EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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We study the existence, multiplicity, bifurcation and the stability of the solutions to semilinear elliptic boundary value problems. These problems are motivated both by the mathematical structure and the numerous applications in fluid mechanics, chemical reactions, nuclear reactors, Riemannian geometry and elasticity theory. Existence and multiplicity of positive solutions is important because in many applications the solution represents quantities such as concentration or density and hence has to be nonnegative. We consider the problem for different classes of nonlinearities and obtain the existence and multiplicity of positive solutions. For a certain family of problems, we obtain a complete structure for the branches of radial solutions.

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CHAPTER 1

INTRODUCTION TO THE STUDY

This paper is mainly devoted to the study of solutions to semilinear elliptic boundary value problems of the form:

$$-\Delta u(x) = h(x, u, \lambda) \quad \text{for} \quad x \in \Omega, \tag{1.1}$$

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{1.2}$$

where Ω is an open bounded region in $\mathbb{R}^N (N \geq 1)$ with smooth boundary and $\lambda \in \mathbb{R}$. Main questions in the study of these problems are the existence, multiplicity, bifurcation and the stability of the solutions. These questions are motivated both by the mathematical structure and the numerous applications in fluid mechanics, chemical reactions, nuclear reactors, Riemannian geometry and elasticity theory (see [6], [9–12], [15]). Methods involving spectral theory, monotone iterations, fixed point theorems, variational methods, apriori estimates and the degree theory arguments have proven useful in the study (see [1], [2], [7], [8], [16–19]). Existence and multiplicity of positive solutions are important because in many applications u represents quantities such as concentration or density and thus has to be nonnegative.

A considerable amount of work has been done regarding the positive solutions of (1.1)–(1.2) when $h(x,\ u,\ \lambda) = \lambda f(u)$. That is for the problem:

$$-\Delta u(x) = \lambda f(u) \quad \text{for} \quad x \in \Omega, \tag{1.3}$$

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{1.4}$$

The existence of a solution depends significantly on the assumptions made on f. The reader is refered to [17] for a survey of the results when $f(0) = 0$ and $f(0) > 0$. In the recent past, extensive efforts have been concentrated on the study of (1.3)–(1.4) in the presence of symmetries. In particular, the case when Ω and f are invariant under rotations has attracted a great deal of attention. If Ω is a ball in $\mathbb{R}^N$ the positive solutions to (1.3)–(1.4) are proved to be radially symmetric [13]. Very recently, the case $f(0) < 0$ (f semipositone) is studied in [14] and [20] in connection with the symmetry breaking properties of the degenerate positive solutions. A series of existence, multiplicity, and stability results for the positive solutions of semipositone ($f(0) < 0$) nonlinearities can be found in [3–5]. My work is mainly devoted to the study of the radial solutions to (1.1)–(1.2) for various classes of semipositone nonlinearities. The statements of the main theorems I obtained are presented in the next section.

Main theorems

1. Let $h(x,u,\lambda) = f(u) + \lambda\ g(\|x\|)$ where f is superlinear with $f(0) = 0$ and $g > 0$ in $B_1(0) \subset \mathbb{R}^N$, where $B_1(0)$ denotes the unit ball centered at the origin.

That is,

$$(2.1) \qquad -\Delta u(x) = f(u) + \lambda\ g(\|x\|) \quad \text{for} \quad x \in B_1(0),$$

$$(2.2) \qquad u(x) = 0 \quad \text{for} \quad x \in \partial B_1(0).$$

When $g : [0,\ 1] \to (0,\ \infty)$ is continuous nondecreasing, the following nonexistence result is obtained.

Theorem A : There exist $\alpha < 0$ and $\beta > 0$ such that for $\lambda \notin [\alpha,\ \beta]$ problem (2.1)–(2.2) has no positive solutions.

2. Let $h(x, u, \lambda) = u^p(x) + \lambda\ g(x)$, $1 < p < (N+2)/(N-2)$, and $g(x) = u_0^p(x)$, where u_0 denotes the positive solution to (1.1)–(1.2) with $\lambda = 0$.

Theorem B : There exists a $\lambda^* > 0$ such that for $\lambda > \lambda^*$, (1.1)–(1.2) has no positive solutions; for $\lambda \in (0,\ \lambda^*)$ it has at least two positive solutions; for $\lambda \leq 0$ it has at least one positive solution. In addition, when Ω is a ball in $\mathbb{R}^N$ there exists a sequence $\lambda_n \to -\infty$ such that at $\lambda = \lambda_n$ nonradial solutions bifurcate from the radial one.

This is in contrast with the nonexistence result in Theorem A when g is increasing. For this case a complete bifurcation diagram when N = 1 is obtained.

3. Let $h(x,\ u, \lambda) = \lambda\ f(u)$ where f is semipositone, monotonically increasing

and convex. That is

$$(2.3) \qquad -\Delta u(x) = \lambda f(u) \quad \text{for} \quad x \in B_1(0),$$

$$(2.4) \qquad u(x) = 0 \quad \text{for} \quad x \in \partial B_1(0).$$

Here $B_1(0)$ denotes the unit ball centered at the origin and $\lambda > 0$. The following theorem extends the results in [4].

Theorem C : If f is as above and is such that $(f(t)/(tf'(t) - f(t)))$ is nondecreasing, then for any $\lambda > 0$, (2.3)–(2.4) has at most one positive solution u such that $(\lambda,\ u)$ belongs to the ***unbounded*** branch of positive solutions.

Radial solutions of (2.3)–(2.4) are solutions to

$$(2.5) \qquad u'' + ((N-1)/r)u' + \lambda f(u) = 0,$$

$$(2.6) \qquad u'(0) = 0 \text{ and } u(1) = 0.$$

A complete bifurcation diagram consisting of all the radial branches of solutions to (2.3)–(2.4) is obtained in the following theorems by studying the variations of solutions to (2.5)–(2.6) with respect to parameters and rescaling arguments.

Theorem D : Let S be a connected component of solutions $(\lambda,\ u)$ to (2.3)–(2.4) with u radially symmetric and $u(0) > 0$. Then there exists a positive integer k such that if $(\lambda,\ u) \in S$ then u has 2k or $2k+1$ interior zeros. Moreover if $(\lambda,\ u)$ is a solution to (2.3)–(2.4) with u radial, $u(0) > 0$, and 2k or $2k+1$ interior zeros then $(\lambda,\ u) \in S$.

Theorem E : There exists $\mu_k \to \infty$ such that for each nonnegative integer k, if $\lambda > \mu_k$ then the problem (2.5)–(2.6) has no solutions with $u(0) > 0$ and such that u has exactly k zeroes in (0, 1).

4. Let $h(x, u, \lambda) = \lambda\ f(u)$ where f is semipositone ($f(0) < 0$) with $f(u) > 0$ for some $u > 0$. Assuming that f is monotonically increasing and concave for $u > 0$ the following theorem is proved.

Theorem F : If f is as above and if $f'(t) \to 0$ as $t \to \infty$, then there exist λ_1 and λ_2 with $\lambda_1 > \lambda_2$ such that for $\lambda > \lambda_1$ the problem (2.3)–(2.4) has a unique positive solution, which is stable; for $\lambda \in (\lambda_1,\ \lambda_2]$ it has exactly two positive solutions, one stable and one unstable; and for $\lambda = \lambda_2$ it has a unique positive solution.

Theorem G : There exists $\mu_k \to \infty$ such that for each nonnegative integer k, if $\lambda < \mu_k$ then the problem (2.5)–(2.6) has no solutions with $u(0) > 0$ and such that u has exactly k zeroes in (0, 1).

Theorem H : If, in addition, f is convex for $u < 0$ (i.e., $uf''(u) \leq 0$) then for any $\lambda > 0$ there exists at most one solution u satisfying (2.5)–(2.6) with $u(0) > 0$ and such that u has exactly one zero in (0, 1).

Theorem C and Theorem F are in contrast with the uniqueness results for the case when f is positone ($f(0) > 0$) where concavity of f implies the uniqueness of positive solution and the convexity of f allows multiple positive solutions.

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CHAPTER 2

MONOTONICITY OF THE FORCING TERM AND THE EXISTENCE OF POSITIVE SOLUTIONS

In this chapter we study about the influence of the monotinicity of the forcing term on the existence of positive solutions for semilinear elliptic problems. We mainly concentrate on the problems when the domain is the unit ball in $\mathbb{R}^N$. Because of the radial symmetry of the positive solutions in a ball, we restrict ourselves to the case when the forcing term is a function of $r = \|x\|$.

Let $g : [0,\ 1] \to (0,\ \infty)$ be a continuous, nondecreasing function and $\lambda \in \mathbb{R}$. We study the existence of positive solutions to the equation

$$-\Delta u = f(u) + \lambda g(\|x\|) \quad \text{in } B_1(0), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial B_1(0), \tag{1.2}$$

as λ varies over $\mathbb{R}$, where $B_1(0)$ denotes the open ball in $\mathbb{R}^N$ centered at the origin and $f : [0,\ \infty) \to [0,\ \infty)$ is differentiable, satisfying

$$f(0) = 0, \qquad f''(t) \geq 0 \qquad \text{with} \lim_{t \to \infty} (f(t)/t) = \infty. \tag{1.3}$$

We prove that for λ's with $|\lambda|$ large, (1.1)–(1.2) does not have positive solutions. This contrasts with the case in which g is monotonically decreasing. Indeed, taking $f(t) = t^p$, with $1 < p < (N+2)/(N-2)$, u_0 the positive solution of

$$(1.4) \qquad -\Delta w = w^p \quad \text{in } B_1(0),$$

$$(1.5) \qquad w = 0 \quad \text{on } \partial B_1(0),$$

and then taking $g(r) = u_0{}^p(r)$, it is easily seen that (1.1)–(1.2) has a positive solution for each $\lambda \leq 0$. The main result is

Theorem ***A*****:** There exist $\alpha < 0$ and $\beta > 0$ such that for $\lambda \notin [\alpha,\ \beta]$ problem (1.4)–(1.5) has no positive solutions.

Proof: The existence of β follows from a simple integration-by-parts argument using the fact that f is superlinear.

Let λ be negative. From Gidas, Ni, and Nirenberg [3] it follows that any positive solution to (1.1)–(1.2) is radially symmetric and decreasing. Hence (1.1)–(1.2) is equivalent to the singular ordinary differential equation

$$(1.6) \qquad u'' + (n/r)u' + f(u) - \lambda g(r) = 0 \quad \text{in } (0,1),$$

$$(1.7) \qquad u'(0) = 0, \qquad u(1) = 0,$$

where $\lambda > 0$ and $n = N - 1$. If it is assumed, on the contrary, that such an α does not exist, then there exists a sequence (u_j, λ_j) satisfying (1.6)–(1.7) with $\lambda_j \to \infty$. Since $u = u_j$ is radially decreasing on [0, 1], $\Delta u \leq 0$ at 0 and $\Delta u > 0$ at 1. Thus there exists $a_j \in [0,1)$ such that $\Delta u = 0$ at a_j. That is,

$$(1.8) \qquad f(u(a_j)) = \lambda_j g(a_j).$$

To prove that $a_j \to 1$ as $\lambda_j \to \infty$, the monotonicity assumption on g and the convexity of f are explored. Since u is radially decreasing, there exists a $b_j \in (a_j, 1)$ such that

$$2f(u(b_j)) = \lambda_j g(a_j). \tag{1.9}$$

Since g is nondecreasing, $a_j < b_j$. This and (1.9) imply that

$$\lambda_j g(0) \le 2f(u(b_j)) = \lambda_j g(a_j) \le \lambda_j g(b_j) \le \lambda_j g(1). \tag{1.10}$$

Because of (1.10), for $t \in [b_j, 1]$, we have

$$(t^n u')' = t^n(\lambda_j g(t) - f(u(t))) \ge \frac{1}{2} t^n \lambda_j g(t),$$

using the fact that f, g are nondecreasing and u is decreasing. (Note that (1.4) forces f to be nondecreasing.) Now, integrating on $[b_j,\ 1]$, yields

$$u'(1) - {b_j}^n u'(b_j) \ge \lambda_j g(b_j)(1 - {b_j}^N)/2N.$$

Thus

$$g(0)\lambda_j(1 - {b_j}^N) \le 2N({b_j}^n |u'(b_j)| - |u'(1)|), \tag{1.11}$$

using (1.10) and, since $u'(b_j) < 0, u'(1) \le 0$. Also note that the choice of a_j as in (1.8) implies that $u'' > 0$ in $(a_j, 1)$. Now, multiplying (1.6) by $r^{2n}u'$ and integrating on $[b_j,\ 1]$ yields

$$\begin{aligned}(u'(1))^2 - {b_j}^{2n}(u'(b_j))^2 &\ge 2{b_j}^{2n}F(u(b_j)) + 4n\int_{b_j}^1 r^{2n-1}F(u) + 2\lambda_j \int_{b_j}^1 r^{2n} g u' \\ &> 2\lambda_j \int_{b_j}^1 r^{2n} g u' \ge -2\lambda_j g(1) u(b_j).\end{aligned}$$

This and (1.9) would give

$$b_j^{2n}(u'(b_j))^2 - (u'(1))^2 < cf(u(b_j))u(b_j), \tag{1.12}$$

where c is a constant independent of j. From (1.11) and (1.12), it follows that

$$(1 - b_j^{N})^2 < c_1 u(b_j)/f(u(b_j)) \to 0$$

as $j \to \infty$ (see (1.8) and (1.9)).

Now, by the convexity of $f \circ g$ in $[a_j,\ 1]$, it follows that $f(u((1+a_j)/2)) \leq f(u(a_j))/2 = f(u(b_j))$. Thus $(1+a_j) \geq b_j$, proving that $a_j \to 1$ as $j \to \infty$. Let $\epsilon > 0$, and choose j large enough such that $1 - a_j < \epsilon$. Let $\mu_1 = \mu_1(-\Delta,\ B_{1-\epsilon})$ denote the first eigenvalue of $-\Delta$ in $B_{1-\epsilon}$, a ball of radius $(1-\epsilon)$ around the origin. That is

$$-\Delta\varphi_1 = \mu_1\varphi_1, \quad \text{in } B_{1-\epsilon}(0), \tag{1.13}$$

$$\varphi_1 = 0 \quad \text{on } \partial B_{1-\epsilon}(0), \tag{1.14}$$

where φ_1 is chosen to be positive in $B_{1-\epsilon}(0)$. By virtue of (1.4), along with (1.8) and (1.10) j can be chosen large enough so that $f'(u(a_j)) > \mu_1$. If $v(x) := u(x) - u(a_j)$, then v satisfies

$$-\Delta v > \mu_1 v \quad \text{in } B_{a_j}(0),$$

$$v = 0 \quad \text{on } \partial B_{a_j}(0),$$

with $v > 0$ in B_{a_j}, which yields a contradiction if compared with the eigenvalue problem (1.13)–(1.14) in B_{a_j}. Hence the theorem is proven. □

Remark 1. In contrast, when g is decreasing the problem (1.1)–(1.2) might have positive solutions for all $\lambda < 0$. For example, when $f(t) = t|t|^{p-1}$, $1 < p < (N+2)/(N-2)$ for $N > 3$, and $g(x) = {u_0}^p(x)$ for any $\lambda < 0$, there is a positive radial solution of (1.1)–(1.2) given by cu_0, where c is such that $c - c^p + \lambda$. Thus the monotonicity of g and the positivity of g on the boundary ∂B_1 play an important role in the existence results.

Remark 2. It can be easily seen that there is no nonradial degeneracy all along the positive solution curve for negative λ's as opposed to symmetry breaking in the case when g is decreasing (see [2]).

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CHAPTER 3

SYMMETRY BREAKING AND THE BIFURCATION DIAGRAM FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

In this chapter we study the existence and the bifurcation of positive solutions to the equation

$$-\Delta u = u|u|^{p-1} + \lambda g(x) \quad \text{in } B_1(0), \tag{1.1}$$

$$u = 0 \quad \text{on } \partial B_1(0), \tag{1.2}$$

as λ varies over $\mathbb{R}$, where $B_1(0)$ denotes the open ball in $\mathbb{R}^N$ centered at the origin and $p < (N+2)/(N-2)$. Here g is assumed to be smooth with $g(x) > 0$ in $B_1(0)$ and is decreasing as a function of $r = \|x\|$. We concentrate our study only for a special class of functions g, namely, $g(x) = {u_0}^p(x)$ where u_0 is the unique positive solution to (1.1)–(1.2) when $\lambda = 0$. That is u_0 is the positive solution to the problem:

$$-\Delta w = w^p \quad \text{in } B_1(0), \tag{1.3}$$

$$w = 0 \quad \text{on } \partial B_1(0). \tag{1.4}$$

This particular choice of g gives the existence of positive solutions for all $\lambda < 0$. Also, the transversality condition holds whenever an eigenvalue of the corresponding linearized operator vanishes, guaranteeing that it is a bifurcation point. As a result, we get the nonuniquneness of positive solutions in a small neighbourhood of these λ's as opposed to uniqueness in the case of a negative constant. When $N = 1$ we obtain the complete bifurcation diagram. For $N > 1$, we prove that for certain values of λ nonradial solutions bifurcate from radially symmetric positive solutions. We state our results more precisely, in the following propositions and theorems.

PRELIMINARIES

The following proposition makes use of the techniques in [1] in giving us a positive solution curve near 0. We sketch the proof in brief for the sake of completeness.

Proposition 1 : There exists λ_1^* such that there is a positive solution branch (u_λ, λ) in $(0,\ \lambda_1^*)$ satisfying:

(i) for fixed λ, $u_\lambda(x)$ is the minimal solution,

(ii) $u_\lambda(x)$ is nondecreasing as a function of λ for $x \in B_1(0)$,

(iii) for $0 < \lambda < 1$, u_λ is a C^1 map from $[0,\ \lambda_1^*) \to C^{2,\alpha}(B_1(0))$.

Proof: We define the operator $G : H_0^1(B_1(0)) \times \mathbb{R} \to H^{-1}(B_1(0))$ by

$$G(u, \lambda) = -\Delta u - f(u) - \lambda g(x). \tag{1.5}$$

Thus

$$G_u(u, \lambda)v = -\Delta v - f'(u)v. \tag{1.6}$$

At $(0,\ 0)$ $G_u = -\Delta$ is invertible and hence by the implicit function theorem we get a smooth curve $(u_\lambda(x),\ \lambda)$ in a small neighbourhood of $(0,\ 0)$ and the derivative $du/d\lambda$ satisfying:

$$-\Delta(du/d\lambda) = g(x) \quad \text{in } B_1(0),$$

$$du/d\lambda = 0 \quad \text{on } \partial B_1(0).$$

By the maximum principle, $du/d\lambda \geq 0$ at $\lambda = 0$. Thus we conclude that there exists a solution branch for $\lambda > 0$ near 0. We can continue this branch, by repeating the above argument, till we encounter a point λ_1^* where G_u is singular. This branch of positive solutions u_λ in $(0,\ \lambda_1^*)$ has to be the minimal branch of positive solutions. □

Notation : Let $\mu_1(h,\ B_1(0))$ denote the eigenvalues of

$$-\Delta u + hu = \mu_i u \quad \text{in } B_1(0),$$

$$u = 0 \quad \text{on } \partial B_1(0),$$

for any continuous function h on $B_1(0)$. The following proposition gives the behaviour of the positive solution curve in $[0,\ \lambda_1^*]$.

Proposition 2 : The minimal branch obtained in the previous proposition satisfies:

(i) $\lim_{\lambda \to \lambda_1^*} u_\lambda = u_1^*$ exists and is the only positive solution at $\lambda = \lambda_1^*$.

(ii) near λ_1^* the positive solutions of (1.1)–(1.2) form a smooth curve which is given by $(u(s),\ \lambda(s))$ for $|s| < \epsilon$ with

$$(u(0),\ \lambda(0)) = (u_1^*, \lambda_1^*),$$

$$\lambda'(0) = 0 \quad \text{and } \lambda''(0) < 0.$$

(iii) for $\lambda \in (0,\ \lambda_1^*)$ the problem (1.1)–(1.2) has two positive solutions.

Proof : By the results of de Figueirdo, Lions, and Nussbaum [3], positive solutions to (1.1)–(1.2) are bounded in $[0,\ \lambda_1^*]$. This, with the left continuity of u_λ at λ_1^* (see Proposition 1), gives

$$\lim_{\lambda \to \lambda_1^*} = u_1^*.$$

This u_1^*, indeed, is a positive solution λ_1^* and hence

$$\mu_1 = \mu_1(-f'(u_1^*),\ B_1(0)) = 0.$$

Now, to show the uniqueness of positive solutions at $\lambda = \lambda_1^*$, let there exist another positive solution u such that $u \neq u_1^*$. Thus, by Proposition 1, $w = u - u_1^* > 0$ satisfies

$$-\Delta w = \frac{f(u) - f(u_1^*)}{u - u_1^*} w \quad \text{in } B_1(0),$$

and hence

$$-\Delta w > f'(u_1^*) w \quad \text{in } B_1(0),$$

$$w = 0 \quad \text{on } \partial B_1(0).$$

This leads to a contradiction since $\mu_1 = 0$.

To prove (ii), we observe that at $\lambda_1^*, G_u^0 = G_u(\lambda_1^*,\ u_1^*)$ is a self-adjoint linear operator satisfying

$$\mathrm{Range}(G_u^0) = \mathrm{Ker}(G_u^0)^{\perp} = \{f \in H^{-1} : \langle f,\ \varphi_1\rangle_{H^{-1},H_0^1} = 0\},$$

where φ_1 is the eigenfunction corresponding go μ_1, chosen to be positive in $B_1(0)$. Also, we observe that $G_\lambda^0 = G_\lambda(\lambda_1^*,\ u_1^*) \notin \mathrm{Ra}(G_u^0)$ since

$$\langle -g(x),\ \varphi_1\rangle_{H^{-1},H_0^1} = -\int_{B_1(0)} g(x)\varphi_1(x) < 0.$$

Hence we have

$$\dim \mathrm{Ker} G_u^0 = \mathrm{codim}\ \mathrm{Ra}(G_u^0) = 1,$$

$$G_\lambda^0 \notin \quad \mathrm{Ra}(G_u^0),$$

which fits into the framework of Crandall and Rabinowitz [2] giving us the following C^1 branch fo $|s| < \epsilon$,

$$u(s) = u_1^* + s\varphi_1 + z(s),$$

$$\lambda(s) = \lambda_1^* + \tau(s),$$

satisfying

$$u(0) = u_1^*, \qquad z(s) \in \mathrm{Ra}(G_u^0), \qquad z'(0) = 0,$$

$$\lambda(0) = \lambda_1^*, \qquad \lambda'(0) = 0.$$

Differentiating $G(u(s),\ \lambda(s)) = 0$ twice with respect to s, at $s = 0$, we get

$$G_u^* z''(0) = -[G_\lambda^0 \tau''(0) + G_{uu}^0 {\varphi_1}^2].$$

This, with the Fredholm alternative (for the existence of $z''(0)$), implies that

$$\tau''(0) = -\frac{\int_{B_1(0)} f''(u_1^*){\varphi_1}^3}{\int_{B_1(0)} g(x)\varphi_1} < 0,$$

showing that the solution curve turns back.

Since we have apriori bounds on solutions, a standard degree theory argument gives that there exist at least two positive solutions for each $\lambda \in (0,\ \lambda_1^*)$, proving part (iii) of the lemma. □

Remark 1: Using the convexity of f, it can be easily seen that for any fixed λ there can not exist three solutions with $u_1 < u_2 < u_3$.

From now on we restrict ourselves to the case when $g(x) = u_0^p(x)$. u_0 is known to be radial and decreasing as a function of $\|x\|$ in $B_1(0)$ and hence g satisfies (iii).

Proposition 3: There exists a positive radial solution for (1.1) for any $\lambda \leq \lambda_1^*$ where $\lambda_1^* = p^{-(p-1)^{-1}} - p^{-p(p-1)^{-1}}$. For any $\lambda \in (0,\ \lambda_1^*]$ the minimal positive solution is given by cu_0 where $c \in (0,\ p^{-(p-1)^{-1}}]$ with $\lambda = c - c^p$.

Proof: For any constant $c > 0, u = cu_0$ is a positive radial solution for (1.1) at $\lambda = c - c^p$. Choosing $c > 0$ such that $\lambda = c - c^p$ we get a positive solution for any

$\lambda \in (0,\ \lambda_1^*]$. Indeed, the positive solution branch given by $(cu_0,\ (c - c^p))$ where $c > 0$ turns back at $c = p^{-(p-1)^{-1}}$. The following observation using weighted eigenvalue probems shows that the branch given by cu_0 for $0 < c \leq p^{-(p-1)^{-1}}$ is the minimal one. Let us fix a c in $(0,\ p^{-(p-1)^{-1}}]$ and set $\lambda = c - c^p$. Let $u \neq cu_0$ be a positive solution at λ. Then $w = u - cu_0$ satisfies

$$-\Delta w = \frac{u^p - c^p u_0^p}{u - cu_0} w \qquad \text{in} \qquad B_1(0),$$

$$w = 0 \qquad \text{on} \qquad \partial B_1(0).$$

Setting $m(x) = pc^{p-1}u_0^{p-1}(x)$, we notice that

$$-\Delta(cu_0) = \frac{1}{pc^{p-1}} m(cu_0)$$

and hence $\rho_1 = \left(pc^{p-1}\right)^{-1} > 1$ is the principal eigenvalue of the weighted eigenvalue problem:

$$-\Delta v = \rho_i m v \qquad \text{in} \qquad B_1(0),$$

$$v = 0 \qquad \text{on} \qquad \partial B_1(0),$$

with $v = cu_0$ as the corresponding positive eigenfunction. This implies that $w \geq 0$. Now an application of the strong maximum principle gives that $w = u - cu_0 > 0$. $\square$

We study the bifurcation phenomenon along this positive solution branch given by cu_0. If $\mu_i := \mu_i(-f'(u),\ B_1(0))$ then we have seen that μ_1 vanishes at λ_1^*, i.e., when $c = p^{-(p-1)^{-1}}$ where cu_0 itself serves as the first eigenfunction for the linearized operator. The solution curve turns back at this point and c increases on this upper branch as λ decreases. Hence as we move along this curve to the left μ_i decreases and we encounter a point where $\mu_2 = \mu_2(-f'(u),\ B_1(0))$ vanishes. Let us denote that point by $(u_2^*,\ \lambda_2^*)$. To get a better insight into the problem, we now study the problem when $N = 1$ and then generalize the results for the unit ball in $\mathbb{R}^N$, to obtain the symmetry breaking of the positive solutions.

ONE DIMENSIONAL PROBLEM

Here we consider the corresponding Dirichlet problem in (0, 1):

$$\begin{cases} -u'' &= u^p + \lambda u_0^p \quad \text{in} \quad (0,\ 1), \\ u(0) &= u(1) = 0, \end{cases} \tag{2.1}$$

and study the behaviour of the positive branch given by cu_0. We recall that u_0 is symmetric with respect to $x = 1/2$.

Theorem 1. *(u_2^*, λ_2^*) is a bifurcation point and the bifurcating branch is made of solutions to (2.1) with two local maxima in (0, 1).*

Proof. All the arguments in the previous section go through and at λ_2^* where G_u has its second eigenvalue vanishing, let φ_2 be the eigenfunction corresponding to $\mu_2 = 0$. Then φ_2 is antisymmetric with respect to $1/2$ and without loss of generality we can take $\varphi_2'(0) > 0$. Considering $G : C_0^2((0,1)) \to C_0((0,1))$ and denoting G_u and G_λ at (u_2^*, λ_2^*) by G_u^0 and G_λ^0 this means (by the self-adjointness of G_u^0)

$$\begin{aligned} &\mathrm{Ker} G_u^0 = \mathrm{span}\{\varphi_2\}, \\ (2.2) \qquad &\mathrm{Ra} G_u^0 = \Big\{ w \in C_0((0,1)) : \int_0^1 w\varphi_2 = 0 \Big\}, \\ &\mathrm{codim}\ \mathrm{Ra} G_u^0 = \dim\ \mathrm{Ker} G_u^0 = 1. \end{aligned}$$

We derive the bifurcation equations using standard techniques (see [4]) as follows. If c is such that $u_2^* = cu_0$, then $\varphi := \left(1 - pc^{p-1}\right)^{-1} u_0$ satisfies

$$(2.3) \qquad \begin{cases} G_u^0 \varphi \ + \ G_\lambda^0 & = 0 \text{ in } (0,1), \\ \varphi(0) = \varphi(1) & = 0, \end{cases}$$

proving that $G_\lambda^0 \in \mathrm{Ra} G_u^0$. Let us assume that the solution branch through (u_2^*, λ_2^*) is given by $(u(s), \lambda(s))$, i.e.,

$$G(u(s), \lambda(s)) = 0 \text{ for } s \in (s_0 - \epsilon, s_0 + \epsilon)$$

with $(u(s_0), \lambda(s_0)) = (u_2^*, \lambda_2^*)$. Hence we get at s_0,

$$(2.4) \qquad G_u^0 u'(s_0) + G_\lambda^0 \lambda'(s_0) = 0,$$

which along with (2.2) and (2.3) implies that

$$u'(s_0) - \lambda'(s_0)\varphi = \alpha_1 \varphi_2$$

for some scalar α_1. We set

$$(2.5) \qquad \begin{cases} \lambda'(s_0) & = \alpha_0, \\ u'(s_0) & = \alpha_0 \varphi + \alpha_1 \varphi_2. \end{cases}$$

By differentiating $G(u(s), \lambda(s)) = 0$ twice at $s = s_0$, we get

$$G_u^0 u''(s_0) = [G_{uu}^0 u'(s_0)^2 + 2G_{u\lambda}^0 u'(s_0)\lambda'(s_0) + G_{\lambda\lambda}^0 \lambda'(s_0)^2] - G_\lambda^0 \lambda''(s_0).$$

Since $G_u^0 u''(s_0)$ and $G_\lambda^0 \in$ Ra (G_u^0) we obtain that $G_u^0 u''(s_0) + G_\lambda^0 \in$ Ra (G_u^0). This yields

$$a\alpha_0^2 + 2b\alpha_0\alpha_1 + c\alpha_1^2 = 0, \tag{2.6}$$

where

$$a = \int_0^1 \varphi_2 G_{uu}^0(\varphi, \varphi),$$
$$b = \int_0^1 \varphi_2 G_{uu}^0(\varphi_2, \varphi),$$
$$c = \int_0^1 \varphi_2 G_{uu}^0(\varphi_2, \varphi_2).$$

By the symmetry of φ and the antisymmetry of φ_2 we conclude that both $a = 0$ and $c = 0$. Hence if we prove that $b \neq 0$ then (0, 1) and (1, 0) would give the solutions of (2.6). Obviously,

$$b = \int_0^1 \varphi_2^2 f''(u_2^*)\varphi \neq 0,$$

since the integrand does not change sign. Thus $(du, d\lambda) = (\varphi, 1)$ or $(\varphi_2, 0)$ give the tangent directions to the curve $(u(s), \lambda(s))$ at s_0. Thus at (u_2^*, λ_2^*) we have

(i) Ker G_u^0 = span $\{\varphi_2\}$,

(ii) codim RaG_u^0 = dim Ker $G_u^0 = 1$,

(iii) $G_\lambda^0 \in$ Ra G_u^0,

(iv) the nondegeneracy condition $b \neq 0$ holds.

Hence (u_2^*, λ_2^*) is a potential bifurcation point and the distinct solution curves through (u_2^*, λ_2^*) are given by

$$\begin{cases} u(t) &= u_2^* + \varphi t + O(t^2), \\ \lambda(t) &= \lambda_2^* + A_1 t + O(t^2), \end{cases} \tag{2.7}$$

and

$$\begin{cases} u(t) &= u_2^* + \varphi_2 t + O(t^2), \\ \lambda(t) &= \lambda_2^* + A_2 t + O(t^2). \end{cases} \tag{2.8}$$

The branch given by (2.7) continues to be the positive solution branch to the left of λ_2^* whereas the bifurcating one given by (2.8) is a branch of solutions to (2.1) which are not symmetric with respect to $x = 1/2$. In fact, near λ_2^* they are positive solutions with two local maxima in (0, 1). □

Remark 2: Let S^k denote the solutions of

$$\begin{cases} -u'' &= u^p + \lambda u_0^p \quad \text{in} \quad (0,\ 1), \\ u(0) &= u(1) = 0, \end{cases} \tag{2.9}$$

with k local maxima in (0, 1). As λ decreases on the positive solution branch, we encounter a point where μ_3 becomes zero and then a point where μ_4 becomes zero etc. By the same argument as above, at a point where μ_k becomes zero, a branch of S^k emanates as in the bifurcation diagram below. Since $f(t) = t|t|^{p-1}$ is odd if (u, λ) is a solution of (2.1) so is $(-u, -\lambda)$.

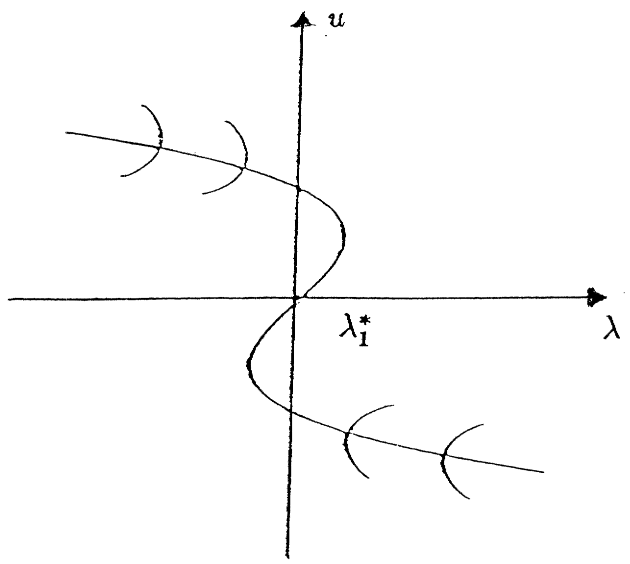

FIGURE 1. Bifurcation of positive solutions

SYMMETRY BREAKING

Theorem 2. *For any $N \geq 2$, there exists a sequence $\lambda_{k_n} \to -\infty$ such that at each (u_{k_n}, λ_{k_n}) a branch of nonradial solutions bifurcates, i.e., there is symmetry breaking of positive radial solutions.*

Proof. In section 1, we have seen that on the positive solution curve cu_0 for $c > p^{-(p-1)^{-1}}$, μ_k decreases as λ decreases and hence we encounter a point (u_k, λ_k) such that $\mu_k = 0$ at (u_k, λ_k). Let k_n be chosen such that (u_{k_n}, λ_{k_n}) is a point of purely nonradial degeneracy. It is well-known that the eigenvalues μ_k are distributed asymptotically as $k^{2/N}$, whereas those which are radial are distributed as k^2. Thus the existence of the sequence k_n as above follows. We rewrite the operator equation as

$$F(u,\ \lambda) = 0 \tag{3.1}$$

with

$$F(u,\ \lambda) = \Delta(\rho(\lambda)(u_0 + u)) + (\rho(\lambda)(u_0 + u))^p + \lambda u_0^p \tag{3.2}$$

where $\rho(\lambda)$ is the unique real number c with $c - c^p = \lambda$. Then along the trivial branch $(0, \lambda)$ of solutions of (3.1) we have

$$F_u(0,\ \lambda)v = \rho(\lambda)(\Delta v + p(\rho(\lambda)u_0)^{p-1}v) \tag{3.3}$$

and hence $F_u(0,\ \lambda)$ is singular whenever $\lambda = \lambda_k$. Since F_u is self-adjoint, at each of these degenerate points there is bifurcation (see Theorem 11.4 in [7]). In particular at (u_{k_n}, λ_{k_n}), by Lyapunov - Schmidt argument, the zeroes of F bifurcation from (u_{k_n}, λ_{k_n}) have the form $((u_{k_n} + \varphi + \mathcal{L}(\varphi)),\ \lambda)$, where φ is a nonradial eigenfunction corresponding to the ${k_n}^{\text{th}}$ eigenvalue $\mu_{k_n} = 0$ and $\mathcal{L}(\varphi) = o(\varphi)$. Since $\mathcal{L}(\varphi) = o(\varphi)$, the first component here is nonradial and hence there is symmetry breaking of positive solutions at (u_{k_n}, λ_{k_n}). □

Concluding Remarks: Several numerical experiments suggest that in the case of a general function g satisfying (iii) problem (1.1)–(1.2) should have atleast one positive solution for every $\lambda < 0$. Also, since we do not use apriori bounds for this special choice of g all the above arguments go through when $B_1(0)$ is replaced with $\mathbb{R}^N$. In fact, we do not explicitly use the assumption that the region under consideration is a ball. So, all the arguments go through for any radially symmetric domains in $\mathbb{R}^N$ with a smooth boundary for which (1.1)–(1.2) has a unique positive solution.

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CHAPTER 4

BRANCHES OF RADIAL SOLUTIONS FOR SEMIPOSITONE PROBLEMS

In this chapter we establish the structure of radial solution branches for semipositone semilinear elliptic problems. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable, monotone function such that

$$f(0) < 0 \quad \text{(semipositone)}, \tag{1.1}$$

$$\lim_{d \to \infty} f(d)/d = \infty \quad \text{(superlinear)}, \tag{1.2}$$

$$F(d) - ((N-2)/2N) df(d) \quad \text{is bounded below}, \tag{1.3}$$

and for some $k \in (0,\ 1)$

$$\Lambda = \lim_{d \to \infty} (d/f(d))^{N/2} \{F(kd) - ((N-2)/2N) df(d)\} = \infty, \tag{1.4}$$

where $F(t) = \int_0^t f(s)ds$. We consider the set of radial solutions to the equation

$$-\Delta u(x) = \lambda f(u(x)) \quad \text{for} \quad x \in \Omega, \tag{1.5}$$

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{1.6}$$

with $u(0) > 0$, where Ω denotes the unit ball in $\mathbb{R}^N (N > 1)$ centered at the origin, $\lambda > 0$, and Δ is the Laplacian operator. The monotonicity of f along with (1.2), implies that there exist unique positive numbers β and θ with $f(\beta) = F(\theta) = 0$. We assume that

$$f'(\beta) > 0. \tag{1.7}$$

Assumption (1.4) implies that f grows subcritically on $[0, \infty)$ (i.e., $|f(t)| \leq A(1 + t^p)$ with $p < (N+2)/(N-2)$). On the other hand, (1.3) ensures that f can have even the critical growth ($p = (N+2)/(N-2)$) on $(-\infty, 0]$. These assumptions are needed to prove that, for a given nonnegative integer k, if $u(0)$ is large and u has k interior nodal hypersurfaces then the parameter λ is small. If (1.4) is weakened to allow critical or supercritical growth on $[0, \infty)$, the latter might not hold (see [1]). While assumptions (1.1)–(1.4) and (1.7) can be relaxed we do not pursue in that direction in order to keep the technicalities as simple as possible. We state our main results in the following theorems.

THEOREM 1. *Let $S \subset \mathbb{R} \times C(\bar{\Omega})$ be a connected component of radially symmetric solutions to (1.5)–(1.6) with $u(0) > 0$. Here $C(\bar{\Omega})$ denotes the set of all continuous functions from $\bar{\Omega}$ into $\mathbb{R}$.*

(a) *If S is nonempty then there exists a nonnegative integer k such that if $(\lambda,\ u) \in S$ then u has $2k$ or $2k+1$ nodal hypersurfaces in Ω.*

(b) *Suppose* $(\lambda_0,\ u_0) \in S$. *The function* u_0 *has* $2k$ *nodal hypersurfaces in* Ω *and* $\nabla u_0(x) \neq 0$ *for* $x \in \partial\Omega$ *iff there exists* $(\lambda_1,\ u_1) \in S$ *satisfying* $u_0(0) = u_1(0)$ *and* u_1 *has* $2k+1$ *nodal hypersurfaces in* Ω.

THEOREM 2.

(a) *For any nonnegative integer* k*, there exists a unique unbounded branch of solutions* S_k *where* $S_k = \{(\lambda,\ u) : (\lambda,\ u)$ *is a solution to (1.5)–(1.6) with* u *radial,* $u(0) > 0$ *and such that* u *has either* $2k$ *or* $2k+1$ *nodal hypersurfaces in* $\Omega\ \}$.

(b) *There exists* $\bar{\lambda} > 0$ *such that if* $(\lambda,\ u)$ *is a solution to (1.5)–(1.6) with* $\lambda < \bar{\lambda}$ *then* $(\lambda,\ u) \in S_k$ *for some nonnegative integer* k*. Moreover,* $\nabla u(x) \neq 0$ *for* $x \in \partial\Omega$.

THEOREM 3. *There exists* $\mu_k \to \infty$ *such that for each nonnegative integer* k*, if* $\lambda > \mu_k$ *then the problem (1.5)–(1.6) does not have a solution with* $u(0) > 0$ *and such that* u *has exactly* k *zeroes in* $(0,\ 1)$.

THEOREM 4. *If, in addition,* f *is convex and* $f(t)/(tf'(t) - f(t))$ *is a nondecreasing function then for each* $\lambda > 0$ *there exists at most one positive solution* u *such that* $(\lambda,\ u) \in S_0$.

Figure 1 summarizes the above theorems and Lemma 2.4 given below. Since we concentrate on the radial solutions to (1.5)–(1.6), we use an O.D.E. approach and our techniques include rescaling, variations with respect to parameters, energy analysis and the implicit function theorem. For other studies on the positive solutions to semipositone problems, the reader is referred to [2]–[4], [6], and [8]–[11]. However, in the direction of sign changing solutions for semipositone problems we know only of [5], where the one dimensional problem is considered.

Preliminaries

We first note that radial solutions to (1.5)–(1.6) correspond to solutions to the singular problem

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \quad \text{for } r \in [0, 1], \tag{2.1}$$

$$u'(0) = 0, \tag{2.2}$$

$$u(1) = 0, \tag{2.3}$$

where $'$ denotes the differentiation with respect to $r = \|x\|$. For $d > 0$ we define $u(\cdot, \lambda, d) := u(\cdot)$ as the solution to (2.1), (2.2) and $u(0) = d$. For future reference, we note that $S = \{(\lambda, u) \in \mathbb{R} \times C(\bar{\Omega}) : (\lambda, u)$ satisfies (2.1)–(2.3)$\}$ is connected iff $\{(\lambda, u(0)) : (\lambda, u) \in S\}$ is connected. This is an immediate consequence of the continuous dependence of solutions to (2.1)–(2.3) on the initial conditions. *To facilitate the proofs of above theorems, we identify S with the latter set in $\mathbb{R}^2$.*

We define $E(r) := E(r, \lambda, d) = (u'(r))^2 + 2\lambda F(u(r))$. From (2.1) it follows that $E'(r) = -2(N-1)(u'(r))^2/r$. Hence, by the uniqueness of solutions to (2.1) subject to initial conditions we see that

$$\text{(2.4)} \qquad \text{if } u(s) = 0 \text{ then } E(r) > 0 \text{ for all } r \in [0,\ s).$$

Thus if u is a solution to (2.1)–(2.3) then $(u,\ u') \neq (0,\ 0)$ for all $r \in [0,\ 1)$ (see (2.4)).

For any $\rho > 0$, if w is defined by $w(r) = u(r\rho,\ \lambda,\ d)$, then w satisfies

$$w''(r) + ((N-1)/r)w'(r) + \lambda\ \rho^2\ f(w(r)) = 0 \text{ for } \ r > 0,$$

$$w'(0) = 0 \text{ and } \quad w(0) = d.$$

By the uniqueness of the solution to an initial value problem, this implies that

$$\text{(2.5)} \qquad u(r\rho,\ \lambda,\ d) = u(r,\ \lambda\rho^2,\ d).$$

Differentiating with respect to ρ and taking $\rho = 1$, we obtain

$$\text{(2.6)} \qquad u_\lambda(r,\ \lambda,\ d) = ru'(r,\ \lambda,\ d)/2\lambda,$$

where u_λ denotes the derivative of u with respect to λ.

An immediate consequence of (2.5) is:

Lemma 2.1. *Given a nonnegative integer j, for each $d > \beta$ there exists at most one $\lambda > 0$ such that $u(\cdot,\ \lambda,\ d)$ has exactly j zeroes in (0, 1) and satisfies (2.1)–(2.3).*

Proof. Suppose, on the contrary that, for some $d > \beta$ we have $u(\cdot,\ \lambda_1,\ d)$ and $u(\cdot,\ \lambda_2,\ d)$ having exactly j zeroes in (0, 1) and satisfying (2.1)–(2.3). Without loss of generality we can assume that $\lambda_1 < \lambda_2$. From (2.5) we have

$$u(r\rho,\ \lambda_1,\ d) = u(r,\ \lambda_1\rho^2,\ d).$$

Taking $\rho^2 = \lambda_2/\lambda_1$, we get

$$u(r\sqrt{\lambda_2/\lambda_1},\ \lambda_1,\ d) = u(r,\ \lambda_2,\ d),$$

from which we get that $u(\cdot,\ \lambda_1,\ d)$ has $j+1$ zeroes in $(0,\ \sqrt{\lambda_1/\lambda_2}] \subset (0,1)$. This contradicts our assumption that $u(\cdot,\ \lambda_1,\ d)$ has exactly j zeroes in (0, 1) and hence proves the lemma. $\square$

We let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$ to denote the eigenvalues of the problem:

$$\phi'' + ((N-1)/r)\phi' + \lambda\phi = 0 \quad \text{for } r \in [0,\ 1], \tag{2.7}$$

$$\phi'(0) = 0, \tag{2.8}$$

$$\phi(1) = 0. \tag{2.9}$$

Let ϕ_k denote the eigenfunction corresponding to the eigenvalue λ_k such that $\phi_k(0) = 1$. The following lemma describes the β–levels of $u(1,\ \lambda,\ d)$.

Lemma 2.2. *For each nonnegative integer k there exists a differentiable function $\sigma_k : (\beta,\ \infty) \to \mathbb{R}$ such that $u(1,\ \sigma_k(d),\ d) - \beta = 0$ and $u(\cdot,\ \sigma_k(d),\ d) - \beta$ has*

k zeroes in (0, 1). Moreover, $\lim_{d\to\infty}\sigma_k(d) = 0$ and $\lim_{d\to\beta}\sigma_k(d) = \lambda_{k+1}/f'(\beta)$. Conversely, if $u(1,\ \lambda,\ d) = \beta$ and $u(\cdot,\ \lambda,\ d) - \beta$ has k zeroes in (0, 1) then $\lambda = \sigma_k(d)$.

Proof. Since $f'(\beta) > 0$, and the eigenvalues to

$$\phi'' + ((N-1)/r)\phi' + \mu f'(\beta)\phi = 0 \quad \text{for } r \in [0,\ 1], \tag{2.10}$$

$$\phi'(0) = 0, \tag{2.11}$$

$$\phi(1) = 0, \tag{2.12}$$

are simple, from standard bifurcation arguments, it follows that $(\lambda_k/f'(\beta),\ \beta)$ are the points of bifurcation for the equation $u(1,\ \lambda,\ d) - \beta = 0$. Note that, because $f(\beta) = 0$, the ray $\{(\lambda,\ \beta) : \lambda > 0\}$ satisfies $u(1,\ \lambda,\ \beta) = \beta$. Also, for $d > \beta$ close to β and λ close to $\lambda_k/f'(\beta)$, well established result on bifurcation from simple eigenvalues implies that, $u(\cdot,\ \lambda,\ d)$ is of the form $s\phi_k + o(s)$, which in particular, implies that

$$u(\cdot,\ \lambda,\ d) - \beta \text{ has } k-1 \text{ zeroes in } (0,\ 1). \tag{2.13}$$

By the uniqueness of solutions to the initial value problem (2.1), (2.2), $u(0) = d$, we see that if $d > \beta$ and $u(t,\ \lambda,\ d) = \beta$ then $u'(t,\ \lambda,\ d) \neq 0$. In particular, from (2.6) we have

$$u_\lambda(t,\ \lambda,\ d) \neq 0 \text{ if } u(t,\ \lambda,\ d) = \beta. \tag{2.14}$$

This and the implicit function theorem imply that if $J \subset \{(\lambda,\ d) : \lambda > 0, d > \beta\}$ is a connected component of solutions to $u(1,\ \lambda,\ d) = \beta$, then there exists a nonnegative integer k such that if $u(\cdot,\ \lambda,\ d) - \beta$ has k zeroes in $(0,\ 1)$ for each $(\lambda,\ d) \in J$. Also, from (2.14) we infer that $J = \{(\sigma_k(d),\ d) : d \in (\beta,\ \infty)$ and $\sigma_k : (\beta,\ \infty) \to \mathbb{R}$ is a differentiable function $\}$.

Conversely, suppose $u(1,\ \lambda,\ d) = \beta$ and $u(\cdot,\ \lambda,\ d) - \beta$ has k zeroes in $(0,\ 1)$. If we assume that $\lambda > \sigma_k(d)$, then by taking $\rho^2 = \sigma_k(d)/\lambda$ in (2.5) we get that $u(\cdot,\ \lambda,\ d)$ has at least $k+1$ zeroes in $(0,\ 1)$ which contradicts our assumption that $u(\cdot,\ \lambda,\ d) - \beta$ has exactly k zeroes in $(0,\ 1)$. Similarly the possibility that $\lambda < \sigma_k(d)$ is ruled out. Since (2.13) implies that

$$\lim_{d \to \beta} \sigma_k(d) = \lambda_{k+1}/f'(\beta), \tag{2.15}$$

we have $J_k := J = \{(\lambda,\ d) : u(1,\ \lambda,\ d) = \beta$ and $u(\cdot,\ \lambda,\ d) - \beta$ has exactly k zeroes in $(0,\ 1)\}$. Finally, the superlinearity of f along with (1.4) imply that $\lim_{d \to \infty} \sigma_k(d) = 0$ (for details, see [7]). Hence the lemma is proven. $\square$

Lemma 2.3. *For any $d > \beta$ and for any nonnegative integer k, one of the following holds.*

(a) *There exists no $\lambda \in (\sigma_{2k}(d),\ \sigma_{2k+1}(d))$ for which $u(\cdot,\ \lambda,\ d)$ satisfies (2.1)–(2.3).*

(b) *There exists exactly one $\lambda_0 \in (\sigma_{2k}(d),\ \sigma_{2k+1}(d))$ for which $u(\cdot,\ \lambda_0,\ d)$ satisfies (2.1)–(2.3). In this case $u(\cdot,\ \lambda_0,\ d)$ has $2k$ zeroes in (0, 1) and*

$u'(1,\ \lambda_0,\ d) = 0$.

(c) *There exist* $\lambda_0,\ \lambda_1 \in (\sigma_{2k}(d),\ \sigma_{2k+1}(d))$ *with* $\lambda_0 < \lambda_1$ *and* $u(\cdot,\ \lambda_i,\ d)$ *satisfying (2.1)–(2.3) for* $i = 0, 1$. *In this case* $u_i := u(\cdot,\ \lambda_i,\ d)$ *has exactly* $2k + i$ *zeroes in* $(0,\ 1)$ *and* $u'(1,\ \lambda_i,\ d) \neq 0$ *for* $i = 0, 1$.

Proof. Suppose $u(\cdot,\ \lambda,\ d)$ satisfies (2.1)–(2.3). First, we observe that

(2.16) $\quad u(\cdot,\ \lambda,\ d)$ has $2k$ or $2k+1$ interior zeroes iff $\sigma_{2k}(d) < \lambda < \sigma_{2k+1}(d)$.

Indeed, suppose u has $2k$ or $2k+1$ zeroes in $(0,\ 1)$ and $\lambda < \sigma_{2k}(d)$. Hence $u(\cdot,\ \lambda,\ d) - \beta$ has $2k+1$ zeroes in $(0,\ 1)$. Since

$$u(r\sqrt{\sigma_{2k}(d)/\lambda},\ \lambda,\ d) - \beta = u(r,\ \sigma_{2k}(d),\ d) - \beta,$$

we see that $u(\cdot,\ \sigma_{2k}(d),\ d) - \beta$ has $2k+1$ zeroes in $(0,\ \sqrt{\lambda/\sigma_{2k}(d)})$. This contradicts the fact that $u(\cdot,\ \sigma_{2k}(d),\ d) - \beta$ has $2k$ zeroes in $(0,\ 1)$ and hence proves that $\sigma_{2k}(d) < \lambda$. The proof that $\lambda < \sigma_{2k+1}(d)$ follows along the same lines and we leave it to the reader.

Conversely, suppose $u(\cdot,\ \lambda,\ d)$ satisfies (2.1)–(2.3) and $\sigma_{2k}(d) < \lambda < \sigma_{2k+1}(d)$. Since, (2.5) implies,

$$u(r\sqrt{\sigma_{2k}(d)/\lambda},\ \lambda,\ d) = u(r,\ \sigma_{2k}(d),\ d),$$

$u(\cdot,\ \lambda,\ d)$ has $2k$ zeroes in $(0,\ \sqrt{\sigma_{2k}(d)/\lambda}) \subset (0,\ 1)$. Similarly, since

$$u(r\sqrt{\sigma_{2k+1}(d)/\lambda},\ \lambda,\ d) = u(r,\ \sigma_{2k+1}(d),\ d),$$

$u(\cdot,\ \lambda,\ d)$ has $2k+2$ zeroes in $(0,\ \sqrt{\sigma_{2k+1}(d)/\lambda}) \supset (0,\ 1]$. Since $u(1,\ \lambda,\ d)=0$ we conclude that $u(\cdot,\ \lambda,\ d)$ has at most $2k+1$ zeroes in $(0,\ 1)$ which completes the proof of (2.16). From Lemma 2.1 and (2.16) we see that for any $d > \beta$ there exists at most one $\lambda_i \in (\sigma_{2k}(d),\ \ \sigma_{2k+1}(d))$ for which $u(\cdot,\ \lambda_i,\ d)$ satisfies (2.1)–(2.3) and has exactly $2k+i$ zeroes in $(0,\ 1)$ for $i=0,1$.

Suppose there exists a λ^* with $u(1,\ \lambda^*,\ d) = 0$ and $u'(1,\ \lambda^*,\ d) \neq 0$. If $u'(1,\ \lambda^*,\ d) < 0$, then by the intermediate value theorem, there exists $T \in (1,\ \ \sqrt{\sigma_{2k+1}(d)/\lambda^*})$ such that $u(T,\ \lambda^*,\ d) = 0$. Thus $u(1,\ \lambda^*T^2,\ d) = 0$ and $\lambda^* < \lambda^*T^2 < \sigma_{2k+1}(d)$. Taking $\lambda_0 = \lambda^*$ and $\lambda_1 = \lambda^*T^2$, we see that (c) holds. On the other hand, if $u'(1,\ \lambda^*,\ d) > 0$, there exists $T \in (\sqrt{\sigma_{2k}(d)/\lambda^*},\ 1)$ such that $u(T,\ \lambda^*,\ d) = 0$. Thus $u(1,\ \lambda^*T^2,\ d) = 0$ and $\sigma_{2k}(d) < \lambda^*T^2 < \lambda^*$. Taking $\lambda_0 = \lambda^*T^2$ and $\lambda_1 = \lambda^*$ we see that (c) holds. On the other hand, if such a λ^* does not exist (i.e., if (c) does not hold), then either (a) holds, or else for every $\lambda \in (\sigma_{2k}(d),\ \sigma_{2k+1}(d))$ that $u(1,\ \lambda,\ d) = 0$ we have $u'(1,\ \lambda,\ d) = 0$. Since in the latter case (see (2.4)) there can be at most one such λ, we see that (b) holds. $\square$

Lemma 2.4. *For any nonnegative integer k, there exists a differentiable function τ_k such that $\{(\lambda,\ d) : \sigma_{2k}(d) < \lambda < \sigma_{2k+1}(d)$ and $u'(1,\ \lambda,\ d) = 0\} = \{(\tau_k(d),\ d) : d > \beta\}$. If S is a connected component of solutions to (2.1)–(2.3) then $S = U \cup V$ where $U = \{(\lambda,\ d) \in S : u(\cdot,\ \lambda,\ d)$ has $2k$ zeroes in (0, 1) $\}$ and $V = \{(\lambda,\ d) \in S : u(\cdot,\ \lambda,\ d)$ has $2k+1$ zeroes in $(0,\ 1)\}$. Moreover, there exists a $\sigma : I \to \mathbb{R}$*

such that $U = \{(\sigma(d),\ d) : d \in I\}$, *with* $I = [a,\ b]$ *if* $b < \infty$ and $I = [a,\ b)$ *if* $b = \infty$ *where* $a =$ *min* $\{d : (\lambda,\ d) \in S\}$ and $b =$ *sup* $\{d : (\lambda,\ d) \in S\}$.

Proof. For each $d > \beta$, we have $u'(1,\ \sigma_{2k}(d),\ d) < 0$ and $u'(1,\ \sigma_{2k+1}(d),\ d) > 0$. Hence there exists a $\lambda \in (\sigma_{2k}(d),\ \sigma_{2k+1}(d))$ such that $u'(1,\ \lambda,\ d) = 0$. Since, from (2.6)

$$u_{\lambda\lambda}(1,\ \lambda,\ d) = u''(1,\ \lambda,\ d)/2\lambda, \tag{2.17}$$

we get that $u_{\lambda\lambda}(1,\ \lambda,\ d) > 0$ because $f(t) < 0$ if $t < \beta$. Hence such a λ is unique and we write $\lambda = \tau_k(d)$. By the implicit function theorem (see (2.17)) τ_k is a differentiable function. Moreover, if $\lambda < \tau_k(d)$ then $u'(1,\ \lambda,\ d) < 0$ and $u'(1,\ \lambda,\ d) > 0$ for $\lambda > \tau_k(d)$. Thus if u is a solution to (2.1)–(2.3) with $2k+1$ interior zeroes then $\lambda > \tau_k(d)$ whereas if u is a solution to (2.1)–(2.3) with $2k$ interior zeroes then $\lambda \leq \tau_k(d)$. Observe that if S is unbounded then $b = \infty$. We claim that for each $d \in [a,\ b]$ there exists a unique $\sigma(d) \in (\sigma_{2k}(d),\ \tau_k(d)]$ such that $(\sigma(d),\ d) \in S$ and σ is a continuous function. The uniqueness follows from Lemma 2.3. Suppose that $S \cap \{(\lambda,\ d) : \sigma_{2k}(d) < \lambda \leq \tau_k(d)\} = \phi$ for some $d \in (a,\ b)$. From Lemma 2.3 again, this implies that, $S \cap \{(\lambda,\ d) : \sigma_{2k}(d) < \lambda \leq \sigma_{2k+1}(d)\} = \phi$, which contradicts that S is connected. That σ is also defined at a, or b if $b < \infty$, and that σ is continuous follow from the continuous dependence of u on the parameters. $\square$

Proof of theorem 1

Part (a):

Let Γ be a connected compenent of $\{(\lambda,\ d)\colon u(\cdot,\ \lambda,\ d)$ is a solution to (2.1)–(2.3)$\}$. Let $(\lambda_0,\ d_0) \in \Gamma$. By (2.4) $u_0 := u(\cdot,\ \lambda_0,\ d_0)$ has (say) j zeroes in $(0,\ 1)$. Let $\Sigma = \{(\lambda,\ d) \in \Gamma : u(\cdot,\ \lambda,\ d)$ has j or $j+1$ zeroes in $(0,\ 1)\}$ if j is even, and $\Sigma = \{(\lambda,\ d) \in \Gamma : u(\cdot,\ \lambda,\ d)$ has $j-1$ or j zeroes in $(0,\ 1)\}$ if j is odd. We claim that in either case $\Gamma = \Sigma$.

We prove that Σ is both open and closed. In fact, let $(\lambda_1,\ d_1) \in \Sigma$. If $u'(1,\ \lambda_1,\ d_1) \neq 0$, by continuous dependence on parameters we see that if $(\lambda,\ d)$ is close to $(\lambda_1,\ d_1)$ then $u(\cdot,\ \lambda,\ d)$ has as many zeroes as $u(\cdot,\ \lambda_1,\ d_1)$. On the other hand, if $u'(1,\ \lambda_1,\ d_1) = 0$, from (2.1) we see that $u(\cdot,\ \lambda_1,\ d_1)$ has a local minimum at 1. Hence, since $u(0,\ \lambda_1,\ d_1) > 0$ we see that $u(\cdot,\ \lambda_1,\ d_1)$ has an even number of zeroes in $(0,\ 1)$. That is, $u(\cdot,\ \lambda_1,\ d_1)$ has $2k \in \{j-1,\ j\}$ zeroes in $(0,\ 1)$. Let $a_1 < a_2 < \cdots < a_{2k} \in (0,\ 1)$ be the critical points of $u(\cdot,\ \lambda_1,\ d_1)$. Since $d > 0$, we have that $u(a_i,\ \lambda_1,\ d_1) < 0$ if i is odd and $u(a_i,\ \lambda_1,\ d_1) > 0$ if i even. By the continuous dependence of u on parameters we see that if $(\lambda,\ d)$ is close to $(\lambda_1,\ d_1)$ then $u(\cdot,\ \lambda,\ d)$ has a zero in each interval of the form $(a_i,\ a_{i+1})$, $i = 0,\ \ldots,\ 2k$ where $a_0 = 0$ and $a_{2k+1} = 1$. In addition, $u(a_i,\ \lambda_1,\ d_1) \cdot u(a_i,\ \lambda,\ d) > 0$ for $i = 0,\ \ldots,\ 2k$. Thus $u(\cdot,\ \lambda,\ d)$ has at least $2k$ zeroes in $(0,\ 1)$. Let us see that $u(\cdot,\ \lambda,\ d)$ can not have more than $2k+1$ zeroes in $(0, 1)$. Suppose there is a sequence (λ_n, d_n) converging to (λ_1, d_1) and such that each $u(\cdot, \lambda_n, d_n)$ has at least $2k+2$ zeroes in $(0, 1)$. Hence, by taking a subsequence if necessary, we can assume that there exists $l \in \{0, 1, \ldots, 2k+1\}$ such that $u(\cdot, \lambda_n, d_n)$ has two zeroes $\alpha_n, \beta_n \in (a_l, a_{l+1})$. Since $u(a_l, \lambda_n, d_n) \cdot u(a_{l+1}, \lambda_n, d_n) \leq 0$, without loss of generality we can assume that

$$u'(\alpha_n, \lambda_n, d_n)[u(a_{l+1}, \lambda_1, d_1) - u(a_l, \lambda_1, d_1)] < 0. \tag{3.1}$$

Since $\{\alpha_n\} \subset (a_l, a_{l+1})$, without loss of generality, we can assume that $\alpha_n \to \alpha \in [a_l, a_{l+1}]$. By the continuous dependence on parameters, we have

$$u(\alpha, \lambda_1, d_1) = \lim_{n\to\infty} u(\alpha_n, \lambda_n, d_n) = 0.$$

Hence $\alpha \in (a_l, a_{l+1})$. From (3.1) we see that

$$u'(\alpha, \lambda_1, d_1)[u(a_{l+1}, \lambda_1, d_1) - u(a_l, \lambda_1, d_1)] < 0, \tag{3.2}$$

which contradicts the fact that $u(\cdot, \lambda_1, d_1)$ has exactly one zero in (a_l, a_{l+1}). Thus, for (λ, d) close to (λ_1, d_1), we conclude that $u(\cdot, \lambda, d)$ has either $2k$ or $2k+1$ zeroes in $(0, 1)$ and hence Σ is an open subset of Γ.

Also, if $\{\lambda_n, d_n\} \subset \Sigma$ converges to (λ^*, d^*), then by the continuous dependence of u on parameters, $u(\cdot, \lambda^*, d^*)$ is a solution to (2.1)–(2.3). By the definition of Σ, we can assume that for all n, $u(\cdot, \lambda_n, d_n)$ has the same number of zeroes (say j) in $(0, 1)$. If j is even, then $u(\cdot, \lambda^*, d^*)$ has j zeroes in $(0, 1)$ and hence $(\lambda^*, d^*) \in \Sigma$. If j is odd, then $u(\cdot, \lambda^*, d^*)$ has j zeroes or $j-1$ zeroes in $(0, 1)$ and hence $(\lambda^*, d^*) \in \Sigma$. This proves that Σ is closed and hence $\Gamma = \Sigma$ as claimed, which concludes the proof of part (a).

Part (b) :

Let $(\lambda_i, u_i) \in S$ with u_i having $2k+i$ zeroes in $(0, 1)$ and such that $u'(1, \lambda_i, d_0) \neq 0$. Here $d_0 = u_i(0)$, for $i = 0, 1$; that is $u_i := u(\cdot, \lambda_i, d_0)$. Suppose $S \cap \{(\lambda, d_0) : \lambda \in \mathbb{R}\} = \{(\lambda_i, d_0)\}$. Since $u_\lambda(1, \lambda_i, d_0) \neq 0$ (see (2.6)), by the implicit function theorem, there exists a differentiable function $\chi : (d_0 - \epsilon, d_0 + \epsilon) \to \mathbb{R}$ such that $u(1, \chi(d), d) = 0$ with $\chi(d_0) = \lambda_i$. Then, by the definition of χ, $S - \{(\lambda_i, d_0)\}$ is disconnected. Let us see that this is not possible.

If 0 is a regular value of $u(1, \lambda, d)$, then S is homeomorphic to either the unit circle in $\mathbb{R}^2$ or the open interval $(-1, 1)$. If S is homeomorphic to the unit circle S^1 and $h : S^1 \to S$ denotes a homeomorphism, then $S - \{(\lambda_i, d_0)\}$ is homeomorphic to $S^1 - \{h^{-1}(\lambda_i, d_0)\}$. This is a contradiction since $S - \{(\lambda_i, d_0)\}$ is not connected whereas $S^1 - \{h^{-1}(\lambda_i, d_0)\}$ is connected. On the other hand, if S is homeomorphic to $(-1, 1)$, and $S \cap \{(\lambda, d_0) : \lambda > 0\} = \{(\lambda_i, d_0)\}$, we let $h : (-1, 1) \to S$ to denote a homeomorphism with $h(0) = (\lambda_i, d_0)$. Without loss of generality, we can assume that $S \cap \{(\lambda, d_0) : \lambda \leq \lambda_i\} = h([0, 1))$. Since $\{(\lambda, d) : \beta \leq d \leq d_0, \ \sigma_{2k}(d) \leq \lambda \leq \sigma_{2k+1}(d)\}$ is compact, we see that

$\{h(n/(n+1)) : n \in \mathbb{N}\}$ has a limit point $(\hat{\lambda},\ \hat{d})$. By the continuity of u we see that $(\hat{\lambda},\ \hat{d}) \in S$. Hence $(\hat{\lambda},\ \hat{d}) = h(x)$ for some $x \in [0,\ 1)$. By the implicit function theorem $(\hat{\lambda},\ \hat{d}) \neq (\lambda_i,\ d_0)$. Thus, $(\hat{\lambda},\ \hat{d}) = h(x)$ for some $x \in (0,\ 1)$ and hence $h([x,\ 1))$ is a closed loop, which contradicts the fact that S is homeomorphic to $(-1,\ 1)$. Thus there exists a λ_j ($j \neq i$, and $i,\ j \in \{0,\ 1\}$) with $u(1,\ \lambda_j,\ d_0) = 0$ and by Lemma 2.3 (c) we have $u'(1,\ \lambda_j,\ d_0) \neq 0$.

If 0 is not a regular value, we let $\{\epsilon_n : n \geq 1\}$ to denote regular values of $u(1,\ \lambda,\ d)$ converging to 0. We let G_n to denote the connected component of $\{(\lambda,\ d) : u(1,\ \lambda,\ d) - \epsilon_n = 0\}$ containing $(\lambda_n,\ d_0)$ with λ_n converging to λ_i. Arguing as above, we see that for each n there exists $(\bar{\lambda}_n,\ d_0) \in G_n$ with $\bar{\lambda}_n \neq \lambda_n$ and $u(1,\ \bar{\lambda}_n,\ d_0) - \epsilon_n$ has $2k+j$ zeroes in $(0,\ 1)$; $j \neq i$ and $i,\ j \in \{0,\ 1\}$. Taking $\lambda_j = \lim_{n\to\infty} \bar{\lambda}_n$, we see that $(\lambda_j,\ d_0) \in S$ and u_j has $2k+j$ zeroes in $(0,\ 1)$. This completes the proof of the theorem. □

PROOF OF THEOREM 2

Part (a):

Suppose now that S_k and S'_k are two unbounded components of solutions containing $2k$ and $2k+1$ zeroes in $(0,\ 1)$. By (3.3) we know that S_k and S'_k lie between J_{2k} and J_{2k+1} and hence are bounded in the λ direction (see lemma 2.2). Thus S_k and S'_k are unbounded in the d direction and hence we can choose $\lambda_1 \neq \lambda_2$ and d such that $(\lambda_1,\ u_1) \in S_k$ and $(\lambda_2,\ u_2) \in S'_k$, where $u_i = u(\cdot,\ \lambda_i,\ d)$

for $i = 1, 2$. If both u_1 and u_2 have $2k$ zeroes in $(0,\ 1)$, then it is a contradiction to Lemma 2.1. So, we may assume that u_1 has $2k$ interior zeroes and u_2 has $2k+1$ interior zeroes. From Lemma 2.3, we have $u_1'(1) \neq 0$. Then, by part (b) of Theorem 1, there exists a $\bar{\lambda}_1 > \lambda_1$ such that $(\bar{\lambda}_1,\ \bar{u}_1) \in S_k$ with $\bar{u}_1(0) = u_2(0) = d$ and such that $\bar{u}_1 := u(\cdot,\ \bar{\lambda}_1,\ d)$ has $2k+1$ zeroes. By virtue of (2.5) $\bar{\lambda}_1 \neq \lambda_2$. This is a contradiction to Lemma 2.1 since both $(\bar{\lambda}_1,\ \bar{u}_1)$ and $(\lambda_2,\ u_2)$ satisfy (2.1)–(2.3) with $\bar{u}_1(0) = u_2(0) = d$ and $\bar{u}_1$, u_2 have $2k+1$ interior zeroes. This proves the uniqueness of the unbounded branch S_k.

Part (b):

Since the basic idea of this part of the proof can be found in Lemma 3.2 of [9], we omit the details given in [9]. By (1.3) and (1.4) we see that

$$(4.1) \qquad t^N E(t) + ((N-2)/2)t^{N-1}u(t)u'(t) \geq \lambda\{C(d)\lambda^{-N/2} + B/N\},$$

where $B \leq 0$ is a lower bound for the expression in (1.3) and $C(d) \to \infty$ as $d \to \infty$ (see (1.4)). From (4.1) we see that there exists $\bar{\lambda} > 0$ and $\bar{d} > 0$ such that if $\lambda < \bar{\lambda}$ and $d > \bar{d}$ then $u_\lambda(1,\ \lambda,\ d) \neq 0$ (see (2.5)). Thus, by the implicit function theorem, if $u(1,\ \lambda,\ d) = 0$ then there exists a differentiable function $\tau : [d-\epsilon,\ \infty) \to \mathbb{R}$ such that $u(1,\ \tau(d),\ d) = 0$. In particular, the component containing $(\lambda,\ d)$ is unbounded, which proves part (b). $\square$

Proof of theorem 3 and 4

Proof of theorem 3. Follows directly from Lemma 2.2 and the rescaling in (2.5).

To prove that for any $\lambda > 0$ there is at most one positive solution $u(r,\ \lambda,\ d)$ such that $(\lambda,\ d) \in S_0$, we concentrate our study on the variations of $u(r,\ \lambda,\ d)$ with respect to the parameters λ and d. The following lemma on the zeroes of u_d, the derivative of u with respect to d, is crucial and is an extension of lemma 3.1 of [4].

Lemma 5.1. *If $u(1,\ \lambda,\ d) = 0, u > 0$ on [0, 1), and $u_d(r,\ \lambda,\ d)$ has exactly one zero in (0, 1) then $u_d(1) < 0$.*

Proof. Let $r_0 \in (0, 1)$ be such that $u(r_0) = \beta$. Such an r_0 exists since $u(0,\ \lambda,\ d) > \theta$. Let s denote the zero of u_d in $(0,\ 1)$. Let us see that $s < r_0$. For any $\gamma \in (0,\ \beta)$, let $r_\gamma \in\ (0,\ 1)$ be such that $u(r_\gamma) = \gamma$. We define $g(t) = f(t+\gamma)$ and $w(r) = u(r) - \gamma$. Thus w satisfies (see (2.1))

$$(5.1) \qquad (r^{N-1}w')' + \lambda\, r^{N-1} g(w(r)) = 0.$$

Differentiating with respect to d we have

$$(5.2) \qquad (r^{N-1}w_d')' + \lambda\, r^{N-1} g'(w) w_d = 0,$$

$$(5.3) \qquad w_d(0) = 1 \text{ and } \ w_d'(0) = 0.$$

Multiplying (5.1) by w_d, (5.2) by w, subtracting and integrating by parts, we get

$$(5.4)\qquad {r_\gamma}^{N-1} w'(r_\gamma) w_d(r_\gamma) = \lambda \int_0^{r_\gamma} r^{N-1}(wg'(w) - g(w)) w_d \, dr.$$

Since $w'(r_\gamma) < 0$ and g is convex with $g(0) < 0$, w_d has a zero in $(0,\ r_\gamma]$. Since this holds for any $\gamma \in (0,\ \beta)$ and since $w_d = u_d$, we conclude that $s \le r_0$.

Since ψ is nondecreasing , for each $k \in [0,\ 2\psi(u(0)))$ the function $\varphi(u(r), k) := k\ f'(u(r))u(r) - (k+2)f(u(r))$ has exactly one zero $r^*(k) \in [0,\ 1]$. Indeed, r^* is a continuous function of k and is given by $k = 2\ \psi(u(r^*))$. Moreover,

$$(5.5)\qquad \varphi(u(r),\ k) < 0 \text{ for } r < r^*(k), \text{ and } \varphi(u(r),\ k) > 0 \text{ for } r > r^*(k).$$

Since $r^*(0) = r_0$ and $r^*(2\psi(d)) = 0$, by the continuity of r^*, there exists a $k^* \in [0,\ 2\psi(d))$ such that $r^*(k^*) = s$. Thus we have

$$(5.6)\qquad \varphi(u(r),\ k^*) u_d(r) \le 0 \text{ in } [0,\ 1].$$

Now, defining $v(r) = r\ u'(r) + k^* u(r)$, it can be seen that

$$(5.7)\qquad (r^{N-1}v')' + \lambda\ r^{N-1} f'(u) v = \lambda\ r^{N-1} \varphi(u(r),\ k^*).$$

Differentiating (2.1) with respect to d we get

$$(5.8)\qquad (r^{N-1}u_d')' + \lambda\ r^{N-1} f'(u) u_d = 0,$$

$$(5.9)\qquad u_d(0) = 1 \text{ and } u_d'(0) = 0.$$

Now, multiplying (5.7) by u_d, (5.9) by v, subtracting and integrating by parts, we get

$$v'(1)u_d(1) - u_d'(1)v(1) = \lambda \int_0^1 r^{N-1}\varphi(u(r),\, k^*)u_d(r)dr. \tag{5.10}$$

If we assume that $u_d(1) = 0$, then the integral in (5.10) is nonnegative, which is a contradiction to (5.6). This proves the lemma. □

Proof of Theorem 4. By Lemma 2.4 we know that $S_0 = U_0 \cup V_0$, with $U_0 = \{(\sigma(d),\, d) : d \geq a\}$. From Lemma 3.1 of [4] we know that if d is large (i.e., λ small) then $u_d(\cdot,\, \sigma(d),\, d)$ vanishes exactly once in $[0,\, 1]$ and $u_d(1,\, \sigma(d),\, d) < 0$. We claim that this holds for all $d \geq a$. If not, taking $d_1 = \inf \{d : u_d(\cdot,\, \sigma(d),\, d)$ has exactly one zero in $[0,\, 1]\}$, we see that $u_d(1,\, \sigma(d_1),\, d_1) = 0$ and $u_d(\cdot,\, \sigma(d_1),\, d_1)$ has exactly one zero in $(0,\, 1)$. Since by Lemma 5.1 this is impossible, we have $u_d(1,\, \sigma(d),\, d) < 0$ for all $d \geq a$, which proves that σ is invertible and that for any λ there exists a unique d with $(\lambda,\, d) \in S_0$ and $u(\cdot,\, \lambda,\, d) > 0$ on $(0,\, 1)$. □

REMARK 1. *The assumption that $\psi(t)$ is nondecreasing which played a significant role in lemma 5.1 is not very restrictive. It covers the model family of superlinear nonlinearities $f(t) = t^p - k$, $p > 1$, and $k > 0$. Another example which satisfies the requirement is*

$$f(t) = \begin{cases} t-1 & \text{for } t \leq 1, \\ (t^2-t)e^{1/t} & \text{for } t \geq 1. \end{cases}$$

In fact, many more functions which satisfy the requirement can be constructed.

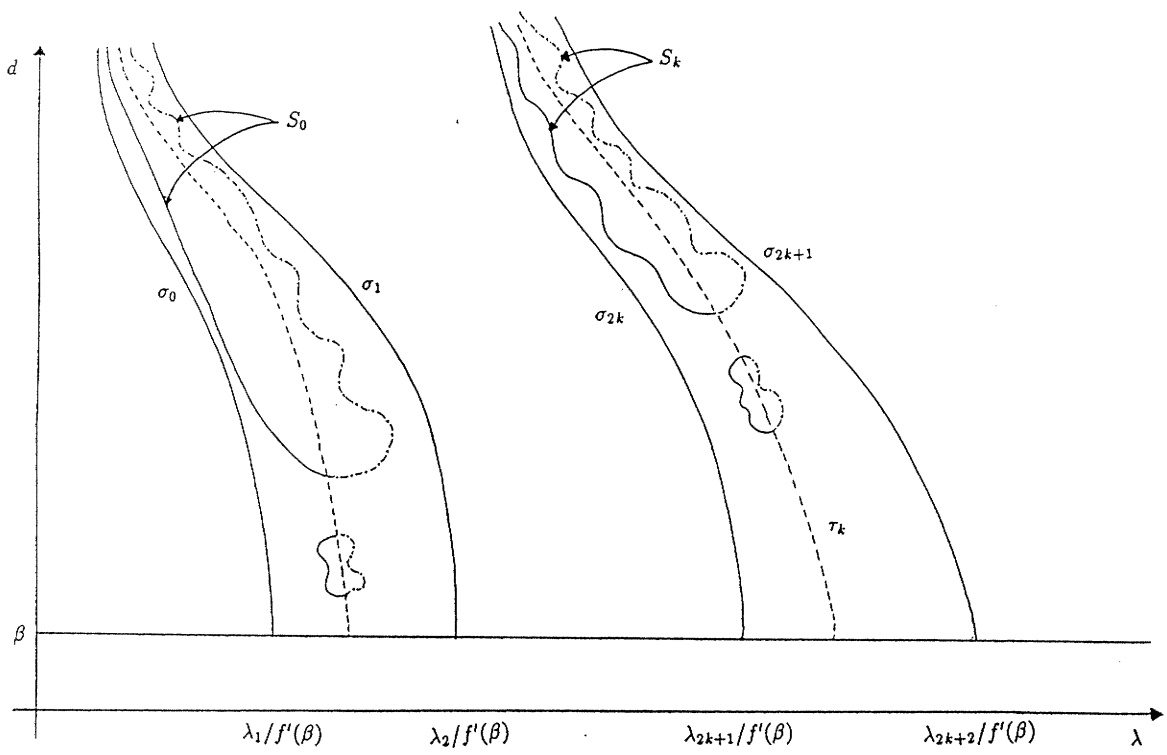

FIGURE 2. Solution branches for f convex

REMARK 2. *The question of whether or not there are bounded branches of solutions to (2.1)–(2.3) remains open.*

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CHAPTER 5

STABLE AND UNSTABLE POSITIVE SOLUTIONS FOR SEMIPOSITONE PROBLEMS

INTRODUCTION

In this chapter we consider positive solutions to the equation

$$-\Delta u(x) = \lambda f(u(x)) \quad \text{for} \quad x \in \Omega \tag{1.1}$$

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega \tag{1.2}$$

where Ω denotes the unit ball in $\mathbb{R}^N (N > 1)$, centered at the origin and $\lambda > 0$. Here $f : [0, \infty) \to \mathbb{R}$ is assumed to be monotonically increasing, concave and such that

$$f(0) < 0 \quad \text{(semipositone)}, \quad f(u) > 0 \text{ for some } u > 0. \tag{1.3}$$

Let F be defined by $F(t) = \int_0^t f(s)ds$. We let β and θ to denote the unique positive zeros of f and F, respectively. It can be easily seen that if u is a positive solution to (1.1)–(1.2) then $u(0) > \theta$. We prove that for each λ there is at most one stable and one unstable positive solution to the problem (1.1)–(1.2). If $f : \mathbb{R} \to \mathbb{R}$ is such that $uf''(u) \leq 0$ we establish the structure of radial solutions to the problem (1.1)–(1.2) by doing a similar analysis as in [2]. More precisely, we prove:

Theorem A. *If f is as above and $\lim_{t\to\infty} f'(t) = 0$, then there exist λ_1 and λ_2 with $\lambda_1 > \lambda_2$ such that for $\lambda > \lambda_1$ the problem (1.1)–(1.2) has a unique positive solution, which is stable; for $\lambda \in (\lambda_2, \lambda_1]$ it has exactly two positive solutions, one stable and one unstable; and for $\lambda = \lambda_2$ it has a unique positive solution.*

Thoerem B. *There exists $\mu_k \to \infty$ such that for each nonnegative integer k, if $\lambda < \mu_k$ then the problem (1.7)–(1.8) has no solutions with $u(0) > 0$ and such that u has exactly k zeroes in (0, 1).*

Theorem C. *If, in addition, f is convex for $u < 0$ (i.e., $uf''(u) \leq 0$) then for any $\lambda > 0$ there exists at most one solution u satisfying (1.7)–(1.8) with $u(0) >$ and such that u has exactly one zero in* (0, 1).

We note that the hypothesis $\lim_{t\to\infty} f'(t) = 0$ is also necessary for the existence of positive solutions for large values of λ (see [1, Lemma 2.3]). The case N=1 can be found in [3]. For other results in the radial case the reader is referred to [1]–[3], and [5]. In contrast with the positone case ($f(0) > 0$), Theorem A shows the nonuniqueness of positive solutions for the semipositone case. In a forthcoming paper we prove that for any $\lambda > 0$, the problem (1.1)–(1.2) has at most one positive solution when f is convex.

The positive solutions to (1.1)–(1.2) are known to be radially symmetric (see [6]). Thus the problem reduces to the study of solutions to the ordinary differential equation:

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \text{ in } (0,1), \tag{1.4}$$

$$u'(0) \; = 0 \text{ and } u(1) = 0, \tag{1.5}$$

where $'$ denotes differentiation with respect to $r = \|x\|$. Our proofs use the one-dimensional maximum principle, eigenvalue comparision arguments, and the bifurcation analysis at a degenerate solution.

Variations with respect to parameters

For any $d > 0$, let $u(r, \lambda, d)$ denote the solution to the initial value problem:

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \tag{2.1}$$

$$u'(0) = 0 \text{ and } u(0) = d. \tag{2.2}$$

Thus $u(r, \lambda, d)$ is a positive solution to (1.1)–(1.2) if, in addition, it satisfies

$$u(r, \lambda, d) > 0 \quad \text{for} \quad r \in (0,1) \quad \text{and} \quad u(1, \lambda, d) = 0 \tag{2.3}$$

Let v denote the solution to the corresponding linearized problem:

$$v'' + ((N-1)/r)v' + \lambda f'(u)v = 0 \tag{2.4}$$

$$v(0) = 1 \text{ and } v'(0) = 0. \tag{2.5}$$

Let w denote the solution to the problem:

$$w'' + ((N-1)/r)w' + \lambda f'(u)w = -\lambda f''(u)v^2 \tag{2.6}$$

$$w(0) = 0 \text{ and } w'(0) = 0 \tag{2.7}$$

That is, v is the derivative of $u(r, \lambda, d)$ with respect to d and w is the second derivative of $u(r, \lambda, d)$ with respect to d .

Lemma 1: If u is a solution to (2.1)– (2.3) then v has at most one zero in $[0, 1]$.

Proof: Let u be a solution to $(2.1) - (2.3)$ such that v changes sign in $[0,\ 1]$. Let s be the first zero of v. That is $v > 0$ in $[0, s)$ and $v(s) = 0$. Let $r_0 \in (0, 1)$ be such that $u(r_0) = \beta$. Such an r_0 exists since $u(0) > \theta$ and $u(1) = 0$. We first prove that $s \notin (0, r_0)$. Suppose, on the contrary, that $s \in (0, r_0)$. By setting

$$\varphi(r) = v(r)/f(u(r)),$$

we obtain that in $(0, s), \varphi$ satisfies

$$\varphi'' + (((N-1)/r) + (2f'(u)u'/f(u)))\varphi' + (f''(u)(u')^2/f(u))\varphi = 0, \tag{2.8}$$

$$\varphi'(0) = 0,\ \varphi(0) > 0,\ \varphi(s) = 0. \tag{2.9.}$$

Since $f(u(r)) > 0$ in $(0, s)$, and $f'' \leq 0$, by the maximum principle (see[7, Theorem 4, p7]) we conclude that φ attains its maximum at 0 and $\varphi'(0) < 0$. Since this contradicts (2.9) we see that $s \geq r_0$.

Now we rule out the possibility of v having a second zero in $[r_0, 1]$. Suppose v has two zeros in $[r_0, 1]$. Let s_1 and s_2 be the first two zeros of v. Let $t \in (s_1, s_2)$ be such that $v'(t) = 0$ and $v' > 0$ in (t, s_2). Multiplying (2.4) by u' and integrating over (t, s_2) we get

$$u'(s_2)v'(s_2) - \int_t^{s_2} v'u'' + \int_t^{s_2} ((N-1)/r)v'u' + \lambda \int_t^{s_2} f'(u)u'v = 0.$$

This, with (2.1), yields

$$u'(s_2)v'(s_2) + 2\int_t^{s_2} ((N-1)/r)v'u' + \lambda \int_t^{s_2} (f(u)v)' = 0.$$

Hence we get

$$u'(s_2)v'(s_2) - \lambda f(u(t))v(t) > 0,$$

which is a contradiction to our assumption that $t \in (r_0, 1)$. Thus v can not have a second zero in (0, 1]. This proves the lemma. □

Lemma 2: If $u(\cdot, \lambda_0, d_0)$ is a solution to (2.1)–(2.3) and $v(1, \lambda_0, d_0) = 0$, then $u_\lambda(1, \lambda_0, d_0) < 0$ and $u_{dd}(1, \lambda_0, d_0) > 0$. Moreover, there exists an $\epsilon > 0$ and a differentiable function $\Lambda : (d_0 - \epsilon, d_0 + \epsilon) \to \mathbb{R}$ such that for any $d \in (d_0 - \epsilon, d_0 + \epsilon)$, $u(\cdot, \Lambda(d), d)$ is a solution to (2.1)–(2.3), $\Lambda'(d_0) = 0$ and $\Lambda''(d_0) > 0$. In addition, if $u(1, \lambda, d) = 0$ with $|d - d_0| < \epsilon$, $|\lambda - \lambda_0| < \epsilon$ then $\lambda = \Lambda(d)$. In particular, if $|\lambda - \lambda^*| < \epsilon, |d - d^*| < \epsilon$ then $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$.

Proof: An elementary rescaling gives that

$$u(r/\rho, \lambda, d) = u(r, \lambda/\rho^2, d)$$

for all $\rho > 0$. Differentiating this with respect to ρ and evaluating at $\rho = 1$ we obtain

$$ru'(r, \lambda, d) = 2\lambda u_\lambda(r, \lambda, d). \tag{2.10}$$

By Lemma 1, $v > 0$ on [0, 1). Thus v is an eigenfunction corresponding to the smallest eigenvalue of the problem :

$$-\Delta\varphi - \lambda f'(u)\varphi = \mu_i\varphi \quad \text{in} \quad \Omega,$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega.$$

If $u'(1, \lambda, d_0) = 0$, then $\partial u/\partial x_i, 1 \le i \le N$ are also eigenfunctions corresponding to the eigenvalue $\mu_1 = 0$. Since this contradicts the fact that μ_1 is simple we have $u'(1, \lambda_0, d_0) < 0$. This and (2.10) give $u_\lambda(1, \lambda_0, d_0) < 0$.

By implicit function theorem there exists an $\epsilon > 0$ and a differentiable function $\Lambda : (d_0 - \epsilon, d_0 + \epsilon) \to \mathbb{R}$ such that for any $d \in (d_0 - \epsilon, d_0 + \epsilon)$, $u(1, \Lambda(d), d) = 0$. In particular $u(\cdot, \Lambda(d), d)$ satisfies (2.1)–(2.3). Differentiating $u(1, \Lambda(d), d) = 0$ with respect to d, we have

$$u_\lambda(1, \Lambda(d_0), d_0)\Lambda'(d_0) + u_d(1, \Lambda(d_0), d_0) = 0 \tag{2.11}$$

This, (2.10) and the assumption $v(1, \lambda_0, d_0) = 0$ imply that $\Lambda'(d_0) = 0$. Differentiating again with respect to d, we obtain

$$u_\lambda(1, \Lambda(d_0), d_0)\Lambda''(d_0) + u_{dd}(1, \Lambda(d_0), d_0) = 0, \tag{2.12}$$

where we have used that $\Lambda'(d_0) = 0$. Multiplying (2.4) by $r^{N-1}w$ and (2.6) by $r^{N-1}v$, subtracting one from the other and integrating by parts on $[0, t)$ we see that

$$t^{n-1}(v'(t)w(t) - w'(t)v(t)) = \lambda \int_0^t r^{N-1} f''(u(r))v^3(r)dr$$

Now since by Lemma 1 $u_d(r, \lambda_0, d_0) > 0$ on $[0, 1)$, it follows that $u_{dd}(1, \lambda_0, d_0) > 0$. This, in turn, implies that $\Lambda''(d_0) > 0$ which proves the lemma. □

Lemma 3: If $\Gamma \subset \mathbb{R}^2$ is a component of $\{(\lambda, d) : u(\cdot, \lambda, d)$ is a solution to (2.1)–(2.3)$\}$, then Γ is unbounded in the λ direction.

Proof: Let $(\lambda_0, d_0) \in \Gamma$. We distinguish three cases, namely $u_d(1, \lambda_0, d_0) > 0, u_d(1, \lambda_0, d_0) = 0$ and $u_d(1, \lambda_0, d_0) < 0$.

If $u_d(1, \lambda_0, d_0) > 0$, by the implicit function theorem there exists a $\delta > 0$ and an increasing differentiable function $\zeta : (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$ such that

$$u(1, \lambda, \zeta(\lambda)) = 0 \tag{2.13}$$

Hence $u(\cdot, \lambda, \zeta(\lambda))$ satisfies (2.1)–(2.3). We claim that ζ defined by (2.13) can be extended to (λ, ∞). In fact, suppose $\lambda^* = \sup \{\lambda : \zeta$ can be extended to (λ_0, λ) with $u(\cdot, s, \zeta(s))$ satisfying (2.1)–(2.3)$\} < \infty$. Letting $d^* := \sup \{\zeta(\lambda) : \lambda < \lambda^*\}$, we see that $u_d(1, \lambda^*, d^*) = 0$. Taking ϵ as in Lemma 2 we have a contradiction because if $|\lambda - \lambda^*| < \epsilon, |d - d^*| < \epsilon$ are such that $u(\cdot, \lambda, d)$ satisfies (2.1)–(2.3) then $\lambda > \lambda^*$. Thus $\lambda^* = \infty$, which proves that Γ is unbounded in the λ direction.

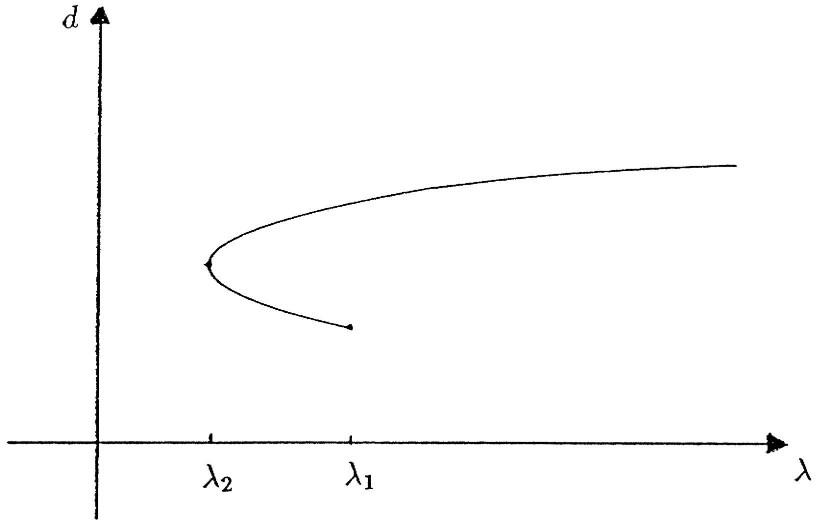

FIGURE 3. Branch of positive solutions

If $u_d(1, \lambda_0, d_0) = 0$, applying Lemma 2 we see that $u(\cdot, \Lambda(d_0 + \epsilon/2), d_0 + \epsilon/2)$ is in Γ and $u_d(1, \Lambda(d_0 + \epsilon/2), d_0 + \epsilon/2) > 0$. Thus, by the arguments in the latter paragraph, we conclude that Γ is unbounded in the λ direction.

Finally, if $u_d(1, \lambda_0, d_0) < 0$, then by the implicit function theorem there exists a $\delta > 0$ and a decreasing differentiable function $\eta : (\lambda_0 - \delta, \lambda_0 + \delta) \to \mathbb{R}$ such that $u(1, \lambda, \eta(\lambda)) = 0$ and $u(\cdot, \lambda, \eta(\lambda))$ satisfies (2.1)–(2.3). Let $\lambda^* = \inf \{\lambda : \eta$ can be extended to (λ, λ_0) with $u(\cdot, s, \eta(s))$ satisfying (2.1)–(2.3)$\}$. Letting $d^* = \sup \{\eta(\lambda) : \lambda < \lambda_0\}$ we see that $u(\cdot, \lambda^*, d^*)$ satisfies (2.1)–(2.3). Because $f(0) < 0$ for $\lambda > 0$ near zero the problem (2.1)–(2.3) does not have a solution. In particular, $\lambda^* > 0$ and $d^* < \infty$. Since $u_d(1, \lambda^*, d^*) = 0$, arguing as in the previous case, we see that Γ is unbounded in the λ direction which proves the lemma. $\square$

PROOF OF THEOREM A

First we show that $\Gamma = \{(\lambda, d) : u(\cdot, \lambda, d)$ satisfies (2.1)–(2.3)$\}$ is connected. In fact, if Γ_1 and Γ_2 are two connected components then by Lemma 3 both contain elements of the form (λ, d) with $\lambda > 0$ large. However, by Theorem A of [1] for λ large (2.1)–(2.3) has a unique solution. Since this contradicts that Γ_1 and Γ_2 are disjoint, we have $\Gamma_1 = \Gamma_2 = \Gamma$.

Next, we show that there exists a unique $(\lambda_2, d_2) \in \Gamma$ such that $u_d(1, \lambda, d) = 0$. Suppose, on the contrary, that there exist (λ_2, d_2) and (λ_2', d_2') on Γ with $v(1, \lambda_2, d_2) = v(1, \lambda_2', d_2') = 0$. Let $J = \gamma([0,1])$ be an arc on Γ connecting (λ_2, d_2) and (λ_2', d_2') with $\gamma(0) = (\lambda_2, d_2)$ and $\gamma(1) = (\lambda_2', d_2')$. Let γ_1 and γ_2 denote the components of γ. Because J is compact, there are only finitely many points (λ, d) in J with $v(1, \lambda, d) = 0$. Let $t_1 = \min\,\{t \in (0,1) : v(1, \gamma_1(t), \gamma_2(t)) = 0\}$ and let $\gamma(t_1) = (\lambda_2'', d_2'')$. Then for $t \in (0, t_1)$ either $v > 0$ or $v < 0$. Let us assume, for definiteness, that for $t \in (0, t_1)$ we have $v > 0$. On $\gamma((0, t_1))$, by the implicit function theorem, we have $d = \zeta(\lambda)$, with $\zeta' > 0$. Hence $\lambda_2 \neq \lambda_2''$. Suppose $\lambda_2 < \lambda_2''$. Then taking $\lambda_n \to \lambda_2''$ with $\lambda_n < \lambda_2''$ we see that $(\lambda_n, \zeta(\lambda_n)) \to (\lambda_2'', d_2'')$ which contradicts Lemma 2. Similarly, $\lambda_2 > \lambda_2''$ also leads to a contradiction. Thus, we conclude that there exists a unique $(\lambda_2, d_2) \in \Gamma$ such that $u_d(1, \lambda, d) = 0$.

Now we prove that for each $\lambda > 0$ the problem (2.1)–(2.3) has at most one stable and one unstable solution. Suppose not. Let (λ_0, d_0) and (λ_0, d_0') be two points in Γ such that $v(1,\lambda_0,d_0)\cdot v(1,\lambda_0,d_0') > 0$. Let $K = \psi([0,1])$ be a path in Γ connecting (λ_0,d_0) and (λ_0,d_0'); we also let ψ_1,ψ_2 denote the components of ψ. Without loss of generality we can assume that ψ is one to one. Because $v(1,\lambda_0,d_0)\cdot v(1,\lambda_0,d_0') > 0$, and ψ is one to one we see that there exists an $\epsilon > 0$ such that $\psi_1((0,\epsilon)) \subset (0,\lambda_0)$ or $\psi_1((0,\epsilon)) \subset (\lambda_0,\infty)$. Let us assume that $\psi_1((0,\epsilon)) \subset (0,\lambda_0)$; the other case can be treated in a similar way. Let t_1 be such that $\psi_1(t_1) = \min\ \{\psi(t) : t \in (0,1)\}$. Hence $0 < \psi_1(t_1) < \lambda_0$. By the implicit function theorem $v(1,\psi_1(t_1),\psi_2(t_1)) = 0$. By Lemma 2 there exist t_2 and t_3 near t_1 such that $v(1,\psi_1(t_2),\psi_2(t_2))\cdot v(1,\psi_1(t_3),\psi_2(t_3)) < 0$ and $t_2 < t_1 < t_3$. Without loss of generality we can assume that $v(1,\psi_1(t_2),\psi_2(t_2)) > 0$. By the intermediate value theorem there exists $t_4 \in (0,t_2)$ such that $v(1,\psi_1(t_4),\psi_2(t_4)) = 0$ which contradicts that Γ contains only one point (λ,d) with $v(1,\lambda,d) = 0$. This contradiction shows that for each λ the problem (2.1)–(2.3) has at most one stable and one unstable solution. $\square$

Branches of radial solutions

The bifurcation analysis here is on the same lines as it was for the case when f is convex(see Chapter 4). The only difference here is that the unbounded branches of radial solutions are unbounded both in the λ and in the d direction. Also, there are no bounded branches of positive solutions for any semipositone concave nonlinearity. Moreover, it can be easily seen that there is no wiggling on

S_0. i.e., we prove that $u_d(1)$ does not become zero if u has exactly one interior zero(see Lemma 4.2 below). Since most of the analysis is very similar to the one done in Chapter 4, we do not repeat it here.

Lemma 4.1. *For each nonnegative integer k there exists a differentiable function $\sigma_k : (\beta,\ \infty) \to \mathbb{R}$ such that $u(1,\ \sigma_k(d),\ d) - \beta = 0$ and $u(\cdot,\ \sigma_k(d),\ d) - \beta$ has k zeroes in (0, 1). Moreover, $\lim_{d\to\infty} \sigma_k(d) = \infty$ and $\lim_{d\to\beta} \sigma_k(d) = \lambda_{k+1}/f'(\beta)$. Conversely, if $u(1,\ \lambda,\ d) = \beta$ and $u(\cdot,\ \lambda,\ d) - \beta$ has k zeroes in (0, 1) then $\lambda = \sigma_k(d)$.*

Proof. The proof of this lemma follows on the same lines as in Lemma 2.2 of chapter 4.

Lemma 4.2. *If $u(\cdot,\ \lambda,\ d)$ is a solution to (2.1)–(2.3) and has exactly one zero, say, $t_1 \in (0,\ 1)$, then $u_d(\cdot,\ \lambda,\ d)$ has exactly one zero in $(0,\ 1)$ and $u_d(1) \neq 0$.*

Proof. Let $t_2 \in (0,\ 1)$ be such that $u'(t_2) = 0$. We first prove that u_d has exactly one zero in $(0,\ t_2)$. Suppose, on the contrary that, $s_1, s_2 \in (0,\ t_2)$ are the first two zeroes of u_d. From Lemma 1, we know that $s_1 \in (r_0,\ t_1)$ where $u(r_0) = \beta$. From (2.1) we see that u_d satisfies

$$(r^{N-1}u_d')' + \lambda r^{N-1} f'(u) u_d = 0 \tag{4.1}$$

and u_λ satisfies

$$(r^{N-1}u_\lambda')' + \lambda r^{N-1} f'(u) u_\lambda = -r^{N-1} f(u). \tag{4.2}$$

Multiplying (4.1) by u_λ and (4.2) by u_d, subtracting and integrating over $(s_1,\ s_2)$ we get

$$(4.3)\qquad s_2^{N-1}u_d'(s_2)u_\lambda(s_2) - s_1^{N-1}u_d'(s_1)u_\lambda(s_1) = \int_{s_1}^{s_2} r^{N-1}f(u(r))u_d(r)dr.$$

This leads to a contradiction since the left side is negative and the integral on the right side is positive. Thus u_d has exactly one zero in $[0,\ t_2]$.

If s_2 in $(t_2,\ 1)$ is the second zero of u_d in $(0,\ 1)$, then because $u_\lambda(1) > 0$ and $\sigma_1(d) \to \infty$ as $d \to \infty$, it has to have at least one more zero, say s_3, in $(t_2,\ 1]$. A simple computation shows that $z = f(u)$ satisfies

$$(4.4)\qquad (r^{N-1}z')' + \lambda r^{N-1}f'(u)z = -r^{N-1}f''(u)u'^2.$$

Multiplying (4.1) by z and (4.4) by u_d, subtracting and integrating over $(s_2,\ s_3)$ we get

$$(4.5)\quad s_3^{N-1}u_d'(s_3)f(u(s_3)) - s_2^{N-1}u_d'(s_2)f(u(s_2)) = -\int_{s_2}^{s_3} r^{N-1}f''(u)u'^2u_d dr.$$

This leads to a contradiction since the left side is positive and the right side is negative.

Proof of Theorem B. From Lemma 4.1, we have $\lim_{d\to\infty}\sigma_k(d) = \infty$ and $\lim_{d\to\beta}\sigma_k(d) = \lambda_{k+1}/f'(\beta)$. If (λ, d) is such that $u(\cdot,\ \lambda,\ d)$ satisfies (2.1)–(2.3) and has exactly k zeroes in $(0,\ 1)$ then from the rescaling in Lemma 2, we obtain that $\lambda \in (\sigma_{2k}(d),\ \sigma_{2K+1}(d))$ and since σ_k's are bounded below, the existence of μ_k's follows. Thus the theorem is proven.

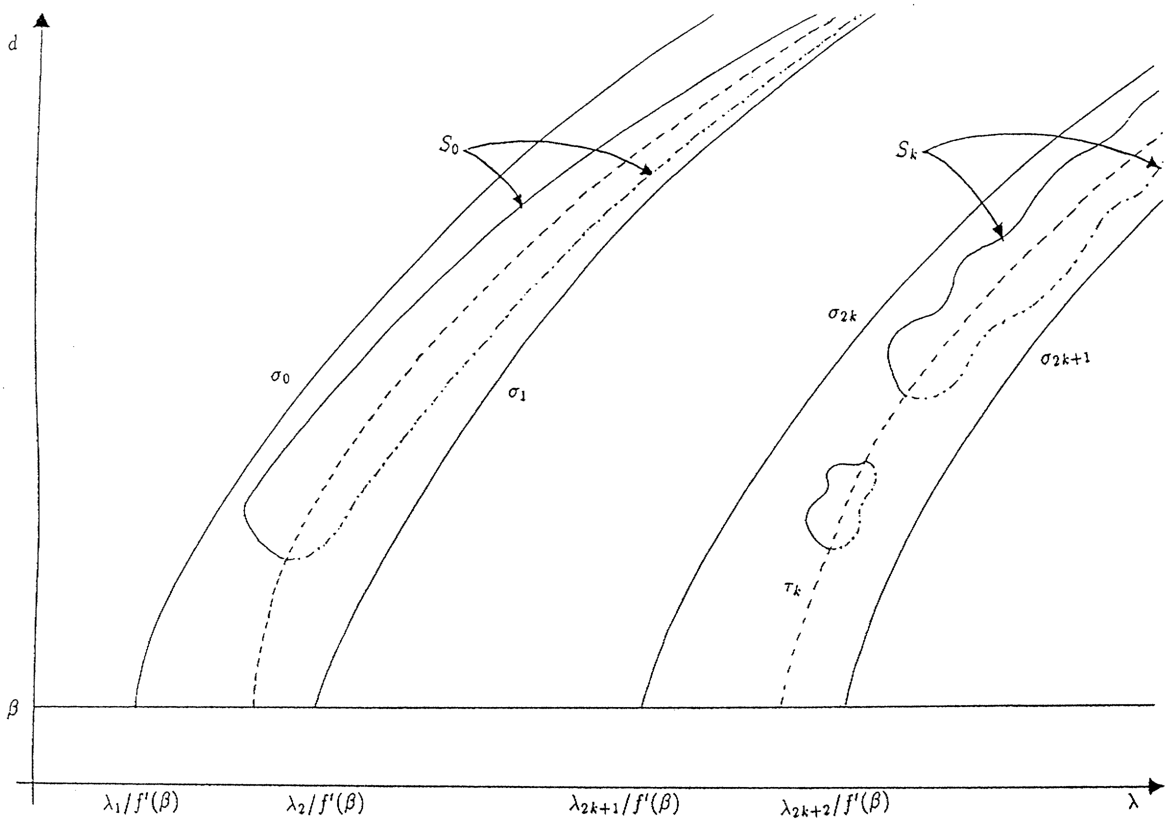

FIGURE 4. Solution branches for f concave

Proof of Theorem C. From Theorem A, we infer that there is exactly one branch $S_0 := \{(\lambda,\ u) : (\lambda,\ u)$ satisfies (2.1)–(2.3), u radial and $u > 0$ or u has exactly one zero in $(0,\ 1)\}$(see Theorem 2 in [2]). Also, from Lemma 2.4 of [2] we can write $S_0 = U_0 \cup V_0$ where $U_0 = \{(\lambda,\ u) : (\lambda,\ u)$ satisfies (2.1)–(2.3), u radial and $u > 0$ in $[0,\ 1)\}$ and $V_0 = \{(\lambda,\ u) : (\lambda,\ u)$ satisfies (2.1)–(2.3), u radial and u has exactly one zero in $[0,\ 1)\}$. From (2.10), since $f(0) < 0$, we see that $u_\lambda(1) > 0$. Hence by the implicit function theorem there exists a differentiable function $\Lambda : (a,\ \infty) \to (b,\ \infty)$ such that $u(1, \Lambda(d), d) = 0$ for $d \in (a,\ \infty)$ where $a = \min\ \{d : (\lambda,\ d) \in S_0\}$. This Λ can be extended as a continuous function to

$[a,\ \infty)$. Now, by the Lemma 4.2 above, since $u_d(1, \Lambda(d), d) \neq 0$ for $d \in [a,\ \infty)$ we conclude that Λ is invertible, which proves the uniqueness of radial solutions with one interior zero. $\square$

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CHAPTER 6

UNIQUENESS OF POSITIVE SOLUTIONS IN SEMIPOSITONE PROBLEMS

INTRODUCTION

We consider the positive solutions to the semilinear equation:

$$-\Delta u(x) = \lambda f(u(x)) \quad \text{for} \quad x \in \Omega, \tag{1.1}$$

$$u(x) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{1.2}$$

where Ω denotes the unit ball in $\mathbb{R}^N (N > 1)$ centered at the origin and $\lambda > 0$. Here $f : [0, \infty) \to \mathbb{R}$ is assumed to be monotonically increasing, concave and such that

$$f(0) < 0 \quad \text{(semipositone)}, \quad f(u) > 0 \text{ for some } u > 0. \tag{1.3}$$

We define F by $F(t) = \int_0^t f(s)ds$ and let β and θ denote the unique positive zeros of f and F, respectively.

Since positive solutions to (1.1)–(1.2) are known to be radially symmetric, we restrict ourselves to the radial solutions of (1.1)–(1.2). We first note that radial solutions to (1.1)–(1.2) correspond to solutions to the singular problem:

$$u'' + ((N-1)/r)u' + \lambda f(u) = 0 \quad \text{for } r \in [0,\ 1], \tag{1.4}$$

$$u'(0) = 0, \tag{1.5}$$

$$u(1) = 0, \tag{1.6}$$

where $'$ denotes the differentiation with respect to $r = \|x\|$. For $d > 0$ we define $u(\cdot,\ \lambda,\ d) := u(\cdot)$ as the solution to (1.4), (1.5) and $u(0) = d$. It can be easily seen that if u is a positive solution to (1.4)–(1.6) then $u(0) > \theta$.

In chapter 5 we obatined that for each λ there is at most one stable and one unstable positive solution to the problem (1.1)–(1.2) if, in addition, we have $f'(\infty) := \lim_{t\to\infty} f'(t) = 0$. We note that this hypothesis is also necessary for the existence of positive solutions for large values of λ. It also implies that $d \to \infty$ as $\lambda \to \infty$ (see [4]). In fact, if $f'(\infty) := \lim_{t\to\infty} f'(t) = 0$ we obtain that if λ belongs to a certain interval then (1.4)–(1.6) has a unique stable positive solution and a unique unstable positive solution. Here we study how the existence, multiplicity and the stability of positive solutions could drastically change if $f'(\infty) > 0$ by considering various classes of nonlinearities f. That is, we assume

$$f'(\infty) := \lim_{t\to\infty} f'(t) > 0. \tag{1.7}$$

Our main result is

Theorem 1. *Let* $S_0 := \{(\lambda,\ u) \in (0,\ \infty) \times \mathcal{C}(\bar{\Omega}) : u > 0$ *in* Ω *and* u *satisfies (1.1)–(1.2)*$\}$. *If* f *satisfies (1.7) and is such that* $f(t)/t$ *is nondecreasing then*

(i) S_0 *is connected and bounded in the* λ*–direction.*

(ii) *for any* $\lambda > 0$ *the problem (1.1)–(1.2) has at most one positive solution.*

Our proofs use the one-dimensional maximum principle, eigenvalue comparison arguments, and the bifurcation analysis at a degenerate solution. For several results about the solutions of (1.1)–(1.2) the reader is referred to [1]–[7].

PRELIMINARY LEMMAS

We denote the derivatives of u with respect to λ and d by u_λ and u_d respectively. Using a rescaling and the uniqueness of the solution to an initial value problem, we obtain

$$u(r\rho,\ \lambda,\ d) = u(r,\ \lambda\rho^2,\ d). \tag{2.1}$$

A differentiation of (2.1) with respect to ρ results in

$$u_\lambda(r,\ \lambda,\ d) = ru'(r,\ \lambda,\ d)/2\lambda. \tag{2.2}$$

Differentiating (1.4) with respect to d we see that u_d satisfies the corresponding linearized problem:

$$u_d'' + ((N-1)/r)u_d' + \lambda f'(u)u_d = 0, \tag{2.3}$$

$$u_d(0) = 1 \text{ and } u_d'(0) = 0. \tag{2.4}$$

The following result on the zeroes of u_d is from [1]. We include it here as a lemma for the sake of completeness.

Lemma 2.1. *If u is a positive solution to (1.4)–(1.6) then u_d has at most one zero in $[0,1]$.*

Lemma 2.2. *Let $(\lambda_0,\ d_0)$ be such that $u(\cdot,\ \lambda_0,\ d_0)$ is a positive solution to (1.4)–(1.6) satisfying $u_0'(1) = 0$. If $(\lambda,\ d)$ is such that $u(\cdot,\ \lambda,\ d)$ is a positive solution to (1.4)–(1.6) with $\lambda > \lambda_0$ then $d > d_0$.*

Proof. Let $\lambda_1 > \lambda_0$ and $u(\cdot,\ \lambda_1,\ d_1)$ be a positive solution to (1.4)–(1.6). Defining $u_i(r) := u(r/\sqrt{\lambda_i},\ \lambda_i,\ d)$, from (2.2) we infer that

$$u_i'' + ((N-1)/r)u_i' + f(u_i) = 0, \tag{2.5}$$

$$u_i'(0) = 0,\ \ u_i(\sqrt{\lambda_i}) = 0, \tag{2.6}$$

for $i = 0,\ 1$. Let $d_0 > d_1$. We first prove that u_0 and u_1 can not meet above β–level. For, let $u_0(r) > u_1(r)$ for $r \in [0,\bar{r})$ and let $u_0(\bar{r}) = u_1(\bar{r}) = a > \beta$. Thus $u_0'(\bar{r}) < u_1'(\bar{r}) < 0$. Since $u_0(r) > u_1(r) > \beta$ for $r \in [0,\bar{r})$, the concavity of f gives

$$(u_1(r)-\beta)f(u_0(r)) \leq (u_0(r)-\beta)f(u_1(r)) \ \text{ on } [0,\bar{r}).$$

$$\text{i.e., } (u_1(r)-\beta)(r^{N-1}u_0')' \geq (u_0(r)-\beta)(r^{N-1}u_1')' \ \text{ on } [0,\bar{r}).$$

$$\text{i.e., } [(u_1(r)-\beta)r^{N-1}u_0']' \geq [(u_0(r)-\beta)r^{N-1}u_1']' \ \text{ on } [0,\bar{r}).$$

Integrating this over $(0,\ \bar{r})$ we get $u_0'(\bar{r}) \geq u_1'(\bar{r})$, which is a contradiction. Now, let $u_1(r) > u_0(r)$ for $r \in (\bar{r},\ \sqrt{\lambda_0})$ and $u_0(\bar{r}) = u_1(\bar{r}) = a \leq \beta$. Thus $u_0'(\bar{r}) < u_1'(\bar{r}) < 0$. Multiplying (2.5) by $r^{2N-2}u_i'$ and integrating over $(\bar{r},\ \sqrt{\lambda_i})$ we obatin

$$\sqrt{\lambda_i}^{2N-2}\left(u_i'(\sqrt{\lambda_i})\right)^2 - \bar{r}^{2N-2}(u_i'(\bar{r}))^2 = 2\int_0^a r_i^{2N-2}(u)f(u)du$$

where $r_i(u)$ represents the inverse function to u_i. (i.e., $r_i : [0,\ d_i] \to [0,\ \sqrt{\lambda_i}]$ with $u_i(r_i(u)) = u$ for $0 \leq u \leq d_i, i = 0,\ 1$.) This, in turn, implies that

$$\begin{aligned} 2\int_0^a [r_1^{2N-2}(u) - r_0^{2N-2}(u)]f(u)du = &\sqrt{\lambda_1}^{2N-2}\left(u_1'(\sqrt{\lambda_1})\right)^2 \\ &+ \bar{r}^{2N-2}[(u_0'(\bar{r}))^2 - (u_1'(\bar{r}))^2]. \end{aligned}$$

(Note that here we have used $u_0'(\sqrt{\lambda_0}) = 0$.) This is a contradiction since the left side of the above equation is positive and the right side is negative. Thus we conclude that $d_1 > d_0$ and hence the lemma is proven. □

Remark 2.1. *Lemma 2.1 and 2.2 hold for any semipositone concave nonlinearity f. In fact, in Lemma 2.2 we have proved that if $\lambda_0 < \lambda_1$ then $u_0(r) < u_1(r)$ in $[0,\ \sqrt{\lambda_0}]$.*

Let μ_i denote the eigenvalues of the problem:

$$\phi'' + ((N-1)/r)\phi' + \mu\phi = 0 \ \text{ in } (0,\ 1), \tag{2.7}$$

$$\phi'(0) = 0 \text{ and } \phi(1) = 0. \tag{2.8}$$

Using a comparision argument and the rescaling in (2.1) we obtain the following nonexistence result.

Lemma 2.3. *If f satisfies (1.7) then (1.4)–(1.6) does not have positive solutions for λ large.*

Proof. We extend f to the left of 0 in such way that $f''(t) \leq 0$ for all $t \in \mathbb{R}$. Let $(\lambda,\ d)$ be such that $u(\cdot,\ \lambda,\ d)$ is a positive solution to (1.4)–(1.6). Using (2.1) we can choose $\zeta > \lambda$ such that $u(1,\ \zeta,\ d) = \beta$ ans $u(\cdot,\ \zeta,\ d)$ has exactly two zeroes in $(0,\ 1)$. Thus $v(r) := u(\cdot,\ \zeta,\ d) - \beta$ satisfies

$$v'' + ((N-1)/r)v' + \zeta\left(\frac{f(u) - f(\beta)}{u - \beta}\right)v = 0, \tag{2.9}$$

$$v'(r) = 0 \text{ and } v(1) = 0, \tag{2.10}$$

and v has exactly one zero in $(0,\ 1)$. Comparing this with (2.7) for $\mu = \mu_2$, by Sturmian theory we conclude that there exists an $r \in (0,\ 1)$ such that

$$\zeta\left(\frac{f(u(r)) - f(\beta)}{u(r) - \beta}\right) < \mu_2. \tag{2.11}$$

Since $\frac{f(u(r))-f(\beta)}{u(r)-\beta} = f'(a)$ for some a and hence is bounded below by $cf'(\infty)$ for some constant c. This with (2.11) gives that

$$\lambda < \zeta < \mu_2/cf'(\infty)$$

and hence the lemma is proven. □

Lemma 2.4. *If f satisfies (1.7) and is such that $f(t)/t$ is nondecreasing then (1.4)–(1.6) has at most one positive solution with $u'(1, \lambda, d) = 0$.*

Proof. Let $u(\cdot, \lambda_i, d_i)$ be positive solutions to (1.4)–(1.6) satisfying $u'(1, \lambda_i, d_i) = 0$ for $i = 0, 1$. From the uniqueness of solution to the initial value problem (1.4) subjected to the conditions $u(1) = u'(1) = 0$ it follows that $\lambda_0 \neq \lambda_1$. Without loss of generality, we may assume that $\lambda_0 < \lambda_1$. Defining $u_i(r) := u(r/\sqrt{\lambda_i}, \lambda_i, d)$, from Remark 2.1, we have $u_0(r) < u_1(r)$ for $r \in [0, \sqrt{\lambda_0}]$. Since $f(t)/t$ is nondecreasing, $f(u_0(r))/u_0(r) \leq f(u_1(r))/u_1(r)$ and hence by Sturm's comparision theorem u_2 must have a zero in $(0, \sqrt{\lambda_0})$ which is a contradiction. This proves the lemma. □

Lemma 2.5. *If f is as in Lemma 2.4 and $u(\cdot, \lambda, d)$ is a positive solution to (1.4)–(1.6) then u is unstable. Moreover, u_d has exactly one zero in* (0, 1) *and* $u_d(1) < 0$.

Proof. Multiplying (1.4) by $r^{N-1}u_d$ and (2.3) by $r^{N-1}u$, subtracting and integrating by parts we get

$$u'(1)u_d(1) + \int_0^1 (f'(u(r))u(r) - f(u(r)))u_d(r)u^{N-1}dr = 0.$$

Since $f'(u)u - f(u) > 0$ (by the monotonicity of $f(t)/t$) and $u'(1) \leq 0$ we conclude that u_d has at least one zero in (0, 1) and hence u is unstable. By Lemma 2.1 u_d can have at most one zero in [0, 1]. Therefore, u_d has exactly one zero in (0, 1) and $u_d(1) < 0$. This proves the lemma. □

PROOF OF THEOREM 1

Let $S \subset \{(\lambda,\ u) \in \mathbb{R} \times \mathcal{C}(\bar{\Omega}) : (\lambda,\ u)$ satisfies (1.4)–(1.6)$\}$. We note that S is connected iff $\{(\lambda,\ u(0)) : (\lambda,\ u) \in S\}$ is connected. This follows from the continuous dependence of solutions to (1.4)–(1.6) on the initial conditions. *We identify S with the latter subset of $\mathbb{R}^2$.* From Lemma 2.4 there exists a unique $(\lambda_0,\ d_0)$ such that $u'(1,\ \lambda_0,\ d_0) = 0$ and hence S_0 is connected and unbounded (see [2] where a similar argument is used). From Lemma 2.3 we conclude that S_0 is bounded in the $\lambda-$direction and hence unbounded in the $d-$direction. Since $u_\lambda(1,\ \lambda,\ d) \neq 0$ for any $(\lambda,\ d) \in S_0$ with $\lambda \neq \lambda_0$(see 2.2), by the implicit function theorem there exists a continuous function $\sigma : I \to \mathbb{R}$ such that $S_0 = \{(\sigma(d),\ d) : d \in I\}$, with $I = [d_0,\ \infty)$. Since, by Lemma 2.5, $u_d(1,\ \sigma(d),\ d)$ does not vanish for $d \geq d_0$, we conclude that σ is invertible. Thus, for any λ there is a unique d with $(\lambda,\ d) \in S_0$ and this proves the theorem. □

Remark 2. *When f is concave and $f'(\infty) = 0$ the results in [4] show that there is also only one branch of positive solutions, which is also unbounded. For a class of semipositone convex nonlinearities the existence of bounded branches of positive solutions is ruled out in [2].*

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