{\nu}) \partial f_0 / \partial E}{E_{\nu'} - E_{\nu} - \hbar\omega + i\hbar/\tau} \quad (28a)$$

and

$$\frac{f_0(E_{\nu'}) - f_0(E_{\nu})}{E_{\nu'} - E_{\nu}} = \frac{\partial f_0}{\partial E}, \quad (28b)$$

but

$$E_{\nu} = (n + \frac{1}{2}) \hbar\omega_c + \hbar^2 k_z^2 / 2M,$$

$$(E_{\nu'} - E_{\nu}) \delta_{n', n+1} = \hbar\omega_c + (\hbar^2 k_o'^2 / 2M) - (\hbar k_o^2 / 2M)$$

With k_o written as $2m\pi/a$,

$$(E_{\nu'} - E_{\nu}) \delta_{n', n+1} = \hbar\omega_c + (\hbar^2 \pi^2 / 2Ma^2) 4(m'^2 - m^2)$$

Define $E_L = (2\hbar^2 \pi^2 / Ma^2)$, then

$$(E_{\nu'} - E_{\nu}) \delta_{n', n+1} = \hbar\omega_c + E_L (m'^2 - m^2) \quad (28c)$$

and

$$(E_{\nu'} - E_{\nu}) \delta_{n', n-1} = -\hbar\omega_c + E_L(m'^2 - m^2). \quad (28d)$$

The sums $\sum_{k'_y k_y}$ give $(L_x L_y M \omega_c / \hbar)$ since the expression for σ_{xy} doesn't contain k_y or k'_y .

For case number one the information in equations (28) can be substituted into equation (22) to get

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar\omega_c)^2}{E_L} \frac{L_x L_y V i}{\pi^3 N (1+i\omega\tau)} \sum_{n, k'_z k_z} \int dq' \sin qa/2 \sin q'a/2 qq' \\ & \cdot \frac{\partial f_0}{\partial E} \left[\frac{m'^2 m^2}{(m'^2 - m^2)^4} \right] \left\{ (n+1) \left[\frac{E_L(m'^2 - m^2) + \hbar\omega_c}{E_L(m'^2 - m^2) + \hbar\omega_c - \hbar\omega + i\hbar/\tau} i\omega\tau + 1 \right] \right. \\ & \left. - n \left[\frac{E_L(m'^2 - m^2) - \hbar\omega_c}{E_L(m'^2 - m^2) - \hbar\omega_c - \hbar\omega + i\hbar/\tau} i\omega\tau + 1 \right] \right\}. \end{aligned}$$

Let $(\hbar\omega_c/E_L) = \alpha$ and $\hbar(\omega - i\hbar/\tau)/(E_L) = \beta$, so that σ_{xy} becomes

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar\omega_c)^2}{E_L} \frac{L_x L_y V i}{\pi^3 N (1+i\omega\tau)} \sum_{n, k'_z k_z} \int dq' \sin qa/2 \sin q'a/2 qq' \\ & \cdot \frac{\partial f_0}{\partial E} \left[\frac{m'^2 m^2}{(m'^2 - m^2)^4} \right] \left\{ (n+1) \left[\frac{i\omega\tau (m'^2 - m^2 + \alpha)}{m'^2 - m^2 + \alpha - \beta} + 1 \right] \right. \\ & \left. - n \left[\frac{i\omega\tau (m'^2 - m^2 - \alpha)}{m'^2 - m^2 - \alpha - \beta} + 1 \right] \right\}. \quad (29a) \end{aligned}$$

For case number two, the form of σ_{xy} is the same as for case number one with m and m' replaced by $m^{-\frac{1}{2}}$ and $m'^{-\frac{1}{2}}$ respectively.

This gives

$$\begin{aligned}
 \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar \omega_c)^2}{E_L} \frac{L_x L_y V i}{\pi^3 N (1+i\omega\tau)} \sum_{n, k_z^i k_z} \int dq' \sin qa/2 \sin q'a/2 qq' \\
 & \cdot \frac{\partial f_0}{\partial E} \left\{ \frac{(m' - \frac{1}{2})^2 (m - \frac{1}{2})^2}{[(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2]^4} \right\} \left\{ (n+1) \left[\frac{i\omega\tau [(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 + \alpha]}{(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 + \alpha - \beta} + 1 \right] \right. \\
 & \left. - n \left[\frac{i\omega\tau [(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 - \alpha]}{(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 - \alpha - \beta} + 1 \right] \right\}. \quad (29b)
 \end{aligned}$$

For case number three the evaluation of the matrix elements maybe used in equation (22) to give

$$\begin{aligned}
 \sigma_{xy} = & (\omega_p^2 / 4\pi i \omega) (VM / \pi N) \sum_{\nu\nu'} \int dq' \cos(qa/2) \cos(q'a/2) qq' \\
 & \cdot (i\hbar \omega_c a^2 / 2M \pi^4) \left\{ m'^2 (m - \frac{1}{2})^2 / [m'^2 - (m - \frac{1}{2})^2]^4 \right\} \\
 & \cdot [(n+1) \delta_{n', n+1} - n \delta_{n', n-1}] \Lambda_{\nu\nu'}.
 \end{aligned}$$

The form here differs from case number one in the q and q' dependence and with m replaced by $m - \frac{1}{2}$. So σ_{xy} for the odd state to even state transition becomes

$$\begin{aligned}
 \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar \omega_c)^2}{E_L} \frac{L_x L_y V - i}{\pi^3 N (1+i\omega\tau)} \sum_{n, k_z^i k_z} \int dq' \cos qa/2 \cos q'a/2 qq' \\
 & \cdot \frac{\partial f_0}{\partial E} \left\{ \frac{m'^2 (m - \frac{1}{2})^2}{[m'^2 - (m - \frac{1}{2})^2]^4} \right\} \left\{ (n+1) \left[\frac{m'^2 - (m - \frac{1}{2})^2 + \alpha}{m'^2 - (m - \frac{1}{2})^2 + \alpha - \beta} i\omega\tau + 1 \right] \right. \\
 & \left. - n \left[\frac{m'^2 - (m - \frac{1}{2})^2 - \alpha}{m'^2 - (m - \frac{1}{2})^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\}. \quad (29c)
 \end{aligned}$$

For case number four, the even state to odd state transition, σ_{xy} is the same as for case number three with m and $m-\frac{1}{2}$ interchanged. Thus

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar \omega_c)^2}{E_L} \frac{L_x L_y V(-i)}{\pi^3 N(1+i\omega\tau)} \sum_{n, k'_z, k_z} \int dq' \cos(qa/2) \cos(q'a/2) qq' \\ & \cdot \frac{\partial f_o}{\partial E} \frac{m'^2 (m-\frac{1}{2})^2}{[(m'-\frac{1}{2})^2 - m^2]^4} \left\{ n+1 \left[\frac{(m'-\frac{1}{2})^2 - m^2 + \alpha}{(m'-\frac{1}{2})^2 - m^2 + \alpha - \beta} i\omega\tau + 1 \right] \right. \\ & \left. - n \left[\frac{(m'-\frac{1}{2})^2 - m^2 - \alpha}{(m'-\frac{1}{2})^2 - m^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\}. \end{aligned} \quad (29d)$$

Summation Over Magnetic Quantum Numbers

If $n+1$ is written as $(n+\frac{1}{2})+\frac{1}{2}$ and n is written as $(n+\frac{1}{2})-\frac{1}{2}$ in equations (29), the Poisson sum rule (3), given as

$$\sum_{n=0}^{\infty} f(n+\frac{1}{2}) = \int_0^{\infty} f(x) dx + 2 \sum_{s=1}^{\infty} (-1)^s \int_0^{\infty} f(x) \cos 2\pi x s dx \quad (30)$$

can be used to evaluate the sum over the magnetic quantum numbers in σ_{xy} for each of the four cases. The oscillatory part represented by the second term is substituted into equation (29a) to give,

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{(\hbar \omega_c)^2}{E_L} \frac{L_x L_y V i}{\pi^3 N(1+i\omega\tau)} \sum_{n, k'_z, k_z} \int dq' \sin(q'a/2) \sin(qa/2) qq' \\ & \cdot \frac{m'^2 m^2}{[m'^2 - m^2]^4} \sum_{s=1}^{\infty} (-1)^s \sum_{n=0}^{\infty} \cos 2\pi n s \frac{\partial f_o}{\partial E} \end{aligned}$$

$$\cdot \left\{ (n+\frac{1}{2}) \left[\frac{(m'^2 - m^2 + \alpha)}{m'^2 - m^2 + \alpha - \beta} i\omega\tau + 1 \right] - (n-\frac{1}{2}) \left[\frac{(m'^2 - m^2 - \alpha)}{m'^2 - m^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\} dn$$

for case number one.

The change of variables from n to E , is accomplished with use of equation (5), the expression for the energy eigenvalues of the free electron. For $k = 2\pi m/a$,

$$E = (n+\frac{1}{2})\hbar\omega_c + E_L m^2$$

so

$$n = (E/\hbar\omega_c) - (m^2/\hbar\omega_c) - \frac{1}{2},$$

$$n = (E/\hbar\omega_c) - (m^2/\alpha) - \frac{1}{2},$$

$$n+\frac{1}{2} = E/\hbar\omega_c - m^2/\alpha,$$

$$n-\frac{1}{2} = E/\hbar\omega_c - m^2/\alpha - 1,$$

and

$$dn = dE/\hbar\omega_c.$$

Thus σ_{xy} becomes

$$\sigma_{xy} = \frac{\omega_p^2}{4\pi i\omega} \frac{L_x L_y V(-i)}{\pi^3 N(1+i\omega\tau)} \sum_{k'_z k_z} \int dq' \sin(q'a/2) \sin(qa/2) qq'$$

$$\cdot \frac{m'^2 m^2}{[m'^2 - m^2]^4} \sum_{s=1}^{\infty} (-1)^s \int \cos 2s(E/\hbar\omega_c - m^2/\alpha - \frac{1}{2}) \frac{\partial f_0}{\partial E_v}$$

$$\cdot \left\{ (E/\hbar\omega_c - m^2/\alpha) \left[\frac{(m'^2 - m^2 + \alpha)}{m'^2 - m^2 + \alpha - \beta} i\omega\tau + 1 \right] - (E/\hbar\omega_c - m^2/\alpha - 1) \right.$$

$$\cdot \left. \left[\frac{(m'^2 - m^2 - \alpha)}{m'^2 - m^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\} dE.$$

The integral over E can be evaluated by use of the integral theorem of Blankenbecler (1) which gives

$$\int_0^{\infty} dE (-\partial f_0 / \partial E) E^{\alpha} \cos 2\pi s (E/\hbar\omega_c + \gamma)$$

$$= E_F \cos [2\pi s (E_F/\hbar\omega_c + \gamma - \pi/4)] \frac{X}{\sinh X}$$

where E_F is the Fermi energy and $X = 2\pi^2 s kT / \hbar\omega_c$. The conductivity becomes

$$\sigma_{xy} = \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V(-i)}{\pi^3 N (1+i\omega\tau)} \sum_{k'_z k_z} \left\{ dq' \sin(q'a/2) \sin(qa/2) qq' \right.$$

$$\cdot \frac{m'^2 m^2}{[m'^2 - m^2]^4} \sum_{s=1}^{\infty} (-1)^s \cos [2\pi s (E_F/\hbar\omega_c - m^2/\alpha - \frac{1}{2} - \pi/4)] \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} \right]$$

$$\cdot \left[\frac{(m'^2 - m^2 + \alpha)}{m'^2 - m^2 + \alpha - \beta} i\omega\tau + 1 \right] - \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - 1 \right] \left[\frac{m'^2 - m^2 - \alpha}{m'^2 - m^2 - \alpha - \beta} i\omega\tau + 1 \right] \Bigg\}$$

$$\cdot \frac{X}{\sinh X} \quad (31a)$$

Case number two will have the same form as case number one with m' and m everywhere replaced by $m' - \frac{1}{2}$ and $m - \frac{1}{2}$ respectively.

Thus

$$\sigma_{xy} = \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V(-i)}{\pi^3 N (1+i\omega\tau)} \sum_{k'_z k_z} \left\{ dq' \sin(qa/2) \sin(q'a/2) qq' \right. \quad (31b)$$

$$\cdot \frac{(m' - \frac{1}{2})^2 (m - \frac{1}{2})^2}{[(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2]^4} \sum_s (-1)^s \cos [2\pi s (E_F/\hbar\omega_c - (m - \frac{1}{2})^2/\alpha$$

$$- \frac{1}{2} - \pi/4)] \left\{ \left[\frac{E_F}{\hbar\omega_c} - (m - \frac{1}{2})^2/\alpha \right] \left[\frac{(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 + \alpha}{(m - \frac{1}{2})^2 - (m - \frac{1}{2})^2 + \alpha - \beta} i\omega\tau + 1 \right] \right.$$

$$- \left[\frac{E_F}{\hbar\omega_c} - (m - \frac{1}{2})^2 / \alpha - 1 \right] \left[\frac{(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 - \alpha}{(m' - \frac{1}{2})^2 - (m - \frac{1}{2})^2 - \alpha - \beta} i\omega\tau + 1 \right] .$$

With the appropriate replacement of m and $m - \frac{1}{2}$, case number three will have the same form except for the q and q' dependence.

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V i}{\pi^3 N (1 + i\omega\tau)} \sum_{k'_z k_z} \left\{ dq' \cos(q'a/2) \cos(qa/2) qq' \right. \\ & \cdot \frac{(m'^2)(m + \frac{1}{2})^2}{[m'^2 - (m + \frac{1}{2})^2]^4} \sum_{s=1} (-1)^s \cos 2\pi s \left[\frac{E_F}{\hbar\omega_c} - \frac{(m + \frac{1}{2})^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4} \right] \\ & \cdot \left\{ \left[\left(\frac{E_F}{\hbar\omega_c} \right) - (m + \frac{1}{2})^2 / \alpha \right] \left[\frac{m'^2 - (m + \frac{1}{2})^2 + \alpha}{m'^2 - (m + \frac{1}{2})^2 + \alpha - \beta} i\omega\tau + 1 \right] \right. \\ & \left. \left. - \left[\left(\frac{E_F}{\hbar\omega_c} \right) - (m + \frac{1}{2})^2 / \alpha - 1 \right] \left[\frac{m'^2 - (m + \frac{1}{2})^2 - \alpha}{m'^2 - (m + \frac{1}{2})^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\} \frac{x}{\sinh x} \right. \end{aligned} \quad (31c)$$

Case number four will be the same as case number three if m and $m - \frac{1}{2}$ are interchanged.

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V i}{\pi^3 N (1 + i\omega\tau)} \sum_{k'_z k_z} \left\{ dq' \cos(q'a/2) \cos(qa/2) qq' \right. \\ & \cdot \frac{(m' - \frac{1}{2})^2 m^2}{[(m' - \frac{1}{2})^2 - m^2]^4} \sum_s (-1)^s \cos 2\pi s \left(\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4} \right) \\ & \cdot \left\{ \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} \right] \left[\frac{(m' - \frac{1}{2})^2 - m^2 + \alpha}{(m' - \frac{1}{2})^2 - m^2 + \alpha - \beta} i\omega\tau + 1 \right] \right. \\ & \left. \left. - \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - 1 \right] \left[\frac{(m' - \frac{1}{2})^2 - m^2 - \alpha}{(m' - \frac{1}{2})^2 - m^2 - \alpha - \beta} i\omega\tau + 1 \right] \right\} . \end{aligned} \quad (31d)$$

Summation Over Final Momentum States

The evaluation of the sum over k'_z can be handled by the method outlined by Morse and Feshbach (7, p. 413) which says

$$\sum_{-\infty}^{\infty} f(n) = - \sum \text{Residues } [\pi f(z) \cot \pi z]$$

where the residues are evaluated at the poles of $f(z)$ which are not on the real axis. For case number one let $m' = z$ in equation (31a) and evaluate the sum at the poles corresponding to zeros of the denominators (i) and (ii) which are as follows:

$$(i) \ m'^2 - m^2 + \alpha - \beta = z^2 - z_0^2 = (z + z_0)(z - z_0),$$

$$(ii) \ m'^2 - m^2 - \alpha - \beta = z^2 - z_0^2 = (z + z_0)(z - z_0).$$

The zeros of the denominators $(m'^2 - m^2)^4$ are not real poles. The condition $m'^2 = m^2$ would have given zero for the integral I_1 as evaluated earlier. Therefore the matrix elements of the velocity operator in the small q limit, will allow only transition which represent a change in momentum. Therefore there are no residues associated with the matrix elements of the velocity operator. With these substitutions the sum becomes

$$\sum_{k'_z} f(m') = -\pi \frac{z^2}{[z^2 - z_0^2 - \alpha + \beta]^4} \frac{(z^2 - z_0^2 + \beta)}{z^2 - z_0^2} \cot \pi z \quad (32a)$$

evaluated at $z = \pm z_0$ and

$$\sum_{k_z'} f(m') = -\pi \frac{Z^2}{[Z^2 - Z_0'^2 + \alpha + \beta]^4} \frac{(Z^2 - Z_0'^2 + \beta)}{Z^2 - Z_0'^2} \cot \pi Z \quad (32b)$$

evaluated at $Z = \pm Z_0'$. The first residue can be obtained if $Z^2 - Z_0'^2$ is written as $(Z+Z_0')(Z-Z_0')$, the right hand side of equation (32a) is multiplied by $Z-Z_0'$ and evaluated at $Z-Z_0'=0$, which is $Z = Z_0'$. Thus

$$R_1 = -\pi \frac{Z_0'^2}{(-\alpha + \beta)^4} \frac{i\omega\tau\beta}{2Z_0'} \cot \pi Z_0'.$$

The second residue is obtained by multiplying the right hand side of equation (32a) by $Z+Z_0'$ and evaluating at $Z+Z_0'=0$, which is $Z = -Z_0'$. This gives

$$R_2 = -\pi \frac{Z_0'^2}{(-\alpha + \beta)^4} \frac{i\omega\tau\beta}{-2Z_0'} \cot \pi(-Z_0').$$

The third and fourth residues are obtained from equation (32b) by the same procedure except that Z_0 is replaced by Z_0' . This gives

$$R_3 = -(-\pi) \frac{Z_0'^2}{(\alpha + \beta)^4} \frac{i\omega\tau\beta}{2Z_0'} \cot \pi Z_0'$$

and

$$R_4 = -(-\pi) \frac{Z_0'^2}{(\alpha + \beta)^4} \frac{i\omega\tau\beta}{-2Z_0'} \cot \pi(-Z_0').$$

Because $\cot(-x) = -\cot x$, $R_1=R_2$, and $R_3=R_4$. These may be combined and substituted back into equation (31a) to give the conductivity for case number one, the odd state to odd state transition, as

$$\sigma_{xy} = \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V(-i)}{\pi^3 N(1+i\omega\tau)} \sum_{k_z} \int dq' \sin(q'a/2) \sin(qa/2) qq' \quad (33)$$

$$\cdot \sum_s (-1)^s \cos 2\pi s \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4} \right] [-\pi m^2 i \omega \tau \beta]$$

$$\cdot \left\{ \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} \right] \frac{Z_0 \cot \pi Z_0}{(-\alpha + \beta)^4} - \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - 1 \right] \frac{Z'_0 \cot \pi Z'_0}{(\alpha + \beta)^4} \right\} \frac{X}{\sinh X}.$$

For case number two, the even state to even state transition, the form of the residues will be similar except for m replaced by $m - \frac{1}{2}$ everywhere, and

$$(Z - Z_0^+)(Z + Z_0^-) = (m - \frac{1}{2})^2 - (m - \frac{1}{2})^2 + \alpha - \beta$$

where $Z_0^+ = Z_0 + \frac{1}{2}$, $Z_0^- = Z_0 - \frac{1}{2}$, and

$$(Z - Z_0^{+'})(Z + Z_0^{-'}) = (m - \frac{1}{2})^2 - (m - \frac{1}{2})^2 - \alpha - \beta.$$

The sums become

$$\sum_{k_z} f(m') = -\pi \frac{(Z - \frac{1}{2})^2}{[(Z + Z_0^-)(Z - Z_0^+) - \alpha + \beta]^4} \frac{i\omega\tau[(Z + Z_0^-)(Z - Z_0^+) + \beta]}{(Z + Z_0^-)(Z - Z_0^+)} \cot \pi Z$$

evaluated at $Z = -Z_0^-$ and $Z = Z_0^+$, and

$$\sum_{k_z} f(m') = -(-\pi) \frac{(Z - \frac{1}{2})^2}{[(Z + Z_0^{-'})(Z - Z_0^{+'}) + \alpha + \beta]^4} \frac{i\omega\tau[(Z + Z_0^{-'})(Z - Z_0^{+'}) + \beta]}{(Z + Z_0^{-'})(Z - Z_0^{+'})} \cot \pi Z \quad (34b)$$

evaluated at $Z = -Z_0^{-'}$ and $Z = Z_0^{+'}$. The first residue can be obtained if the right hand side of equation (33a) is multiplied by $Z - Z_0^+$ and evaluated at $Z = Z_0^+$. This gives

$$R_1 = -\pi \frac{(Z_0^+ - \frac{1}{2})^2}{(-\alpha + \beta)^4} \frac{i\omega\tau\beta}{Z_0^+ + Z_0^-} \cot \pi Z_0^+.$$

The second residue is obtained by multiplying the right hand side of equation (34a) by $Z + Z_0^-$ and evaluating at $Z = -Z_0^-$.

This gives

$$R_2 = -\pi \frac{(Z_0^- + \frac{1}{2})^2 i\omega\tau\beta}{(-\alpha + \beta)^4 (-Z_0^- - Z_0^+)} \cot \pi(-Z_0^-).$$

The third and fourth residues are obtained from equation (34b) by the same procedure. This gives

$$R_3 = -(-\pi) \frac{(Z_0^{+'} - \frac{1}{2})^2}{(\alpha + \beta)^4} \frac{i\omega\tau\beta}{(Z_0^{+'} + Z_0^{-'})} \cot \pi Z_0^{+'},$$

and

$$R_4 = -(-\pi) \frac{(Z_0^{-'} + \frac{1}{2}) i\omega\tau\beta}{(\alpha + \beta)^4 (-Z_0^{-'} - Z_0^{+'})} \cot \pi(-Z_0^{-'}).$$

If

$$m' = Z, \quad Z_0^+ = [(m - \frac{1}{2})^2 - \alpha + \beta]^{\frac{1}{2}} + \frac{1}{2}$$

and

$$Z_0^- = [(m - \frac{1}{2})^2 - \alpha + \beta]^{\frac{1}{2}} - \frac{1}{2}.$$

Let

$$[(m - \frac{1}{2})^2 - \alpha + \beta]^{\frac{1}{2}} = Z_0$$

so

$$Z_0^+ = Z_0 + \frac{1}{2}$$

and

$$Z_0^- = Z_0 - \frac{1}{2}.$$

Therefore

$$\cot \pi(-Z_0^-) = -\cot \pi(Z_0^-) = -\cot(\pi Z_0 - \pi/2) = \tan \pi Z_0,$$

$$\cot \pi Z_0^+ = \cot(\pi Z_0 + \pi/2) = -\tan \pi Z_0$$

$$\cot \pi(-Z_0^-) = \tan \pi Z_0$$

and

$$\cot \pi Z_0^{+'} = -\tan \pi Z_0'.$$

Therefore $R_1 = R_2$, $R_3 = R_4$ and all four may be substituted into equation (31b) to give the conductivity for case number two, the even state to even state transition, in the same form as for case number one except for the cotangent being replaced by the negative of the tangent and Z_0 depending on $m-\frac{1}{2}$ rather than m . This gives

$$\begin{aligned} \sigma_{xy} = & \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V(-i)}{\pi^3 N(1+i\omega\tau)} \sum_{k_z} \left\{ dq' \sin(q'a/2) \sin(qa/2) qq' \right. \\ & \cdot \sum_s (-1)^s \cos 2\pi s \left[\frac{E_F}{\hbar\omega_c} - \frac{(m-\frac{1}{2})^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4} \right] \\ & \cdot -\pi(m-\frac{1}{2})^2 i\omega\tau\beta \left\{ \left[\frac{E_F}{\hbar\omega_c} - \frac{(m-\frac{1}{2})^2}{\alpha} \right] \frac{Z_0(-\tan \pi Z_0)}{(-\alpha+\beta)^4} \right. \\ & \left. \left. - \left[\frac{E_F}{\hbar\omega_c} - \frac{(m-\frac{1}{2})^2}{\alpha} - 1 \right] \frac{Z_0'(-\tan \pi Z_0')}{(\alpha+\beta)^4} \right\} \frac{X}{\sinh X} \right\}. \end{aligned} \quad (35)$$

For case number three, the even state to odd state transition, the form of the residues will be the same as case number one except m is replaced by $m-\frac{1}{2}$ with

$$Z_0 = [(m-\frac{1}{2})^2 - \alpha + \beta]^{\frac{1}{2}}$$

and

$$Z'_0 = (m^2 + \alpha + \beta)^{\frac{1}{2}}$$

For this case

$$\sigma_{xy} = \frac{\omega_p^2}{4\pi i \omega} \frac{L_x L_y V_i}{\pi^3 N(1+i\omega\tau)} \sum_{k_z} \int dq' \cos(q'a/2) \cos(qa/2) qq' \quad (37)$$

$$\begin{aligned} & \cdot \sum_s (-1)^s \cos 2\pi s \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4} \right] \\ & \cdot \left[-\pi m^2 i \omega \tau \beta \left\{ \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} \right] \frac{Z_0 (-\tan \pi Z'_0)}{(-\alpha + \beta)^4} - \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - 1 \right] \right. \right. \\ & \left. \left. \cdot \frac{Z'_0 (-\tan \pi Z'_0)}{(\alpha + \beta)^4} \right\} \frac{X}{\sinh X} \right] \cdot \end{aligned}$$

Summation Over Initial Momentum States

Because the sum over k'_z was done exactly in order to pick up the contributions from the poles in the denominators, the sum over k_z can be changed to an integral. The main contribution from an integral of the type containing oscillatory functions comes where there are extremal points and at the end points (2). The first non-zero term in the asymptotic expansion of the integral at the end points are of the form

$$\begin{aligned} \int_{-m_F}^{m_F} \cos [\nu f(m)] \phi(m) dm &= \frac{2}{\nu^2} \left\{ \frac{\phi'(m_F)}{[f'(m_F)]^2} \right. \\ & \left. - \frac{\phi(m_F) f''(m_F)}{[f'(m_F)]^3} \right\} \cos \nu f(m_F). \end{aligned} \quad (38)$$

At the extremal points of the energy as a function of k_z , the matrix element and the first derivative vanish so that the contribution is found by using an integral of the form

$$\int_{-m_F}^{m_F} \cos [\nu f(m)] \phi(m) dm = - \frac{1}{4} \frac{(2\pi)^{\frac{1}{2}} \phi''(m_0)}{[-\nu f''(m_0)]^{\frac{3}{2}}} \cos [\nu f(m_0) - \frac{\pi}{4}] \quad (39)$$

for $-m_F < m_0 < m_F$, where m_0 is the value of m where $f(m)$ is an extremal, namely $m = 0$.

The evaluations by the Blankenbecler method leading to equations (31) is a low temperature expansion in which the upper bound on the magnetic quantum number n is the one for which $E_F = (n_F + \frac{1}{2})\hbar\omega_c$ and the upper and lower bound on k_z is set by $E_F = (\hbar^2 k_F^2 / 2M)$, where $k_F = 2m_F\pi/a$. The sum becomes

$$\sum_{k_z} f(k_z) = \sum_{k_z=-k_F}^{k_F} f(k_z) = \int_{-m_F}^{m_F} 2\pi/a \, dm.$$

For use of equation (38) and (39) to evaluate equation (33), the following substitutions have to be made:

$$\nu = 2\pi s,$$

$$f(m) = \frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - \frac{1}{2} - \frac{\pi}{4},$$

$$f'(m) = -\frac{2m}{\alpha},$$

$$f''(m) = -\frac{2}{\alpha},$$

$$\phi(m) = m^2 \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} \right] \frac{Z_o \cot Z_o}{(-\alpha + \beta)^4} - \left[\frac{E_F}{\hbar\omega_c} - \frac{m^2}{\alpha} - 1 \right] \frac{Z'_o \cot Z'_o}{(\alpha + \beta)^4},$$

$$f(m_F) = -\frac{\pi}{4},$$

$$f(m_o) = \frac{E_F}{\hbar\omega_c} - \frac{1}{2} - \frac{\pi}{4}$$

$$\begin{aligned} \phi'(m_F) = & \left[(m_F Z_{oF} - 2m_F^3 Z_{oF}/\alpha + \frac{1}{2}m_F^3/Z_{oF})/(-\alpha + \beta)^4 \right] \cot \pi Z_{oF} \\ & - \left[\frac{\pi}{2} m_F^3/(-\alpha + \beta)^4 \right] \csc^2 \pi Z_{oF} \\ & - \left[(m_F Z'_{oF} - 2m_F^3 Z'_{oF}/\alpha + \frac{1}{2}m_F^3/Z'_{oF})/(\alpha + \beta)^4 \right] \cot \pi Z'_{oF} \\ & + \left[\frac{\pi}{2} m_F^3/(\alpha + \beta)^4 \right] \csc^2 \pi Z'_{oF}, \end{aligned}$$

$$\text{where } Z_{oF} = (m_F^2 - \alpha + \beta)^{\frac{1}{2}} \text{ and } Z'_{oF} = (m_F^2 + \alpha + \beta)^{\frac{1}{2}},$$

$$\begin{aligned} \phi''(m_o) = & 2 \frac{E_F}{\hbar\omega_c} \left[Z_{oo}/(-\alpha + \beta)^4 \right] \cot \pi Z_{oo} - \left[\frac{E_F}{\hbar\omega_c} - 1 \right] \\ & \cdot \left[2 Z'_{oo}/(\alpha + \beta)^4 \right] \cot \pi Z'_{oo}, \end{aligned}$$

$$\text{where } Z_{oo} = (-\alpha + \beta)^{\frac{1}{2}} \text{ and } Z'_{oo} = (\alpha + \beta)^{\frac{1}{2}}.$$

For samples where $m_F^2 \gg \alpha \gg \beta$,

$$Z_{oF} \approx m_F \left(1 + \frac{\beta - \alpha}{2m_F^2} \right),$$

$$Z'_{oF} \approx m_F \left(1 + \frac{\beta + \alpha}{2m_F^2} \right),$$

$$(-\alpha + \beta)^4 \approx \alpha^{-4} (1 + 4\beta/\alpha),$$

and

$$(\alpha + \beta)^4 \approx \alpha^{-4} (1 - 4\beta/\alpha)$$

are used to evaluate case number one. This leads to

$$\cot \pi Z_{oF} \approx \cot \left[\pi m_F + \frac{\pi}{2} \frac{\beta - \alpha}{m_F} \right].$$

The term πm_F represents a phase shift so that

$$\cot \pi Z_{oF} \approx \cot \frac{\pi}{2} \frac{\beta - \alpha}{m_F}$$

and

$$\cot \pi Z'_{oF} \approx \cot \frac{\pi}{2} \frac{\beta + \alpha}{m_F}.$$

The approximations also lead to

$$\frac{Z_{oF}}{(-\alpha + \beta)^4} \approx \frac{m_F}{\alpha^4} \left\{ \left[1 - \frac{3\beta}{2m_F} \right] + \left[\frac{4\beta}{\alpha^2} - \frac{\alpha}{2m_F^2} \right] \right\} = \frac{m_F}{\alpha^4} [x+y]$$

$$\frac{Z'_{oF}}{(\alpha + \beta)^4} \approx \frac{m_F}{\alpha^4} \left\{ \left[1 - \frac{3\beta}{2m_F} \right] - \left[\frac{4\beta}{\alpha^2} - \frac{\alpha}{2m_F^2} \right] \right\} = \frac{m_F}{\alpha^4} [x-y].$$

This gives, to highest order in m_F ,

$$\begin{aligned} \phi'(m_F) &= \frac{-2m_F^4}{\alpha^5} \left\{ (x+y) \cot \left[\frac{\pi}{2} \frac{\beta - \alpha}{m_F} \right] - (x-y) \cot \left[\frac{\pi}{2} \frac{\beta + \alpha}{m_F} \right] \right\} \\ &\quad \cdot \frac{\pi \alpha}{4m_F} \left\{ \left[1 + \frac{4\beta}{\alpha^2} \right] \csc^2 \left[\frac{\pi}{2} \frac{\beta - \alpha}{m_F} \right] - \left[1 - \frac{4\beta}{\alpha^2} \right] \csc^2 \left[\frac{\pi}{2} \frac{\beta + \alpha}{m_F} \right] \right\}. \end{aligned}$$

The first term in $\phi'(m_F)$ can be rearranged to give a sum and difference of cotangents. If $\pi\beta/m_F$ is small, the

contribution of $\phi'(m_F)$ to the real part of σ_{xy} becomes

$$\phi'(m_F) = \frac{2m_F^4}{\alpha^5} \frac{\sin \frac{\alpha\pi}{m_F} + \frac{4\pi\beta^2}{m_F\alpha}}{\sin^2 \frac{\pi\alpha}{2m_F} - \left[\frac{\pi\beta}{2m_F}\right]^2} \cos\left(-\frac{\pi^2 s}{2}\right)$$

Substitution of this back into equation (38) gives the end point contribution to the sum over k_z as

$$\int_{-m_F}^{m_F} \cos[\nu f(m)] \phi(m) dm = \frac{m_F^2}{(2\pi s)^2 \alpha^3} \frac{\sin \frac{\alpha\pi}{m_F} + \frac{4\pi\beta^2}{m_F\alpha}}{\sin^2 \frac{\pi\alpha}{2m_F} - \left[\frac{\pi\beta}{2m_F}\right]^2} \cos\left(-\frac{\pi^2 s}{2}\right)$$

Therefore the end point contribution to equation (33) for case number one becomes,

$$\sigma_{xy} = 2\sigma_0 \frac{L_x L_y h m_F^2}{\pi^3 n a \alpha^2 E_L} \frac{\sin \frac{\alpha\pi}{m_F} + \frac{4\pi\beta^2}{m_F\alpha}}{\sin^2 \frac{\pi\alpha}{2m_F} - \left[\frac{\pi\beta}{2m_F}\right]^2} \sum_s (-1)^s \frac{\cos(-\pi^2 s/2)}{s^2} \cdot \frac{X}{\sinh X} \int dq' \sin(q'a/2) \sin(qa/2) qq' , \quad (40)$$

where $\sigma_0 = \omega_p^2 \tau / 4$, $N/V = n$, the density of carriers, and

$\beta = (i\hbar/E_L \tau)(1+i\omega\tau)$. For case number two

$$-\tan \pi Z_0 \approx -\tan \left[(m_F \pi - \pi/2) + \pi/2 \frac{\beta - \alpha}{m_F - \frac{1}{2}} \right]$$

because

$$\tan(x - \pi/2) = -\cot x$$

and

$$\cot(m\pi + x) = \cot x ,$$

$$-\tan Z_0 = \cot Z_0 .$$

Therefore for large m_F the results for case number two will be the same as for case number one.

The sum over k_z at the extremal point, is given by equation (39). If $\alpha \gg \beta$, the expression of $\phi''(m_0)$ becomes

$$\phi''(m_0) = -2 \frac{E_F}{\hbar \omega_c} (+\alpha)^{-1/2} [i \cot \pi(-\alpha + \beta)^{1/2} + \cot \pi(\alpha + \beta)^{1/2}].$$

The extremal point contribution to the sum over k_z for case number one becomes

$$\int_{-m_F}^{m_F} \cos[\nu f(m)] \phi(m) dm = \frac{1}{s^{3/2} 8(2^{1/2}) \pi \alpha^2} \frac{E_F}{\hbar \omega_c} [i \cot \pi(-\alpha + \beta)^{1/2} + \cot \pi(\alpha + \beta)^{1/2}] \cos 2\pi s \left[\frac{E_F}{\hbar \omega_c} - \frac{1}{2} - \frac{\pi}{4} \right].$$

Therefore the extremal point contribution to equation (33) for case number one becomes

$$\sigma_{xy} = -\sigma_0 \frac{L_x L_y \hbar}{\pi^2 n_a 2^{5/2} E_L \tau \alpha} \frac{E_F}{\hbar \omega_c} [\cot \pi(\alpha + \beta)^{1/2} + i \cot \pi(-\alpha + \beta)^{1/2}]$$

$$\cdot \sum_s \frac{(-1)^s}{s^{3/2}} \cos 2\pi s \left[\frac{E_F}{\hbar \omega_c} - \frac{1}{2} - \frac{\pi}{4} \right] \frac{X}{\sinh X} \int dq'$$

$$\cdot \sin(q'a/2) \sin(qa/2) qq' . \quad (41)$$

For case number three the evaluation of the sum over k_z differs from case number one in that m is replaced by $m^{-1/2}$ so that $\cot Z_0$ for case three is $-\tan Z_0$ if the Z_0

of case one is used. The difference in the evaluation lies in $\phi'(m_F)$ where the sum and difference of tangents is used in place of the sum and difference of cotangents. Therefore

$$\phi'(m_F) = \frac{-2m_F^4}{\alpha^s} \frac{\sin \frac{\alpha\pi}{m_F} - \frac{4\pi\beta^2}{m_F\alpha}}{\cos^2 \frac{\pi\alpha}{2m_F} - \left[\frac{\pi\beta}{2m_F}\right]^2} \cos\left(-\frac{\pi^2 s}{2}\right).$$

The end point contribution to the conductivity for case number three, the even state to odd state transition, is

$$\sigma_{xy} = 2 \sigma_0 \frac{L_x L_y \hbar m_F^2}{\pi^3 n a E_L \tau \alpha^2} \frac{\sin \frac{\alpha\pi}{m_F} - \frac{4\pi\beta^2}{m_F\alpha}}{\cos^2 \frac{\pi\alpha}{2m_F} - \left[\frac{\pi\beta}{2m_F}\right]^2} \sum_s (-1)^s \frac{\cos(-\pi^2 s/2)}{s^2} \\ \cdot \frac{X}{\sinh X} \int dq' \cos q'a/2 \cos qa/2 q'q. \quad (42)$$

For case number four the results will be identical to case number three.

At the extremal point, the contribution for case number three differs from case number one in that cotangents are replaced by the negative tangents. The results become,

$$\sigma_{xy} = \sigma_0 \frac{L_x L_y \hbar}{\pi^2 n a E_L \tau \alpha^{5/2}} \frac{E_F}{\hbar \omega_c} \left[\tan \pi(\alpha+\beta)^{\frac{1}{2}} + i \tan \pi(-\alpha+\beta)^{\frac{1}{2}} \right] \\ \cdot \sum_s \frac{(-1)^s}{s^{3/2}} \cos 2\pi s \left[\frac{E_F}{\hbar \omega_c} - \frac{1}{2} - \frac{\pi}{4} \right] \frac{X}{\sinh X} \\ \cdot \int dq' \cos q'a/2 \cos qa/2 q'q. \quad (43)$$

Case number four will give the same results as case number three.

CHAPTER BIBLIOGRAPHY

1. Blankenbecler, R., "Integrals Over the Fermi Function," American Journal of Physics, 25(May, 1957), 279.
2. Copson, E., Asymptotic Expansions, Cambridge Tracts in Mathematics and Mathematical Physics," Cambridge Cambridge University Press, 1949.
3. Courant, R. and D. Hilbert, Methods of Mathematical Physics, Vol. 1, New York, Interscience Publishers, Inc., 1953.
4. Green, M.P., H.K. Lee, J.J. Quinn, and S. Rodriguez, "Linear Response Theory for a Degenerate Electron Gas in a Strong Magnetic Field," Physical Review, 177(January, 1969), 1019-1036.
5. Karplus, R., and J. Schwinger, "A Note on Saturation in Microwave Spectroscopy," The Physical Review, 73 (May, 1948), 1020-1026.
6. Kittel, C., Elementary Statistical Physics, New York, John Wiley and Sons, Inc., 1958.
7. Morse, P.M., and H. Feshback, Methods of Theoretical Physics, Vol. 1, New York, McGraw Hill, Inc., 1953.
8. Schiff, L.I., Quantum Mechanics, New York, McGraw Hill, Inc., 1949.

CHAPTER III

DISCUSSION

The long wavelength contributions to the xy component of the magnetoconductivity in wave number space are given by equations (40), (41), (42), and (43). Magnetic field dependent oscillations are found in all four contributions. Equations (40) and (42) each show the same size dependence as the classical size effect found by Sondheimer, with periodicity $B = 2 \hbar_c k_F / ae$. The two contributions, equations (40) and (42) however are out of phase with each other. To add the two contributions would be inconsistent with experimental results for the direct current effects. For the direct current experiments a nearly uniform electric field exists across the sample. This corresponds to an infinite wavelength and a zero wave vector. Only unlike parity transitions contribute. For the radio frequency effects, both contributions should be present since the disturbance is in the form of a plane wave. Previously unexplained data supports this. Low field data reported by Gantmakher and Kaner (1) show two sets of peaks. Equation (43) exhibits the Shubnikov-de Haas effect with size modulations. The effects of the modulations require samples of thickness of less than one tenth of a millimeter to be observable. The amplitudes for the size effect contributions

and the Shubnikov-de Haas contributions are comparable. Equation (42) exhibits size oscillations which will be referred to as Quantum Size Oscillations (QSO) because the amplitude goes to zero as h approaches zero and because the oscillation is not found when either the sum over the z momentum states or the sum over the Landau levels is replaced by an integral. The QSO is due then to the discrete nature of the magnetic levels and the z momentum states at the Fermi momentum.

The Quantum Size Effect (QSE) of Sandomirskii (2) can also be obtained from the above formalism. Equation (37) is a periodic function of $\pi(m^2 - \alpha + \beta)^{\frac{1}{2}}$. In a zero limit magnetic field, $\pi(m^2 - \alpha + \beta)^{\frac{1}{2}} \approx \pi m$. For a thin film for which there are only one or two allowed z -momentum states, $m = 1$ or 2 the de Broglie wavelength, $\lambda = h/p$, becomes $\lambda = h/\hbar k = a/m$ where a is the thickness of the sample. Therefore the period of equation (37) goes like the thickness of the sample. Sandomirskii assumed that the QSE oscillations occur when the thickness of the sample is on the order of the de Broglie wavelength. Thus QSE can be calculated by evaluating Equation (37) at the end points.

Mathematical Basis of Quantum Size Oscillations

Mathematically the QSO occurs when there is a pole energy term in Equation (22). For the situation where $\omega\tau \ll 1 \ll \omega_c\tau$, this can only occur when the initial and final z momentum

states are different. For low temperatures these initial and final states must also be on either side of the Fermi energy level.

For an infinite sample the matrix elements of the velocity operator are non zero only where the initial and final z momentum states differ by q_z . At the Fermi surface, the energy denominator will then have the form $\hbar\omega_c - (\hbar^2 k_F q / M) - (\hbar^2 q^2 / 2M)$. If q_z is small ($q_z v_F / \omega_c \ll 1$, where $v_F = \hbar k_F / M$) there will be no pole for an infinite sample. For larger q_z ($q_z \approx \omega_c / v_F$) there is one pole for a specific value of q_z . For a perturbing field prepared in such a way that only one Fourier component of q_z is present, there is a resonant effect for only one particular value of B . However, if a continuous spectrum of wave numbers is present, there is a resonance for every value of the magnetic field and no effect is observed as the magnetic field is swept.

For a finite sample the initial and final z momentum states are no longer coupled by q_z . Because the z momentum states are discrete the energy denominator takes on the form $\hbar\omega_c - [E_L [m'^2 - (m - \frac{1}{2})^2]]$. Poles can occur only at values of B_0 which match $\hbar\omega_c$ with the difference between the energies represented by the initial and final z momentum values. As a result of the end point contribution to the sum over the z momentum states only these transitions involving the Fermi momentum contribute.

Physical Basis for Quantum Size Oscillations

A more physical basis for the QSO can be found in terms of the intersection of Landau levels with the discrete z momentum states associated with a small sample. The points of intersection correspond to allowed electron states. A Landau level diagram, a plot of E vs. k_z , can be used to illustrate the QSO. A resonance occurs for the condition matching $\hbar\omega_c$, the energy difference between two adjacent Landau levels at the Fermi energy with the energy between two allowed z momentum states with one being the Fermi momentum. Therefore the resonance corresponds to a horizontal transition at the Fermi level between adjacent Landau levels which intersect initial and final z momentum states at the Fermi momentum.

For an infinite sample the z momentum states are quasi-continuous and the matching condition is always satisfied if there is a continuous spectrum of q_z . Therefore no oscillations will be observed in an infinite sample. Figures 1 and 2 are used to illustrate the resonance associated with a finite sample. If it is assumed for simplicity that the Landau level $n=0$ intersects k_F at E_F , the resonance occurs when the Landau level $n=1$ intersects at E_F each allowed momentum state less than k_F . For a finite sample there are only certain values of the magnetic field at which this occurs. Therefore an oscillation will be observed as the magnetic field is swept. For $k_F = 2 m_F/a$, the first

Fig. 1--First Resonance

Fig. 2--Second Resonance

resonance occurs when the field is increased until the $n = 1$ level intersects the momentum state $2\pi(m_F - \frac{1}{2})/a$ as in Figure 1. The value of B_0 needed to do this is obtained from

$$\hbar\omega_c = Ek_F = \frac{\hbar(2\pi/a)^2}{2e} [m_F^2 - (m_F - \frac{1}{2})^2].$$

This gives

$$B_0 = \frac{\hbar(2\pi/a)^2 c}{2e} [m_F^2 - (m_F - \frac{1}{2})^2].$$

The next resonance as shown in Figure 2 occurs when

$$\hbar\omega_c = \frac{\hbar^2(2\pi/a)^2}{2M} [m_F^2 - (m_F - 1 - \frac{1}{2})^2],$$

where $m_F - 1 - \frac{1}{2}$ is used in order to illustrate only the even-odd state transition. The value of B_0 needed is

$$B_0 = \frac{\hbar(2\pi/a)^2 c}{2e} [m_F^2 - (m_F - 3/2)^2].$$

The change in magnetic field, corresponding to the period of the oscillation is

$$\Delta B = \frac{\hbar 2\pi c}{2e} [m_F^2 - (m_F - 1)] 2/a$$

but for large values of m_F

$$\Delta B \approx (2\pi\hbar c/ae)k_F$$

which is the Sondheimer period.

Scattering and The Uncertainty Principle

The resonance occurs when electrons with energy E_F leaving momentum states at k_F are scattered into another allowed momentum state at E_F . This represents a condition corresponding to diffuse scattering at the surface, with momentum being conserved but only within the limits of the uncertainty principle. Off of the resonance, the scattering is essentially specular because there is no allowed state available. This assumption is consistent with the application of the uncertainty principle to electrons confined by the planes at $z = \pm a/2$. The momentum is considered to have components P_z and $P_\perp$, where $P_z \approx P_F$. The electron is effectively traveling through a slit of width a normal to the component $P_\perp$. The sequence of allowed momentum changes ΔP_z occur according to the relation for the intensity peaks of the electron diffraction pattern

$$(m + \frac{1}{2})\pi = (\pi a / \lambda) \sin \theta .$$

The analogy here is for $\sin \theta = \Delta P_z / P_\perp$ for small θ and

$\lambda = h / P_\perp$. These substitutions give

$$(m + \frac{1}{2}) = (\pi a / h) \Delta P_z .$$

Due to energy conservation, change in P_z is accompanied by a change in Landau level given by

$$\hbar \omega_c = \Delta(P_z^2 / 2M) = P_z \Delta P_z / M .$$

This gives

$$(m+\frac{1}{2})\pi = \pi M\hbar\omega_c/\hbar P_z .$$

Substitution of $P_z = P_F$ and $\omega_c = eB/MC$ gives

$$B = (2\pi P_F c/ea)(m+\frac{1}{2})$$

The change in B becomes

$$\Delta B = (2\pi c/ea) P_F$$

which is the Sondheimer period.

The above argument is useful for showing why the Sondheimer oscillations can be obtained from a quantum-mechanical treatment utilizing specular reflection at the surface when the semiclassical calculation must utilize diffuse surface scattering. The scattering at the surface is essentially specular until there is an allowed state for the electron to scatter into. The scattering then becomes diffuse within the limits of the uncertainty principle.

Comparison with Experiments

There are two contributions to the magnetoconductivity which exhibit the QSO. However, they are out of phase with each other and if added would give a result with a period one half that of the observed direct current size effect. In the direct current experiments a very nearly uniform electric field exists across the thickness of the sample. This corresponds to an infinite wavelength and a zero q . If this is approximated by the first term in the series

expansion of the q dependent amplitude, the contribution to the QSO from the like parity transitions may be neglected. The $q = 0$ calculation will then be in agreement with the experimental observations of the direct current size effect.

The periodicity of the radio frequency size effect is reported to be the same as the periodicity of the direct current size effect. Because the disturbance is the form $\exp(iqz - i t)$, both $\sin qz$ and $\cos qz$ components should be present. Equation (40) is the result obtained from the component $\sin qz$, and Equation (42) is the result obtained from the component $\cos qz$. Again, the presence of both contributions would lead to a period which is only one half that usually reported. However, early data reported by Gantmakher and Kaner (1) clearly show two sets of oscillations which are spike-like in shape at low magnetic fields. Each set shows the usual periodicity but they are 180 degrees out of phase. One set has an amplitude about one tenth of the other. For small values of q, q' equation (40) goes like $q^2 q'^2$ while equation (42) goes like qq' . Therefore the contribution from equation (40) would be of considerably small amplitude. This would give a result in agreement with the observations.

The spike-like nature of the oscillations at low fields is in agreement with the physical model presented above. When there is a large number of momentum states between the

intersection of adjacent Landau levels with the Fermi energy level, the QSO will be spread out and approach the limit of an infinite sample as the number of momentum states increase. This condition can exist for a finite sample if the magnetic field is very large. But for a low field the discreteness of the momentum values is very pronounced when referred to the spacing between adjacent Landau levels. Therefore spike-like resonance are to be expected at low fields and more sinusoidal resonance are to be expected at higher fields. This is in agreement with the nature of the QSO in equations (40) and (42).

The calculation gives good agreement with observations of size effect experiments and therefore supports the model on which the calculation is based. It suggests that the use of the infinite square well potential to represent the surfaces is justifiable.

CHAPTER BIBLIOGRAPHY

1. Gantmakher, V.F., and E.A. Kaner, "Dimension Effect in the Presence of Electron Drift inside a Metal," Soviet Physics-JETP, 18(April, 1964), 988-997.
2. Sandomirskii, V.B., "Quantum Size Effect in a Semimetal Film," Soviet Physics-JETP, 25(July, 1967), 101-106.

APPENDIX

The expression for $j_2(\vec{q}, \omega)$ as given by equation (18) can be investigated for size dependent oscillations by calculating $\langle \nu | \exp [i(\vec{q}-\vec{q}') \cdot \vec{r}] | \nu \rangle$. The integrals become

$$\int_{-\infty}^{\infty} dy \exp [i(k_y - k'_y)y] = L_y,$$

and

$$\int_{-\infty}^{\infty} U_n^* U_n = 1$$

for $\vec{q} = (0, 0, q)$. With $q - q'$ written as Δq ,

$$\exp[i(q - q') \cdot r] = \exp[i \Delta q z]$$

which can be written as

$$\exp[i \Delta q z] = \cos \Delta q z + i \sin \Delta q z.$$

Because the integration over z has symmetric limits, only two contributions will be non-zero. These are

$$I_1 = \int_{-a/2}^{a/2} \sin^2 k_z z \cos \Delta q z dz$$

and

$$I_2 = \int_{-a/2}^{a/2} \cos^2 k_z z \cos \Delta q z dz.$$

Use of the standard identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

and

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

gives

$$I_1 = \frac{\sin \Delta q a/2}{\Delta q} - \frac{\sin(\Delta q - 2k_z)a/2}{2(\Delta q - 2k_z)} - \frac{\sin(\Delta q + 2k_z)a/2}{2(\Delta q + 2k_z)}$$

and

$$I_2 = \frac{\sin \Delta q a/2}{\Delta q} + \frac{\sin(\Delta q - 2k_z)a/2}{2(\Delta q - 2k_z)} - \frac{\sin(\Delta q + 2k_z)a/2}{2(\Delta q + 2k_z)}.$$

Because q is small, there will be no resonance from the denominators and hence no size dependent oscillation due to j_2 .

BIBLIOGRAPHY

Books

- Blakemore, J.S., Solid State Physics, Philadelphia, W.B. Saunders Company, 1969.
- Copson, E., "Asymptotic Expansions," Cambridge Tracts in Mathematics and Mathematical Physics, Cambridge, Cambridge University Press, 1949.
- Courant, R. and D. Hilbert, Methods of Mathematical Physics, Vol. 1, New York, Interscience Publishers, Inc., 1953.
- Kittel, C., Elementary Statistical Physics, New York, John Wiley and Sons, Inc., 1958.
- Morse, P.M., and H. Feshback, Methods of Theoretical Physics, Vol. 1, New York, McGraw Hill, Inc., 1953.
- Ziman, J.M., Electrons in Metals, London, Taylor and Francis Ltd., 1963.

Articles

- Adams, E.N., and T.D. Holstein, "Quantum Theory of Transverse Galvanomagnetic Phenomena," Journal of Physics and Chemistry of Solids, 10(February, 1959), 254-276.
- Blankenbecler, R., "Integrals Over the Fermi Function," American Journal of Physics, 25(May, 1957), 279.
- Bommel, H.E., "Ultrasonic Attenuation in Superconducting and Normal-Conducting Tin at Low Temperature," Physical Review, 100(October, 1955), 758-759.
- Cohen, M.H., M.J. Harrison and W.A. Harrison, "Magnetic Field Dependence of the Ultrasonic Attenuation in Metals," Physical Review, 117(February, 1960), 937-952.
- Drude, P., "Zur Elektronen Theorie der Metalle," Annalen der Physik, 1(June, 1900), 566.
- Gantmahker, V.F., "A Method of Measuring the Momentum of Electrons in a Metal," Soviet Physics-JETP, 15 (November, 1962), 982-983.

-
- "Investigation of the Fermi Surface of Tin by the Size Effect," Soviet Physics-JETP, 17 (September, 1963), 549-560.
- Gantmakher, V.F., and E.A. Kaner, "Dimension Effect in the Presence of Electron Drift inside a Metal," Soviet Physics-JETP, 18(April, 1964), 988-997.
- Green, M.P., H.K. Lee, J.J. Quinn, and S. Rodriguez, "Linear Response Theory for a Degenerate Electron Gas in a Strong Magnetic Field," Physical Review, 177(January, 1969), 1019-1036.
- Kapitza, P., "The Change of Electrical Conductivity in Strong Magnetic Fields," Proceedings of the Royal Society of London, A123(1929), 292.
- Karplus, R., and J. Schwinger, "A Note on Saturation in Microwave Spectroscopy," The Physical Review, 73 (May, 1948), 1020-1026.
- Landau, L., "Diamagnetismus der Metalle," Zeitschrift fur Physik, 64(1930), 629.
- Munarin, J.A., J.A. Marcus and P.E. Bloomfield, "Size-Dependent Oscillatory Magnetoresistance Effect in Gallium," Physical Review, 172(August, 1968), 718.
- Pippard, A.B., "Metallic Conduction at High Frequencies and Low Temperatures," Advances in Electronics-Electron Physics, 6(1954), 1.
-
- "Proposal for Determining the Fermi Surface by Magneto-Acoustic Resonance," Philosophical Magazine, Ser. 8, 2(September, 1957), 1147.
- Sandomirskii, V.B., "Quantum Size Effect in a Semimetal Film," Soviet Physics-JETP, 25(July, 1967), 101-106.
- Shubnikov, L. and W.J. de Haas, "Neue Erscheinungen bei der Widerstandsänderung bei der Temperatur von flüssigen Wasserstoff," Leiden Commun, 207d(1930).
- Sommerfeld, A., "Zur Elektronen Theorie der Metalle auf Grund der Fermischen Statistik," Zeitschrift fur Physik, 47(1928), 1.
- Sondheimer, E.H., "The Influences of a Transverse Magnetic Field on the Conductivity of Thin Metallic Films," Physical Review, 80(November, 1950), 401-406.

Zebouni, N.H., R.E. Hamburg and H.J. Mackey, "Magneto-
morphic Oscillations in the Hall Effect and
Magnetoresistance in Cadmium," Physical Review Letters,
11(August, 1963), 260-263.