MINIMIZATION OF A NONLINEAR ELASTICITY FUNCTIONAL USING STEEPEST DESCENT

DISSERTATION

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The method of steepest descent is used to minimize typical functionals from elasticity. These functionals are coercive and bounded below but are not defined everywhere on an appropriate Sobolev space $H_0^s(\Omega)$ or product of such spaces. For the case $\Omega = [a,b]$, suppose $w \in H^2(\Omega)$, $u \in H_0^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ $w(a) < w(b)$, and $u'+w' > 0$ on $[a,b]$. The function $u + w$ is called a deformation of the interval $[a,b]$. The functional I is defined as follows:

$$I(u) = \int_a^b \left[[u'(x)+w'(x)]^2 + \frac{1}{|u'(x)+w'(x)|} \right] dx$$

$$= \int_a^b W(x,u'(x))\, dx$$

where W is the stored-energy, $I(u)$ is the total stored-energy and the potential energy is assumed to be zero. We are interested in minimizing I while preserving the boundary conditions of w. In the case $\Omega = [a,b]$ and $s = 2$, I is convex; but for bounded regions Ω in $\mathbb{R}^2$ with appropriate smoothness on $\partial\Omega$ and $s = 3$, the corresponding I is not convex. The choice of s in both cases is made to insure I is Fréchet differentiable at many points in $H_0^2(\Omega)$

and $H_0^2(\Omega)xH_0^2(\Omega)$ respectively.

The following results are proved. The gradient ∇I and a "modified gradient" Q are derived. It is shown that Q is locally Lipschitz, Q(u) is a descent direction and Q(u) = 0 if and only if $\nabla I(u) = 0$. The existence of solutions y and z respectively to the equations $y'(t) = -\nabla I(y(t))$ with $y(0) = u_0$ and $z'(t) = -Q(z(t))$ with $z(0) = u_0$, is proved for $t \in [0,\delta)$ where $\delta > 0$ and $u_0' + w' > 0$ on Ω. If z exists for all t in $[0,\infty)$, then $\lim_{t\to\infty} z(t)$ exists and is a local minimum of I. If $\Omega \subset \mathbb{R}^2$, if z exists for all t in $[0,\infty)$ and if the range of z is bounded in $H_0^3xH_0^3$, then there is an increasing sequence (t_i) of numbers such that $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} \| Q(z(t_i)) \| = 0$, and $\lim_{i\to\infty} z(t_i) = \phi$ exists in $H_0^1(\Omega)$ where $\phi \in H_0^3(\Omega)$. Under certain conditions, $\nabla I(\phi) = 0$.

Finally, this steepest descent process is implemented in a True Basic code for the $\mathbb{R}^2$ case and graphical output is displayed.

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CHAPTER I

INTRODUCTION

Solutions to many mathematical problems involve finding a minimum or critical point of a functional. Such functionals are often derived as variational principles for differential equations. Proving the existence of critical points can be difficult and has required the development of many sophisticated mathematical methods, such as the "mountain pass lemma" [14, pp. 7-20] and the theory developed for convex functionals, earlier by Rosenbloom [16] and later much more extensively by Brezis and others [5].

However, some problems involve functionals which are nonlinear, not defined everywhere and not convex. Such functionals arise in elasticity. In this paper, a typical functional I from elasticity will be investigated, where $I(\phi)$ is the total stored-energy for the deformation ϕ of the set $\Omega \subset \mathbb{R}^n$. The study will be confined to $\mathbb{R}^1$ and $\mathbb{R}^2$, where I is convex in $\mathbb{R}^1$ and I is not convex in $\mathbb{R}^2$. A constructive method using steepest descent will be used with an emphasis on applicability to computing. The setting will be Sobolev spaces (see Adams, [1]). Great use will be made of the Sobolev gradients, an idea used by Neuberger [10, 11, 12, 13], Richardson [15] and Glowinski, Keller and Reinhart [8].

Because many of the obstacles to overcome are present in the one dimensional case, the key ideas will be developed and implemented for $\Omega = [0,1]$ in this introduction and Chapter II. The more complicated, nonconvex case will be dealt with in Chapter III.

Let $H^2(\Omega)$ denote the Sobolev space $H^{2,2}(\Omega)$ of square integrable functions on the closed, bounded region Ω with L^2 second derivatives. When it is clear, $H^2(\Omega)$ will be written as H^2. Similarly, let H_0^2 denote the subspace $H^2 \cap H_0^1$ of H^2 which contains all the elements of H^2 which satisfy zero boundary conditions. Denote by $C(\Omega)$ the space of continuous functions on Ω and by $C^1(\Omega)$ the subspace of $C(\Omega)$ with continuous first derivatives.

For the remainder of this chapter, Ω is assumed to be $[0,1]$. Let w be a fixed function in $H^2(\Omega)$ such that $w(0) < w(1)$ and u be a function in $H_0^2(\Omega)$ such that $w'(x) + u'(x) > 0$ for each $x \in \Omega$. The function $u + w$ is called a deformation of the interval $[0,1]$. The functional I is defined as follows:

$$(1.1) \qquad I(u) = \int_0^1 \left[(u'(x)+w'(x))^2 + \frac{1}{|u'(x)+w'(x)|} \right] dx = \int_0^1 W(x,u'(x))\,dx$$

where W is the stored-energy density, $I(u)$ is the total stored-energy and the potential energy is assumed to be zero. The goal is to minimize I while preserving the boundary conditions of w. One of the characteristics of elasticity problems is that minima are not unique. In the case of $\Omega = [0,1] \subset \mathbb{R}^1$, I is convex (see Ball, [2]). Another way to show the convexity of I is to calculate the first and second Frechet derivatives of I on a certain natural open subset BZ of H_0^2. A function $v \in H_0^2$ is said to belong to the set BZ if and only if $v'(x)+w'(x) > 0$ for all $x \in \Omega$.

<u>LEMMA 1.1</u> BZ is an open set.

PROOF: Since H_0^2 is compactly embedded in C^1, there is an $M > 0$ such that if $v \in H_0^2$ then $\|v\|_{C^1} \leq M \cdot \|v\|_{H_0^2}$, [1, pp. 97-98]. Suppose $v \in H_0^2$ and v has property BZ, that is $v'+w' > 0$ on Ω. Since v is in C^1, there is a $\delta > 0$ such that $\delta = \min_{x \in \Omega} \{v'(x)+w'(x)\}$. Let $r = \frac{\delta}{2 \cdot M}$. Let $q \in H_0^2$ such that $\|v-q\|_{H_0^2} < r$. We know that $\|v'-q'\|_{\text{Sup}} \leq \|v-q\|_{C^1} \leq M\ \|v-q\|_{H_0^2} \leq M \cdot r = M \cdot \frac{\delta}{2 \cdot M} = \frac{\delta}{2}$. Thus, $|v'(x)-q'(x)| < \frac{\delta}{2}$ for each x in Ω and so $v'(x) - \frac{\delta}{2} < q'(x)$ and $v'(x)+w'(x) - \frac{\delta}{2} < q'(x)+w'(x)$. Since $v'(x)+w'(x) \geq \delta$ then $q'(x)+w'(x) \geq \frac{\delta}{2}$. Therefore q belongs to BZ and BZ is an open set.

The open set BZ is a natural set with which to begin a study of I. The functional I is defined for each u in BZ since the second term of the integral is bounded. Of course there are functions for which I can be defined and which are not in H_0^2 and so not in BZ. For example, let $w(x) = x$ for each $x \in [0,1]$ and define u by the following: let $u(x) = x$ for $x \in [0, {}^1/_3)$, let $u(x) = -2x+1$ for $x \in ({}^1/_3, {}^2/_3]$ and let $u(x) = x-1$ for $x \in ({}^2/_3, 1]$. This function u belongs to H_0^1 and u+w is said to have cracks or buckles. When following the steepest descent process using an initial member of BZ, it is possible that the trajectory leaves the set BZ and descends to a region of H_0^1 containing functions with such cracks. From computations, this is not likely in $\mathbb{R}^1$ but has been observed for many initial ϕ in the $\mathbb{R}^2$ case. This seems to indicate that some functions ϕ have boundary conditions which force a buckling when the method of steepest descent is used to minimize I.

Returning attention to I and the calculation of its first and second Frechet derivatives, suppose u is in BZ. Since BZ is an open subset of H_0^2 and H_0^2 is embedded in C^1, then for each g and h in H_0^2

$$(1.2) \qquad I'(u)h = \int_0^1 \left[2(u'+w')h' - \frac{\operatorname{sign}(u'+w')\ h'}{|u'+w'|^2}\right]$$

and

$$(1.3) \qquad I''(u)(h,g) = 2\int_0^1 \left[h'g' + \frac{h'\cdot g'}{|u'+w'|^3}\right] .$$

Setting $g = h$,

$$I''(u)(h,h) = 2\int_0^1 \left[(h')^2 + \frac{(h')^2}{|u'+w'|^3}\right],$$

which is positive if $h \neq 0$. Hence, I is convex, although not strongly convex as defined by Rosenbloom [16, p. 128]. Note also that I is twice continuously differentiable.

In order to calculate the gradient of I, first note that for each $h \in H_0^2$, $\|h\|^2_{H_0^2} = \int_0^1 \left[h^2+(h')^2+(h'')^2\right]$, the H^2 norm for the subspace H_0^2. Let P_2 be the projection of $L^2\times L^2\times L^2$ onto the subspace $\mathcal{K}_2$ consisting of triples of the form (h,h',h'') and $h(0) = h(1) = 0$. This subspace is isomorphic to H_0^2. Since $u \in BZ$, $\text{sign}(u'(x)+w'(x)) = 1$ for each x in Ω. Thus,

$$I'(u)h = \int_0^1 \left[2(u'+w')h' + \frac{1}{|u'+w'|^2}\right]$$

$$= \int_0^1 \left[\ 0\cdot h + \left[2(u'+w') + \frac{1}{|u'+w'|^2}\right]h' + 0\cdot h''\ \right]$$

$$= \left\langle \begin{bmatrix} 0 \\ 2(u'+w') - \frac{1}{|u'+w'|^2} \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ h' \\ h'' \end{bmatrix} \right\rangle_{L^2\times L^2\times L^2}$$

$$= \left\langle \begin{bmatrix} 0 \\ 2(u'+w') - \frac{1}{|u'+w'|^2} \\ 0 \end{bmatrix}, P_2\begin{bmatrix} h \\ h' \\ h'' \end{bmatrix} \right\rangle_{L^2\times L^2\times L^2}$$

$$= \left\langle P_2 \begin{bmatrix} & 0 & \\ 2(u'+w') & - & \dfrac{1}{|u'+w'|^2} \\ & 0 & \end{bmatrix}, P_2 \begin{bmatrix} h \\ h' \\ h'' \end{bmatrix} \right\rangle_{L^2\times L^2\times L^2}$$

$$= \left\langle \Pi_1 P_2 \begin{bmatrix} & 0 & \\ 2(u'+w') & - & \dfrac{1}{|u'+w'|^2} \\ & 0 & \end{bmatrix}, h \right\rangle_{H^2}$$

where $\Pi_1(q,q',q'') = q$ for each triple in the subspace H of $L^2\times L^2\times L^2$. Thus, $\Pi_1 P_2(0,\ 2(u'+w')-1/|u'+w'|^2\ ,0)$ is the H_0^2 gradient of I at u, denoted by $(\nabla_{H_0^2} I)u$ or $(\nabla I)u$

Since I is twice continously differentiable at $u \in BZ$ and ∇I is locally Lipschitz, then the differential equation

$$(1.4) \qquad y'(t) = -(\nabla I)(y(t)) \quad \text{and } y(0) = u_0 \in BZ$$

has a local solution using the usual Picard iteration technique. Since ∇I is a function from H_0^2 to H_0^2, an examination of the iteration technique used to prove the existence of y yields that trajectories y stay (locally) in H_0^2. This contrasts with the not uncommon difficulty in nonlinear semi-group theory in which one starts with a generator A and has to prove that the composite $(I - \frac{t}{n} A)^{-n} p$ stays in the domain of A. The process generates a nonlinear semi-group which is not defined everywhere, i.e. defines a local semi-group.

One may argue that insistence on the use of H_0^2 requires too strong an hypothesis. Use of H_0^1 would allow for the buckling observed in elasticity problems in higher dimensions [2, 3]. The functional I is only lower semi-continuous on H_0^1, Gateau differentiable in some directions and not Frechet differentiable anywhere. However, the intent here is to establish some theorems, the hypotheses for which are based largely upon observations of numerical work. With this foundation, it is hoped to eventually extend the numerical methods to boundary conditions associated with buckling.

However, the space H_0^1 will be used in a surprising way to develop a modified steepest descent. Again starting with an $h \in H_0^2$, $u \in BZ$ and using the H^1 norm defined by

$$\| h \|_{H_0^1}^2 = \int_0^1 (h)^2+(h')^2 \quad ,$$

it follows that

$$\begin{aligned} I'(u)h &= \int_0^1 \left[2(u'+w') - \frac{1}{|u'+w'|^2}\right]h' \\ &= \left\langle \left[2(u'+w') \overset{0}{-} \frac{1}{|u'+w'|^2}\right], \begin{bmatrix} h \\ h' \end{bmatrix} \right\rangle_{L^2\times L^2} \\ &= \left\langle \Pi_1 P_1 \left[2(u'+w') \overset{0}{-} \frac{1}{|u'+w'|^2}\right], h \right\rangle_{H_0^1} \end{aligned}$$

where P_1 is the projection of $L^2\times L^2$ onto the subspace $\mathcal{H}$ consisting of pairs (h,h') for which $h(0) = h(1) = 0$, which is isomorphic to H_0^1. Define $Q(u)$ to be

$\Pi_1 P_1 (0, 2(u'+w')-1/|u'+w'|^2)$.

Three facts will now be established about Q that will be useful in the steepest descent scheme.

<u>LEMMA 1.2</u> If $u \in BZ$ then $Q(u) \in H_0^2$.

PROOF: Suppose $u \in BZ$. $Q(u)$ is the element h of H_0^1 for which the distance between $(0,\ 2(u'+w')-1/|u'+w'|^2\ )$ and (h,h') is minimum. Let $g = 2(u'+w')-1/|u'+w'|^2$. Since u and w are both in H^2 and $u'+w'$ is in C, g' is seen to be $2(u''+w'') + \dfrac{2(u''+w'')}{|u'+w'|^3}$ and $g' \in L^2$. Computing $Q(u)$ is thus equivalent to minimizing the functional $G(v) = \int_0^1 (v - 0)^2+(v'- g)^2$. Taking the Frechet derivative $G'(v)h$ and using a standard integration by parts argument, one gets $G'(v)h = 2\cdot\int_0^1 (v - v'' + g')h$, where $v \in H_0^1$ and $h \in C_0^\infty$. In order for v to be the minimum of G, it must be the case that $G'(v)h = 0$ for each $h \in C_0^\infty$. This is equivalent to v satisfying the equation

$$v - v'' + g' = 0 \text{ and } v(0) = v(1) = 0\ . \tag{1.5}$$

This is an elliptic second order differential equation for which it is well known that there is a unique solution. Since $g' \in L^2$, it is known that the solution to this equation belongs H_0^2 [8, pp. 169-175]. Hence $Q(u)$ is the solution to (1.5) and $Q(u)$ is in H_0^2.

The preservation of the smoothness by Sobolev gradients is a major factor in the decision to use a steepest descent method. The advantage of using a Sobolev gradient instead of an L^2 gradient is quite evident computationally and has been confirmed theoretically by Neuberger [13]. The nature and form of the projections used in these gradients have also been investigated by Neuberger ([10, 11, 12]).

The second fact about Q is that for each $u \in BZ$, $-Q(u)$ is a descent direction in H_0^2. In order to show this one assumes Q is locally Lipschitz (this will be proved in Chapter II). With this assumption, the differential equation

$$z'(t) = -Q(z(t)) \quad \text{and} \quad z(0) = u_0 \in BZ \tag{1.6}$$

has a local solution, that is, z is defined on $[0,\partial)$ for some $\partial > 0$. For $t \in [0,\partial)$, let $f(t) = I(z(t))$. Taking the derivative of f,

$$\begin{aligned} f'(t) &= I'(z(t))z'(t) \\ &= \langle Q(z(t)), z'(t)\rangle_{H_0^1} \\ &= \langle Q(z(t)), -Q(z(t))\rangle_{H_0^1} \\ &= -\| Q(z(t)) \|^2_{H_0^1}. \end{aligned}$$

Thus, f is a non-increasing function. Unless there is a t such that $Q(z(t)) = 0$, f is a decreasing function and as t increases, $f = I(z(\cdot))$ decreases. The function I provides a Lyapunov function for solutions z to $z' = -Q(z)$ (see Brauer and Nohel, [4]). Considerable use will be made of Lyapunov functions throughout this work.

Finally, a close relationship will be established between $(\nabla I)u$ and $Q(u)$ for $u \in BZ$.

<u>LEMMA 1.3</u> If $u \in BZ$, then $(\nabla I)u = 0$ if and only if $Q(u) = 0$.

PROOF: Suppose $u \in BZ$ and $(\nabla I)u = 0$. Let $g = 2(u'+w') - \frac{1}{|u'+w'|^2}$. Then,

$$
\begin{aligned}
0 &= \langle (\nabla I)u, Q(u) \rangle_{H_0^2} = I'(u)O(Q(u)) \\
&= \int_0^1 \left\langle \mathcal{P}_2 \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \begin{bmatrix} Qu \\ (Qu)' \\ (Qu)'' \end{bmatrix} \right\rangle \\
&= \int_0^1 g \cdot (Qu)' \\
&= \int_0^1 \left\langle \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} Qu \\ (Qu)' \end{bmatrix} \right\rangle \\
&= \int_0^1 \left\langle \mathcal{P}_1 \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} Qu \\ (Qu)' \end{bmatrix} \right\rangle \\
&= \langle Q(u), Q(u) \rangle_{H_0^1}
\end{aligned}
$$

This implies that $\| Q(u) \|_{H_0^1}^2 = 0$ and $Q(u) = 0$ in H_0^1. Thus, $Q(u) = 0$ in H_0^2.

Similarly, suppose $Q(u) = 0$. Then,

$$0 = \langle (\nabla I)u, Q(u) \rangle = \int_0^1 \left\langle \begin{bmatrix} (\nabla I)u \\ [(\nabla I)u]' \end{bmatrix}, P_1 \begin{bmatrix} 0 \\ g \end{bmatrix} \right\rangle$$

$$= \int_0^1 \left\langle \begin{bmatrix} (\nabla I)u \\ [(\nabla I)u]' \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix} \right\rangle$$

$$= \int_0^1 \left\langle \begin{bmatrix} (\nabla I)u \\ [(\nabla I)u]' \\ [(\nabla I)u]'' \end{bmatrix}, P_2 \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \right\rangle$$

$$= \langle (\nabla I)u, (\nabla I)u \rangle_{H_0^2} \ .$$

Hence, $(\nabla I)u = 0$.

As a result of this lemma, the search for critical points for I becomes a search for zeroes of Q. Of course a critical point of a nonconvex functional I is not necessarily a minimum of I. But in using steepest descent methods to minimize functionals, critical points that are not local minimums are usually very unstable numerically. Critical points computed in numerous examples have shown no signs of such instability. This is a good indication that a zero gradient obtained in a stable manner is a local minimum. In computing for the $\mathbb{R}^2$ case in which I is not convex, there is numerical evidence of local convexity near stable functions where the gradient is close to zero.

In Chapter II, the existence of a local minimum of I, namely $\lim_{t\to\infty} z(t)$, is proved assuming only that the function z exists for each $t \in [0,\infty)$. In Chapter III, it is assumed that Ω is a closed, bounded region of $\mathbb{R}^2$ with a smooth boundary. A corresponding function Q is defined and shown to be locally Lipschitz. Again under the assumption that the solution z to (1.6) exists for all $t \in [0,\infty)$, it is shown that there exists an infinite, unbounded and increasing sequence of positive numbers $(t_i)_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \left\{ \|Q(z(t_i))\|_{H_0^1\times H_0^1} \right\} = 0$. Under the condition that the range of z is bounded [10, 11] in $H_0^3\times H_0^3$, it is shown that there is a limit point ϕ of $\{z(t_i)\}$ in $H_0^3\times H_0^3$ so that if ϕ is in BZ then $Q(\phi) = 0$.

Chapter IV describes how to implement the steepest descent for these problems on a computer. A finite difference scheme is used, employing the Sobolev norms. Key parts of a True Basic code are discussed, including a Gauss-Seidel type linear solver for the $\mathbb{R}^2$ case. The output on several test problems is presented, including some graphical output to illustrate the time-stepping process.

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CHAPTER II

THE ONE DIMENSIONAL CONVEX PROBLEM

The condition of convexity on functionals has been widely utilized in minimization problems. Extensive work has been done on convex elasticity problems by S. S. Antman [2] and J. M. Ball [3, 4]. Using the strong assumption of convexity, an extensive theory of semi-groups has been developed by H. Brezis and others [6]. An early development by P. C. Rosenbloom [10] proved asymptotic convergence of trajectories of a semi-group satisfying equations of the form $y'(t) = -(\nabla J)y(t)$ where $y(0) = u$ and J satisfies the strong convexity condition that $J''(u)(h,h) \geq C\cdot\|h\|_H^2$ for each u and h in the Hilbert space H.

The steepest descent method has been adapted by J. W. Neuberger for the following least squares problem [7, 8]. Suppose each of H and K is a Hilbert space, F is a twice continuously differentiable function from H to K and $\phi:H \to K$ such that $\phi(u) = (^1/_2)\|F(u)\|_K^2$ for each $u \in H$. Under the assumption that

$$\|(\nabla\phi)u\|_H \geq M\cdot\|F(u)\|_K \tag{2.1}$$

for some $M > 0$ and every $u \in B \subsetneq H$, Neuberger proves there exists a solution $y:[0,\infty) \to H$ to the equation

$y'(t) = -(\nabla\phi)y(t)$ and $y(0) = u_0$, and if Range(y) $\subseteq$ B then $\lim_{t\to\infty} y(t)$ exists and is a zero of ϕ and of F. This gradient inequality is satisfied by many nonconvex functionals. The set B is typically a bounded subset of a Sobolev space H and the above result requires that the trajectory y remain bounded. The results of this paper are an adaptation of Neuberger's approach to the variational problem (1.1), but without the advantage of a gradient inequality. In this chapter, it is assumed that $[0,1] = \Omega \subseteq \mathbb{R}^1$. In this case the functional I is convex. Although much is known for such convex functionals, our constructive method may shed some new light on non affine minimizers for I (see Ball [4, pp. 4-5]). Most of the key difficulties in higher dimensions are present and may be clearly seen in the one dimensional case.

Attention will be focused on the "modified" gradient Q. Recall from chapter I that $-Q(u)$ is a descent direction, $Q(u) = 0$ if and only if $(\nabla I)u = 0$ and if $u \in BZ$ then $Q(u) \in H_0^2$. An important question arises as to the existence of a solution z to the differential equation

$$z'(t) = -Q(z(t)) \text{ and } z(0) = u_0 \in H_0^2 \tag{2.2}$$

where $t \in [0,\infty)$. In order to prove that there are local solutions to such equations, it is sufficient to prove that Q satisfies a Lipschitz condition.

<u>THEOREM 2.1</u> Q is locally Lipschitz on BZ.

PROOF: Recall that I and I' are given by

$$I(u) = \int_0^1 \left[(u'+w')^2 + \frac{1}{|u'+w'|}\right]$$

and

$$I'(u)h = \int_0^1 \left[2(u'+w')h' - \frac{\operatorname{sign}(u'+w')\cdot h'}{|u'+w'|^2}\right]$$

for each u and h in H_0^2 and $\operatorname{sign}(u'+w') = 1$ since $u \in BZ$. Let $u \in BZ$. Since H_0^2 is embedded in C^1 and $u'+w' > 0$ on $[0,1]$, there is a neighborhood $B(u,r_1)$ and positive numbers M_1 and M_2 such that for each $v \in B(u,r_1)$, $\|1/(v'+w')\| \le M_1$ and $\|v'+w'\| \le M_2$. Let $v \in B(u,r_1)$.

Let $f = 2(u'+w') - \dfrac{1}{|u'+w'|^2}$ and

$g = 2(v'+w') - \dfrac{1}{|v'+w'|^2}$ and note that f' and g' are in L^2.

Let $F = Q(u) = \Pi_1 P_1 \begin{bmatrix} 0 \\ f \end{bmatrix}$ and $G = Q(v) = \Pi_1 P_1 \begin{bmatrix} 0 \\ g \end{bmatrix}$. From the proof of Lemma 1.2, F and G satisfy the differential equations $-F''+F+f' = 0$ and $-G''+G+g' = 0$, respectively. Using the equivalent norm of H_0^2, $\|h\|^2_{H_0^2} = \int_0^1 (h'')^2$, it follows that

$$\|Q(u)-Q(v)\|^2_{H_0^2} =$$

$$\left\| \Pi_1 P_1 \begin{bmatrix} 0 \\ f \end{bmatrix} - \Pi_1 P_1 \begin{bmatrix} 0 \\ g \end{bmatrix} \right\|^2_{H_0^2}$$

$$= \int_0^1 (F''-G'')^2 = \int_0^1 (F+f'-G-g')^2$$

$$= \int_0^1 (F-G)^2 + 2(F-G)(f'-g') + (f'-g')^2$$

$$\leq 2\int_0^1 (F-G)^2 + 2\int_0^1 (f'-g')^2 . \tag{2.2}$$

The two terms of the right side of (2.2) will be considered separately. Since H_0^1 is embedded in L^2 and H_0^2 is embedded in H_0^1, there are positive constants K_1 and K_2 such that for each $h_1 \in H_0^1$ and $h_2 \in H_0^2$, $\|h_1\|^2_{L^2} \leq K_1 \cdot \|h_1\|^2_{H_0^1}$ and $\|h_2\|^2_{H_0^1} \leq K_2 \cdot \|h_2\|^2_{H_0^2}$. For the first term from the right side of (2.2), it follows that

$$\int_0^1 (F-G)^2 = \|F-G\|^2_{L^2} \leq K_1 \cdot \|F-G\|^2_{H_0^1} \tag{2.3}$$

$$= K_1 \cdot \left\| P_1 \begin{bmatrix} 0 \\ f \end{bmatrix} - P_1 \begin{bmatrix} 0 \\ g \end{bmatrix} \right\|^2_{L^2 \times L^2}$$

$$\leq K_1 \cdot 1 \cdot \left\| \begin{bmatrix} 0 \\ f \end{bmatrix} - \begin{bmatrix} 0 \\ g \end{bmatrix} \right\|^2_{L^2 \times L^2}$$

$$= K_1 \int_0^1 (f-g)^2$$

$$= K_1 \int_0^1 \left[2(u'-v') - \frac{(u'-v')(u'+v'+2w')}{(u'+w')^2(v'+w')^2} \right]^2$$

$$\leq 2K_1 \int_0^1 (u'-v')^2 \left\{ 4 + \left[\frac{|u'+w'|+|v'+w'|}{(u'+w')^2(v'+w')^2} \right]^2 \right\}$$

$$\le 2K_1 \int_0^1 (u'-v')^2 [2^2+(2M_1M_2{}^4)^2]$$

$$\le 2K_1 \cdot \|u-v\|^2_{H_0^1} \;\; [2^2+(2M_1M_2{}^4)^2]$$

$$\le 2K_1K_2 \cdot \|u-v\|^2_{H_0^2} \;\; [4+4M_1^2M_2^8] \le K \cdot \|u-v\|^2_{H_0^2}$$

where $K = 2K_1K_2[4+4M_1^2M_2^8]$.

Consider now the second term $\int_0^1 (f'-g')^2$ from the right side of (2.2). Differentiating f and g, gives that $f' = 2u''+2w''+\dfrac{2(u''+w'')}{|u'+w'|^3} \in L^2$ and $g' = 2v''+2w''+\dfrac{2(v''+w'')}{|v'+w'|^3} \in L^2$.

Thus,

$$(2.4) \qquad \int_0^1 (f'-g')^2$$

$$= \int_0^1 \left[2(u''-v'')+2\left[\frac{(u''+w'')(v'+w')^3-(v''+w'')(u'+w')^3}{|u'+w'|^3\,|v'+w'|^3}\right]\right]^2$$

$$\le 8\int_0^1 (u''-v'')^2 + \left[\frac{(u''+w'')(v'+w')^3-(v''+w'')(u'+w')^3}{|u'+w'|^3\,|v'+w'|^3}\right]^2$$

(2.5)

$$\le 8 \cdot \|u-v\|^2_{H_0^2}$$

$$+ \; 8 \cdot M_2^{12} \int_0^1 [(u''+w'')(v'+w')-(v''+w'')(u'+w')]^2 .$$

Let $\delta = u'-v'$, $\sigma = u'+w'$ and $\alpha = v'+w'$. It is clear that $\|\alpha\|_{Sup} \le M_2$ and $\|\sigma\|_{Sup} \le M_2$. Since H_0^2 is embedded in C^1, there is a $K_3 > 0$ such that $\|\delta\|_{Sup} \le K_3 \cdot \|\delta\|_{H_0^1} = K_3 \cdot \|u-v\|_{H_0^2}$

$\leq K_3 \cdot r_1$. Let $L = K_3 \cdot r_1$. Focusing on the integral of the second term of the right side of (2.5), the function pointwise satisfies the following:

$$
\begin{aligned}
&[(u''+w'')(v'+w')-(v''+w'')(u'+w')]^2 \\
&= \left| u''(v'+w')^3 + w''(v'+w')^3 - v''(u'+w')^3 - w''(u'+w')^3 \right|^2 \\
&\leq \left[\left| -v''\sigma^3 + u''(\sigma-\delta)^3 \right| + \left| w''[\alpha^3-\sigma^3] \right| \right]^2 \\
&\leq \Big[|u''-v''|\,|\sigma|^3 + |u''|\,|(-3\sigma^2\delta+3\sigma\delta^2-\delta^3)| \\
&\qquad + |w''|\,|\delta|\,|(\alpha^2+\alpha\sigma+\sigma^2)| \Big]^2 \\
&\leq \left[|u''-v''| M_2^3 + |u''|\,|\delta|\,(3M_2^2+3M_2L+L^2) + |w''|\,|\delta|\, 3M_2^2 \right]^2 \\
&\leq 3\Big[|u''-v''|^2 M_2^6 + |u''|^2|\delta|^2(3M_2^2+3M_2L+L^2)^2 \\
&\qquad + |w''|^2|\delta|^2 9M_2^4 \Big]
\end{aligned}
$$

Integrating, it is clear that

$$
\begin{aligned}
&\int_0^1 [(u''+w'')(v'+w')-(v''+w'')(u'+w')]^2 \\
&\leq 3M_2^6 \int_0^1 (u''-v'')^2 + 3(3M_2^2+3M_2L+L^2)^2 \left[\int_0^1 (u'')^2\right] \cdot \|\delta\|_{\text{Sup}}^2 \\
&\qquad + 9M_2^4 \left[\int_0^1 (w'')^2\right] \cdot \|\delta\|_{\text{Sup}}^2 \\
&\leq 3M_2^6 \cdot \|u-v\|_{H_0^2}^2 + 3(3M_2^2+3M_2L+L^2)^2 \|u\|_{H_0^2}^2 \cdot K_3^2 \cdot \|u-v\|_{H_0^2}^2 \\
&\qquad + 9M_2^4 \cdot \|w\|_{H_0^2}^2 K_3^2 \cdot \|u-v\|_{H_0^2}^2 \\
&= \Big[3M_2^6 + 3(3M_2^2+3M_2L+L^2)^2 \|u\|_{H_0^2}^2 K_3^2 \\
&\qquad\qquad + 9M_2^4 \|w\|_{H_0^2}^2 K_3^2 \Big] \|u-v\|_{H_0^2}^2
\end{aligned}
$$

$$= M\cdot\|u-v\|^2_{H_0^2}$$

where M is the constant from the previous step. Using this inequality in (2.4), and then using (2.3) and (2.4), it follows that

$$2\int_0^1 (F-G)^2 + 2\int_0^1 (f'-g')^2$$
$$\le 2\cdot K\cdot\|u-v\|^2_{H_0^2} + 2\cdot\left[8\cdot\|u-v\|^2_{H_0^2} + 8\cdot M_2^{1\,2}\cdot M\cdot\|u-v\|^2_{H_0^2}\right]$$
$$= (2\cdot K + 16 + 16\cdot M_2^{1\,2}\cdot M\ )\cdot\|u-v\|^2_{H_0^2}.$$

Thus,

$$\|Q(u)-Q(v)\|^2_{H_0^2} \le (2\cdot K + 16 + 16\cdot M_2^{1\,2}\cdot M\ )\cdot\|u-v\|^2_{H_0^2}$$

for each $v \in B(u,r_1)$. Therefore, Q is locally Lipschitz on BZ and the theorem is proved.

Using this result and a standard existence argument (see Brauer and Nohel, [5, pp. 364-417]), for each $u_0 \in BZ$ there is an $\epsilon > 0$ and a function $z:[0,\epsilon) \to H_0^2$ such that $z'(t) = -Q(z(t))$ and $z(0) = u_0$. This is of course a local result. But in numerical experiments for this steepest descent problem, the global existence of solutions z to such differential equations has always been observed. This phenomenon can hold even for problems which have no solution. For example, the least squares problem where $F:H_0^1([0,1]) \to L^2([0,1])$ such that $F(u) = u'-1-u^2$ and

$\phi(u) = (^1/_2)\|F(u)\|^2_{H^1_0}$ produces a globally defined trajectory z, but F has no zero for an interval Ω of length more than π. The global existence of trajectories shall be part of the hypothesis of the following theorem.

<u>THEOREM 2.2</u> Suppose $z:[0,\infty) \to H^2_0$ such that for each $t \in [0,\infty)$ $z'(t) = -Q(z(t))$ and $z(0) = u_0$. Then, $\lim_{t\to\infty} z(t)$ exists and is a member u of H^1_0 and $\lim_{t\to\infty} \|Q(z(t))\|_{H^1_0} = 0$.

PROOF: Let $f:[0,\infty) \to \mathbb{R}$ such that for each $t \in [0,\infty)$, $f(t) = I(z(t))$. Using the differentiability of I,

$$
\begin{aligned}
(2.6)\qquad f'(t) &= I'(z(t))\circ z'(t) = \langle Q(z(t)),z'(t)\rangle_{H^1_0} \\
&= \langle Q(z(t)),-Q(z(t))\rangle_{H^1_0} = -\|Q(z(t))\|^2_{H^1_0}.
\end{aligned}
$$

and

$$
\begin{aligned}
(2.7)\qquad f''(t) &= I''(z(t))[Q(z(t)),Q(z(t))] \\
&= 2\int_0^1\left[1+\frac{1}{|Q(z(t))'+w'|^3}\right][Q(z(t))']^2 \\
&\geq 2\int_0^1 [Q(z(t))']^2 = 2\cdot\|Q(z(t))\|^2_{H^1_0} \\
&= 2\cdot(-f'(t)).
\end{aligned}
$$

Therefore,

$$(2.8)\qquad f''(t) \geq -2\cdot f'(t)$$

for each $t \in [0,\infty)$. Dividing both sides of (2.8) by the negative quantity $f'(t)$ and integrating, it is clear that f satisfies the inequalities:

$$\text{(2.9)} \qquad \frac{f''(t)}{f'(t)} \leq -2 \quad \text{and} \quad \frac{-f'(t)}{-f'(0)} \leq e^{-2t}.$$

and hence,

$$\text{(2.10)} \qquad 0 < -f'(t) \leq -f'(0)e^{-2t}.$$

The following is an argument that $\lim_{t\to\infty} z(t)$ exists in H_0^1. Suppose s and t are in $[0,\infty)$ and $s < t$. Let k and n be the non-negative integers such $k \leq s < k+1$ and $n-1 < t \leq n$. Thus,

$$\begin{aligned}
\|z(t)-z(s)\|_{H_0^1} &= \left\| \int_s^t z' \right\|_{H_0^1} \\
&\leq \sum_{i=k}^{n-1} \int_i^{i+1} \|z'\|_{H_0^1} \\
&\leq \sum_{i=k}^{n-1} \left[\int_i^{i+1} \|z'\|^2_{H_0^1} \right]^{1/2} \\
&= \sum_{i=k}^{n-1} \left[\int_i^{i+1} -f' \right]^{1/2} \\
&\leq \sum_{i=k}^{n-1} \left[\int_i^{i+1} -f'(0)e^{-2j} \right]^{1/2} \\
&\leq \sum_{i=k}^{n-1} \left[f'(0)\right]^{1/2} \left[\frac{-e^{-2i-2}}{2} + \frac{e^{-2i}}{2}\right]^{1/2} \\
&\leq \left[-f'(0)\right]^{1/2} \left[\frac{-e^{-2}+1}{2}\right]^{1/2} \cdot \sum_{i=k}^{n-1} e^{-i}.
\end{aligned}$$

Since the series $\sum_{i=0}^{\infty} e^{-i}$ converges, then the trajectory $z(t)$

is Cauchy in H_0^1 as t approaches ∞. Therefore, the $\lim_{t\to\infty} z(t)$ exists and is in H_0^1. Call this limit u. Observe that $\lim_{t\to\infty} f'(t)$ and is zero because of (2.10). Since

$f'(t) = -\|Q(z(t))\|^2_{H_0^1}$ for each $t \in [0,\infty)$, then

$\lim_{t\to\infty} \|Q(z(t))\|^2_{H_0^1} = 0.$

Even though the functional I is coercive in the H_0^1 norm but not in the H_0^2 norm, it has been observed that for many trajectories z the Range(z) is bounded. The following theorem uses the hypothesis that z remains bounded in H_0^2.

<u>THEOREM 2.3</u> Suppose z satisfies the hypothesis of Theorem 2.2 and $u = \lim_{t\to\infty} z(t) \in H_0^1$. Then,

(i) if Range(z) is bounded in H_0^2, then $u \in H_0^2$, and

(ii) if $u \in BZ$ then $\nabla I(u) = 0$ and u is a local minimum of I.

<u>PROOF</u>: (i) Suppose the Range(z) is bounded in H_0^2. Let $\{t_i\}_{i=1}^{\infty}$ be an increasing sequence in $[0,\infty)$ such that $\lim_{i\to\infty} \{t_i\} \to \infty$. There is a subsequence $\{z(s_i)\}$ of $\{z(t_i)\}$ which converges weakly in H_0^2 to a point $v \in H_0^2$. Thus, $\{z(s_i)\}$ converges weakly to v in H_0^1. But, u is the strong limit of z(t) in H_0^1 and so is the weak limit of $\{z(s_i)\}$. Since weak limits are unique, $v = u$ and $u \in H_0^2$.

(ii) Suppose $u \in BZ$. Since H_0^2 is compactly embedded in C^1 [1], there is a subsequence of $\{z(s_i)\}$ which converges strongly to an element v_1 in C^1. Strong convergence in C^1 implies strong convergence in H_0^1, so that $v_1 = u$. Suppose $\{u_k\}$ is the convergent subsequence in C^1. Since $u \in BZ$ and $\{u_k\}$ converges strongly to u in C^1, there is an $M_1 > 0$ and a positive integer n such that if $k \geq n$ then $\frac{1}{|u_k'(x)+w'(x)|} \leq M_1$ and $\frac{1}{|u'(x)+w'(x)|} \leq M_1$ for each $x \in [0,1]$. Let M_2 be the maximum of $\{\|u_k+w\|_{C^1}\}$ and $\|u+w\|_{C^1}$. Let $h \in H_0^1$ such that $\|h\|_{H_0^1} = 1$. For each $k \geq n$,

$$\left|\langle Q(u_k)-Q(u),h\rangle_{H_0^1}\right|$$

$$\leq \left|\int_0^1 2(u_k'-u')h' + \left[\frac{(u'-u_k')(u'+u_k'+2w')}{(u_k'+w')^2(u'+w')^2}\right]h'\right|$$

$$\leq 2\cdot\left|\int_0^1 (u_k'-u')h'\right| + \left|\int_0^1 \left[\frac{|u'-u_k'|\,|u'+u_k'+2w'|}{(u_k'+w')^2(u'+w')^2}\right]|h'|\right|$$

$$\leq 2\cdot\|u_k-u\|_{H_0^1}\,\|h\|_{H_0^1} + \int_0^1 |u'-u_k|\cdot 2M_1\cdot|h'|\cdot M_2^4$$

$$\leq 2\cdot\|u_k-u\|_{H_0^1}\,\|h\|_{H_0^1} + 2M_1M_2^4\cdot\|u-u_k\|_{H_0^1}\,\|h'\|_{H_0^1}$$

$$= (2+2M_1M_2^4)\cdot\|u-u_k\|_{H_0^1}\ .$$

Since $\|u-u_k\|_{H_0^1} \leq \|u'-u_k'\|_{C^1}$, then $\left\{\left|\langle Q(u_k)-Q(u),h\rangle_{H_0^1}\right|\right\}$ converges to 0. Hence, $\{Q(u_k)\}$ converges weakly to $Q(u)$ in H_0^1. Since $\left\{\|Q(u_k)\|_{H_0^1}\right\} \to 0$ and $\|Q(u)\| \leq \liminf \left\{\|Q(u_k)\|_{H_0^1}\right\}$, then $\|Q(u)\|_{H_0^1} = 0$ (see [9, p. 60]). Therefore, $Q(u) = 0$.

By lemma 1.3, this implies $(\nabla I)u = 0$. Since I is convex, then u is a local minimum for I on H_0^2.

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CHAPTER III

THE TWO DIMENSIONAL NONCONVEX PROBLEM

This elasticity problem in two dimensions has a nonconvex functional I. However the main difficulties encountered in two dimensions are of the same nature as those in Chapter II for the one dimensional case. Let Ω be a closed and bounded region in $\mathbb{R}^2$ with a smooth boundary. Let $H_0^3(\Omega) = H^3(\Omega) \cap H_0^1(\Omega)$. Suppose $(w,y) \in H^3(\Omega) \times H^3(\Omega)$, $(u,v) \in H_0^3(\Omega) \times H_0^3(\Omega)$ and $\alpha_{u,v}(x) = \alpha(x) = \det D\begin{bmatrix} u+w \\ v+y \end{bmatrix}(x)$ $= \det\begin{bmatrix} u_1+w_1 & u_2+w_2 \\ v_1+y_1 & v_2+y_2 \end{bmatrix}(x)$ for each $x \in \Omega$. For each appropriate (u,v), define $I(u,v)$ by

$$I(u,v) = \int_\Omega (u_1+w_1)^2+(u_2+w_2)^2+(v_1+y_1)^2+(v_2+y_2)^2+ \frac{1}{|\alpha_{u,v}|}$$

$$= \int_\Omega W(D\phi)$$

where $\phi = (u,v)$. Note that I is not defined for all $(u,v) \in H_0^3 \times H_0^3$.

An affine deformation ϕ is one of the form $\phi(x) = A_0(x) + x_0$ where $A_0 \in M_+^{n\times n}$ and $x \in \mathbb{R}^n$ (see Ball, [4, page 4]). W as defined above is $H^{1,p}$ quasiconvex for $1 \leq p < \infty$. The condition of quasiconvexity, combined with certain growth conditions on W, implies I is sequentially weakly lower semicontinuous (swlsc). This implies by standard arguments that I has minimizers (Ball [3, p. 190]). Ball has pointed out that, although non-affine minimizers supposedly abound, little is known about such minimizers or their smoothness [4, p. 5]. L. C. Evans [5, p. 3] has proven partial regularity for functionals which satisfy a uniformly strictly quasiconvexity condition, that is,

$$\int_\Omega W(A)+\lambda\,|D\phi|\,dy \leq \int_\Omega W(A+D\phi)\,dy \tag{3.1}$$

for some $\lambda > 0$, every smooth, bounded open $\Omega \subset \mathbf{R}^2$, each 2×2 matrix A, and every $\phi \in C^1(\Omega,\mathbf{R}^2)$ with $\phi = 0$ on $\partial\Omega$. If $\lambda = 0$, this is the definition of quasiconvexity. As Ball points out, this does not allow for the singular behavior of $W(A)$ as $\det(A) \to 0+$ [4, p. 5]. In this paper the approach is constructive with hypotheses suggested by observations of computations. Conditions will be given that imply existence of ϕ in $C^1\times C^1$ for which $(\nabla I)(\phi) = 0$. Many such functions observed in computing are non-affine. Initial boundary conditions which are planar do give rise, as expected, to affine solutions.

Let BZ be the subset of $H_0^3\times H_0^3$ such that $(u,v) \in$ BZ if and only if $\sigma_{u,v}(x) > 0$ for each x in Ω.

<u>LEMMA 3.1</u> BZ is an open subset of $H_0^3\times H_0^3$.

PROOF: Using the fact that $H_0^3\times H_0^3$ is compactly embedded in

$C^1(\Omega)\times C^1(\Omega)$, the proof is similar to that of Lemma 1.1.

The choice of $H_0^3\times H_0^3$ is made so as to insure that I is Frechet differentiable on BZ. The strategy for investigating I will be the same as in Chapter II. For each (u,v) and (g,h) in BZ, differentiating I yields

$$
\begin{aligned}
&I'(u,v)(g,h)\\
&= 2\int_\Omega (u_1+w_1)g_1+(u_2+w_2)g_2+(v_1+y_1)h_1+(v_2+y_2)h_2\\
&\quad+\int_\Omega \frac{-\mathrm{sign}(\alpha)}{|\alpha_{u,v}|^2}\cdot[(u_1+w_1)h_2+(v_2+y_2)g_1\\
&\qquad\qquad -(v_1+y_1)g_2-(u_2+w_2)h_1]
\end{aligned}
$$

$$
(3.2)\qquad = \int_\Omega\left\langle 2\begin{pmatrix}(u_1+w_1)\\(u_2+w_2)\\(v_1+y_1)\\(v_2+y_2)\end{pmatrix}+\frac{1}{\alpha^2}\begin{pmatrix}-(v_2+y_2)\\+(v_1+y_1)\\+(u_2+w_2)\\-(u_1+w_1)\end{pmatrix},\begin{pmatrix}g_1\\g_2\\h_1\\h_2\end{pmatrix}\right\rangle
$$

$$
= \left\langle \Pi_{1,1}\mathcal{P}\left\{2\begin{pmatrix}(u_1+w_1)\\(u_2+w_2)\\(v_1+y_1)\\(v_2+y_2)\end{pmatrix}+\frac{1}{\alpha^2}\begin{pmatrix}-(v_2+y_2)\\+(v_1+y_1)\\+(u_2+w_2)\\-(u_1+w_1)\end{pmatrix}\right\},\begin{pmatrix}g\\h\end{pmatrix}\right\rangle_{H_0^1\times H_0^1}
$$

where $\mathcal{P}$ is the projection from $L^2\times L^2\times L^2\times L^2\times L^2\times L^2$ onto the subspace consisting of elements of the form (g,h,g_1,g_2,h_1,h_2) and $\Pi_{1,1}:\mathbb{R}^6\to\mathbb{R}$ such that $\Pi_{1,1}(x_1,x_2,x_3,x_4,x_5,x_6) = (x_1,x_2)$. Note that the first two components in the left side of the above inner product expression have been deleted because they are (0,0) and the two equivalent norms of $H_0^1\times H_0^1$ have the same value in this case. Let Q be the function from BZ to $H_0^1\times H_0^1$ defined by

$$Q(u,v) = \Pi_{1,1}\mathcal{P}\left\{2\begin{pmatrix}(u_1+w_1)\\(u_2+w_2)\\(v_1+y_1)\\(v_2+y_2)\end{pmatrix}+\frac{1}{\alpha^2}\begin{pmatrix}-(v_2+y_2)\\+(v_1+y_1)\\+(u_2+w_2)\\-(u_1+w_1)\end{pmatrix}\right\}$$

$$(3.3)\qquad = \Pi_{1,1}\mathcal{P}\begin{pmatrix}a_\phi\\b_\phi\\c_\phi\\d_\phi\end{pmatrix} = \left(\pi_1\mathcal{P}_1\begin{bmatrix}0\\a_\phi\\b_\phi\end{bmatrix},\pi_1\mathcal{P}_1\begin{bmatrix}0\\c_\phi\\d_\phi\end{bmatrix}\right) = (F,G)$$

where $\mathcal{P}_1$ is the projection of $L^2\times L^2\times L^2$ onto the subspace consisting of elements of the form (g,g_1,g_2), $\mathcal{P} = \mathcal{P}_1\times\mathcal{P}_1$ and π_1 is the function on $\mathbb{R}^3$ such that $\pi_1(x_1,x_2,x_3) = x_1$. Since Q is a projection, finding Q(u,v) is equivalent to minimizing the functional $\eta:BZ \to \mathbb{R}$ defined by

$$\eta(p,q) = \int_\Omega (p-0)^2+(p_1-a_\phi)^2+(p_2-b_\phi)^2 +(q-0)^2+(q_1-c_\phi)^2+(q_2-d_\phi)^2.$$

A variational argument gives that a minimum of η is a pair (F,G) satisfying the pair of differential equations

$$(3.4)\qquad \Delta F = 2\Delta(u+w) + \frac{2}{|\alpha|^3}\det D\begin{bmatrix}\alpha\\v+y\end{bmatrix} + F$$

with $F = 0$ on $\partial\Omega$, and

$$(3.5)\qquad \Delta G = 2\Delta(v+w) + \frac{2}{|\alpha|^3}\det D\begin{bmatrix}u+w\\\alpha\end{bmatrix} + G$$

with $G = 0$ on $\partial\Omega$. By the theory of second order elliptic partial differential equations (see Miranda [6]), for each $g \in H^1$ there exists a unique solution y to the equation

$$\Delta y = y + g \text{ and } y = 0 \text{ on } \partial\Omega$$

such that $y \in H_0^3$. Since $(u,v) \in BZ$, then for each equation (3.4) and (3.5) the corresponding g is in H_0^1. Thus, there exists a unique pair of solutions (F,G) to (3.4) and (3.5) such that $Q(u,v) = (F,G) \in H_0^3 \times H_0^3$. Hence, Q is a function from BZ to $H_0^3 \times H_0^3$.

Following the same strategy as in Chapter II, the next step is to show that Q is locally Lipschitz on BZ. Several technical lemmas are required to begin this effort.

<u>LEMMA 3.2</u> Suppose $(u,v) = \phi \in BZ$. Suppose A_1, A_2, A_3 and A_4 are functions defined on $H_0^3 \times H_0^3$ by the following:

(i) $A_1(\phi) = \dfrac{1}{|{}^{\alpha}\phi|^q} = \dfrac{1}{|\alpha|^q}$ for $q = 2$, 3 or 4

(ii) $A_2(\phi) = [\det D(\phi)]_p = \alpha_p$, $p = 1$ or 2, $\alpha_p = \dfrac{\partial(\alpha)}{\partial(x_p)}$

(iii) $A_3(\phi) = \det D\begin{bmatrix} \alpha \\ v+y \end{bmatrix}$

(iv) $A_4(\phi) = [A_3(\phi)]_p$, $p = 1$ or 2 .

Then, each of A_1, A_2, A_3 and A_4 is locally Lipschitz as a function from $H_0^3 \times H_0^3$ to $C(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$ and $L^2(\Omega)$ respectively.

PROOF: Since $\Omega \subset \mathbb{R}^2$, $H^1(\Omega)$ and $H_0^3(\Omega)$ are algebras (see Adams, [1]) and H_0^3 is embedded in C^1. By elementary calculations, one has that each of A_1, A_2, A_3 and A_4 is

Fréchet differentiable. Thus, they are each locally Lipschitz.

In showing the desired properties of Q, the norm of $H_0^3 \times H_0^3$ must be considered carefully. As is well known, an equivalent norm for $H_0^2(\Omega)$ is given by

$$\|u\|_{H_0^2}^2 = \int_\Omega (u_{11}+u_{22})^2 = \int_\Omega (\Delta u)^2,$$

where Δ is the Laplacian operator (see Agmon, Douglis and Nirenberg, [2]). A typical calculation shows that for H_0^3, the usual norm is equivalent to one that depends only on the non-cross partial derivatives u_{111} and u_{222}. Thus, it is easy to see that a third equivalent norm for H_0^3 is given by

$$\|u\|_{H_0^3}^2 = \int_\Omega (\Delta u)^2+([\Delta u]_1)^2+([\Delta u]_2)^2 \quad ,$$

with the first term being necessary. Use of this equivalent norm will be made in showing that Q is locally Lipschitz.

Another technical but elementary lemma is needed to show the consequences of products of such functions as A_1, A_2, A_3 and A_4.

<u>LEMMA 3.3</u> Suppose $(u,v) = \phi \in BZ$, $\beta > 0$, $\mathcal{F}:BZ \to H^2$, $\mathcal{G}:BZ \to H^1$, $\mathcal{H}:BZ \to H^1$ and each of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ is Lipschitz on the open ball $B[\phi,\beta] \subset BZ$. Then, there is a $N > 0$ such that for each $\sigma \in B[\phi,\beta]$,

$$\int_{\Omega}[\mathcal{F}(\sigma)\mathcal{G}(\sigma)\mathcal{H}(\sigma)-\mathcal{F}(\phi)\mathcal{G}(\phi)\mathcal{H}(\phi)]^2 \leq N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}.$$

PROOF: Let $M > 0$ be a Lipschitz constant for each of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ on $B[\phi,\beta]$. Since the ranges of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are contained in $H^1(\Omega)$ and $H^1(\Omega)$ is embedded in $L^4(\Omega)$, then there is a $M_1 > 0$ such that $\|\mathcal{A}(\sigma)\|^2_{L^4} \leq M_1$ for each σ in $B[\phi,\beta]$ and $\mathcal{A} = \mathcal{F}$, $\mathcal{G}$ or $\mathcal{H}$. Since range of $\mathcal{F} \subset H^2(\Omega)$ and $H^2(\Omega)$ is embedded in $C(\bar{\Omega})$, then $\mathcal{F}(B[\phi,\beta])$ is uniformily bounded on $\hat{\Omega}$ by a constant M_2. Hence, for each $\sigma \in B[\phi,\beta]$

$$\begin{aligned}
&\int_{\Omega}[\mathcal{F}(\sigma)\mathcal{G}(\sigma)\mathcal{H}(\sigma)-\mathcal{F}(\phi)\mathcal{G}(\phi)\mathcal{H}(\phi)]^2 \\
&\leq 3\int_{\Omega}[\mathcal{F}(\sigma)-\mathcal{F}(\phi)]^2[\mathcal{G}(\sigma)\mathcal{H}(\sigma)]^2 \\
&\quad + 3\int_{\Omega}|\mathcal{F}(\phi)|^2[\mathcal{G}(\sigma)\mathcal{H}(\sigma)-\mathcal{G}(\sigma)\mathcal{H}(\phi)]^2 \\
&\quad + 3\int_{\Omega}|\mathcal{F}(\phi)|^2[\mathcal{G}(\sigma)\mathcal{H}(\phi)-\mathcal{G}(\phi)\mathcal{H}(\phi)]^2 \\
&\leq 3\|\mathcal{F}(\sigma)-\mathcal{F}(\phi)\|^2_{C}\left(\int_{\Omega}[\mathcal{G}(\sigma)]^4\right)^{\frac{1}{2}}\left(\int_{\Omega}[\mathcal{H}(\sigma)]^4\right)^{\frac{1}{2}} \\
&\quad + 3M_2\left(\int_{\Omega}[\mathcal{G}(\sigma)]^4\right)^{\frac{1}{2}}\left(\int_{\Omega}[\mathcal{H}(\sigma)-\mathcal{H}(\phi)]^4\right)^{\frac{1}{2}} \\
&\quad + 3M_2\left(\int_{\Omega}[\mathcal{H}(\sigma)]^4\right)^{\frac{1}{2}}\left(\int_{\Omega}[\mathcal{G}(\sigma)-\mathcal{G}(\phi)]^4\right)^{\frac{1}{2}} \\
&\leq 3M\|\phi-\sigma\|^2_{H_0^3\times H_0^3}(M_1)^2(M_1)^2 \\
&\quad + 3M_2(M_1)^2\|\mathcal{H}(\sigma)-\mathcal{H}(\phi)\|^2_{H_0^1\times H_0^1}
\end{aligned}$$

$$+ \; 3M_2(M_1)^2\|\mathscr{G}(\sigma)-\mathscr{G}(\phi)\|^2_{H_0^1\times H_0^1}$$

$$\leq \; 3M(M_1)^4\|\phi-\sigma\|^2_{H_0^3\times H_0^3} \; + \; 3M_2(M_1)^2M\|\sigma-\phi\|^2_{H_0^3\times H_0^3}$$

$$+ \; 3M_2(M_1)^2M\|\sigma-\phi\|^2_{H_0^3\times H_0^3}$$

$$= \; [3M(M_1)^4 \; + \; 6M_2(M_1)^2M]\|\sigma-\phi\|^2_{H_0^3\times H_0^3}.$$

Thus, $N = 3M(M_1)^4 + 6M_2(M_1)^2M$ and the lemma is proved.

A similar result can be proved in which $\mathscr{F}:BZ \to H^2$, $\mathscr{G}:BZ \to H^2$ and $\mathscr{H}:BZ \to L^4$ and each locally Lipschitz. These results are essential for the following theorem.

<u>THEOREM 3.4</u> Q is locally Lipschitz on BZ.

PROOF: Let $(u,v) = \phi \in BZ$. In analyzing Q, functions such as A_1, A_2, A_3 and A_4 from lemma 3.2 and similar ones in which u+w and v+y are interchanged, must be considered. Choose $\beta > 0$ and $N > 0$ such that $B[(\phi),\beta] \subset BZ$ and this ball of radius β and the constant N satisfies the hypotheses and conclusions for each of the ten applications of lemma 3.3 that will used in the next paragraph. Let $(r,s) = \sigma \in B[(\phi),\beta]$. Let $Q(u,v) = (F,G)$ and $Q(r,s) = (H,K)$. Let $\alpha_{u,v} = \alpha$ and $\alpha_{r,s} = \theta$. Thus,

$$\left\|Q\begin{bmatrix}u\\v\end{bmatrix}-Q\begin{bmatrix}r\\s\end{bmatrix}\right\|^2_{H_0^3\times H_0^3} = \left\|\begin{bmatrix}F\\G\end{bmatrix}-\begin{bmatrix}H\\K\end{bmatrix}\right\|^2_{H_0^3\times H_0^3}$$

$$= \|F-H\|^2_{H_0^3} + \|G-K\|^2_{H_0^3}$$

Considering the first term from above, it follows that

$$\text{(3.6)} \qquad \|F-H\|^2_{H_0^3} = \int_\Omega [(\Delta F)-(\Delta H)]^2 + \int_\Omega [(\Delta F)_1-(\Delta H)_1]^2 + \int_\Omega [(\Delta F)_2-(\Delta H)_2]^2.$$

Considering the first term on the right side of (3.6) and using (3.4) for each of F and H,

$$\int_\Omega [(\Delta F)-(\Delta H)]^2$$

$$= \int_\Omega \left[2\Delta(u+w)+\frac{2}{|\alpha|^3} \det D\begin{bmatrix}\alpha \\ v+y\end{bmatrix}+F - 2\Delta(r+w)-\frac{2}{|\theta|^3} \det D\begin{bmatrix}\theta \\ s+y\end{bmatrix}-H\right]^2$$

$$\text{(3.7)} \qquad \leq 3\int_\Omega 4(\Delta u-\Delta r)^2 + 3\int_\Omega 4\left[\frac{1}{|\alpha|^3} \det D\begin{bmatrix}\alpha \\ v+y\end{bmatrix} - \frac{1}{|\theta|^3} \det D\begin{bmatrix}\theta \\ s+y\end{bmatrix}\right]^2 + 3\int_\Omega (F-H)^2.$$

If for each $(p,q) \in B[\phi,\beta]$, $\mathcal{F}(p,q) = \dfrac{1}{|\alpha_{p,q}|^3}$, $\mathcal{G}(p,q)(x) = 1$ and $\mathcal{H}(p,q) = \det D\begin{bmatrix}\alpha_{p,q} \\ q+y\end{bmatrix}$, then lemma 3.3 applies and the right side of (3.7) is less than or equal to

$$12\cdot\|u-r\|^2_{H_0^3} + 12\int_\Omega \left[\mathcal{F}(\phi)\mathcal{H}(\phi) - \mathcal{F}(\sigma)\mathcal{H}(\sigma)\right]^2 + 3\cdot\|F-H\|^2_{L^2}$$

$$\leq 12\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} + 12\cdot N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} + 3\cdot\|F-H\|^2_{L^2}$$

$$(3.8) \qquad \leq 12(1+N)\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} + 3\cdot\|F-H\|^2_{H^1}$$

The second term of the right side of (3.8) will be dealt with as it appears again in the second term of $\|F-H\|^2_{H_0^3}$.

Next, considering the second term of $\|F-H\|^2_{H_0^3}$ from (3.6), it follows that

$$\int_\Omega [(\Delta F)_1-(\Delta H)_1]^2$$

$$\leq \int_\Omega \left[2\Delta(u+w)_1+\left[\frac{2}{|\alpha|^3}\det D\begin{bmatrix}\alpha\\ v+y\end{bmatrix}\right]_1+F_1 - 2\Delta(r+w)_1-\left[\frac{2}{|\theta|^3}\det D\begin{bmatrix}\theta\\ s+y\end{bmatrix}\right]_1-H_1\right]^2$$

$$(3.9) \qquad \leq 4\int_\Omega [\Delta(u+w)_1-\Delta(r+w)_1]^2+ 4\int_\Omega (F_1-H_1)^2$$
$$+ 4\int_\Omega \left[\frac{6}{|\alpha|^4}\cdot\alpha_1\cdot\det D\begin{bmatrix}\alpha\\ v+y\end{bmatrix} - \frac{6}{|\theta|^4}\cdot\theta_1\cdot\det D\begin{bmatrix}\theta\\ s+y\end{bmatrix}\right]^2$$
$$+ 4\int_\Omega \left[\frac{2}{|\alpha|^3}\left[\det D\begin{bmatrix}\alpha\\ v+y\end{bmatrix}\right]_1 - \frac{2}{|\theta|^3}\left[\det D\begin{bmatrix}\theta\\ s+y\end{bmatrix}\right]_1\right]^2.$$

Focusing on the third term of the right side of (3.9), if $A_1(r,s) = \frac{6}{|\theta_{r,s}|^4}$, $A_2(r,s) = \left[\det D\begin{bmatrix}r+w\\ s+y\end{bmatrix}\right]_1 = \alpha_1$, and

$A_3(r,s) = \det D\begin{bmatrix}\theta\\ s+y\end{bmatrix}$ for each $(r,s) \in B[\phi,\beta]$, then these three functions satisfy the hypotheses of lemmas 3.2 and 3.3 with β and N as the constants from lemma 3.3. Thus, the third term of the right side of (3.9) is less than or equal to $4\cdot N\cdot\|\phi-\sigma\|_{H_0^3\times H_0^3}$.

Similarly, for the fourth term of the right side of (3.9), if $A_1(r,s) = \frac{2}{|\theta_{r,s}|^3}$, $A_2(r,s)(x) = 1$, and $A_4(r,s) = \left[\det D\begin{bmatrix}\theta \\ s+y\end{bmatrix}\right]_1$ for each $(r,s) \in B[\phi,\beta]$, then these three functions satisfy the hypotheses of lemmas 3.2 and 3.3 with β and N as the constants from lemma 3.3. Thus, the third term of the right side of (3.9) is less than or equal to $4\cdot N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}$.

The first term of the right side of (3.9) satisfies

$$4\int_\Omega [\Delta(u+w)_1-\Delta(r+w)_1]^2 = 4\int_\Omega [\Delta(u)_1-\Delta(r)_1]^2$$
$$\leq 4\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}.$$

This leaves only the second term of the right side of (3.9) to be analyzed. Recall from (3.3) that Q(u,v) is defined as

$$Q(u,v) = \Pi_{1,1}\mathcal{P}\begin{pmatrix}a_\phi \\ b_\phi \\ c_\phi \\ d_\phi\end{pmatrix} = \left(\pi_1\mathcal{P}_1\begin{bmatrix}0 \\ a_\phi \\ b_\phi\end{bmatrix}, \pi_1\mathcal{P}_1\begin{bmatrix}0 \\ c_\phi \\ d_\phi\end{bmatrix}\right) = (F,G).$$

Thus,

$$4\int_\Omega (F_1-H_1)^2 \leq 4\|F-H\|^2_{H^1}$$
$$= 4\left\| \Pi_1\mathcal{P}_1\begin{bmatrix}0 \\ a_\phi \\ b_\phi\end{bmatrix} - \Pi_1\mathcal{P}_1\begin{bmatrix}0 \\ a_\sigma \\ b_\sigma\end{bmatrix} \right\|^2_{H^1}$$
$$= 4\left\| \mathcal{P}_1\begin{bmatrix}0 \\ a_\phi \\ b_\phi\end{bmatrix} - \mathcal{P}_1\begin{bmatrix}0 \\ a_\sigma \\ b_\sigma\end{bmatrix} \right\|^2_{L^2\times L^2\times L^2}$$

$$\leq 4\cdot 1\cdot \left\| \begin{bmatrix} 0 \\ a_\phi \\ b_\phi \end{bmatrix} - \begin{bmatrix} 0 \\ a_\sigma \\ b_\sigma \end{bmatrix} \right\|^2_{L^2\times L^2\times L^2}$$

$$= 4\int_\Omega \left[2(u_1+w_1)-\tfrac{2}{\alpha^2}(v_2+y_2)-2(r_1+w_1)+\tfrac{2}{\beta^2}(s_2+y_2)\right]^2$$

$$+\left[2(u_2+w_2)+\tfrac{2}{\alpha^2}(v_1+y_1)-2(r_2+w_2)+\tfrac{2}{\beta^2}(s_1+y_1)\right]^2$$

$$\leq 32\int_\Omega (u_1-r_1)^2+(u_2-r_2)^2+\left[\tfrac{1}{\alpha^2}(v_2+y_2) - \tfrac{1}{\beta^2}(s_2+y_2)\right]^2$$

$$+\left[\tfrac{1}{\alpha^2}(v_1+y_1) - \tfrac{1}{\beta^2}(s_1+y_1)\right]^2$$

$$\leq 32\cdot\|u-r\|^2_{H^1_0} + 32\cdot\|v-s\|^2_{H^1_0} \tag{3.10}$$

$$+ 32\int_\Omega \left[\tfrac{1}{\alpha^2}(v_2+y_2) - \tfrac{1}{\beta^2}(s_2+y_2)\right]^2$$

$$+ 32\int_\Omega \left[\tfrac{1}{\alpha^2}(v_1+y_1) - \tfrac{1}{\beta^2}(s_1+y_1)\right]^2.$$

Focusing on the third term of the right side of (3.10), if $A_1(r,s) = \frac{1}{|\theta_{r,s}|^2}$, $A_2(r,s)(x) = 1$ and $A_3(r,s) = (s_2+y_2)$ for each $(r,s) \in B[\phi,\beta]$, then these three functions satisfy the hypotheses of lemmas 3.2 and 3.3 with β and N as the constants from lemma 3.3. Thus, the third term of the right side of (3.10) satisfies

$$32\int_\Omega \left[\tfrac{1}{\alpha^2}(v_2+y_2) - \tfrac{1}{\beta^2}(s_2+y_2)\right]^2$$

$$\leq 32\int_\Omega \left[A_1(\phi)A_2(\phi)A_3(\phi) - A_1(\sigma)A_2(\sigma)A_3(\sigma)\right]^2$$

$$\leq 32\cdot N\cdot\|\phi-\sigma\|^2_{H^3_0\times H^3_0}.$$

Similarly, the fourth term of the right side of (3.10) satisfies the following inequality

$$32\int_{\Omega}\left[\tfrac{1}{\alpha^2}(v_1+y_1) - \tfrac{1}{\beta^2}(s_1+y_1)\right]^2 \leq 8\cdot N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}.$$

Combining these inequalities involving the right side of (3.10), it has been shown that

$$\begin{aligned}
&4\int_{\Omega}(F_1-H_1)^2 \\
&\leq 32\cdot\|u-r\|^2_{H_0^1} + 32\cdot\|v-s\|^2_{H_0^1} + 64\cdot N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} \\
&\leq 32\cdot N_1\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} + 64\cdot N\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3} \\
&= (32\cdot N_1 + 64\cdot N)\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}
\end{aligned}$$

where $N_1 > 0$ is an embedding constant for $H_0^3\times H_0^3$ in $H_0^1\times H_0^1$. These calculations and inequalities hold also for the second term of the right side of (3.8).

The third term of the right side of (3.6) can be bounded in the same way as the second term. Combining all of these inequalities, it follows that there exists an $M_F > 0$ such that for each $\sigma \in B[\phi,\beta]$

$$\|F-H\|^2_{H_0^3} \leq M_F\cdot\|\phi-\sigma\|^2_{H_0^3\times H_0^3}.$$

In an analogous way, there exists an $M_G > 0$ such that for each $\sigma \in B[\phi,\beta]$

$$\|G-K\|^2_{H^3_0} \le M_G \cdot \|\phi-\sigma\|^2_{H^3_0\times H^3_0}.$$

Combining these two inequalities, it follows that

$$\left\|Q\begin{bmatrix}u\\v\end{bmatrix}-Q\begin{bmatrix}r\\s\end{bmatrix}\right\|^2_{H^3_0\times H^3_0} \le (M_F+M_G)\cdot\|\phi-\sigma\|^2_{H^3_0\times H^3_0}.$$

Therefore, Q is locally Lipschitz on BZ and the theorem is proved.

Using a standard existence argument, for each (u_0,v_0) in BZ there exists an $\epsilon > 0$ and a function $z:[0,\epsilon) \to BZ$ such that $z'(t) = -Q(z(t))$ and $z(0) = (u_0,v_0)$. In many numerical experiments using steepest descent and an initial pair (w,y), evidence of the global existence of z has been observed. This has been the case even for (w,y) with nonplanar boundary conditions. A more complete discussion of these results will be given in Chapter IV. The global existence of trajectories shall be part of the hypothesis of the following theorem.

<u>THEOREM 3.4</u> Suppose $(u_0,v_0) = \phi_0 \in BZ$ and $z:[0,\infty)\to BZ$ such that for each $t \in [0,\infty)$ $z'(t) = -Q(z(t))$ and $z(0) = \phi_0$. Then,

(i) There exists an increasing, unbounded sequence $\{t_i\}_{i=1}^{\infty} \subset [0,\infty)$ such that $\lim_{i\to\infty} \|Q(z(t_i))\|_{H^1_0\times H^1_0} = 0$,

(ii) If Range(z) is bounded in $H^3_0\times H^3_0$, then there is a sequence $\{t_i\}_{i=1}^{\infty}$ satisfying part (i) such that $\{z(t_i)\}_{i=1}^{\infty}$ converges strongly in $C_0^1 \times C_0^1$ to a point ϕ in $H_0^3 \times H_0^3$, and

(iii) If ϕ from part (ii) is in BZ, then $\nabla I(\phi) = 0$.

PROOF: Suppose the hypothesis of the theorem is true. If there is a $t_0 \in [0,\infty)$ such that $\|Q(z(t_0))\|_{H_0^1 \times H_0^1} = 0$, then for each $t \geq t_0$, $z'(t) = 0$ and $z(t) = z(t_0)$. In this case the theorem in clearly true. Therefore, suppose that $\|Q(z(t_0))\|_{H_0^1 \times H_0^1} > 0$ for each $t \in [0,\infty)$.

Part (i): Let $f:[0,\infty) \to \mathbb{R}$ such that $f(t) = I(z(t))$. Differentiating, it follows that

$$\begin{aligned} f'(t) &= I'(z(t))z'(t) = \langle Q(z(t)), z'(t) \rangle_{H_0^1 \times H_0^1} \\ &= \langle Q(z(t)), -Q(z(t)) \rangle_{H_0^1 \times H_0^1} = -\|Q(z(t))\|^2_{H_0^1 \times H_0^1}. \end{aligned}$$

Thus, f is a decreasing function. Note also that f is bounded below by 0 and $-f'(t) > 0$ for each $t \in [0,\infty)$. Assume $-f'$ is bounded away from zero by some $\delta > 0$. Then for each $t \in [0,\infty)$, $f'(t) \leq -\delta$ and $f(t) \leq -\delta t + f(0)$. But then there must be an $s \in [0, \frac{f(0)}{\delta}]$ such that $f(s) = 0$. The fact that $f(t) \geq 0$ for each t implies $f(t) = 0$ for each $t \geq s$. Thus, $f'(s+1) = 0$. But $f'(s+1) = -\|Q(z(s+1))\|^2_{H_0^1 \times H_0^1} < 0$. This is a contradiction and so $-f'$ is not bounded away from 0. Since $- f'(t) > 0$ for each $t \in [0,\infty)$, an increasing unbounded sequence $\{t_i\}_{i=1}^{\infty}$ can be constructed such that $\lim_{i\to\infty} - f'(t_i) = \lim_{i\to\infty} \|Q(z((t_i))\|^2_{H_0^1 \times H_0^1}$ exists and is 0, which is the conclusion of part (i).

Part (ii): Suppose the Range(z) is bounded in $H_0^3 \times H_0^3$. There exists a bounded sequence in $H_0^3 \times H_0^3$ satisfying part (i). By a well known property of Hilbert Spaces [8, p. 89], there exists a subsequence $\{\sigma_i\}$ which converges weakly to an element ϕ in $H_0^3 \times H_0^3$. Since this sequence is also bounded in $H_0^3 \times H_0^3$ and $H_0^3 \times H_0^3$ is compactly embedded in $C_0^1 \times C_0^1$, there exists a subsequence $\{\phi_i\}$ of $\{\sigma_i\}$ which converges strongly in $C_0^1 \times C_0^1$ to an element $\bar{\phi}$ of $C_0^1 \times C_0^1$. Since $\{\phi_i\}$ converges weakly in $H_0^3 \times H_0^3$ to ϕ, it converges weakly to ϕ in $H_0^1 \times H_0^1$. But, since $\{\phi_i\}$ converges strongly to $\bar{\phi}$ in $C_0^1 \times C_0^1$, it converges weakly to $\bar{\phi}$ in $C_0^1 \times C_0^1$ and hence it converges weakly to $\bar{\phi}$ in $H_0^1 \times H_0^1$. Thus, $\{\phi_i\}$ converges weakly to both ϕ and $\bar{\phi}$ in $H_0^1 \times H_0^1$, implying that $\bar{\phi} = \phi$. Therefore, $\{\phi_i\}$ converges strongly in $C_0^1 \times C_0^1$ to an element $\phi \in H_0^3 \times H_0^3$, which is the conclusion of part (ii).

Part (iii): Suppose $\{\phi_i\} = \{z(t_i)\} = \{(u^i, v^i)\}$ is the sequence generated in part (ii) and $\phi = (u,v) \in BZ$. The strategy is to show that $Q(\phi)$ is the weak limit of the sequence $\{Q(z(t_i))\} = \{Q(u^i, v^i)\}$ in $H_0^1 \times H_0^1$. Let $\epsilon > 0$.

For each $g \in C_0^1$, $\|g\|_{C_0^1} = \sup_{x\in\Omega} \left\{|g_1(x)|, |g_2(x)|\right\}$. Define

$F:\mathbb{R}^5 \to \mathbb{R}$ by $F(x_1,x_2,x_3,x_4,x_5) = (x_1x_3-x_2x_4)^2x_5$. Since $((u^i,v^i))$ converges to (u,v) in $C_0^1\times C_0^1$, there is a compact interval S containing the ranges of u_1+w_1, u_2+w_2, v_1+y_1, v_2+y_2, $u_1^m+w_1$, $u_2^m+w_2$, $v_1^m+y_1$ and $v_2^m+y_2$ for each $m \in Z^+$. Thus, F is uniformly continuous on S^5. Since F is uniformly continuous on S^5 and $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $\|u^m-u\|_{C_0^1} < \delta_1$ and $\|v^m-v\|_{C_0^1} < \delta_1$, then for each $x \in \Omega$

$$(3.11)\qquad \begin{aligned}&|F(u_1+w_1,u_2+w_2,v_1+y_1,v_2+y_2,a)(x)\\ &\quad -F(u_1^m+w_1,u_2^m+w_2,v_1^m+y_1,v_2^m+y_2,b)(x)| < \frac{\epsilon\cdot\beta^4}{4}\end{aligned}$$

where (a,b) is one of the pairs $(v_2^m+y_2,v_2+y_2)$, $(v_1^m+y_1,v_1+y_1)$, $(u_2^m+w_2,u_2+w_2)$ or $(u_1^m+w_1,u_1+w_1)$. Note that (3.11) is the same as

$$(3.12)\qquad |(\alpha^2a)(x)-((\alpha^m)^2b)(x)| < \frac{\epsilon\cdot\beta^4}{4}$$

for each appropriate pair (a,b). Let δ be the minimum of δ_1 and ϵ.

Since $\phi \in BZ$, $\{z(t_i)\}$ converges to ϕ strongly in $C_0^1\times C_0^1$ and $\frac{\delta}{2} > 0$, there exist positive numbers β and n such that if $m \in Z^+$ and $m > n$, then

$$|\alpha(x)| = \left|\det\begin{bmatrix}u_1+w_1 & u_2+w_2\\ v_1+y_1 & v_2+y_2\end{bmatrix}(x)\right| > \beta,$$

$$|\alpha^m(x)| = \left|\det\begin{bmatrix}u_1^m+w & u_2^m+w_2\\ v_1^m+y_1 & v_2^m+y_2\end{bmatrix}(x)\right| > \beta \text{ and}$$

$$\left\|\begin{bmatrix} u^m-u \\ v^m-v \end{bmatrix}\right\|^2_{H_0^1\times H_0^1} \le 2\cdot\left\|\begin{bmatrix} u^m-u \\ v^m-v \end{bmatrix}\right\|^2_{C_0^1\times C_0^1} < \frac{\delta}{4} .$$

Let $(g,h) \in H_0^1\times H_0^1$ such that $\left\|\begin{bmatrix} g \\ h \end{bmatrix}\right\|^2_{H_0^1\times H_0^1} = 1$. For each $m \ge n$,

$$\left|\left\langle Q(z(t_m))-Q(\phi), \begin{bmatrix} g \\ h \end{bmatrix}\right\rangle_{H_0^1\times H_0^1}\right|$$

$$= \left|\left\langle 2\begin{pmatrix} (u_1^m+w_1) \\ (u_2^m+w_2) \\ (v_1^m+y_1) \\ (v_2^m+y_2) \end{pmatrix} + \frac{1}{(\alpha^m)^2}\begin{pmatrix} -(v_2^m+y_2) \\ +(v_1^m+y_1) \\ +(u_2^m+w_2) \\ -(u_1^m+w_1) \end{pmatrix} , \begin{pmatrix} g \\ h \end{pmatrix}\right\rangle_{H_0^1\times H_0^1}\right.$$

$$\left. - \left\langle 2\begin{pmatrix} (u_1+w_1) \\ (u_2+w_2) \\ (v_1+y_1) \\ (v_2+y_2) \end{pmatrix} + \frac{1}{\alpha^2}\begin{pmatrix} -(v_2+y_2) \\ +(v_1+y_1) \\ +(u_2+w_2) \\ -(u_1+w_1) \end{pmatrix} , \begin{pmatrix} g \\ h \end{pmatrix}\right\rangle_{H_0^1\times H_0^1}\right|$$

$$= \left| 2\int_\Omega (u_1^m-u_1)g_1+(u_2^m-u_2)g_2+(v_1^m-v_1)h_1+(v_2^m-v_2)h_2\right.$$

$$+ \int_\Omega \left[\frac{-1}{(\alpha^m)^2}(v_2^m-y_2) + \frac{1}{\alpha^2}(v_2+y_2)\right]g_1$$

$$+ \int_\Omega \left[\frac{1}{(\alpha^m)^2}(v_1^m-y_1) - \frac{1}{\alpha^2}(v_1+y_1)\right]g_2$$

$$+ \int_\Omega \left[\frac{1}{(\alpha^m)^2}(v_2^m-y_2) - \frac{1}{\alpha^2}(v_2+y_2)\right]h_1$$

$$\left. + \int_\Omega \left[\frac{-1}{(\alpha^m)^2}(v_1^m-y_1) + \frac{1}{\alpha^2}(v_1+y_1)\right]h_2\right|$$

$$\le 2\left|\left\langle \begin{bmatrix} u^m-u \\ v^m-v \end{bmatrix}, \begin{bmatrix} g \\ h \end{bmatrix}\right\rangle_{H_0^1\times H_0^1}\right|$$

$$+ \int_\Omega \left|\frac{\alpha^2(v_2^m+y_2)-\alpha^{m\,2}(v_2+y_2)}{(\alpha^m)^2\alpha^2}\right| |g_1|$$

$$+ \int_\Omega \left| \frac{\alpha^2(v_1^m+y_1)-\alpha^{m2}(v_1+y_1)}{(\alpha^m)^2\alpha^2} \right| |g_2|$$

$$+ \int_\Omega \left| \frac{\alpha^2(u_2^m+w_2)-\alpha^{m2}(u_2+w_2)}{(\alpha^m)^2\alpha^2} \right| |h_1|$$

$$+ \int_\Omega \left| \frac{\alpha^2(u_1^m+w_1)-\alpha^{m2}(u_1+w_1)}{(\alpha^m)^2\alpha^2} \right| |h_2|$$

$$\leq 2\left\| \begin{bmatrix} u^m-u \\ v^m-v \end{bmatrix} \right\|_{H_0^1\times H_0^1} \cdot \left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|_{H_0^1\times H_0^1} + \int_\Omega \frac{\epsilon\beta^4}{4}\cdot\frac{1}{\beta^4}\cdot|g_1|$$

$$+ \int_\Omega \frac{\epsilon\beta^4}{4}\cdot\frac{1}{\beta^4}\cdot|g_2| + \int_\Omega \frac{\epsilon\beta^4}{4}\cdot\frac{1}{\beta^4}\cdot|h_1| + \int_\Omega \frac{\epsilon\beta^4}{4}\cdot\frac{1}{\beta^4}\cdot|h_2|$$

$$< 2\cdot\frac{\epsilon}{4}\cdot 1 + \frac{\epsilon}{4}\left[\int_\Omega |g_1|^2 + \int_\Omega |g_2|^2 + \int_\Omega |h_1|^2 + \int_\Omega |h_2|^2\right]$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{4}\cdot\left\| \begin{bmatrix} g \\ h \end{bmatrix} \right\|^2_{H_0^1\times H_0^1} < \epsilon \ .$$

Thus $\{Q(z(t_m))\}$ converges weakly to $Q(\phi) = Q(u,v)$. Using the fact that $\|Q(u,v)\|_{H_0^1\times H_0^1} \leq \liminf \{\|Q(z(t_m))\|_{H_0^1\times H_0^1}\}$, [8, p. 60], and $\lim_{m\to\infty} \|Q(z(t_m))\|_{H_0^1\times H_0^1} = 0$, it follows that $\|Q(u,v)\|_{H_0^1\times H_0^1} = 0$. Thus, $Q(u,v) = 0$. By a result analogous to lemma 1.3, this implies $(\nabla I)(u,v) = 0$. Thus, $(u,v) = \phi$ is a critical point of I.

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CHAPTER IV

NUMERICAL SOLUTIONS TO THE TWO DIMENSIONAL PROBLEM FROM ELASTICITY

Motivation for most of the results in the first three chapters comes from numerical experiments using a finite dimensional version of the problem. In some respects, the theorems of chapters II and III are an indication of the reliability of the numerical method used. In this chapter, an explanation of this method will be given, as well as results of computer runs using several initial pair (w,y). A self contained True Basic program for the two dimensional setting is given in appendix A.

Let n be a positive integer such that $n \geq 4$, Ω be $[0,1]\times[0,1] \subset \mathbb{R}^2$ and $\delta = \frac{1}{n}$. Let $G = \left\{(i,j):\ i,j\in\{0,1,\cdots,n\}\right\}$ and $G' = \left\{(i,j):\ i,j\in\{1,2,\cdots,n-1\}\right\}$. Let $H^1(n+1)$ denote the space of functions from G to $\mathbb{R}^{(n+1)^2}$ and $L^2(n-1)$ denote the space of functions from G' to $\mathbb{R}^{(n-1)^2}$. If $u \in H^1(n+1)$, $u(i,j) = u_{i,j}$ will represent the value of the corresponding function $\overline{u}$ at the point $(\frac{i}{n},\frac{j}{n}) \in \Omega$, where $i,j\in\{0,1,\cdots,n\}$.

In order to define a norm for $H^1(n+1)$, it is necessary to define two difference operators. Let $D1:H^1(n+1)\to L^2(n-1)$ such that for each $u \in H^1(n+1)$

$$(4.1) \qquad D1u(i,j) = \frac{u(i+1,j) - u(i-1,j)}{2\delta}.$$

Let $D2:H^1(n+1) \to L^2(n-1)$ such that for each $u \in H^1(n+1)$

$$(4.2) \qquad D1u(i,j) = \frac{u(i,j+1) - u(i,j-1)}{2\delta}.$$

The norm for $H^1(n+1)$ shall be the Sobolev norm defined in the following way: if $u \in H^1(n+1) = H^1$ then

$$(4.3) \qquad \|u\|^2_{H^1} = \delta^2 \left[\sum_{i,j=0}^{n} [u(i,j)]^2 + \sum_{i,j=1}^{n-1} [D1u(i,j)]^2 + [D2u(i,j)]^2 \right].$$

The norm for $L^2(n-1)$ is the Euclidean norm giveen by

$$\|u\|^2_{L^2(n-1)} = \delta^2 \sum_{i,j=1}^{n-1} [u(i,j)]^2.$$

It is easy to see that each of $L^2(n-1)$ and H^1 is a finite dimensional Hilbert space with the appropriate inner product. A special subspace of H^1 is $H^1_0(n+1) = \left\{ u \in H^1 : u(i,j) = 0 \text{ if } i \text{ or } j \text{ belongs to } (o,n) \right\}$. Denote $H^1_0(n+1)$ by H^1_0.

The problem to be solved is to minimize a functional $I: H^1_0 \times H^1_0 \to \mathbb{R}$. Suppose $(w,y) \in H^1 \times H^1$. For each $(u,v) \in H^1_0 \times H^1_0$, define I by

$$(4.4) \qquad I(u,v) = \delta^2 \sum_{i,j=1}^{n-1} \Big[[D1(u+w)(i,j)]^2$$

$$+ [D2(u+w)(i,j)]^2 + [D1(v+y)(i,j)]^2$$

$$+ [D2(v+y)(i,j)]^2 + \frac{1}{|\alpha_{i,j}(u,v)|}\Bigg]$$

where $\alpha_{ij}(u,v) = \det\begin{bmatrix} D1(u+w)(i,j) & D2(u+w)(i,j) \\ D1(v+y)(i,j) & D2(v+y)(i,j) \end{bmatrix}$. The only problem with the differentiability of I at a point (u,v) in this finite dimensional setting is if $\alpha_{ij}(u,v) = 0$ for any $(i,j) \in G'$, which would also imply that I is not defined at (u,v). Let BZ be the subset of $H_0^1 \times H_0^1$ such that $(u,v) \in BZ$ if and only if $\alpha_{ij}(u,v) > 0$ for each $(i,j) \in G'$. Again, BZ is an open subset of $H_0^1 \times H_0^1$. It is easy to see that I is Fréchet differentiable on BZ. Performing this calculation, one gets

$$I'(u,v)(g,h)$$

$$(4.5) \qquad = \delta^2 \sum_{i,j=1}^{n-1} \left\langle \begin{bmatrix} D1g_{ij} \\ D2g_{ij} \\ D1h_{ij} \\ D2h_{ij} \end{bmatrix}, 2\begin{bmatrix} D1u+w_{ij} - \lambda_{ij} D2v+y_{ij} \\ D2u+w_{ij} + \lambda_{ij} D1v+y_{ij} \\ D1v+y_{ij} + \lambda_{ij} D2u+w_{ij} \\ D2v+y_{ij} - \lambda_{ij} D1u+w_{ij} \end{bmatrix} \right\rangle_{\mathbb{R}^4}$$

$$(4.6) \qquad = \delta^2 \sum_{i,j=1}^{n-1} \left\langle \begin{bmatrix} D1g_{ij} \\ D2g_{ij} \end{bmatrix}, \begin{bmatrix} A_{ij} \\ B_{ij} \end{bmatrix} \right\rangle_{\mathbb{R}^2} + \left\langle \begin{bmatrix} D1h_{ij} \\ D2h_{ij} \end{bmatrix}, \begin{bmatrix} C_{ij} \\ F_{ij} \end{bmatrix} \right\rangle_{\mathbb{R}^2}$$

where for each $(i,j) \in G'$, $\lambda_{ij} = \frac{1}{(\alpha_{ij})^2}$ and A_{ij}, B_{ij}, C_{ij} and F_{ij} are the obvious values from the right side of (4.5) depending on (u,v). Let $A_{uv} = \{A_{ij}\}$, $B_{uv} = \{B_{ij}\}$, $C_{uv} = \{C_{ij}\}$ and $F_{uv} = \{F_{ij}\}$, noting that each of these is a member of $L^2(n-1)$.

Let $\mathcal{D}$ be the linear transformation from $H^1(n+1)$ to the space $K = L^2(n+1) \times L^2(n-1) \times L^2(n-1)$ such that $\mathcal{D}(u)$ is the triple $(u, D1u, D2u)$. Let $\mathcal{P}$ be the orthogonal projection of K onto its subspace K_0 consisting of elements of the form $(g, D1g, D2g)$. Note that K_0 is isomorphic to H_0^1. Then (4.6) becomes

$$I'(u,v)(g,h)$$

$$(4.7) \qquad = \delta^2 \sum_{i,j=1}^{n-1} \left\langle \mathcal{D}g_{ij}, \begin{bmatrix} 0 \\ A_{ij} \\ B_{ij} \end{bmatrix} \right\rangle_{\mathbb{R}^2} + \left\langle \mathcal{D}h_{ij}, \begin{bmatrix} 0 \\ C_{ij} \\ F_{ij} \end{bmatrix} \right\rangle_{\mathbb{R}^2}$$

$$= \left\langle \mathcal{D}g, \begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix} \right\rangle_K + \left\langle \mathcal{D}h, \begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix} \right\rangle_K$$

$$= \left\langle \mathcal{D}g, \mathcal{P}\begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix} \right\rangle_K + \left\langle \mathcal{D}h, \mathcal{P}\begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix} \right\rangle_K$$

$$= \left\langle g, \pi_1 \mathcal{P}\begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix} \right\rangle_{H_0^1} + \left\langle h, \pi_1 \mathcal{P}\begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix} \right\rangle_{H_0^1}.$$

Thus, the gradient $\left[\nabla_{(H_0^1 \times H_0^1)} I\right]\begin{bmatrix} u \\ v \end{bmatrix} = \left[\pi_1 \mathcal{P}\begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix}, \ \pi_1 \mathcal{P}\begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix} \right]$.

Denote this gradient by $(\nabla_H I)(u,v)$.

The strategy for the numerical scheme is to investigate the differential equation

$$(4.8) \qquad z'(t) = -(\nabla_H I)(z(t)) \quad \text{and} \quad z(0) = (u_0, v_0)$$

for $(u_0, v_0) \in BZ$ and $z:[0,\infty) \to H_0^1 \times H_0^1$. Following the notation used by Neuberger in [5], let $E = \pi_0 D^t D|_{H_0^1}$ where π_0 is the orthogonal projection of H^1 onto H_0^1. E is non-negative and invertible. The following theorem gives an explicit formula for $(\nabla_H I)(u,v)$.

<u>THEOREM 4.1</u> Suppose $(u,v) \in H_0^1 \times H_0^1$. Then $P = DE^{-1}\pi_0 D^t$.

The proof involves showing that P satisfies the necessary and sufficient conditions for an orthogonal projection, that is, $P^2 = P$, $P(u, D1u, D2u) = (u, D1u, D2u)$ for $(u, D1u, D2u) \in K_0$, $P^* = P$ and the $R(P) \subset K_0$. A careful examination of P reveals that

$$(4.9) \qquad \pi_1 P \begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix} = E^{-1}\pi_0 D^t \begin{bmatrix} 0 \\ A_{uv} \\ B_{uv} \end{bmatrix}$$

and

$$(4.10) \qquad \pi_1 P \begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix} = E^{-1}\pi_0 D^t \begin{bmatrix} 0 \\ C_{uv} \\ F_{uv} \end{bmatrix}$$

An equivalent norm for the subspace $H_0^1 \times H_0^1$ is given by the following: for each $(u,v) \in H_0^1 \times H_0^1$

$$(4.11) \qquad \|(u,v)\|^2_{H_0^1 \times H_0^1} = \delta^2 \sum_{i,j=1}^{n-1} [D1u(i,j)]^2 + [D2u(i,j)]^2$$

$$+ \delta^2 \sum_{i,j=1}^{n-1} [D1v(i,j)]^2 + [D2v(i,j)]^2.$$

Consequently, I is coercive and as well as bounded below on BZ. Although I is not defined on all of $H_0^1 \times H_0^1$, in particular the boundary of BZ, steepest descent avoids these places because I is lower semi-continuous [2] and blows up near boundary points of BZ. From the theory of ordinary differential equations arguments, there exists a positive number δ and a function $z:[0,\delta) \to H_0^1 \times H_0^1$ such that z satisfies (4.8). Using these ideas and an argument similar to that of lemma 3 from [4, p. 191], the following result can be proved.

<u>THEOREM 4.2</u> Suppose $(u_0, v_0) \in BZ$. There exists a solution z to (4.8) such that $z:[0,\infty) \to BZ$.

A key to this argument is that I(z) is a decreasing function for every such z. This means that I has some properties of a Lyapunov function.

The problem of looking for the asymptotic limit of z is reformulated numerically as constructing the sequence

$$(4.12) \qquad Z_{n+1} = Z_n - h_n \cdot (\nabla_H I)(Z_n) \qquad n = 0,1,2,3,\cdots$$

or equivalently, using $Z_n = (U_n, V_n) \in H_0^1 \times H_0^1$ and equalities (4.9) and (4.10),

$$(4.13) \qquad U_{n+1} = U_n - h_n \cdot [F_n]$$

and

$$(4.14) \qquad V_{n+1} = V_n - h_n \cdot [G_n]$$

where for each n

$$F_n = E^{-1}\pi_0 \mathcal{D}^t(0, A_{U_nV_n}, B_{U_nV_n})$$

and

$$G_n = E^{-1}\pi_0 \mathcal{D}^t(0, C_{U_nV_n}, F_{U_nV_n})$$

and $\{h_i\}$ is an sequence of positive time steps which must carefully selected.

The program in Appendix A is divided into six sections. Section 1 defines (W,Y), the initial (U_0, V_0) and the desired accuracy for several processes in the code. Section 2 contains the Call statement for the subrountines which compute $F_n = E^{-1}\pi_0 \mathcal{D}^t(0, A_{U_nV_n}, B_{U_nV_n})$ and $G_n = E^{-1}\pi_0 \mathcal{D}^t(0, C_{U_nV_n}, F_{U_nV_n})$. The description of Section 6 will give more details on each subroutine. Section 3 contains the step defining the new pair (U_{n+1}, V_{n+1}) using (4.13) and (4.14). It also provides an opportunity to monitor the choice of the timestep for the first three steps in the main process. This section calls a subroutine to check the convexity of I at (U_n, V_n), using the quantity

$I''(Z_n)[(\nabla_H I)(Z_n),(\nabla_H I)(Z_n)]$. Section 4 implements the graphic subroutines which help monitor the process. Section 5 prints the approximate solution whenever the error tolerances have been achieved. This section contains the END statement.

Section 6 contains 4 subroutines, three of which are used in computing F_n and G_n. The subroutine "ROUGH_GRAD" computes $(0,A_{U_nV_n},B_{U_nV_n})$ and $(0,C_{U_nV_n},F_{U_nV_n})$. The subroutine "TRANS_D" computes $\mathcal{D}^t(0,A_{U_nV_n},B_{U_nV_n})$ and $\mathcal{D}^t(0,C_{U_nV_n},F_{U_nV_n})$. The subroutine "INV_DTD" computes an approximation to $E^{-1}\pi_0\mathcal{D}^t(0,A_{U_nV_n},B_{U_nV_n})$ and $E^{-1}\pi_0\mathcal{D}^t(0,C_{U_nV_n},F_{U_nV_n})$. This is done using the Gauss-Siedel iteration technique (see Dahlquist [7, pp. 1016-1026]). In this technique, E is decomposed into the sum of three matrices, a strictly lower triangular matrix L, a diagonal matrix M and the strictly upper triangular matrix L^t. Letting $\eta = \mathcal{D}^t(0,A_{U_nV_n},B_{U_nV_n})$, the iterative scheme is generally given by

$$M(X_{k+1}) - M(X_k) = \eta - E(X_k) \tag{4.15}$$

and

$$X_{k+1} = X_k + M^{-1}[\eta - E(X_k)] \tag{4.16}$$

with an updating occuring after the calculation of each component of X_k and (X_k) converges to χ where $E(\chi) = \eta$. The other component of $(\nabla_H I)(U_{n+1},V_{n+1})$ is calculated in the same way.

Table I contains the graphic output of two runs of the program in Appendix A. Each pair of pictures depicts $z(t_k)$ for some k and the value of the norm of the gradient at that point. The first pair of initial functions (W,Y) is a discretization of a pair (w,y) that satisfies the hypotheses in Chapter III and are given by $w(S,T) = S^2+S-T$ and $y(S,T) = S+T+T^2$. Using the initial $(U_0,V_0) = (0,0)$, Figure 1 on page 62 is the initial pair (W,Y). Letting $\lim z(t) = (U_1,V_1)$, Figure 2 is the approximation of the pair $(W+U_1,Y+V_1)$. Figures 3 and 4 on page 63 are the middle cross sections of functions W and $W+U_1$, respectively. Figures 5 and 6 on page 64 are the middle cross sections of the functions Y and $Y+V_1$, respectively. Notice that the graphs of $W+U_1$ and $Y+V_1$ indicate that the critical point is not affine. The square of the norm of the gradient of this approximate solution is 7.212×10^{-5}.

The second pair of initial functions considered is a discretization of a pair (w,y) that does not satisfy the hypotheses of Chapter III. The trajectory z in this case can be observed to crack and the $\lim z(t) = (U_2,V_2)$ belongs to $H_0^1 \times H_0^1$ but not $H_0^2 \times H_0^2$. The formulas for this second pair (w,y) are $w(S,T) = S-S^3$ and $y(S,T) = T$. Again using the initial pair $(U_0,V_0) = (0,0)$, Figure 7 on page 65 is the initial pair (W,Y). Figure 8 is the deformation of (W,Y) after 15 time steps ($k = 15$). Figure 9 is the approximation of the pair $(W+U_2,Y+V_2)$, for which $k = 100$. Figures 10, 11, 12 on page 66 are the corresponding middle cross sections of the functions W, its deformation at step $k = 15$ and $W+U_1$, respectively. The square of the norm of the gradient of this approximate solution is 5.813×10^{-5}.

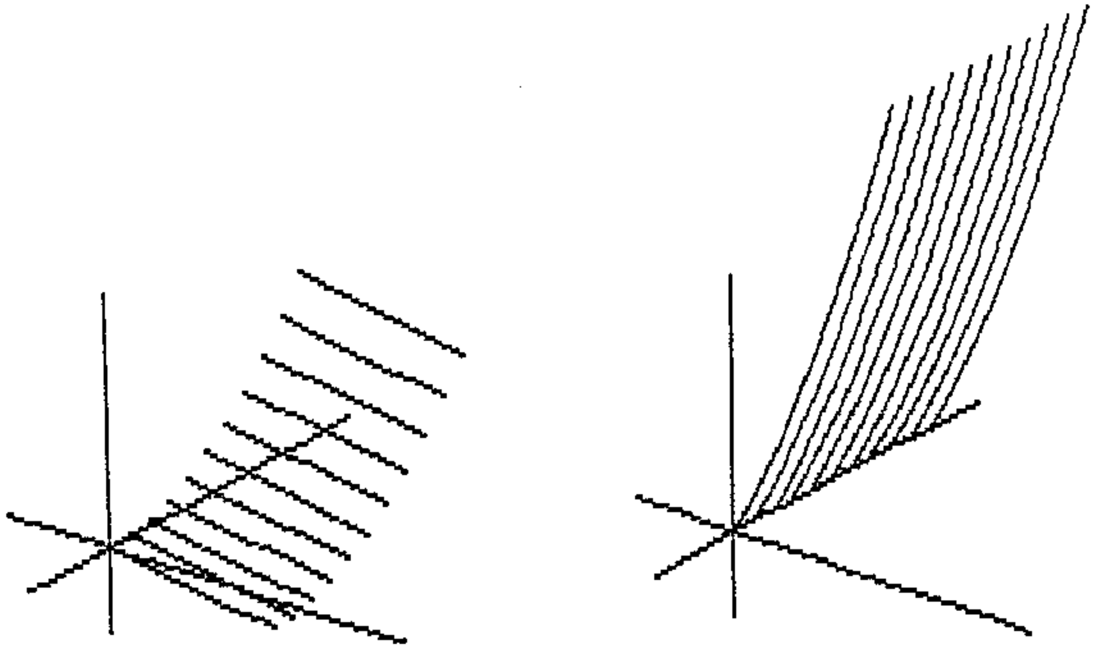

Fig. 1--Graph of initial pair W and Y

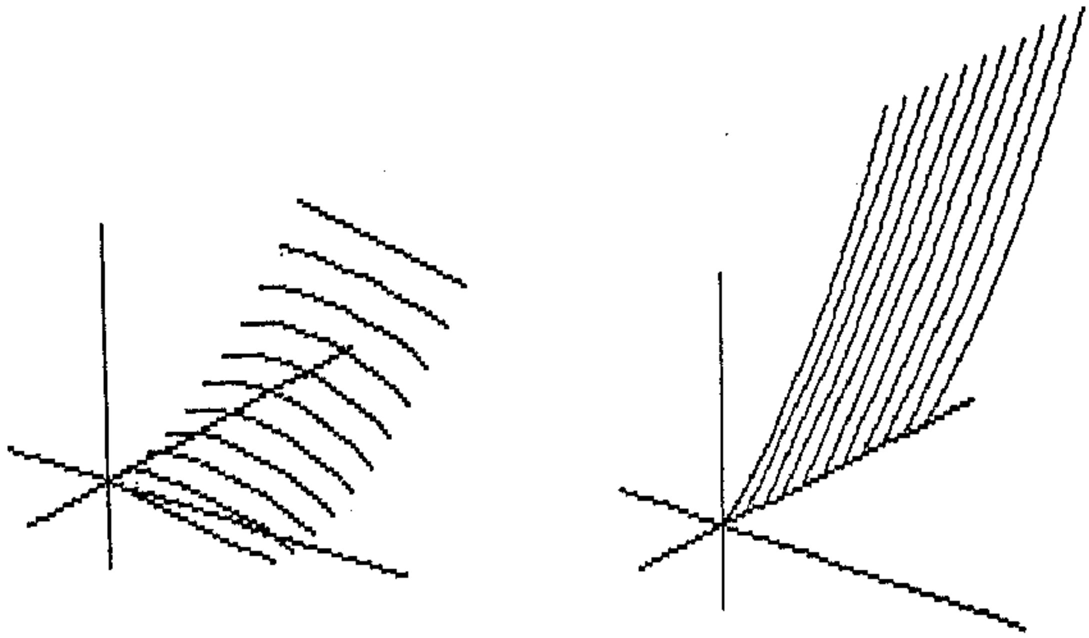

Fig. 2--Graph of approximate solution pair $W+U_t$ and $Y+V_t$

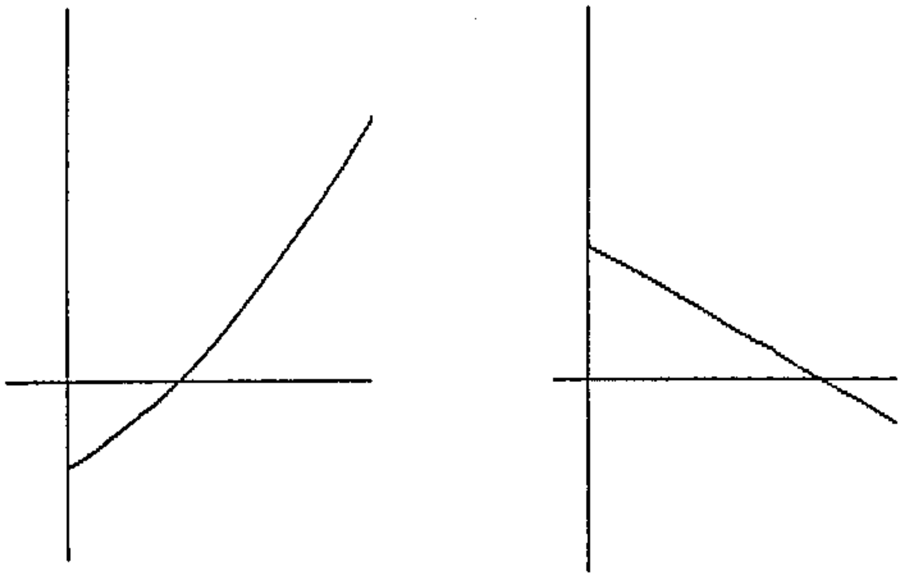

Fig. 3--Cross sections of initial W, W(I,5) and W(5,J)

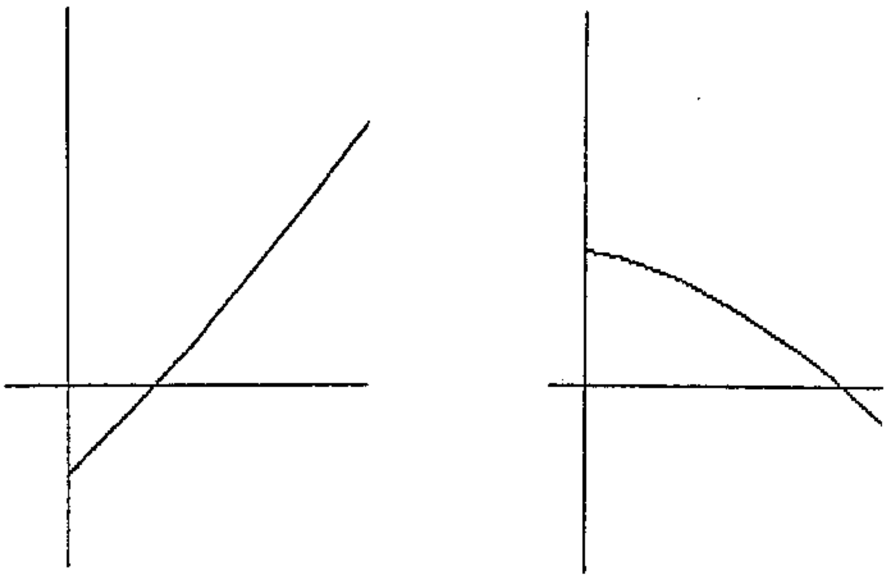

Fig. 4--Cross sections of solutions $(W+U_1)(I,5)$

and $(W+U_1)(5,J)$

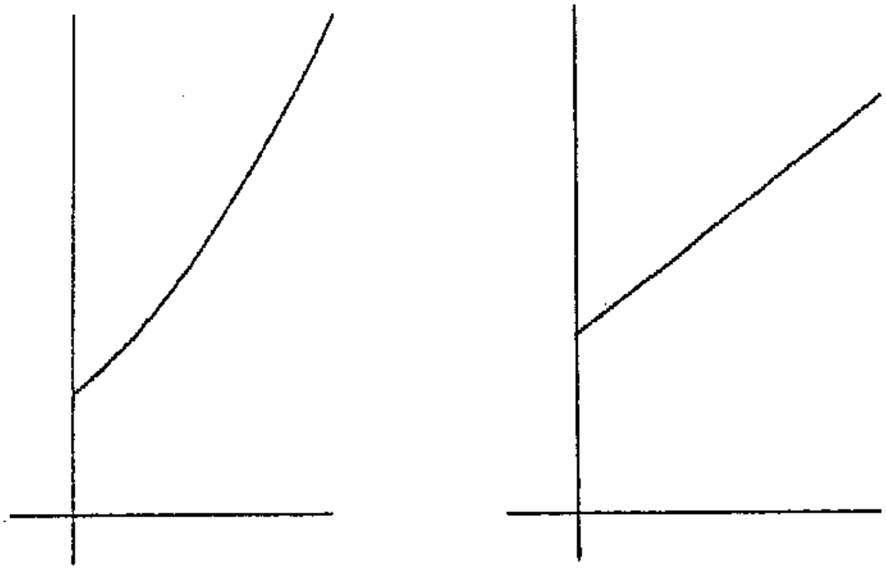

Fig. 5--Cross sections of initial Y, Y(5,J) and Y(I,5)

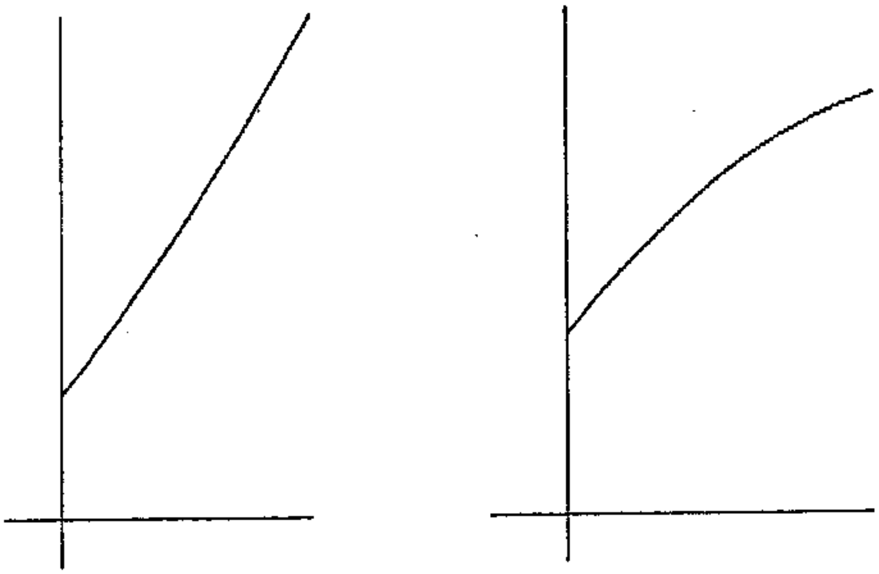

Fig. 6--Cross sections of solutions $(Y+V_1)(5,J)$ and $(Y+V_1)(I,5)$

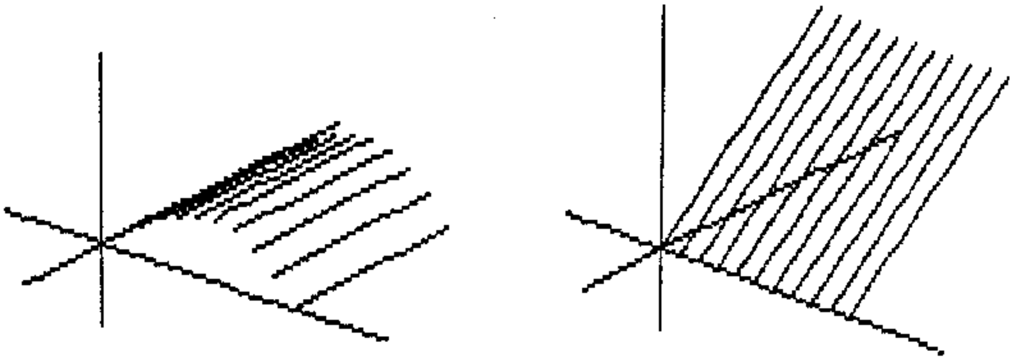

Fig. 7--Initial pair W and Y

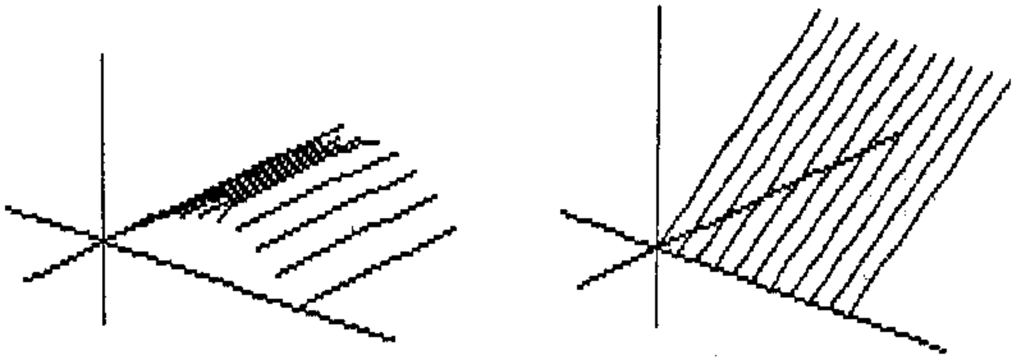

Fig. 8--Deformations of W and Y at $k = 15$

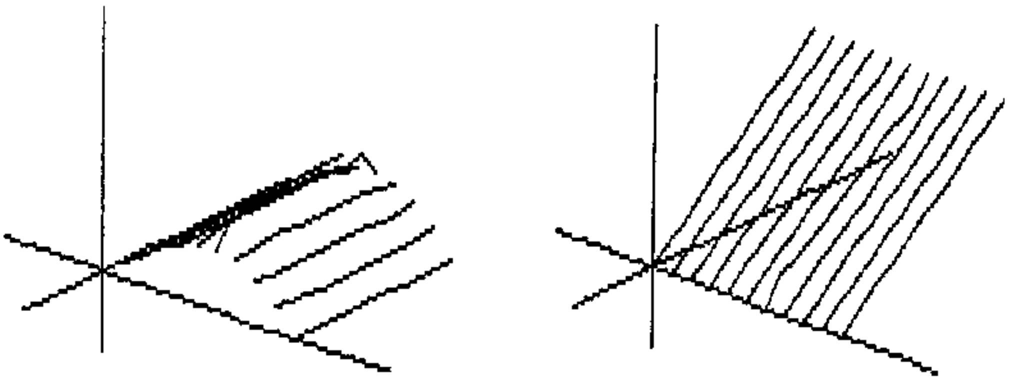

Fig. 9--Graphs of approximate solutions $W+U_2$ and $Y+V_2$

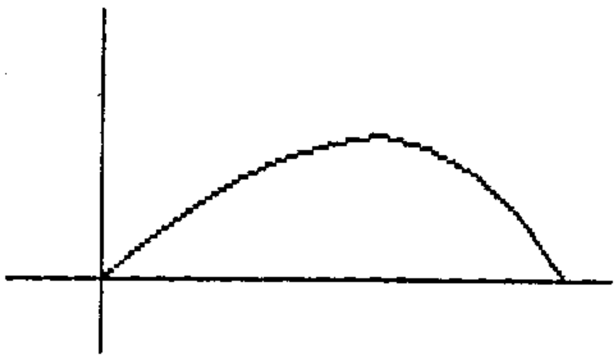

Fig. 10--Cross section W(I,5)

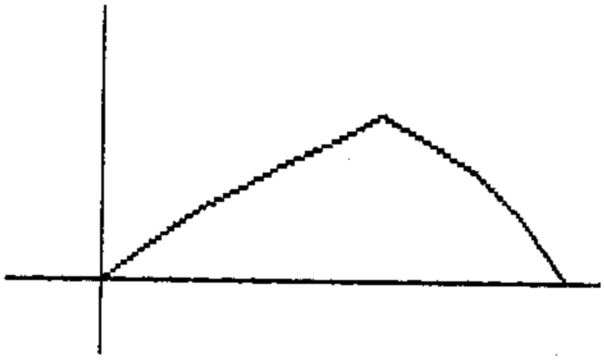

Fig. 11--Cross section of intermediate stage: (W+U)(I,5)

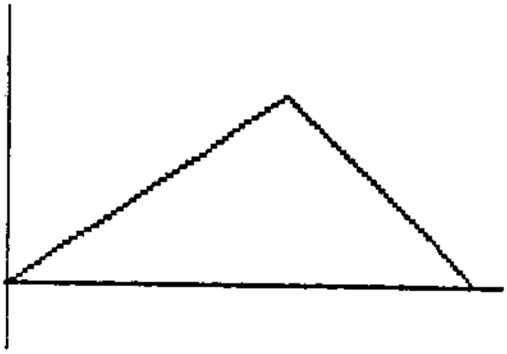

Fig.12--Cross section of solution: $(W+U_2)(I,5)$ at k = 100

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APPENDIX

APPENDIX

```
! PROGRAM NAME: 2D ELST
! ELASTICITY - PHANTOM GRID,ITERATIVE LINEAR SOLVER,
GRAPHICS
! J(U,V)= INTEGRAL [ ||D(U,V)||^2 - 1/(DET(U+W,V+Y)) ]

!------------------------------------------------- SECTION 1

LIBRARY "3DLIB"
PRINT "WHAT SIZE GRID?  N=10"
LET N=10
PRINT "WHAT ACCURACY FOR SOLUTION?  E1=.00000001"
LET E1=.00000001
PRINT "WHAT ACCURACY FOR GRADIENT? E2=.00001"
LET E2=.00001
PRINT "WHAT MESH IN T-DIRECTION";
INPUT H
LET K=1
DIM U(0 TO 10,0 TO 10),V(0 TO 10,0 TO 10),W(0 TO 10,0 TO 10)
DIM Y(0 TO 10,0 TO 10)
DIM X1(0 TO 10,0 TO 10),DT_DU(9,9),RG1(10,10),RG2(10,10)
DIM RG3(10,10),RG4(10,10),X2(0 TO 10,0 TO 10)
LET S=N/2
FOR I=1 TO N-1
    FOR J=1 TO N-1
        LET U(I,J)=0
        LET V(I,J)=0
    NEXT J
NEXT I
FOR I=0 TO N
    FOR J=0 TO N
        LET W(I,J)=(I/N)^2+(I/N)-(J/N)
        LET Y(I,J)=(I/N)++(J/N)+(J/N)^2
        ! LET W(I,J)=(I/N)-(I/N)^3
        ! LET Y(I,J)=(J/N)
    NEXT J
NEXT I
LET ERR1=-1
LET ERR2=-1

!------------------------------------------------- SECTION 2

DO UNTIL MAX(ABS(ERR1),ABS(ERR2))<E1
   CALL ROUGH_GRAD(N,S,U(,),V(,),W(,),Y(,),
                        RG1(,),RG2(,),RG3(,),RG4(,))
```

```
! COMPUTE FIRST COMPONENT OF ROUGH GRADIENT
CALL TRANS_D(N,S,RG1(,),RG2(,),DT_DU(,))
! COMPUTE FIRST COMPONENT OF SMOOTH GRADIENT
LET ERROR=1
MAT X1=0     ! RESET X1=0
CALL INV_DTD(N,S,DT_DU(,),X1(,),E2)
LET ERR1=-1
LET GRAD1=0

! COMPUTE SECOND COMPONENT OF ROUGH GRADIENT
CALL TRANS_D(N,S,RG3(,),RG4(,),DT_DU(,))
! COMPUTE SECOND COMPONENT OF SMOOTH GRADIENT
LET ERROR=1
MAT X2=0     ! RESET X2=0
CALL INV_DTD(N,S,DT_DU(,),X2(,),E2)
LET ERR2=-1
LET GRAD2=0

!-------------------------------------------------------- SECTION 3

! SELECT TIMESTEP IN RELATIONSHIP TO SIZE OF GRADIENT
SELECT CASE K
CASE 1, 2     ! CHECK SIZE OF GRADIENT
     PRINT "K=";K,"GRADIENT OF FIRST VECTOR - START"
     MAT PRINT X1
     PRINT "K=";K,"GRADIENT OF FIRST VECTOR - FINISH"
     PRINT "K=";K,"GRADIENT OF SECOND VECTOR - START"
     MAT PRINT X2
     PRINT "K=";K,"GRADIENT OF SECOND VECTOR - FINISH"
     ! CHECK CONVEXITY OF I AT (U(K),V(K))
     CALL CONVEX(X1(,),X2(,),U(,),W(,),V(,),Y(,),N,S)
     PRINT "WHAT INITIAL TIME STEP";
     INPUT H3
     FOR I=1 TO N-1
         FOR J=1 TO N-1
             LET U(I,J)=U(I,J)-H3*X1(I,J)
             LET T=X1(I,J)
             LET ERR1=MAX(ERR1,ABS(T))
             LET V(I,J)=V(I,J)-H3*X2(I,J)
             LET T=X2(I,J)
             LET ERR2=MAX(ERR2,ABS(T))
         NEXT J
     NEXT I
CASE 3
     PRINT "WHAT TIME STEP AT STEP 3";
     INPUT H
CASE ELSE
     CALL CONVEX(X1(,),X2(,),U(,),W(,),V(,),Y(,),N,S)
     FOR I=1 TO N-1
         FOR J=1 TO N-1
             LET T=X1(I,J)
```

```
                LET ERR1=MAX(ERR1,ABS(T))
                LET T=X2(I,J)
                LET ERR2=MAX(ERR2,ABS(T))
            NEXT J
        NEXT I
        LET TIMESTEP=MIN(1/ERR2,1/ERR1)
        LET H1=MIN(TIMESTEP*.001,H)
        FOR I=1 TO N-1
            FOR J=1 TO N-1
                LET U(I,J)=U(I,J)-H1*X1(I,J)
                LET V(I,J)=V(I,J)-H1*X2(I,J)
            NEXT J
        NEXT I
    END SELECT

!--------------------------------------------------- SECTION 4

    ! PLOT GRAPHS OF U(K)+W AND V(K)+Y
    SET MODE "HIRES"
    CALL PARAWINDOW (0, 1, 0, 1, 0, .5, WORK$)
    SETCAMERA (1, -2, 2)
    FOR I=0 TO 10
        FOR J=0 TO 10
            CALL PLOTON3 (I/10, J/10, W(I,J)+U(I,J), WORK$)
        NEXT J
        PLOT
    NEXT I

    CALL PARAWINDOW (0, 1, 0, 1, -.5, 1, WORK$)
    FOR I=0 TO 10
        FOR J=0 TO 10
            CALL PLOTON3 (I/10, J/10, Y(I,J)+V(I,J), WORK$)
        NEXT J
        PLOT
    NEXT I

    ! PRINT OUT U(K)+W AND V(K)+Y IN NUMERICAL TABLE
    IF INT(K/5)=K/5 THEN
       PRINT "U(I,J) IS THE FOLLOWING AT STEP =";K
       FOR J=N TO 0 STEP -1
           FOR I=0 TO N STEP 1
                PRINT USING "###.####":U(I,J)+W(I,J);
           NEXT I
           PRINT
       NEXT J
       PRINT "________________"
       PRINT
       PRINT "V(I,J) IS THE FOLLOWING AT STEP =";K
       FOR J=N TO 0 STEP -1
           FOR I=0 TO N STEP 1
               PRINT USING "###.####":V(I,J)+Y(I,J);
```

```
            NEXT I
            PRINT
         NEXT J
         PRINT "________________"
      END IF
      PRINT "STEP = ";K;"MAX GRAD1";ERR1;"MAX GRAD2";ERR2
      LET K=K+1

   LOOP

   !------------------------------------------ SECTION 5

   ! PRINT THE SOLUTION PAIR (U+W,V+Y)
   PRINT
   PRINT "U(I,J) IS THE FOLLOWING"
   FOR J=N TO O STEP -1
       FOR I=0 TO N STEP 1
           PRINT USING "###.####":U(I,J)+W(I,J);
       NEXT I
       PRINT
   NEXT J
   PRINT "________________"
   PRINT
   PRINT "V(I,J) IS THE FOLLOWING"
   FOR J=N TO O STEP -1
       FOR I=O TO N STEP 1
           PRINT USING "###.####":V(I,J)+Y(I,J);
       NEXT I
       PRINT
   NEXT J
   PRINT "________________"
   PRINT
   END

   !------------------------------------------ SECTION 6

   SUB ROUGH_GRAD (N, S, U(,), V(,), W(,), Y(,), RG1(,),
   RG2(,),
                                  RG3(,), RG4(,))
     FOR I=1 TO N
       FOR J=1 TO N
         ! A=U1+W1  C=V1+Y1  B=U2+W2  D=V2+Y2
         LET A=S*(U(I,J-1)+W(I,J-1)-U(I-1,J-1)-W(I-1,J-1))
         LET A=A+S*(U(I,J)+W(I,J)-U(I-1,J)-W(I-1,J))
         LET B=S*(-U(I,J-1)-W(I,J-1)-U(I-1,J-1)-W(I-1,J-1))
         LET B=B+S*(U(I,J)+W(I,J)+U(I-1,J)+W(I-1,J))
         LET C=S*(V(I,J-1)+Y(I,J-1)-V(I-1,J-1)-Y(I-1,J-1))
         LET C=C+S*(V(I,J)+Y(I,J)-V(I-1,J)-Y(I-1,J))
         LET D=S*(-V(I,J-1)-Y(I,J-1)-V(I-1,J-1)-Y(I-1,J-1))
         LET D=D+S*(V(I,J)+Y(I,J)+V(I-1,J)+Y(I-1,J))
         LET DET3=(A*D-B*C)^2
```

```
      LET RG1(I,J)=2*(A-(D/DET3))
      LET RG2(I,J)=2*(B-(C/DET3))
      LET RG3(I,J)=2*(C-(B/DET3))
      LET RG4(I,J)=2*(D-(A/DET3))
    NEXT J
  NEXT I
END SUB

SUB CONVEX (X1(,), X2(,), U(,), W(,), V(,), Y(,), N, S)
  LET DET2=1E+6
  LET SUM1=0
  LET SUM2=0
  LET SUM3=0
  LET SUM4=0
  FOR I=1 TO N
    FOR J=1 TO N
      ! A=U1+W1  C=V1+Y1  B=U2+W2  D=V2+Y2
      ! D1X1=FIRST PARTIAL OF GRAD OF X1 ,ETC.
      LET A=S*(U(I,J-1)+W(I,J-1)-U(I-1,J-1)-W(I-1,J-1))
      LET A=A+S*(U(I,J)+W(I,J)-U(I-1,J)-W(I-1,J))
      LET B=S*(-U(I,J-1)-W(I,J-1)-U(I-1,J-1)-W(I-1,J-1))
      LET B=B+S*(U(I,J)+W(I,J)+U(I-1,J)+W(I-1,J))
      LET C=S*(V(I,J-1)+Y(I,J-1)-V(I-1,J-1)-Y(I-1,J-1))
      LET C=C+S*(V(I,J)+Y(I,J)-V(I-1,J)-Y(I-1,J))
      LET D=S*(-V(I,J-1)-Y(I,J-1)-V(I-1,J-1)-Y(I-1,J-1))
      LET D=D+S*(V(I,J)+Y(I,J)+V(I-1,J)+Y(I-1,J))
      LET DET1=(A*D-B*C)
      ! PRINT I;J;"=";DET1,
      LET D1X1=S*(X1(I,J-1)-X1(I-1,J-1)+X1(I,J)-X1(I-1,J))
      LET D2X1=S*(X1(I,J)-X1(I,J-1)+X1(I-1,J)-X1(I-1,J-1))
      LET D1X2=S*(X2(I,J-1)-X2(I-1,J-1)+X2(I,J)-X2(I-1,J))
      LET D2X2=S*(X2(I,J)-X2(I,J-1)+X2(I-1,J)-X2(I-1,J-1))
      LET DETGRAD=D1X1*D2X2-D2X1*D1X2
      ! PRINT "DETGRAD(";I;J;")=";DETGRAD
      LET
SUM1=SUM1+(2/DET1^3)*(D1X1*D-D2X1*C+A*D2X2-B*D1X2)^2
      LET SUM2=SUM2+(2/DET1^2)*DETGRAD
      LET SUM3=SUM3+(A^2+B^2+C^2+D^2)+1/DET1^2
      LET SUM4=SUM4+(D1X1)^2+(D2X1)^2+(D1X2)^2+(D2X2)^2
      LET DET2=MIN(DET1,DET2)
    NEXT J
  NEXT I
  PRINT
  PRINT "MIN DET =";DET2
  PRINT "SUM1 =";SUM1;
  PRINT "SUM2 =";SUM2
  PRINT "NORM OF GRAD";SUM4
  PRINT "I''(U(K),V(K))(GRAD,GRAD) = ";2*SUM4+SUM1-SUM2
  PRINT "I(U(K),V(K)) =";SUM3
END SUB
```

```
SUB TRANS_D (N, S, A(,), B(,), DT_DU(,))
  ! D TRANSPOSE WITH ZERO BOUNDARY CONDITION
  ! THIS IS THE ROUGH GRADIENT
  FOR I=1 TO N-1
    FOR J=1 TO N-1
      LET T=S*(A(I,J)+A(I,J+1)-A(I+1,J)-A(I+1,J+1))
      LET
DT_DU(I,j)=T+S*(B(I,J)-B(I,J+1)+B(I+1,J)-B(I+1,J+1))
    NEXT J
  NEXT I
END SUB

SUB INV_DTD (N, S, DT_DU(,), X(,), E2)

  ! (DTD) = L + M + N
  ! M(X(K)-X(K-1)) = Y - (DTD)(X(K-1)) : ITERATION TO SOLVE
LINEAR
                                            SYSTEM
  ! A = X(K) = INV(M)[ X(K-1) + Y - DTD(X(K-1)]
  !          = INV(M)[ Y - (L+N)(X(K-1))]
  ! X(I,J) = THE GRAD OF U(I,J) OR V(I,J)
  LET L=1
  LET E=10
  DO UNTIL E<E2
    LET ERROR=0
    IF INT(L/2)<L/2 THEN
      FOR I=1 TO N-1
        FOR J=1 TO N-1
          LET A=DT_DU(I,J)+S*S*2*(X(I-1,J-1)+X(I-1,J+1))
          LET A=(A+S*S*2*(X(I+1,J-1)+X(I+1,J+1)))/(8*S*S)
          LET B=A-X(I,J)
          LET X(I,J)=A
          LET ERROR=MAX(ERROR,ABS(B))
        NEXT J
      NEXT I
      LET E=ERROR
    ELSE
      FOR I=N-1 TO 1 STEP -1
        FOR J=N-1 TO 1 STEP -1
          LET A=DT_DU(I,J)+S*S*2*(X(I-1,J-1)+X(I-1,J+1))
          LET A=(A+S*S*2*(X(I+1,J-1)+X(I+1,J+1)))/(8*S*S)
          LET B=A-X(I,J)
          LET X(I,J)=A
          LET ERROR=MAX(ERROR,ABS(B))
        NEXT J
      NEXT I
      LET E=ERROR
    END IF
    LET L=L+1
  LOOP
END SUB
```

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