BOREL SETS WITH CONVEX SECTIONS AND EXTREME POINT SELECTORS

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In this dissertation we present some separation and selection theorems. We begin by presenting a detailed proof of the Inductive Definability Theorem of D. Cenzer and R.D. Mauldin, including their boundedness principle for monotone coanalytic operators.

By a faithful separation property we mean a property P such that if A and E are disjoint analytic subsets of the product X×Y of two Polish spaces X and Y and for each x, $A_x=\{y: (x,y)\in A\}$ has property P, then there is a Borel subset B of X×Y such that $A\subset B$, $B\cap E=\emptyset$ and for each x, B_x has property P.

In Chapter III, using the boundedness principle for monotone coanalytic operators, we reprove a portion of J. Saint-Raymond's argument that σ-compactness is a faithful separation property. Furthermore, we show in Chapter IV that convexity is a faithful separation property in the case $Y=\mathbb{R}^k$.

In Chapter V, we prove a selection theorem involving compact-valued upper semi-continuous multifunctions with values in the unit ball of the dual of a separable normed linear space.

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CHAPTER I

INTRODUCTION

In this dissertation we present some separation and selection theorems. In particular, we prove that convexity is a faithful separation property, and we prove that under certain conditions a multifunction with compact convex values has a Borel class 1 selector which selects extreme points. In addition, we give a detailed proof of the Inductive Definability Theorem of Cenzer and Mauldin [3], including their boundedness principle.

Before proceeding, we define some basic terms and set some notation. A Polish space is a separable completely metrizable topological space. Unless otherwise stated, we assume that all spaces are Polish. By $\mathbb{N}^{\mathbb{N}}$ we mean the space of all infinite sequences of positive integers with the product topology. For a space X, we denote by $\mathcal{B}(X)$ the family of all Borel subsets of X, that is the smallest family including the open subsets of X and closed under countable unions and complementation. By $\mathcal{G}_\delta$, $\mathcal{F}_\sigma$, and $\mathcal{K}_\sigma$ we mean the collections of all countable intersections of open subsets of X, all countable unions of closed subsets of X, and all countable unions of compact subsets of X respectively. We say a set A is analytic if it is the continuous image of $\mathbb{N}^{\mathbb{N}}$. We denote by $\mathcal{A}(X)$ the family of analytic subsets of X.

By a faithful separation property we mean a property P such that if A and E are disjoint analytic subsets of the product X×Y of two Polish spaces X and Y and for each x, $A_x=\{y: (x,y)\in A\}$ has property P, then there is a Borel subset B of X×Y such that $A\subset B$, $B\cap E=\emptyset$ and for each x, B_x has property P. Some examples of faithful separation properties include compactness [8], σ-compactness [10], first category [2], countable [5,6,7] and measure zero [2]. In Chapter III we reprove a portion of J. Saint-Raymond's argument that σ-compactness is a faithful separation property. Furthermore, we show in Chapter IV that convexity is a faithful separation property in the case $Y=\mathbb{R}^k$. Our argument given is a parameterization of an argument of D. Preiss [9].

By a multifunction $F:X\to Y$ we mean a function whose domain is X and whose values are nonempty subsets of Y. (Here, our spaces need not be Polish). By a selector for a multifunction we mean a function $f:X\to Y$ such that for, each $x\in X$, $f(x)\in F(x)$. By the Axiom of Choice, there exists a selector for a given multifunction; however, one is usually interested in a selector that possesses a certain property, for example continuity, Borel measurability, etc. In the literature, there are a good number of results concerning what sort of selector one can obtain under certain assumptions on the multifunction $F:X\to Y$ and on the spaces X and Y. We refer the reader to the surveys of Wagner [11,12] for a listing of many of these results.

In Chapter V, we prove a selection theorem involving compact-valued upper semi-continuous multifunctions with values in the unit ball of the dual of a separable normed linear space. This result is closely related to a previous theorem of Jayne and Rogers [4]. One difference is that our theorem deals with the selection of extreme points. As a corollary of our theorem, we give an alternative proof of a selection theorem of L. Baggett [1].

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CHAPTER II

THE INDUCTIVE DEFINABILITY THEOREM

In this chapter we aim to give a detailed proof of the Inductive Definability Theorem of Cenzer and Mauldin [1]. In particular, we verify a boundedness principle for analytic derivations. The approach here is the same as that in [1]. We begin by recalling some definitions.

By an operator over X, we mean a map from the power set $\mathcal{P}(X)$ to itself. An operator Γ is said to be monotone if for any $K \subset M \subset X$, $\Gamma(K) \subset \Gamma(M)$. An operator is said to be inductive if for any $K \subset X$, $K \subset \Gamma(K)$. The dual operator D of an operator Γ over X is defined by

$$D(A) = X - \Gamma(X - A).$$

Let $A \subset X$ and let Γ be an operator on X. We define

$$\Gamma^0(A) = A,$$

$$\Gamma^{\alpha+1}(A) = \Gamma(\Gamma^\alpha(A)) \quad \text{for all ordinals } \alpha,$$

$$\Gamma^\lambda(A) = \bigcup_{\alpha<\lambda} \Gamma^\alpha(A) \quad \text{for limit ordinals } \lambda.$$

The set $Cl(\Gamma;A) = \bigcup_{\alpha} \Gamma^\alpha(A)$ where the union is over the set of all ordinals is called the closure of Γ on A. For some ordinal $\alpha < card(X)^+$, $\Gamma^{\alpha+1}(A) = \Gamma^\alpha(A) = Cl(\Gamma;A)$, and we denote the least such ordinal by $|\Gamma;A|$. Also, we let $|\Gamma|$ =

$|\Gamma;\emptyset|$, and we let $Cl(\Gamma) = Cl(\Gamma;\emptyset)$.

An operator Δ over a Polish space X is said to be Borel (or $\underset{\sim}{\Delta}^1_1$) if it is defined in one of the following ways:

(a) $\Delta(K) = B$, where B is a fixed Borel subset of X;

(b) $\Delta(K) = f^{-1}(K)$, where f is a fixed Borel map from X to X;

(c) $\Delta(K) = X - K$;

(d) $\Delta(K) = \Delta_1(\Delta_2(K))$, where Δ_1 and Δ_2 are previously defined Borel operators;

(e) $\Delta(K) = \bigcup_{n=1}^{\infty} \Delta_n(K)$, where the Δ_n are previously defined Borel operators.

An operator Γ over a Polish space X is analytic or $\underset{\sim}{\Sigma}^1_1$ (respectively coanalytic or $\underset{\sim}{\Pi}^1_1$) if there is a Polish space Y and a Borel operator Δ over X×Y such that for all x and K:

$$x \in \Gamma(K) \quad \text{iff} \quad (\exists y)\ (x,y) \in \Delta(K\times Y),$$

$$\text{(respectively)} \quad (\forall y)\ (x,y) \in \Delta(K\times Y).$$

Note that Γ is an analytic operator if and only if its dual is coanalytic. By an analytic derivation, we mean an operator whose dual operator is monotone and coanalytic. The notion of analytic derivation is due to Dellacherie (see [2]).

Example. Let X be a Polish space. Let Γ denote the closure operator over X; i.e., $\Gamma(M)=\bar{M}$. Then Γ is an example of an analytic operator which is not necessarily Borel.

Proof. Note that $\Gamma(M)=\bar{M}=\{x:(\forall k)(\exists y\in M)(d(x,y)<1/k\}$. Let $Y=X^{\mathbb{N}}$. Then Y is Polish. For each $k\in\mathbb{N}$, define a subset B_k of $X\times Y$ by $B_k=\{(x,(y_n)): d(x,y_k)<1/k\}$. Each B_k is open. Also, for each $k\in\mathbb{N}$, define $f_k:X\times Y\to X\times Y$ by $f_k(x,(y_n))=(y_k,(y_n))$. Note that each f_k is continuous. Next, define an operator Δ over $X\times Y$ by

$$\Delta(A)=\bigcap_k (B_k\cap f_k^{-1}(A)).$$

Then Δ is a Borel operator, and $x\in\Gamma(M)$ iff $(\exists(y_n))$ $(x,(y_n))\in\Delta(M\times Y)$. Therefore, Γ is analytic.

Next, we show that Γ need not be Borel. Let $A\subset[0,1]$ be an analytic nonBorel set. There is a Borel set $B\subset[0,1]\times[0,1]$ such that $\pi_1(B)=A$. Now assume that Γ is a Borel operator. Then the operator Γ^* over $[0,1]\times[0,1]$ given by $\Gamma^*(E) = \bigcup_y \Delta(E^y)\times\{y\}$ is also Borel (see Lemma 2.4). By Theorem 2.1(a), $\Gamma^*(B)$ is a Borel set. Now the sections of $\Gamma^*(B)$ are compact. Therefore, $\pi_1(\Gamma^*(B))$ is a Borel subset of [0,1] [3,p.392]. However, $\pi_1(\Gamma^*(B))=\pi_1(B)=A$. This is a contradiction. □

Theorem 2.1 *Inductive Definability* [1,p. 58].

(a) If Δ is a Borel operator over X, then Δ^α is also a Borel operator for each $\alpha<\omega_1$ and $\Delta(B)$ is a Borel subset of X, if B is.

(b) If the monotone operator Γ and the set A are both analytic (resp. coanalytic), then for each countable ordinal α, $\Gamma^\alpha(A)$ is analytic (resp. coanalytic).

(c) If the monotone operator Γ and the set A are both coanalytic, then $Cl(\Gamma;A)$ is coanalytic.

(d) For any coanalytic subset C of X, there is a monotone Borel operator Δ over $X\times\mathbb{N}^{\mathbb{N}}$ and a real $r\in\mathbb{N}^{\mathbb{N}}$ such that $C=\{x:(x,r)\in Cl(\Delta)\}$.

(e) If Γ is a coanalytic monotone operator with closure C, on the coanalytic subset P of X, then for any analytic subset A of X with $A \subset C$, there is some countable ordinal α such that $A \subset \Gamma^\alpha(P)$.

(f) If the inductive operator Γ is either (1) Borel or (2) monotone and either analytic or coanalytic, then $|\Gamma|\leq\omega_1$.

We refer to part (e) of the theorem as the boundedness principle for coanalytic monotone operators.

Preliminary Results

Before proving the Inductive Definability Theorem, we first prove several preliminary results. The first lemma,which is mentioned in [1],but not proved, allows us in a natural way to associate an inductive operator with any given operator.

Lemma 2.2. Let Γ be a monotone $\underset{\sim}{\Pi}_1^1$ (resp. $\underset{\sim}{\Delta}_1^1$ or $\underset{\sim}{\Sigma}_1^1$) operator over X. Define the operator Ψ over X by $\Psi(K)=K\cup\Gamma(K)$. Then Ψ is a monotone inductive $\underset{\sim}{\Pi}_1^1$ (resp. $\underset{\sim}{\Delta}_1^1$ or $\underset{\sim}{\Sigma}_1^1$) operator and for each ordinal α, $\Gamma^\alpha(\emptyset)=\Psi^\alpha(\emptyset)$). Consequently, $Cl(\Gamma)=Cl(\Psi)$.

Proof. Suppose Γ is a monotone $\underset{\sim}{\Pi}_1^1$ operator over X (the proof where Γ is $\underset{\sim}{\Delta}_1^1$ or $\underset{\sim}{\Sigma}_1^1$ is similar). Let $K\subset M\subset X$. Since Γ is monotone, $\Psi(K)=K\cup\Gamma(K)\subset M\cup\Gamma(M)=\Psi(M)$. Therefore, Ψ is monotone. Note also that $K\subset K\cup\Gamma(K)=\Psi(K)$. Hence, Ψ is inductive.

Let Δ_1 be a Borel operator over $X\times Y$ where Y is Polish such that $x\in\Gamma(K)$ iff $(\forall y)(x,y)\in\Delta(K\times Y)$. Define Δ_2 over $X\times Y$ by $\Delta_2(L)=f^{-1}(L)$ where $f:X\times Y\to X\times Y$ is the identity map $(f(x,y)=(x,y))$. Then Δ_2 is Borel. Set $\Delta=\Delta_1\cup\Delta_2$. Then Δ is Borel. Now $x\in\Psi(K)$ iff $(\forall y)(x,y)\in\Delta(K\times Y)$. Thus, Ψ is $\underset{\sim}{\Pi}_1^1$.

Clearly, $\Gamma(\emptyset)=\Psi(\emptyset)$. Suppose that for some α, $\Gamma^\alpha(\emptyset)\neq\Psi^\alpha(\emptyset)$. Let λ be the smallest ordinal such that $\Gamma^\lambda(\emptyset)\neq\Psi^\lambda(\emptyset)$. Either λ is a successor ordinal or a limit ordinal. Suppose λ is a successor. Then $\lambda=\beta+1$ for some ordinal β. Since $\beta<\lambda$, $\Gamma^\beta(\emptyset)=\Psi^\beta(\emptyset)$. Also, since Γ is monotone, for all γ, $\Gamma^\gamma(\emptyset)\subset\Gamma^{\gamma+1}(\emptyset)$. Thus, $\Psi^\lambda(\emptyset)=\Psi(\Psi^\beta(\emptyset))=\Psi(\Gamma^\beta(\emptyset))=\Gamma^\beta(\emptyset)\cup\Gamma(\Gamma^\beta(\emptyset))=\Gamma^\beta(\emptyset)\cup\Gamma^{\beta+1}(\emptyset)=\Gamma^{\beta+1}(\emptyset)$ $=\Gamma^\lambda(\emptyset)$, a contradiction. Hence, λ must be a limit ordinal. Now, $\Psi^\lambda(\emptyset)=\bigcup_{\tau<\lambda}\Psi^\tau(\emptyset)=\bigcup_{\tau<\lambda}\Gamma^\tau(\emptyset)=\Gamma^\lambda(\emptyset)$, a

contradiction. Therefore, for each ordinal α, $\Gamma^{\alpha}(\emptyset)=\Psi^{\alpha}(\emptyset)$.
□

Recall from [1], the next lemma which says that if Δ is a Borel operator and if $x\in X$ and $K\subset X$ are fixed, then the determination of $\Delta(K)$ at x depends on only a countable amount of information.

Lemma 2.3 [1,p. 66].
(a) If Δ is a Borel operator over the Polish space X, then for any $x\in X$ and $K\subset X$, there are countable sets $U\subset K$ and $V\subset X-K$ such that, for any set M with $U\subset M$ and $V\subset X-M$, $x\in\Delta(K)$ iff $x\in\Delta(M)$.
(b) If Γ is a $\underset{\sim}{\Sigma}^1_1$ monotone operator over X, then for any $x\in X$ and $K\subset X$, $x\in\Gamma(K)$ iff (for some countable $U\subset K$) $x\in\Gamma(U)$.
(c) If Γ is a $\underset{\sim}{\Pi}^1_1$ monotone operator over X, then for any $x\in X$ and $K\subset X$, $x\in\Gamma(K)$ iff (for all countable $V\subset X-K$) $x\in\Gamma(X-V)$.

Proof of (a). Let Ω denote the collection of all operators over X such that for every $x\in X$ and $K\subset X$, there are countable sets $U\subset K$ and $V\subset X-K$ such that, for any set M with $U\subset M$ and $V\subset X-M$, $x\in\Delta(K)$ iff $x\in\Delta(M)$. We assert that Ω contains all Borel operators over X.

Suppose $\Delta(K)=B$ where B is a fixed Borel subset of X. Fix $x\in X$ and $K\subset X$. Set $U=V=\emptyset$. Then for any M with $U\subset M\subset X-V$,

$x \in \Delta(K)=B$ iff $x \in \Delta(M)=B$. Thus, $\Delta \in \Omega$.

Suppose $\Delta(L)=f^{-1}(L)$ where f is Borel measurable. Fix $x \in X$ and $K \subset X$. If $f(x) \in K$, set $U=\{f(x)\}$ and $V=\emptyset$. Otherwise, set $U=\emptyset$ and $V=\{f(x)\}$. In either case, if $U \subset M \subset X-V$ we have $x \in \Delta(K)$ iff $x \in \Delta(M)$. Therefore, $\Delta \in \Omega$.

Now suppose $\Delta(L)=X-L$. Fix $x \in X$ and $K \subset X$. If $x \in K$, set $U=\{x\}$ and $V=\emptyset$. Otherwise, set $U=\emptyset$ and $V=\{x\}$. Then for any M with $U \subset M \subset X-V$ we have $x \in \Delta(K)$ iff $x \in \Delta(M)$. Thus, $\Delta \in \Omega$.

Next, assume $\Delta_1, \Delta_2 \in \Omega$. We will show that $\Delta_1 \circ \Delta_2 \in \Omega$. Fix $x \in X$ and $K \subset X$. There are countable sets U_1 and V_1 such $U_1 \subset K$ and $V_1 \subset X-K$ and if $U_1 \subset L \subset X-V_1$ then $x \in \Delta_1(\Delta_2(K))$ iff $x \in \Delta_1(L)$. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be enumerations of U_1 and V_1 respectively. For each n, there are countable A_n and B_n such that if $A_n \subset J \subset X-B_n$, then $u_n \in \Delta_2(K)$ iff $u_n \in \Delta_2(J)$. Also, for each n, there are countable C_n and D_n where $C_n \subset K \subset X-D_n$ such that if $C_n \subset J \subset X-D_n$, then $v_n \in \Delta_2(K)$ iff $v_n \in \Delta_2(J)$. Set $U = (\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} C_n)$ and $V = (\bigcup_{n=1}^{\infty} B_n) \cup (\bigcup_{n=1}^{\infty} D_n)$. Then U and V are countable and $U \subset K \subset X-V$. Suppose $U \subset M \subset X-V$. Then $U_1 \subset \Delta_2(M) \subset X-V_1$. Consequently, $x \in \Delta_1(\Delta_2(K))$ iff $x \in \Delta_1(\Delta_2(M))$. Thus $\Delta_1 \circ \Delta_2 \in \Omega$.

Lastly, suppose that for each n, $\Delta_n \in \Omega$. We assert that $\bigcup_{n=1}^{\infty} \Delta_n \in \Omega$. Fix $x \in X$ and $K \subset X$. For each n, there are countable U_n and V_n where $U_n \subset K \subset X-V_n$ such that if $U_n \subset M \subset X-V_n$ then $x \in \Delta_n(K)$ iff $x \in \Delta_n(M)$. Set $U = \bigcup_{n=1}^{\infty} U_n$ and $V = \bigcup_{n=1}^{\infty}$

V_n. Suppose $U \subset M \subset X - V$. Then $x \in \bigcup_{n=1}^{\infty} \Delta_n(K)$ iff $x \in \bigcup_{n=1}^{\infty} \Delta_n(M)$. Therefore, $\bigcup_{n=1}^{\infty} \Delta_n \in \Omega$. This completes the proof of part (a).

Proof of part (b). Suppose Γ is a $\utilde{\Sigma}_1^1$ monotone operator over X. Let $x \in X$ and $K \subset X$. Let Δ be a Borel operator on $X \times Y$ such that $x \in \Gamma(L)$ iff $(\exists y)(x,y) \in \Delta(L \times Y)$. Suppose $x \in \Gamma(K)$. Choose y so that $(x,y) \in \Delta(K \times Y)$. By part (a), there are countable sets $U \subset K \times Y$ and $V \subset (X-K) \times Y$ such that whenever $U \subset M$ and $V \subset (X \times Y) - M$, $(x,y) \in \Delta(M)$. Let $T = \pi_1(U)$. Then T is countable and $T \subset K$. Also, $U \subset T \times Y$ and $V \subset (X-T) \times Y$. Therefore, $(x,y) \in \Delta(T \times Y)$. Consequently, $x \in \Gamma(T)$.

Conversely, assume that for some countable $T \subset K$, $x \in \Gamma(T)$. Since Γ is monotone, $x \in \Gamma(K)$. This completes the proof of part (b).

Proof of part (c). Let D be the dual operator of Γ. Then D is a $\utilde{\Sigma}_1^1$ monotone operator. Fix $x \in X$, $K \subset X$. $x \in \Gamma(K)$ iff $x \notin D(X-K)$. By part (b), $x \notin D(X-K)$ iff for all countable $V \subset X-K$, $x \notin D(V)$. Equivalently, $x \notin D(X-K)$ iff for all countable $V \subset X-K$, $x \in \Gamma(X-V)$. This completes the proof of part (c). □

Let Δ be an operator over X. Define the operator Δ^* on $X \times Y$ by

$$\Delta^*(E) = \bigcup_y \Delta(E^y) \times \{y\}.$$

By E^y, we mean $\{x:(x,y)\in E\}$. An analysis of this section-wise operator Δ^* is useful in verifying some missing details in [1].

Lemma 2.4. If Δ is a Borel operator then Δ^* is a Borel operator.

Proof. The proof is by induction on the class of Borel operators on X. Set $\Omega = \{\Delta:2^X\to 2^X:\Delta^* \text{ is Borel}\}$.

Suppose $\Delta(K)=B$ where B is a fixed Borel subset of X. Let $E\subset X\times Y$. Then $\Delta^*(E) = \bigcup_y \Delta(E^y)\times\{y\} = \bigcup_y B\times\{y\} = B\times Y$, a fixed Borel set. Thus, $\Delta\in\Omega$.

Suppose $\Delta(K) = f^{-1}(K)$ where $f:X\to X$ is a Borel measurable map. Define $g:X\times Y\to X\times Y$ by $g(x,y)=(f(x),y)$. Then g is Borel measurable. $\Delta^*(E) = \bigcup_y f^{-1}(E^y)\times\{y\} = g^{-1}(E)$. Thus, $\Delta\in\Omega$.

Now assume $\Delta(K)=X-K$. Then $\Delta^*(E)=\bigcup_y (X-E^y)\times\{y\} = (X\times Y)-E$. Hence, $\Delta\in\Omega$.

Next, suppose Δ_1 and Δ_2 belong to Ω. Let $\Delta=\Delta_1\circ\Delta_2$. By assumption, $\Delta_1^*\circ\Delta_2^* = \Delta^*$. For

$$\begin{aligned}\Delta_1^*\ \Delta_2^*(E) &= \Delta_1^*((\textstyle\bigcup_y \Delta_2(E^y)\times\{y\})\\ &=\textstyle\bigcup_z \Delta_1((\bigcup_y \Delta_2(E^y)\times\{y\})^z)\times\{z\}\\ &= \textstyle\bigcup_z \Delta_1(\Delta_2(E^z))\times\{z\} = \Delta^*(E).\end{aligned}$$

Hence Δ^* is Borel, and consequently $\Delta\in\Omega$.

Lastly, assume that $\Delta_n\in\Omega$ for each $n\in N$. Let $\Delta=\bigcup_n \Delta_n$. Then

$$\Delta^*(E) = \bigcup_y \Delta(E^y)\times\{y\}$$

$$= \bigcup_y (\bigcup_n \Delta_n(E^y))\times\{y\}$$
$$= \bigcup_y \bigcup_n (\Delta_n(E^y)\times\{y\})$$
$$= \bigcup_n \bigcup_y (\Delta_n(E^y)\times\{y\})$$
$$= \bigcup_n \Delta_n^*(E).$$

Therefore, $\Delta\in\Omega$.

Consequently, Ω is the class of all Borel operators over X. This completes the proof of the lemma. □

Lemma 2.5. If Δ is a Borel operator over X, then $B_\Delta = \{(x,(y_n))\in X\times X^{\mathbb{N}}: x \in\Delta(\{y_1,y_2,\ldots\})\}$, where $Y=X^{\mathbb{N}}$. Then B_Δ is a Borel subset of $X\times X^{\mathbb{N}}$.

Proof. The proof is by induction on the family of Borel operators. For each m, set $G_m=\{(x,(y_n)): x\neq y_m\}$ and $F_m=\{(x,(y_n)): x=y_m\}$. Note that G_m is open and F_m is closed for each m.

Suppose $\Delta(K)=C$, where C is a fixed Borel subset of $X\times X^{\mathbb{N}}$. Then $B_\Delta=C\times X^{\mathbb{N}}$, a Borel subset of $X\times X^{\mathbb{N}}$.

Suppose $\Delta(K)=X-K$. Then $B_\Delta = \bigcap_m G_m$. Thus, B_Δ is a G_δ, and hence, B_Δ is Borel.

Next, assume $\Delta(K)=f^{-1}(K)$ where $f:X\to X$ is Borel measurable. Define $g:X\times X^{\mathbb{N}}\to X\times X^{\mathbb{N}}$ by $g(x,(y_n))=(f(x),(y_n))$. Then g is Borel measurable. Now $B_\Delta=\bigcup_m g^{-1}(F_m)$. Consequently, B_Δ is Borel.

Let Δ_1 and Δ_2 be Borel operators such that B_{Δ_1} and B_{Δ_2} are Borel subsets of $X\times X^{\mathbb{N}}$. Set $\Delta=\Delta_1\circ\Delta_2$. Now $(x,(y_n))\in B_\Delta$

iff $x\in\Delta_1(\Delta_2(\{y_1,y_2,\ldots\}))$ iff $x\in\{u:(u,(y_n))\in B_{\Delta_2}\}$ iff $x\in\Delta_1{}^*(B_{\Delta_2})$. Thus, $B_\Delta=\Delta_1{}^*(B_{\Delta_2})$. By lemma 2.4, $\Delta_1{}^*$ is a Borel operator. Therefore, by theorem 2.1(a) B_Δ is Borel.

Lastly, assume that $\{\Delta_n\}$ is a sequence of Borel operators such that for each n, B_{Δ_n} is Borel. Suppose $\Delta(K)=\bigcup_n\Delta_n(K)$. Then $B_\Delta=\bigcup_n B_{\Delta_n}$. Hence B_Δ is Borel. This completes the proof. □

Corollary 2.6. Let Δ be a Borel operator over $X\times Z$. Then the set

$C_\Delta=\{(x,(y_n),z)\in X\times X^{\mathbb{N}}\times Z:\ (x,z)\in\Delta(\{y_1,y_2,\ldots\}\times\{z\})\ \}$ is Borel.

Proof. By lemma 2.5,

$B_\Delta=\{((x,u),((y_n,z_n)):(x,u)\in\Delta(\{(y_1,z_1),(y_2,z_2),\ldots\})\}$ is a Borel subset of $(X\times Z)\times(X\times Z)^{\mathbb{N}}$. Define $f:X\times X^N\times Z\to(X\times Z)\times(X\times Z)^N$ by $f(x,z,(y_n))=((x,z),((y_1,z),(y_2,z),\ldots))$. Then f is continuous. Now $C_\Delta=f^{-1}(B_\Delta)$. Thus, C_Δ is Borel. □

Lemma 2.7 [1,p.67]. The family of $\underset{\sim}{\Sigma}{}^1_1$ (resp. $\underset{\sim}{\Pi}{}^1_1$) subsets of a Polish space is closed under $\underset{\sim}{\Sigma}{}^1_1$ (resp. $\underset{\sim}{\Pi}{}^1_1$) monotone operators.

Proof. Let Γ be a $\underset{\sim}{\Sigma}{}^1_1$ monotone operator. Suppose $M=\{(x,(y_n))\in X\times X^{\mathbb{N}}:\ x\in\Gamma(\{y_1,y_2,\ldots\})\ \}$. Let Δ be a Borel operator over $X\times Z$ such that $x\in\Gamma(K)$ iff $\exists z(x,z)\in\Delta(K\times Z)$. Then

$M = \pi_{12}(C_\Delta)$ where C_Δ is defined in corollary 2.6. Since, C_Δ is Borel, M is $\underset{\sim}{\Sigma}^1_1$. Now, suppose $A \subset X$ is $\underset{\sim}{\Sigma}^1_1$. By lemma 2.3, $\Gamma(A)=\{x:\exists y[(x,y)\in M \text{ and } \forall n\ y_n \in A]\}$. Thus, $\Gamma(A)$ is $\underset{\sim}{\Sigma}^1_1$.

On the other hand, let Γ be a $\underset{\sim}{\Pi}^1_1$ monotone operator and let D be its dual. Suppose $C \subset X$ is $\underset{\sim}{\Pi}^1_1$. Then $\Gamma(C)=X-D(X-C)$. Therefore, by the first part of the theorem $\Gamma(C)$ is $\underset{\sim}{\Pi}^1_1$. □

Lemma 2.8 [1,p.67]. $Cl(\Gamma)=\bigcap$ {K: K is a cocountable fixed point of Γ}.

Proof. Let $x \in C = Cl(\Gamma)$. Suppose K is a fixed point of Γ. There is an ordinal α such that $\Gamma^\alpha(\emptyset)=C$. Since K is a fixed point, $\Gamma^\alpha(K)=K$. Also, since Γ is monotone, $C=\Gamma^\alpha(\emptyset)\subset\Gamma^\alpha(K)=K$. Hence, $x \in K$. Therefore, $x\in\bigcap\{K$: K is a cocountable fixed point of $\Gamma\}$ = D. Consequently, $C \subset D$.

For the other inclusion, suppose $x \notin C$. By lemma 2.3(c), there is some cocountable K_1 such that $C \subset K_1$ and $x \notin \Gamma(K_1)$. If $K_1=\Gamma(K_1)$, then we are done. Thus assume $K_1 \neq \Gamma(K_1)$, and let $\{y_i\}_{i=1}^\infty$ enumerate $\Gamma(K_1)-K_1$. For each i, $y_i \notin C$. Hence, for each i, there is a cocountable $J_i \supset C$ such that $y_i \notin \Gamma(J_i)$. Set $K_2=K_1 \cap (\bigcap_i J_i)$. Then K_2 is cocountable and $C \subset K_2 \subset K_1$. Also, for each i, $y_i \in \Gamma(K_2)$, i.e., $\Gamma(K_2) \subset K_1$. By induction, we get a decreasing sequence $\{K_n\}_{n=1}^\infty$ of cocountable sets such that for each n, $C \subset \Gamma(K_{n+1}) \subset K_n$ and $x \notin K_n$. Let $K = \bigcap_n K_n$. Then K is cocountable, $x \notin K$ and $\Gamma(K)\subset\bigcap_n\Gamma(K_n)\subset\bigcap_n K_n=K$. Hence, K is a cocountable fixed point. Consequently, $x \notin D = \bigcap \{K: K \text{ is a cocountable fixed point of } \Gamma\}$. Therefore, $D \subset C$. □

The following will be used in proving the boundedness principle for monotone coanalytic operators.

For an inductive operator Γ over X, set

$$|x|_\Gamma = (\text{least } \alpha)\ x \in \Gamma^{\alpha+1}(\emptyset) = \Gamma^{\alpha+1}, \text{ if } x \in Cl(\emptyset)$$
$$= \infty \qquad\qquad\qquad\qquad , \text{ otherwise.}$$

Suppose Γ and Δ are inductive operators over X. Define

$$R(x,y) \text{ iff } |x|_\Gamma \leq |y|_\Delta \text{ and } x \in Cl(\Gamma)$$
$$\text{and } S(x,y) \text{ iff } |x|_\Gamma < |y|_\Delta \text{ and } x \in Cl(\Delta).$$

Lemma 2.9. (a) $R(x,y)$ iff $x \in \Gamma(\{x' : S(x',y)\})$ and (b) $S(x,y)$ iff $y \notin \Delta(\{y' : \neg R(x,y)\})$.

Proof of (a). Fix $y \in Y$. Set $K = \{x' : S(x',y)\}$. First note that $K = \bigcup_{\alpha < |y|_\Delta} \Gamma^{\alpha+1}$. Suppose $R(x,y)$. Then $|x|_\Gamma \leq |y|_\Delta$ and $x \in Cl(\Gamma)$. If $|x|_\Gamma < |y|_\Delta$, then $x \in K \subset \Gamma(K)$. Thus, assume $|x|_\Gamma = |y|_\Delta$. We assert that $\Gamma^\beta = K$ where $\beta = |y|_\Delta$. Clearly, $K \subset \Gamma^\beta$. Suppose $x' \in \Gamma^\beta$. Either β is a limit ordinal or not. Suppose β is a limit ordinal. In which case, $x' \in \Gamma^\beta = \bigcup_{\alpha<\beta} \Gamma^\alpha = \bigcup_{\alpha<\beta} \Gamma^{\alpha+1} = K$. Next, consider the case where β is a successor ordinal. In which case, $x' \in \Gamma^{(\beta-1)+1} \subset K$. Therefore, $\Gamma^\beta = K$. Now, $x \in \Gamma^{\beta+1} = \Gamma(\Gamma^\beta) = \Gamma(K)$.

Conversely, suppose that $x \in \Gamma(K) = \Gamma(\{x' : S(x',y)\})$. Either $y \in Cl(\Delta)$ or $y \notin Cl(\Delta)$. Assume that $y \in Cl(\Delta)$. Then

$x\in\Gamma(K)=\Gamma(\bigcup_{\alpha<\beta}\Gamma^{\alpha+1})=\Gamma(\Gamma^{\beta})=\Gamma^{\beta+1}$. Consequently, $x\in Cl(\Gamma)$ and $|x|_{\Gamma}\leq|y|_{\Delta}$. Therefore $\mathbb{R}(x,y)$. Now suppose that $y\notin Cl(\Delta)$. $K=Cl(\Gamma)$. Thus, $x\in Cl(\Gamma)$, and hence, $\mathbb{R}(x,y)$. □

Proof of (b). First note that if $x\in Cl(\Gamma)$, then $\bigcup_{\alpha<|x|_{\Gamma}}\Delta^{\alpha+1} = \{y':\neg\mathbb{R}(x,y)\}$. Assume $S(x,y)$. Then $|x|_{\Gamma}<|y|_{\Delta}$ and $x\in Cl(\Gamma)$. Thus, $y\notin\Delta^{\beta+1}=\Delta(\bigcup_{\alpha<\beta}\Delta^{\alpha+1})=\Delta(\{y':\neg\mathbb{R}(x,y)\})$ where $\beta=|x|_{\Gamma}$.

Conversely, suppose $y\notin\Delta(\{y':\neg\mathbb{R}(x,y)\})$. Then $y\notin\{y':\neg\mathbb{R}(x,y)\}$ and hence $\mathbb{R}(x,y)$. Consequently, $x\in Cl(\Gamma)$. If $|x|_{\Gamma}<|y|_{\Delta}$, then $S(x,y)$. Thus, suppose $|x|_{\Gamma}=|y|_{\Delta}$. Then $y\in\Delta^{\beta+1}=\Delta(\{y':\neg\mathbb{R}(x,y)\})$ where $\beta=|x|_{\Gamma}$, a contradiction. □

Denote by $\mathbb{N}^*$ the set of all finite sequences of nonnegative integers. Consider the relation $\leq$ on $\mathbb{N}^*$ given by

$$s=(s_1,\dots,s_m)\leq t=(t_1,\dots,t_n)$$

iff s extends t ($s\supset t$) or $(\exists j)(\forall i<j)(s_i=t_i$ and $s_j<t_j)$. $\leq$ is called the Brouwer-Kleene ordering, and it defines a linear ordering of $\mathbb{N}^*$. Let us mention that $(\mathbb{N}^*,\leq)$ is isomorphic to the set of dyadic rationals in $[0,1)$ with the usual ordering reversed. Define $W = \{x\in\{0,1\}^{\mathbb{N}^*}:x^{-1}(1)$ is well ordered by $\leq\}$. For $x\in W$, let $\sigma(x)$ be the order type of $x^{-1}(1)$. Also, for $x\in\{0,1\}^{\mathbb{N}^*}$ and $p\in\mathbb{N}^*$, define $x|_p\ \in\{0,1\}^{\mathbb{N}^*}$

by $x|_p(s)=1$ iff $x(s)=1$ and $s<p$.

Lemma 2.10. There is a Borel operator Δ over $\{0,1\}^{\mathbb{N}^*}$ such that for each ordinal α, $\Delta^\alpha=\{x:\sigma(x)<\alpha\}$. Consequently, $W=\mathrm{Cl}(\Delta)$, $|\Delta|=\omega_1$ and W is coanalytic. Furthermore, W is not a Borel set.

Proof. Define Δ over $\{0,1\}^{\mathbb{N}^*}$ by

$x\in\Delta(K)$ iff $x\in K$ or $(\forall s)x(s)=0$ or $(\forall p)(x(p)=1\rightarrow x|_p\in K)$.

Then Δ is Borel.

To show that $\Delta^\alpha=\{x:\sigma(x)<\alpha\}$ for each ordinal α, we use transfinite induction. First, note that $\Delta^1=\{x_\emptyset\}$. Thus, $\Delta^1=\{x:\sigma(x)<1\}$. Now let α be an ordinal greater than 1, and assume that $\Delta^\beta=\{x:\sigma(x)<\beta\}$ for all $\beta<\alpha$. Now, $\Delta^\alpha=\Delta(\bigcup_{\beta<\alpha}\Delta^\beta)$. Let $x_R\in\Delta^\alpha$. If $x_R\in\bigcup_{\beta<\alpha}\Delta^\beta$, then $\sigma(x_R)<\alpha$. Thus, assume that $x_R\notin\bigcup_{\beta<\alpha}\Delta^\beta$. Then there is $p\in R$ such that $x_R|_p\in\bigcup_{\beta<\alpha}\Delta^\alpha$. Consequently, $\sigma(x_R)=\sigma(x_R|_p)+1<\beta+1<\alpha$. Therefore, $\Delta^\alpha\subset\{x:\sigma(x)<\alpha\}$. Suppose $\sigma(x_R)<\alpha$. If there is an ordinal $\beta<\alpha$ such that $\sigma(x_R)<\beta<\alpha$, then $x_R\in\Delta^\alpha$ by assumption. Thus, we can assume that $\sigma(x_R)+1=\alpha$. Suppose that for some p, $x_R(p)=1$ and $x_R|_p\notin\bigcup_{\beta<\alpha}\Delta^\alpha$. Now $\sigma(x_R|_p)<\sigma(x_R)$, thus $x_R|_p\in\bigcup_{\beta<\alpha}\Delta^\alpha$. This is a contradiction. Hence, $x_R\in\Delta^\alpha$. Consequently, $\Delta^\alpha=\{x:\sigma(x)<\alpha\}$.

By Theorem 1.2(c), W is coanalytic. Also, since there exist analytic nonBorel sets, W is not Borel [1,p.64].

Proof of the Inductive Definability Theorem

We now are ready to prove the Inductive Definability Theorem.

(a) If Δ is a Borel operator over X, then Δ^{α} is also a Borel operator for each $\alpha<\omega_1$ and $\Delta(B)$ is a Borel subset of X, if B is.

Proof. Suppose that for some $\alpha<\omega_1$, Δ^{α} is not Borel. Let λ be the smallest such ordinal. Either λ is a successor ordinal or a limit ordinal. Suppose that λ is a successor ordinal. Then $\lambda=\beta+1$ for some ordinal β. Now $\Delta^{\lambda}=\Delta^{\beta+1}=\Delta\circ\Delta^{\beta}$ and Δ^{β} is a Borel operator. Thus, Δ^{λ} is Borel, a contradiction. Therefore, λ must be a limit ordinal. Hence, $\Delta^{\lambda} = \bigcup_{\tau<\lambda} \Delta^{\tau}$. For each $\tau<\lambda$, Δ^{τ} is Borel. Thus, since λ is countable, Δ^{λ} must be Borel. Again, we have a contradiction. Therefore, for every $\alpha<\omega_1$, Δ^{α} is Borel.

Next, we verify that if Δ is a Borel operator, then for each Borel subset B of X, $\Delta(B)$ is a Borel subset of X. The proof is by induction on the family of Borel operators. Clearly, if Δ defined by $\Delta(K)=D$ where D is a fixed Borel set, then $\Delta(B)$ is Borel if B is. If $\Delta(K)=f^{-1}(K)$ where f is a Borel map from X to X, then $\Delta(B)$ is Borel if Δ is. Also, if Δ is given by $\Delta(K)=X-K$, the $\Delta(B)$ is Borel if B is. Now assume that $\Delta=\Delta_1\circ\Delta_2$ where for each Borel set C, $\Delta_1(C)$ and

$\Delta_2(C)$ are Borel. Let B be Borel. Then $\Delta_2(B)$ is Borel, and consequently, $\Delta_1(\Delta_2(B))=\Delta(B)$ is Borel. Lastly, suppose that $\Delta(K) = \bigcup_n \Delta_n(K)$, where for each n, Δ_n is a Borel operator such that $\Delta_n(D)$ is Borel if D is. Then if B is Borel, certainly, $\Delta(B) = \bigcup_n \Delta_n(B)$ is Borel. This completes the proof. □

(b) If the monotone operator Γ and the set A are both analytic (resp. coanalytic), then for each countable ordinal α, $\Gamma^\alpha(A)$ is analytic (resp. coanalytic).

Proof. Let Γ be a monotone $\underset{\sim}{\Sigma}^1_1$ operator. By lemma 2.7, $\Gamma(A)$ is analytic if A is analytic. Suppose there is some countable ordinal α such that for some analytic subset A of X, $\Gamma^\alpha(A)$ is not analytic. Let λ be the smallest ordinal such that for some analytic set A, $\Gamma^\lambda(A)$ is not analytic. Then $\lambda>1$. Either λ is a successor ordinal or a limit ordinal. Suppose that λ is a successor ordinal. Then $\lambda=\beta+1$ for some ordinal β. Now $\Gamma^\lambda=\Gamma^{\beta+1}=\Gamma\circ\Gamma^\beta$ and $\Gamma^\beta(A)$ is a analytic if A is. Thus, $\Gamma^\lambda(A) = \Gamma(\Gamma^\beta(A))$ is analytic if A is, a contradiction. Therefore, λ must be a limit ordinal. Hence, $\Gamma^\lambda = \bigcup_{\tau<\lambda} \Gamma^\tau$. For each $\tau<\lambda$, $\Delta^\tau(A)$ is analytic if A is. Therefore, since λ is countable, if A is analytic, $\Gamma^\lambda(A) = \bigcup_{\tau<\lambda} \Gamma^\tau(A)$ is analytic. Again, we have a contradiction. Therefore, for every $\alpha<\omega_1$ and every analytic A, $\Gamma^\alpha(A)$ is analytic.

The proof is the same if Γ is $\underset{\sim}{\Pi}^1_1$ and the set A is coanalytic. □

(c) If the monotone operator Γ and the set A are both coanalytic, then $Cl(\Gamma;A)$ is coanalytic.

Proof. We first consider the case where $A=\emptyset$. Consider the operator ϕ given by $\phi(K)=K\cup\Gamma(K)$. Then by lemma 2.2, ϕ is a monotone inductive $\underset{\sim}{\Pi}^1_1$ operator. Also, for each α, $\phi^\alpha(\emptyset)=\Gamma^\alpha(\emptyset)$. Consequently, $Cl(\phi)=Cl(\Gamma)$. Therefore, we can assume that Γ is inductive. Let D be the dual of Γ. Define

$$\begin{aligned}\hat{M} &= \{(y_n)\in X^{\mathbb{N}}: \Gamma(X-\{y_1,y_2,\ldots\})\subset X-\{y_1,y_2,\ldots\})\} \\ &= \{\{(y_n)\in X^{\mathbb{N}}: (\forall n)y_n\notin\Gamma(X-\{y_1,y_2,\ldots\})\}\end{aligned}$$

Now, $\hat{M}=\pi_2(M)$ where $M=\{(x,(y_n))\in X\times X^{\mathbb{N}}: x\in D(\{y_1,y_2,\ldots\})$ and $(\forall n)y_n=x\}$. Since M is analytic (see the proof of lemma 2.7), $\hat{M}$ is analytic. Next, define $P = \{x\in X : (\forall y\in X^{\mathbb{N}})(y\in M \rightarrow (\forall n)(y_n\neq x))\}$. Then P is $\underset{\sim}{\Pi}^1_1$. Also, by Lemma 2.8, $Cl(\Gamma)=P$.

Lastly, consider the case where A is an arbitrary coanalytic set. Define the operator Ψ by $\Psi(K)=\Gamma(K\cup A)$. Then Ψ is a monotone $\underset{\sim}{\Pi}^1_1$ operator, and for each α, $\Psi^\alpha(\emptyset) = \Gamma^\alpha(A)$. Consequently, $Cl(\Psi)=Cl(\Gamma;A)$. Thus, $Cl(\Gamma;A)$ is $\underset{\sim}{\Pi}^1_1$. □

(d) For any $\underset{\sim}{\Pi}^1_1$ subset C of X, there is a monotone $\underset{\sim}{\Delta}^1_1$ operator Δ over $X\times N^{\mathbb{N}}$ and $r\in N^{\mathbb{N}}$ such that $C=\{x:(x,r)\in Cl(\Delta)\}$.

Proof. See [1,p.65].

(e) If Γ is a coanalytic monotone operator with closure C, on the coanalytic subset P of X, then for any analytic subset A of X with $A \subset C$, there is some countable ordinal α such that $A \subset \Gamma^{\alpha}(P)$.

Proof. Let Γ be a monotone $\underset{\sim}{\Pi}^1_1$ operator over X and, let Δ be a monotone $\underset{\sim}{\Delta}^1_1$ operator over Y. Considering Lemma 2.2, we can assume that both Γ and Δ are inductive. Define an operator Λ over $\{0,1\}\times X\times Y$ by

$$(0,x,y)\in\Lambda(K) \quad \text{iff} \quad x\in\Gamma(\{x':(1,x',y)\in K\});$$
$$(1,x,y)\in\Lambda(K) \quad \text{iff} \quad y\notin\Delta(\{y':(0,x,y')\notin K\}).$$

Then Λ is a monotone $\underset{\sim}{\Pi}^1_1$ operator. We claim that R(x,y) iff $(0,x,y)\in Cl(\Lambda)$ and S(x,y) iff $(1,x,y)\in Cl(\Lambda)$. Consequently, by theorem 2.1(c) both R and S are coanalytic subsets of $X\times Y$.

Claim 1. For each $x\in X$ and $y\in Y$, $R(x,y) \Rightarrow (0,x,y)\in Cl(\Lambda)$ and $S(x,y) \Rightarrow (1,x,y)\in Cl(\Lambda)$.

Proof of claim. The proof is by induction. First assume that $|x|_{\Gamma}=0$. Then $x\in\Gamma(\emptyset)$. Suppose R(x,y). We have $x\in\Gamma(\{x':(1,x',y)\in\emptyset\})$. Thus, $(0,x,y)\in\Lambda(\emptyset)$, and consequently, $(0,x,y)\notin Cl(\Lambda)$. Therefore, $R(x,y) \Rightarrow (0,x,y)\in Cl(\Lambda)$. Now suppose S(x,y). By the above, $\{y':(0,x,y')\notin Cl(\Lambda)\} = \emptyset$. Since S(X<Y) and $|x|_{\Gamma}=0$, $|y|_{\Delta}>0$. Hence, $y\notin\Delta(\{y': (0,x,y')\notin Cl(\Lambda)\})$. Therefore, $(1,x,y)\in\Lambda(Cl(\Lambda))=Cl(\Lambda)$.

Hence, $S(x,y) \Rightarrow (1,x,y)\in Cl(\Lambda)$.

Now assume that the claim holds for all x' and y such that $|x'|_\Gamma<|x|_\Gamma$. Suppose $R(x,y)$. Now $x\in\Gamma(\{x':|x'|_\Gamma<|x|_\Gamma\})$. By assumption, $\{x':|x'|_\Gamma<|x|_\Gamma\}\subset\{x':(1,x',y)\in Cl(\Lambda)\}$. Therefore, $x\in\Gamma(\{x':(1,x',y)\in Cl(\Lambda)\})$. Hence, $(0,x,y)\in\Lambda(Cl(\Lambda))=Cl(\Lambda)$. Therefore, $R(x,y) \Rightarrow (0,x,y)\in Cl(\Lambda)$. Next, assume $S(x,y)$. Then $\{y':(0,x,y')\notin Cl(\Lambda)\}\subset\{y':\neg R(x,y')\}$ by the above. Thus, $y\notin\Delta(\{y': (0,x,y')\notin Cl(\Lambda)\})$. Consequently, $S(x,y) \Rightarrow (1,x,y)\in Cl(\Lambda)$. This completes the proof of claim 1.

Claim 2. For each $x\in X$ and $y\in Y$, $(0,x,y)\in Cl(\Lambda) \Rightarrow R(x,y)$ and $(1,x,y)\in Cl(\Lambda) \Rightarrow S(x,y)$.

Proof of claim. First assume $|(0,x,y)|_\Lambda=0$. Then $x\in\Gamma(\{x':(1,x',y)\in\emptyset\})$. Hence, $x\in\Gamma(\emptyset)$, and thus, $|x|_\Gamma=0$. Therefore $R(x,y)$. Thus, $(0,x,y)\in Cl(\Lambda) \Rightarrow R(x,y)$. Now suppose $|(1,x,y)|_\Lambda=0$. Then $y\notin\Delta(\{y': (0,x,y')\notin\emptyset\})$. Since $Y\subset\Delta(Y)$, this is a contradiction. Thus, $(1,x,y)\in Cl(\Lambda) \Rightarrow S(x,y)$.

Now assume that the claim holds for all (s,x',y'), $s\in\{0,1\}$, such that $|(s,x',y')|_\Lambda<\alpha$. Suppose $|(0,x,y)|_\Lambda=\alpha$. Set $K=\{(1,x',y):|(1,x',y)|_\Lambda<\alpha\}$. Now $(0,x,y)\in\Lambda(K)$. Hence, $x\in\Gamma(\{x':(1,x',y)\in K\})$. Thus, by assumption, $x\in\Gamma(\{x':S(x',y)\})$. Therefore, $R(x,y)$ by Lemma 2.9. Therefore, $(0,x,y)\in Cl(\Lambda) \Rightarrow R(x,y)$. Lastly, assume

$(1,x,y)|_\Lambda=\alpha$. Set $L=\{(0,x,y'):|(1,x,y')|_\Lambda<\alpha\}$. Now $(1,x,y)\in\Lambda(L)$. Thus, $y\notin\Delta(\{y': (0,x,y')\notin L\})$. Therefore, $y\notin\Delta(\{y':\neg R(x,y')\})$. Hence, by Lemma 2.9 $S(x,y)$. Thus, $(1,x,y)\in Cl(\Lambda) \Rightarrow S(x,y)$. This completes the proof of the claim.

Now assume that $A\subset X$ is analytic and $A\subset Cl(\Gamma)$. Let $\alpha=\sup\{|x|_\Gamma+1:x\in A\}$. Then

$$y\in\Delta^\alpha \text{ iff } (\exists x)(x\in A \text{ and } \neg S(x,y)).$$

Therefore, Δ^α is analytic. Since $W=Cl(\Delta)$ is not analytic and $|\Delta|=\omega_1$, α is countable. Since $A\subset\Gamma^\alpha$, this completes the proof when $P=\emptyset$.

To complete the proof, let P be an arbitrary coanalytic subset of X. Define Ψ by $\Psi(K)=\Gamma(P\cup K)$. Then Ψ is $\underset{\sim}{\Pi}^1_1$ monotone, and $\Psi^\alpha(\emptyset)=\Gamma^\alpha(P)$ for all α. Suppose A is analytic and $A\subset Cl(\Gamma;P)$. Then $A\subset Cl(\Psi)$. Hence, there is a countable α such that $A\subset\Psi^\alpha(\emptyset)$. Thus, $A\subset\Gamma^\alpha(P)$. □

(e) If the inductive operator Γ is either (1) Borel or (2) monotone and either analytic or coanalytic, then $|\Gamma|\leq\omega_1$.

Proof. First assume that Γ is $\underset{\sim}{\Delta}^1_1$. Let $K=\Gamma^{\omega_1}(\emptyset)$, and suppose $x\in\Gamma(K)$. By lemma 2.3, there are countable sets U and V where $U\subset K$ and $V\subset X-K$ such that for any set M with $U\subset M$ and $V\subset X-M$, $x\in\Gamma(K)$ iff $x\in\Gamma(M)$. Since U is countable, there is a countable ordinal α such that $U\subset\Gamma^\alpha(\emptyset)$. Then $U\subset\Gamma^\alpha(\emptyset)$

and $V\subset X-\Gamma^{\alpha}(\emptyset)$. Hence $x\in\Gamma^{\alpha+1}(\emptyset)$. Thus $|\Gamma|\leq\omega_1$.

Next, assume that Γ is $\underset{\sim}{\Sigma}^1_1$. Let $K=\Gamma^{\omega_1}(\emptyset)$, and suppose $x\in\Gamma(K)$. By lemma 2.3, there is a countable set $U\subset K$ such that $x\in\Gamma(U)$. Since U is countable, there is a countable ordinal α such that $U\subset\Gamma^{\alpha}(\emptyset)$. Hence, $x\in\Gamma(\Gamma^{\alpha}(\emptyset))=\Gamma^{\alpha+1}(\emptyset)$. Thus, $|\Gamma|\leq\omega_1$.

Lastly, assume that Γ is $\underset{\sim}{\Pi}^1_1$. Let $x\in Cl(\Gamma)$. Then $\{x\}$ is an analytic subset of $Cl(\emptyset)$. By theorem 2.1(e), there is some countable α such that $\{x\}\subset\Gamma^{\alpha}(\emptyset)$. Thus, $|\Gamma|\leq\omega_1$. □

This completes the proof of the Inductive Definability Theorem. Next, we give as a corollary to theorem 2.1(e) a boundedness principle for analytic derivations. Recall that by an analytic derivation we mean an operator whose dual operator is monotone and coanalytic. If D is an analytic derivation, the set $\bigcap_{\alpha<\omega_1} D^{\alpha}(A)$ is called the kernel of D on A.

Corollary 2.10. *Boundedness Principle for Analytic Derivations.* If D is an analytic derivation on the analytic set A with kernel K, then for any coanalytic subset C of X with $K\subset C$ there is some countable ordinal β such that $D^{\beta}(A)\subset C$. In particular, if D is an analytic derivation on X with $\bigcap_{\alpha<\omega_1} D^{\alpha}(X)=\emptyset$, then there exists a countable ordinal β

such that $D^{\beta}(X)=\emptyset$.

Proof. Suppose D is an analytic derivation on the analytic set A with kernel K, and assume that C is a coanalytic subset of X with $K\subset C$. Let Γ be the dual of D, i.e., $\Gamma(M)=X-D(X-M)$. Then Γ is $\underset{\sim}{\Pi}^1_1$ monotone. Also, for each α, $\Gamma^{\alpha}(X-A)=X-D^{\alpha}(A)$. Since $K\subset C$, $X-C\subset Cl(\Gamma;X-A)$. Therefore, by theorem 2.1(f) there is a countable ordinal α such that $X-C\subset\Gamma^{\alpha}(X-A)$. Consequently, $D^{\alpha}(A)\subset C$. □

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CHAPTER III

THE BOUNDEDNESS PRINCIPLE AND AN ARGUMENT OF SAINT RAYMOND

In this chapter we give an application of the boundedness principle for analytic derivations. Namely, we reprove a portion of a faithful separation theorem of J. Saint-Raymond [4]. By a faithful separation property we mean a property P such that if A and E are disjoint analytic subsets of the product X×Y of two Polish spaces X and Y and for each x, $A_x=\{y:(x,y)\in A\}$ has property P, then there is a Borel subset B of X×Y such that $A\subset B$, $B\cap E=\emptyset$ and for each x, B_x has property P. In 1939, Novikov [3] proved that compactness is faithful separation property. Some thirty-seven years later, Saint-Raymond proved that σ-compactness is a faithful separation theorem.

Theorem 3.1. (Saint-Raymond, 1976 [4]) Define C to be the collection of all Borel subsets of X×Y with compact sections. Let A, E $\in \mathcal{A}(X\times Y)$ and assume that $\forall x$ there is a σ-compact subset K_x of Y such that $A_x \subset K_x$ and $K_x\cap E_x= \emptyset$. Then there are Borel sets $B_n \in C$ such that $A \subset B = \bigcup B_n$ and $B \cap E = \emptyset$.

Corollary 3.2. If B is a Borel subset of X×Y whose sections are K_σ, then $B \in C_\sigma$.

Proof of theorem. In demonstrating this, Saint-Raymond uses a derivation operator which we define below. Let A and E be two disjoint analytic subsets of X×Y. Let φ be a continuous surjection of some Polish space P onto A.

For each subset Z of P define D(Z) to be the set of points z of Z such that for each neighborhood V of z,

$$\overline{\varphi(V\cap Z)\cap(\{x\}\times Y)} \cap E \neq \emptyset, \text{ where } x = \pi_X(\varphi(z)).$$

Saint-Raymond then gives the following recursion:

$Z^0 = P$, $Z^{\alpha+1} = D(Z^\alpha)$, and $Z^\lambda = \bigcap_{\alpha<\lambda} Z^\alpha$ if λ is a limit ordinal and then proves the following lemma and corollary.

Lemma 3.3. [4,p. 393] If B is a Borel subset of P which contains Z^α, $\alpha < \omega_1$, then there is $H \in C_\sigma$ containing $\varphi(P-B)$ and disjoint from E.

Corollary 3.4. [4,p. 394] If $\exists\ \alpha < \omega_1$ such that $Z^\alpha = \emptyset$, then there is $H \in C_\sigma$ such that $A \subset H$ and $H \cap E = \emptyset$.

Consequently to prove the above theorem, it suffices to show that for some $\alpha < \omega_1$, $Z^\alpha = \emptyset$ given that for each $x \in X$, the section A_x is contained in a K_σ disjoint from E. In order to prove this, Saint-Raymond gives an indirect argument by showing that if the Z^α are nonempty then there is a compact set K contained in a section of A∪E and such that no K_σ can contain K∩A without meeting K∩E. Below we give a different argument which involves the boundedness principle for monotone coanalytic operators and the Baire category theorem.

Claim 1. D is an analytic operator. Consequently, if Z is analytic, then Z^α is analytic for $\alpha < \omega_1$.

Proof. For each $m \in \mathbb{N}$, define the operator $\Lambda_m : 2^P \to 2^P$ as follows:

$$x \in \Lambda_m(Z) \quad \text{IFF}$$

$$x \in \pi_1 \left\{ (z,(z_n),y) \in Z \times Z^{\mathbb{N}} \times E \,\middle|\, \forall n [d(z,z_n) < 1/m \wedge \pi_1(\varphi(z_n)) = \pi_1(\varphi(z))] \wedge \varphi(z_n) \to y \right\},$$

where d is a metric for the topology on P.
We then have

$$z \in D(Z) \quad \text{IFF} \quad \forall m \; z \in \Lambda_m(Z).$$

Consequently, it suffices to show that each Λ_m is analytic. Let ψ be a continuous surjection of some Polish space Q onto E. Fix $m \in \mathbb{N}$. For each $k \in \mathbb{N}$, set

$$B_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, d(z_k, z) < 1/m \right\},$$

$$C_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, \pi_1(\varphi(z_k)) = \pi_1(\varphi(z)) \right\} \text{ and}$$

$$D_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, \rho(\varphi(z_k), \psi(w)) < 1/k \right\}.$$

For each k, B_k is open, C_k is closed and D_k is open. Next define for each k,

$$f_k\colon P\times P^{\mathbb{N}}\times Q\to P\times P^{\mathbb{N}}\times Q \quad \text{by}$$

$$f_k(z,(z_n),w) = (z_k,(z_n),w).$$

Note that for each k, f_k is continuous. Now define Δ: $2^P\to 2^P$ by

$$\Delta(K) = \bigcap_{k=1}^{\infty} (B_k\cap C_k\cap D_k\cap f_k^{-1}(K)).$$

Since for each k, B_k,C_k and D_k are Borel and since for each k, f_k is Borel measurable, it follows that Δ is a Borel operator. Finally,

$$z\in\Lambda_m(Z) \quad \text{IFF} \quad (\exists((z_n),w))(z,(z_n),w)\in\Delta(Z\times P^{\mathbb{N}}\times Q).$$

Therefore, Λ_m is a $\underset{\sim}{\Sigma}_1^1$ operator. Q.E.D.

Now let Γ be the dual operator of D, i.e., $\Gamma(B)$ = P-D(P-B). Note that $\forall\alpha < \omega_1$, $\Gamma^{\alpha}(\emptyset)$ = P-Z^{α}.

Claim 2. Γ is an inductive, monotone $\underset{\sim}{\Pi}_1^1$ operator.

Proof. Suppose $B \subset P$. Then $D(P-B) \subset P-B$. Thus,

$$B = P-(P-B) \subset P-D(P-B) = \Gamma(B).$$

Therefore, Γ is inductive.

To show Γ is monotone, suppose that $B \subset C$. Then $P-C \subset P-B$. Hence $D(P-C) \subset D(P-B)$. Thus, $\Gamma(B) = B-D(P-B) \subset C-D(P-C) = \Gamma(C)$.

Lastly, since D is $\underset{\sim}{\Sigma}_1^1$, Γ is $\underset{\sim}{\Pi}_1^1$. Q.E.D.

Next, we make use of the Baire category theorem.

Claim 3. If for each x∈X, A_x is contained in a K_σ disjoint from E_x, then for each nonempty $Z \subset P$, $D(Z) \subsetneq Z$.

Proof. Fix x∈X such that $\varphi(Z)_x \neq \emptyset$. There is a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that $A_x \subset \bigcup_{n=1}^{\infty} K_n$ and $(\bigcup K_n) \cap E_x = \emptyset$. Thus,

$$\varphi^{-1}(A_x) \subset \varphi^{-1}(\bigcup K_n) = \bigcup \varphi^{-1}(K_n)$$

Since $\varphi^{-1}(A_x) = \varphi^{-1}(\{x\}\times Y)$, $\varphi^{-1}(A_x)$ is a closed subset of P. Also note that for each n, $\varphi^{-1}(K_n)$ is closed. Now set

$$C = \overline{Z \cap \varphi^{-1}(A_x)} \ .$$

Since $\varphi(Z)_x \neq \emptyset$, $C \neq \emptyset$. Furthermore, C is Polish and $C \subset \bigcup \varphi^{-1}(K_n)$. Therefore by the Baire category theorem, there is $n \in N$ such that $\text{int}_C\, \varphi^{-1}(K_n) \neq \emptyset$. Consequently, there is an open subset V of P such that $C \cap V \neq \emptyset$ and $C \cap V \subset \varphi^{-1}(K_n)$. Choose $z \in Z \cap V \cap \varphi^{-1}(A_x)$. Since $\overline{\varphi(Z \cap V) \cap (\{x\}\times Y)} \subset K_n$, $z \notin D(Z)$. Thus $D(Z) \subsetneq Z$. Q.E.D.

Claim 4. If for each $x \in X$, A_x is contained in a K_σ disjoint from E_x, then there is $\alpha < \omega_1$ such that $\Gamma^\alpha(\emptyset) = P$.

Proof. Since Γ is an inductive, monotone, coanalytic operator, $|\Gamma| \leq \omega_1$. Thus $\Gamma(\bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset)) = \bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset)$. Consequently by the claim, $\bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset) = P$. By the boundedness principle, there is $\alpha<\omega_1$ such that $P \subset \Gamma^\alpha(\emptyset)$. Hence $\Gamma^\alpha(\emptyset) = P$. Q.E.D.

An immediate consequence of claim 4 is: If for each $x \in X$, A_x is contained in a K_σ disjoint from E_x, then there is $\alpha < \omega_1$ such that $Z^\alpha = \emptyset$. This completes the proof of Saint-Raymond's theorem.

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CHAPTER IV

A FAITHFUL SEPARATION THEOREM FOR ANALYTIC SETS WITH CONVEX SECTIONS

Once again, by a faithful separation property, we mean a property P such that if A and E are disjoint analytic subsets of the product $X \times Y$ of two Polish spaces X and Y and for each x, $A_x = \{y:(x,y) \in A\}$ has property P, then there is a Borel subset B of $X \times Y$ such that $A \subset B$, $B \cap E = \emptyset$ and for each x, B_x has property P. In this chapter we show that convexity is a faithful separation property in the case $Y = \mathbb{R}^k$. Our proof resembles the method of the proof of Theorem 2 in [7]. Let us mention that in [11], Sarbadhikari and Srivastava prove that convexity is a faithful separation property in the case $Y = \mathbb{R}$. However, their technique heavily uses the order structure of $\mathbb{R}$ and does not seem to generalize.

Given a collection Γ of sets, the monotone family generated by Γ is the smallest family of sets containing Γ and closed under countable monotone limits. Let $\mathcal{M}$ be the monotone family generated by the sets $B \in \mathcal{B}(X \times \mathbb{R}^k)$ such that for each x, B_x is compact and convex.

Theorem 4.1. Suppose A,E$\in$ $\mathcal{A}(X\times\mathbb{R}^k)$ and A$\cap$E=$\emptyset$. Furthermore, suppose that for each x$\in$X, A_x is convex. Then there exists a set B$\in\mathcal{M}$ such that A$\subset$B and B$\cap$E=$\emptyset$.

First, let us give a consequence of this theorem. Also, let us note that if X is a singleton, then this is the result of Preiss [7].

Corollary 4.2. $\mathcal{M} = \{B\in\mathcal{B}(X\times\mathbb{R}^k): \forall x\ B_x \text{ is convex}\}$

Proof. Let $\mathcal{C} = \{B\in\mathcal{B}(X\times\mathbb{R}^k): \forall x\ B_x \text{ is convex}\}$. Then $\mathcal{C}$ is a monotone class and $\mathcal{C}$ contains all Borel sets with compact, convex sections. Thus, $\mathcal{M} \subset \mathcal{C}$. To verify the opposite inclusion, suppose C$\in\mathcal{C}$. By the above theorem, there is B$\in\mathcal{M}$ such that C$\subset$B and $B\cap[(X\times\mathbb{R}^k)\setminus C]=\emptyset$. We must have B=C, and consequently, $\mathcal{M}=\mathcal{C}$. □

Before giving the proof we first verify a few preliminary lemmas.

Lemma 4.3. The family $\mathcal{M}$ is closed under finite intersections.

Proof. Let $\mathcal{R} = \{B\in\mathcal{B}(X\times\mathbb{R}^k): \forall x\ B_x \text{ is compact and convex}\}$. Let $\mathcal{S} = \{B\in\mathcal{M}: \forall\ A\in\mathcal{R}\ ,\ A\cap B\in\mathcal{M}\}$. Then $\mathcal{S}\subset\mathcal{M}$ and $\mathcal{R}\subset\mathcal{S}$. Furthermore, $\mathcal{S}$ is a monotone family. Hence $\mathcal{S}=\mathcal{M}$. Next, consider the collection $\mathcal{T} = \{B\in\mathcal{M}: \forall\ A\in\mathcal{M}\ ,\ A\cap B\in\mathcal{M}\}$. $\mathcal{T}$ is a monotone family containing $\mathcal{R}$. Thus, $\mathcal{M} = \mathcal{T}$. Therefore, if $B, C \in \mathcal{M}$ then $B \cap C \in \mathcal{M}$. By induction, $\mathcal{M}$ is closed under finite intersections. □

Lemma 4.4. $\mathcal{M}$ is closed under countable intersections.

Proof. Suppose $\{B_n\}_{n=1}^{\infty} \subset \mathcal{M}$. Then by lemma 1, for each m , $\bigcap_{n=1}^{m} B_n \in \mathcal{M}$. Since $\bigcap_{n=1}^{\infty} B_n = \bigcap_{m=1}^{\infty} \left[\bigcap_{n=1}^{m} B_n\right]$ and $\mathcal{M}$ is a monotone family, $\bigcap_{n=1}^{\infty} B_n \in \mathcal{M}$. □

Definition. An ordered pair A,E of subsets of $X \times \mathbb{R}^k$ is called separated if there exists a set $B \in \mathcal{M}$ such that $A \subset B$ and $E \cap B = \emptyset$.

Lemma 4.5. Let A and E be subsets of $X \times \mathbb{R}^k$ and let $A = \bigcup A_n$, $A_n \subset A_{n+1}$ and $E = \bigcup E_n$. If the pair A,E is not separated then there exist n and m such that the pair A_n, E_m is not separated.

Proof. Suppose not. Then for each $(n,m) \in \mathbb{N} \times \mathbb{N}$ there is $C_{nm} \in \mathcal{M}$ such that $A_n \subset C_{nm}$ and $E_m \cap C_{nm} = \emptyset$. Set $C_n = \bigcap_{i=n}^{\infty}\left[\bigcap_{m=1}^{\infty} C_{im}\right]$. Then each $C_n \in \mathcal{M}$. Also, $C = \bigcup_{n=1}^{\infty} C_n \in \mathcal{M}$. Since for each n, $A_n \subset A_{n+1}$, $A_n \subset C_n$. Hence $A \subset C$. Furthermore, for each pair (n,m), $E_m \cap C_n = \emptyset$. Hence $C \cap E = \emptyset$. Therefore, we have a contradiction. □

Definition. Let $D \subset X \times \mathbb{R}^k$. $\text{sconv}(D) \equiv \{(x,y) \in X \times \mathbb{R}^k$:

$y \in \text{conv } D_x\}$ where conv(S) denotes the convex hull of S.

Lemma 4.6. If $K \subset X \times \mathbb{R}^k$ is compact, then sconv(K) is compact.

Proof. Since K is compact, $\pi_2(K)$ is compact. Hence, there is $n \in \mathbb{N}$ such that $\pi_2(K) \subset B_n$ where $B_n = \{y \in \mathbb{R}^k : \|y\| \le n\}$. We assert that sconv(K) is contained in the compact set $\pi_1(K) \times B_n$. To this end, suppose $(x,y) \in$ sconv(K), i.e., $y \in \text{conv}(K_x)$. Then $y \in \pi_1(K)$. In addition, by Carthéodory's Theorem [8,p.155], $y = \sum_{i=1}^{k+1} s_i y_i$ where for each $1 \le i \le k+1$, $s_i \in [0,1]$ and $y_i \in K_x$, and $\sum_{i=1}^{k+1} s_i = 1$. For each i, $y_i \in B_n$. Since B_n is convex, $y \in B_n$. Therefore, sconv(K) $\subset \pi_1(K) \times B_n$. Consequently, it suffices to show that sconv(K) is closed.

Suppose $\{(x_n,y_n)\}_{n=1}^{\infty} \subset$ sconv(K) and $\lim_{n\to\infty} (x_n,y_n) = (x,y)$. For each n, $y_n \in \text{conv}(K_{x_n})$. Thus, we can write $y_n = \sum_{i=1}^{k+1} s_n^i y_n^i$ where for each $1 \le i \le k+1$, $s_n^i \in [0,1]$ and $y_n^i \in K_{t_n}$, and $\sum_{i=1}^{k+1} s_i = 1$. For $1 \le i \le k+1$, consider the sequences $\{y_m^i\}_{m=1}^{\infty}$ and $\{s_m^i\}_{m=1}^{\infty}$. Using the compactness of B_n and of [0,1], we may assume without loss of generality by taking subsequences that each of these sequences converges. For $1 \le i \le k+1$, let $\lim_{m\to\infty} y_m^i = y^i$ and $\lim_{m\to\infty} s_m^i = s^i$. We have for each $1 \le i \le k+1$, $\lim_{j\to\infty} (x_m, y_m^i) = (x, y^i)$. For each $1 \le i \le k+1$, since for each m,

$(x_m, y_m^i) \in K$, $(x, y^i) \in K$. Since $\lim_{n\to\infty} y_n = y$, we have $y = \sum_{i=1}^{k+1} s^i y^i$. Also, since for each $1 \leq i \leq k+1$, $\lim_{j\to\infty} s_m^i = s^i$, $\sum_{i=1}^{k+1} s^i = 1$. Therefore, $y \in \operatorname{conv}(K_x)$, i.e., $(x,y) \in \operatorname{sconv}(K)$. This completes the proof. □

Proof of theorem 4.1. Assume that the pair A,E cannot be separated. Let $f: \mathbb{N}^{\mathbb{N}} \twoheadrightarrow A$ and $g: \mathbb{N}^{\mathbb{N}} \twoheadrightarrow E$ be continuous maps of the space $\mathbb{N}^{\mathbb{N}}$ (the space of sequences of positive integers with the product topology) onto A and E respectively. Given a finite sequence of integers $n_1, \ldots, n_i$ let

$$\langle n_1, \ldots, n_i \rangle = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma(j) \leq n_j,\ 1 \leq j \leq i\}$$

$$(n_1, \ldots, n_i) = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma(j) = n_j,\ 1 \leq j \leq i\}.$$

Also for each finite sequence $n_1, \ldots, n_i$ let

$$A(n_1, \ldots, n_i) = f(\langle n_1, \ldots, n_i \rangle)$$

$$E(n_1, \ldots, n_i) = g((n_1, \ldots, n_i)).$$

Now $A = \bigcup_n A(n)$, and for each n, $A(n) \subset A(n+1)$. Also, $E = \bigcup_m E(m)$. Therefore, by Lemma 4.5, there are positive integers n_1 and m_1 such that $A(n_1)$ and $E(m_1)$ cannot be separated. $A(n_1) = \bigcup_n A(n_1, n)$,and for each n, $A(n_1, n) \subset A(n_1, n+1)$. Also, $E(m_1) = \bigcup_m E(m_1, m)$. Therefore, by Lemma 4, there are positive integers n_2 and m_2 such that $A(n_1, n_2)$ and $E(m_1, m_2)$ cannot be separated. Continuing, repeated use of Lemma 4.5 gives us two sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of positive integers such that for each i,

$A(n_1,\ldots,n_i)$ and $E(n_1,\ldots,n_i)$ cannot be separated.

Set

$$\tilde{A} = \bigcap_i A(n_1,\ldots,n_i) \text{ and}$$
$$\tilde{E} = \bigcap_i E(n_1,\ldots,n_i).$$

Then $\tilde{A}$ is compact and $\tilde{E}$ is a singleton. Let $\tilde{E} = \{(x_0,y_0)\}$, and $A^* = \mathrm{sconv}(\tilde{A})$. Then by Lemma 4.6, A^* is compact. Also, since the sections of A are convex, $A^* \subset A$. Consequently, $A^* \cap \tilde{E} = \emptyset$. For $\epsilon > 0$, let

$$U_\epsilon = \{(x,y) \in X \times \mathbb{R}^k : d((x,y),A^*) \leq \epsilon\},$$
$$V_\epsilon = \{(x,y) \in X \times \mathbb{R}^k : d((x,y),\tilde{E}) \leq \epsilon\} \text{ and}$$
$$W_\epsilon = \mathrm{sconv}(U_\epsilon).$$

Claim 1. For each $\epsilon > 0$, $W_\epsilon \in B\,(X \times \mathbb{R}^k)$ and each section of W_ϵ is compact and convex.

Proof of claim. Fix $x \in X$. Consider $(U_\epsilon)_x = \{y: d((x,y),A^*) \leq \epsilon\}$. This set is closed and bounded and therefore compact. Hence $(W_\epsilon)_x$ is compact and convex. It remains to show that W_ϵ is Borel. Now

$$W_\epsilon = \pi_{12}\{(x,y,y^1,\ldots,y^{k+1},s^1,\ldots,s^{k+1}) \in X \times (\mathbb{R}^k)^{k+2} \times [0,1]^{k+1} : \sum_{j=1}^{k+1} s^j y^j = y,\ \sum_{j=1}^{k+1} s^j = 1 \text{ and for each } j,\ (x,y^j) \in U_\epsilon\}.$$

Hence, W_ϵ is the projection of a Borel set whose sections are σ-compact (See [10]). Therefore, W_ϵ is Borel. □

Claim 2. There is $n \in \mathbb{N}$ such that $W_{1/n} \cap V_{1/n} = \emptyset$.

Proof of claim. Assume not. Then for each n, let $(x_n,y_n) \in W_{1/n} \cap V_{1/n}$. For each n, since $(x_n,y_n) \in W_{1/n}$, $y_n \in$

conv $((U_{1/n})_{x_n})$. Thus, for each n, $y_n=\sum_{j=1}^{k+1} s_n^j y_n^j$, where for each j, $s_n^j\in[0,1]$ and $y_n^j\in(U_{1/n})_{x_n}$, and $\sum_{j=1}^{k+1} s^j y^j = 1$. Now for each n and j, $1\le j\le k+1$, there is $(u_n^j,z_n^j)\in A^*$ such that $d((x_n,y_n^j),(u_n^j,z_n^j))<1/n$. By the compactness of A^* and [0,1] let us assume without loss of generality that for each j, the sequences $\{(u_n^j,z_n^j)\}_{n=1}^{\infty}$ and $\{s_n^j\}_{n=1}^{\infty}$ converge. For each j, let $\lim_{n\to\infty} (u_n^j,z_n^j) = (u^j,z^j)$ and $\lim_{n\to\infty} s_n^j = s^j$. Note that $\sum_{j=1}^{k+1} s^j = 1$. Then $\lim_{n\to\infty} (x_n,y_n^j) = (u^j,z^j)$. However, $\lim_{n\to\infty} (x_n,y_n)=(x_0,y_0)$. Thus, $u^j=x_0$ for $1\le j\le k+1$. $(x_0,y_0)\in A^*$, which is a contradiction. □

Choose $\delta>0$ such that $W_\delta \cap V_\delta = \emptyset$. By compactness there is $i\in\mathbb{N}$ such that $A(n_1,\ldots,n_i) \subset U_\delta$ and $E(m_1,\ldots,m_i)\subset V_\delta$. Thus, W_δ separates $A(n_1,\ldots,n_i)$, $E(m_1,\ldots,m_i)$. This is a contradiction. This completes the proof of the theorem. □

P. Holicky has shown in [3] that every infinite dimensional locally convex space, X, contains a convex Borel set which is not in the monotone family generated by the compact convex subsets of X. Therefore, Theorem 4.1 does not hold if $\mathbb{R}^k$ is replaced by any infinite dimensional locally convex space. However, we do pose the following question.

Question. Is convexity a faithful separation property in the case where Y is an infinite dimensional locally convex space?

Finally, we would like to add that while typing this dissertation it was discovered that the main result of this chapter, namely Theorem 4.1, has already been proven by J. Saint-Pierre [9].

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CHAPTER V

EXTREME POINT SELECTIONS

In [4], Jayne and Rogers studied upper semi-continuous multifunctions from a metric space to a Banach space with its weak topology. Among other things, the authors prove that an upper semi-continuous map from a metric space to the unit ball of a Banach space with compact values has a Borel class 1 selector. In this chapter we deal with upper semi-continuous multifunctions with values in a dual Banach space with the weak* topology.

Theorem 5.1. Let T be a metric space, X a separable normed linear space. Let $B=\{x^*\in X^*:\|x^*\|\leq 1\}$ where X^* denotes the dual of X and give B the relative weak* topology. Suppose $F:T\to B$ is an upper semi-continuous multifunction with compact values. Then F has a Borel class 1 selector f with respect to the weak* topology on X^* such that for each t, $f(t)\in \text{ext}(\text{cl conv } F(t))$.

Recall that X is reflexive if and only if the weak and weak* topologies on X^* coincide. Thus, by our main result, an upper semi-continuous map of a metric space to the unit ball of a separable, reflexive Banach space with compact values has a Borel class 1 selector. This is the Jayne-Rogers theorem. In addition, our selection picks extreme points. Let us mention that the selection given by Jayne and Rogers does not necessarily involve extreme points.

We apply our theorem to improve the Borel class of a selection lemma of L. Baggett [1].

A multifunction $F:T\to Y$ is a function whose domain is T and whose values are nonempty subsets of Y. If $E\subset Y$, $F^{-1}(E) \equiv \{t\in T:F(t)\cap E\neq\emptyset\}$. F is said to be lower semi-continuous (lsc) if $F^{-1}(V)$ is open for all open V, and F is said to be upper semi-continuous (usc) if $F^{-1}(K)$ is closed for all closed K. The graph of a multifunction, denoted by Gr(F), is the set $\{(t,x)\in T\times X:x\in F(t)\}$. A function $f:T\to X$ is a selector for the multifunction $F:T\to X$ if for all t, $f(t)\in F(t)$. Also, a function $f:T\to X$ is said to be of Borel class 1 if $f^{-1}(V)$ is an F_σ for all open V.
By ext(K), conv(K) and cl conv(K), we mean the set of extreme points of K, the convex hull of K and the closed convex hull of K respectively.

Let Y be a topological space, and let $\mathcal{F}(Y)$ and $\mathcal{K}(Y)$ denote the collection of all nonempty closed subsets of Y and all nonempty compact subsets of Y respectively. For each open $U \subset Y$, define

$$C(U) = \{F \in \mathcal{F}(Y) : F \subset U\}, \text{ and}$$
$$I(U) = \{F \in \mathcal{F}(Y) : F \cap U \neq \emptyset\}.$$

The collection $\{C(U) : U \text{ open in } Y\} \cup \{I(U) : U \text{ open in } Y\}$ forms a subbase for a topology on $\mathcal{F}(Y)$. This topology is called the exponential or Vietoris topology on $\mathcal{F}(Y)$. For a detailed discussion of this topology, see [6]. Now suppose the topology on Y is given by a bounded metric d. We can then define a metric ρ_H (called the Hausdorff metric) on $\mathcal{F}(Y)$ as follows:

$$\rho_H(A,B) = \max \{D_A(B), D_B(A)\}$$

where $D_A(B)=\sup\{d(A,b):b \in B\}$ and $d(A,b)=\inf\{d(a,b):a \in A\}$. An important fact is that the Hausdorff metric topology on $\mathcal{K}(Y)$ coincides with the relative exponential topology on $\mathcal{K}(Y)$ [7,p.47].

Proof of Theorem 5.1

Assume the hypotheses of the theorem are satisfied. Define the multifunction $H:T \to B$ by $H(t)=\text{ext cl conv}(F(t))$. We note that if $K \subset B$ is compact then ext cl conv$(K) \subset K$ [5,p.132]. Thus, it suffices to show that H has a Borel class 1 selector. The proof of this will follow from a theorem of G. Debs.

Theorem 5.2 [2]. Let T be a metric space, Y be a Polish space, and $G:T\to Y$ be a multifunction. Suppose Gr(G) is a $\mathcal{G}_\delta$, $\alpha>0$ is an ordinal, and $G^{-1}(U)$ is of additive class α for open $U\subset Y$. Then G has a selection which is of additive class α. In particular, if $G^{-1}(U)$ is an $\mathcal{F}_\sigma$ for each open U, then G has a class 1 selector.

We check that the multifunction H satisfies the conditions of Theorem 5.2 via a sequence of lemmas.

Let $\mathcal{C}(B)$ denote the collection of all nonempty, convex, weak*-compact subsets of B. We give $\mathcal{C}(B)$ the relative Vietoris topology. Since B is a compact metric space, this topology agrees with the Hausdorff metric topology.

Lemma 5.3. Let E: $\mathcal{C}(B)\to B$ be the multifunction given by E(K)=ext(K). Then E is lsc.

Proof. It suffices to show that $E^{-1}(N)$ is open for each basic open set N. Let

$$N = \{y\in B: |x(x_i)-y(x_i)|<\epsilon,\ 1\leq i\leq m\}$$

be a basic open set in B. Let C be the complement of $E^{-1}(N)$. It suffices to show that C is closed. Let us define $G:[\mathcal{C}(B)]^{2m}\to\mathcal{C}(B)$ by

$$G(A_1,A_2,\dots,A_{2m}) = \text{conv}\ (A_1\cup A_2\cup\dots\cup A_{2m}).$$

Then G is continuous. We claim that $K\in C$ if and only if

$K\in G([\mathcal{C}(B)-I(N)]^{2m})$. We prove this as follows:
Suppose $K\in C$. Then set for $1\leq i\leq m$,

$$K_i^+ = \{k\in K \mid k(x_i)\geq \epsilon+y(x_i)\} \text{ and}$$
$$K_i^- = \{k\in K \mid k(x_i)\leq -\epsilon+y(x_i)\}.$$

Let $L_1,\ldots,L_p$ be a listing of the K_i^+ and K_i^- which are nonempty. Then $L_j\in\mathcal{C}(B)-I(N)$ for $1\leq j\leq p$. Also, $\mathrm{ext}(K)\subset\bigcup L_j$. By the Krein-Milman theorem [8,p. 242], $K=$ cl conv(ext K). Consequently, $K = \mathrm{conv}(L_1\cup\ldots\cup L_p)$. Therefore, $K\in G([\mathcal{C}(B)-I(N)]^{2m})$.

Conversely, suppose $K\in G([\mathcal{C}(B)-I(N)]^{2m})$. Then $K = \mathrm{conv}(A_1,\ldots,A_{2m})$ where for each $1\leq j\leq 2m$, $A_j\in\mathcal{C}(B)-I(N)$. Hence, if $k\in$ ext K, $k\in A_q$ for some q. Thus, $K\in C$. Therefore, the claim holds.

Since $[\mathcal{C}(B)-I(N)]^{2m}$ is compact, C is compact. □

Lemma 5.4. $\mathcal{C}(B)$ is a compact subset of $\mathcal{K}(B)$.

Proof. Suppose $L_n\to L$ in the Hausdorff metric where for every n, $L_n\in\mathcal{C}(B)$. To show that $L\in\mathcal{C}(B)$, it suffices to show that L is convex. To this end, let $h,k\in L$ and let $t\in[0,1]$. Since $L_n\to L$, there are sequences $\{h_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ such that for every n, h_n and $k_n\in L_n$, and $h_n\to h$ and $k_n\to k$. Since each L_n is convex, $s_n=th_n+(1-t)k_n\in L_n$ for all n. We have that $s_n\to th+(1-t)k$. Since $L_n\to L$, we must have $\{s_n\}_{n=1}^{\infty}$ converging to a point in L. Thus, $th+(1-t)k\in L$. Therefore, L is convex. □

Lemma 5.5. The map $J:\mathcal{K}(B)\to\mathcal{C}(B)$ given by $J(K)=\text{cl conv}(K)$ is continuous.

Proof. Suppose $\{K_n\}_{n=1}^{\infty} \subset \mathcal{K}(B)$ and $K_n\to K$. We show that $J(K_n)\to J(K)$. Since $\mathcal{C}(B)$ is compact, there is a subsequence $\{K_{n_p}\}_{p=1}^{\infty}$ of $\{K_n\}_{n=1}^{\infty}$ and $L\in\mathcal{C}(B)$ such that $J(K_{n_p})\to L$. We assert that $L = \text{cl conv}(K)$. Suppose $x \in \text{ext cl conv}(K)$. Then $x\in K$. Since $K_{n_p}\to K$, there is a sequence $\{k_{n_p}\}_{p=1}^{\infty}$ with $k_{n_p}\in K_{n_p}$ for each p such that $k_{n_p}\to x$. For each p, $k_{n_p}\in J(K_{n_p})$. Thus, $x\in L$. Consequently, $\text{ext cl conv}(K) \subset L$. Hence, $\text{cl conv}(K) \subset L$. Next, suppose $y\in L-J(K)$. There is a continuous linear functional f on B such that $f(y)<\inf_{k\in J(K)} f(k)$ [8,p.241]. Choose $a\in\mathbb{R}$ such that $f(y)<a<\inf_{k\in J(K)} f(k)$. $f^{-1}([a,\infty))$ is closed and convex. Furthermore, $K\subset f^{-1}((a,\infty))$. Thus, there is P such that if $P\leq p$ then $K_{n_p}\subset f^{-1}((a,\infty))\subset f^{-1}([a,\infty))$. Therefore, for $P\leq p$, $J(K_{n_p})\subset f^{-1}([a,\infty))$. Since $J(K_{n_p})\to L$, $L\subset f^{-1}((a,\infty))$. This contradicts the fact that $y\in L$. Hence, $L\subset\text{cl conv}(K)$. Consequently, $L=\text{cl conv}(K)$. Now assume $\{J(K_n)\}_{n=1}^{\infty}$ does not converge to $J(K)$. Then there is some $\epsilon>0$ and some subsequence $\{K_{n_q}\}_{q=1}^{\infty}$ of $\{K_n\}_{n=1}^{\infty}$ such that for each q, $\rho_H(J(K_{n_q}),J(K)) > \epsilon$. However, $\{J(K_{n_q})\}_{q=1}^{\infty}$ has a convergent subsequence. By the above argument, this subsequence must converge to $J(K)=\text{cl conv}(K)$. This is a contradiction.

Therefore $J(K_n) \to J(K)$. □

Lemma 5.6. Gr(H) is a $\mathcal{G}_\delta$.

Proof. For each $n \in \mathbb{N}$, set

$$A_n = \left\{(t,x) \mid \exists\ s \in [1/n, 1-1/n] \text{ and } \exists u,v \in J(F(t))\ [d(u,v) \geq 1/n \text{ and } x = su+(1-s)v]\right\}$$

where d is a metric for the topology on B and J is the map $J(K)=\text{cl conv}(K)$. (We remind the reader that since X is separable, B is a compact metric space). We assert that each A_n is closed. Fix n. Suppose $(t_m,x_m) \in A_n$ and $(t_m,x_m) \to (t,x)$. For each m, $x_m = s_m u_m + (1-s_m)v_m$ where $s_m \in [1/n, 1-1/n]$, $u_m, v_m \in J(F(t_m))$ and $d(u_m,v_m) \geq 1/n$. By the compactness of $[1/n,1-1/n]$ and of B, the sequences $\{s_m\}_{m=1}^{\infty}$, $\{u_m\}_{m=1}^{\infty}$ and $\{v_m\}_{m=1}^{\infty}$ all have convergent subsequences. Without loss of generality, assume $s_m \to s$, $u_m \to u$ and $v_m \to v$. Then clearly, $s \in [1/n, 1-1/n]$. Also, since J is continuous (lemma 4), $J \circ F$ is usc. Thus, $Gr(J \circ F)$ is closed [6,p.175]. Hence, $u,v \in J(F(t))$. Furthermore, $d(u,v) \geq 1/n$ and $x=su+(1-s)v$. Therefore, $(t,x) \in A_n$. Consequently, A_n is closed. Now $Gr(H) = Gr(J \circ F) \setminus \bigcup_n A_n$, and since $Gr(J \circ F)$ is closed, $Gr(J \circ F)$ is a $\mathcal{F}_\delta$. Thus, Gr(H) is a $\mathcal{G}_\delta$. □

Lemma 5.7. For each weak* open set U in B, $H^{-1}(U)$ is an $\mathcal{F}_\sigma$.

Proof. Let U be a weak* open set in B. By lemma 3,

$E^{-1}(U)$ is open in $\mathcal{C}(B)$. We can write $E^{-1}(U) = \bigcup_n \mathcal{U}_n$ where each $\mathcal{U}_n$ is a basic open set. We have

$$\begin{aligned} H^{-1}(U) &= \{t \mid H(t) \cap U \neq \emptyset\} \\ &= \{t \mid \text{ext } J(F(t)) \cap U \neq \emptyset\} \\ &= \{t \mid J(F(t)) \in E^{-1}(U)\} \\ &= \{t \mid J(F(t)) \in \bigcup_n \mathcal{U}_n\} \\ &= \bigcup_n \{t \mid J(F(t)) \in \mathcal{U}_n\}. \end{aligned}$$

Next consider a basic open set $C(U_0) \cap I(U_1) \cap \ldots \cap I(U_k)$ in $\mathcal{C}(B)$. Let

$$A = \{t \mid J(F(t)) \in C(U_0) \cap I(U_1) \cap \ldots \cap I(U_k)\}.$$

Then

$$A = \{t \mid J(F(t)) \subset U_0\} \cap \{t \mid J(F(t)) \cap U_1 \neq \emptyset\} \cap \ldots \cap \{t \mid J(F(t)) \cap U_k \neq \emptyset\}.$$

Since $J \circ F$ is usc. Thus, the set $\{t \mid J(F(t)) \subset U_0\}$ is open in T (and hence an F_σ). Also, the sets $\{t \mid J(F(t)) \cap U_i \neq \emptyset\}$, $1 \leq i \leq k$, are F_σ's. Therefore, the set A is an F_σ. Consequently, $H^{-1}(U)$ is an F_σ. □

This completes the proof of the theorem.

One corollary of the above theorem is a selection lemma due to L. Baggett [1]. Baggett uses this lemma to prove a selection theorem which he asserts that "together with its immediate consequences, should suffice for most needs within functional analysis" [1,p.2].

Corollary 5.7. [1] Let X be a separable normed linear space, let Y be a closed subspace of X and let $\mathbb{R}$ denote the restriction map of X^* onto Y^*. Let K be a compact subset of (X^*,w^*), and let $L=\mathbb{R}(K)$. Then there exists a Borel map (in fact a Borel class 1 map) $s:L\to K$ such that

(1) $\mathbb{R}(s(y)) = y$ for all $y\in L$.

(2) $s(y)$ is an extreme point of $\mathbb{R}^{-1}(y)$.

(3) If $y\in ext(L)$, then $s(y)\in ext(K)$.

Proof. Consider the multifunction F: $L\to K$ given by $F(y)=\mathbb{R}^{-1}(y)$. Then F is usc with compact, convex values. By theorem 5.1, there is a Borel class 1 selector, $s:L\to K$ for $H(y)=ext(F(y))$. Clearly, $\mathbb{R}(s(y))=y$, and $s(y)$ is an extreme point of $\mathbb{R}^{-1}(y)$. Now suppose that $y\in ext(L)$. Assume that $s(y) = tk+(1-t)h$ where $k,h\in K$ and $0<t<1$. Consider $\mathbb{R}(s(y))$. $\mathbb{R}(s(y))=y=t\mathbb{R}(k)+(1-t)\mathbb{R}(h)$. Since $y\in ext(L)$, we have $h,k\in\mathbb{R}^{-1}(y)$. Therefore, since $s(y)\in ext\ \mathbb{R}^{-1}(y)$, we must have $k=h$. Thus, $s(y)\in ext(K)$. □

Corollary 5.8. The multifunction $E:\mathcal{C}(B)\to B$ given by $E(K)=ext(K)$ has a Borel class 1 selector.

Proof. Define $F:\mathcal{C}(B)\to B$ by $F(K)=K$. Then F is usc, and $E(K)=ext(cl\ conv F(K))$. Therefore, by our main theorem, E has a class 1 selector. □

Let $\mathcal{C}(X^*)$ denote the collection of all nonempty, convex, weak*-compact subsets of X^*. We give $\mathcal{C}(X^*)$ the relative Vietoris topology. Let us mention that if X is infinite dimensional then X^* is not metrizable [3,p.10]. Consequently, $\mathcal{C}(X^*)$ would not be metrizable. However, X^* is a Lusin space, i.e., X^* is a continuous one-to-one image of N^N. Hence, $\mathcal{C}(X^*)$ is an analytic space.

Corollary 5.9. The multifunction $E:\mathcal{C}(X^*)\to X^*$ given by $E(K)=\text{ext}(K)$ has a Borel class 1 selector. Moreover, E has a continuous selector if and only if $X=\mathbb{R}$

Proof. For each $n\in\mathbb{N}$, set $B_n=n\cdot B$ where $B=\{x\in X^*:\|x\|\leq 1\}$. Let $\mathcal{C}(B_n)$ denote the collection of all nonempty, convex, compact subsets of B_n. We give $\mathcal{C}(B_n)$ the relative exponential topology. Define for each n, $E_n:\mathcal{C}(B_n)\to B_n$ by $E_n(K)=\text{ext } K$. It follows from corollary 5.8 that for each n there is a Borel class 1 selector f_n for E_n. Define $f:\mathcal{C}(X^*)\to X^*$ as follows:

$$f(K) = f_n(K) \text{ where } n=\text{least}\{m:K\subset B_m\}.$$

Then $\forall K\in\mathcal{C}(X^*)$, $f(K)\in E(K)$. Furthermore, f is of Borel class 1, since

$$f^{-1}(A) = f_1^{-1}(A) \cup \Big[\bigcup_{i\geq 2} f_i^{-1}(A)\cap(\mathcal{C}(B_{i-1}))^c\Big]$$

for any $A\subset X^*$ and since each f_n is of Borel class 1. Hence, f is a Borel class 1 selector for E.

For the second assertion, suppose $X=\mathbb{R}$. Define $f:\mathcal{C}(\mathbb{R})\to\mathbb{R}$

by $f([a,b]) = b$. Then f is continuous. For the converse, consider $\mathbb{R}^2$ and suppose f is a continuous selector for $E:\mathcal{C}(\mathbb{R}^2) \to \mathbb{R}^2$ where $E(K) = \mathrm{ext}(K)$. For $0 \leq \theta \leq \pi$, let I_θ denote the closed interval with endpoints $(\cos\theta, \sin\theta)$ and $(-\cos\theta, -\sin\theta)$. Without loss of generality, assume $f(I_0) = (1,0)$. Since $\mathcal{C}(B)$ is compact, $f|_{\mathcal{C}(B)}$ is uniformly continuous. Therefore, $\exists\ \delta>0$ such that if $\rho_H(I_\theta, I_\alpha)<\delta$ then $d(f(I_\theta), f(I_\alpha))<1$. Now choose $k>2$, such that $\rho_H(I_0, I_{\pi/k})<\delta$. Then we have $d(f(I_0), f(I_{\pi/k}))<1$. Consequently, $f(I_{\pi/k}) = (\cos \pi/k, \sin \pi/k)$. Since $\rho_H(I_0, I_{\pi/k})<\delta$, $\rho_H(I_{\pi/k}, I_{2\pi/k})<\delta$. Hence, $d(f(I_{\pi/k}), f(I_{2\pi/k}))<1$. Therefore, $f(I_{2\pi/k}) = (\cos 2\pi/k, \sin 2\pi/k)$. Continuing, we get that $f(I_{m\pi/k}) = (\cos m\pi/k, \sin m\pi/k)$ for $1 \leq m \leq k$. Thus, in particular, $f(I_\pi) = (-1,0)$. However, $I_0 = I_\pi$ and $f(I_0) = (0,1)$. Therefore, we have a contradiction. Hence, f cannot be continuous. □

Remarks. An alternative proof to Baggett's lemma is given by considering the map $s(y)=f(R^{-1}(y))$ where f is the selector above. Also, the above shows that Debs's theorem does not hold in the case $\alpha=0$.

Next, we give an example of a class 1 selector for $E:\mathcal{C}([0,1])\to[0,1]^2$ given by $E(K)=\mathrm{ext}(K)$.

Example. Define $f:\mathcal{C}([0,1]^2) \to [0,1]^2$ by letting $f(K)$

be the smallest element of (K,<') where (a,b)<'(c,d) if and only if a<c, or a=c and b<d. Note that f is a selector for E(K) = ext(K). We assert that f is of Borel class 1. Consider the basic open set U = (a,b)×(c,d) where a,b,c, and d are rational. It suffices to show that $f^{-1}(U)$ is an F_σ. For r∈ℚ∩[0,1], define $V_r = [0,r)\times[0,1]$ and $H_r = [0,1]\times[0,r)$. We have that $K \in f^{-1}(U)$ if and only if $K\in I(U) \cap C((a,1]\times[0,1]) \cap A \cap B$, where

$$A = \bigcap_{r\in\mathbb{Q}}\left[\ I(\overline{V}_r\cap\overline{U}) \cup I(((a,b)\times([0,c)\cup(d,1])) \cap V_r)^c\ \right] \text{ and}$$

$$B = \bigcup_{\substack{r,q\in\mathbb{Q}\\ q\geq c}} \left[\ I(\overline{H}_q\cap\overline{V}_r\cap([a,b]\times[0,c]))^c \cap I(U\cap\overline{H}_q^c\cap V_r)\ \right].$$

Since I(U), C((a,1]×[0,1]), and B are all open, and A is closed, $f^{-1}(U)$ is an F_σ. Therefore, f is of class 1.

Another application of our theorem is

Corollary 5.10. There is a Borel class 1 selector for $F:\mathcal{K}(X^*)\to X^*$ given by F(K) = ext cl conv (K) .

Proof. Define $G:\mathcal{K}(B)\to B$ by G(K) = ext cl conv (K). It suffices to show that G has a class 1 Borel selector. Define the multifunction $F:\mathcal{K}(B)\to B$ by F(K)=K. Then F is usc. Therefore, by our main theorem G has a class 1 selector. □

To conclude, we wish to communicate that while typing this dissertation it was pointed out by a referee that

Theorem 5.1 is essentially an application of a theorem of G. Debs which is stated in [9] and proven in his thesis at the University of Paris VI.

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