R₀ SPACES, R₁ SPACES, AND HYPERSPACES

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Charles I. Dorsett, B. S., M. S.

Denton, Texas

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The purpose of this paper is to further investigate R₀ spaces, R₁ spaces, and hyperspaces. The R₀ axiom was introduced by N. A. Shanin in 1943. Later, in 1961, A. S. Davis investigated R₀ spaces and introduced R₁ spaces. Then, in 1975, William Dunham further investigated R₁ spaces and proved that several well-known theorems can be generalized from a T₂ setting to an R₁ setting. In Chapter II R₀ and R₁ spaces are investigated and additional theorems that can be generalized from a T₂ setting to an R₁ setting are obtained.

In 1975 William Dunham used T₀-identification spaces to characterize R₁ spaces. In Chapter III of this paper characterizations of R₀ spaces, R₁ spaces, regular spaces, completely regular spaces, and normal R₀ spaces are obtained using T₀-identification spaces, T₀-identification spaces are investigated for R₀ spaces, and the relationships between \((2^X, E(X))\) and \((2^{X_0}, E(X_0))\) are investigated for \((X, T)\), an R₀ space, and \((X_0, S_0)\), the T₀-identification of \((X, T)\). This information is combined with Ernest Michael's results to generalize many of Michael's results from a T₁ setting to an R₀ setting.
In 1974 Jack T. Goodykoontz, Jr. characterized point-wise local connectedness and connectedness im kleinen of \((2^X, E(X))\) at points of \(C(X)\) for \((X, T)\) a metric continuum as follows: if \((X, T)\) is a metric continuum and \(A \in C(X)\), then \(2^X\) is locally connected (connected im kleinen) at \(A\) if and only if for \(U\) open in \(X\) such that \(A \subseteq U\), there exists an open connected set \(V\) such that \(A \subseteq V \subseteq U\) (the component of \(U\) containing \(A\) contains \(A\) in its interior). In Chapter IV the same result is obtained, where metric continuum is replaced by \(R_0\) and the same characterizations for local connectedness and connectedness im kleinen are obtained for \((K(X), E(X))\) at elements of \(C(X) \cap K(X)\) for \((X, T)\), an \(R_0\) space.

Also, in 1974 Jack T. Goodykoontz, Jr. characterized point-wise local connectedness and connectedness im kleinen of \((2^X, E(X))\) for \((X, T)\) a metric continuum as follows: if \((X, T)\) is a metric continuum and \(A \in 2^X\), then the statements (I) \(2^X\) is locally connected (connected im kleinen) at \(A\) and (II) \(2^X\) is locally connected (connected im kleinen) at each component of \(A\) are equivalent. In Chapter V the relationships between statements (I) and (II) are investigated for \((X, T)\) weaker than a metric continuum and the following result is obtained. If \((X, T)\) is a locally compact \(R_1\) space, then the following are equivalent: (a) \(K(X)\) is locally connected (connected im kleinen) at \(A\), (b) \(K(X)\) is locally connected (connected
im kleinen) at each component of $A$, (c) $2^X$ is locally connected (connected im kleinen) at each component of $A$, and (d) $2^X$ is locally connected (connected im kleinen) at $A$. 
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CHAPTER I

INTRODUCTION

The purpose of this paper is to further investigate the properties of $R_0$ topological spaces, $R_1$ topological spaces, and $(2^X, E(X))$ where $2^X = \{F \subseteq X | F$ is a nonempty closed subset of $X\}$ and $E(X)$ is the Vietoris topology on $2^X$. In this paper, as in Stephen Willard's book General Topology [6], the $T_1$ axiom is not included in the definitions of regular spaces, completely regular spaces, and normal spaces. Regular $T_1$ spaces will be denoted by $T_3$, completely regular $T_1$ spaces will be denoted by $T_{3\frac{1}{2}}$, and normal $T_1$ spaces will be denoted by $T_4$. Notation used in this paper and not defined agrees with notation used in Willard's book General Topology [6].

Listed below are definitions and theorems that will be utilized in subsequent chapters.

**Definition 1.1.** A topological space $(X, T)$ is an $R_0$ space if and only if for each closed set $C$ and for each $x \notin C$, $C \cap \text{Cl}(x) = \emptyset$ [1].

**Theorem 1.1.** Let $(X, T)$ be a topological space. Then the following are equivalent:

(a) $(X, T)$ is an $R_0$ space;

(b) Every open set contains the closure of each of its points;
(c) for all $x \in X$, $y \in X$, either $\text{Cl}(x) = \text{Cl}(y)$ or $\text{Cl}(x) \cap \text{Cl}(y) = \emptyset$ [1].

**Definition 1.2.** A topological space $(X, T)$ is an $R_1$ space if and only if for $x \in X$ and $y \in X$ such that $\text{Cl}(x) \neq \text{Cl}(y)$ there exist disjoint open sets $U$ and $V$ such that $\text{Cl}(x) \subseteq U$ and $\text{Cl}(y) \subseteq V$ [1].

**Definition 1.3.** A topological space $(X, T)$ is a weakly Hausdorff space if and only if $\text{Cl}(x) = \text{Cl}(y)$ whenever there is a net $S : D \rightarrow X$ such that $\lim S = x$ and $\lim S = y$ [2].

**Theorem 1.2.** A topological space $(X, T)$ is weakly Hausdorff if and only if it is $R_1$ [2].

**Definition 1.4.** A topological space $(X, T)$ is rim-compact if and only if each of its points has a base of neighborhoods with compact frontiers [6].

**Definition 1.5.** A topological space $(X, T)$ is locally compact if and only if each point of $X$ has a neighborhood base consisting of compact sets [6].

**Definition 1.6.** A topological space $(X, T)$ is para-compact if and only if every open cover of $X$ has a locally finite open refinement [4].

**Definition 1.7.** If $\sim$ is an equivalence relation on the topological space $(X, T)$, then the $\sim$ identification space on $X$ is $(\mathcal{D}_\sim, Q(\mathcal{D}_\sim))$ where $\mathcal{D}_\sim = \{C_x | x \in X \text{ and } C_x \text{ is the equivalence class of } \sim \text{ containing } x\}$ and $Q(\mathcal{D}_\sim)$ is the decomposition topology on $\mathcal{D}_\sim$ [6].
Definition 1.8. Let \((X,T)\) be a topological space and let \(\sim^0\) be the equivalence relation on \(X\) defined by \(x \sim^0 y\) if and only if \(\text{Cl}\{x\} = \text{Cl}\{y\}\). Then the \(T_0\)-identification space of \((X,T)\) is the \(\sim^0\) identification space on \(X\) and \((\mathcal{D}^{\sim^0}, \mathcal{Q}(\mathcal{D}^{\sim^0}))\) is a \(T_0\) space [6].

Definition 1.9. Let \((X,T)\) be a topological space, \(A \subseteq X\), and define \(K(X)\), \(C(X)\), \(S(A)\), and \(I(A)\) as follows:

\[
K(X) = \{F \in 2^X \mid F \text{ is compact}\},
\]
\[
C(X) = \{F \in 2^X \mid F \text{ is connected}\},
\]
\[
S(A) = \{F \in 2^X \mid F \subseteq A\}, \text{ and}
\]
\[
I(A) = \{F \in 2^X \mid F \cap A \neq \emptyset\}.
\]

Denote by \(E(X)\) the smallest topology on \(2^X\) which satisfies the following conditions:

1. If \(G \in T\), then \(S(G) \in E(X)\),
2. If \(G \in T\), then \(I(G) \in E(X)\).

The topology \(E(X)\) is called the Vietoris topology on \(2^X\).

If \(G_1, \ldots, G_n\) are subsets of \(X\), let \(\langle G_1, \ldots, G_n \rangle = \{F \in 2^X \mid F \subseteq \bigcup_{i=1}^n G_i \text{ and } F \cap G_i \neq \emptyset \text{ for all } i \in \{1, \ldots, n\}\} [3]\).

Theorem 1.3. Let \((X,T)\) be a topological space.

Then \(\mathcal{B} = \{\langle G_1, \ldots, G_n \rangle \mid n \in \mathbb{N} \text{ and } G_i \text{ is open in } X \text{ for all } i \in \{1, \ldots, n\}\}\) is a base for the Vietoris topology \(E(X)\) on \(2^X\) [5].

Definition 1.10. Let \((X,T)\) be a topological space and let \(x \in X\). Then \((X,T)\) is locally connected (l.c.) at \(x\) if and only if for each open set \(U\) containing \(x\) there exists an open connected set \(V\) such that \(x \in V \subseteq U\) [6].
Definition 1.11. Let \((X,T)\) be a topological space and let \(x \in X\). Then \((X,T)\) is connected in kleinen (c.i.k.) at \(x\) if and only if for each open set \(U\) containing \(x\) the component of \(U\) containing \(x\) contains \(x\) in its interior [6].

In Chapter II \(R_0\) and \(R_\perp\) spaces are further investigated and the following information is obtained. For \(R_0\) spaces closures of points are compact, subspaces of \(R_0\) spaces are \(R_0\), the product of spaces is \(R_0\) if and only if each factor space is \(R_0\), and continuous closed images of \(R_0\) spaces are \(R_0\). A topological space \((X,T)\) is \(R_0\) if and only if for \(x,y \in X\), \(y \in \text{Cl}(x)\) if and only if every net in \(X\) converging to \(y\) converges to \(x\), and a non-compact space is \(R_0\) if and only if the one-point compactification is \(R_0\).

The conditions (a) \((X,T)\) is normal \(R_0\), (b) \((X,T)\) is completely regular, (c) \((X,T)\) is regular, (d) \((X,T)\) is \(R_1\), and (e) \((X,T)\) is \(R_0\) are related as follows:

(1) For any space \((X,T)\), (a) implies (b), (b) implies (c), (c) implies (d), and (d) implies (c).

(2) For a rim-compact space \((X,T)\), (c) and (d) are equivalent.

(3) For a locally compact space \((X,T)\), (b), (c), and (d) are equivalent and (d) does not imply (a).

(4) For a Lindelöf space \((X,T)\), (a), (b), and (c) are equivalent, and (d) does not imply (a).

(5) For a paracompact space \((X,T)\), (a), (b), (c), and (d) are equivalent.
The following results are generalizations of well known theorems where $T_2$ is replaced by $R_1$.

1. If $(X,T)$ is compact $R_1$, then components equal quasi-components.

2. If $(X,T)$ is compact $R_1$, $D$ is a closed subset of $X$, $C$ is a component of $D$, and $U$ is open in $X$ such that $C \subseteq D$, then there exists an open set $V$ such that $C \subseteq V \subseteq U$ and $\text{Fr}(V) \cap D = \emptyset$. These results are used to obtain two additional results that will be utilized in Chapter V.

In Chapter III characterizations of $R_0$ spaces, $R_1$ spaces, regular spaces, completely regular spaces, and normal $R_0$ spaces are obtained using $T_0$-identification spaces and hyperspaces, $T_0$-identification spaces are investigated for $R_0$ spaces, and the relationship between $(2^X, E(X))$ and $(2^{\mathcal{D}_0^0}, E(\mathcal{D}_0^0))$ is investigated for $R_0$ spaces. This information is combined with Ernest Michael's work [5] to generalize many of Michael's results from a $T_1$ setting to an $R_0$ setting. The main results from Chapter III are listed below.

1. Let $(X,T)$ be a topological space. Then the following are equivalent: (a) $(X,T)$ is $R_0$, (b) $\sim$ defined in $X \times X$ by $x \sim y$ if and only if $x \in \text{Cl}(y)$ is an equivalence relation on $X$, (c) $(\mathcal{D}_0^0, Q(\mathcal{D}_0^0)) = (\mathcal{D}_0, Q(\mathcal{D}_0))$, $\mathcal{D}_0 = \{ \text{Cl}(x) \mid x \in X \}$, $Q(\mathcal{D}_0^0) = E(X) \mathcal{D}_0^0$, $(\mathcal{D}_0, Q(\mathcal{D}_0))$ is an upper semi-continuous decomposition of $X$, a lower semi-continuous decomposition of $X$, and a $T_1$ space, and if
0 \in T$, then $\bigcup_{B \in P(0)} B = 0$, where \( P: (X, T) \rightarrow (\mathcal{D}_0, Q(\mathcal{D}_0)) \) is the natural map, (d) \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is a \( T_1 \) space, (e) \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is \( R_0 \), and (f) \( \mathcal{D}_0 \subseteq 2^X \).

(2) Let \((X, T)\) be a topological space. Then the following are equivalent: (a) \((X, T)\) is \( R_1 \), (b) \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is \( R_1 \), and (c) \( \mathcal{D}_0 \) is a closed subset of \( 2^X \).

(3) Let \((X, T)\) be an \( R_0 \) space. Then (a) \((X, T)\) is separable if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is separable, (b) \((X, T)\) is second countable if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is second countable, (c) \((X, T)\) is first countable if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is first countable, (d) \((X, T)\) is compact if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is compact, (e) \((X, T)\) is connected if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is connected, and (f) \((X, T)\) is locally connected if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is locally connected.

(4) Let \((X, T)\) be a topological space. Then \((X, T)\) is regular if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is \( T_3 \), \((X, T)\) is completely regular if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is \( T_{3\frac{1}{2}} \), and \((X, T)\) is normal \( R_0 \) if and only if \((\mathcal{D}_0, Q(\mathcal{D}_0))\) is \( T_4 \).

(5) If \((X, T)\) is an \( R_0 \) space, then \((2^X, E(X))\) is homeomorphic to \((2^\mathcal{D}_0, E(\mathcal{D}_0))\).

In Chapter IV point-wise local connectedness and connectedness im kleinen of \( 2^X \) and \( K(X) \) at elements of \( C(X) \) and \( K(X) \cap C(X) \) for \((X, T)\) an \( R_0 \) space are investigated. The main results from Chapter IV are listed below.
(1) If \((X, T)\) is a topological space and \(A\) is a connected subset of \(2^X\) such that \(A \cap C(X) \neq \emptyset\), then \(\bigcup A\) is connected in \(X\).

(2) If \((X, T)\) is an \(R_0\) space and \(U_i \neq \emptyset\) and open for all \(i \in \{1, \ldots, n\}\), then
\[
\bigcup_{i=1}^{n} U_i = \bigcup_{A \in \langle U_1, \ldots, U_n \rangle} A \in \langle U_1, \ldots, U_n \rangle \cap K(X).
\]

(3) If \((X, T)\) is an \(R_0\) space and \(U_i \neq \emptyset\) and open for all \(i \in \{1, \ldots, n\}\), then \(\langle U_1, \ldots, U_n \rangle \cap \mathcal{F}(X)\) is dense in \(\langle U_1, \ldots, U_n \rangle\) and \(\langle U_1, \ldots, U_n \rangle \cap K(X)\), where \(\mathcal{F}(X) = \{ \cup_{i=1}^{p} \text{Cl} \{ x_i \} | p \in \mathbb{N} \text{ and } x_i \in X \text{ for all } i \in \{1, \ldots, p\} \}\).

(4) If \((X, T)\) is an \(R_0\) space, for each \(i \in \{1, \ldots, n\}\), \(C_i \subseteq X\) such that \(C_i \subseteq \bigcup_{j=1}^{n} U_{ij}\), where \(U_{ij}\) is open and \(C_i \cap U_{ij} \neq \emptyset\) for all \(j \in \{1, \ldots, p_i\}\), \(\mathcal{F}(C_1, \ldots, C_n) = \{ \bigcup_{i=1}^{p} \text{Cl} \{ x_i \} \in \mathcal{F}(X) | \bigcup_{i=1}^{p} \{ x_i \} \subseteq \bigcup_{i=1}^{n} C_i \text{ and also } \bigcup_{i=1}^{p} \{ x_i \} \cap C_j \neq \emptyset \text{ for all } j \in \{1, \ldots, n\} \}\), \(\bigcup_{i=1}^{p} \text{Cl} \{ x_i \} \in \mathcal{F}(C_1, \ldots, C_n) \cap \langle U_1, \ldots, U_{np_n} \rangle\), and \(f: \prod_{i=1}^{n} C_i \rightarrow 2^X\) is defined by \(f((C_1, \ldots, C_n)) = (\bigcup_{i=1}^{p} \text{Cl} \{ x_i \}) \bigcup (\bigcup_{i=1}^{p} \text{Cl} \{ C_i \})\) for all \((C_1, \ldots, C_n) \in \prod_{i=1}^{n} C_i\), then \(f\) is a continuous function from \(\prod_{i=1}^{n} C_i\) into \(\langle U_1, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)\).

(5) Let \((X, T)\) be an \(R_0\) space and for each \(i \in \{1, \ldots, n\}\), let \(C_i\) be a connected subset of \(X\) such that
C_i \subseteq \bigcup_{j=1}^{P_i} U_{ij}$, where $U_{ij}$ is open and $C_i \cap U_{ij} \neq \emptyset$ for all $j \in \{1, \ldots, P_i\}$. Then $\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)$ and $\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np_n} \rangle \cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]$ are connected subsets of $2^X$, and $\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)$ and $[\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np_n} \rangle \cap K(X)] \cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]$ are connected subsets of $K(X)$.

(6). Let $(X,T)$ be an $\mathcal{R}_0$ space and, for each $i \in \{1, \ldots, n\}$, let $C_i$ be connected such that $C_i \subseteq \bigcup_{j=1}^{P_i} U_{ij}$, where $U_{ij}$ is open and $C_i \cap U_{ij} \neq \emptyset$ for all $j \in \{1, \ldots, P_i\}$, where $C_i^0 = \text{Int}(C_i)$. Then $\langle C_i^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle$ $\cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]$ is a connected subset of $\langle U_{11}, \ldots, U_{np_n} \rangle$ and $[\langle C_i^0 \cap U_{11}, \ldots, C_n \cap U_{np_n} \rangle \cap K(X)]$ $\cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]$ is a connected subset of $\langle U_{11}, \ldots, U_{np_n} \rangle \cap K(X)$.

(7) Let $(X,T)$ be an $\mathcal{R}_0$ space and let $B \in C(X)$. Then $2^X$ is connected im kleinen at $B$ if and only if for each open set $U$ in $X$ such that $B \subseteq U$, the component of $U$ containing $B$ contains $B$ in its interior.

(8) Let $(X,T)$ be an $\mathcal{R}_0$ space and let $B \in K(X) \cap C(X)$. Then the following are equivalent: (a) $2^X$ is connected im kleinen at $B$, (b) for each open set $U$ in $X$ such that
B ⊂ U, the component of U containing B contains B in its interior, and (C) K(X) is connected im kleinen at B.

(9) Let (X,T) be an $R_0$ space and let $B \in C(X)$. Then $2^X$ is locally connected at B if and only if for each open set $U$ in $X$ such that $B \subset U$, there exists an open connected set $C$ such that $B \subset C \subset U$.

(10) Let $(X,T)$ be an $R_0$ space and let $B \in C(X) \cap K(X)$. Then the following are equivalent: (a) $2^X$ is locally connected at B, (b) for each open set $U$ in $X$ such that $B \subset U$, there exists an open connected set $C$ such that $B \subset C \subset U$, and (c) $K(X)$ is locally connected at B.

(11) Let $(X,T)$ be an $R_0$ space and let $C$ be a component of $X$. Then the following are equivalent: (a) $2^X$ is connected im kleinen at $C$, (b) $C$ is a closed-open subset of $X$, and (c) $2^X$ is locally connected at $C$.

In Chapter V point-wise local connectedness and connectedness im kleinen of $(2^X,E(X))$ and $(K(X),E(X))_{|K(X)}$, and the relationships between statements (I)$(2^X,E(X))$ is locally connected (connected im kleinen) at $A \in 2^X$ and (II) $(2^X,E(X))$ is locally connected (connected im kleinen) at each component of $A \in 2^X$ are investigated for $(X,T)$ weaker than a metric continuum. The main results from Chapter V are listed below.

(1) (I) does not imply (II) for $(X,T)$ a compact connected $T_1$ space.
(2) (II) does not imply (I) for $(X,T)$ a locally connected, locally compact, connected, metric space.

(3) Let $(X,T)$ be an $R_0$ space and let $B \in K(X)$. If $2^X$ or $K(X)$ is connected im kleinen (locally connected) at each component of $B$, then $2^X$ and $K(X)$ are connected im kleinen (locally connected) at $B$.

(4) Let $(X,T)$ be an $R_0$ space and let $B \in 2^X$ such that $B$ has only finitely many components. If $2^X$ is connected im kleinen (locally connected) at each component of $B$, then $2^X$ is connected im kleinen (locally connected) at $B$.

(5) Let $(X,T)$ be an $R_0$ space and let $A \in 2^X$ such that $2^X$ is locally connected at $A$. If $C$ is a component of $A$ such that if $U$ is open in $X$, and $C \subset U$, then there exists an open set $V$ in $X$ such that $C \subset V \subset U$ and $\text{Fr}(V) \cap A = \emptyset$, then $2^X$ is locally connected at $C$.

(6) Let $(X,T)$ be an $R_0$ space and let $A \in K(X)$ such that $2^X$ or $K(X)$ is locally connected at $A$. If $C$ is a component of $A$ such that if $U$ is open in $X$ and $C \subset U$, then there exists an open set $V$ in $X$ such that $C \subset V \subset U$ and $\text{Fr}(V) \cap A = \emptyset$, then $2^X$ and $K(X)$ are locally connected at $C$.

(7) Let $(X,T)$ be a locally compact $R_1$ space and let $A \in 2^X$ such that $A$ has a compact component $C$. If $2^X$ is locally connected (connected im kleinen) at $A$, then $2^X$ and $K(X)$ are locally connected (connected im kleinen) at $C$. 
(8) Let \((X,T)\) be a locally compact \(R_1\) space and let \(A \in K(X)\). Then the following are equivalent: (a) \(K(X)\) is locally connected (connected im kleinen) at \(A\), (b) \(K(X)\) is locally connected (connected im kleinen) at each component of \(A\), (c) \(2^X\) is locally connected (connected im kleinen) at each component of \(A\), and (d) \(2^X\) is locally connected (connected im kleinen) at \(A\).

(9) Let \((X,T)\) be an \(R_0\) space and let \(x \in X\). Then the following are equivalent: (a) \(X\) is locally connected (connected im kleinen) at \(x\), (b) \(2^X\) is locally connected (connected im kleinen) at \(\text{Cl}\{x\}\), and (c) \(K(X)\) is locally connected (connected im kleinen) at \(\text{Cl}\{x\}\).
CHAPTER BIBLIOGRAPHY


CHAPTER II

$R_0$ AND $R_1$ TOPOLOGICAL SPACES

According to the book General Topology and Its Relations to Modern Analysis and Algebra [3], the $R_0$ axiom was first introduced by N. A. Shanin [6] in 1943. Later, in 1961, A. S. Davis [1] rediscovered the $R_0$ axiom. Among the contributions in A. S. Davis' paper [1] are five characterizations of $R_0$ spaces, two of which are given in Theorem 1.1 of Chapter I, and the following characterization of $T_1$ spaces: A space $(X,T)$ is $T_1$ if and only if it is $T_0$ and $R_0$. In 1966, M. G. Murdeshwar and S. A. Naimpally [4] gave characterizations of $R_0$ quasi-uniform spaces and used quasi-uniformity to give characterizations of $R_0$ and in 1967 S. A. Naimpally [5] contributed another characterization of $R_0$ spaces using quasi-uniformity. In this chapter $R_0$ spaces are further investigated and two additional characterizations are given for $R_0$ spaces.

If $(X,T)$ is a $T_1$ space or a regular space, then $(X,T)$ is an $R_0$ space. Note that $\{(X,T)| (X,T)$ is a $T_1$ space or a regular space$\} \subsetneq \{(X,T)| (X,T)$ is an $R_0$ space$\}$ as the following example illustrates.

Example 2.1: Let $Y = \{x \in [0,1] | x$ is rational$\}$ and let $S$ be the usual metric topology on $Y$. Let $X = Y \cup \{2,3\}$,
let \( W = \{ 0 \subseteq X \mid 2,3 \in 0 \text{ and } X \setminus 0 \text{ is finite} \} \), and let \( T = \bigcup W \).

Then \((X,T)\) is a compact, \( R_0 \), connected space that is not regular or \( T_1 \).

**Lemma 2.1.** If \((X,T)\) is an \( R_0 \) space, then \( \text{Cl}\{x\} \) is compact for all \( x \in X \).

**Proof.** Let \( x \in X \). Let \( \mathcal{O} \) be an open cover for \( \text{Cl}\{x\} \). Then there exists an \( O_x \in \mathcal{O} \) such that \( x \in O_x \) and since \((X,T)\) is an \( R_0 \) space, then \( \text{Cl}\{x\} \subseteq O_x \), which implies \( \{O_x\} \) is a finite subcover of \( \mathcal{O} \). Hence, if \( \mathcal{O} \) is an open cover for \( \text{Cl}\{x\} \), then there exists a finite subcover, which implies \( \text{Cl}\{x\} \) is compact.

Therefore, if \( x \in X \), then \( \text{Cl}\{x\} \) is compact.

**Theorem 2.1.** Every subspace of an \( R_0 \) space is an \( R_0 \) space.

**Proof.** Let \((X,T)\) be an \( R_0 \) and let \((Y,T_Y)\) be a subspace of \((X,T)\). Let \( y \in Y \) and let \( O \in T_Y \) such that \( y \in O \). Then there exists \( V \in T \) such that \( O = V \cap Y \). Since \( X \) is an \( R_0 \) space, then \( \text{Cl}_X\{y\} \subseteq V \) and since \( y \in Y \), then

\[
\text{Cl}_Y\{y\} = \text{Cl}_X\{y\} \cap Y,
\]

which implies \( \text{Cl}_Y\{y\} = \text{Cl}_X\{y\} \cap Y \subseteq V \cap Y = 0 \).

Therefore, if \( y \in Y \) and \( O \in T_Y \) such that \( y \in O \), then \( \text{Cl}_Y\{y\} = 0 \), which implies \((Y,T_Y)\) is an \( R_0 \) space.

**Theorem 2.2.** For each \( \alpha \in A \) let \((X_{\alpha},T_{\alpha})\) be a topological space. Then \((X_{\alpha},T_{\alpha})\) is an \( R_0 \) space for all \( \alpha \in A \) if and only if \(( \prod_{\alpha \in A} X_{\alpha},P)\) is an \( R_0 \) space, where \( P \) is the product topology on \( \prod_{\alpha \in A} X_{\alpha} \).

**Proof.** Suppose that \((X_{\alpha},T_{\alpha})\) is an \( R_0 \) space for all
$\alpha \in A$. Let $\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} X_\alpha$ and let $\emptyset$ be open in $\prod_{\alpha \in A} X_\alpha$ such that $\{x_\alpha\}_{\alpha \in A} \subseteq \emptyset$. Then there exists $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha$ is open in $X_\alpha$ and $U_\alpha = X_\alpha$ except for finitely many $\alpha \in A$, such that $\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} U_\alpha \subseteq \emptyset$. For each $\alpha \in A$, $x_\alpha \in U_\alpha$. Thus

\[ \text{Cl}\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} U_\alpha \subseteq \emptyset, \]

for all $\alpha \in A$ and $\text{Cl}\{x_\alpha\}_{\alpha \in A} = \prod_{\alpha \in A} \text{Cl}\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} U_\alpha \subseteq \emptyset$.

Therefore, if $\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} X_\alpha$ and $\emptyset$ is open in $\prod_{\alpha \in A} X_\alpha$ such that $\{x_\alpha\}_{\alpha \in A} \subseteq \emptyset$, then $\text{Cl}\{x_\alpha\}_{\alpha \in A} \subseteq \emptyset$, which implies $(\prod_{\alpha \in A} X_\alpha, P)$ is $R_0$.

Conversely, suppose that $(\prod_{\alpha \in A} X_\alpha, P)$ is an $R_0$ space. Let $\beta \in A$, let $x_\beta \in X_\beta$, and let $\emptyset$ be open in $X_\beta$ such that $x_\beta \in \emptyset$. For each $\alpha \in A \setminus \{\beta\}$, let $x_\alpha \in X_\alpha$. Then $\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} U_\alpha$, where

\[ U_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \neq \beta \\ 0 & \text{if } \alpha = \beta, \end{cases} \]

which is open in $\prod_{\alpha \in A} X_\alpha$. Thus

\[ \text{Cl}\{x_\alpha\}_{\alpha \in A} \subseteq \prod_{\alpha \in A} U_\alpha, \]

which implies $\text{Cl}\{x_\beta\}_{\alpha \in A} \subseteq \emptyset$.

Therefore, if $\beta \in A$, $x_\beta \in X_\beta$, and $\emptyset$ is open in $X_\beta$ such that $x_\beta \in \emptyset$, then $\text{Cl}\{x_\beta\} \subseteq \emptyset$, which implies $(X_\beta, T_\beta)$ is an $R_0$ space for all $\beta \in A$.

**Theorem 2.3.** If $(X,T)$ is an $R_0$ space, $(Y,S)$ is a topological space, and $f:(X,T) \to (Y,S)$ is continuous, closed, and onto, then $(Y,S)$ is an $R_0$ space.

**Proof.** Let $y \in Y$ and let $\emptyset$ be open in $Y$ such that $y \in \emptyset$. Let $x \in X$ such that $f(x) = y$. Then $x \in f^{-1}(\emptyset)$,
which is open in \(X\). Thus \(\operatorname{Cl}\{x\} \subseteq f^{-1}(0)\), which implies \(y \in f(\operatorname{Cl}\{x\}) \subseteq 0\). Since \(f(\operatorname{Cl}\{x\})\) is closed in \(Y\), then \(\operatorname{Cl}\{y\} \subseteq f(\operatorname{Cl}\{x\}) \subseteq 0\).

Therefore, if \(y \in Y\) and \(0\) is open in \(Y\) such that \(y \in 0\), then \(\operatorname{Cl}\{y\} \subseteq 0\), which implies \((Y,S)\) is an \(R_0\) space.

**Lemma 2.2.** Let \((X,T)\) be a topological space. If \(x,y \in X\) such that every net in \(X\) converging to \(y\) converges to \(x\), then \(x \in \operatorname{Cl}\{y\}\).

**Proof.** Let \(x,y \in X\) such that every net in \(X\) converging to \(y\) converges to \(x\). For each \(n \in \mathbb{N}\), let \(x_n = y\). Then \(\{x_n\}_n \in \mathbb{N}\) is a net in \(X\) converging to \(y\). Thus \(\{x_n\}_n \in \mathbb{N}\) converges to \(x\). Then \(\{x_n\}_n \in \mathbb{N}\) is a net in \(\{y\}\) converging to \(x\), which implies \(x \in \operatorname{Cl}\{y\}\).

**Theorem 2.4.** Let \((X,T)\) be a topological space. Then the following are equivalent:

(a) \((X,T)\) is an \(R_0\) space;

(b) if \(x,y \in X\), then \(y \in \operatorname{Cl}\{x\}\) if and only if every net in \(X\) converging to \(y\) converges to \(x\).

**Proof.** (a) implies (b): Suppose \(x,y \in X\) such that \(y \in \operatorname{Cl}\{x\}\). Let \(\{x_\alpha\}_\alpha \in A\) be a net in \(X\) such that \(\{x_\alpha\}_\alpha \in A\). Now \(x \in \operatorname{Cl}\{y\}\) since if \(x \notin \operatorname{Cl}\{y\}\), then \(x \in X\setminus \operatorname{Cl}\{y\}\), which is open, and \(\operatorname{Cl}\{x\} \subseteq X\setminus \operatorname{Cl}\{y\}\), which contradicts \(y \in \operatorname{Cl}\{x\}\). Let \(0\) be open in \(X\) such that \(x \notin 0\). Since \(x \in \operatorname{Cl}\{y\}\), then \(y \in 0\) and since \(\{x_\alpha\}_\alpha \in A\), then there exists \(\alpha_0 \in A\) such that if \(\alpha \in A\) and \(\alpha \geq \alpha_0\), then \(x_\alpha \notin 0\). Hence, if \(0\) is open in \(X\)
such that \( x \in 0 \), then there exists \( \alpha_0 \in A \) such that if \( \alpha \in A \) and \( \alpha \geq \alpha_0 \), then \( x \in 0 \), which implies \( \{ x_\alpha \} \alpha \in A \rightarrow x \).

Therefore, every net in \( X \) converging to \( y \) converges to \( x \).

Conversely, suppose \( x, y \in X \) such that every net in \( X \) converging to \( y \) converges to \( x \). By Lemma 2.2, \( x \in Cl\{y\} \).

Then by the argument above, every net in \( X \) converging to \( x \) converges to \( y \). Thus by Lemma 2.2, \( y \in Cl\{x\} \).

(b) implies (a): Let \( x, y \in X \). Then \( Cl\{x\} = Cl\{y\} \) or \( Cl\{x\} \neq Cl\{y\} \). Consider the case that \( Cl\{x\} \neq Cl\{y\} \).

Then \( Cl\{x\} \cap Cl\{y\} = \emptyset \), for suppose not. Let \( z \in Cl\{x\} \cap Cl\{y\} \). Since \( z \in Cl\{x\} \), then there exists a net \( S \) in \( Cl\{x\} \) such that \( S \rightarrow z \). Since \( z \in Cl\{y\} \) and \( S \) is a net in \( X \) such that \( S \rightarrow z \), then \( S \rightarrow y \). Thus there exists a net in \( Cl\{x\} \) converging to \( y \), which implies \( y \in Cl\{x\} \) and \( Cl\{y\} \subset Cl\{x\} \).

By a similar argument, \( Cl\{x\} \subset Cl\{y\} \). Thus \( Cl\{x\} = Cl\{y\} \), which is a contradiction.

Therefore, if \( x, y \in X \), then \( Cl\{x\} = Cl\{y\} \) or \( Cl\{x\} \cap Cl\{y\} = \emptyset \), which implies \( (X, T) \) is an \( R_0 \) space.

**Theorem 2.5.** Let \( (X, T) \) be a non-compact space, let \( p \notin X \), and let \( (X^*, i) \) be the one-point compactification of \( X \), where \( X^* = X \cup \{ p \} \). If \( (X, T) \) is \( R_0 \) or \( (X^*, T^*) \) is \( R_0 \), then for each \( x \in X \), \( Cl_{X^*}\{x\} = Cl_{X^*}\{x\} \).

**Proof.** Since \( i: (X, T) \rightarrow (i(X), T^* i(X)) \) is a homeomorphism, then \( Cl_{X^*}\{x\} = i(Cl_{X}\{x\}) = Cl_{i(X)}\{x\} \).
Suppose that \((X,T)\) is an \(R_0\) space. Let \(x \in X\). Since \((X,T)\) is an \(R_0\) space, then \(\text{Cl}_X\{x\}\) is closed compact in \((X,T)\). Thus \(X^* \setminus \text{Cl}_X\{x\}\) is open in \(X^*\), \(p \in X^* \setminus \text{Cl}_X\{x\}\), and \(x \notin X^* \setminus \text{Cl}_X\{x\}\), which implies \(\text{Cl}_{X^*}\{x\} \subset X^* \setminus \{p\} = X = i(X)\). Then \(\text{Cl}_X\{x\} = \text{Cl}_i(X)\{x\} = \text{Cl}_{X^*}\{x\} \cap i(X) = \text{Cl}_{X^*}\{x\}\). Hence, if \(x \in X\), then \(\text{Cl}_X\{x\} = \text{Cl}_{X^*}\{x\}\).

Now suppose that \((X^*,T^*)\) is an \(R_0\) space. If \(x \in X\), then \(x \in X\), which is open in \(X^*\), and \(\text{Cl}_{X^*}\{x\} \subset X = i(X)\), which implies \(\text{Cl}_X\{x\} = \text{Cl}_i(X)\{x\} = \text{Cl}_{X^*}\{x\} \cap i(X) = \text{Cl}_{X^*}\{x\}\).

**Theorem 2.6.** Let \((X,T)\) be a non-compact space, let \(p \notin X\), and let \((X^*,i)\) be the one-point compactification of \(X\), where \(X^* = X \cup \{p\}\). Then \((X,T)\) is an \(R_0\) space if and only if \((X^*,T^*)\) is an \(R_0\) space.

**Proof.** Suppose \((X,T)\) is an \(R_0\) space. Let \(x,y \in X^*\). Consider the case that \(\text{Cl}_{X^*}\{x\} \neq \text{Cl}_{X^*}\{y\}\). Since \(x,y \in X^* = X \cup \{p\}\), then both \(x,y \in X\), or one of \(x\) and \(y\) is in \(X\) and the other equals \(p\).

Suppose both \(x,y \in X\). By Theorem 2.5, \(\text{Cl}_X\{x\} = \text{Cl}_{X^*}\{x\}\) and \(\text{Cl}_X\{y\} = \text{Cl}_{X^*}\{y\}\). Thus \(x,y \in X\) such that \(\text{Cl}_X\{x\} \neq \text{Cl}_X\{y\}\), which implies \(\text{Cl}_{X^*}\{x\} \cap \text{Cl}_{X^*}\{y\} = \text{Cl}_X\{x\} \cap \text{Cl}_X\{y\} = \emptyset\).

Now suppose that one of \(x\) and \(y\) is in \(X\) and the other equals \(p\), say \(x \in X\) and \(y = p\). By Theorem 2.5, \(\text{Cl}_{X^*}\{x\} = \text{Cl}_X\{x\} \subset X\), and since \(\text{Cl}_{X^*}\{p\} = \{p\} = X^* \setminus X\), then \(\text{Cl}_{X^*}\{x\} \cap \text{Cl}_{X^*}\{p\} = \emptyset\).

Therefore, \(\text{Cl}_{X^*}\{x\} = \text{Cl}_{X^*}\{y\}\) or \(\text{Cl}_{X^*}\{x\} \cap \text{Cl}_{X^*}\{y\} = \emptyset\).
Therefore, if \( x, y \in X^* \), then \( \text{Cl}_{X^*}(x) = \text{Cl}_{X^*}(y) \) or \( \text{Cl}_{X^*}(x) \cap \text{Cl}_{X^*}(y) = \emptyset \), which implies \((X^*, T^*)\) is an \( R_0 \) space.

Conversely, suppose \((X^*, T^*)\) is an \( R_0 \) space. By Theorem 2.5, if \( x, y \in X \), then \( \text{Cl}_X(x) = \text{Cl}_X^*(x) \) and \( \text{Cl}_X(y) = \text{Cl}_X^*(y) \). Thus, if \( x, y \in X \), then \( \text{Cl}_X(x) = \text{Cl}_X(y) \) or \( \text{Cl}_X(x) \cap \text{Cl}_X(y) = \emptyset \), which implies \((X, T)\) is an \( R_0 \) space.

**Theorem 2.7.** If \((X, T)\) is a normal \( R_0 \) space, then \((X, T)\) is completely regular.

**Proof.** Let \( A \) be closed in \( X \) and let \( x \notin A \). Then \( x \in X \setminus A \), which is open in \( X \), which implies \( \text{Cl}(x) \subset X \setminus A \). Since \( A \) and \( \text{Cl}(x) \) are disjoint closed subsets of the normal space \((X, T)\), then by Urysohn's Lemma, there exists a continuous function \( f : (X, T) \to [0,1] \) such that \( f(\text{Cl}(x)) = 0 \) and \( f(A) = 1 \), which implies \( f(x) = 0 \) and \( f(A) = 1 \).

Therefore, if \( A \) is a closed subset of \( X \) and \( x \notin A \), then there exists a continuous function \( f : (X, T) \to [0,1] \) such that \( f(x) = 0 \) and \( f(A) = 1 \), which implies \((X, T)\) is completely regular.

The \( R_1 \) axiom was introduced in 1961 by A. S. Davis [1] and was used to obtain the following characterization of \( T_2 \) spaces: a topological space is \( T_2 \) if and only if it is \( T_1 \) and \( R_1 \). Then in 1975 William Dunham [2] explored the \( R_1 \) axiom more fully and several of his results are utilized in this paper. Theorem 1.2 is one of Dunham's
contributions and it allows the weakly Hausdorff axiom and the $R_1$ axiom to be used interchangeably.

Lemma 2.6. of Dunham's paper [2] states that $R_1$ spaces are $R_0$ and Theorem 3.5 of Dunham's paper [2] states that regular spaces are $R_1$. Combining these results with Theorem 2.6 of this paper yields the following corollary.

Corollary 2.1. \((X,T) | (X,T) \text{ is normal } R_0 \} \subset \{(X,T) | (X,T) \text{ is completely regular} \} \subset \{(X,T) | (X,T) \text{ is regular} \} \subset \{(X,T) | (X,T) \text{ is } R_1 \} \subset \{(X,T) | (X,T) \text{ is } R_0 \} \).

Theorem 2.8. Let \((X,T)\) be a rim-compact space. Then \((X,T)\) is $R_1$ if and only if \((X,T)\) is regular.

Proof. Suppose \((X,T)\) is $R_1$. Let $A$ be closed in $X$ and let $x \notin A$. Then $x \in X \setminus A$, which is open. Thus there exists a neighborhood $O$ of $x$ such that $O \subset X \setminus A$ and $Fr(O)$ is compact. Let $y \in Fr(O)$. Then $x, y \in X$ such that $Cl\{x\} \neq Cl\{y\}$. Thus there exist disjoint open sets $O_y$ and $V_y$ such that $Cl\{x\} \subset O_y$ and $Cl\{y\} \subset V_y$. Let $U_y = O_y \cap Int(O)$. Then $U_y$ and $V_y$ are disjoint open sets such that $Cl\{x\} \subset U_y \subset Int(O)$ and $Cl\{y\} \subset V_y$. For each $y \in Fr(O)$ let $U_y$ and $V_y$ be disjoint open sets such that $Cl\{x\} \subset U_y \subset Int(O)$ and $Cl\{y\} \subset V_y$. Then $\{V_y\}_{y \in Fr(O)}$ is an open cover for $Fr(O)$. Thus there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$. Then $x \in \bigcap_{i=1}^n U_{y_i}$, which is open, $A \subset X \setminus Cl(\bigcap_{i=1}^n U_{y_i})$, which is open, and $\bigcap_{i=1}^n U_{y_i} \cap [X \setminus Cl(\bigcap_{i=1}^n U_{y_i})] = \emptyset$. 
Therefore, if \( A \) is closed in \( X \) and \( x \notin A \), then there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( A \subseteq V \), which implies \((X,T)\) is regular.

Conversely, suppose \((X,T)\) is regular. Then by Corollary 2.2., \((X,T)\) is \( R_1 \).

Combining Theorem 3.5 and Corollary 6.4 in Dunham's paper [2] yields the following corollary.

**Corollary 2.2.** If \((X,T)\) is a locally compact space, then the following are equivalent: (a) \((X,T)\) is regular, (b) \((X,T)\) is \( R_1 \), and (c) \((X,T)\) is completely regular.

A locally compact \( R_1 \) space need not be normal since the Tychonoff plank is locally compact \( R_1 \) but not normal.

Combining Theorem 16.8 of Willard's book [7], which states that a regular Lindelöf space is normal, and Corollary 2.1 of this paper yields the following corollary.

**Corollary 2.3.** If \((X,T)\) is a Lindelöf space, then the following are equivalent: (a) \((X,T)\) is regular, (b) \((X,T)\) is normal \( R_0 \), and (c) \((X,T)\) is completely regular.

The following example shows that "\((X,T)\) is an \( R_1 \) space" is not an equivalent statement in Corollary 2.3.

**Example 2.2.** Let \( X = [0,1] \cap \{ x \mid x \text{ is rational} \} \), let \( T \) be the usual topology on \( X \), and let \( d \) be the usual metric on \([0,1]\), let \( A = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \), and let \( \mathcal{B} = \{ 0 \subseteq X \mid 0 = D \setminus B, \text{where } D \in T \text{ and } B \subseteq A \} \). Then \( \mathcal{B} \) is a base for a topology on \( X \). Let \( S \) be the topology on \( X \) generated by \( \mathcal{B} \). Then
TCS and since (X,T) is $T_2$, then (X,S) is $T_2$, which implies (X,T) is $R_1$. Since A is closed in (X,S), $\emptyset \notin A$, and there do not exist disjoint open sets U and V such that $0 \in U$ and $A \subseteq V$, then (X,S) is not regular and since (X,S) is second countable, then (X,S) is Lindelöf.

Combining Corollary 5.4 of Dunham's paper (2, p. 47), which states that paracompact $R_1$ spaces are normal, and Corollary 2.1 of this paper yields the following corollary.

**Corollary 2.4.** If (X,T) is a paracompact space, then the following are equivalent: (a) (X,T) is regular, (b) (X,T) is $R_1$, (c) (X,T) is normal $R_0$, and (d) (X,T) is completely regular.

Corollary 2.4 and the fact that every compact space is paracompact yield the next corollary.

**Corollary 2.5.** If (X,T) is a compact space, then the following are equivalent: (a) (X,T) is regular, (b) (X,T) is $R_1$, (c) (X,T) is normal $R_0$, and (d) (X,T) is completely regular.

**Theorem 2.9.** If X is a compact $R_1$ space, $x \in X$, $C_x$ is the component of X containing x, and $Q_x$ is the quasi-component of X containing x, then $C_x = Q_x$.

**Proof.** For any space $C_x \subseteq Q_x$. Since (X,T) is compact $R_1$, then $Q_x$ is connected, for suppose not. Then $Q_x = M \cup N$, where M and N are non-empty separated sets.
Since \( M \) and \( N \) are disjoint closed sets in \( Q_x \), which is closed in \( X \), then \( M \) and \( N \) are disjoint closed sets in \( X \) and since \((X,T)\) is compact \( R_1 \), then \((X,T)\) is normal. Thus there exist disjoint open sets \( U \) and \( V \) such that \( M \subset U \) and \( N \subset V \). Now \( x \in M \) or \( x \in N \), say \( x \in M \). For each \( y \in X \setminus (U \cup V) \), there exists a closed open set \( W_y \) such that \( y \in W_y \subset X \setminus Q_x \). Then \( \{W_y\}_{y \in X \setminus (U \cup V)} \) is an open cover of \( X \setminus (U \cup V) \), which is compact. Thus there exists a finite subcover \( \{W_{y_1}\}_{i=1}^n \). Then \( X = U \cup V \cup (\bigcup_{i=1}^n W_{y_i}) \). Since \( U \) and \( V \) are disjoint open sets, then \( \text{Cl}(V) \cap U = \emptyset \) and

\[
\overline{V} \subset X \setminus U \subset V \cup (\bigcup_{i=1}^n W_{y_i}).
\]

Then

\[
\text{Cl}(V \cup (\bigcup_{i=1}^n W_{y_i})) = \text{Cl}(V) \cup (\bigcup_{i=1}^n W_{y_i}) \subset V \cup (\bigcup_{i=1}^n W_{y_i}).
\]

Thus,

\[
\text{Cl}(V \cup (\bigcup_{i=1}^n W_{y_i})) = V \cup (\bigcup_{i=1}^n W_{y_i}),
\]

which implies \( V \cup (\bigcup_{i=1}^n W_{y_i}) \) is closed-open, but then

\[
x \in X \setminus [V \cup (\bigcup_{i=1}^n W_{y_i})],
\]

which is closed-open. Thus

\[
Q_x \subset X \setminus [V \cup (\bigcup_{i=1}^n W_{y_i})],
\]

which contradicts \( Q_x \cap V \neq \emptyset \).

Therefore, \( Q_x \) is connected, and thus \( Q_x \subset C_x \), which implies \( Q_x = C_x \).
Theorem 2.10. Let \( X \) be a compact \( R_1 \) space, let \( D \) be a closed subset of \( X \), let \( C \) be a component of \( D \), and let \( U \) be open in \( X \) such that \( C \subseteq U \). Then there exists an open set \( V \) such that \( C \subseteq V \subseteq U \) and \( \text{Fr}(V) \cap D = \emptyset \).

Proof. If \( \text{Fr}(U) \cap D = \emptyset \), let \( V = U \).

Suppose \( \text{Fr}(U) \cap D \neq \emptyset \). Since \( (D, T_D) \) is compact \( R_1 \), then the components of \( D \) and the quasi-components of \( D \) are the same. Let \( x \in \text{Fr}(U) \cap D \). Then \( x \notin C \). Thus there exists a closed-open set \( A_x \) in \( D \) such that \( x \in A_x \) and \( C \cap A_x = \emptyset \).

Since \( A_x \) is open in \( D \), then there exists an open set \( O_x \) in \( X \) such that \( A_x = O_x \cap D \). Now \( O_x \cap D \) is open in \( D \), and \( D \setminus (O_x \cap D) \) is closed in \( D \). Let \( M = C \cup \{\{D \setminus (O_x \cap D) \} \cap \text{Fr}(O_x)\} \), which is closed in \( D \). Then \( M \) and \( O_x \cap D \) are disjoint closed in \( D \), which is closed in \( X \). Thus \( M \) and \( O_x \cap D \) are disjoint closed in \( X \) and there exist disjoint open sets \( U_x \) and \( V_x \) in \( X \) such that \( M \subseteq U_x \) and \( O_x \cap D \subseteq V_x \). Then \( \text{Cl}(U_x) \cap V_x = \emptyset \) and \( O_x \cap D \subseteq V_x \), which implies

\[ x \in O_x \cap D \subseteq (O_x \setminus \text{Cl}(U_x)) \cap D \]
\[ \subseteq O_x \setminus \text{Cl}(U_x) \subseteq \text{Cl}(O_x) \setminus U_x. \]

Since \( C \subseteq U_x \), then \( (\text{Cl}(O_x) \setminus U_x) \cap C = \emptyset \). Now
\[ O_x \cap D \subseteq (\text{Cl}(O_x) \setminus U_x) \cap D \]
and if \( y \in (\text{Cl}(O_x) \setminus U_x) \cap D \), then \( y \notin U_x \) and
\[ y \in (\text{Cl}(O_x) \cap D) \setminus [C \cup \{\{D \setminus (O_x \cap D) \} \cap \text{Fr}(O_x)\}] \]
\[ \subseteq (\text{Cl}(O_x) \cap D) \setminus \{\{D \setminus (O_x \cap D) \} \cap \text{Fr}(O_x)\} \]
\[ = (\text{Cl}(O_x) \cap D) \setminus \{D \setminus (O_x \cap D) \} \cup \{(\text{Cl}(O_x) \cap D) \setminus \text{Fr}(O_x)\} \]
\[ = (O_x \cap D) \cup \{(O_x \cup \text{Fr}(O_x)) \cap D \setminus \text{Fr}(O_x)\} = O_x \cap D. \]
Thus \((\text{Cl}(O'_x) \setminus U'_x) \cap D = O'_x \cap D\), which implies
\[(\text{Cl}(O'_x) \setminus U'_x) \cap D = 0_x \cap D\).

Then \(O'_x\) and \(U'_x\) are open sets in \(X\) such that
\[x \in O'_x \cap D \subset O'_x \setminus \text{Cl}(U'_x) \subset \text{Cl}(O'_x) \setminus U'_x,\]
\[(\text{Cl}(O'_x) \setminus U'_x) \cap C = \emptyset,\]
and
\[(\text{Cl}(O'_x) \setminus U'_x) \cap D = 0_x \cap D.\]

For each \(x \in \text{Fr}(U) \cap D\), let \(O'_x\) and \(U'_x\) be open sets in \(X\) such that
\[x \in O'_x \cap D \subset O'_x \setminus \text{Cl}(U'_x) \subset \text{Cl}(O'_x) \setminus U'_x,\]
\[(\text{Cl}(O'_x) \setminus U'_x) \cap D = \emptyset,\]
and
\[(\text{Cl}(O'_x) \setminus U'_x) \cap D = 0_x \cap D.\]

Then \(\{\text{Cl}(O'_x) \setminus U'_x \mid x \in \text{Fr}(U) \cap D\}\) is a cover of \(\text{Fr}(U) \cap D\) by neighborhoods of \(X\). Thus there exists a finite subcover
\[\{\text{Cl}(O'_{x_i}) \setminus U'_{x_i} \mid i = 1, \ldots, n\},\]
which is open in \(X\). Also \(D \cap \text{Fr}(U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})) = \emptyset\), for suppose not. Let \(z \in \text{Fr}(U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})) \cap D\). Since
\[U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})\]
is open, then \(z \notin U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})\).

Now
\[z \in \text{Fr}(U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})) \cap D \subset U \cap D\]
\[= (U \cup \text{Fr}(U)) \cap D = (U \cap D) \cup (\text{Fr}(U) \cap D)\]
\[\subset \left[\left(\bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})\right) \cup \left(\bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})\right)\right] \cap D\]
\[\cup \left(\bigcup_{i=1}^{n} (\text{Cl}(O'_{x_i}) \setminus U'_{x_i})\right) \cap D\].
which implies \( z \in \bigcup_{i=1}^{n} (\text{Cl}(O_{x_{i}}) \setminus U_{x_{i}})) \cap D. \) Then there exists \( j \in \{1,...,n\} \) such that

\[ z \in (\text{Cl}(O_{x_{j}}) \setminus U_{x_{j}}) \cap D = 0 \quad \cap D \subset 0 \quad \cap \text{Cl}(U_{x_{j}}) \subset \bigcup_{i=1}^{n} (\text{Cl}(O_{x_{i}}) \setminus U_{x_{i}}). \]

But then \( z \in O_{x_{j}} \setminus \text{Cl}(U_{x_{j}}) \), which is open in \( X \), and

\[ (O_{x_{j}} \setminus \text{Cl}(U_{x_{j}})) \cap [U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O_{x_{i}}) \setminus U_{x_{i}})] = \emptyset, \]

which contradicts \( z \in \text{Fr}(U \cup (\text{Cl}(O_{x_{i}}) \setminus U_{x_{i}})). \) Let

\[ V = U \setminus \bigcup_{i=1}^{n} (\text{Cl}(O_{x_{i}}) \setminus U_{x_{i}}). \]

**Theorem 2.11.** Let \( (X,T) \) be a locally compact \( R_{1} \) space, let \( D \) be a closed subset of \( X \) with compact component \( C \), and let \( U \) be open in \( X \) such that \( C \subset U. \) Then there exists an open set \( V \) such that \( \text{Cl}(V) \) is compact, \( C \subset V \subset \text{Cl}(V) \subset U \), and \( \text{Fr}(V) \cap D = \emptyset. \)

**Proof.** By Corollary 2.2 of this paper, \( (X,T) \) is regular. Let \( x \in C \). Then there exists a compact neighborhood \( U_{x} \) of \( x \) such that \( U_{x} \subset U. \) Now \( x \in \text{Int}(U_{x}) \) and there exists an open set \( W_{x} \) such that \( x \in W_{x} \subset \text{Cl}(W_{x}) \subset \text{Int}(U_{x}). \) Then \( \text{Cl}(W_{x}) \) is a closed subset of the compact set \( U \), which implies \( \text{Cl}(W_{x}) \) is compact. For each \( x \in C \), let \( W_{x} \) be open such that \( \text{Cl}(W_{x}) \) is compact and \( x \in W_{x} \subset \text{Cl}(W_{x}) \subset U. \) Then \( \{W_{x}\} \}_{x \in C} \) is an open cover of \( C \). Thus there exists a finite subcover \( \{W_{x_{i}}\}_{i=1}^{n} \). Let \( Y = \bigcup_{i=1}^{n} \text{Cl}(W_{x_{i}}), \) which is compact. Then \( C \) is a component of \( D \cap Y \), which is a closed subset of the compact \( R_{1} \) space \( (Y,T_{Y}), \) and \( C \subset \bigcup_{i=1}^{n} W_{x_{i}}, \) which
is open in $Y$. By Theorem 2.10, there exists an open set $V$ in $Y$ such that $C \cap V \subseteq \bigcup_{i=1}^{n} W_i$ and

$$\text{Fr}_Y(V) \cap (D \cap \bigcup_{i=1}^{n} \text{Cl}(W_i)) = \emptyset.$$ 

Since $V$ is open in $Y$, then there exists an open set $0$ in $X$ such that $V = 0 \cap Y$. Then $V \subseteq 0 \cap \bigcup_{i=1}^{n} W_i \subseteq 0 \cap Y = V$.

Thus $V = 0 \cap \bigcup_{i=1}^{n} W_i$, which is open in $X$. Since $\text{Cl}_X(V) \subseteq Y$, which is compact, then $\text{Cl}(V)_X$ is compact and since $\text{Fr}_X(V) = \text{Fr}_Y(V) \subseteq \bigcup_{i=1}^{n} \text{Cl}(W_i)$, then

$$\text{Fr}_X(V) \cap D = \text{Fr}_Y(V) \cap (D \cap \bigcup_{i=1}^{n} \text{Cl}(W_i)) = \emptyset.$$

**Theorem 2.12.** Let $(X,T)$ be an $R_1$ space, let $A \in \text{K}(X)$, let $C$ be a component of $A$, and let $U$ be open in $X$ such that $C \subseteq U$. Then there exists an open set $W$ in $X$ such that $C \subseteq W \subseteq U$ and $\text{Fr}(X) \cap A = \emptyset$.

**Proof.** Since $(A,T_A)$ is a compact $R_1$ space, $A$ is a closed subset of $A$, $C$ is a component of $A$, and $C \subseteq U \cap A$, which is open in $A$, then by Theorem 2.10, there exists an open set $V$ in $A$ such that $C \subseteq V \subseteq U \cap A$ and $\text{Fr}_A(V) \cap A = \emptyset$.

Then $V$ is closed-open in $A$ and $V = 0 \cap A$, where $0$ is open in $X$ and $0 \subseteq U$. If $\text{Fr}_X(0) \cap A = \emptyset$, then $0$ is open in $X$ such that $C \subseteq 0 \subseteq U$ and $\text{Fr}_X(0) \cap A = \emptyset$. Thus consider the case that $\text{Fr}_X(0) \cap A \neq \emptyset$. Since $\text{Fr}_X(0) \cap A$ is closed in $A$ and $A$ is compact, then $\text{Fr}_X(0) \cap A$ is compact and since $V$ is
closed in A and A is closed compact in X, then V is closed compact in X. If \( x \in \text{Fr}_X(0) \cap A \), then \( x \notin V \) and since \((X,T)\) is \( R_1 \) and V is closed compact in X, then there exists an open set \( O_x \) in X such that \( x \in O_x \subset \text{Cl}(O_x) \subset X \setminus V \). For each \( x \in \text{Fr}_X(0) \cap A \), let \( O_x \) be open in X such that \( x \in O_x \subset \text{Cl}(O_x) \subset X \setminus V \). Then \( \{O_x\}_{x \in \text{Fr}_X(0) \cap A} \) is an open cover for \( \text{Fr}_X(0) \cap A \). Thus there exists a finite subcover \( \{O_{x_1} \}_{i=1}^n \).

Then \( O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i}) \) is open in X and \( C \subset V \subset O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i}) \subset U \).

Also, \( \text{Fr}_X(O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i})) \cap A = \emptyset \), for suppose not. Let

\[ x \in \text{Fr}_X(O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i})) \cap A. \]

Then \( x \in A \), \( x \in \text{Cl}(O) \), and \( x \notin V \).

Since \( x \notin V \), then \( x \notin 0 \), which implies \( x \in \text{Fr}_X(0) \), but then

\[ x \in \text{Fr}_X(0) \cap A \subset \bigcup_{i=1}^n O_{x_i}, \text{ which is open, and } \]

\[ \bigcap_{i=1}^n \left( \bigcup_{i=1}^n O_{x_i} \right) \cap (0 \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i})) = \emptyset, \]

which contradicts

\[ x \in \text{Fr}_X(O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i})). \]

Then \( W = O \setminus \bigcup_{i=1}^n \text{Cl}(O_{x_i}) \) is open in X such that \( C \subset W \subset U \) and

and \( \text{Fr}(W) \cap A = \emptyset \).
CHAPTER BIBLIOGRAPHY


CHAPTER III

$T_0$-IDENTIFICATION SPACES AND $(2^X,E(X))$

William Dunham used $T_0$-identification spaces to obtain the following characterization of $R_1$ spaces:
A topological space $(X,T)$ is $R_1$ if and only if the $T_0$-identification space is $T_2$ [1]. In this chapter $T_0$-identification spaces are used to obtain additional results.

Definition 3.1. For a topological space $(X,T)$, let $X_0 = \mathcal{D}^0$ and let $S_0 = Q(\mathcal{D}^0)$.

Definition 3.2. For a topological space $(X,T)$ let $J$ be the relation in $X \times X$ defined by $x J y$ if and only if $x \in Cl\{y\}$. Then $J$ is not always an equivalence relation on $X$ and $\sim^0 \subset J$.

Definition 3.3. For a topological space $(X,T)$ let
$$\mathcal{B} = \{<G_1, \ldots, G_n>|n \in \mathbb{N} \text{ and } G_i \text{ is open in } X \text{ for all } i \in \{1, \ldots, n\}\},$$
which by Theorem 1.3, is a base for the Vietoris topology on $2^X$.

Lemma 3.1. If $(X,T)$ is a topological space and $A \in 2^X$, then $Cl\{A\} = \{B \in 2^X | B \subset A \text{ and every open set containing } B \text{ contains } A\}$.

Proof. Let $\mathcal{A} = \{B \in 2^X | B \subset A \text{ and every open set containing } B \text{ contains } A\}$. Let $B \in \mathcal{A}$. Let $\mathcal{O}$ be open in
$\mathcal{B}\subset 2^X$ such that $B\in \mathcal{B}$. Then there exists $\langle U_1, \ldots, U_n \rangle \in \mathcal{B}$ such that $B\in \langle U_1, \ldots, U_n \rangle \subset \mathcal{O}$. Since $B\subset \bigcup_{i=1}^n U_i$ and since $A\cap U_i$ for all $i \in \{1, \ldots, n\}$, then $A\in \langle U_1, \ldots, U_n \rangle \subset \mathcal{O}$. Hence, if $\mathcal{O}$ is open in $2^X$ such that $B\in \mathcal{O}$, then $\mathcal{O}\cap \{A\} \neq \emptyset$, which implies $B\in \text{Cl}\{A\}$.

Therefore, if $B\in \mathcal{A}$, then $B\in \text{Cl}\{A\}$, which implies $\mathcal{A}\subset \text{Cl}\{A\}$.

Let $D\in \text{Cl}\{A\}$. Then $D\in \text{Cl}\{A\}\subset \text{Cl}\{S(A)\} \subset S(\text{Cl}\{A\}) = S(A)$, which implies $D\subset A$. If $\mathcal{O}$ is open in $X$ such that $D\subset \mathcal{O}$, then $D\in S(\mathcal{O})$, which is open in $2^X$, and thus $\{A\} \cap S(\mathcal{O}) \neq \emptyset$, which implies $A\subset \mathcal{O}$. Then $D\in \mathcal{A}$.

Therefore, if $D\in \text{Cl}\{A\}$, then $D\in \mathcal{A}$, which implies $\text{Cl}\{A\}\subset \mathcal{A}$.

Therefore, $\text{Cl}\{A\} = \mathcal{A}$.

In 1951 Ernest Michael proved that if $\mathcal{B}$ is a decomposition of a $T_1$ space into closed sets, then the decomposition topology on $\mathcal{B}$ is coarser than the relative Vietoris topology on $\mathcal{B}$ [3]. This result is true for any topological spaces and will be used in the next theorem for an arbitrary topological space.

**Theorem 3.1.** Let $(X,T)$ be a topological space. Then the following are equivalent:

(a) $(X,T)$ is an $R_0$ space,

(b) $\sim$ is an equivalence relation on $X$,

(c) $(\mathcal{B}, \mathcal{Q}(\mathcal{B})) = (X_0, S_0)$, $X_0 = \{\text{Cl}(x) | x \in X\}$, $S_0 = E(X)_{X_0} (X_0, S_0)$ is an upper semi-continuous decomposition
of $X$, a lower semi-continuous decomposition of $X$, and a $T_1$ space, and if $0 \in T$, then $\bigcup B = 0$, where $B \in P(0)$

$P: (X,T) \rightarrow (X_0,S_0)$ is the natural map,

(d) the $T_0$-identification space on $X$ is a $T_1$ space.

Proof. (a) implies (b): If $(x,y) \in \sim'$, then $x \in \text{Cl}\{y\}$ and $\text{Cl}\{x\} \cap \text{Cl}\{y\} \neq \emptyset$, which implies $\text{Cl}\{x\} = \text{Cl}\{y\}$ and $(x,y) \in \sim^0$.

Therefore, if $(x,y) \in \sim'$, then $(x,y) \in \sim^0$. Thus $\sim' \subseteq \sim^0$ and since $\sim^0 \subseteq \sim'$, then $\sim' = \sim^0$, which is an equivalence relation on $X$.

(b) implies (c): Since $\sim'$ is an equivalence relation on $X$, then $\mathcal{B}'$ is a decomposition of $X$. Let $x \in X$ and let $C_x$ be the equivalence class for $\sim'$ containing $x$. If $y \in C_x$, then $y \sim' x$, and $y \in \text{Cl}\{x\}$, which implies $C_x \subseteq \text{Cl}\{x\}$, and if $y \in \text{Cl}\{x\}$, then $y \sim' x$ and $y \in C_x$, which implies $\text{Cl}\{x\} \subseteq C_x$. Thus $C_x = \text{Cl}\{x\}$.

Therefore, if $x \in X$, then $C_x = \text{Cl}\{x\}$, which implies $\mathcal{B}' = \{\text{Cl}\{x\} | x \in X\}$.

Since $\{\text{Cl}\{x\} | x \in X\}$ is a decomposition of $X$, then $(X,T)$ is an $R_0$ space and by the argument above, $\sim^0 = \sim'$.

Thus $X_0 = \mathcal{B}'$ and $(X_0,S_0) = (\mathcal{B}',Q(\mathcal{B}'))$. If $0$ is open in $X$, then $P(0) = \{\text{Cl}\{x\} | x \in 0\}$ and $\bigcup B = \bigcup_{B \in P(0)} \text{Cl}\{x\} = 0 \in T$, $x \in 0$ which implies $P(0) \in S_0$. Then $P$ is open, which implies
$(X_0, S_0)$ is a lower semi-continuous decomposition of $X$. If $C$ is closed in $X$, then $P(C) \cup P(X \setminus C) = X_0$, $P(X \setminus C)$ is open, and $P(C) = X_0 \setminus P(X \setminus C)$, which is closed in $(X_0, S_0)$, which implies $P$ is closed. Thus $(X_0, S_0)$ is an upper semi-continuous decomposition of $X$.

Let $\emptyset \in E(X)_{X_0}$. Let $\text{Cl}(x) \in \emptyset$. Then there exists $\langle U_1, \ldots, U_n \rangle \in \mathcal{B}$ such that $\text{Cl}(x) \in \langle U_1, \ldots, U_n \rangle \cap X_0 \subset \emptyset$. Since $(X, T)$ is an $R_0$ space, then

$$\langle U_1, \ldots, U_n \rangle \cap X_0 = \{ \text{Cl}(y) \mid y \in \bigcap_{i=1}^{n} U_i \}$$

and

$$\bigcup_{B \in \langle U_1, \ldots, U_n \rangle \cap X_0} U_i = \bigcap_{i=1}^{n} U_i,$$

which is open in $X$. Thus $\langle U_1, \ldots, U_n \rangle \cap X_0 \in S_0$. Hence, if $\text{Cl}(x) \in \emptyset$, then there exists $\emptyset_{\text{Cl}(x)} \in S_0$ such that

$$\text{Cl}(x) \in \emptyset_{\text{Cl}(x)} \subset \emptyset,$$

which implies $\emptyset \in S_0$.

Therefore, if $\emptyset \in E(X)_{X_0}$, then $\emptyset \in S_0$, which implies $E(X)_{X_0} \subset S_0$, and since $S_0 \subset E(X)_{X_0}$, then $E(X)_{X_0} = S_0$.

If $\text{Cl}(x) \in X_0$, then by Lemma 3.1, $\text{Cl}_2 X(\text{Cl}(x)) = \{ \text{Cl}(x) \}$, and thus

$$\text{Cl}_{X_0} \{ \text{Cl}(x) \} = \text{Cl}_2 X(\text{Cl}(x)) \cap X_0 = \{ \text{Cl}(x) \},$$

which implies singleton sets are closed in $(X_0, S_0)$. Thus $(X_0, S_0)$ is $T_1$.

(c) implies (d): (d) is included in (c).

(d) implies (a): For each $x \in X$, let $C_x$ be the
equivalence class of \( \mathcal{I} \) containing \( x \). Let \( x, y \in X \). Then
\[ Cl\{x\} = Cl\{y\} \text{ or } Cl\{x\} \neq Cl\{y\}. \]
Suppose \( Cl\{x\} \neq Cl\{y\} \).
Then \( Cl\{x\} \cap Cl\{y\} = \emptyset \), for suppose not. Let
\[ z \in Cl\{x\} \cap Cl\{y\}. \]
Then \( Cl\{z\} \neq Cl\{x\} \) or \( Cl\{z\} \neq Cl\{y\} \), say \( Cl\{z\} \neq Cl\{x\} \). Then \( (z, x) \not\in \mathcal{I} \), which implies
\[ C_z \cap C_x = \emptyset. \]
Since \( (X_0, S_0) \) is \( T_1 \), then \( \{C_x\} \) is closed in \( X_0 \). Thus \( X_0 \setminus \{C_x\} \) is open in \( X_0 \), \( z \in C_z \subseteq \bigcup B \]
\[ B \in X_0 \setminus \{C_x\} \]
which is open in \( X \), and \( x \notin \bigcup B \), which contradicts \( z \in Cl\{x\} \).

Therefore, if \( x, y \in X \), then \( Cl\{x\} = Cl\{y\} \) or
\[ Cl\{x\} \cap Cl\{y\} = \emptyset, \]
which implies \( (X, T) \) is an \( R_0 \) space.

**Corollary 3.1.** Let \( (X, T) \) be a topological space.

Then

(a) \( (X, T) \) is \( R_0 \) if and only if \( (X_0, S_0) \) is \( R_0 \), and

(b) \( (X, T) \) is \( R_1 \) if and only if \( (X_0, S_0) \) is \( R_1 \).

**Proof.** (a) Suppose \( (X, T) \) is \( R_0 \). By Theorem 3.1, \( (X_0, S_0) \) is \( T_1 \), which implies \( (X_0, S_0) \) is \( R_0 \).

Conversely, suppose \( (X_0, S_0) \) is \( R_0 \). Then \( (X_0, S_0) \) is \( R_0 \) and \( T_0 \), which implies \( (X_0, S_0) \) is \( T_1 \). Then by Theorem 3.1, \( (X, T) \) is \( R_0 \).

(b) Suppose \( (X, T) \) is \( R_1 \). Then \( (X_0, S_0) \) is \( T_2 \), which implies \( (X_0, S_0) \) is \( R_1 \).

Conversely, suppose \( (X_0, S_0) \) is \( R_1 \). Since \( (X_0, S_0) \) is \( R_1 \), then \( (X_0, S_0) \) is \( R_0 \). Then \( (X_0, S_0) \) is \( R_0 \) and \( T_0 \),
which implies \((X_0, S_0)\) is \(T_1\). Since \((X_0, S_0)\) is \(R_1\) and \(T_1\), then \((X_0, S_0)\) is \(T_2\), which implies \((X, T)\) is \(R_1\).

**Corollary 3.2.** A topological space \((X, T)\) is \(R_0\) if and only if \(X_0 \subseteq 2^X\).

**Proof.** Suppose \((X, T)\) is \(R_0\). By Theorem 3.1, 
\[ X_0 = \{ \text{Cl}(x) \mid x \in X \} \subseteq 2^X. \]

Conversely, suppose \(X_0 \subseteq 2^X\). Let \(x \in X\) and let \(C_x\) be the equivalence class of \(x\) containing \(x\). Then 
\[ C_x \subseteq \text{Cl}(x). \]
Since \(C_x \subseteq 2^X\), then \(C_x\) is closed in \(X\) and \(x \in C_x\) which implies \(\text{Cl}(x) \subseteq C_x\). Thus \(C_x = \text{Cl}(x)\). Hence, if \(x \in X\), then \(\text{Cl}(x) = C_x\), which implies \(X_0 = \{ \text{Cl}(x) \mid x \in X \}\).

Since \(X_0\) is a decomposition of \(X\), then if \(x, y \in X\), then \(\text{Cl}(x) = \text{Cl}(y)\) or \(\text{Cl}(x) \cap \text{Cl}(y) = \emptyset\), which implies \((X, T)\) is \(R_0\).

**Theorem 3.2.** A topological space \((X, T)\) is \(R_1\) if and only if \(X_0\) is a closed subset of \(2^X\).

**Proof.** Suppose \((X, T)\) is \(R_1\). Then \((X, T)\) is \(R_0\). Thus by Theorem 3.1, 
\[ X_0 = \{ \text{Cl}(x) \mid x \in X \} \subseteq 2^X. \]
Let \(F \subseteq 2^X \setminus X_0\). Then \(F \neq \text{Cl}(x)\) for all \(x \in X\). Let \(y \in F\). Then there exists \(z \in F\) such that \(\text{Cl}(z) \neq \text{Cl}(y)\), since otherwise \(F = \bigcup \text{Cl}(x) = \text{Cl}(y)\), which is a contradiction.

Since \(\text{Cl}(y) \neq \text{Cl}(z)\) and \((X, T)\) is \(R_1\), then there exist disjoint open sets \(U\) and \(V\) such that \(\text{Cl}(y) \subseteq U\) and \(\text{Cl}(z) \subseteq V\). If \(\text{Cl}(y) \cup \text{Cl}(z) \not\subseteq F\), then 
\[ F \subseteq \langle U, V, X \setminus (\text{Cl}(y) \cup \text{Cl}(z)) \rangle, \]
which is open in \(2^X\), and \(\langle U, V, X \setminus (\text{Cl}(y) \cup \text{Cl}(z)) \rangle \cap X_0 = \emptyset\), and if
\( \text{Cl}\{y\} \cup \text{Cl}\{z\} = F, \text{then } F \in \langle U,V \rangle, \text{which is open in } 2^X, \text{and } \langle U,V \rangle \cap X_0 = \emptyset. \)

Therefore, if \( F \in 2^X \setminus X_0 \), then there exists an open set \( \mathcal{O} \) in \( 2^X \) such that \( F \in \mathcal{O} \) and \( \mathcal{O} \cap X_0 = \emptyset. \) Thus \( 2^X \setminus X_0 \) is open in \( 2^X \), which implies \( X_0 \) is closed in \( 2^X \).

Conversely, suppose \( X_0 \) is a closed subset of \( 2^X \). Then \( X_0 \subseteq 2^X \). Thus by Corollary 3.2., \( (X,T) \) is an \( R_0 \) space. Then by Theorem 3.1, \( X_0 = \{\text{Cl}\{x\} | x \in X\} \). Let \( x,y \in X \) such that \( \text{Cl}\{x\} \neq \text{Cl}\{y\} \). Then \( \text{Cl}\{x\} \cup \text{Cl}\{y\} \not\subseteq X_0 \) and there exists \( \langle U_1, \ldots, U_n \rangle \not\subseteq \mathcal{B} \) such that

\[
\text{Cl}\{x\} \cup \text{Cl}\{y\} \in \langle U_1, \ldots, U_n \rangle.
\]

and

\[
\langle U_1, \ldots, U_n \rangle \cap X_0 = \emptyset.
\]

Let

\[
I_x = \{i \in \{1, \ldots, n\} | x \in U_i\}
\]

and let

\[
I_y = \{i \in \{1, \ldots, n\} | y \in U_i\}.
\]

Then

\[
I_x \cup I_y = \{1, \ldots, n\}
\]

and

\[
\text{Cl}\{y\} \cup \text{Cl}\{x\} \in \langle \bigcap_{i \in I_x} U_i, \bigcap_{i \in I_y} U_i \rangle \subseteq \langle U_1, \ldots, U_n \rangle.
\]

Thus

\[
\langle \bigcap_{i \in I_x} U_i, \bigcap_{i \in I_y} U_i \rangle \cap X_0 = \emptyset,
\]

which implies

\[
(\bigcap_{i \in I_x} U_i) \cap (\bigcap_{i \in I_y} U_i) = \emptyset.
\]
Then \( \bigcap_{i \in I_x} U_i \) and \( \bigcap_{i \in I_y} U_i \) are disjoint open sets in \( X \) such that \( \text{Cl}\{x\} \subseteq \bigcap_{i \in I_x} U_i \) and \( \text{Cl}\{y\} \subseteq \bigcap_{i \in I_y} U_i \).

Therefore, if \( x, y \in X \) such that \( \text{Cl}\{x\} \neq \text{Cl}\{y\} \), then there exist disjoint open sets \( U \) and \( V \) such that \( \text{Cl}\{x\} \subseteq U \) and \( \text{Cl}\{y\} \subseteq V \), which implies \( (X, T) \) is \( R_1 \).

**Theorem 3.3.** Let \( (X, T) \) be an \( R_0 \) space. Then

(a) \( (X, T) \) is rim-compact if and only if \( (X_0, S_0) \) is rim-compact,

(b) \( (X, T) \) is separable if and only if \( (X_0, S_0) \) is separable,

(c) \( (X, T) \) is second countable if and only if \( (X_0, S_0) \) is second countable,

(d) \( (X, T) \) is first countable if and only if \( (X_0, S_0) \) is first countable,

(e) \( (X, T) \) is connected if and only if \( (X_0, S_0) \) is connected,

(f) \( (X, T) \) is compact if and only if \( (X_0, S_0) \) is compact, and

(g) \( (X, T) \) is locally connected (l.c.) if and only if \( (X_0, S_0) \) is l.c.

**Proof.** Since \( (X, T) \) is \( R_0 \), then by Theorem 3.1, \( (X_0, S_0) \) is a \( T_1 \) space, \( X_0 = \{\text{Cl}\{x\}|x \in X\} \), and the natural map \( P:(X, T) \rightarrow (X_0, S_0) \) is closed-open such that if \( O \) is open in \( X \), then \( \bigcup \text{Cl}\{y\} = O \) if \( \text{Cl}\{y\} \in P(O) \).
(a) Suppose \((X,T)\) is rim-compact. Let \(\text{Cl}\{x\} \in X_0\). Let \(\mathcal{O}\) be open in \(X_0\) such that \(\text{Cl}\{x\} \in \mathcal{O}\). Then
\[x \in \bigcup \text{Cl}\{y\},\] which is open in \(X\). Thus there exists a neighborhood \(V\) of \(x\) in \(X\) such that \(V \subseteq \bigcup \text{Cl}\{y\}\) and \(\text{Fr}(V)\) is compact. Then \(\text{Cl}\{x\} \in \text{P}(V)\), which is a neighborhood of \(\text{Cl}\{x\}\) in \(X_0\), and \(\text{Fr}(\text{P}(V)) = \text{P}(\text{Fr}(V))\), which is compact in \(X_0\).

Therefore, if \(\mathcal{O}\) is open in \(X_0\) such that \(\text{Cl}\{x\} \in \mathcal{O}\), then there exists a neighborhood \(\mathcal{V}\) of \(\text{Cl}\{x\}\) in \(X_0\) such that \(\mathcal{V} \in \mathcal{O}\) and \(\text{Fr}(\mathcal{V})\) is compact, which implies \(\text{Cl}\{x\}\) has a base of neighborhoods with compact frontiers.

Therefore, if \(\text{Cl}\{x\} \in X_0\), then \(\text{Cl}\{x\}\) has a base of neighborhoods with compact frontiers, which implies \((X_0,S_0)\) is rim-compact.

Conversely, suppose \((X_0,S_0)\) is rim-compact. Let \(x \in X\). Let \(\mathcal{O}\) be open in \(X\) such that \(x \in \mathcal{O}\). Then \(\text{P}(\mathcal{O})\) is open in \(X_0\) and \(\text{Cl}\{x\} \in \text{P}(\mathcal{O})\). Thus there exists a neighborhood \(\mathcal{V}\) of \(\text{Cl}\{x\}\) such that \(\mathcal{V} \subseteq \text{P}(\mathcal{O})\) and \(\text{Fr}(\mathcal{V})\) is compact.

Then \(\bigcup \text{Cl}\{y\}\) is a neighborhood of \(x\), \(\bigcup \text{Cl}\{y\} \subseteq \mathcal{O}\), and \(\text{Cl}\{y\} \in \mathcal{V}\).

\[\text{Fr}(\bigcup \text{Cl}\{y\}) = \bigcup \text{Cl}\{y\}, \quad \text{Cl}\{y\} \in \mathcal{V}\]

Let \(\mathcal{U}\) be an open cover of \(\text{Fr}(\bigcup \text{Cl}\{y\})\). Then \(\{\text{Cl}\{y\} \in \mathcal{V}\}\) is an open cover of \(\text{Fr}(\mathcal{V})\), which is compact.

Thus there exists a finite subcover \(\{\text{P}(U_i)\}_{i=1}^{n}\). Then \(\{U_i = \text{P}^{-1}(\text{P}(U_i))\}_{i=1}^{n}\) is a finite subcover of \(\mathcal{U}\).
Therefore, if \( U \) is an open cover of \( \text{Fr}(\bigcup \text{Cl}\{y\}) \), then there exists a finite subcover, which implies
\[
\text{Fr}(\bigcup \text{Cl}\{y\}) \text{ is compact.}
\]

Therefore, if \( O \) is open such that \( x \in O \), then there exists a neighborhood \( V \) of \( x \) such that \( V \subset O \) and \( \text{Fr}(V) \) is compact.

Therefore, each point of \( X \) has a base of neighborhoods with compact frontiers, which implies \( (X,T) \) is rim-compact.

(b) Suppose \( (X,T) \) is separable. Then \( (X,T) \) has a countable dense subset \( \{x_i\}_{i=1}^{\infty} \). Then \( \{\text{Cl}\{x_i\}\}_{i=1}^{\infty} \) is a countable subset of \( X_0 \). If \( O \) is open in \( X_0 \), then
\[
\bigcup \text{Cl}\{y\} \text{ is open in } X \text{ and there exists } n \in \mathbb{N} \text{ such that } \text{Cl}\{y\} \in O
\]
\[
x_n \in \bigcup \text{Cl}\{y\}, \text{ which implies Cl}\{x_n\} \in O. \text{ Thus } \{\text{Cl}\{x_i\}\}_{i=1}^{\infty} \text{ is dense in } (X_0,S_0).
\]

Therefore, there exists a countable dense subset of \( (X_0,S_0) \), which implies \( (X_0,S_0) \) is separable.

Conversely, suppose \( (X_0,S_0) \) is separable. Then \( (X_0,S_0) \) has a countable dense subset \( \{\text{Cl}\{x_i\}\}_{i=1}^{\infty} \). Then \( \{x_i\}_{i=1}^{\infty} \) is a countable subset of \( X \). If \( O \) is open in \( X \), then \( P(O) \) is open in \( X_0 \) and there exists \( n \in \mathbb{N} \) such that \( \text{Cl}\{x_n\} \in P(O) \), which implies \( x_n \in O \). Thus \( \{x_i\}_{i=1}^{\infty} \) is dense in \( (X,T) \).

Therefore, \( (X,T) \) has a countable dense subset, which implies \( (X,T) \) is separable.
(c) Suppose \((X,T)\) is second countable. Then \((X_0,S_0)\) is an upper semi-continuous decomposition of \((X,T)\) into nonempty compact sets and \((X,T)\) is second countable, which implies \((X_0,S_0)\) is second countable [2].

Conversely, suppose \((X_0,S_0)\) is second countable. Then \((X_0,S_0)\) has a countable base \(\{\mathcal{O}_i\}_{i=1}^\infty\). For each \(i \in \mathbb{N}\), let \(O_i = \bigcup B_i\), which is open in \(X\). Then \(\{O_i\}_{i=1}^\infty \subset T\). Let \(x \in X\) and let \(O \in T\) such that \(x \in O\). Then \(\text{Cl}\{x\} \in \mathcal{P}(O)\), which is open in \(X_0\), and there exists \(n \in \mathbb{N}\) such that \(\text{Cl}\{x\} \in \mathcal{O}_n \subset \mathcal{P}(O)\), which implies

\[
x \in \text{Cl}\{x\} \subset \bigcup \text{Cl}\{y\} = O_n \subset \bigcup \text{Cl}\{y\} = O.
\]

Therefore, if \(x \in X\) and 0 is open in \(X\) such that \(x \in O\), then there exists \(n \in \mathbb{N}\) such that \(x \in O_n \subset 0\) and \(\{O_i\}_{i=1}^\infty \subset T\), which implies \(\{O_i\}_{i=1}^\infty\) is a base for \(T\).

Therefore, \((X,T)\) has a countable base, which implies \((X,T)\) is second countable.

(d) Part (d) is obtained by an argument similar to that for (c).

(e) Suppose \((X,T)\) is connected. Since the natural map \(P:(X,T) \rightarrow (X_0,S_0)\) is continuous, then \((X_0,S_0)\) is connected.

Conversely, suppose \((X_0,S_0)\) is connected. Then \((X,T)\) is connected, for suppose not. Then \(X = M \cup N\),
where M and N are non-empty disjoint closed-open sets in X, but then \( P(M) \) and \( P(N) \) are non-empty disjoint closed open sets in \( X_0 \) and \( X_0 = P(M) \cup P(N) \), which is a contradiction.

(f) Suppose \((X,T)\) is compact. Since the natural map \( P : (X,T) \to (X_0,S_0) \) is continuous, then \((X_0,S_0)\) is compact.

Conversely, suppose \((X_0,S_0)\) is compact. Let \( \mathcal{O} \) be an open cover for \( X \). Then \( \{P(O) \mid O \in \mathcal{O}\} \) is an open cover for \( X_0 \). Thus there exists a finite subcover \( \{P(O_i)\}_{i=1}^n \). Then \( \{O_i = P_i^{-1}(P(O_i))\}_{i=1}^n \) is a finite subcover of \( \mathcal{O} \).

Therefore, if \( \mathcal{O} \) is an open cover of \( X \), then there exists a finite subcover, which implies \((X,T)\) is compact.

(g) Suppose \((X,T)\) is l.c.. Let \( \text{Cl}\{x\} \in X_0 \) and let \( \mathcal{O} \) be open in \( X_0 \) such that \( \text{Cl}\{x\} \in \mathcal{O} \). Then \( x \in \bigcup \text{Cl}\{y\} \), \( \text{Cl}\{y\} \in \mathcal{O} \), which is open in \( X \), and there exists an open connected set \( V \) in \( X \) such that \( x \in V \subseteq \bigcup \text{Cl}\{y\} \). Then \( P(V) \) is open connected in \( X_0 \) and \( \text{Cl}\{x\} \in P(V) \subseteq \mathcal{O} \).

Therefore, if \( \text{Cl}\{x\} \in X_0 \) and \( \mathcal{O} \) is open in \( X_0 \) such that \( \text{Cl}\{x\} \in \mathcal{O} \), then there exists an open connected set \( \mathcal{V} \) in \( X_0 \) such that \( \text{Cl}\{x\} \in \mathcal{V} \subseteq \mathcal{O} \), which implies \((X_0,S_0)\) is l.c.

Conversely, suppose \((X_0,S_0)\) is l.c.. Let \( x \in X \) and let \( \mathcal{O} \) be open in \( X \) such that \( x \in \mathcal{O} \). Then \( \text{Cl}\{x\} \in P(\mathcal{O}) \), which is open in \( X_0 \), and there exists an open connected
set \( \mathcal{V} \) in \( X_0 \) such that \( \text{Cl}\{x\} \in \mathcal{V} \subset \mathcal{P}(O) \). Then 
\[
\bigcup_{\text{Cl}\{y\} \in \mathcal{V}} \text{Cl}\{y\} \in \mathcal{V}
\]
is open in \( X \) and \( x \in \bigcup \text{Cl}\{y\} \subset O \). Also, \( \bigcup_{\text{Cl}\{y\} \in \mathcal{V}} \text{Cl}\{y\} \in \mathcal{V} \) is con-
nected, for suppose not. Then \( V = \bigcup \text{Cl}\{y\} = M \cup N \), where 
\( \text{Cl}\{y\} \in \mathcal{V} \) and \( M \) and \( N \) are non-empty disjoint closed-open sets in \( V \).

Since \( V \) is open in \( X \), then \( M \) and \( N \) are open in \( X \) and 
P(\( M \)) and P(\( N \)) are open in \( X_0 \), but then \( \mathcal{V} = \mathcal{P}(M) \cup \mathcal{P}(N) \), 
where P(\( M \)) and P(\( N \)) are disjoint open sets, which contra-
dicts \( \mathcal{V} \) is connected.

Therefore, if \( x \in X \) and \( O \) is open in \( X \) such that 
\( x \in O \), then there exists an open connected set \( V \) such that 
\( x \in V \subset O \), which implies \((X,T)\) is 1.c.

**Theorem 3.4.** A topological space \((X,T)\) is regular 
if and only if \((X_0, S_0)\) is \( T_3 \).

**Proof.** Suppose \((X,T)\) is regular. Then \((X,T)\) is \( R_0 \). By Theorem 3.1, \((X_0, S_0)\) is a \( T_1 \) upper semi-continuous 
decomposition of \( X \) into nonempty compact sets and since 
\( (X,T) \) is regular, then \((X_0, S_0)\) is regular [2]. Then \((X,T)\) 
is a regular \( T_1 \) space, which implies \((X,T)\) is \( T_3 \).

Conversely, suppose \((X_0, S_0)\) is \( T_3 \). Then \((X_0, S_0)\) is 
\( T_1 \). Thus by Theorem 3.1, \((X,T)\) is \( R_0 \), \( X_0 = \{\text{Cl}\{x\} | x \in X\} \), 
and if \( O \) is open in \( X \), then \( \bigcup_{B \in \mathcal{P}(O)} B = O \), where \( P \) is the 
natural map from \((X,T)\) onto \((X_0, S_0)\). Let \( O \) be open in \( X \) 
and let \( x \in O \). Then \( \text{Cl}\{x\} \subset O \) and \( \bigcup_{B \in \mathcal{P}(O)} B = O \). Thus 
\( \text{Cl}\{x\} \in \mathcal{P}(O) \), which is open in \( X_0 \). Since \((X_0, S_0)\) is regular,
then there exists an open set \( V \) in \((X_0, S_0)\) such that 
\( Cl(x) \in V \subseteq Cl(V) \subseteq P(0) \). Then \( P^{-1}(V) \) is open in \( X \) and 
\( x \in P^{-1}(V) \subseteq P^{-1}(V) \cap P^{-1}(V) \subseteq 0 \).

Therefore, if \( 0 \) is open in \( X \) and \( x \in 0 \), then there 
extists an open set \( V \) such that \( x \in V \subseteq V \subseteq 0 \), which implies 
\((X, T)\) is regular.

**Theorem 3.5.** A topological space \((X, T)\) is completely 
regular if and only if \((X_0, S_0)\) is \( T_{3\frac{1}{2}} \).

**Proof.** Suppose \((X, T)\) is completely regular. Then 
\((X, T)\) is \( R_0 \). Thus by Theorem 3.1, \((X_0, S_0)\) is a \( T_1 \) lower 
semi-continuous decomposition of \( X \), where \( X_0 = \{Cl(x) | x \in X \} \).
Let \( \mathcal{C} \) be closed in \( X_0 \) and let \( Cl(x) \notin \mathcal{C} \). Then 
\( P^{-1}(\mathcal{C}) = \bigcup_{B \in \mathcal{C}} B \in \mathcal{C} \) 
is closed in \( X \) and \( x \notin P^{-1}(\mathcal{C}) \). Since \( X \) is completely 
regular, there exists a continuous function \( f: (X, T) \rightarrow [0, 1] \) 
such that \( f(x) = 0 \) and \( f(P^{-1}(\mathcal{C})) = 1 \). If \( y_1, y_2 \in X \) such 
that \( Cl(y_1) = Cl(y_2) \), then \( f(y_1) = f(y_2) \), since otherwise 
\( f(y_1) \neq f(y_2) \) in \([0, 1]\), say \( f(y_1) < f(y_2) \), and 
\[ y_2 \in f^{-1}(\frac{f(y_1) + f(y_2)}{2}, 1), \]
which is open in \( X \), and \( y_1 \notin f^{-1}(\frac{f(y_1) + f(y_2)}{2}, 1) \), 
which contradicts \( y_2 \in Cl(y_1) \). Let \( g: (X_0, S_0) \rightarrow [0, 1] \) be 
defined by \( g(Cl(y)) = f(y) \). Then \( g(Cl(x)) = f(x) = 0 \) and 
\( g(\mathcal{C}) = 1 \). If \( 0 \) is open in \([0, 1]\), then \( f^{-1}(0) \) is open in \( X \) 
and \( g^{-1}(0) = \{Cl(y) | y \in f^{-1}(0)\} = P(f^{-1}(0)) \) is open in \( X_0 \), 
which implies \( g \) is continuous.
Therefore, if $C$ is closed in $(X_0,S_0)$ and $\operatorname{Cl}\{x\} \notin C$, then there exists a continuous function $g:(X_0,S_0) \to [0,1]$ such that $g(x) = 0$ and $g(C) = 1$, which implies $(X_0,S_0)$ is completely regular.

Therefore, $(X_0,S_0)$ is a completely regular $T_1$ space, which implies $(X_0,S_0)$ is $T_{3\frac{1}{2}}$.

Conversely, suppose $(X_0,S_0)$ is $T_{3\frac{1}{2}}$. Then $(X_0,S_0)$ is $T_1$. Thus by Theorem 3.1, $(X,T)$ is $R_0$ and $(X_0,S_0)$ is an upper semi-continuous decomposition of $X$, where $X_0 = \{\operatorname{Cl}\{x\} | x \in X\}$. Let $C$ be closed in $X$ and let $x \notin C$.

Then $P(C) = \{\operatorname{Cl}\{y\} | y \in C\}$ is closed in $X_0$ and $P(x) = \operatorname{Cl}\{x\} \notin P(C)$. Thus there exists a continuous function $f:(X_0,S_0) \to [0,1]$ such that $f(P(x)) = f(\operatorname{Cl}\{x\}) = 0$ and $f(P(C)) = 1$. Then $f\cdot P:(X,T) \to [0,1]$ is continuous and $(f\cdot P)(x) = 0$ and $(f\cdot P)(C) = 1$.

Therefore, if $C$ is closed in $X$ and $x \notin C$, then there exists a continuous function $g:(X,T) \to [0,1]$ such that $g(x) = 0$ and $g(C) = 1$, which implies $(X,T)$ is completely regular.

**Theorem 3.6.** A topological space $(X,T)$ is normal $R_0$ if and only if $(X_0,S_0)$ is $T_4$.

**Proof.** Suppose $(X,T)$ is normal $R_0$. By Theorem 3.1, $(X_0,S_0)$ is a $T_1$ upper semi-continuous decomposition of $X$ into compact sets and since $(X,T)$ is normal, then $(X_0,S_0)$ is normal.
Therefore, \((X_0, S_0)\) is a normal \(T_1\) space, which implies \((X_0, S_0)\) is \(T_4\).

Conversely, suppose \((X_0, S_0)\) is \(T_4\). Then \((X_0, S_0)\) is \(T_1\). Thus by Theorem 3.1, \((X, T)\) is \(R_0\) and \((X_0, S_0)\) is an upper semi-continuous decomposition of \(X\), where \(X_0 = \{\text{Cl}\{x\} | x \in X\}\). Let \(C_1\) and \(C_2\) be disjoint closed sets in \(X\). Then \(P(C_1)\) and \(P(C_2)\) are disjoint closed sets in \((X_0, S_0)\) and there exist disjoint open sets \(U\) and \(V\) in \(X_0\) such that \(P(C_1) \subseteq U\) and \(P(C_2) \subseteq V\). Then \(C_1 \subseteq P^{-1}(U) \in T\), \(C_2 \subseteq P^{-1}(V) \in T\), and \(P^{-1}(U) \cap P^{-1}(V) = \emptyset\).

Therefore, if \(C_1\) and \(C_2\) are disjoint closed sets in \(X\), then there exist disjoint open sets \(U\) and \(V\) in \(X\) such that \(C_1 \subseteq U\) and \(C_2 \subseteq V\), which implies \((X, T)\) is normal.

Therefore, \((X, T)\) is normal \(R_0\).

**Corollary 3.3.** A topological space \((X, T)\) is regular second countable if and only if \((X_0, S_0)\) is separable and metrizable.

**Proof.** Suppose \((X, T)\) is regular second countable. By Theorem 3.3 (c), \((X_0, S_0)\) is second countable and by Theorem 3.4, \((X_0, S_0)\) is a \(T_3\) space. Thus by Urysohn's Metrization Theorem, \((X_0, S_0)\) is separable and metrizable.

Conversely, suppose \((X_0, S_0)\) is separable and metrizable. By Urysohn's Metrization Theorem, \((X_0, S_0)\) is \(T_3\) and second countable. By Theorem 3.4, \((X, T)\) is regular. Then by Theorem 3.3 (c), \((X, T)\) is second countable.
Theorem 3.7. If $(X,T)$ is an $R_0$ space, then $(2^X,E(X))$ is homeomorphic to $(2^{X_0},E(X_0))$.

Proof. By Theorem 3.1, $(X_0,S_0)$ is an upper semi-continuous decomposition of $X$, a lower semi-continuous decomposition of $X$, a $T_1$ space such that if $0 \in T$, then

$\cup B = 0$, where $X_0 = \{\text{Cl}(x) \mid x \in X\}$ and $P:(X,T) \to (X_0,S_0)$ $B \in P(0)$

is the natural map, and $S_0 = E(X)X_0$. Let

$f:(2^X,E(X)) \to (2^{X_0},E(X_0))$

be defined by

$f(F) = P(F) \in 2^{X_0}$.

If $\mathcal{C} \in 2^{X_0}$, then $\mathcal{C}$ is closed in $X_0$, $P^{-1}(\mathcal{C})$ is closed in $X$, and $f(P^{-1}(\mathcal{C})) = \mathcal{C}$, which implies $f$ is onto. If $F_1$ and $F_2$ are elements of $2^X$ such that $f(F_1) = f(F_2)$, then

$F_1 = P^{-1}(P(F_1)) = P^{-1}(f(F_1))$

$= P^{-1}(f(F_2)) = P^{-1}(P(F_2)) = F_2$

which implies $f$ is 1-1. A subbase for $(2^{X_0},E(X_0))$ is

$\mathcal{S} = \{\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle \mid n \in \mathbb{N} \text{ and } \mathcal{O}_i \text{ is open in } X_0 \text{ for all } i \in \{1, \ldots, n\} \text{ and if } \langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle \in \mathcal{S}, \text{ then } P^{-1}(\mathcal{O}_i) \in T \text{ for all } i \in \{1, \ldots, n\} \text{ and } f^{-1}(\langle \mathcal{O}_1, \ldots, \mathcal{O}_n \rangle) = \langle P^{-1}(\mathcal{O}_1), \ldots, P^{-1}(\mathcal{O}_n) \rangle$

is open in $2^X$, which implies $f$ is continuous. A base for $(2^X,E(X))$ is $\mathcal{B} = \{\langle O_1, \ldots, O_n \rangle \mid n \in \mathbb{N} \text{ and } O_i \text{ is open in } X \text{ for all } i \in \{1, \ldots, n\} \text{ and if } \langle O_1, \ldots, O_n \rangle \in \mathcal{B}, \text{ then } P(O_i) \in S_0 \text{ for all } i \in \{1, \ldots, n\} \text{ and } f(\langle O_1, \ldots, O_n \rangle) = \langle P(O_1), \ldots, P(O_n) \rangle$

is open in $2^{X_0}$, which implies $f$ is open.
Therefore, $f$ is a homeomorphism, which implies $(2^X, E(X))^X_0$ is homeomorphic to $(2^X_0, E(X_0))$.

The following corollaries are generalizations of many of Ernest Michael's results [3], where $T_1$ is replaced by $R_0$ and $T_2$ is replaced by $R_1$.

**Corollary 3.4.** If $(X,T)$ is an $R_0$ space, then $(2^X, E(X))$ is $T_1$.

**Proof.** By Theorem 3.1, $(X_0, S_0)$ is $T_1$. Thus by Theorem 4.9.2 of Michael's paper [3], $(2^{X_0}, E(X_0))$ is $T_1$. By Theorem 3.7, $(2^{X_0}, E(X_0))$ is homeomorphic to $(2^X, E(X))$, which implies $(2^X, E(X))$ is $T_1$.

**Corollary 3.5.** For an $R_0$ space $(X,T)$, each of the following are true:

(a) $(X,T)$ is separable if and only if $(2^X, E(X))$ is separable,

(b) $(X,T)$ is second countable if and only if $(K(X), E(X)^K(X))$ is second countable,

(c) if $(K(X), E(X)^K(X))$ is first countable, then $(X,T)$ is first countable.

**Proof.** (a) Suppose $(X,T)$ is separable. By Theorem 3.1, $(X_0, S_0)$ is a $T_1$ space and by Theorem 3.3 (b), $(X_0, S_0)$ is separable. Then by Proposition 4.5.1 of Michael's paper [3], $(2^{X_0}, E(X_0))$ is separable. By Theorem 3.7, $(2^{X_0}, E(X_0))$ is homemorphic to $(2^X, E(X))$, which implies $(2^X, E(X))$ is separable.
Conversely, suppose \((2^X, E(X))\) is separable. By Theorem 3.7, \((2^{X_0}, E(X_0))\) is homeomorphic to \((2^X, E(X))\), which implies \((2^{X_0}, E(X_0))\) is separable. By Theorem 3.1, \((X_0, S_0)\) is \(T_1\) and thus by Proposition 4.5.1 of Michael's paper [3], \((X_0, S_0)\) is separable. Then by Theorem 3.3 (b), \((X, T)\) is separable.

(b) Suppose \((X, T)\) is second countable. By Theorem 3.1, \((X_0, S_0)\) is \(T_1\) and by Theorem 3.3 (c), \((X_0, S_0)\) is second countable. Then by Proposition 4.5.2 of Michael's paper [3], \((K(X_0), E(X_0))_{K(X_0)}\) is second countable. By Theorem 3.7, \((2^{X_0}, E(X_0))\) is homeomorphic to \((2^X, E(X))\). Thus \((K(X_0), E(X_0))_{K(X_0)}\) is homeomorphic to \((K(X), E(X))_{K(X)}\), which implies \((K(X), E(X))_{K(X)}\) is second countable.

Conversely, suppose \((K(X), E(X))_{K(X)}\) is second countable. By Theorem 3.7, \((2^{X_0}, E(X_0))\) is homeomorphic to \((2^X, E(X))\). Thus \((K(X_0), E(X_0))_{K(X_0)}\) is homeomorphic to \((K(X), E(X))_{K(X)}\), which implies \((K(X_0), E(X_0))_{K(X_0)}\) is second countable. By Theorem 3.1, \((X_0, S_0)\) is \(T_1\). Thus by Proposition 4.5.2 of Michael's paper [3], \((X_0, S_0)\) is second countable. Then by Theorem 3.3 (c), \((X, T)\) is second countable.

(c) Part (c) follows from an argument similar to that for the second part of (b).

Corollary 3.6. Let \((X, T)\) be an \(R_0\) space. If \((2^X, E(X))\) is second countable, then \((X, T)\) is compact.
Proof. By Theorem 3.1, \((X_0, S_0)\) is T₁. By Theorem 3.7, \((2^X, E(X))\) is homeomorphic to \((2^{X_0}, E(X_0))\), which implies \((2^{X_0}, E(X_0))\) is second countable. Then by Theorem 4.6 of Michael's paper [3], \((X_0, S_0)\) is compact. Thus by Theorem 3.3 (f), \((X, T)\) is compact.

Definition 3.4. For a topological space \((X, T)\) and \(n \in \mathbb{N}\), let \(F_n(X) = \{ \bigcup_{i=1}^{n} \text{Cl}\{x_i\} | x_i \in X \text{ for all } i \in \{1, \ldots, n\} \}\) and let \(F(X) = \bigcup_{i=1}^{\infty} F_i(X)\).

Corollary 3.7. Let \((X, T)\) be an R₀ space and let \(F(X) \subset J \subset 2^X\). If one of \((X, T)\), \((F_n(X), E(X) F_n(X))\), \(n \in \mathbb{N}\), or \((J, E(X)J)\) is connected, then all of them are connected.

Proof. By the argument of Theorem 3.7, \(f: (2^X, E(X)) \to (2^{X_0}, E(X_0))\) defined by \(f(F) = \{ \text{Cl}\{x\} | x \in F \}\) is a homeomorphism. If \(n \in \mathbb{N}\), then \(f(F_n(X)) = F_n(X_0)\). Thus \(f(F(X)) = F(X_0)\) and \(F(X_0) = f(J) \subset 2^{X_0}\).

Suppose \((X, T)\) is connected. By Theorem 3.1, \((X_0, S_0)\) is T₁ and by Theorem 3.3 (e), \((X_0, S_0)\) is connected. Then by Theorem 4.10 of Michael's paper [3], \((F_n(X_0), E(X_0) F_n(X_0))\), \(n \in \mathbb{N}\), and \((J, E(X)J)\) are connected, which implies \((F_n(X), E(X) F_n(X))\), \(n \in \mathbb{N}\), and \((J, E(X)J)\) are connected.

Suppose \((F_p(X), E(X) F_p(X))\) is connected for some \(p \in \mathbb{N}\). Then \((F_p(X_0), E(X_0) F_p(X_0))\) is connected and \((X_0, S_0)\) is T₁.
Thus by Theorem 4.10 of Michael’s paper [3], \((X_0, S_0)\) is connected. By Theorem 3.3 (e), \((X, T)\) is connected. Then by the argument above, all of \((\mathcal{F}_n(X), E(X) \mathcal{F}_n(X))\), \(n \in \mathbb{N}\), and \((\mathcal{G}, E(X) \mathcal{G})\) are connected.

If \((\mathcal{G}, E(X) \mathcal{G})\) is connected, then by an argument similar to that above, all of \((X, T)\), \((\mathcal{F}_n(X), E(X) \mathcal{F}_n(X))\), \(n \in \mathbb{N}\), and \((\mathcal{G}, E(X) \mathcal{G})\) are connected.

**Corollary 3.8.** Let \((X, T)\) be an \(R_0\) space and let \(\mathcal{F}(X) \subseteq \mathcal{G} \subseteq \mathcal{K}(X)\). Then \((X, T)\) is locally connected (l.c.) if and only if \((\mathcal{G}, E(X) \mathcal{G})\) is l.c.

**Proof.** This corollary follows by using Theorem 4.12 of Michael’s paper [3], Theorem 3.3. (g), and an argument similar to that for Corollary 3.7.

**Corollary 3.9.** Let \((X, T)\) be an \(R_0\) space. Then the following are equivalent: (a) \((X, T)\) is regular, (b) \((2^X, E(X))\) is \(T_2\), and (c) \((\mathcal{K}(X), E(X) \mathcal{K}(X))\) is \(T_3\).

**Proof.** Since \((X, T)\) is \(R_0\), then by Theorem 3.1, \((X_0, S_0)\) is \(T_1\), \(X_0 = \{Cl\{x\} | x \in X\} \subseteq \mathcal{K}(X)\), and \(S_0 = E(X)X_0\), and by Theorem 3.7, \((2^{X_0}, E(X_0))\) is homeomorphic to \((2^X, E(X))\).

(a) implies (b): by Theorem 3.4, \((X_0, S_0)\) is \(T_3\). Then by Theorem 4.9.3 of Michael’s paper [3], \((2^{X_0}, E(X_0))\) is \(T_2\), which implies \((2^X, E(X))\) is \(T_2\).

(b) implies (c): since \((2^X, E(X))\) is \(T_2\), then \((2^{X_0}, E(X_0))\) is \(T_2\). Then \((X_0, S_0)\) is \(T_1\) and \((2^{X_0}, E(X_0))\) is \(T_2\).
Thus by Theorem 4.9.3 of Michael's paper [3], \((X_0, S_0)\) is T\(_3\). Then by Theorem 4.9.10 of Michael's paper \((K(X_0), E(X_0), K(X_0))\) is T\(_3\), which implies \((K(X), E(X), K(X))\) is T\(_3\).

(c) implies (a): since \((X_0, S_0)\) is a subspace of \((K(X), E(X), K(X))\), then \((X_0, S_0)\) is a T\(_3\) and by Theorem 3.4 \((X, T)\) is regular.

Definition 3.5. A topological space \((X, T)\) is a Stone space if and only if for \(x, y \in X\) such that \(x \neq y\), there exists a continuous real-valued function for \(X\) such that \(f(x) = 0\) and \(f(y) = 1\) [3].

Using Theorem 4.9.4 and Theorem 3.9.11 of Michael's paper [3], and an argument similar to that for Corollary 3.9 yields Corollary 3.10, using Theorem 4.9.5 of Michael's paper [3], Theorem 3.6, and an argument similar to that for Corollary 3.9 yields Corollary 3.11, and using Theorem 4.9.8 of Michael's paper [3] and an argument similar to that for Corollary 3.9 yields Corollary 3.12.

**Corollary 3.10.** Let \((X, T)\) be an \(R_0\) space. Then the following are equivalent: (a) \((X, T)\) is completely regular, (b) \((2^X, E(X))\) is a Stone space, and (c) \((K(X), E(X), K(X))\) is T\(_{3\frac{1}{2}}\).

**Corollary 3.11.** Let \((X, T)\) be an \(R_0\) space. Then the following are equivalent: (a) \((X, T)\) is normal, (b) \((2^X, E(X))\) is T\(_{3\frac{1}{2}}\), and (c) \((2^X, E(X))\) is T\(_3\).
Corollary 3.12. Let \((X, T)\) be an \(R_0\) space. Then \((X, T)\) is \(R_1\) if and only if \((K(X), E(X)_{K(X)})\) is \(T_2\).

Using Corollary 3.9 and Theorem 2.7 yields the next corollary.

Corollary 3.13. Let \((X, T)\) be a rim-compact \(R_0\) space. Then the following are equivalent: (a) \((X, T)\) is \(R_1\), (b) \((2^X, E(X))\) is \(T_2\), and (c) \((K(X), E(X)_{K(X)})\) is \(T_3\).

Corollary 3.14. If \((X, T)\) is an \(R_0\) space, then \((2^X, E(X))\) is homeomorphic to \(((2^X)_0, E(2^X)(2^X)_0)\), which implies \((2^{X_0}, E(X_0))\) is homeomorphic to \(((2^X)_0, E(2^X)(2^X)_0)\).

Proof. By Corollary 3.4, \((2^X, E(X))\) is \(T_1\). Let \(P:(2^X, E(X)) \to ((2^X)_0, Q)\) be the natural map, where \(Q\) is the decomposition topology on \((2^X)_0\). Since \((2^X, E(X))\) is \(T_1\), then \((2^X, E(X))\) is \(R_0\). Then by Theorem 3.1, \(P\) is closed, open, \(((2^X)_0, Q)\) is a \(T_1\) space,

\[(2^X)_0 = \{\text{Cl}\{F\} | F \in 2^X\} \subset 2^{2^X},\]

and \(Q = E(2^X)(2^X)_0\). Since \(P(F) = \text{Cl}\{F\} = \{F\}\) for all \(F \in 2^X\), then \(P\) is 1-1. Then \(P:(2^X, E(X)) \to ((2^X)_0, E(2^X)(2^X)_0)\) is a homeomorphism.
CHAPTER BIBLIOGRAPHY


CHAPTER IV

LOCAL CONNECTEDNESS AND CONNECTEDNESS IM KLEINEN
OF $2^X$ AND $K(X)$ AT ELEMENTS OF $C(X)$ AND $C(X) \cap K(X)$ FOR $R_0$ SPACES

By Corollary 3.8, if $(X,T)$ is an $R_0$ space and $\mathcal{F}(X) \subseteq \mathcal{G} \subseteq K(X)$, then $(X,T)$ is locally connected (l.c.) if and only if $(\mathcal{G}, E(X)_{\mathcal{G}})$ is locally connected. In this chapter point-wise local connectedness and connectedness im kleinen (c.i.k.) of $2^X$ and $K(X)$ at elements of $C(X)$ and $K(X) \cap C(X)$ for $(X,T) \text{ an } R_0 \text{ space}$ are investigated.

In 1974 Jack T. Goodykoontz, Jr. [1] obtained the following characterizations of point-wise local connectedness and connectedness im kleinen of $2^X$ at elements of $C(X)$ for $(X,T) \text{ a metric continuum}$: if $(X,T)$ is a metric continuum and $M \in C(X)$, then $2^X$ is l.c. (c.i.k.) at $M$ if and only if for $U$ open in $X$ such that $M \subseteq U$, there exists an open connected set $V$ such that $M \subseteq V \subseteq U$ (for $U$ open in $X$ such that $M \subseteq U$, the component of $U$ containing $M$ contains $M$ in its interior). These characterizations are generalized by replacing metric continuum by the $R_0$ axiom and similar characterizations are obtained for point-wise local connectedness and connectedness im kleinen of $2^X$ and $K(X)$ at elements of $K(X) \cap C(X)$.
In 1942, J. L. Kelley [2] proved that if \((X, T)\) is a metric continuum, \(\mathcal{A} \subseteq 2^X\), \(\mathcal{A} \cap C(X) \neq \emptyset\), and \(\mathcal{A}\) is connected in \((2^X, \mathcal{E}(X))\), then \(\bigcup A\) is connected in \((X, T)\). The next result shows that this statement is true for any topological space \((X, T)\).

**Theorem 4.1.** Let \((X, T)\) be a topological space. If \(\mathcal{A}\) is connected in \(2^X\) and \(\mathcal{A} \cap C(X) \neq \emptyset\), then \(\bigcup A\) is connected in \(X\).

**Proof.** Assume \(\bigcup A\) is not connected in \(X\). Then \(\bigcup A = M \cup N\), where \(M\) and \(N\) are nonempty separated sets.

Let \(K \in \mathcal{A} \cap C(X)\). Then \(K \subseteq \bigcup A = M \cup N\) and thus \(K \subset M\) or \(K \subset N\), say \(K \subset M\). Since \(\bigcup A = M \cup N\), then there exists \(A \in \mathcal{A}\) such that \(A \cap N \neq \emptyset\). Let \(B = \{A \in \mathcal{A} | A \cap M \neq \emptyset\}\). Let \(A \in B\). Now \(A \subset M \cup N \neq \emptyset\). Thus \(A \subset N\) or \(A \cap M \neq \emptyset\). If \(A \subset N\), then \(A \in S(N)\) and if \(A \cap M \neq \emptyset\), then \(A \in \langle M, N\rangle\). Hence, if \(A \in B\), then \(A \in S(N) \cup \langle M, N\rangle\). Thus

\[ A = [A \cap S(M)] \cup \{[S(N) \cup \langle M, N\rangle] \cap A\}, \]

where \(A \cap S(M)\) and \([S(N) \cup \langle M, N\rangle] \cap A\) are nonempty separated sets, which contradicts \(\mathcal{A}\) is a connected subset of \(2^X\).

Hence, \(\bigcup A\) is a connected subset of \(X\).

**Theorem 4.2.** If \((X, T)\) is an \(R_0\) space and \(U_i \neq \emptyset\) and open for all \(i \in \{1, \ldots, n\}\), then
\[ \bigcup_{i=1}^{n} U_i = \bigcup A \quad A \in \langle U_1, \ldots, U_n \rangle \quad A \in \langle U_1, \ldots, U_n \rangle \cap K(X). \]

Hence, if \((X,T)\) is an \(R_0\) space and \(\emptyset\) is open in \(2^X\) or \(K(X)\), then \(\bigcup A\) is open in \(X\).

**Proof.** Let \(y \in \bigcup U_i\). For each \(i \in \{1, \ldots, n\}\), let \(x_i \in U_i\). Then \(y \in \bigcup U_i\), which is open in \(X\), and thus \(\text{Cl}[y] \subseteq \bigcup U_i\), and \(x_i \in U_i\) for all \(i \in \{1, \ldots, n\}\), and thus \(\text{Cl}[x_i] \subseteq U_i\) for all \(i \in \{1, \ldots, n\}\), which implies

\[ \text{Cl}[y] \cup \bigcup_{i=1}^{n} \text{Cl}[x_i] \in \langle U_1, \ldots, U_n \rangle \cap K(X) \]

and

\[ y \in \bigcup A \quad A \in \langle U_1, \ldots, U_n \rangle \cap K(X) \quad A \in \langle U_1, \ldots, U_n \rangle. \]

Therefore, if \(y \in \bigcup U_i\) then

\[ y \in \bigcup A \quad A \in \langle U_1, \ldots, U_n \rangle \cap K(X) \quad A \in \langle U_1, \ldots, U_n \rangle, \]

which implies

\[ \bigcup_{i=1}^{n} U_i \subseteq \bigcup A \quad A \in \langle U_1, \ldots, U_n \rangle \cap K(X) \quad A \in \langle U_1, \ldots, U_n \rangle, \]

and since

\[ \bigcup A \subseteq \bigcup_{i=1}^{n} U_i \]

then

\[ \bigcup_{i=1}^{n} U_i = \bigcup A \quad A \in \langle U_1, \ldots, U_n \rangle \cap K(X) \quad A \in \langle U_1, \ldots, U_n \rangle. \]
Definition 4.1. If \((X, T)\) is a topological space and \(A \subset X\), then \(A^0 = \text{Int}(A)\).

Corollary 4.1. If \((X, T)\) is an \(R_0\) space, \(\mathcal{A}\) is a connected subset of \(2^X\) such that \(\mathcal{A} \cap C(X) \neq \emptyset\), and \(B \in \mathcal{A}^0\), then \(B \subset (\bigcup A)^0 \subset \bigcup A\), which is connected in \(X\).

Proof. By Theorem 4.1, \(\bigcup A\) is connected in \(X\). By Theorem 4.2, \(\bigcup A^0\) is open in \(X\) and since \(B \subset \bigcup A\), then

\[
B \subset (\bigcup A)^0 \subset \bigcup A.
\]

Corollary 4.2. If \((X, T)\) is an \(R_0\) space, \(\mathcal{A}\) is a connected subset of \(K(X)\) such that \(\mathcal{A} \cap C(X) \neq \emptyset\), and \(B \in \mathcal{A}^0\), then

\[
B \subset (\bigcup A)^0 \subset \bigcup A,
\]

which is connected in \(X\).

Proof. Since \(\mathcal{A}\) is connected in \(K(X)\), then \(\mathcal{A}\) is connected in \(2^X\). Then \(\mathcal{A}\) is connected in \(2^X\) and \(\mathcal{A} \cap C(X) \neq \emptyset\). Thus by Theorem 4.1, \(\bigcup A\) is connected in \(X\). By Theorem 4.2, \(\bigcup A^0\) is open in \(X\) and since \(B \subset \bigcup A\), then

\[
B \subset (\bigcup A)^0.
\]

The next corollary follows immediately from Corollary 4.1 and Corollary 4.2.

Corollary 4.3. If \((X, T)\) is an \(R_0\) space, \(\mathcal{A}\) is open connected in \(2^X\) or \(K(X)\) such that \(\mathcal{A} \cap C(X) \neq \emptyset\), and \(B \in \mathcal{A}\), then
Theorem 4.3. If $(X,T)$ is an $R_0$ space and $U_i \neq \emptyset$ and open for all $i \in \{1, \ldots, n\}$, then $\langle U_1, \ldots, U_n \rangle \cap \mathcal{F}(X)$ is dense in $\langle U_1, \ldots, U_n \rangle \cap K(X)$.

Proof. Let $\mathcal{O}$ be a nonempty open set in $\langle U_1, \ldots, U_n \rangle$. Then there exists $\langle V_1, \ldots, V_m \rangle \in \mathcal{B}$ such that

$$\langle V_1, \ldots, V_m \rangle \subset \mathcal{O} \subset \langle U_1, \ldots, U_n \rangle.$$ 

For each $i \in \{1, \ldots, m\}$, let $x_i \in V_i$. For each $i \in \{1, \ldots, m\}$, let $x_i \in V_i$, which is open in $X$. Thus $\text{Cl}[x_i] \subset V_i$ for all $i \in \{1, \ldots, m\}$ and

$$\bigcup_{i=1}^{m} \text{Cl}[x_i] \in \langle V_1, \ldots, V_m \rangle \cap \mathcal{F}(X) \subset \mathcal{O} \cap \mathcal{F}(X).$$

Therefore, if $\mathcal{O}$ is a nonempty open set in $\langle U_1, \ldots, U_n \rangle$, then $\mathcal{O} \cap \mathcal{F}(X) \neq \emptyset$, which implies $\langle U_1, \ldots, U_n \rangle \cap \mathcal{F}(X)$ is dense in $\langle U_1, \ldots, U_n \rangle$.

By a similar argument, $\langle U_1, \ldots, U_n \rangle \cap \mathcal{F}(X)$ is dense in $\langle U_1, \ldots, U_n \rangle \cap K(X)$.

Definition 4.2. If $(X,T)$ is a topological space and $C_i \subset X$ for all $i \in \{1, \ldots, n\}$, then let

$$\mathcal{F}(C_1, \ldots, C_n) = \{ \bigcup_{i=1}^{P} \text{Cl}[x_i] \in \mathcal{F}(X) \mid \bigcup_{i=1}^{P} \{x_i\} \subset \bigcup_{i=1}^{n} C_i \}$$

and

$$(\bigcup_{i=1}^{P} \{x_i\}) \cap C_j \neq \emptyset \text{ for all } j \in \{1, \ldots, n\}.$$ 

Theorem 4.4. Let $(X,T)$ be an $R_0$ space, for each $i \in \{1, \ldots, n\}$, let $C_i \subset X$ such that $C_i \subset \bigcup_{j=1}^{P} U_{ij}$, where $U_{ij}$ is
is open and $C_i \cap U_{ij} \neq \emptyset$ for all $j \in \{1, \ldots, p_i\}$, let

$$\bigcup_{i=1}^{r} \text{Cl}(x_i) \in \mathcal{F}(C_1, \ldots, C_n) \cap \langle U_{11}, \ldots, U_{np_n} \rangle,$$

and let $f: \prod C_i \rightarrow 2^X$ be defined by

$$f((c_1, \ldots, c_n)) = \left( \bigcup_{i=1}^{r} \text{Cl}(x_i) \right) \cup \left( \bigcup_{i=1}^{n} \text{Cl}(c_i) \right)$$

for all $(c_1, \ldots, c_n) \in \prod C_i$. Then $f$ is a continuous function from $\prod C_i$ into $\langle U_{11}, \ldots, U_{1p_1}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)$.

**Proof.** If $(c_1, \ldots, c_n) \in \prod C_i$, then

$$\bigcup_{i=1}^{n} \text{Cl}(c_i) \in \mathcal{F}(C_1, \ldots, C_n)$$

and for each $i \in \{1, \ldots, n\}$, $c_i \in C_i \subset \bigcup_{j=1}^{p_i} (\bigcup_{i=1}^{i} U_{ij})$, which is open in $X$, and thus $\text{Cl}(c_i) \subset \bigcup_{j=1}^{i} (\bigcup_{i=1}^{i} U_{ij})$ and since

$$\bigcup_{i=1}^{r} \text{Cl}(x_i) \in \mathcal{F}(C_1, \ldots, C_n) \cap \langle U_{11}, \ldots, U_{np_n} \rangle,$$

then

$$\left( \bigcup_{i=1}^{r} \text{Cl}(x_i) \right) \cup \left( \bigcup_{i=1}^{n} \text{Cl}(c_i) \right) \in \mathcal{F}(C_1, \ldots, C_n) \cap \langle U_{11}, \ldots, U_{np_n} \rangle.$$

Let $\{(z_{1\alpha}, \ldots, z_{n\alpha})\}_{\alpha \in A}$ be a net in $\prod C_i$ such that

$$\{(z_{1\alpha}, \ldots, z_{n\alpha})\}_{\alpha \in A} \rightarrow (z_1, \ldots, z_n) \in \prod C_i.$$

Then for each $i \in \{1, \ldots, n\}$, $[z_{i\alpha}]_{\alpha \in A} \rightarrow z_i$. Let

$$\langle 0_1, \ldots, 0_p \rangle \subset \langle U_{11}, \ldots, U_{np_n} \rangle.$$
such that \( f((z_1, \ldots, z_n)) \in \langle 0_1, \ldots, 0_p \rangle \). Let \( i \in \{1, \ldots, n\} \).

Then \( z_i \in O_j \) for some \( j \in \{1, \ldots, p\} \). Thus

\[
\text{Cl}\{z_i\} \subseteq \cap \{O_j \mid j \in \{1, \ldots, p\} \}
\]

and \( z_i \in O_j \) = M_i,

which is open in \( X \). Since \( \{z_i\}_{\alpha} \in A \ast z_i \), then there exists \( \alpha \in A \) such that if \( \alpha \geq \alpha_i \), then \( Z_{i\alpha} \in M_i \). Thus if \( \alpha \geq \alpha_i \), then

\[
\text{Cl}\{Z_{i\alpha}\} \subseteq M_i.
\]

For each \( i \in \{1, \ldots, n\} \), let \( \alpha_i \in A \) such that if \( \alpha \geq \alpha_i \), then \( \text{Cl}\{Z_{i\alpha}\} \subseteq M_i = \cap \{O_j \mid j \in \{1, \ldots, p\} \} \) and \( z_i \in O_j \).

Let \( \alpha_0 \in A \) such that \( \alpha_0 \geq \alpha_i \) for all \( i \in \{1, \ldots, n\} \). If \( \alpha \geq \alpha_0 \), then

\[
\left( \bigcup_{i=1}^{r} \text{Cl}\{x_i\} \right) \cup \left( \bigcup_{i=1}^{n} \text{Cl}\{z_{i\alpha}\} \right) \subseteq \bigcup_{j=1}^{p} O_j.
\]

Let \( \alpha \geq \alpha_0 \). If \( B = \{ j \in \{1, \ldots, p\} \mid \bigcup_{i=1}^{r} \text{Cl}\{x_i\} \cap O_j \neq \emptyset \} = \{1, \ldots, p\} \), then

\[
\left( \bigcup_{i=1}^{r} \text{Cl}\{x_i\} \right) \cup \left( \bigcup_{i=1}^{n} \text{Cl}\{z_{i\alpha}\} \right) \in \langle 0_1, \ldots, 0_p \rangle.
\]

Consider the case that \( B \subseteq \{1, \ldots, p\} \). Let \( l \in \{1, \ldots, p\} \setminus B \).

Then there exists \( t \in \{1, \ldots, n\} \) such that \( \text{Cl}\{z_t\} \cap O_l \neq \emptyset \).

Then \( z_t \in O_l \) and since \( \alpha \geq \alpha_0 \), then

\[
\text{Cl}\{z_t\} \subseteq M_t = \cap \{O_j \mid j \in \{1, \ldots, p\} \} \text{ and } z_t \in O_j \subseteq O_l.
\]

Hence, if \( l \in \{1, \ldots, p\} \setminus B \), then there exists \( t \in \{1, \ldots, n\} \) such that \( \text{Cl}\{z_t\} \cap O_l \neq \emptyset \), which implies

\[
f((z_1, \ldots, z_n)) = \left( \bigcup_{i=1}^{r} \text{Cl}\{x_i\} \right) \cup \left( \bigcup_{i=1}^{n} \text{Cl}\{z_{i\alpha}\} \right) \in \langle 0_1, \ldots, 0_p \rangle.
\]
Therefore, if \( a \geq a_0 \), then \( f((z_1, \ldots, z_n)) \in (0_1, \ldots, 0_p) \).

Therefore, if \( \Theta \) is open in \( \langle U_{11}, \ldots, U_{np_n} \rangle \) such that
\[
f((z_1, \ldots, z_n)) \in \Theta,
\]
then there exists \( a_0 \in A \) such that if \( a \geq a_0 \), then \( f((z_1, \ldots, z_n)) \in \Theta \), which implies
\[
\{ f((z_{\alpha_1}, \ldots, z_{n_{\alpha}})) \}_{\alpha \in A} \supset f((z_1, \ldots, z_n)).
\]

Therefore, if \( \{ (z_{\alpha_1}, \ldots, z_{n_{\alpha}}) \}_{\alpha \in A} \) is a net in \( \Pi_{i=1}^{n} C_i \) such that
\[
\{ (z_{\alpha_1}, \ldots, z_{n_{\alpha}}) \}_{\alpha \in A} \rightarrow (z_1, \ldots, z_n) \in \Pi_{i=1}^{n} C_i,
\]
then
\[
\{ f((z_{\alpha_1}, \ldots, z_{n_{\alpha}})) \}_{\alpha \in A} \rightarrow f((z_1, \ldots, z_n)),
\]
which implies \( f \) is continuous.

**Theorem 4.5.** Let \((X, T)\) be an \( R_0 \) space and for each \( i \in \{1, \ldots, n\} \), let \( C_i \) be a connected subset of \( X \) such that
\[
P_i \supset \bigcup_{j=1}^{P_i} U_{ij}, \text{ where } U_{ij} \text{ is open and } C_i \cap U_{ij} \neq \emptyset \text{ for all } j \in \{1, \ldots, P_i\}.
\]
Then \( \langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n) \) and
\[
\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np_n} \rangle \cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]
\]
are connected subsets of \( 2^X \) and \( \langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n) \) and \([\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np_n} \rangle \cap K(X)] \cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n)]\)
are connected subsets of \( K(X) \).

**Proof.** Let \( \bigcup_{i=1}^{r} Cl\{x_i\}, \bigcup_{i=1}^{S} Cl\{y_i\} \in \langle U_{11}, \ldots, U_{np_n} \rangle \cap \mathcal{F}(C_1, \ldots, C_n) \). For each \( i \in \{1, \ldots, n\} \), let \( n_i \) be the number of elements \( y_j \) such that \( y_j \in C_i \) and let \( p = \max\{n_i | i = 1, \ldots, n\} \).
For each \((i,j) \in \bigcup_{i=1}^{n} \{(i) \times (i-p+1,\ldots,i-p)\} = D\), let 

\[ C(i,j) = C_i. \]

Then \(F(C(1,1),\ldots,C(1,p),\ldots,C(n,n-p)) = F(C_1,\ldots,C_n)\). Let \(f: \prod_{(i,j) \in D(i,j)} C(i,j) \rightarrow 2^X\) be defined by 

\[ f((c(i,j))_{(i,j) \in D}) = \bigcup_{i=1}^{r} Cl[x_i] \cup \bigcup_{(i,j) \in D(i,j)} (c(i,j)). \]

By Theorem 4.4., \(f\) is a continuous function from \(\prod_{(i,j) \in D(i,j)} C(i,j)\) into \(\langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\) and since \(\prod_{(i,j) \in D(i,j)} C(i,j)\) is connected, then \(f((C_1,\ldots,C_n))\) is connected in \(\langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\) and contains \(\bigcup_{i=1}^{r} Cl[x_i]\) and 

\[ \bigcup_{i=1}^{r} Cl[y_i]. \]

By a similar argument, there exists a connected subset of \(\langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\) containing 

\[ \bigcup_{i=1}^{r} Cl[x_i] \cup \bigcup_{i=1}^{r} Cl[y_i], \]

which implies there exists a connected subset of \(\langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\) containing 

\[ \bigcup_{i=1}^{r} Cl[x_i] \cup \bigcup_{i=1}^{r} Cl[y_i]. \]

Therefore, if \(\bigcup_{i=1}^{r} Cl[x_i], \bigcup_{i=1}^{r} Cl[y_i] \in \langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\), then there exists a connected subset of 

\[ \langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n) \]

containing \(\bigcup_{i=1}^{r} Cl[y_i]\), which implies \(\langle U_{11},\ldots,U_{np_n}\rangle \cap F(C_1,\ldots,C_n)\) is connected.
a connected subset of $2^X$ and since

$$\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n) \subseteq K(X),$$

then

$$\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n)$$

is a connected subset of $K(X)$.

Since $\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle \cap F(X)$ is dense in

$$\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle,$$

then

$$\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle$$

$$\subseteq C1(\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle \cap F(X))$$

$$\subseteq C1(\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n))$$

and since $\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n)$ is connected in $2^X$

and $\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n) \subseteq \langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n)$

$$\cup \langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle \subseteq C1(\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n)),$$

then

$$\langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n) \cup \langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle$$

is a connected subset of $2^X$. By a similar argument,

$$\langle C_1 \cap U_{11}, \ldots, C_n \cap U_{np} \rangle \cap K(X)$$

$$\cup \langle U_{11}, \ldots, U_{np} \rangle \cap F(C_1, \ldots, C_n)$$

is a connected subset of $K(X)$.

**Theorem 4.6.** Let $(X,T)$ be an $R_0$ space and for each

$i \in \{1, \ldots, n\}$, let $C_i$ be connected such that $C_i \subseteq \bigcup_{j=1}^{p_i} U_{ij}$, where
U_{ij} is open and $C_i^0 \cap U_{ij} \neq \varnothing$ for all $j \in \{1, \ldots, p_i\}$. Then

$$\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle \cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)]$$

is a connected subset of $\langle U_{11}, \ldots, U_{np_n} \rangle$ and

$$[\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle \cap K(X)]$$

$$\cup [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)]$$

is a connected subset of $\langle U_{11}, \ldots, U_{np_n} \rangle \cap K(X)$.

**Proof.** By Theorem 4.3, $\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle \cap \overline{F}(X)$

is dense in $\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle$. Thus

$$\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle$$

$$\subset \text{Cl}(\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle \cap \overline{F}(X))$$

$$\subset \text{Cl}(\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)),$$

which implies

$$\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)$$

$$\subset [\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)]$$

$$\cup \langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle$$

$$\subset \text{Cl}(\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)),$$

and since $\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)$ is connected, then

$[\langle U_{11}, \ldots, U_{np_n} \rangle \cap \overline{F}(C_1, \ldots, C_n)] \cup \langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{np_n} \rangle$ is

a connected subset of $\langle U_{11}, \ldots, U_{np_n} \rangle$. By a similar argument,
\((\bigcup_{11}^{nPn} \cap \mathcal{F}(c_1, ..., c_n)) \cup (\bigcup_{1}^{0} \cap \bigcup_{11}^{nPn} \cap K(X))\)

is a connected subset of \((\bigcup_{11}^{nPn} \cap K(X))\).

**Corollary 4.4.** Let \((X, T)\) be an \(R_0\) space and for each \(i \in \{1, ..., n\}\), let \(C_i \in C(X)\) such that \(C_i \subset \bigcup_{j=1}^{p_i} U_{ij}\), where \(U_{ij}\) is open and \(U_{ij} \cap C_i \neq \emptyset\) for all \(j \in \{1, ..., p_i\}\) and \(C_i \subset K_{i}^0\), where \(K_i\) is a connected subset of \(\bigcup_{j=1}^{p_i} U_{ij}\) containing \(C_i\).

Then \((\bigcup_{1}^{0} \cap \bigcup_{11}^{nPn} \cap K_{1}^0) \cup (\bigcup_{11}^{nPn} \cap \mathcal{F}(K_1, ..., K_n))\)

is connected in \((\bigcup_{11}^{nPn} \cap K(X))\), which implies \(\bigcup_{i=1}^{n} C_i \subset \mathcal{F}^0\), where \(\mathcal{F}^0\) is the component of \((\bigcup_{11}^{nPn} \cap K(X))\) containing \(\bigcup_{i=1}^{n} C_i\).

**Proof.** By Theorem 4.6, \((K_1^0 \cap \bigcup_{11}^{nPn} \bigcup_{1}^{0} \cap \bigcup_{11}^{nPn} \cap K(X))\)

\(\cup (\bigcup_{11}^{nPn} \cap \mathcal{F}(K_1, ..., K_n))\)

is a connected subset of \((\bigcup_{11}^{nPn} \cap K(X))\) and since \(\bigcup_{i=1}^{n} C_i \in (K_1^0 \cap \bigcup_{11}^{nPn} \cap K(X))\),

which is open in \((\bigcup_{11}^{nPn} \cap K(X))\), then \(\bigcup_{i=1}^{n} C_i \subset \mathcal{F}^0\), where \(\mathcal{F}^0\) is the component of \((\bigcup_{11}^{nPn} \cap K(X))\) containing \(\bigcup_{i=1}^{n} C_i\).

The next corollary follows by an argument similar to that for Corollary 4.4.

**Corollary 4.5.** Let \((X, T)\) be an \(R_0\) space and for each \(i \in \{1, ..., n\}\), let \(C_i \in C(X) \cap K(X)\) such that \(C_i \subset \bigcup_{j=1}^{p_i} U_{ij}\), where
$U_{ij}$ is open and $U_{ij} \cap C_i \neq \emptyset$ for all $j \in \{1, \ldots, p_i\}$ and $p_i$
\[ C_i \subset K_i^0 \], where $K$ is a connected subset of $\bigcup_{j=1}^{p_i} U_{ij}$ containing $C_i$. Then
\[
\{K_0 \cap U_{11}, \ldots, K_0 \cap U_{n_0} \} \cap K(X)
\]
\[
\cup \{U_{11}, \ldots, U_{n_0} \cap \mathcal{F}(K_1, \ldots, K_n)\}
\]
is connected in $\langle U_{11}, \ldots, U_{n_0} \rangle \cap K(X)$, which implies
\[
\bigcup_{i=1}^{n} C_i \in \mathcal{K}^0_K(X) \), where $\mathcal{K}$ is the component of $\langle U_{11}, \ldots, U_{n_0} \rangle \cap K(X)$ containing $\bigcup_{i=1}^{n} C_i$.

**Corollary 4.6.** Let $(X, T)$ be an $R^0$ space and for each $i \in \{1, \ldots, n\}$, let $C_i$ be connected in $X$ such that
\[ C_i = \bigcup_{j=1}^{p_i} U_{ij} \], where $U_{ij} \neq \emptyset$ and open for all $j \in \{1, \ldots, p_i\}$.
Then $\langle U_{11}, \ldots, U_{n_0} \rangle$ is open connected in $2^X$ and
\[ \langle U_{11}, \ldots, U_{n_0} \cap K(X) \rangle \] is open connected in $K(X)$.

**Proof.** Since $C_i = \bigcup_{j=1}^{p_i} U_{ij}$ for all $i \in \{1, \ldots, n\}$, where $U_{ij}$ is open for all $j \in \{1, \ldots, p_i\}$, then $C_i = C_i^0$ and by Corollary 4.4.,
\[
\langle C_1^0 \cap U_{11}, \ldots, C_n^0 \cap U_{n_0} \rangle
\]
\[
\cup \{U_{11}, \ldots, U_{n_0} \cap \mathcal{F}(C_1, \ldots, C_n)\}
\]
\[ = \langle U_{11}, \ldots, U_{n_0} \rangle \]
is connected. By a similar argument, \( \langle U_{11}, \ldots, U_{n_p n} \rangle \cap K(X) \) is open connected in \( K(X) \).

**Theorem 4.7.** Let \((X,T)\) be an \( R_0 \) space and let \( B \in C(X) \). Then \( 2^X \) is c.i.k. at \( B \) if and only if for each open set \( U \) in \( X \) such that \( B \in U \), the component of \( U \) containing \( B \) contains \( B \) in its interior.

**Proof.** Suppose \( 2^X \) is c.i.k. at \( B \). Let \( U \) be open in \( X \) such that \( B \in U \). Then \( S(U) \) is open in \( 2^X \) and \( B \in S(U) \). Thus the component \( A \) of \( S(U) \) containing \( B \) contains \( B \) in its interior. By Corollary 4.1, \( B \subseteq (\bigcup A)^{\emptyset} \subseteq \bigcup A \), which is connected, and \( \bigcup A \subseteq U \), which implies that the component of \( U \) containing \( B \) contains \( B \) in its interior.

Therefore, if \( U \) is open in \( X \) such that \( B \in U \), then the component of \( U \) containing \( B \) contains \( B \) in its interior.

Conversely, suppose that if \( U \) is open in \( X \) such that \( B \in U \), then the component of \( U \) containing \( B \) contains \( B \) in its interior. Let \( \emptyset \) be open in \( 2^X \) such that \( B \in \emptyset \). Then there exists \( \langle U_1, \ldots, U_n \rangle \in \emptyset \) such that \( B \in \langle U_1, \ldots, U_n \rangle \subseteq \emptyset \).

By Theorem 4.2, \( \bigcup_{i=1}^{n} U_i = \bigcup A \). Since \( B \subseteq \bigcup_{i=1}^{n} U_i \), which is open in \( X \), then the component \( K \) of \( \bigcup_{i=1}^{n} U_i \) containing \( B \) contains \( B \) in its interior. By Corollary 4.4, \( B \in K^{\emptyset} \), where \( K \) is the component of \( \langle U_1, \ldots, U_n \rangle \) containing \( B \), which implies \( B \subseteq L^{\emptyset} \), where \( L \) is the component of \( \emptyset \) containing \( B \).
Therefore, if \( \mathcal{O} \) is open in \( 2^X \) such that \( B \in \mathcal{O} \), then the component of \( \mathcal{O} \) containing \( B \) contains \( B \) in its interior, which implies \( 2^X \) is c.i.k. at \( B \).

**Theorem 4.8.** Let \( (X,T) \) be an \( R_0 \) space and let \( B \in K(X) \cap C(X) \). Then the following are equivalent: (a) \( 2^X \) is c.i.k. at \( B \), (b) for each open set \( U \) in \( X \) such that \( B \subseteq U \), the component of \( U \) containing \( B \) contains \( B \) in its interior, and (c) \( K(X) \) is c.i.k. at \( B \).

**Proof.** By Theorem 4.7, (a) and (b) are equivalent. (b) implies (c): Let \( \mathcal{O} \) be open in \( K(X) \) such that \( B \in \mathcal{O} \). Then there exists \( \langle U_1, \ldots, U_n \rangle \in \mathcal{B} \) such that \( B \in \langle U_1, \ldots, U_n \rangle \cap K(X) \subseteq \mathcal{O} \).

By Theorem 4.2., \( \bigcup_{i=1}^n U_i = \bigcup_{A \in \langle U_1, \ldots, U_n \rangle \cap K(X)} A \). Then the component \( K \) of \( \bigcup_{i=1}^n U_i \) containing \( B \) contains \( B \) in its interior. Thus by Corollary 4.5, \( B \in \mathcal{Z}^0 \subseteq K(X) \), where \( \mathcal{Z} \) is the component of \( \langle U_1, \ldots, U_n \rangle \cap K(X) \) containing \( B \), which implies \( B \in \mathcal{Z}^0 \), where \( \mathcal{Z} \) is the component of \( \mathcal{O} \) containing \( B \).

Therefore, if \( \mathcal{O} \) is open in \( K(X) \) such that \( B \in \mathcal{O} \), then the component of \( \mathcal{O} \) containing \( B \) contains \( B \) in its interior, which implies \( K(X) \) is c.i.k. at \( B \).

(c) implies (b): Let \( U \) be open in \( X \) such that \( B \subseteq U \). Then \( S(U) \cap K(X) \) is open in \( K(X) \) and \( B \in S(U) \cap K(X) \). Thus the component \( \mathcal{A} \) of \( S(U) \cap K(X) \) containing \( B \) contains \( B \) in its interior. Then by Corollary 4.2, \( B \subseteq \bigcup_{A \in \mathcal{A}}^0 \subseteq \bigcup_{A \in \mathcal{A}} A \).
which is connected, and \( \bigcup A \subset U \), which implies that the component of \( U \) containing \( B \) contains \( B \) in its interior.

Therefore, if \( U \) is open in \( X \) such that \( B \subset U \), then the component of \( U \) containing \( B \) contains \( B \) in its interior.

**Theorem 4.9.** Let \((X,T)\) be an \( R_0 \) space and let \( B \in C(X) \). Then \( 2^X \) is l.c. at \( B \) if and only if for each open set \( U \) in \( X \) such that \( B \subset U \), there exists an open connected set \( C \) such that \( B \subset C \subset U \).

**Proof.** Suppose \( 2^X \) is l.c. at \( B \). Let \( U \) be open in \( X \) such that \( B \subset U \). Then \( S(U) \) is open in \( 2^X \) and \( B \in S(U) \).

Thus there exists an open connected subset \( A \) of \( S(U) \) containing \( B \). Then \( A \) is an open connected subset of \( 2^X \) such that \( A \cap C(X) \neq \emptyset \) and \( B \in A \). Thus by Corollary 4.3, \( B \subset \bigcup A \), \( A \in \mathcal{A} \) which is open connected in \( X \). Hence, if \( U \) is open in \( X \) such that \( B \subset U \), then there exists an open connected set \( C \) such that \( B \subset C \subset U \).

Conversely, suppose that for each open set \( U \) in \( X \) such that \( B \subset U \), there exists an open connected set \( C \) such that \( B \subset C \subset U \). Let \( U \) be open in \( 2^X \) such that \( B \in U \). Then there exists \( \langle U_1, \ldots, U_n \rangle \in \mathcal{B} \) such that \( B \in \langle U_1, \ldots, U_n \rangle \subset \mathcal{B} \).

By Theorem 4.2., \( \bigcup_{i=1}^n U_i = \bigcup A \). Since \( B \subset \bigcup_{i=1}^n U_i \), which is open in \( X \), then there exists an open connected set \( C \) such that \( B \subset C \subset \bigcup_{i=1}^n U_i \). Since \( B \cap U_i \neq \emptyset \) for all
If \( i \in \{1, \ldots, n\} \) and \( B \subseteq C \subseteq \bigcup_{i=1}^{n} U_i \), then \( C \cap U_i \neq \emptyset \) and open for all \( i \in \{1, \ldots, n\} \) and \( C = \bigcup_{i=1}^{n} (C \cap U_i) \). Thus by Corollary 4.6, \( \langle C \cap U_1, \ldots, C \cap U_n \rangle \) is open connected in \( 2^X \). Then \( B \in \langle C \cap U_1, \ldots, C \cap U_n \rangle \), which is open connected in \( \langle U_1, \ldots, U_n \rangle \subseteq \emptyset \). Hence, if \( \emptyset \) is open in \( 2^X \) such that \( B \in \emptyset \), then there exists an open connected set \( A \) such that \( B \in A \subseteq \emptyset \), which implies \( 2^X \) is l.c. at \( B \).

Using Theorem 4.9, Theorem 4.2, Corollary 4.6, Corollary 4.3, and an argument similar to that for Theorem 4.8 yields the next theorem.

**Theorem 4.10.** Let \((X,T)\) be an \( R_0 \) space and let \( B \in K(X) \cap C(X) \). Then the following are equivalent: (a) \( 2^X \) is l.c. at \( B \), (b) for each open set \( U \) in \( X \) such that \( B \subseteq U \), there exists an open connected set \( C \) such that \( B \subseteq C \subseteq U \), and (c) \( K(X) \) is l.c. at \( B \).

**Corollary 4.7.** Let \((X,T)\) be an \( R_0 \) space and let \( C \) be a component of \( X \). Then the following are equivalent: (a) \( 2^X \) is c.i.k. at \( C \), (b) \( C \) is a closed-open subset of \( X \), and (c) \( 2^X \) is l.c. at \( C \).

**Proof.** (a) implies (b): since \((X,T)\) is \( R_0 \), \( C \in C(X) \) such that \( 2^X \) is c.i.k. at \( C \), and \( X \) is open in \( X \) such that \( C \subseteq X \), then by Theorem 4.7, the component of \( X \) containing \( C \) contains \( C \) in its interior. Thus \( C \subseteq C^0 \subseteq C \), which implies \( C = C^0 \) and \( C \) is closed-open in \( X \).
(b) implies (c): since \((X,T)\) is \(R_0\) and \(C \in C(X)\) such that if \(U\) is open in \(X\) such that \(C \subseteq U\), then \(C\) is open connected in \(X\) such that \(C \subseteq C \subseteq U\), then by Theorem 4.9, \(2^X\) is l.c. at \(C\).

(c) implies (a): since \(2^X\) is l.c. at \(C\), then \(2^X\) is c.i.k. at \(C\).
CHAPTER BIBLIOGRAPHY


CHAPTER V

POINT-WISE LOCAL CONNECTEDNESS AND CONNECTEDNESS IM KLEINEN OF \((2^X,\mathcal{E}(X))\) AND \((K(X),\mathcal{E}(X))_{K(X)}\)

One of the earliest results about \((2^X,\mathcal{E}(X))\) for \((X,T)\) a metric continuum, due to Wajdyslowski \[2\], is that \((2^X,\mathcal{E}(X))\) is l.c. if and only if \((X,T)\) is l.c. In this chapter one of the results is that if \((X,T)\) is an \(R_0\) space and \((2^X,\mathcal{E}(X))\) is l.c., then \((X,T)\) is l.c. The converse is not true even if \((X,T)\) is a l.c., locally compact, connected, metric space.

In 1974, Jack T. Goodykoontz, Jr. \[1\] characterized point-wise local connectedness and connectedness im kleinen of \((2^X,\mathcal{E}(X))\) for \((X,T)\) a metric continuum as follows: if \(A \in 2^X\), then \((2^X,\mathcal{E}(X))\) is l.c. (c.i.k.) at \(A\) if and only if \((2^X,\mathcal{E}(X))\) is l.c. (c.i.k.) at each component of \(A\). In this chapter the relationships between statements (I) \((2^X,\mathcal{E}(X))\) is l.c. (c.i.k.) at \(A \in 2^X\) and (II) \((2^X,\mathcal{E}(X))\) is l.c. (c.i.k.) at each component of \(A \in 2^X\) are investigated for \((X,T)\) weaker than a metric continuum.

The next example shows that (I) does not imply (II) for \((X,T)\) a compact connected \(T_1\) space.
Example 5.1. Let $Y = \{x \in [0,1] \mid x \text{ is rational}\}$, let $S$ be the usual topology on $Y$, let $X = Y \cup \{2\}$, let $W = \{O \subset X \mid 2 \in O \text{ and } X \setminus O \text{ is finite}\}$, and let $T = S \cup W$. Then $(X,T)$ is a compact, $T_1$, connected space. Let $A = \{0,2\} \subset 2^X$. Let $\mathcal{O}$ be open in $2^X$ such that $A \in \mathcal{O}$. Then there exists $\langle U_1, \ldots, U_n \rangle \in \mathcal{B}$ such that $A \in \langle U_1, \ldots, U_n \rangle \subset \mathcal{O}$. By Theorem 4.2, $\bigcup_{i=1}^{n} U_i = \bigcup B \in \langle U_1, \ldots, U_n \rangle$. Since $2 \in \bigcup U_i$, then $\bigcup_{i=1}^{n} U_i$ is connected. By Corollary 4.6, $\langle U_1, \ldots, U_n \rangle$ is open connected in $2^X$. Hence, if $\mathcal{O}$ is open in $2^X$ such that $A \in \mathcal{O}$, then there exists an open connected set $\mathcal{O}'$ such that $A \in \mathcal{O}' \subset \mathcal{O}$, which implies $2^X$ is l.c., and hence c.i.k., at $A$.

The components of $A$ are $\{0\}$ and $\{2\}$. Let $O = \{y \in Y \mid |y| < \frac{1}{2}\}$. Then $O$ is open in $X$ such that $0 \in O$ and the component of $O$ containing $0$ is $\{0\}$, which has empty interior. Hence, $(X,T)$ is an $R_0$ space and $\{0\} \in C(X)$ such that there exists an open set $O$ in $X$ containing $0$ such that there does not exist a connected subset of $O$ containing $0$ in its interior. Then by Theorem 4.7 and Theorem 4.9, $2^X$ is not c.i.k. or 1.c. at $\{0\}$.

The next example shows that (II) does not imply (I) for $(X,T)$ at $T_2$ space.

Example 5.2. Let $X$ be an infinite set and let $T$ be the set of all subsets of $X$. Then $X \in 2^X$ and components of
X are singleton sets. Let \( x \in X \). If \( U \) is open in \( X \) such that \( \{x\} \subset U \), then \( \{x\} \) is open connected in \( X \) such that \( \{x\} \subset \{x\} \subset U \), and thus by Theorem 4.7 and Theorem 4.9, \( 2^X \) is c.i.k. and l.c. at \( \{x\} \). Hence, if \( C \) is a component of \( X \), then \( 2^X \) is c.i.k. and l.c. at \( C \). Now \( 2^X \) is not c.i.k. or l.c. at \( X \), for suppose \( 2^X \) is c.i.k. or l.c. at \( X \).

Then there exists a connected subset \( \mathcal{A} \) of \( 2^X \) such that \( X \in \mathcal{A}^0 \). Thus there exists \( \langle U_1, \ldots, U_n \rangle \in \mathcal{A} \) such that \( X \in \langle U_1, \ldots, U_n \rangle \subset \mathcal{A}^0 \). For each \( i \in \{1, \ldots, n\} \), let \( x_i \in U_i \).

Then \( \langle \{x_1\}, \ldots, \{x_n\} \rangle \) is closed-open in \( 2^X \) and

\[
\langle \{x_1\}, \ldots, \{x_n\} \rangle \subsetneq \langle U_1, \ldots, U_n \rangle \subset \mathcal{A}.
\]

Thus

\[
\mathcal{A} = [\mathcal{A} \setminus \langle \{x_1\}, \ldots, \{x_n\} \rangle] \cup \langle \{x_1\}, \ldots, \{x_n\} \rangle,
\]

where

\[
\mathcal{A} \setminus \langle \{x_1\}, \ldots, \{x_n\} \rangle \text{ and } \langle \{x_1\}, \ldots, \{x_n\} \rangle
\]

are nonempty separated sets, which is a contradiction. Hence, there does not exist a connected subset of \( 2^X \) containing \( X \) in its interior, which implies \( 2^X \) is not l.c. or c.i.k. at \( X \).

Even if \( (X,T) \) is a l.c., locally compact, connected, metric space, (II) does not imply (I).

Example 5.3. Let \( X = \{(0,y) | y \geq 0\} \cup \{(x,n) | x \in [0,1] \text{ and } n \in \mathbb{N}\} \), and let \( T \) be the usual metric topology on \( X \). Then \( (X,T) \) is a l.c., locally compact, connected, metric space. Let \( A = \{(x,n) | x \in [\frac{1}{2},1] \text{ and } n \in \mathbb{N}\} \). Then
\[ A \in 2^X \] and the components of \( A \) are \( \{(x,n) | x \in [\frac{1}{2}, 1] \} | n \in \mathbb{N} \).

Let \( C = \{(x,n) | x \in [\frac{1}{2}, 1] \} \) be a component of \( A \). If \( U \) is open in \( X \) such that \( C \subseteq U \), then there exists an open connected set \( K \) such that \( C \subseteq K \subseteq U \). Thus by Theorem 4.7 and Theorem 4.9, \( 2^X \) is c.i.k. and l.c. at \( C \). Hence, if \( C \) is a component of \( A \), then \( 2^X \) is c.i.k. and l.c. at \( C \).

Now \( 2^X \) is not l.c. or c.i.k. at \( A \), for suppose \( 2^X \) is l.c. or c.i.k. at \( A \). Let \( U = \{(x,n) | x \in (\frac{1}{2}, 1] \text{ and } n \in \mathbb{N} \} \). Then \( A \in \langle U \rangle \), and thus there exists a connected subset \( \mathcal{A} \) of \( \langle U \rangle \) such that \( A \in \mathcal{A}^0 \). Let \( S \) be the discrete topology on \( \mathbb{N} \), let \( f : (C\text{\textnormal{l}}(U), T_{C\text{\textnormal{l}}(U)}) \rightarrow (\mathbb{N}, S) \) be defined by \( f((x,n)) = n \) for all \( (x,n) \in C\text{\textnormal{l}}(U) \), and let \( \mathcal{E}(C\text{\textnormal{l}}(U)) \)

be defined by \( f^*(D) = f(D) \) for all \( D \in 2^{C\text{\textnormal{l}}(U)} \). Then \( f \) is a continuous closed-open function from \( C\text{\textnormal{l}}(U) \) onto \( \mathbb{N} \) and \( f^* \) is a continuous open function from \( 2^{C\text{\textnormal{l}}(U)} \) onto \( 2^\mathbb{N} \).

Since \( \langle U \rangle \) is open in \( 2^{C\text{\textnormal{l}}(U)} \) and \( A \in \mathcal{A}^0 \subseteq \langle U \rangle \), then \( A \in \mathcal{A}^0 \), which is open in \( 2^{C\text{\textnormal{l}}(U)} \). Since \( \mathcal{A} \) is a connected subset of \( \langle U \rangle \) and \( \langle U \rangle \subseteq 2^{C\text{\textnormal{l}}(U)} \), then \( \mathcal{A} \) is a connected subset of \( 2^{C\text{\textnormal{l}}(U)} \). Then \( f^*(\mathcal{A}) \) is connected in \( 2^\mathbb{N} \) and \( f^*(A) = \mathbb{N} \in f^*(\mathcal{A}^0) \subseteq (f^*(\mathcal{A}))^0 \), but by the argument in Example 5.2, there does not exist a connected subset of \( 2^\mathbb{N} \) containing \( \mathbb{N} \) in its interior, which is a contradiction. Hence \( 2^X \) is not l.c. or c.i.k. at \( A \).
Theorem 5.1. Let \((X,T)\) be an R\(_0\) space and let \(B \in K(X)\). If \(2^X\) or \(K(X)\) is c.i.k. (l.c.) at each component of \(B\), then \(2^X\) and \(K(X)\) are c.i.k. (l.c.) at \(B\).

Proof. Suppose \(2^X\) is c.i.k. at each component of \(B\). Let \(\mathcal{O}\) be open in \(2^X\) such that \(B \in \mathcal{O}\). Then there exists \(\langle U_1, \ldots, U_n \rangle \in \mathcal{B}\) such that \(B \in \langle U_1, \ldots, U_n \rangle \subset \mathcal{O}\). By Theorem 4.2, \(\bigcup_{i=1}^{n} U_i = \bigcup_{A \in \langle U_1, \ldots, U_n \rangle} A\). Then \(B \subset \bigcup_{i=1}^{n} U_i\), which is open in \(X\). For each component \(C\) of \(B\), let

\[I_C = \{1, \ldots, p_C\} = \{i \in \{1, \ldots, n\} | C \cap U_i \neq \emptyset\}\]

and for each \(j \in I_C\), let \(U_{c_j} = U_j\). Then for each component \(C\) of \(B\), \(C \subset C(X)\), \(C \subset \bigcup_{j=1}^{p_C} U_{c_j}\) such that \(C \cap U_{c_j} \neq \emptyset\) for all \(j \in I_C\). Then \(\langle U_{c_1}, \ldots, U_{c_{p_C}} \rangle \) is an open cover of \(B\), which is compact. Then there exists a finite subcover \(\{K_{c_i}^0 | i = 1, \ldots, m\}\). Since \(B \subset \bigcup_{i=1}^{m} K_{c_i}^0 \subset \bigcup_{i=1}^{n} U_i\) and \(B \cap U_i \neq \emptyset\) for all \(i \in \{1, \ldots, n\}\), then

\(\bigcup_{i=1}^{m} I_{c_i} = \{1, \ldots, n\}\). Then \((X,T)\) is an R\(_0\) space and for each \(i \in \{1, \ldots, m\}\), \(C_{i} \subset \bigcup_{j=1}^{p_{c_i}} U_{c_{ij}}\)

where \(U_{c_{ij}}\) is open and \(U_{c_{ij}} \cap C_{i} \neq \emptyset\) for all \(j \in \{1, \ldots, p_{c_i}\}\).
and $C_i \subset K_c^0$, where $K_c$ is a connected subset of $P_c \bigcup_{j=1}^{c_i} U c_j$. Thus by Corollary 4.4,

$$\langle K_c^0 \cap \bigcup_{j=1}^{c_i} U c_j, \ldots, K_c^0 \cap U c_m P_c_m \rangle$$

is in a connected subset of

$$\langle U c_1 \ldots U c_m P_c_m \rangle = \langle U_1, \ldots, U_n \rangle$$

and by Corollary 4.5,

$$\langle K_c^0 \cap \bigcup_{j=1}^{c_i} U c_j, \ldots, K_c^0 \cap U c_m P_c_m \rangle \cap K(X)$$

is in a connected subset of

$$\langle U c_1 \ldots U c_m P_c_m \rangle \cap K(X) = \langle U_1, \ldots, U_n \rangle \cap K(X).$$

Then $B \in \mathcal{K}^0$, where $\mathcal{K}$ is the component of $\mathcal{O}$ containing $B$, and $B \in \mathcal{C}^0_{K(X)}$, where $\mathcal{C}$ is the component of $\mathcal{O} \cap K(X)$ containing $B$. Hence, if $\mathcal{O}$ is open in $2^X$ such that $B \in \mathcal{O}$, then $B \in \mathcal{K}^0$, where $\mathcal{K}$ is the component of $\mathcal{O}$ containing $B$, and $B \in \mathcal{C}^0_{K(X)}$, where $\mathcal{C}$ is the component of $\mathcal{O} \cap K(X)$ containing $B$, which implies $2^X$ and $K(X)$ are c.i.k. at $B$.

By using Theorem 4.2, Theorem 4.9, Corollary 4.6, and an argument similar to that above, if $2^X$ is l.c. at each component of $B$, then $2^X$ and $K(X)$ are l.c. at $B$.

If $K(X)$ is c.i.k. (l.c.) at each component of $B$, then by Theorem 4.8 (Theorem 4.10), $2^X$ is c.i.k. (l.c.) at each component of $B$ and by the argument above, $2^X$ and $K(X)$ are c.i.k. (l.c.) at $B$. 
The next corollary is an immediate result from Theorem 5.1.

**Corollary 5.1.** Let \((X,T)\) be a compact \(R_0\) space and let \(B \in 2^X\). If \(2^X\) is c.i.k. (l.c.) at each component of \(B\), then \(2^X\) is c.i.k. (l.c.) at \(B\).

**Theorem 5.2.** Let \((X,T)\) be an \(R_0\) space and let \(B \in 2^X\) such that \(B\) has only finitely many components. If \(2^X\) is c.i.k. (l.c.) at each component of \(B\), then \(2^X\) is c.i.k. (l.c.) at \(B\).

**Proof.** By an argument similar to that for Theorem 5.1, \(2^X\) is c.i.k. (l.c.) at \(B\).

Since Example 5.1 shows that (I) does not imply (II) even if \((X,T)\) is compact connected \(T_1\) and \(B\) has only two components, then the converse of Theorem 5.1, Corollary 5.1, and Theorem 5.2 is false.

**Theorem 5.3.** Let \((X,T)\) be an \(R_0\) space and let \(A \in 2^X\) such that \(2^X\) is l.c. at \(A\). If \(C\) is a component of \(A\) such that if \(U\) is open in \(X\) and \(C \subseteq U\), then there exists an open set \(V\) in \(X\) such that \(C \subseteq V \subseteq U\) and \(\text{Fr}(V) \cap A = \emptyset\), then \(2^X\) is l.c. at \(C\).

**Proof.** Let \(C\) be a component of \(A\) such that if \(U\) is open in \(X\) and \(C \subseteq U\), then there exists an open set \(V\) in \(X\) such that \(C \subseteq V \subseteq U\) and \(\text{Fr}(V) \cap A = \emptyset\). If \(C = A\), then \(2^X\) is l.c. at \(C\). Thus consider the case that \(C \neq A\). Let \(O\) be open such that \(C \subseteq O\). Let \(C_1\) be a component of \(A\) such that \(C_1 \neq C\). Then \(U = O \setminus C_1\) is open in \(X\) and \(C \subseteq U\). Thus
there exists an open set \( V \) such that \( C \subset V \subset U \) and
\[
\text{Fr}(U) \cap A = \emptyset.
\]
Then \( A \cap V = A \cap \text{Cl}(V) \) is nonempty closed and \( A \setminus \text{Cl}(V) = A \setminus V \) is nonempty closed. Since \( V \) and \( X \setminus \text{Cl}(V) \) are open, \( A \subset V \cup (X \setminus \text{Cl}(V)) \), and \( A \cap V \neq \emptyset \neq A \cap (X \setminus \text{Cl}(V)) \), then \( A \in \langle V, X \setminus \text{Cl}(V) \rangle \) and since \( 2^X \) is l.c.
at \( A \), then there exists an open connected set
\[
\mathcal{C} \subset \langle V, X \setminus \text{Cl}(V) \rangle
\]
containing \( A \). By Theorem 4.2., \( B = \cup \{ K | K \in \mathcal{C} \} \) is open in \( X \) and
\[
B \subset \cup \{ K | K \in \langle V, X \setminus \text{Cl}(V) \rangle \} = V \cup (X \setminus \text{Cl}(V)).
\]
Let \( D \) be the quasi-component of \( B \) containing \( C \). Since 
\( B \cap V \) is closed-open in \( B \) and \( C \subset B \cap V \), then \( D \subset B \cap V \). Now 
\( D \) is open in \( X \), for suppose not. Let \( x \in D \setminus D^0 \). Let \( F \in \mathcal{C} \) 
such that \( x \in F \). Then there exists \( \langle V_1, \ldots, V_q \rangle \in \mathcal{B} \) such that 
\( F \in \langle V_1, \ldots, V_q \rangle \subset \mathcal{C} \). Let
\[
\{ V_{r_1}^1, \ldots, V_{r_t}^t \} = \{ V_i \in \{ V_1, \ldots, V_q \} | F \cap V_i \cap V \neq \emptyset \}.
\]
and let
\[
\{ V_{s_1}^1, \ldots, V_{s_u}^u \} = \{ V_i \in \{ V_1, \ldots, V_q \} | F \cap V_i \cap (X \setminus \text{Cl}(V)) \neq \emptyset \}.
\]
Then
\[
F \subset ( \bigcup_{i=1}^t (U_{r_i}^i \cap V)) \cup (\bigcup_{i=1}^u (V_{s_i}^i \cap X \setminus \text{Cl}(V))) = \bigcup_{i=1}^q V_i
\]
and
\[
\{ V_{r_i}^i | i=1, \ldots, t \} \cup \{ V_{s_i}^i | i=1, \ldots, u \} = \{ V_1, \ldots, V_q \}.
\]
For each \( i \in \{ 1, \ldots, t \} \), let \( U_i = V_{r_i}^i \cap V \) and for each
i \in \{t+1, \ldots, t+u\}, let U_i = V_{s_i-t} \cap (X \setminus \text{Cl}(V)). Then

F \in \langle U_1, \ldots, U_{t+u} \rangle \subset \langle V_1, \ldots, V_q \rangle \subset \langle V, X \setminus \text{Cl}(V) \rangle, where

\bigcap_{i=1}^{t+u} F \cap (\bigcup_{i=1}^{t+u} U_i) \subset V \text{ and } F \cap (\bigcup_{i=1}^{t+u} U_i) \subset X \setminus \text{Cl}(V). \text{ Since } x \in F \cap D \subset V,

then there exists 1 \in \{1, \ldots, t\} such that x \in U_1. \text{ If } n \in \mathbb{N},

then there exists a finite sequence \{O_i\}_{i=1}^n such that

\begin{enumerate}
\item \{O_i\}_{i=1}^{j+1} is an extension of \{O_i\}_{i=1}^j for all \ j \in \{1, \ldots, n-1\},
\item \emptyset \neq O_i \cap U_1, O_i is closed-open \textbf{B} \cap V, and O_i \cap D = \emptyset for all \ i \in \{1, \ldots, n\},
\item if \ j \in \{1, \ldots, n-1\} and \{i \in \{1, \ldots, t\} | U_i \subset O_j \} \neq \emptyset,
\text{ then } O_{j+1} \subset \bigcap_{i=1}^j O_i, \text{ and}
\item if \ j \in \{1, \ldots, n-1\} and \{i \in \{1, \ldots, t\} | U_i \subset O_j \} = \emptyset,
\text{ then } O_{j+1} = O_j.
\end{enumerate}

Proof. Since \ x \in U_1, which is open, then U_1 \neq D.

Let y_1 \in U_1 \setminus D. \text{ Then there exists a closed-open set } O_1 \text{ in } B \cap V \text{ such that } y_1 \in O_1 \text{ and } O_1 \cap D = \emptyset. \text{ Thus } \{O_i\}_{i=1}^1 \text{ satisfies the desired properties and the statement is true for } n = 1.

Assume the statement is true for \ n = k. \text{ Then there exists a finite sequence } \{O_i\}_{i=1}^k \text{ satisfying properties (1) through (4). Consider the case that }

\{i \in \{1, \ldots, t\} | U_i \subset O_k \} \neq \emptyset.
Then \( \bigcup_{i=1}^{k} O_i \) is closed-open in \( B \cap V \) and \( (\bigcup_{i=1}^{k} O_i) \cap D = \emptyset \).

Since \( x \in U_1 \setminus \bigcup_{i=1}^{k} O_i \), which is open in \( X \), then

\[
U_1 \setminus \bigcup_{i=1}^{k} O_i \notin D
\]

and there exists \( y_{k+1} \in [U_1 \setminus \bigcup_{i=1}^{k} O_i] \setminus D \). Then there exists a closed-open set \( O \) in \( B \cap V \) such that \( y_{k+1} \in O \) and \( O \cap D \neq \emptyset \). Thus \( y_{k+1} \in O \cap [(B \cap V) \setminus \bigcup_{i=1}^{k} O_i] = O_{k+1} \), which is closed-open in \( B \cap V \), and \( O_{k+1} \cap D = \emptyset \). Then \( \{O_i\}_{i=1}^{k+1} \) satisfies properties (1) through (4). If \( \{i \in [1, \ldots, t] \mid U_i \subset O_k \} = \emptyset \), then let \( O_{k+1} = O_k \) and \( \{O_i\}_{i=1}^{k+1} \) satisfies properties (1) through (4). Hence, if the statement is true for \( n = k \), then the statement is true for \( n = k + 1 \).

Therefore, if \( n \in \mathbb{N} \), then there exists a finite sequence \( \{O_i\}_{i=1}^{n} \) satisfying properties (1) through (4). Hence, there exists an infinite sequence \( \{O_i\}_{i=1}^{\infty} \) satisfying properties (2) through (4). Let

\[
\mathcal{S} = \{n \in \mathbb{N} \mid O_n = O_{n+1}\}
\]

Then \( \mathcal{S} \neq \emptyset \), since \( t-1 \in \mathcal{S} \). Let \( m \) be the least element of \( \mathcal{S} \). Then \( O_m \) is closed-open in \( B \cap V \), which is closed-open in \( B \), and thus \( O_m \) is closed-open in \( B \). Thus \( O_m \) and \( B \setminus O_m \) are open in \( X \). Then \( B = O_m \cup (B \setminus O_m) \), where \( O_m \) and \( B \setminus O_m \) are nonempty separated sets. Thus \( (O_m, B \setminus O_m) \) and \( S(B \setminus O_m) \) are nonempty separated sets and \( \mathcal{S} \subset (O_m, B \setminus O_m) \cup S(B \setminus O_m) \).
Since $O_m$ is closed-open in $B \cap V$, $O_m \cap U_1 \neq \emptyset$, and

\[ \{i \in \{1, \ldots, t\} | U_i \subset O_m\} = \emptyset, \]

then for each $i \in \{1, \ldots, t+u\}$, there exists $x_i \in U_i \setminus O_m \subset B \setminus O_m$. For each $i \in \{1, \ldots, t+u\}$, $x_i \in U_i \setminus O_m$, which is open in $X$, and thus

\[ \text{Cl}\{x_i\} \subset U_i \setminus O_m \subset B \setminus O_m, \]

which implies

\[ \bigcup_{i=1}^{t+u} \text{Cl}\{x_i\} \in \langle U_1, \ldots, U_{t+u} \rangle \cap S(B \setminus O_m) = \mathcal{E} \cap S(B \setminus O_m). \]

Let $y \in O_m \cap U_1$, which is open in $X$. Then $\text{Cl}\{y\} \subset O_m \cap U_1 \subset O_m$,

which implies

\[ \bigcup_{i=1}^{t+u} \text{Cl}\{x_i\} \cup \text{Cl}\{y\} \in \langle U_1, \ldots, U_{t+u} \rangle \cap \langle O_m, B \setminus O_m \rangle \]

\[ \subset \mathcal{E} \cap \langle O_m, B \setminus O_m \rangle. \]

Thus $\mathcal{E} = \langle O_m, B \setminus O_m \rangle \cup \langle (S(B \setminus O_m) \cap \mathcal{E}) \rangle$,

where $\langle O_m, B \setminus O_m \rangle \cap \mathcal{E}$ and $S(B \setminus O_m) \cap \mathcal{E}$ are nonempty separated sets, which contradicts $\mathcal{E}$ is connected. Hence, $D$ is open in $X$, which implies $D$ is closed-open in $B$, and since $D$ is a quasi-component of $B$, then $D$ is connected. Hence, if $O$ is open in $X$ such that $C \subset O$, then there exists an open connected set $D$ in $X$ such that $C \subset D \subset O$, and thus by Theorem 4.9, $2^X$ is l.c. at $\mathcal{E}$.

**Theorem 5.4.** Let $(X,T)$ be a normal $R_0$ space and let $A \in 2^X$ such that $A$ has only finitely many components. Then $2^X$ is l.c. at $A$ if and only if $2^X$ is l.c. at each component of $A$.

**Proof.** Suppose $2^X$ is l.c. at each component of $A$. By Theorem 5.2, $2^X$ is l.c. at $A$. 

Theorem 5.4. Let $(X,T)$ be a normal $R_0$ space and let $A \in 2^X$ such that $A$ has only finitely many components. Then $2^X$ is l.c. at $A$ if and only if $2^X$ is l.c. at each component of $A$.

**Proof.** Suppose $2^X$ is l.c. at each component of $A$. By Theorem 5.2, $2^X$ is l.c. at $A$. 

**Theorem 5.4.** Let $(X,T)$ be a normal $R_0$ space and let $A \in 2^X$ such that $A$ has only finitely many components. Then $2^X$ is l.c. at $A$ if and only if $2^X$ is l.c. at each component of $A$.
Conversely, suppose $2^X$ is l.c. at $A$. Let $C_1$ be a component of $A$. If $C_1 = A$, then $2^X$ is l.c. at $C_1$. Thus consider the case that $C_1 \subsetneq A$. Let $U$ be open in $X$ such that $C_1 \subset U$ and let \( \{C_1, \ldots, C_n\} = \{C \subseteq A | C \text{ is a component of } A\} \). Since $C_1$ and $\bigcup_{i=2}^{n} C_i$ are disjoint closed sets, then there exist disjoint open sets $O_1$ and $O_2$ such that $C_1 \subseteq O_1$ and $\bigcup_{i=2}^{n} C_i \subseteq O_2$. Then

$$
\overline{O_1} \cap \bigcup_{i=2}^{n} C_i = \emptyset.
$$

Let $V = O_1 \cap U$. Then $C_1 \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$. Thus $C_1$ is a component of $A$ such that if $U$ is open in $X$ and $C_1 \subseteq U$, then there exists an open set $V$ in $X$ such that $C_1 \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$. Then by Theorem 5.3, $2^X$ is l.c. at $C$. Hence, if $C$ is a component of $A$, then $2^X$ is l.c. at $C$.

**Theorem 5.5.** Let $(X, T)$ be an $R_0$ space and let $A \subseteq K(X)$ such that $2^X$ or $K(X)$ is l.c. at $A$. If $C$ is a component of $A$ such that if $U$ is open in $X$ and $C \subseteq U$, then there exists an open set $V$ in $X$ such that $C \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$, then $2^X$ and $K(X)$ are l.c. at $C$.

**Proof.** Let $C$ be a component of $A$ such that if $U$ is open in $X$ and $C \subseteq U$, then there exists an open set $V$ in $X$ such that $C \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$.

Suppose $2^X$ is l.c. at $A$. By Theorem 5.3, $2^X$ is l.c. at $C$. Then by Theorem 4.10, $K(X)$ is l.c. at $C$. 


Now suppose $K(X)$ is l.c. at $A$. By an argument similar to that for Theorem 5.3, $K(X)$ is l.c. at $C$. Then by Theorem 4.10, $2^X$ is l.c. at $C$.

**Corollary 5.2.** Let $(X,T)$ be a locally compact $R_1$ space and let $A \in 2^X$ such that $A$ has a compact component $C$. If $2^X$ is l.c. at $A$, then $2^X$ and $K(X)$ are l.c. at $C$.

**Proof.** By Theorem 2.11, if $U$ is open in $X$ such that $C \subseteq U$, then there exists an open set $V$ such that $C \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$. If $2^X$ is l.c. at $A$, then by Theorem 5.3, $2^X$ is l.c. at $C$, and thus by Theorem 4.10, $K(X)$ is l.c. at $C$.

The next corollary follows immediately from Corollary 5.2.

**Corollary 5.3.** Let $(X,T)$ be a locally compact $R_1$ space and let $A \in 2^X$ such that all components of $A$ are compact. If $2^X$ is l.c. at $A$, then $2^X$ and $K(X)$ are l.c. at each component of $A$.

Example 5.3 shows that the converse of Corollary 5.2 and Corollary 5.3 is false even if the space is a l.c., locally compact, connected, metric space.

**Corollary 5.3.** Let $(X,T)$ be a compact $R_1$ space and let $A \in 2^X$. Then $2^X$ is l.c. at $A$ if and only if $2^X$ is l.c. at each component of $A$.

**Proof.** Suppose $2^X$ is l.c. at each component of $A$. Then by Theorem 5.1, $2^X$ is l.c. at $A$. 
Conversely, suppose $2^X$ is l.c. at $A$. Since $(X,T)$ is compact $\mathbb{R}_1$, then $(X,T)$ is locally compact $\mathbb{R}_1$ and by Corollary 5.3, $2^X$ is l.c. at each component of $A$.

**Corollary 5.5.** Let $(X,T)$ be a $\mathbb{R}_1$ space and let $A \in K(X)$. Then the following are equivalent: (a) $2^X$ is l.c. at $A$, (b) $2^X$ is l.c. at each component of $A$, (c) $K(X)$ is l.c. at each component of $A$, and (d) $K(X)$ is l.c. at $A$.

**Proof.** By Theorem 2.12, if $C$ is a component of $A$ and $U$ is open in $X$ such that $C \subseteq U$, then there exists an open set $V$ in $X$ such that $C \subseteq V \subseteq U$ and $\text{Fr}(V) \cap A = \emptyset$.

(a) implies (b): by Theorem 5.3, if $C$ is a component of $A$, then $2^X$ is l.c. at $C$.

(b) implies (c): by Theorem 4.10, $K(X)$ is l.c. at each component of $A$.

(c) implies (d): by Theorem 5.1, $K(X)$ is l.c. at $A$.

(d) implies (a): by Theorem 5.5, $2^X$ is l.c. at each component of $A$ and then by Theorem 5.1, $2^X$ is l.c. at $A$.

**Theorem 5.6.** Let $(X,T)$ be a locally compact $\mathbb{R}_1$ space and let $A \in 2^X$ with compact component $C$. If $2^X$ is c.i.k. at $A$, then $2^X$ and $K(X)$ are c.i.k. at $C$.

**Proof.** If $C = A$, then $2^X$ is c.i.k. at $C$ and by Theorem 4.8, $K(X)$ is c.i.k. at $C$. Thus consider the case that $C \not\subseteq A$. Let $C_1$ be a component of $A$ such that $C_1 \neq C$. Let $O$ be open in $X$ such that $C \subseteq O$. Then $C \subseteq O \setminus C_1$, which
is open in $X$, and by Theorem 2.11, there exists an open set $V$ in $X$ such that $\text{Cl}(V)$ is compact, $C \subseteq V \subseteq \text{Cl}(V) \subseteq O \setminus C_1$, and $\text{Fr}(V) \cap A = \emptyset$. Since $C \subseteq V$, which is open in $X$, then by Theorem 2.11, there exists an open set $W$ such that $\text{Cl}(W)$ is compact, $C \subseteq W \subseteq \text{Cl}(W) \subseteq V$, and $\text{Fr}(W) \cap A = \emptyset$.

Then $(A \cap \text{Cl}(V)) \setminus \text{Cl}(W) = (A \cap \text{Cl}(V)) \setminus W = (A \cap V) \setminus W$ is closed compact. Let $x \in (A \setminus V) \setminus W$. Then $x \in (X \setminus \text{Cl}(X)) \cap V$, which is open, and thus there exists an open set $O_x$ such that $x \in O_x \subseteq \text{Cl}(O_x) \subseteq (X \setminus \text{Cl}(W)) \cap V$. For each $x \in (A \setminus V) \setminus W$, let $O_x$ be open such that $x \in O_x \subseteq \text{Cl}(O_x) \subseteq (X \setminus \text{Cl}(W)) \cap V$. Then $\{O_x\}_{x \in (A \setminus V) \setminus W}$ is an open cover of $(A \setminus V) \setminus W$, and thus there exists a finite subcover $\{O_{x_i}\}_{i=1}^n$. Then

$$A \subseteq W \cup \left( \bigcup_{i=1}^n O_{x_i} \right) \cup (X \setminus \text{Cl}(V)),$$

where

$$\text{Cl}(W) \cap \text{Cl}(O_{x_i}) = \emptyset$$

for all $i \in \{1, \ldots, n\}$,

$$\text{Cl}(W) \cap \text{Cl}(X \setminus \text{Cl}(V)) = \emptyset,$$

$A \cap W \neq \emptyset$, $A \cap O_{x_i} \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, and

$$A \cap (X \setminus \text{Cl}(V)) \neq \emptyset.$$

Thus

$$A \in \langle W, O_{x_1}, \ldots, O_{x_n}, X \setminus \text{Cl}(V) \rangle.$$

Since $2^X$ is c.i.k. at $A$, then the component $\mathcal{C}$ of

$$\langle W, O_{x_1}, \ldots, O_{x_n}, X \setminus \text{Cl}(V) \rangle$$

containing $A$ contains $A$ in its
interior. Let $B = \bigcup \{ K \cap W | K \in \mathcal{E} \}$. Then $\text{Cl}(B) \subset \text{Cl}(W)$. Let $D$ be the component of $\text{Cl}(B)$ containing $C$. Then $C \subset D^0$, for suppose not. Let $z \in C \setminus D^0$. Since $A \in \mathcal{E}^0$, then there exists $(V_1, \ldots, V_q) \in \mathcal{B}$ such that $A \in (V_1, \ldots, V_q) \subset \mathcal{E}^0$. Let $\mathcal{V}_1 = \{ V_i \in (V_1, \ldots, V_q) | V_i \cap A \cap W \neq \emptyset \}$ and

$$V_i \cap A \cap [(\bigcup_{i=1}^n O_{x_i}) \cup (X \setminus \text{Cl}(V))] = \emptyset \rightarrow \{ V_{n_1}, \ldots, V_{n_r} \},$$

let

$$\mathcal{V}_2 = \{ V_i \in (V_1, \ldots, V_q) | V_i \cap A \cap W \neq \emptyset \}$$

$$V_i \cap A \cap [(\bigcup_{i=1}^n O_{x_i}) \cup (X \setminus \text{Cl}(U))] \neq \emptyset \rightarrow \{ V_{m_1}, \ldots, V_{m_s} \},$$

and let

$$\mathcal{V}_3 = \{ V_i \in (V_1, \ldots, V_q) | V_i \cap A \cap W = \emptyset \}$$

and

$$V_i \cap A \cap [(\bigcup_{i=1}^n O_{x_i}) \cup (X \setminus \text{Cl}(V))] \neq \emptyset \rightarrow \{ V_{p_1}, \ldots, V_{p_t} \}.$$

Then $\{ \mathcal{V}_i \}_{i=1}^3$ is a collection of disjoint sets such that

$$\bigcup_{i=1}^3 \mathcal{V}_i = \{ V_1, \ldots, V_q \}.$$

For each $i \in \{1, \ldots, r\}$, let $W_i = V_{n_i}$ for each $i \in \{r+1, \ldots, r+s\}$, let $W_i = V_{m_{i-r}}$ for each $i \in \{r+s+1, \ldots, r+2s\}$, let

$$W_i = V_{m_{i-r-s}} \cap [(\bigcup_{i=1}^n O_{x_i}) \cup (X \setminus \text{Cl}(V))].$$
and for each \( i \in \{r+2s+1, \ldots, r+2s+t\} \), let

\[
W_i = V_{p_{i-r-2s}} \cap \left( \bigcup_{i=1}^{n} O_{x_i} \right) \cup (X \setminus C_1(V)).
\]

Then \( A \in \langle W_1, \ldots, W_{r+2s+t} \rangle \subset \langle V_1, \ldots, V_q \rangle \subset C^0 \) such that

\[
\bigcup_{i=1}^{r+s} W_i \subset W, \quad \bigcup_{i=r+s+1}^{r+2s+t} W_i \subset \left( \bigcup_{i=1}^{n} O_{x_i} \right) \cup (X \setminus C_1(V)), \text{ and}
\]

\[
C_1(W_i) \cap C_1(W) = \emptyset
\]

for all \( i \in \{r+s+1, \ldots, r+2s+t\} \). Since \( D \subset C_1(W) \), then

\[
C_1(W_i) \cap D = \emptyset \text{ for all } i \in \{r+s+1, \ldots, r+2s+t\}.
\]

Let

\[
\{W_i | i=1, \ldots, u\} = \{W_i \in \langle W_1, \ldots, W_{r+2s+t} \rangle
\]

\[
|C_1(W_i) \cap D = \emptyset\} \neq \emptyset.
\]

Then \( D \) is a compact component of \( C_1(B) \) and

\[
D \subset X \setminus \left( \bigcup_{i=1}^{u} C_1(W_i) \right) \cup \left( \bigcup_{i=1}^{n} C_1(O_{x_i}) \cup C_1(X \setminus C_1(V)) \right),
\]

which is open in \( X \). Thus by Theorem 2.11, there exists an open set \( O_1 \) such that

\[
D \subset O_1 \subset C_1(O_1) \subset X \setminus \left( \bigcup_{i=1}^{n} C_1(W_i) \right)
\]

\[
\cup \left( \bigcup_{i=1}^{n} C_1(O_{x_i}) \cup C_1(X \setminus C_1(V)) \right)
\]

and \( Fr(O_1) \cap C_1(B) = \emptyset \). Since

\[
z \in O_1 \cap (\cap \{W_i | i \in \{1, \ldots, r+2s+t\} \text{ and } z \in W_i\}),
\]

which is open, then

\[
O_1 \cap (\cap \{W_i | i \in \{1, \ldots, r+2s+t\} \text{ and } z \in W_i\}) \neq \emptyset.
\]
Let
\[ y \in [O_1 \cap (\bigcap \{W_i | i = 1, \ldots, r+2s+t\} \text{ and } z \in W_i}) \setminus D. \]

Let \( C_y \) be the component of \( \text{Cl}(B) \) containing \( y \). Since 
\[ C_y \cap O_1 \neq \emptyset \text{ and } \text{Fr}(O_1) \cap \text{Cl}(B) = \emptyset, \]
then \( C_y \subset O_1 \) and since \( C_y \) is a compact component of \( \text{Cl}(B) \) and \( C_y \subset O_1 \setminus D \), which is open, then there exists an open set \( O_2 \) such that
\[ C_y \subset O_2 \subset \text{Cl}(O_2) \subset O_1 \setminus D \]
and
\[ \text{Fr}(O_2) \cap \text{Cl}(B) = \emptyset. \]

Then \( O_2 \) and \( X \setminus \text{Cl}(O_2) \) are nonempty disjoint open sets in \( X \),
\[ \bigcup \{K | K \in \mathcal{G}\} \subset O_2 \cup (X \setminus \text{Cl}(O_2)), \]
and
\[ K \cap (X \setminus \text{Cl}(O_2)) \neq \emptyset \]
for all \( K \in \mathcal{G} \). Thus \( \langle O_2, X \setminus \text{Cl}(O_2) \rangle \) and \( \langle X \setminus \text{Cl}(O_2) \rangle \) are nonempty separated sets in \( 2^X \) and
\[ \mathcal{G} \subset \langle O_2, X \setminus \text{Cl}(O_2) \rangle \cup \langle X \setminus \text{Cl}(O_2) \rangle. \]

For each \( i \in \{1, \ldots, r+s\} \setminus \{q_1, \ldots, q_u\} \neq \emptyset \), \( \text{Cl}(W_i) \cap D \neq \emptyset \) and \( O_1 \setminus \text{Cl}(O_2) \) is open such that \( D \subset O_1 \setminus \text{Cl}(O_2) \), which implies for each \( i \in \{1, \ldots, r+s\} \setminus \{q_1, \ldots, q_u\} \), there exists
\[ x_i \in W_i \cap (O_1 \setminus \text{Cl}(O_2)) \subset X \setminus \text{Cl}(O_2). \]

For each \( i \in \{q_1, \ldots, q_u\} \), let \( x \in W_i \subset X \setminus \text{Cl}(O_2) \). Then
\[ \bigcup_{i=1}^{r+2s+t} \text{Cl}\{x_i\} \in \langle W_1, \ldots, W_{r+2s+t} \rangle \cap \langle X \setminus \text{Cl}(O_2) \rangle \subset \mathcal{G} \cap \langle X \setminus \text{Cl}(O_2) \rangle \]
and

\[(\bigcup_{i=1}^{r+2s+t} Cl\{x_i\} \cup Cl\{y\} \in \langle W_1, \ldots, \overline{W}_{r+2s+t} \rangle) \cap \langle O_2, X \setminus Cl(O_2) \rangle \subseteq \emptyset \cap \langle O_2, X \setminus Cl(O_2) \rangle,\]

which is a contradiction. Thus \(C \subseteq D \subseteq Cl(W) \subseteq 0\). Hence, if \(0\) is open in \(X\) such that \(C \subseteq 0\), then the component of \(0\) containing \(C\) contains \(C\) in its interior. Then by Theorem 4.8, \(2^X\) and \(K(X)\) are c.i.k. at \(C\).

The next corollary is an immediate result from Theorem 5.6.

**Corollary 5.6.** Let \((X, T)\) be a locally compact \(R^1\) space and let \(A \in 2^X\) such that all components of \(A\) are compact. If \(2^X\) is c.i.k. at \(A\), then \(2^X\) and \(K(X)\) are c.i.k. at each component of \(A\).

**Theorem 5.7.** Let \((X, T)\) be a locally compact \(R^1\) space and let \(A \in K(X)\). Then the following are equivalent:

(a) \(K(X)\) is c.i.k. at \(A\),
(b) \(K(X)\) is c.i.k. at each component of \(A\),
(c) \(2^X\) is c.i.k. at each component of \(A\),
and (d) \(2^X\) is c.i.k. at \(A\).

**Proof.** (a) implies (b): by an argument similar to that for Theorem 5.6, \(K(X)\) is c.i.k. at each component of \(A\).

(b) implies (c): by Theorem 4.8, \(2^X\) is c.i.k. at each component of \(A\).

(c) implies (d): by Theorem 5.1, \(2^X\) is c.i.k. at \(A\).
(d) implies (a): by Corollary 5.6, $K(X)$ is c.i.k. at each component of $A$ and then by Theorem 5.1, $K(X)$ is c.i.k. at $A$.

The next corollary follows from Theorem 5.7 and the fact that every compact $R_1$ space is locally compact $R_1$.

**Corollary 5.7.** Let $X$ be a compact $R_1$ space and let $A \in 2^X$. Then $2^X$ is c.i.k. at $A$ if and only if $2^X$ is c.i.k. at each component of $A$.

**Theorem 5.8.** Let $(X,T)$ be an $R_0$ space and let $x \in X$. Then the following are equivalent: (a) $X$ is l.c. (c.i.k.) at $x$, (b) $2^X$ is l.c. (c.i.k.) at $\text{Cl}\{x\}$, and (c) $K(X)$ is l.c. (c.i.k.) at $\text{Cl}\{x\}$.

**Proof.** Since $(X,T)$ is an $R_0$ space, then $\text{Cl}\{x\} \in K(X) \cap C(X)$. Consider the statement with l.c.

(a) implies (b): let $U$ be open in $X$ such that $\text{Cl}\{x\} \subseteq U$. Then $x \in U$ and since $X$ is l.c. at $x$, then there exists an open connected set $V$ such that $x \in V \subseteq U$. Since $X$ is $R_0$, then $\text{Cl}\{x\} \subseteq V$. Hence, if $U$ is open in $X$ such that $\text{Cl}\{x\} \subseteq U$, then there exists an open connected set $V$ such that $\text{Cl}\{x\} \subseteq V \subseteq U$. Thus by Theorem 4.10, $2^X$ is l.c. at $\text{Cl}\{x\}$.

(b) implies (c): by Theorem 4.10, $K(X)$ is l.c. at $\text{Cl}\{x\}$.

(c) implies (a): Let $U$ be open in $X$ such that $x \in U$. Then $\text{Cl}\{x\} \subseteq U$. Since $K(X)$ is l.c. at $\text{Cl}\{x\}$, then by Theorem 4.10, there exists an open connected set $V$ such
that \( \text{Cl}\{x\} \subset V \subset U \). Hence, if \( U \) is open in \( X \) such that \( x \in U \), then there exists an open connected set \( V \) such that \( x \in V \subset U \), which implies \( X \) is l.c. at \( x \).

By a similar argument, the statement with c.i.k. follows.

**Theorem 5.9.** Let \( (X, T) \) be an \( R_0 \) space and let \( M \in 2^X \). Then the following are equivalent: (a) \( X \) is l.c. (c.i.k.) at each element of \( M \), (b) \( 2^X \) is l.c. (c.i.k.) at each element of \( C(M) \), (c) \( 2^X \) is l.c. (c.i.k.) at each element of \( K(M) \), (d) \( K(X) \) is l.c. (c.i.k.) at each element of \( K(M) \), (e) \( K(X) \) is l.c. (c.i.k.) at each element of \( \{\text{Cl}\{x\} | x \in M\} \), (f) \( 2^X \) is l.c. (c.i.k.) at each element of \( \{\text{Cl}\{x\} | x \in M\} \).

**Proof.** Consider the statement with l.c.

(a) implies (b): let \( A \in C(M) \). Since \( A \in C(M) \) and \( M \in 2^X \), then \( A \in C(X) \). Let \( O \) be open in \( X \) such that \( A \subset O \).

Let \( x \in A \). Since \( X \) is l.c. at \( x \), then there exists an open connected set \( O_x \) such that \( x \in O_x \subset O \). For each \( x \in A \), let \( O_x \) be open connected such that \( x \in O_x \subset O \). Then \( \bigcup_{x \in A} O_x \) is open connected and \( A \subset \bigcup_{x \in A} O_x \subset O \). Hence, if \( 0 \) is open in \( X \) such that \( A \subset 0 \), then there exists an open connected set \( V \) such that \( A \subset V \subset 0 \), and thus by Theorem 4.9, \( 2^X \) is l.c. at \( A \). Hence \( 2^X \) is l.c. at each element of \( C(M) \).

(b) implies (c): let \( A \in K(M) \). Then \( A \in K(X) \). If \( C \) is a component of \( A \), then \( C \in K(X) \cap C(M) \), and thus
2^X is l.c. at C. Then A ∈ K(X) and 2^X is l.c. at each component of A. Thus by Theorem 5.1, 2^X is l.c. at C. Hence, 2^X is l.c. at each element of K(M).

(c) implies (d): Let A ∈ K(M). Then A ∈ K(X). If C is a component of A, then 2^X is l.c. at C. Thus by Theorem 5.1, K(X) is l.c. at A. Hence, K(X) is l.c. at each element of K(M).

(d) implies (e): Since \{Cl(x) | x ∈ M\} ⊆ K(M), then K(X) is l.c. at each element of \{Cl(x) | x ∈ M\}.

(e) implies (f): If x ∈ M, then K(X) is l.c. at Cl{x}, and thus by Theorem 5.8, 2^X is l.c. at Cl{x}, which implies 2^X is l.c. at each element of \{Cl(x) | x ∈ M\}.

(f) implies (a): If x ∈ M, then 2^X is l.c. at Cl{x}, and thus by Theorem 5.8, X is l.c. at x, which implies X is l.c. at each element of M.

By a similar argument, the theorem follows when l.c. is replaced by c.i.k.

**Corollary 5.8.** Let (X,T) be an R_0 space. If 2^X is l.c., then X is l.c.

**Proof.** Since 2^X is l.c., then 2^X is l.c. at each element of \{Cl(x) | x ∈ X\}. Thus by Theorem 5.9, X is l.c.

Example 5.3 shows that the converse of Corollary 5.8 is false even if X is a l.c., locally compact, connected, metric, space.
CHAPTER BIBLIOGRAPHY


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