COMPUTATIONAL ESTIMATION STRATEGIES USED
BY HIGH SCHOOL STUDENTS OF LIMITED
COMPUTATIONAL ESTIMATION ABILITY

DISSERTATION

Presented to the Graduate Council of the
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By

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The problem of this study was to investigate the strategies used by high school students of limited estimation ability for the estimation of the answers to computational problems.

The Assessing Computational Estimation Test was administered to 460 students, and 40 of them were selected for interviews. Each student interviewed was asked to estimate the answers to fourteen computation and application problems.

A comparison of the interview results and the ACE Test results showed that removing the time pressure did improve performance. Students used a wide variety of estimation strategies; however, sometimes they had no strategy for estimation and attempted to use exact calculation.

All but one of the students used some form of the front-number strategies rounding and truncation in making mental estimates. Truncation was replaced by the use of rounding and compensation by the better estimators of the study.
Although many of the estimators seemed to want to use compensation, they were many times not successful in its use. Estimators of limited ability used rounding, but not always consistently or according to the standard rounding rules. Other commonly used strategies were averaging, using compatible or easier numbers, and using the largest number to eliminate choices.

The students in this study were most successful on percent problems when they thought of percents as part of one hundred or in terms of an easier percent. They performed better than expected on division problems. Possibly this is because of the use of estimation in the traditional algorithm.

A major difficulty encountered by estimators of limited ability was the large-number syndrome. Connected to this problem was the power-of-ten error. A student made a power-of-ten error when his answer would have been acceptable if it had been multiplied by an appropriate power of ten.
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CHAPTER I
INTRODUCTION

Problem and Background

It has been generally agreed that the ability to estimate the answer to computational problems is a practical and desirable skill for "everyman" to possess (Bell, 1974). The literature reveals that it is a skill which receives little curricular attention, although it is a skill highly recommended by mathematics educators and researchers and by their organizations, both for its practical value and for its value in developing an understanding of mathematical concepts and number relationships. A limited amount of research has been done to determine what kinds of estimation or mental computation skills students use (Levine, 1982; Olander & Brown, 1959; Paull, 1972; Rubenstein, 1985). Good estimators and their estimation techniques have been described (Reys, Bestgen, Rybolt, & Wyatt, 1980). It has been shown that students can learn techniques for improving their estimates (Schoen, Friesen, Jarrett, & Urbatsch, 1981) and that time taken from regular arithmetic instruction does not negatively affect the achievement of students on tests of exact computation skills (Payne, 1965; Schall, 1973).
Since previous studies have concentrated on elementary or junior high school students, college students, or high school students with better than average mathematical achievement, there was little information about how the average and slightly-below average achievers estimate answers to computational problems. This study described the estimation strategies used by high school students of varying levels of mathematical achievement who were determined to be estimators of limited ability.

Purpose of the Study

The purpose of this study was to describe the estimation strategies used by high school students with limited computational estimation ability.

Significance of the Study

Estimation is currently a topic of concern to mathematics educators and curriculum developers because of its increasing importance in everyday life (Bestgen, Reys, Rybolt, & Wyatt, 1980; Bestgen & Reys, 1982; Kreider, 1980; Maier, 1980) and its importance in the development of mathematical algorithms and concepts (Ibe, 1973; McKillip, 1981; Nelson, 1967; O'Daffer, 1979; Post, 1981). Good estimators and the strategies they use to estimate the
answers to computational problems have been described by Reys, et. al. (1980). This study extended the knowledge about estimation strategies to estimators of limited ability, which had not been done previously, and therefore, added to the knowledge about how these students think about mathematics.

Research Questions

To carry out the purposes of the study, the following research questions were considered:

1. What strategies for estimating the answers to computational problems are used by high school students of limited computational estimation ability?

2. Will high school students of limited computational estimation ability who produce an estimate using pencil and paper use the same strategies that they used to produce a mental estimate for the same problem?

Definition of Terms

For the purposes of carrying out this study, the following terms are defined.

Computational Estimation: The interaction and/or combination of mental computation, number concepts, technical arithmetic skills including rounding, place
value, and less straightforward processes such as mental compensation that rapidly and consistently result in answers that are reasonably close to a correctly computed result. This process is done internally, without the external use of a calculating or recording tool (Reys, et. al., 1980, p. 6).

**Pencil-and-Paper Estimate**: An estimate of the answer to a computation problem arrived at by using a pencil and paper.

**Students of Limited Computational Estimation Ability**: Students who score between the first and second quartiles on the Assessing Computational Estimation Test.

For the purpose of discussing the results of this study, it is necessary to describe three key processes which may be used by students in the interview sample. Specific strategies, such as rounding, may be thought of as a part of these processes. Translation, reformulation, and compensation are defined as they were by Reys, et. al. (1980).

**Translation**: "Changing the equation or mathematical structure of the problem to a more mentally manageable form" (p. 169).

**Reformulation**: "Changing the numerical data into a more mentally manageable form" (p. 170).

**Compensation**: "Adjustments made to reflect numerical variation that came about as a result of translation and/or reformulation of the problem" (p. 172).

**Limitations**

One limitation of this study is that the Assessing Computational Estimation Test (Reys, et. al., 1980) is a series of timed items, and the interview consisted of items that were not timed. This means that students may or may
not have used the same techniques in the interview that they used on the test.

Another limitation is that the sample of students used in the interview represent only students of limited computational estimation ability. Students at this level of ability may not be aware of the processes they use to select strategies (Garofalo & Lester, 1985), and their ability to communicate exactly how they arrived at their answers is a limiting factor in the completeness of the data (Ericsson & Simon, 1980).

Finally, it must be recognized that students who were thinking aloud may not have accurately reported all the processes that they used to produce estimates. This is true both because of the unaccustomed nature of the task and because the process of thinking aloud may have interfered with the accuracy with which they produced estimates. Flaherty (1975) found that subjects who were thinking aloud made significantly more errors than those who worked without verbalizing.
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CHAPTER II

REVIEW OF LITERATURE

Past Trends

Mental arithmetic has enjoyed periods of favor and
disfavor among mathematics educators since Warren Colburn
published *First Lessons in Intellectual Arithmetic* in 1821
(J. V. Hall, 1954). From that date until the beginning of
the twentieth century, mental arithmetic was used as a part
of the training of the mind in keeping with the faculty
psychology and the mental-discipline theory of learning
popular during this time. Not only did the learning theory
of the day support the teaching of mental arithmetic, but
also the demands of everyday life called for the development
of these skills. Shopkeepers and shoppers alike needed to
be able to compute accurately and rapidly. Proficiency in
mental arithmetic was considered to be a sign that an
applicant for a teaching position would be successful
(Pigge, 1967).

Mental arithmetic was unduly emphasized (Smith, 1913;
Suzzallo, 1912) and the reaction against this heavy emphasis
led to almost total neglect of the topic in the curriculum
from the early 1900's until it was revived in the twenties
and thirties. When it returned to favor, educators viewed
mental arithmetic differently. Thorndike (1922) pointed out that the pencil-and-paper versus the no-pencil-and-paper dichotomy was artificial. He said that life would require a range of written work from 0 per cent to recording all of the important intermediate results.

The type of problems used in mental arithmetic drills also changed to reflect the more practical nature of the skills being developed. An early problem, as described by Jordan (1929, p.60), might have read "On the fourth of July, Mr. Brown divided 3/5 of $4 among his children. To the eldest he gave 1/4 of the 3/5, and to each of the others 45 cents. How many children had he?" Jordan encouraged educators to bring back mental arithmetic by the use of problems such as "How much will it cost to screen our front porch if it takes 30 yards of screen wire at 30 cents and a screen door costing $2.25?"

Various writers in the thirties and forties investigated the uses of mental arithmetic or encouraged its use. Dickey (1934) conducted an experimental study to determine whether estimating the answer to a verbal problem would improve student performance in solving verbal problems. Harper (1941) encouraged teachers to allow students to use estimated computation for some involved processes. J. V. Hall (1942) also used estimation as a part of the process of helping students to understand verbal problems and then investigated the degree of difficulty students encountered
in solving verbal problems mentally (1947). By the time
that K. E. Brown (1950) surveyed teachers to find out their
attitudes toward inclusion of mental arithmetic in the
curriculum of a general mathematics class, teachers had come
to view mental arithmetic as important to the curriculum if
they perceived it to mean finding an estimated answer or
using some computational short cut. They saw little or no
value in using mental arithmetic if they perceived it to
mean using traditional computational algorithms to arrive at
an answer without the use of pencil and paper.

Research and writing during the fifties was generally
directed to mental arithmetic viewed as a way to obtain
exact answers (B. I. Brown, 1957; Lentz, 1957; Moser, 1953;
Olander & Brown, 1959; Petty, 1956). There were some
writers who viewed estimation as a part of mental arithmetic
(G. Brown, 1957; Flournoy, 1957; J. V. Hall, 1954; Payne &
Seber, 1959; Sauble, 1955). Wandt and Brown (1957) found
that adults used mental computation in 75 per cent of their
non-occupational uses of mathematics. However, only 27 per
cent of the total number of uses were mental estimates.
Interestingly, adults also reported using paper-and-pencil
estimates 4 per cent of the time.
Current Support for Estimation

More recently, the ability to estimate the results of computation has been generally considered to be of great importance to the individual. Writers who are concerned about the development of good mathematical skills and about the understanding of basic mathematical concepts include estimation as an essential tool (Ibe, 1973; Nelson, 1967; O'Daffer, 1979). Writers who are interested in teaching specific computational skills often include estimation as an essential step in the exact calculation. Both Post, (1981) on teaching fractions, and McKillip (1981), on teaching the division algorithm, list estimation as a first step.

The benefits of teaching mental computation (B. J. Reys, 1985) and estimation (R. E. Reys, 1984) include the opportunity for students to develop and practice many of the skills recognized as important in mathematical reasoning. These include flexibility in thinking and working with different forms of numbers, knowledge of and selection of appropriate algorithms, efficient uses of the properties of numbers, and checks for reasonableness of answers.

Writers who are interested in the use of mathematics in everyday life commonly indicate that estimation is sometimes more useful and practical than exact computation (Bestgen, 1980; Bestgen & Reys, 1982; Kreider, 1980; Maier, 1980; B. J. Reys, 1985; Wolf, 1966). In everyday situations, an estimate may be all that is needed. In other situations, an
estimate may be matched with an exact answer. For example, when estimation is used to check calculations done with a calculator, the estimate and the exact answer are compared to check for keystroking errors or other user or machine errors.

Bell (1974, 1980) has encouraged mathematics educators to include estimation and checks for reasonableness of answers as an essential part of the curriculum. Shulte (1980) says that the basics must be broadened to include more than exact computation and should include rapid mental calculation with simple numbers and calculation with rounded numbers. Many specific examples of how to teach estimation are found in the literature of mathematics education (Bitter, 1979; Easterday, 1978; Holmes, 1975; Johnson, 1977; Koenker, 1961; Levin, 1981; McIntosh, 1980; McKillip, 1981; Musser, 1982; Page, 1970; Payne & Seber, 1959; R. E. Reys, 1983, 1985; Schmid, 1967; Trimble, 1973).

Organizations such as the National Council of Teachers of Mathematics (Edwards, Nichols, & Sharpe, 1972; NCTM, 1980), the National Institute of Education (1975), and the National Council of Supervisors of Mathematics (1977) include the estimation of the results of computation as an essential competency. The Texas Assessment Project included the ability to perform estimation in problems involving the four fundamental operations on whole numbers, simple fractions, and decimals as an essential objective at the
sixth grade level.

Although much support for the topic of estimation can be found from national and state groups, from curriculum leaders, and from researchers, there is little evidence that it has been included in textbooks (Faulk, 1962; Flournoy, 1957; Skvarcius, 1973; Wolf, 1966). Carpenter, Coburn, Reys and Wilson (1976) label the treatment of estimation in the curriculum as "cursory" and say that it appears as a separate topic which most students are poorly motivated to learn. Educators at the Cape Ann Conference on Junior High Mathematics (1973) decided that estimation was used in only three places in the curriculum, in the division algorithm, in checking answers to computation problems, and in the practice of estimation for its own sake. Even when estimation is taught, rounding usually dominates the instruction (R. E. Reys, 1984).

Student Performance on Estimation Tasks

The results of the First National Assessment of Educational Progress (NAEP) (Carpenter, et. al., 1976) do show some improvement in the ability to estimate through the years of schooling. For example, only 31 per cent of the nine-year-olds tested could correctly estimate the sum of four addends when presented in a verbal problem. The percent of correct answers increased to 54 per cent for seventeen-year-olds and to 64 per cent for young adults.
In discussing the results of the second NAEP, Carpenter, Corbitt, Lindquist, Montgomery, and Reys (1980) remark on the fact that the performance of students on an estimation exercise was lower than performance on a comparable exact computation exercise. This may indicate that students who are able to solve computation problems using algorithms have little understanding of the underlying processes and concepts.

Performance on estimation exercises in the second NAEP shows that students have a limited ability to estimate. Seventeen, fifty-two, and sixty-nine per cent of the nine-, thirteen-, and seventeen-year-olds, respectively, tested correctly estimated the answer to a problem which involved finding the difference between two four-digit numbers (Carpenter, Corbitt, Kepner, Lindquist, Reys, 1981b, p. 22). The most common incorrect answer, however, indicates that most of those who were unsuccessful may have used an estimation technique that might be considered to be "primitive." They may have solved the problem by using the first digit of each number without rounding. Schoen (1981) found that students who used front-end numbers without rounding learned to make more accurate estimates by rounding when they were given instruction.

Also included on the second NAEP were estimation problems which showed that students do not completely understand fractional concepts, although they can use basic
algorithms to find exact answers (Carpenter, et. al., 1981b, p. 36). When asked to estimate the answer to the problem $12/13 + 7/8$, only 24 per cent of the thirteen-year-olds and 37 per cent of the seventeen-year-olds answered correctly. This compares to an exact computation problem $7/15 + 4/9$ in which 39 per cent of the thirteen-year-olds and 54 per cent of the seventeen-year-olds were able to choose a correct response.

Problems involving the use of decimals also illustrate that some students do not use estimation techniques to verify their answers. In the multiplication of decimals, about 12 per cent of the thirteen-year-olds and 8 per cent of the seventeen-year-olds who multiplied correctly made an error in placing the decimal. In division using a calculator, 26, 22, and 20 per cent of the nine-, thirteen-, and seventeen-year-olds, respectively, ignored the decimal in the calculator result and gave a totally unreasonable answer (Carpenter, et. al., 1981a, p. 36)

**Improvement of Estimation Skills**

In 1947, J. V. Hall found that the number of correct responses to estimating the answers to verbal problems varied from two to forty-seven. Data from the National Assessments have indicated that some students at all levels are not proficient at estimation. Does this indicate that students are unable to learn estimation techniques? Current
research would indicate that this is not the case. Petty (1965) found that the difference between the number of correct answers produced by students who were provided a work-space and those who were not lessened as those in the second group became more accustomed to the task. Flournoy (1959), Payne (1965) and Schall (1973) found that daily practice increased the ability of students to use mental computation. Hall (1977), Ibe (1973), and Nelson (1967) each found that children given instruction in estimation were more competent estimators than those who were not given the instruction. Bestgen, Reys, Rybolt, and Wyatt (1980) found that both weekly practice and weekly instruction increased the ability of pre-service elementary teachers to estimate. The group receiving the instruction as well as practice also had improved attitudes toward estimation as an important skill. Schoen et. al. (1981) found that each of three different methods of presenting instruction on estimation was effective in increasing the ability of students to estimate. Reys, Trafton, Reys, and Zawojewski (1984) found that materials developed to improve the computational estimation skills of sixth, seventh, and eighth graders were effective in improving these skills. Thus, it would seem that estimation is a skill that can be taught.
Effect on Other Skills

Does estimation interfere with or reinforce other skills? Does the time spent on teaching estimation skills affect the achievement of students in other mathematical areas? Current research seems to have mixed answers to these questions. In an early study, Dickey (1934) found that students who estimated the answers to verbal problems before solving them performed no better than students who did not do so. He did notice that students of superior ability seemed to benefit more from the estimation than did students of lesser ability. Both of these findings were supported by the findings of W. D. Hall (1977). Nelson (1967) found that fourth graders who had been taught to estimate did not score as high as those who had not been taught estimation techniques. She speculated that those who had been taught to estimate might have been using those skills and used more time on the test of exact computation which was timed. This, of course, lowered their scores. This effect was not found among students at the sixth grade level.

Even when the time for instruction came from the regular arithmetic class period, those who received instruction in estimation achieved as well as those who used the full period working on pencil-and-paper computation (Payne & Seber, 1965; Schall, 1973). Austin (1970), Ibe (1973), and Rea and French (1972) found that instruction in estimation and mental computation produced increases in
achievement on standardized paper-and-pencil tests. However, Schoen et. al. (1981) found no evidence that instruction in estimation influenced exact computation.

Relationship to Other Skills and Characteristics

How is estimation related to other skills or characteristics of the estimator? J. V. Hall (1947) and Paull (1972) found no difference between the sexes in the ability to estimate. B. I. Brown (1957) and Reys, et. al. (1980, 1982) found that boys were better estimators than girls. Rubenstein’s work (1985) supported this finding. She noted that boys were markedly better than girls at order of magnitude tasks.

J. V. Hall (1947) found that intelligence was a characteristic possessed by good estimators, but B. I. Brown (1957) found no closer relationship between general intelligence and mental arithmetic ability than between general intelligence and ability to compute with pencil and paper. However, she did find a correlation of .65 between general arithmetic ability and the ability to compute mentally. Levine supported this finding when she wrote that "quantitative ability is most closely related to estimation ability" (1982, p. 358). Rubenstein (1985) found that the mathematical skills which contributed most to the prediction of estimation performance were operating with tens (an especially strong relationship), making comparisons, and
judging relative size. Paull (1972) found that the ability to estimate answers to arithmetic problems was positively correlated with mathematical and verbal ability and with the ability to solve problems by trial and error.

This last finding is likely to be related to the finding of Reys, et. al. (1980) that good estimators were tolerant of error in their estimates in the sense that they did not mind being farther from the exact answer than some other estimators. Other information about good estimators gathered by Reys, et. al. included the following: good estimators (1) can accurately and quickly recall the basic facts for all operations, (2) understand the role of place value in determining the results of computation, (3) can quickly and efficiently compute with multiples of ten or a limited number of digits, (4) possess a knowledge of and use number properties such as the distributive property and the order of operations, (5) are not afraid to use another strategy if the first one chosen seems to be unproductive, and (6) are confident in their own ability to estimate answers. Both Levine (1982) and Reys, et. al. (1980) found that better estimators tend to use a greater number of different strategies.

Comments from Levine (1982) on college-age estimators of lower ability indicate that most of them used a strategy that she labeled "proceeding algorithmically." This would indicate that they had developed few, if any, of the
techniques that better estimators used. Schall (1973) reported a teacher's comment that slower students had trouble keeping up with instruction in mental arithmetic when it was presented on closed-circuit television. Lawson (1977) found that students of the lowest computational ability made the greatest number of mistakes in calculation when using a calculator. He felt that this indicated that students did not use estimation as a means of validating answers produced with a calculator.

Strategies for Estimation

The question of how students produce an estimate has rarely been studied. B. I. Brown (1957) examined the techniques used by students in mental subtraction. She expected students to produce an exact answer rather than an estimate. Buckley (1974) investigated the production of answers to addition and subtraction of one- and two-digit numbers to determine if they were retrieved from memory or constructed. He too expected exact answers.

Two studies have been done which consider the strategies used by students to estimate the results of computation problems. Levine (1982) reported finding eight different strategies for estimating the results to multiplication and division problems involving whole numbers and decimal fractions. These eight strategies involved the use of fractional relationships, exponents, rounding one or
both numbers, powers of ten, known products or quotients, incomplete partial products or quotients, and the traditional algorithms for computation.

Reys, et. al. (1980) identified and discussed three key processes which good estimators use to estimate answers to computational problems involving whole numbers, decimals, and a few fractions in both straight computation and application problems. These three processes are reformulation, translation, and compensation. These researchers also discussed the front-end strategy identified and discussed by Trafton (1978).

Reformulation is the process of changing the numerical data of a problem so that it is more manageable mentally. Examples of strategies related to reformulation are the front-end use of numbers, rounding, use of compatible numbers, and using equivalent forms of numbers.

Translation is the process of changing the form of the problem so that it is more manageable mentally. Examples of strategies related to translation include changing the order of computation and changing to an equivalent operation.

Compensation is the process of adjusting the computation to reflect variations that came about as a result of translation and/or reformulation. This process may be used at two different stages of estimation. Numbers in the problem may be adjusted during the intermediate
stages of estimation, or the final answer may be adjusted so that it is closer to the exact answer.

Reys et. al. found four variations on Trafton's front-end strategy. These were rounding and using the original number of digits, rounding and using an extracted number of digits, truncating and using the original number of digits, and truncating and using an extracted number of digits.

It should be mentioned here that Levine (1981, 1982) used college students who were not mathematics majors in her study, Reys, et. al. (1980) used students and adults who were high achievers and more likely to be identified as good estimators in their study, and Rubenstein (1983, 1985) used eighth grade students.

Summary

Although educators have not always thought that estimation should occupy an important place in the curriculum, it is currently thought to be an essential skill for everyone. In the mathematics classroom, estimation contributes to the development of good mathematical skills and the understanding of basic mathematical concepts and is thought by some educators to be an essential step in exact calculations. Estimation may actually be more useful than exact computation for some situations. Mathematics educators, both as individuals and in formal organizations,
have encouraged the inclusion of the topic in the curriculum. Yet, there is evidence that it is not covered in very many textbooks or curriculum guides.

Measures of student performance such as the National Assessments of Educational Progress have shown that some students at all levels are not proficient at estimation tasks. But these same data show that there is improvement in the skills of estimation over the years of schooling. Researchers who have investigated the possibility of improving the skills of students have found that it is possible to improve both the skills that students possess and their attitude toward estimation simply by providing instruction. The effects of this instruction on other skills have been reported to be mixed. Some researchers found improvement; others found no effect.

The relationship of estimation ability and other characteristics of the estimator has been investigated, and the characteristics of good estimators have been described. Strategies used by good estimators and those used by college-age estimators have been identified. The relationship between the type of estimation task and the degree of difficulty has been explored. The necessity of improving both the quantity and the quality of instruction in estimation skills leads to the question of what skills estimators of limited ability possess when they have had no special instruction in estimation.
CHAPTER BIBLIOGRAPHY


CHAPTER III

PROCEDURE FOR ANALYSIS AND COLLECTION OF DATA

Population and Sample

The students at the Arts Magnet High School at Booker T. Washington are mostly from the Dallas Independent School District. Approximately 12 per cent of those enrolled are tuition students from surrounding suburban school districts. They are all attracted to the school because of their interest in one of the four areas of concentration offered there: music, dance, theater, or visual arts. Although the academic program at the school is varied and its reputation is a factor in the decision of many students to attend the school, superior academic achievement is not a criterion for attendance at the school. When the Test of Achievement and Proficiency was administered in 1985, 53 per cent of the students in grades nine through twelve scored at or above the national median on the mathematics portion of the test. These students are considered by the Dallas Independent School District to be on grade level. Thirty-eight per cent of the students district-wide scored at or above the national median on the mathematics portion of the test.

The necessary permissions were obtained to administer the Assessing Computational Estimation Test (ACE Test) to
all students enrolled in mathematics classes at the Arts Magnet High School. Four hundred sixty students took the test. Sixteen of them produced papers that were considered unacceptable because of incompleteness or because of scratch work on the paper. The students were enrolled in the following mathematics classes: consumer mathematics, pre-algebra, first year algebra, second year algebra, geometry, trigonometry/analysis, computer literacy, computer mathematics, and calculus.

Students whose scores on the ACE Test fell between the first and second quartiles were eligible for the interview sample. A selected sample of twenty students was chosen from this group after a conference with their teachers and counselors during which attempts were made to identify those students who were most likely to be able to communicate their thought processes during the interview. Among the factors considered was the student's ability to communicate mathematics to other students as demonstrated by explanations of problems at the board or in small groups. The student's willingness to cooperate and availability for the interview were also taken into consideration.

Instruments

The Assessing Computational Estimation Test was developed by Reys, et. al. for use in their 1980 study of the characteristics of good estimators. It is made up of
twenty-eight computation problems and twenty-seven application problems. A copy of the test may be found in Appendix A.

It was pilot tested with approximately two hundred students per grade level in grades six through twelve. Items which had a discrimination index of less than .30 were revised or discarded, resulting in a final test on which nearly all the items showed a difficulty level between .30 and .60 during the field tests. Test-retest reliability for the final form of the ACE Test ranged from .74 to .86 in grades seven through twelve. (Reys, et. al., 1980, p. 22)

A copy of the interview problems may be found in Appendix B. It contains fourteen problems, five computation problems and nine application problems. The first five of these problems are from the ACE Test.

Method

The ACE Test was administered by using a slide projector and projecting all items for twelve seconds each except computational items numbered three, nine, ten, fifteen, and sixteen. These were projected for seventeen seconds. A three-second time period was allowed between items for recording of the answers. During this time, the number of the next question was shown. Answers were recorded on a standard answer sheet.

During the interview, the student was shown each
problem and asked to think aloud as he produced an estimated answer. He was allowed to produce the estimate with no further prompting. If he did not think aloud, he was asked to recall his strategy once the estimate had been produced. It was hoped that this combination of think-aloud and stimulated recall would produce more complete data while not interfering with the thought processes of the student.

Twenty interviews were conducted during which the researcher did not ask students to produce a pencil and paper approximation for any of the problems. The data from these interviews were analyzed using the methods described below.

A second sample of twenty students was then selected from the four hundred forty-four students who had produced acceptable papers, and interviews were conducted during which the students produced mental estimates for all problems and pencil-and-paper estimates for six problems. Immediately after a mental estimate had been produced for each of items two, four, five, eight, ten, and twelve, the student was asked to produce an estimate of the correct answer using a pencil and paper. Again, he was asked to think aloud as he performed this task and prompted to recall what he did if he did not think aloud. Students who refused to use the pencil or who could use the pencil only to produce an exact answer were allowed to go on to the next problem.
Each interview was tape-recorded. The estimate and the strategy used to produce the estimate, the pencil-and-paper estimate, and the strategy used to produce the pencil-and-paper estimate, and any comments for each item were written on an interview summary sheet after the interviews were all completed.

Research Design

This study was designed to add to the knowledge of how students think about mathematics. It was not, therefore, quantitative research. It was intended to extend the breadth of knowledge about students' computational estimation skills to a group hitherto not considered.

The data collected in the study included verbal reports. Garfalo and Lester (1985) point out that asking a person to verbalize while performing a task may affect his cognitive processing if he is asked to attend to information not normally heeded in the performance of the task. However, Ericsson and Simon write that when subjects think aloud about information they are already heeding in accomplishing the task they are attempting, the "thinking aloud will not change the course and structure of the cognitive processes" (1980, p. 227). They do mention that the speed of processing may be affected in this situation.

Even when the subject is asked to recall information immediately after the completion of the task, heeded
information will still be available in the short-term memory which allows the direct reporting of the processes used earlier and facilitates the retrieval of additional information stored in the long-term memory in the episodic associations that were formed when the information was originally heeded in performing the task (Ericsson & Simon, 1980).

Some of the subjects interviewed were obviously nervous during the interview. One subject was shaking when the interview began, and more than one exhibited speech patterns and nervous laughter that indicated their tension. This tension may have affected their answers.

The data collected consisted of the scores on the ACE Test, tapes of the interviews, written work produced by the students during the interviews, and the interview summary sheet produced after the interviews were completed.

Procedures for the Analysis of the Data

The students' interview estimates were compared to the ACE Test estimate for the five problems which appeared on both. Two outcomes were of particular interest. When the interview estimate was the same as the ACE Test estimate, it was assumed that the same strategies were used to produce both estimates. When the student gave an estimate during the interview and did not give one during the test, it was assumed that time limitations of the ACE Test were
responsible for the omission rather than the lack of an available strategy.

The recording of each interview was transcribed. Two copies of it were made. One copy was filed intact. The other copy was separated into sections by problem number and stored in folders labeled with the problem numbers.

The techniques used for data analysis in this study were adapted from those discussed by Goetz and LeCompte (1984). The first step in analyzing the data was to read the transcript of each student's attempt at estimating the answer to problem one and to make notes indicating first impressions of the strategies used. These notes were kept on a separate sheet of paper. The transcripts for all students were read before a second reading was started.

On the second reading, notes were made on the transcript itself. The transcripts were separated into stacks according to the strategies used. Making a decision about which stack a transcript belonged in often clarified in the mind of the researcher which strategy a particular student used.

A strategy summary sheet was written to show the number of students who used each type of strategy and whether their answers were in the acceptable range for good estimators. During this process, the strategies used by the subjects in this study were compared to the strategies used by good estimators in the study by Reys, et. al. (1980) and to those
found by Levine (1980, 1982). Transcripts were reread during this process to insure that the strategies used were labeled correctly and completely.

Finally, an interview summary sheet was prepared. This was in the form of a grid with subject names across the top and problem numbers down the side. Strategies used by each subject were recorded in the appropriate block. Any uses of reformulation, translation, or compensation were noted.

After three problems had been entered on the interview summary sheet, a "naive expert" was consulted. She is a teacher who has had considerable experience with students at the remedial level of mathematics, but little or no experience with the literature of mathematical thought as it concerns estimation. The researcher holds her in high regard and listened carefully when there were any differences in interpretation of strategies used. Independently, she described the strategies used by ten students on each of the first three problems. Then she discussed with the researcher her classification of these strategies. This discussion continued until agreement between researcher and expert was reached. There was disagreement in five of the thirty cases considered. The two assessments of strategies used were completely opposite in only one situation. In this case, the researcher decided that the expert was correct. In three of the others, the expert's assessment gave additional insight to what the
researcher had seen in the transcript. And after
discussion, it was decided that the researcher and the
expert were using different terms to describe exactly the
same strategy in the fifth case of disagreement.

After all the problems had been entered on the
interview summary sheet, the strategy summary sheet for each
problem and the interview summary sheet were compared.
Notes were made at the bottom of each subject's column on
the interview summary sheet to indicate the number of
different strategies used, what they were, and any uses of
reformulation, translation, or compensation. This
facilitated the discussion of types and numbers of
strategies and processes used by subjects at this level of
estimation skill.
CHAPTER BIBLIOGRAPHY


CHAPTER IV

ANALYSIS AND DISCUSSION OF DATA

The purpose of this study was to describe the estimation strategies used by high school students with limited computational estimation ability when they made mental estimates and to determine whether they used the same strategies when they used a pencil and paper to make an estimate. The Assessing Computational Estimation Test (ACE Test) was administered to class-sized groups of students by using a slide projector to project the problems on an overhead screen. Students were given from twelve to seventeen seconds to make their estimates.

Students who scored between the first and second quartiles were eligible for the interview sample. Students were grouped by their scores on the ACE Test, and the same number of students from each score level were selected to be interviewed. The students interviewed were chosen after conferences with their teachers during which attempts were made to identify those students who were most likely to be able to communicate their thought processes during the interview. The first group of students interviewed produced only mental estimates; the second group was asked to produce mental estimates for all fourteen problems followed by pencil-and-paper estimates for six of those problems. Both
groups of students interviewed were selected in the same way.

During the interview, the student was shown each problem and asked to think aloud as he produced an estimate for that problem whether the estimate was produced mentally or with pencil and paper. The interviews were tape-recorded and transcribed. The data discussed below were collected from those transcripts and from the papers produced by the students in making pencil-and-paper estimates.

The discussion of these data will begin with a comparison of the performance of the population of this study with the performance of the population studied by Reys, Bestgen, Rybolt, and Wyatt (1980) in their attempt to identify characteristics of good estimators. Then there will be a discussion of the five problems that were common to the Assessing Computational Estimation Test and the interview.

The interview problems will be discussed individually. First, those problems on which all students made only mental estimates will be discussed, and then those on which some students made both mental and pencil-and-paper estimates will be discussed. For each problem, the number of students using each strategy will be given in table form. Then the use of those strategies on that particular problem will be explained.

For problems where pencil-and-paper estimates were also made, a second table will list the number of students using
each strategy, and a third table will show how many students changed strategies to produce their pencil-and-paper estimates. Both of these tables will be discussed.

Four hundred sixty students took the Assessing Computational Estimation Test. Sixteen of these produced papers that were considered unacceptable because of scratch work on the side, because of a confusion about numbering, or because the student did not take the whole test. The mean computation score for the papers considered acceptable was 8.6 with a standard deviation of 4.4. The mean application score for these papers was 10.3 with a standard deviation of 5.5. The lowest score made was a total score of zero, and the highest score was a total score of forty-eight.

Scores of the students chosen to be in the interview sample ranged from a total score of 12 to a total score of 18. In this sample, there were six students with total scores of 12, 15, 16, 17, or 18, and five students with total scores of 13 or 14. In all, there were 40 students who were interviewed. The mean computation score of the interview sample was 6.8 with a standard deviation of 3.7. Their mean application score was 8.3 with a standard deviation of 4.5.

These numbers compare with a computation mean of 13.8 with a standard deviation of 4.4 and an application mean of 15.1 with a standard deviation of 5.2 for ninth grade students in the study of good estimators done by Reys, et.
al. (1980, p. 47). These researchers did a comparison of their sample with ninth graders in the Hazelwood School District (an upper middle class suburb of St. Louis). The Hazelwood ninth graders had a computation mean of 8.8 with a standard deviation of 4.0 and an application mean of 11.3 with a standard deviation of 5.9.

This comparison indicates that the students in this population are similar to those in the Hazelwood School District and below those identified as potential "good estimators" by Reys, et. al. in their ability to estimate the answers to computation and application problems.

The students who were selected to be in the interview sample were enrolled in the following mathematics classes: pre-algebra (9 students), consumer mathematics (1), Computers for Everyday Living (4), first-year algebra (13), second-year algebra (4), geometry (7), computer mathematics (1), elementary analysis (1).

Comparison of ACE Answers and Interview Answers

The first five interview problems were also on the ACE Test. Table I shows a comparison of the responses on the ACE to the responses in the interview. The column labeled "Total" shows the total number of students from the sample who gave an answer to each of these five problems on the ACE Test. The second column shows the number of those who changed from an unacceptable answer on the ACE Test to an
acceptable answer during the interview. The third and fourth columns give the number of students who did not give an answer on the ACE Test but did give an answer during the interview. The third column tells how many gave acceptable answers during the interview; the fourth column tells how many gave unacceptable answers during the interview.

For example, on problem one, thirty students gave an answer on the ACE Test. During the interview, five of them were able to give acceptable estimates to this problem. Of the ten students who gave no answer on the ACE Test, one was able to give an acceptable answer during the interview, and eight gave unacceptable estimates. One student did not give an estimate for problem one either on the ACE Test or during the interview.

As Table I shows, some of the students who gave unacceptable answers on the ACE test were able to change their answers to acceptable estimates during the interview. Two students gave acceptable answers to problem number two on the ACE Test, but gave incorrect answers during the interview. Only one student gave an acceptable answer on the ACE and then gave the same answer during the interview, and she did this for only one problem.

For each of these five problem, 50 per cent or more of those who gave no estimate on the ACE Test subsequently attempted to produce an answer during the interview. Although the number who were able to produce correct answers
in this situation is small for some problems, the number who attempted to answer is worthy of note. It is almost surely an indication that when the time pressure was removed, the students were willing to attempt to estimate. This may have implication for teaching estimation skills to these students.

**TABLE I**

**COMPARISON OF ACE ANSWERS AND INTERVIEW ANSWERS**

<table>
<thead>
<tr>
<th>Interview Problem Number</th>
<th>Gave an Answer on ACE</th>
<th>Gave No Answer On ACE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td>Changed It to an Acceptable Answer in Interview</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>5</td>
</tr>
</tbody>
</table>

With the exception of problem two, allowing the students to have as much time as they needed increased the number of correct answers considerably. Again, students with this level of estimation ability may simply need more time to produce an estimate.

Although many of the students interviewed had attempted to answer problem one on the ACE Test, and all but one of those who had not given an answer on the written test attempted to give one in the interview, most of the students were unsure of themselves early in the interview. Problem two caused confusion because some students were not
comfortable with having three factors in a multiplication problem. This is evidenced by the fact that only one of the forty students gave an acceptable estimate during the interview, and six students gave no estimate at all.

In considering problem three, it was expected that students of limited estimation ability would have difficulty in estimating quotients. That they could not do so rapidly is indicated by the small number who attempted this problem when the test was timed. However, this problem seemed to relieve some of the tension felt by some students. They knew how to attack it. One student in particular settled down to the task of the interview after this problem. Although she produced an unacceptable estimate, she attempted all but one of the remaining problems. She had not produced an estimate for either problem one or problem two.

It was on problem three that the greatest change in the number of answers attempted occurred. This may have been due to the fact that estimation of partial quotients is such an integral part of the division process.

Problem five contained an unfamiliar mix of fractions and decimals and among these five problems resulted in the greatest number of "no answers," both on the ACE test and during the interview.

The information in Table I shows that some students had estimation strategies that they did not use during the timed
test, but did use during the interview. This would indicate that students of this ability level may need to be given more time to make estimates.

The first group of problems that will be discussed are those for which the answers were estimated only mentally. Problem one will be the first such problem discussed.

Problem One
(acceptable interval 400,000 to 500,000)

\[
\begin{align*}
87,419 \\
92,765 \\
90,045 \\
81,974 \\
\hline
\frac{98,102}
\end{align*}
\]

First, a comment about problem one is in order. The numbers in the problem may have overwhelmed some students. It caused comment every time it was projected during the ACE testing. During the interview, some students had difficulty that may have been caused by the size of the numbers. For example, one student added 98 and 81 to give eighteen million. Obviously, he meant 180,000. He changed his answer to 17 or 18 thousand. When he was asked to go through the process a step at a time, he produced an estimate of four hundred thousand. The very size of the numbers seemed to intimidate him.

Even when she was able to use an effective strategy, another student had trouble saying her answer. She gave 460 as the sum of the first two columns of numbers. But her
confusion with numbers of this size was indicated when she tried to give the full answer. She said, "Four hundred thousand and sixty." Then she corrected that to "four hundred thousand sixty thousand." Finally, she was able to produce "four hundred sixty thousand."

Another's answer was expressed once as "twelve hundred thousand" and then as "thirty-two hundred thousand." These both indicate a problem in dealing with numbers of this magnitude.

A look at a fourth student's estimate also indicates some real problems with large numbers. She added the ten thousands column and then guessed the rest. She made a good start which rapidly deteriorated. Her answer was "forty-five, one thirteen, one sixteen." Assuming that she had simply stated her estimate incorrectly, the interviewer wrote the answer as "45113116" on the interview sheet. She did not react at all to its improbability.

Ten of the forty students who were interviewed gave no answer to problem one on the ACE Test. Only six students were able to give an acceptable answer during the interview. Nine students gave answers that reflected a power-of-ten error. A power-of-ten error was made when the answer given would have been acceptable if it had been multiplied by an appropriate power of ten. For example, a student who answered 46,000 for problem one made a power-of-ten error because her answer would have been acceptable if it had been
multiplied by ten. The student who answered 35,000 to this problem also has the wrong order of magnitude, but simply multiplying by a power of ten would not bring the answer into the acceptable range. It was usually the case in this study that a power-of-ten error indicated a combination of the use of an effective estimation strategy and a lack of the understanding of place value needed to produce an acceptable answer.

The number using each strategy can be found in Table II. The number producing an estimate which would be considered acceptable on this problem includes students who produced estimates from 400,000 to 500,000. Although Reys, et. al. considered the range for this problem to be 430,000 to 460,000, this range was expanded after some of the strategies used by estimators in the interview sample were examined closely. For some of the strategies described below, 500,000 or "400 something thousand" seemed logical and acceptable answers.

Further, on this and the other computation problems, students were given no purpose for the estimate and some may, therefore, have felt no reason to give a closer estimate. This conversation illustrates the point.

Student: "You want a really rough estimate of this?"
Interviewer: "However close you can come easily."
Student: "They're all close to a hundred thousand so it would be five hundred thousand."
Also found in Table II is the number of students whose answers were off by a factor of some power of ten. In this problem, this was usually the first power of ten.

\[
\text{TABLE II} \\
\text{STRATEGIES USED ON PROBLEM ONE}
\]

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rounded Up</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>To Nearest 5000</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>To Nearest 1000</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Averaged</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To 100,000</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>With Compensation</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Did Not Round</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grouped</td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Added 1 Column</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Added 2 or 3 Columns</td>
<td>6</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Used Exact Computation</td>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Method Not Clear</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Students who rounded did not all round in the same way. Two rounded to the nearest thousand and used two digits from each number to arrive at their estimate. One rounded the ninety thousands to one hundred thousand and the eighty thousands to ninety thousand. Another rounded to the nearest five thousand and then used the number of digits in the problem to produce her answer. Although all of these student knew about rounding, they did not all round in what might be called the standard way. They chose different numbers of digits to work with and they rather loosely interpreted the rules for rounding. This may show an understanding of the
process as it applies to estimation beyond the rule "Look at the next digit and add one if it's five or more."

When students did not round, they were likely to add the first one or two columns or to group the digits in the first column as \((3 \times 9) + (2 \times 8)\). Often their calculation here was complicated by the fact that some of them did not recognize their own computational limitations. Some of those who arrived at twenty-seven and sixteen by the grouping above could not mentally add these two numbers. Sample results of this addition included thirty-two, thirty-seven, and thirty-four. It should be noted that sixteen plus twenty-seven is forty-three which is thirty-four with the digits reversed. Sometimes the unfamiliar task of talking aloud may have interfered with thought processes.

Students who used averaging looked at the set of numbers and said, "They're all close to one hundred thousand or to ninety thousand and there are five of them so the answer must be five hundred thousand." Only two of these students were able to use compensation.

All of the strategies described above were used by good estimators in the study by Reys, et. al. Five students in this study seemed to have no estimation strategy available and attempted to use exact calculation to produce an estimate. Although two of them were able to produce an acceptable estimate and one more was off by a power of ten, this lack of strategy was not exhibited by good estimators.
The last group of students were those who did not clearly state how they arrived at their answers. On problem one, there were five students whose strategies could not be determined from what they said. Two of them gave acceptable estimates and a third was off by a power of ten. One student gave no answer at all.

There were three general processes used by good estimators described by Reys, et. al. These were reformulation, translation, and compensation. Students of limited estimation ability in this study used strategies which can be classified as examples of reformulation, translation, and compensation.

On the first problem, students who rounded or used the first one or two digits of the numbers given were reformulating the numerical data in the problem. Students who used the averaging strategy translated the form of the problem to a more easily managed form. The three who used compensation used final compensation which means they adjusted their final answers rather than making adjustments in the numbers as they went along. This was best expressed by one of the better estimators interviewed. "Since they are all around a hundred thousand, give or take ten thousand, I'd just add up a hundred thousand and then subtract about forty thousand or sixty thousand." His estimate was 440,000.

A word must be said here about the difference between compensation and adding one more column to adjust an answer.
A number of students found the sum of the first one or two columns and then added the next column of digits to decide how to alter their answers. At least one student added the digits in the ones column and used that sum to adjust her estimate. This was not considered compensation because it was a rote kind of adjustment more closely related to using exact calculation than to using a strategy for estimation.

Of the students interviewed, twenty-two used a strategy which might be classified as a type of reformulation. Seven used the averaging strategy which is a type of translation. And three used compensation.

It is interesting to note that of the students using rounding or truncation, only one used the same number of digits as the problem contained, and she made a calculation error which prevented her from producing an acceptable estimate. All others using these strategies used an extracted number of digits (usually the first one or two) in their calculations.

In comparing the strategies used by estimators in this study to those used by good estimators, it should be noted that no good estimators used the process of trying to add each column of numbers and remember its sum. Good estimators rounded and then grouped; these estimators truncated and then grouped. Fifteen of the thirty-five good estimators in grades nine through twelve used the averaging strategy; only seven out of forty estimators in this study used averaging.
Students made both mental and pencil-and-paper estimates to problem two. That problem will be discussed later in this chapter. Problem three is the next problem on which only a mental estimate was made.

Problem Three

(acceptable interval 50 to 62)

\[ \begin{array}{c}
8127 \\ 474,257
\end{array} \]

For problem three in particular, it was difficult to decide whether a student was merely using exact calculation or was using an estimation strategy. Perhaps this was true because the estimation of the partial quotient is a necessary step in the division process. The decision was made by reading carefully the language of the students as they described what they did.

For example, one student was classified as a student who used exact calculation on the basis of this part of the transcript of her interview: "It would be 8,127 into the first five, one, two, three, four, five, numbers because it wouldn't go into 4,742. So that would probably go about two times. So it would be two and then you would have to bring that down and add a few zeros." Another student was classified as a student who used truncation because she said, "Eight goes into forty-seven about five times....I just took the first number and put it into the first two."
### TABLE III

**STRATEGIES USED ON PROBLEM THREE**

<table>
<thead>
<tr>
<th>Used Division Process</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td>5</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Used Truncation</td>
<td>9</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Used Exact Calculation</td>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Changed to Multiplication</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Used Truncation</td>
<td>7</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Used Different Numbers</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Guessed, Gave No Answer, Used Wrong Process, or Method Not Clear</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>11</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Students who rounded worked sometimes with the same number of digits (four students in all), sometimes with an extracted number of digits (one student). One student named the numbers she worked with as 8,000 and 474,000. She was using the same number of digits found in the original problem. Another student said that he worked the problem "by taking that up to 500,000 and this down to 8000 and then dividing, thinking eight goes into fifty six times." He was using an extracted number of digits to make his estimate.

Some students thought of the division problem in terms of the corresponding multiplication sentence. The most common multiplication sentence used was five times eight equals forty. Students who used this sentence decided that the first digit of the quotient must be five. One student said, "Six times eight is forty-eight, and this problem has
a forty-seven at the beginning. So I said, well, five is
the next to largest."

Another student asked herself, "Eight times what is
forty-five?" This may have been an attempt at intermediate
compensation since she adjusted both the divisor and the
dividend, and forty-five is not a multiple of eight.

Only one student rounded the numbers involved and then
used the same number of digits to answer the question
"8000 x __ = 500,000?" Nine students in all used the
procedure of changing the division problem to the
corresponding multiplication sentence.

Twenty-four students altered the numbers in problem
three in such a way that their strategies could be consid-
ered to be the use of reformulation -- they rounded, used
truncation, or used different numbers. Nine students
thought of the problem as a multiplication problem instead
of a division problem. This procedure might be thought of
as a use of the process of translation, or it might be
considered their standard way of approaching a division
problem.

Only three students used compensation on problem three.
None of them were able to use it successfully. As a matter
of fact, all three of them had a good start at a correct
answer, but ended up with an answer that was unacceptable.

One error that was made more than once deserves
comment. Four students gave answers that indicated that
they might have looked at the problem as eight divided by four instead of forty-seven divided by eight. One said, "Because it's 474,000 and then there's 8,000 approximately. And then you just divide the 400,000 in half." Her answer to this problem was 200,000. It is possible that this confusion was caused by the unaccustomed thinking aloud.

One remarkable difference between good estimators and those in this study was the number who thought of division in terms of the related multiplication sentence. Only two of the thirty-four good estimators in grades nine through twelve used this strategy (Reys, et. al., 1980, p. 186). Nine of the forty interviewed in this study still thought of division in these terms. Other strategies used here were also used by good estimators, and they did not use any strategies that these estimators did not use.

Problems four and five required both mental and pencil-and-paper estimates. They will be discussed later in this chapter. Problem six is the next problem which required only a mental estimate.

Problem Six
(acceptable interval 1000 to 1500)

About how much area does this 28
rectangle have? 47
Students who rounded one factor and not the other all rounded forty-seven to fifty. They did not all handle the twenty-eight in the same way. Two of them thought of it as twenty; the third used it as twenty-eight. This tendency to round some of the factors but not all of them was also exhibited in problem two.

TABLE IV
STRATEGIES USED ON PROBLEM SIX

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded One Factor</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Same Number Digits</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Extracted Number Digits</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Rounded Both Factors</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Same Number Digits</td>
<td>9</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Extracted Number Digits</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Truncated</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Same Number Digits</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Extracted Number Digits</td>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>12</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Method Unclear or</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No Answer Given</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It was not possible to produce an estimate considered acceptable for good estimators by rounding forty-seven to fifty and multiplying by twenty. However, since Reys, et al. accepted 1200 which was produced by rounding forty-seven to forty and twenty-eight to thirty, 1000 was considered acceptable for the purposes of this study.

Students who rounded both factors were generally successful in producing an acceptable estimate. All but one
of those who produced unacceptable answers using this strategy missed by a power of ten.

On problem six, students who used front numbers without rounding were not able to produce an estimate considered acceptable for good estimators. However, five of them produced estimates between 800 and 900, and the other four produced estimates which were out of this range by a power of ten. Since the use of truncation may be thought of as a primitive estimation strategy, these students obviously possess a strategy for dealing with this problem.

In addition, six of these students used final compensation to adjust their answers. It was particularly clear on problem six that students of limited estimation ability sometimes have the feeling that some sort of compensation should be used, but lack the number sense to make them successful in its use. Students in this group who used compensation added from twenty-three to one hundred to their answer of 800 to produce an adjusted estimate. One who started with an estimate of 8000 (off by a power of ten) adjusted his estimate to 8500, and another whose first estimate was eighty changed it to eighty-nine.

Only one other student used compensation on problem six. She was a student who had rounded one factor and not the other. She said that her estimate would be "a little over a thousand" because she had multiplied by twenty and the number was really twenty-eight.
The number of students who attempted to use exact calculation is remarkable. No good estimators attempted to use exact calculation, and some of the students in this study who used it on problem six had already demonstrated that they knew how to use rounding which would have simplified their task considerably.

None of the four who produced an acceptable estimate using this technique produced an exact answer. This shows that the factors were too large for these students to be successful in using mental multiplication.

Finally, a comment must be made about problem six itself. The interviewer quickly began using the question "Do you remember how to find the area of a rectangle?" because she discovered that many of the students were confusing area and perimeter. Of the forty students interviewed, twenty-two indicated some confusion when asked how to find the area of a rectangle. This confusion ranged from simply not remembering how to find the area to confusion with finding the perimeter to statements like "Either it's twenty-eight plus forty-seven, or it's twenty-eight times 3.14."

There were no strategies used on problem six which might be classified as translation of the form of the problem. It was viewed as a straightforward multiplication problem by all students. Those who rounded one or both factors and those who used truncation were using strategies
which might be classified as reformulation of the numerical data.

The next problem which required only a mental estimate was problem seven.

Problem Seven

(acceptable interval 30,000 to 36,000)

If 30% of the fans at the 1979 Superbowl bought one soda, about how many sodas were bought at the game?

<table>
<thead>
<tr>
<th>Year</th>
<th>Attendance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>73,655</td>
</tr>
<tr>
<td>1975</td>
<td>86,421</td>
</tr>
<tr>
<td>1976</td>
<td>91,943</td>
</tr>
<tr>
<td>1977</td>
<td>96,509</td>
</tr>
<tr>
<td>1978</td>
<td>93,421</td>
</tr>
<tr>
<td>1979</td>
<td>106,409</td>
</tr>
</tbody>
</table>

Problem Seven was impossible for those who did not understand the concept of percent as it relates to parts of one hundred. Students who were successful can be divided into two major groups, those who thought of the problem as some form of 30 per cent times 100 or 100,000, and those who related 30 per cent to some other per cent such as 25 per cent or 50 per cent which they evidently found easier to use.

Two students in the first group thought of 30 per cent in its decimal form. Both of them used an extracted number of digits from 106,409 in multiplying. Neither was successful, although one was off only by a power of ten.
A much more successful way of thinking of this problem was finding 30 per cent of 100 or of 100,000. This is a reasonable strategy because it is based on understanding the concept of percent as a part of 100. Students who used an extracted number of digits were likely to make a power-of-ten error.

TABLE V
STRATEGIES USED ON PROBLEM SEVEN

<table>
<thead>
<tr>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Same Number Digits (30% of 100,000)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Extracted Number Digits (30% of 100)</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Used truncation</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Compared to 50% of 100,000</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Compared to 25% of 100,000</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Used wrong procedure</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Changed to 1/3 of 90,000</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Attempted exact calculation</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>No Answer, Guessed, or Method Unclear</td>
<td>16</td>
<td>0</td>
</tr>
</tbody>
</table>

A rudimentary understanding of percent was exhibited by those who compared 30 per cent to 50 per cent or to 25 per cent. Often, they knew that 50 per cent was one half so they could adjust downward from 50,000 or 53,000. That this understanding was not fully developed is illustrated in this excerpt of an interview. "Fifty per cent of those people would be 50,000, about 53,000. And so 30 per cent would be twenty thousand."
Another excerpt clearly shows the thinking of the student who said, "Half of that would be approximately 50,000. And then 30, I might cut 50,000 in half which would be 25 per cent which would be 25,000. And 30 per cent would equal .... I'd say 30,000."

A third student started with 25 per cent and adjusted her answer into the acceptable range. She seemed comfortable with the fact that four 25's make 100.

The difference between these two groups illustrates the unevenness of the development of computational estimation skills. Thinking of this problem in terms of 30 per cent of 100 probably indicated a deeper understanding of the concept of percent than comparing 30 per cent to 50 per cent. Yet a larger percent of the students who used the latter strategy were successful. The others tended to make power-of-ten errors, possibly because of their less than perfect understanding of place value.

Four students attempted to use division to solve this problem. This is probably because they were not able to calculate percent of a number even with pencil and paper. The discussion of problem twelve verifies this statement. This outcome should create more concern than those who simply said that they never could do percent anyway. These students think they possess a procedure for solving percent problems, and they do not. A teacher who wants to teach them a correct procedure must first erase their incorrect
ideas and then teach them the new procedure. Simply overlaying the new procedure will not be enough.

Nine students simply responded that they did not know how to do percent. Five others simply guessed. A teacher who wants to teach them the correct procedure for solving this kind of problem will have to overcome their conviction that they cannot do percent problems, but will not have to correct previous misconceptions about method.

Students who rounded or used truncation and those who changed to the fraction form of the percent were exhibiting strategies which might be considered reformulation. These strategies were commonly used by good estimators. Ten students in this study used these strategies. Four also used compensation.

Seven of the estimators in this study used a strategy not seen in good estimators. It can be classified as a combination of reformulation and compensation. The strategy was starting at 50 per cent (or 25 per cent) of 100,000 and then adjusting to 30 per cent. Students using this strategy started with a percent they could calculate more easily and then adjusted. None of the good estimators used this strategy, probably because they were able to work with "harder" numbers. Some good estimators did start with 10 per cent and then multiplied by three. Starting at 50 per cent is a less efficient way of dealing with this problem, but it seemed to work for most of those who used it.
Problem eight required both a mental and a pencil-and-paper estimate. It will be discussed in the next section of the chapter. Problem nine required only a mental estimate and will be discussed here.

Problem Nine

(acceptable choice 550,000)

Here are 3 estimates for the total attendance for six Superbowl games.

\[
\begin{array}{c|c}
\text{Year} & \text{Attendance} \\
1974 & 73,655 \\
1975 & 86,421 \\
1976 & 91,943 \\
1977 & 96,509 \\
1978 & 93,421 \\
1979 & 106,409 \\
\end{array}
\]

Again the large-number syndrome caused problems for students. Ten of these students made statements that indicated that the size of the number interfered with their solution of the problem. In referring to the sum of the first four numbers, one student said, "Because I know that this right here adds up to more than five, fifty, five million. It's not five million, is it? 55,000." Notice that she did not once correctly name the number to which she was referring.
TABLE VI
STRATEGIES USED ON PROBLEM NINE

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded and added</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Same Number Digits</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Extracted Number Digits</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Truncated and added</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>Looked at largest number</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Averaged</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>Confused by large numbers</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>Guessed</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

After another student had chosen one million, he gave this reason. "Because most of the numbers there are over 50,000 and one of them is 100,000."

A third student expressed her feelings about this problem. "This doesn't make any sense at all because.... Well, maybe it does. But there are so many numbers, it's not just going to one right there. ... If you were to round them off, they would be bigger than that so it would probably be a million." After a question from the interviewer, she continued, "... because you'd have so many you'd have to add on."

The two who were classified as guessers clearly expressed that this was how they arrived at their answers. One said, "Pure speculation. Five is a middle number and it's just easy." The other said, "Because if I add all these together I don't think it would be a million and I
don't think it would be 550,000 or less so I'd choose one in between."

The most successful group was the group which simply added the first one or two columns and based their answer on this sum. They may have been more successful than those who rounded simply because they didn't have to remember as many digits.

The most commonly used strategy was averaging. Unfortunately, this did not produce the greatest number of acceptable answers. Ten of the students who used this strategy chose 600,000 as their answer. The student who did produce an acceptable answer using averaging also used compensation. He was the only one in this group to use compensation, and he clearly stated his use of this procedure: "I just went ahead and rounded all these to 100,000, and there were six of them so I came up with 600,000. But then I started looking at the numbers that were lower, like 73,000 so I figured it would come up with 55."

Interestingly, the twelfth student who used averaging probably suffered from the large-number syndrome. He said, "I just went through and said it's a hundred and I just rounded off to two, three, four, five and with all this raw stuff on the side, I just went on and rounded off to a million."
A strategy related to averaging but different from it was looking at the largest number and choosing an answer based on its size. For the two who used the strategy successfully, it was almost an elimination strategy. Since the largest number was approximately 100,000, then the sum could not be more than 550,000. The other two started with the largest number but were not able to proceed from there successfully. Reys, et. al. found that some good estimators used this strategy.

Twelve students used strategies on this problem which might be considered examples of reformulation - rounding and using truncation. Twelve students used averaging which is an example of translation. Six students used compensation along with their other strategy.

Problem ten required both a mental and a pencil-and-paper estimate. It will be discussed in the next section of the chapter. Problem eleven required only a mental estimate and will be discussed next.

**Problem Eleven**

(acceptable interval $8 to $11)

Three people have dinner. They order:
- Bacon N Cheese Steakburger Platter
- Super Steakburger Platter
- Chili-Mac
- 2 Small Coca-Colas
- 1 Hot Chocolate
- 1 Hot Pie

(A copy of the menu with prices may be found in Appendix B.)
Students did not consistently round to the nearest dollar, half dollar, or ten cents on this problem. Rather, they rounded to the nearest "nice" number. Frequently, the sequence of rounding was $2.89 to $3, $2.64 to $3, $1.47 to $1.50, two $.45's to $1, $.35 to $.35 or $.50, and $.76 to $1. This yielded a total of $10. Some students had better rounding skills than others in that they rounded to numbers that were more usable for them. Rounding $2.89 to $2.90 was really not much help for these students and yet some of them did exactly that.

### TABLE VII

STRATEGIES USED ON PROBLEM ELEVEN

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded item prices</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No compensation</td>
<td>22</td>
<td>19</td>
</tr>
<tr>
<td>Intermediate comp.</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Final compensation</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Rounded running total</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Attempted exact computation</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Added dollar amounts first</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Guessed or method unclear</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Of interest was the use of compensation. One student said, "I'm not even going to add the hot chocolate in because I think it will make up for all the ones I'm rounding off." Another commented that she would round the price of the hot chocolate down to 30 cents because she had "already rounded up a lot." These are examples of intermediate compensation.
One student who had consistently rounded up reached a total of $10, but she reported her answer as $9.50. When asked why she had changed from $10, she replied, "I rounded off the figures. They weren't exactly like two dollars or whatever I rounded them to." This is an example of final compensation.

Sometimes the rounding occurred with the total and not with the item. For example, one student said, "A Bacon N Cheese Steakburger Platter would be $2.89 and a Super Steakburger Platter.... It would be about $5. And the Chili Mac, five, 6.47. About 6.50. Two small Cokes would be about a dollar. So it would be about 7.50. One hot chocolate would be about $8. And the pie.... About 9.76." This rounding made it easier for the student to go on with the addition process.

This problem was the one on which most students exhibited the greatest amount of speed and confidence. All but three of them had a strategy for estimating the answer to this problem. The four who had a strategy and did not produce an acceptable answer all made calculation errors or forgot an item in adding. There were two students who did not even attempt exact calculation. They obviously did not have any way to solve this problem.

Reformulation in the form of rounding or truncating was used by 35 students on this problem. Seven used compensation.
One difference between these students and the good estimators studied by Reys, et. al. was that these students did not change the order of the problem. They all worked straight through the list.

Problem twelve required both a mental and a pencil-and-paper estimate. It will be discussed in the next section of the chapter. Problem thirteen required only a mental estimate and will be discussed next.

Problem Thirteen

(acceptable choice 6 32-ounce Coca-Colas)

Which carton has more soda?

6 32-ounce Coca-Colas
8 16-ounce Pepsi's

Problem thirteen was a relatively easy problem. Of the 29 students who used a correct procedure, 26 had acceptable answers. One answer for problem thirteen which deserves comment was the choice of the eight 16-ounce bottles of Pepsi when the reason given was that eight bottles are more than six bottles. This may speak more of the developmental stage of the students answering this way than of their general ability to estimate. These students stated their choices using phrases such as "Just six of 32 is not as much as eight 16's."

The two students who used exact calculation and did not produce an acceptable estimate both made calculation errors.
Three other students who used this procedure also made calculation errors, but their mistakes were not large enough to affect their choice. Even though these students said they were multiplying exact numbers, there is evidence that some of them were rounding either the factors or the product. The product of 8 and 16 was given as 120 or 120 something, and the product of 6 and 32 was given as 180.

TABLE VIII

<table>
<thead>
<tr>
<th>STRATEGIES USED ON PROBLEM THIRTEEN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Number Using</td>
</tr>
<tr>
<td>-------------------------------------</td>
</tr>
<tr>
<td>Multiplied exact numbers</td>
</tr>
<tr>
<td>Rounded and multiplied</td>
</tr>
<tr>
<td>32 is more than 16</td>
</tr>
<tr>
<td>Ratio approach</td>
</tr>
<tr>
<td>More 16-ounce bottles</td>
</tr>
<tr>
<td>Seemed like/guessed</td>
</tr>
</tbody>
</table>

Why didn't more of these students round or change to easier numbers? The three who did were all successful. One answer may be that these students do not possess number facts such as "8 x 15 = 120." Good estimators would not have to calculate eight times fifteen, but for these students eight times fifteen and eight times sixteen are of equal difficulty. They would have to calculate both answers so they tended to stay with the exact product.

When the student answered "thirty-two is more than sixteen and there are only two more bottles of Pepsi," they
were not specifically comparing amounts. This was more of an intuitive answer than a calculation. It was a computationally immature version of the ratio approach.

The ratio approach used some variation of the idea that thirty-two is twice sixteen and that would mean that eight 16-ounce bottles were equivalent to four 32-ounce bottles. The student who was not successful using this approach decided that there were equal amounts of Coke and Pepsi because sixteen is half of thirty-two. He ignored the difference in the number of bottles.

It may have been the case in three of the four guesses that these students also made their choices based on the number of bottles rather than actual amount. The students did not verbalize this thought, but rather assured the interviewer that it "seemed like" there was more Pepsi than Coke.

Reys, et. al. found that good estimators used the ratio strategy. They also multiplied exact or rounded numbers. Some good estimators used one strategy not found here. They converted to gallons or quarts and compared. This is not a strategy likely to be found among estimators at this level of ability.

Students who rounded either the product or the factors were using strategies which can be classified as compensation. Since students were not given an arithmetic statement to deal with, it would be difficult to say that
they had transformed the form of the equation. Compensation was used by one student who rounded sixteen ounces to twenty and made a mistake in multiplying eight times thirty-two. He was comparing the numbers 160 and 162, and he chose the Cokes because he knew that he had rounded up.

Problem fourteen is the last problem which required only a mental estimate. It was also the last problem of the interview and students may have given it less consideration than some which preceded it.

Problem 14

(acceptable choice 6 32-ounce Coca-Colas for $1.79)

Which soda is cheapest?

6 32-ounce Coca-Colas for $1.79
8 16-ounce Pepsis for $1.29

One student summed up the feelings of all of the students who used the "seems like" strategy on problem fourteen when she said, "I'm not doing it by any math; I'm doing that by sheer logic." Some of them may have been responding to the same feeling that another student expressed: "Because even though this one is more expensive, it's probably a better deal. Just because tests tend to ask trick questions."

There were two subgroups under the "seems like" category. These were students who were a little more specific in their reason for making a particular choice.
The first group said that it was only fifty cents more for more soda. Only two of them checked to see how many more ounces they were getting for their fifty cents. The other group was even less specific. They simply said that the buyer was getting more soda for only a few cents more.

### TABLE IX

**STATEGIES USED ON PROBLEM FOURTEEN**

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seems Like</td>
<td>17</td>
<td>12</td>
</tr>
<tr>
<td>50 Cents for Extra</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>More for a Little More</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Misunderstood Problem</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Error on 13</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>SOP for this Problem</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Guessed/Method Unclear</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>My Mom Buys</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

These students may have had difficulty in estimating an answer for this problem because they had no procedure for solving it at all. Only one student used a strategy that would always produce an acceptable answer if used correctly, and she made a calculation error which prevented her from obtaining an acceptable answer. She said she multiplied the number of bottles times the number of ounces in each bottle and divided the cost by this product. It should be noted that this is not an estimation strategy, but rather an exact calculation.
It is not unreasonable that these students would have difficulty estimating the answer to problem fourteen. This was the only problem on which Reys, et. al. reported that good estimators guessed or said that they did not know. Some of them said that the problem contained too many digits and too many operations to be done mentally. Some even expressed the idea that the large amount for more money is usually the better buy so they chose it. Thus the responses on this problem were about what might have been expected from estimators at this level of ability.

The two who misunderstood the problem were in the first set of students interviewed. When students interviewed after these misunderstood, the interviewer corrected the misconception and allowed them to produce an estimate.

The three students who missed this problem because they had missed problem thirteen probably considered this a silly question. Since they felt that there was more soda in the Pepsis and the price of the Pepsis was lower, they naturally chose it as the better buy. However, some of the students who had chosen Pepsi in problem thirteen changed and chose Coke as the better buy in problem 14. They did not seem to be bothered by this inconsistency. When questioned about her choice, one of them responded that although she felt that she was getting more Coke for her money on this problem, she would not change her answer to problem thirteen.
Pencil and Paper Estimates

After the first set of twenty students were interviewed and asked to produce only a mental estimate, a second set of twenty students was selected using the same procedure that was used to select the first twenty. These students were asked to use a pencil and paper to produce an estimate for problems two, four, five, eight, ten, and twelve immediately after they had attempted to produce a mental estimate for the same problem. It was expected that some students would be unable to produce an estimate mentally, but would be able to produce one using the pencil. This was the case. It was not expected that students would have a mental strategy for producing an estimate and not be able to use the pencil to produce an estimate. For some students this was true.

Students were unable to use a pencil to produce an acceptable estimate for a variety of reasons. Some students refused the pencil when it was offered. This happened when they had no strategy to solve the problem even by exact computation. Problems involving percent were the ones which most commonly fell into this category. Other problems that required the student to do computation that they were not familiar with also fell into this category. For example, problem two with its three factors caused some students difficulty both in producing the mental estimate and in using the pencil. Sometimes a student had made such a good mental estimate that he refused the pencil when it was
offered and said that he could not improve on what had already been done.

Sometimes their pencil estimates were unacceptable because they chose to use the wrong operation to produce the estimate. At other times, a mistake made during the process of mental estimation interfered with the successful production of a pencil estimate.

Finally, some students could use the pencil only to carry out the exact calculation. When the interviewer stopped them from doing this, they usually subsequently refused to take the pencil saying that they could only use it to "work out the problem."

One of the major differences between mental estimates and pencil-and-paper estimates was the number of digits used in making the estimate. Students who had used an extracted number of digits in making the mental estimate wrote rounded or otherwise adjusted numbers which contained the same number of places as the original problem. This frequently led to the correction of a power-of-ten error because the student could see the number of places involved in the problem more clearly. Even when they used exactly the same numbers and the same strategy in finding both estimates, they were less likely to make a power-of-ten error with the pencil. However, the pencil was not a guarantee against a power-of-ten error.
The answers to the problems discussed below were estimated both mentally and with pencil and paper.

Problem Two
(acceptable interval 600,000 to 634,000)
31 x 68 x 296

Problem two was difficult for some students because they were not accustomed to having three factors in a multiplication problem. Of the eighteen students in the last category of Table X, fourteen began working the problem and were unable to finish. Of these fourteen students, ten attempted to multiply the factors in order and four started by using the largest factor, 296, to find the first product. Of those who attempted exact calculation, three used the factors in order and one used them out of order. Of the other eighteen students, twelve multiplied the factors out of order and only six multiplied them in order. This indicates that students who had a strategy to deal with the three factors also felt more comfortable in rearranging the problem to make it easier for themselves, but even among those who guessed or were not able to produce an answer were students who felt free to rearrange the problem.

In this problem, there were only two numbers which could have been rounded up to the next multiple of ten. In order to produce an acceptable estimate at least one of these numbers should have been rounded up. Four students
truncated both of these numbers which meant that they could not produce an acceptable answer even if they carried out their strategy correctly. Four others rounded one factor and not the other. One of these students said that he rounded one up and one down indicating that he was using intermediate compensation. He was able to produce an acceptable estimate. It is possible that other students who rounded one factor and truncated the other did so because they were using intermediate compensation and simply did not express it.

**TABLE X**

MENTAL STRATEGIES USED FOR PROBLEM TWO

<table>
<thead>
<tr>
<th>Truncated</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Numbers</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>One, Rounded Other</td>
<td>4</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Rounded</td>
<td>10</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, or Method Not Clear</td>
<td>18</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Other than making calculation errors and being overwhelmed by the problem, the most common difficulty encountered by students working this problem was producing an answer of the correct order of magnitude. Six students who rounded the factors correctly were off by a power of ten and two of the four who attempted exact calculation made power-of-ten errors.
The good estimators studied by Reys, et. al. used rounding almost exclusively. Thirty-nine used rounding and the same number of digits; fifteen used rounding and an extracted number of digits. This compares with the estimators in this study who used both truncation and exact calculation not found among the good estimators. Fifteen in this study used the same number of digits, and six used an extracted number of digits.

The responses of students to problem two illustrate the lack of structure with which some students operate. When they rounded 68 and did not round 296 or vice versa, they demonstrated that they do not use rounding consistently. Even those who did this as a form of compensation showed that their compensation is done more by rote than by understanding. Both of these numbers were close enough to the next multiple of ten to make compensation unnecessary.

One example of compensation was found. All of the students who used truncation or rounding used a strategy which can be considered reformulation. None of these students used any strategies which can be considered transformation, although they did change the order of the multiplication.

Twenty of the students interviewed were asked to work problem two using pencil and paper. The table below shows the strategies used by these students.
TABLE XI
PENCIL-AND-PAPER STRATEGIES
USED FOR PROBLEM TWO

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power of Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Truncated</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All Numbers</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>One, Rounded Other</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Rounded</strong></td>
<td>6</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Guessed, Gave No Answer,</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>or Method Not Clear</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Of the four who rounded one number and not the other, three produced estimates of the correct order of magnitude. However, they were not large enough to be considered acceptable. All of these students used the same number of digits as the problem contained. Three multiplied the factors in an order different from that of the problem; one multiplied in the order given. Two of these students used different strategies when using a pencil and produced different answers. One student used the same strategy both times and produced the same answer. One had had no mental strategy to solve problem two, but solved it very quickly when offered a pencil.

Of the six who rounded both 68 and 296, three students made mistakes in doing the problem mentally which interfered with the production of an acceptable pencil-and-paper estimate. Because of the nature of their mistakes and because of their subsequent work, it is likely that these
students would have produced acceptable estimates otherwise. For example, two students both said that 30 times 70 was 210 in making a mental estimate and used this same product when using the pencil. Yet both were able to multiply 210 times 300 in the second step of the pencil-and-paper estimate and get the proper number of zeros in the product. Another student said that the product of 300 and 30 was 3000 in the mental estimate and then used the same "number fact" in the pencil-and-paper estimate.

Five of the six multiplied the factors in the order given, and all of them wrote down rounded numbers using the same number of digits contained in the original problem. One student did indicate that she was multiplying the leading digits and annexing the appropriate number of zeros. All six of the students used the same strategy for both estimates and produced different estimates. One student began the mental estimate using rounding but was unable to complete it. She produced an acceptable estimate using a pencil.

The students who had truncated all numbers in making a mental estimate rounded at least one number in working with a pencil. No student in this group or any other wrote an extracted number of digits.

Since this was a search for estimation strategies, students who started exact calculation when offered a pencil were not allowed to finish the problem. Students who used
exact calculation when finding a mental estimate were allowed to continue.

Students who rounded one or more factors or who used the first digits used a strategy that may be considered reformulation. No examples of translation or compensation were found in the pencil-and-paper strategies.

TABLE XII

COMPARISON OF MENTAL STRATEGIES AND PENCIL-AND-PAPER STRATEGIES
PROBLEM TWO

<table>
<thead>
<tr>
<th>Same Strategy</th>
<th>Different Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Same Answer</td>
</tr>
<tr>
<td>Same Strategy</td>
<td>0</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>0</td>
</tr>
</tbody>
</table>

In addition to the students shown in Table XII, there were four students who attempted exact calculation using the pencil. Since they were not allowed to produce an estimate using a pencil, they were not included in these numbers. Further, one student had no mental strategy, but was able to produce a correct estimate using a pencil.

Problem three required only a mental estimate. It has already been discussed. Problem four required both a mental estimate and a pencil-and-paper estimate.

Problem 4

(acceptable interval 42 to 60)

\[347 \times 6\]

\[\frac{347 \times 6}{43}\]
Remarkably few students were unable to produce an estimate of any kind on problem four. This is particularly surprising when the responses here are compared to the responses on problem two which required the multiplication of three factors. Problem four involves three numbers and two operations. Yet only one student was unable to produce an estimate of any kind. Two others produced estimates that may have been guesses, and one used a method that was not clear. This difference may have been due to problem three. As mentioned in the discussion of that problem, those interviewed seemed more comfortable with using estimation on division than on the previous two problems. Once they had attempted an estimate of a problem on which they felt reasonably secure, they may have been willing to try a more difficult problem.

Because truncation and rounding resulted in the same numbers, it was not possible to say which process students were using. The transcripts frequently contained the word "round" but since students were not likely to have the vocabulary to differentiate truncation from rounding, this was not considered adequate evidence for the use of rounding.

Five students who used an extracted number of digits adjusted their product before division. This may be thought of as compensation in some cases, but may have been a use of reformulation in others. At least two students changed the
numbers to make the division process easier, not because they felt the need to adjust for the effects of previous rounding. All of those who were placed in this category multiplied first and then divided.

**TABLE XIII**

MENTAL STRATEGIES USED FOR PROBLEM FOUR

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power-of-Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used Ext. Number of Digits</td>
<td>7</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Used Same Number of Digits</td>
<td>12</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Changed Order</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>13</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Method Unclear</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

One of the students who used the same number of digits changed to compatible numbers and then compensated in his final answer. Four of these students rounded 347 to 350; one rounded to 400; and the others rounded to 300. All of those who were placed in this category multiplied first and then divided.

Once more these students demonstrated both their willingness to work very hard to please the interviewer and their lack of awareness of their computational limitations. Thirteen attempted exact computation at least in the multiplication part of the problem. Only one student gave the exact product of 347 and 6. She continued with an attempt at the exact quotient and was off by a power of ten.
Three students divided before they multiplied. Two of them rounded 347 to 350 and 43 to 50. One successfully multiplied 7 times 6 to get 42. The other attempted to compensate for his previous rounding and thoroughly confused himself. The third student divided 43 by 6 to get 7 and then miscalculated the quotient of 347 and 7. He made a power-of-ten error.

One student used a process that deserves comment. It should serve as a warning to those who would enthusiastically encourage their students to find the number of places in an answer as an aid to producing an estimate. This is his description of the process he used: "I was thinking of 6 times 34, 7. That's going to come out to four numbers. And you have the 43 into the 4 numbers, and that's probably going to be somewhere around 30 or 40."

He assumed that knowing the number of places in the product is enough to let him figure out the quotient. He did not ever estimate what the product would be.

None of the students used an extracted number of digits when the pencil was available, although one of them had previously used this strategy to produce the mental estimate. All of the students worked the problem by multiplying first and then dividing. The student who had previously produced an acceptable estimate by using division first refused a pencil saying she could not get a better estimate using it. Another refused a pencil saying that the
only way she could improve her estimate was to work out the problem.

**TABLE XIV**

PENCIL-AND-PAPER STRATEGIES FOR PROBLEM FOUR

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power-of-Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used Same Number of Digits</td>
<td>8</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Refused Pencil</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Good estimators used rounding with an extracted number of digits or the same number of digits just as the estimators in this study did. They also changed the order of the problem. In addition, they thought of the ratio of six to forty-three as the fraction 1/7 which no student in this study did.

**TABLE XV**

COMPARISON OF MENTAL STRATEGIES AND PENCIL-AND-PAPER STRATEGIES PROBLEM FOUR

<table>
<thead>
<tr>
<th>Same Answer</th>
<th>Different Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same Strategy</td>
<td>1</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>0</td>
</tr>
</tbody>
</table>

In addition to the students shown in Table XV, there were five students who attempted exact calculation and were not allowed to finish their answers. Two students had
mental strategies and refused the pencil so they had no pencil-and-paper estimate for problem four.

Problem five is the next problem which required both a mental estimate and a pencil-and-paper estimate.

Problem Five

(acceptable interval  7.5 to 10)

\[ 1 \frac{7}{8} \times 1.19 \times 4 \]

The fraction \( 1 \frac{7}{8} \) caused problems for many of the students in this study. Fourteen of them changed it to the improper form \( 15/8 \) when making a mental estimate. This made the problem very difficult to finish because they had to multiply by fifteen and then remember to divide by eight. Most of the students who used the improper fraction first multiplied \( 4 \times 1.19 \) and then multiplied by \( 15 \). The most common mistake was forgetting to divide by eight once they had the product of the three numbers.

A number of these students treated the problem as \( 15/8 \times 1 \times 4 \). If carried out correctly, this resulted in the product 7.5. This result was considered an acceptable estimate for those who used this strategy even though Reys, et. al. considered the acceptable range for good estimators to be 8 to 10. None of the good estimators used this strategy.

Very few of these students had strong enough fractional concepts to allow them to change \( 1 \frac{7}{8} \) to 2. One student
who was able to make this change used extra steps which showed that she did not think of 1 7/8 as close to 2, but was able to change it to 2 by using a rote procedure: "And it was fifteen. One times eight is eight, plus seven is fifteen over eight. And then I turned fifteen into sixteen and divided it by eight and got two and multiplied it."

<table>
<thead>
<tr>
<th>MENTAL STRATEGIES FOR PROBLEM FIVE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number</strong></td>
</tr>
<tr>
<td>Used Improper Fractions (Includes Exact Calculation)</td>
</tr>
<tr>
<td>In Order</td>
</tr>
<tr>
<td>Not in Order</td>
</tr>
<tr>
<td>Did Not Use Improper Fractions</td>
</tr>
<tr>
<td>Used Decimal Form</td>
</tr>
<tr>
<td>Rounded to Whole Number</td>
</tr>
<tr>
<td>In Order</td>
</tr>
<tr>
<td>Not in Order</td>
</tr>
<tr>
<td>Truncated</td>
</tr>
<tr>
<td>In Order</td>
</tr>
<tr>
<td>Not in Order</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, Used Wrong Process, or Method Not Clear</td>
</tr>
</tbody>
</table>

This round-and-multiply strategy was by far the most effective one. One student overcompensated and produced an unacceptable answer, and the other made two calculation errors which caused her to miss the estimate.

The difference between the students who rounded and the students who used truncation was in their treatment of 1 7/8 and 1.19. Those who rounded thought of these as 2 and 1 or
1.2 respectively. Those who used truncation used 15/8 (or sometimes 1) and 1 respectively.

The work of another student gives an example of the good logic used by these students and the lack of number sense which resulted in erroneous answers. He said that the product of the three numbers would be about 5. When questioned, he replied, "4 times 1 times 1." The interviewer asked why that was 5 and not 4. He answered, "Because of the point nineteen and the seven-eighths. Kind of rounded it, I guess." He recognized that the fractional parts of the two numbers would increase the product, but he did not have the number sense required to make an acceptable estimate of the amount of increase.

The work of still another student illustrates that what may seem like rounding could be a form of compensation. Her thinking was described like this: "Well, I'm thinking 4 times 1.19 would be more than 4. Maybe about five or six and then times 1 7/8 would be even more than that because you've got the 7/8's. If it were just one, it wouldn't be. So maybe about seven. Six or seven." Again, a stronger number sense might have allowed her to produce an acceptable estimate.

The answers given to this problem did not always follow standard format. Possibly the students are unfamiliar with problems which mix fractions and decimals and could not have worked them at all. Two students gave answers which
indicated some confusion. One gave her answer as 4.67 7/8 and the other gave 4.19 7/8 as an answer. Notice that in both cases the four came from multiplying together the whole number parts of the factors. The second student explained that the .19 came from 1.19 and the 7/8's came from 1 7/8.

Students who rounded or used truncation used strategies which can be classified as reformulation. There were no specific examples of translation or compensation on problem five.

TABLE XVII
PENCIL-AND-PAPER STRATEGIES FOR PROBLEM FIVE

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Used Truncation</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Refused Pencil Because</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Happy with Mental Estimate</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>No Pencil &amp; Paper Strategy</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

Both of the students who rounded had acceptable mental estimates and acceptable pencil-and-paper estimates. One used more decimal places than the other and made a calculation error in his mental estimate. He corrected this mistake in the pencil-and-paper estimate.

One of those who truncated the factors wrote the problem as 15/8 x 1 x 4 and multiplied 4/1 times 15/8 to get 60/8. A calculation error prevented her from obtaining a
correct answer. Another used a similar procedure and obtained an acceptable estimate. The third student who used truncation wrote the problem as 15 x 1.20 x 4. She failed to remember the eight as the denominator of the fifteen. She did the same thing in making a mental estimate. When she began her mental estimate, she indicated some confusion about how to handle the mixed number: "1 7/8. I guess I can work that out. Eight, seven. I'm going to change that mixed number with that whole number and that fraction into just one whole number.... So I guess I'd say eight times seven, I think. Or one times eight. One times eight. So that would be fifteen, I think. Fifteen as a whole number. If that's backwards, I don't know."

Those who attempted exact calculation made calculation errors, forgot to divide by eight, or forgot the decimal places they needed. None were successful.

Four students said that a pencil would not help them improve their mental estimate; three of these had produced acceptable mental estimates. Six simply refused the pencil. Three of them had produced no mental estimate, and the others had produced unacceptable mental estimates.

Students who rounded or used truncation used strategies which can be classified as reformulation. There were no specific examples of translation or compensation on this problem.

Only three students used the same strategy for both
estimates. One who rounded and multiplied obtained the same acceptable estimate both times. The other student who rounded produced a different, acceptable estimate because he was able to correct a calculation error. The third student used exact calculation both times and produced unacceptable estimates both times.

**TABLE XVIII**

**COMPARISON OF MENTAL STRATEGIES AND PENCIL-AND-PAPER STRATEGIES**

**PROBLEM FIVE**

<table>
<thead>
<tr>
<th>Same Answer</th>
<th>Different Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same Strategy</td>
<td>1</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>0</td>
</tr>
</tbody>
</table>

One student who had no mental strategy was able to obtain an acceptable pencil-and-paper estimate by using an improper fraction for 1 7/8 and truncating 1.19. Her strategy approached exact calculation, but she did use truncation.

Problems six and seven required only a mental estimate. They have already been discussed. Problem eight required both a mental estimate and a pencil-and-paper estimate.

**Problem Eight**

(acceptable interval 4200 to 5400)

At the 1979 Superbowl, 8,483 hot dogs were sold for $.60 each. About how much resulted from selling the hot dogs?
The decimal places in $.60 really caused problems for students in this study. Some students who ignored them in finding the product forgot to place them in the final estimate. Students who worked with an extracted number of digits and said "$6 \times 8 = 48$ or 50" added too many zeros or not enough. It should be noted that the form of the money amount given in the problem is such that it should not contribute to this confusion.

TABLE XIX
MENTAL STRATEGIES FOR PROBLEM EIGHT

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power-of-Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used Ext. Number of Digits</td>
<td>14</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Used Same Number of Digits</td>
<td>10</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1/2 of 8483 or 8483/2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Ratio</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Exact Computation</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, Used Wrong Process, or Method Not Clear</td>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

One student admitted that she had added the zeros from 8483 and "one of the zeros from the sixty" to her product of eighty-four times six. Her estimate was 50,400. Another student multiplied 84 times 60 to get 5080. She then placed the decimal in the estimate to change it to 50.80.

Students who used the same number of digits found it equally difficult to deal with the number of decimal places. The number of power-of-ten errors should be noted. No
student was able to use this method to produce an acceptable estimate.

This concentration on the digits without regard for decimal places is different from what was done by the good estimators. They used the decimal form of the number $.60 even when they used an extracted number of digits. They were likely to express the product as ".6 x 84" rather than "6 x 84" as students in this study did. This was the major difference found in strategies used by students in this study and those used by good estimators.

The most bothersome thing about this problem was the lack of reasonableness of the answers that some students were willing to accept. The answers above are at the extremes of answers accepted. Unfortunately, they are not the only such examples.

Occasionally, a student would express doubt about an answer. One student who produced an estimate of $48,288 looked at her answer and said, "It's goofy, but.... Wait a minute, 8483 hot dogs were sold for sixty cents. Okay, I multiplied 60 times 8483." Once she was comfortable with her method, she accepted her estimate as correct, even though she felt that it was "goofy."

Another student showed more insight. After she produced an estimate of $50,000, the interviewer recorded it and moved on. The student said, "That can't be right. I
figured out that can't be right because they're not even a dollar apiece, and there were only 8000 sold. I'll say about $6000." It should be noted that she trusted her feelings about the numbers enough to change her answer, and, even though it is still not in the acceptable range, it is much more reasonable than her first effort.

This student and the one who used the ratio approach may have looked at the problem as if it were 10,000 hot dogs at $.60 each. The student who used the ratio approach mentioned that $.60 is 60 per cent of a dollar and then used $6,000 as his estimate.

One of the students who used exact computation was able to produce an exact answer for this problem. The others were off by a power of ten and also missed some of the digits in the final answer. Students in this study did not demonstrate strong mental arithmetic ability.

Students who used an extracted number of digits or the same number of digits used rounding or truncation in every case. These strategies are examples of reformulation. Changing to division by two and thinking of this problem using the ratio approach are examples of translation. Compensation was used by eight students. This usually took the form of changing the product of eight and six from 48 to 50.

In the pencil-and-paper estimates, the decimal places in $.60 were again a stumbling block. Students who used the
same number of digits rounded 8483 to 8000 or to 8500, but only one student used .60 as the other factor. She did not use the zero when she multiplied, but did use it in determining the decimal places for her estimate. One of those who used an extracted number of digits did count this zero in determining her estimate, but she counted it twice as shown in her conversation.

She wrote the problem as \[8,483 \div 0.60\]

"I saw that the zero would take up one space of the answer. And I counted the rest of the spaces which was 4, 5. And then I multiplied 6 times 8 which is 48 and added 3 more zeros. And I estimated it to be 50,000. And then with decimals, it would be ...." She had written 48000 on the paper. She rewrote the answer as 500.00. It should be noted that she wrote the exact digits but used an extracted number of digits to multiply.

TABLE XX

<table>
<thead>
<tr>
<th>PENCIL-AND-PAPER STRATEGIES FOR PROBLEM EIGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number Using</strong></td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>Used Ext. Number of Digits</td>
</tr>
<tr>
<td>Used Same Number of Digits</td>
</tr>
<tr>
<td>8483/2</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, Used Exact Calculation or Method Not Clear</td>
</tr>
</tbody>
</table>
One of the students who divided by two had produced a mental estimate of 16,000. This probably was the result of multiplying by two in producing this estimate. It was interesting to note that he related $.60 to 50 per cent and then said, "I just cut that in half." His work shows that he used division by two to find his estimate.

The most common reason for refusing a pencil was best expressed by this reply. "No, not unless I could just personally multiply it." Students at this level frequently see the pencil only as an implement for obtaining an exact answer.

Only three students used compensation along with the pencil. Only the two who used division by two used a strategy that might be considered translation. The other students who produced estimates used strategies which might be considered reformulation.

<table>
<thead>
<tr>
<th>TABLE XXI</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMPARISON OF MENTAL STRATEGIES</td>
</tr>
<tr>
<td>AND PENCIL-AND-PAPER STRATEGIES</td>
</tr>
<tr>
<td>PROBLEM EIGHT</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Same Answer</th>
<th>Different Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mental P&amp;P Neither Both</td>
</tr>
<tr>
<td>Same Strategy</td>
<td>1 2 0 0 4 0</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>0 0 0 2 3 0</td>
</tr>
</tbody>
</table>

On this problem, the most common change in strategy was from using an extracted number of digits in the mental
estimate to using the same number of digits in the pencil-and-paper estimate. The power-of-ten error was very common on both the mental estimate and the pencil-and-paper estimate. It was probably caused by the decimal places in the factor $.60.

Problem nine required only a mental estimate. It has already been discussed. Problem ten required both a mental estimate and a pencil-and-paper estimate.

Problem Ten

(acceptable interval 700,000 to 950,000)

The 1979 Superbowl netted $21,319,908 to be equally divided among the 26 NFL teams. About how much did each team receive?

Two of the students who rounded and used the same number of digits rounded incorrectly. One rounded 21,319,908 to 30 million and the other rounded 26 to 20 and 21,319,908 to 20,000,000. They both compounded their rounding errors by making a power-of-ten error in finishing the division. Two of the remaining students rounded correctly to twenty-one million divided by thirty but guessed at the quotient and missed it. The one who was asked to use pencil and paper produced an acceptable estimate using these rounded numbers.

One student who used the same number of digits conducted a search for a suitable multiplier of twenty-six.
He said, "Well, I came up to 800,000 apiece times twenty-six, and I'm trying to get as close as I can to twenty-one. And I came up short so I kept adding. And trying to multiply." His compensated estimate was 950,000. He used this same strategy when he used a pencil and decided on 850,000 as his estimate.

**TABLE XXII**

MENTAL STRATEGIES USED FOR PROBLEM TEN

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power-of-Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Used Same Number Digits</td>
<td>5</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Used Ext. Number Digits</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Changed to Easier Numbers</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Truncated Ext. Number Digits</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Attempted Exact Computation</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Compared 21 and 26</td>
<td>7</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, or Method Not Clear</td>
<td>13</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A strategy related to rounding but different from it is changing to easier numbers. The twenty-six was changed to twenty-five by two of these students to make division easier. One produced an acceptable answer mentally and later used pencil and paper to produce a different acceptable estimate using a different strategy. The other student produced an unacceptable mental estimate, but later used these numbers to produce an acceptable pencil-and-paper estimate.
The third student who changed to easier numbers described his process as "simplifying" twenty-one million. However, his "simplified" problem did not seem simpler to the interviewer. He unsuccessfully tried to estimate the quotient of 20 million and 26. This is another example of an estimator of limited ability who possesses a viable strategy for estimation but not the number sense which would make it work efficiently.

The student who used an extracted number of rounded digits tried to divide 213 by 30. He was unsuccessful. Two of the students who used an extracted number of digits, but not rounding, also conducted a search for a suitable multiplier of twenty-six. Both agreed that 9 times 26 is approximately 213 or 21 million. One then compensated incorrectly and adjusted his answer out of the acceptable range.

Three students extracted too few digits to be successful. They divided twenty-one by two to get an estimate. One of them recognized that the quotient obtained was too large and compensated. A power-of-ten error kept her from producing an acceptable estimate. The others who used an extracted number of digits divided 213 by 26.

Some students compared twenty-six and twenty-one and decided that each team must have received less than one million dollars each. This was one of the most successful
mental strategies used but was not used by pencil-and-paper estimators. Approximately one fifth of the good estimators used a strategy labeled "ratio reasoning" by Reys, et. al. The good estimators are described as having "changed numbers to a ratio form verbalizing their relation to each other, then converting this relationship to a numerical estimate." (p. 200) The students in this study typically expressed their thoughts in this way: "They couldn't be a million apiece because that would be twenty-six million. And I figured 900,000 would still be a little too much so I figured 850,000." Obviously, the strategy used by estimators in this study and that used by good estimators are similar but not identical.

For most of these students, this was almost an educated guess rather than a strategy. One exception was the student who was the best overall estimator in the study. He said, "Well, there's twenty-one here, twenty-one million, so if there were twenty-one NFL teams they would get about a million apiece. Since there's twenty-five, you're going to take about a hundred thousand off each." His estimate was 900,000.

Students who used rounding, truncation, or easier numbers were using strategies that might be considered reformulation. Students who used the comparison strategy used a translation strategy. The students who conducted a
search for a multiplier also used a strategy which might be translation. These students probably use this same strategy to find quotients when doing exact computation. Compensation was used by ten students.

TABLE XXIII

PENCIL-AND-PAPER STRATEGIES FOR PROBLEM TEN

<table>
<thead>
<tr>
<th>Number Using</th>
<th>Acceptable Answers</th>
<th>Power-of-Ten Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded - Same Number Digits</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Changed to Easier Numbers</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Truncated</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extracted Number Digits</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Same Number of Digits</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, or Method Not Clear</td>
<td>11</td>
<td>0</td>
</tr>
</tbody>
</table>

Only two of the students who rounded had done so in their mental estimates. Both of them had acceptable estimates using pencil and paper.

Only one of the students who used easier numbers had done so mentally, but her mental estimate was unacceptable and her pencil estimate was acceptable. The other acceptable estimate using this strategy came from a student who had used the comparison method to produce an acceptable mental estimate.

Six of the nine students who worked on this problem with a pencil changed strategies. This demonstrates that students at this level possess more than one strategy for
estimation. On other problems, there was a tendency to go
from using an extracted number of digits in the mental
estimate to the same number of digits in the pencil-
and-paper estimate. While this was true here also, there
was, in addition, a tendency to change strategies
completely. The most common change occurred when students
changed from comparison to a concentration on the division
process using rounded or truncated numbers.

TABLE XXIV

COMPARISON OF MENTAL STRATEGIES
AND PENCIL-AND-PAPER STRATEGIES
PROBLEM TEN

<table>
<thead>
<tr>
<th>Same Strategy</th>
<th>Different Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same Answer</td>
<td>Different Answer</td>
</tr>
<tr>
<td></td>
<td>Acc.</td>
</tr>
<tr>
<td>Same Strategy</td>
<td>0</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>1</td>
</tr>
</tbody>
</table>

The number of students who changed strategies when they
worked problem ten with a pencil is reason to consider
pencil-and-paper estimation a different process from mental
estimation. Although using pencil and paper to develop
estimation strategies may be helpful, students need to move
away from them into mental estimation. There are some
strategies which will never be developed if estimation stays
at the pencil-and-paper level. The comparison strategy
found on this problem is an example.

Problem eleven required only a mental estimate and has
already been discussed. Problem twelve was the last problem which required both a mental estimate and a pencil-and-paper estimate.

Problem Twelve

(acceptable interval $3 to $5)

The Thompsons' dinner bill totaled $28.75. Mr. Thompson wants to leave a tip of about 15%. About how much should he leave for the tip?

Both students who rounded and multiplied multiplied thirty times fifteen. Both made calculation errors mentally which they corrected when they used the pencil and paper. The student who attempted exact calculation was not successful. Both of these results show the difficulty students at this level have with mental arithmetic.

The students who used percent concepts were all successful. Only one used the distributive strategy described by Reys, et. al. as "operation of 15% handled through two-step distributive procedure using percents, decimals, or whole numbers - 10% of 28 or 30 + half of that." (p. 204). Approximately half of the good estimators used this strategy.

The other students who started with 10 per cent were not as sophisticated in their method. One found 10 per cent plus "a little bit more." The other started with 10 per cent of $1, then $10, then $20, and finally, $28.75.
One student who used percents described his method this way: "I said half of 28 would be $14. That would be 50 per cent. And half of that would be seven. That would be 25 per cent. So approximately 15 per cent, I took off $3." His estimate was $4.

TABLE XXV
MENTAL STRATEGIES FOR PROBLEM TWELVE

<table>
<thead>
<tr>
<th></th>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded and Multiplied</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Attempted Exact Calculation</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Used Percent</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>20%</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>50%</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Used 3/20</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Used Wrong Operation</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, or Method Not Clear</td>
<td>22</td>
<td>12</td>
</tr>
</tbody>
</table>

Not all estimators of limited ability are unsophisticated in their mathematical thinking. This is shown by the work of the student who thought of 15 per cent as 3/20 and was able to produce an acceptable estimate.

Eight students used division rather than multiplication to solve this problem. Most of them divided 30 by 15 to give an estimate of $2. Three were asked to solve the problem using pencil and paper. One changed her strategy to multiplication and only a calculation error kept her from producing an acceptable estimate. The others used division
to solve the problem even with the pencil and paper. Unfortunately, computational skills limit estimation ability.

The guesses on this problem were sometimes based on experience. Three of the students clearly expressed that they had done similar problems before or that their families left a certain amount when they went out to eat. Other guesses showed a total lack of experience with the situation. One student said she would leave $.25 so the bill would be an even $29.

Although none of the guesses had any apparent mathematical basis, the number of acceptable estimates among the guesses indicates that something other than chance was working. The students were not able to verbalize their processes, and the interviewer was unable to probe adequately to uncover the process.

Students who used rounding or equivalent fractions to solve this problem were using strategies that might be considered reformulation. Students who used percent concepts translated the problem from its strict computational interpretation. Two students used compensation to adjust their estimates.

Only eight students produced an estimate using pencil and paper. Two other students had produced mental estimates that they felt they could not improve on by using the
pencil. One student who produced an unacceptable estimate using the pencil rounded and multiplied thirty times fifteen to get forty-five. He did not place a decimal either in the factors or in the product. When he considered forty-five as an estimate, he rejected it and stuck with his mental estimate of $4.

TABLE XXVI

PENCIL-AND-PAPER STRATEGIES
FOR PROBLEM TWELVE

<table>
<thead>
<tr>
<th>Number Using</th>
<th>Acceptable Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rounded and Multiplied</td>
<td>5</td>
</tr>
<tr>
<td>Used Wrong Operation</td>
<td>3</td>
</tr>
<tr>
<td>Guessed, Gave No Answer, or Method Not Clear</td>
<td>12</td>
</tr>
</tbody>
</table>

Consistent with the work on problem eight, students did not record the decimal in 15 per cent when they used it as a factor. More attention should be paid to insuring that students recognize the role of the decimal place.

TABLE XXVII

COMPARISON OF MENTAL STRATEGIES
AND PENCIL-AND-PAPER STRATEGIES
PROBLEM TWELVE

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Same Strategy</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Different Strategy</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
It should be noted that no students used the same strategy to produce the same estimate both mentally and with a pencil and paper, although four used the same strategy both times. This probably is due to the difficulty that students of this ability level have with percent problems. Their performance on the other interview problems involving percents gives further evidence for this statement.
CHAPTER V

Summary, Findings, Conclusions, and Recommendations

Four hundred sixty students took the Assessing Computational Estimation Test. Forty students were selected from the second quarter and were asked to estimate the results of fourteen computation and application problems. Twenty of these students were asked to give pencil-and-paper estimates as well as mental estimates.

Findings and Conclusions

A comparison of the interview results and the ACE results shows that on the five common problems, many students were willing to attempt an estimate during the interview when they had not been able to produce one during the written test. Some of those who had produced unacceptable estimates on the ACE were able to improve those estimates so that they were acceptable. This shows that removing the time pressure does improve performance on estimation tasks.

Students interviewed in this study used a wide variety of estimation strategies. Examples of most of the estimation strategies used by good estimators were found among those used by this group. However, the students in this group were often not successful in their use of these
strategies. This was one of the major differences between them and the good estimators. It was often not lack of strategy that prevented them from producing an acceptable estimate, but rather lack of mental computation skills.

Another major difference between the two groups was in the use of exact calculation. Some students in this study had no strategy for estimation and attempted to use exact calculation. Twenty students attempted to use exact calculation on one or more problems during the interview. These attempts rarely resulted in an exact answer. This is probably because students who were interviewed also had limited mental computation skills. Unfortunately, some of them did not recognize this limitation.

Also, at least one estimator attempted to use exact calculation but tried to disguise it. For example, he would produce a reasonable estimate by multiplying the first digit and then would continue to "adjust" his answer by multiplying by other digits in the problem and adding these products to the first estimate. He seemed to feel that estimation is really just a rearrangement of the algorithm. He was not the only student who expressed some insecurity about the nature of the task of estimation.

All but one of the students interviewed used some form of the front-number strategies rounding and truncation in making mental estimates. These seem to be the universal
estimation strategies. Truncation seemingly precedes rounding in the continuum of estimation strategies, because there were three students who exhibited the use of this strategy and no others. At the other end of the scale, the best estimator (most successful) in this study did not use truncation, but rather used rounding and compensation.

Estimators of limited ability used truncation more often than good estimators. It seems to be the first step in developing estimation strategies. Two problems occurred when students used truncation. First, some of them used too many digits for their mental computation ability. Students who might have been successful with fewer digits made calculation errors.

Second, the use of too few places led to answers judged to be unacceptable for good estimators. Obviously, the solution to this problem is to increase the range of acceptable answers for students in this study. This was done on problems where the researcher felt that it was appropriate. In each of these cases, this adjustment is mentioned in the discussion of the problem. In other cases, the use of too few digits was deemed inappropriate even for estimators at this level and no adjustment in the range was made.

One word of caution must be given here. One of the students interviewed used an estimation strategy which
involved figuring out the number of places that would be contained in the answer and then truncating the numbers to produce the product. This led to the estimate of 8500 as the product of 28 and 47 because he said that there would be four places in the answer (Which is true.) and then said that two times four is eight (Which is true.). He then compounded his mistake by compensating. He consistently used this technique of finding the number of places in the answer and then filling them throughout the interview. Rounding would have helped him considerably.

Estimators of limited ability do not all use rounding consistently or use standard rounding rules. For example, on problem one, some students rounded 92,765 to 100,000 and others rounded it to 90,000. On problem six, some students rounded one factor and not the other. Both Reys, et. al. (1980) in studying good estimators and Levine (1982) in studying college-aged estimators found that some of them rounded one factor and not the other. On problem eleven, students rounded to "nice" numbers, numbers which made addition easier.

This inconsistency in the use of rounding could be considered undesirable. It certainly led to some unacceptable estimates on problems one and six. However, it may have made producing an acceptable estimate easier on problem eleven. It also shows that students have
internalized the rounding process and are using a degree of judgment in applying it. Improvement of the level of judgement is desirable, but strict adherence to a rule that, for the purposes of estimation, all numbers used in a problem should be rounded to the same place is not necessarily desirable. The kind of rounding done in problem eleven facilitates estimation and is done by good estimators as well as those of limited ability.

Other commonly used strategies were averaging and using compatible or easier numbers. Averaging is the technique of identifying a number around which a number of addends cluster and then using multiplication of that number and the number of addends to arrive at an estimate for the sum. For example, in problem one, the addends cluster around 90,000 so a reasonable estimate would be 5 (the number of addends) times 90,000 (the number around which they cluster), or 450,000.

A different strategy closely related to averaging is using the largest number to eliminate possible estimates. This strategy was used on problem nine by students who said that the sum couldn't be as large as 600,000 because the largest addend was 106,000.

The use of compatible numbers is different from the use of "easier" numbers. Compatible numbers are numbers which might be thought of as connected by a number fact. For
example, in problem ten, the divisor might be changed to 30 and the dividend to 24,000,000. These are compatible numbers because they are connected by the number fact "3 times 8 equals 24."

The identification of "easier" numbers is more subjective. When a good estimator chooses to multiply 8 times 15 to estimate the product of 8 and 15, he has chosen "easier" numbers. He probably knows this product as a number fact which makes it an easier combination. Levine (1982, p. 353) labels this strategy the use of "known numbers." The estimator of limited ability may not possess this number fact, and for him, both products are equally difficult to find.

Those estimators of limited ability who were most successful were those who thought of percent of parts of one hundred (problem seven) and those who thought of percent in terms of an "easier" percent (problems seven and twelve). Relating 30 percent of a number to 50 percent of that number is an example of the latter strategy. Students rarely were successful when they simply multiplied the number by the decimal form of the percent because they usually failed to place the decimal correctly in the answer.

It was difficult to separate estimation strategies for division from the algorithm because estimation is an essential part of the algorithm. However, these estimators
of limited ability performed better than expected on the division problems, possibly because of this. They were also more likely to think of division in terms of the corresponding multiplication sentence than the good estimators were. On problem three, nearly one fourth thought of division in this way.

Problem thirteen was solved by some students using the ratio approach. This approach used some variation of the idea that 32 is twice 16 and that would mean that eight 16-ounce bottles were equivalent to four 32-ounce bottles. Some students used a computationally immature version of the ratio approach when they stated that "32 is more than 16 and there are only two more bottles of Pepsi."

Other examples of the ratio approach to estimation occurred on problem eight when $.60 was compared to 60 percent of $1 and on problem ten when students compared 21 and 26.

Estimators in this study did use strategies which might be considered translation of the form of the problem. They changed the order of problems and used multiplication or averaging in place of addition.

Even estimators of limited ability seem to want to use compensation, but some of them lack the number sense needed to use it successfully. They sometimes compensated their answers right out of the acceptable range. This desire to
use compensation may be the result of their not being tolerant of error. Good estimators were found to be tolerant of a good deal of error in their estimates by Reys, et. al. It did not bother them to be fairly far from the exact answer. Estimators in this study were less tolerant of that error which is so natural to estimation.

Some of them were not willing to produce answers in which only the first place or two were nonzero. They would use nonsensical digits in their answer rather than have them filled with zeros. This result is probably due to the lack of experience with estimation and would probably disappear over time with practice in estimation. It supports the comments of the educators at the Cape Ann Conference (1973) that too many students have "exact answer syndrome."

Another problem encountered with estimators of limited ability was the large-number syndrome. When the problem contained numbers with a large number of digits, some of the students in this study had difficulty with it. This difficulty often seemed to be the result of the number of digits and not the process required for estimation.

Connected to this problem was the power-of-ten error. Individual students missed as many as seven problems by a power of ten. This indicates a viable strategy for estimation negated by a lack of ability to operate with powers of ten. Rubenstein (1985, p. 117) says that students
who are unable to work with powers of ten have a "diminished understanding of the size of a number."

More emphasis needs to be placed on dealing with place value, both in whole numbers and in decimal fractions. Students in this study tended to have difficulty in correct placement of the decimal in multiplication problems like problem eight. They tended to ignore it during calculation and then to forget to place it in their answer.

Fractional concepts were not strong in the group of students interviewed. They did not tend to use the fractional equivalent of a number with the exception of 50 percent which they related to one half. They also did not usually round fractions to whole numbers. This was different from good estimators who were able to round them to the nearest whole number.

It was expected that some students would be able to produce a pencil-and-paper estimate when they had been unable to produce a mental estimate. However, the opposite situation was not expected. Yet eight of the 20 students asked to use a pencil and paper had no estimation strategies for this situation. Although most of the strategies used for mental estimates were used when making a written estimate, students frequently changed the number of digits they used when they went from the mental estimate to the written estimate. They also changed strategies. For some
students, use of the pencil led to the correction of a power-of-ten error because they could see the number of digits more clearly.

Students refused the pencil for a variety of reasons. Sometimes it was because they had no strategy to solve the problem. Sometimes it was because they were satisfied with their mental estimate. Sometimes it was because they saw the pencil only as a tool for "working the problem out."

Students who produced both mental and written estimates sometimes used the same strategy to produce both estimates and sometimes changed strategies. They seldom produced the same answer both times even if they used the same strategy.

The strategies used with the pencil were like those used in mental estimation. However, some mental strategies, like the comparison or ratio strategy, were not used with the pencil. Students also tended not to use an extracted number of digits with the pencil. There is sufficient evidence to say that while the use of the pencil can improve estimation and pencil-and-paper estimation should be included in teaching estimation skills, there are skills that may not be developed if only pencil-and-paper estimation practice is used.

In her study of college-aged estimators, Levine (1980) found nine types of difficulties, misconceptions, and mistakes. These were using an incomplete process, loss of
intermediate steps, using an incomplete strategy, not understanding the meaning of an operation, incorrectly adjusting the result of an operation, using an inappropriate algorithm, having trouble with place value in multiplication and division problems, rounding incorrectly, and having order of magnitude problems.

Students in this study exhibited most of these difficulties, particularly the power-of-ten error which involves both place value and order of magnitude. The process of mental estimation may have been interfered with by the thinking aloud process and this could have contributed to using an incomplete process such as forgetting to divide by 8 in problem five. It may also have contributed to forgetting intermediate sums and transposing digits in those sums. Students who used division to solve the percent problems certainly used an inappropriate algorithm. The difficulties with rounding have already been discussed. It seems that in those estimators who would not be classified as good estimators, Levine's categories of difficulties, misconceptions, and mistakes are appropriate.

One of the most prevalent errors that occurred during the interviews was the power-of-ten error. A power-of-ten error was made when the answer given would have been acceptable if it had been multiplied by an appropriate power of ten. An answer like 46,000 for problem one was off by a
power of ten because simply multiplying by ten would have brought it into the acceptable range. An answer like 35,000 also is of the wrong order of magnitude but multiplying by a power of ten would not have brought it into the acceptable range.

These mistakes were sometimes corrected when the student used a pencil and paper to produce an estimate. Almost always the student wrote the numbers in this type of estimate with the same number of digits as the original problem contained. Whether it was changing from the extracted number of digits to the same number of digits or simply being able to see the numbers which led to the improvement in this area is not certain. There was evidence in more than one case that the students wrote the same number of digits but used an extracted number of digits to produce the estimate. There is not doubt, however, that using a pencil to write the numbers used in making an estimate did lessen the number of power-of-ten errors.

The power-of-ten errors made by students in this study support the findings of Rubenstein (1985) that multiplying and dividing by powers of ten had a strong relationship with estimation performance. This may have curriculum implications for those who are interested in teaching estimation to those of limited ability.
Recommendations and Implications for Curriculum

The following recommendations are made for improving the curriculum for teaching estimation. These are based on the findings of the present study and the contemporary literature in teaching estimation.

1. Begin with pencil-and-paper estimates. Students whose mental arithmetic skills are weak may need the support which pencil and paper can provide. Also, seeing how estimation strategies simplify calculations may provide both insight into and motivation to use these processes.

2. Provide more work with compatible numbers. This may help estimators of limited ability choose their numbers more wisely.

3. Springboard on the natural tendency of estimators of limited ability to compare percents to easily found percents like 50 percent. Do more work with ten percent.

4. In teaching rounding for the purpose of estimation, start with truncation. Have students work with the first digits of the numbers in the problem and then let the natural desire to get a better estimate lead to rounding of first one number and then both. It might be desirable for students to estimate the product of two numbers, for example, by using the first digits, by rounding only the first factor, by rounding only the second factor, and then by rounding both factors.
5. Use numbers with five or fewer digits and no decimal places until estimators are comfortable with these. Then move on to numbers with more digits and to decimal fractions. It is particularly important that students recognize the effect of zeros after the decimal and know when they should be used in placing the decimal in the answer and when they should not be used.

6. Use the greatest integer function in working with fractions as suggested by Page (1970). This technique forces students to decide what whole number fractions and sums of fractions are close to. It is particularly important that students learn to recognize when a fraction is more or less than one half.

7. Use problems that ask a student to decide whether or not the answer to a problem is more or less than a given number. This would decrease the emphasis on exact answers.

8. When a student consistently uses an incorrect procedure for an operation such as finding a percent of a number, his teacher must erase that incorrect idea and then teach him the new procedure. Simply overlaying a new procedure will not be enough. It is important that teachers of estimators of limited ability recognize that they may have learned an incorrect procedure which needs to be dealt with before they can successfully learn to estimate using that particular operation.
In summary, it was the finding of this study that estimators of limited ability possess a variety of estimation strategies. They are sometimes not successful in using them because of their limitations in mental computation.

Recommendations for Further Research

Estimators of limited ability exhibited the same strategies for estimating the answers to computational problems as the good estimators studied by Reys, et. al. (1980). However, not only were they not as successful in producing acceptable answers, but they frequently did not recognize that their answers were unacceptable. Research should be done to determine how sensitive students of this ability level should be to unreasonable answers. Reys, et. al. addressed this issue with good estimators in the calculator part of their study.

Some students in this study made comments which led the researcher to believe that they could have produced a closer estimate for the answer to some of the problems. Research should also be done to determine if knowing the reason for the estimate affects the size of the error in the estimate that the student produces.


APPENDIX A

ASSESSING COMPUTATIONAL ESTIMATION TEST
ASSESSING COMPUTATIONAL ESTIMATION TEST

Exercise 1

89 + 382 + 706

Exercise 2

7465 - 572
Exercise 3

\[
\begin{align*}
87,419 \\
92,765 \\
90,045 \\
81,974 \\
+ 98,102
\end{align*}
\]

Exercise 4

\[
37,689 - 18,812
\]

Exercise 5

\[
87 \times 62
\]
Exercise 6

\[ 50 + 200 + 6 \]

Exercise 7

\[ 415 \times 7 \]

Exercise 8

\[ 28 \times 47 \]
Exercise 9

Exercise 10

Exercise 11
Exercise 12

\[ 200 \div 800 \]

Exercise 13

\[ 8 \sqrt{713} \]

Exercise 14

\[ 6809 \times 91 \]
Exercise 15

\[31 \times 68 \times 296\]

Exercise 16

\[\frac{347 \times 6}{43}\]

Exercise 17

\[308 \sim 2.85\]
Exercise 18

400 - 40

Exercise 19

0.7 + 0.002 + 0.81

Exercise 20

327 + 71.8
Exercise 21

835.67 - .526

Exercise 22

648 ÷ 1.06

Exercise 23

1\frac{1}{2} \times 1.67
Exercise 24

\[
1\frac{7}{8} \times 1.19 \times 4
\]

Exercise 25

\[
61.3 \times 0.8
\]

Exercise 26

\[
5.1 \times 4.8 \times 6.3
\]
Exercise 27

.95 \sqrt{17029.6}

Exercise 28

98.6 \times 0.041
Exercise 1

About how much do these cost?

APPLES = 32¢ each

Exercise 2

About how much do these cost?

$7.47 each
Exercise 3

About how much do these cost?

3½ lbs.

23¢ per lb.

Exercise 4

About how many raisins here?

238 raisins in a box

Exercise 5

About how much do these cost?

22 pens in a package

PENS 39¢ each
Exercise 6

About how much do these cost altogether?

39¢

99¢

Exercise 7

About how much do these cost altogether?

$23.67

$11.20

Exercise 8

About how much do these cost altogether?

$7.42

$1.29

$16.43
Exercise 9

![Diagram showing the price difference between two products: $36.95 and $65.65.]

About what is the difference in price?

Exercise 10

![Diagram showing the price difference between a car and another vehicle: $3,788 and $12,367.]

About what is the difference in price?

Exercise 11

![Diagram showing the price difference between two houses: $117,450 and $44,900.]

About what is the difference in price?
Exercise 12

About how far between cities?

ST. LOUIS 76
TIN CUP 165

Exercise 13

About how far between cities?

ST. LOUIS 21502
FORTUNA 23487

Exercise 14

You owe...
$1.29

About how much change will I get?

$5 $5
Five Dollars
Exercise 15

- You owe...
- About how much change will I get?
- $16.34
- $20 $20
- Twenty Dollars

Exercise 16

- About how much does one cost?
- $1.45

Exercise 17

- 24 cans in a case
- About how much does one cost?
- $1.40
Exercise 18

$2.59
12 pens

How much does one cost?

Exercise 19

About how many miles per gallon?

TRAVELED: 1322 miles
USED: 17 gallons gas

Exercise 20

1 need 1½ yards.

About how much will it cost?

MATERIAL
$1.67 per yard
Exercise 21

I need 4\frac{1}{2} yards.

About how much will it cost?

MATERIAL
$3.46 per yard

Exercise 22

About how much for 3 gallons?

GAS: 59¢ per \( \frac{1}{2} \) gallon

Exercise 23

I need 1\frac{2}{3} lbs. of cashews.

About how much will it cost?

CASHEWS
$1.19 per \( \frac{1}{4} \) lb.
Exercise 24

About what is the area?

28 x 47 is about...

Exercise 25

About what is the area?

Exercise 26

**TICKET PRICES**

- Adults: 3.25
- Children: 1.75

About how much do we need?
Exercise 27

**TICKET PRICES**

- Adults: 3.25
- Children: 1.75

About how much do we need?

Exercise 28

**CIRCLE ONE:**

- yes
- no
- not sure

Are you a good estimator?
Hello, my name is Mrs. Brame.
Do you remember the estimation test that you took in
_________________________’s class?

I’m talking to some of the students who took that test to
find out what methods or strategies they used to
estimate the answers to certain questions.

I’m going to show you some problems like those on the test,
and I want you to estimate the answers to them. As you
think about your estimate, I’d like for you to tell me what
you’re thinking. You may not think some of the things are
important, but they may help me understand what you’re
thinking. Please tell me your thoughts as you estimate.

Do you have any questions?
PROBLEM ONE

LOOK AT THIS PROBLEM.

AS YOU ESTIMATE THE ANSWER TO IT, PLEASE TELL ME WHAT YOU ARE THINKING.

87,419
92,765
90,045
81,974

+ 98,102

IF THE STUDENT DOES NOT THINK ALOUD, ASK HIM TO EXPLAIN THE PROCESS HE USED TO ARRIVE AT THE ESTIMATE.
PROBLEM TWO

31 x 68 x 295

USE THE PENCIL AND PAPER TO GET AN APPROXIMATION FOR THIS PROBLEM.

HOW DID YOU GET THAT ANSWER?
PROBLEM THREE

8127 \( \Rightarrow \) 474.257
PROBLEM FOUR

347 x 6
43

Can you get a better estimate using the pencil and paper?

Tell me how you got your answer.
PROBLEM FIVE

\[ 1 \frac{7}{8} \times 1.19 \times 4 \]

Can you get a better estimate using the pencil and paper?
PROBLEM SIX

ABOUT HOW MUCH AREA DOES THIS RECTANGLE HAVE?

28 CM

47 CM

(If a student expresses doubt about how to find the area of a rectangle, s/he will be told that the area is found by multiplying the length times the width.)
PROBLEM SEVEN

If 30% of the fans at the 1979 Superbowl bought one soda, about how many sodas were bought at the game?

<table>
<thead>
<tr>
<th>Year</th>
<th>Attendance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974</td>
<td>73,655</td>
</tr>
<tr>
<td>1975</td>
<td>86,421</td>
</tr>
<tr>
<td>1976</td>
<td>91,943</td>
</tr>
<tr>
<td>1977</td>
<td>96,509</td>
</tr>
<tr>
<td>1978</td>
<td>93,421</td>
</tr>
<tr>
<td>1979</td>
<td>106,409</td>
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</table>
PROBLEM EIGHT

At the 1979 Superbowl, 8,483 hot dogs were sold for $.60 each. About how much resulted from selling the hot dogs?

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Can you get a better estimate using pencil and paper?
PROBLEM NINE

Here are 3 estimates for the total attendance for the past 6 Superbowl games:
1,000,000
600,000
550,000

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Which estimate is the best?
The 1979 Superbowl netted $21,319,908 to be equally divided among the 26 NFL teams. About how much did each team receive?

Can you get a better estimate using pencil and paper?
Problem Eleven

Three people have dinner. They order:

- Bacon n Cheese Steakburger Platter
- Super Steakburger Platter
- Chili-Mac
- 2 small Coca-Cola's
- 1 hot chocolate
- 1 hot pie

About how much money will be needed to pay the bill?

Can you get a better estimate using pencil and paper?
DESSERTS AND DRINKS

STRAWBERRY SUNDAE
Crisp, topped with real strawberries, whipped topping, and a maraschino cherry.

HOT FUDGE NUT SUNDAE
With plenty of rich chocolate fudge, nuts, whipped topping, and a maraschino cherry.

BROWNIE FUDGE SUNDAE
Our brownies topped with ice cream, rich chocolate fudge, whipped topping, and a maraschino cherry.

HOT PIE
Dutch Apple, Dutch Cherry, or Southern Bacon.

CHEESE CAKE
A favorite. Our own creamy delicious.

CHEESE CAKE WITH STRAWBERRY TOPPING
Even better.

OUR FAMOUS VANILLA ICE CREAM

BROWNIE
Our own butter rich fudge brownie.

DANISH
Apple or Cinnamon. Served warm with butter.

TRU-FLAVOR MILK SHAKE
Made the old-fashioned way.

FROZEN ORANGE OR LEMON
A year-round flavor treat.

FLOATS
Coca-Cola, Root Beer, Orange, or Lemon

COCA-COLA 45c and 55c
SODA 45c and 55c
ROOT BEER 45c and 55c
TAB 45c and 55c
FRUIT DRINKS 45c and 55c
COFFEE 35c
HOT TEA 35c
ICED TEA 45c
MILK 45c and 55c
HOT CHOCOLATE 35c

PLATTER SPECIALS

All our sandwich platters are served with our own golden brown French Fries, and your choice of lettuce and tomato salad or baked beans.

BACON N CHEESE STEAKBURGER PLATTER
Featuring a big one-third pound "Steakburger" covered with melted real cheese and lots of crisp real bacon.

STAKES BURGER PLATTER
Features the original "Steakburger" Sandwich FAMOUS SINCE 1934 - served on our own delicious toasted bun with your choice of sandwich dressings.

SUPER STEAKBURGER PLATTER
Double Delicious with two "Steakburger" patties served on our own delicious toasted bun with your choice of sandwich dressings.

BAKED SUGAR CURED HAM (HOT OR COLD)
A generous portion of our own famous oven baked ham served on a toasted bun.

LO-CAL PLATTER
Two "Steakburger" patties served with sliced tomato, lettuce and cottage cheese.

CHILI SPECIALTIES

All our chili specialties start with 100% ground top round steak. Our kidney beans are plum red, simmered for hours. Our chili sauce is a special blend of tomato with spicy spices.

CHILI -- Our Own Genuine You will like it!

CHILI MAC
Liberal order of Italian spaghetti and chilli meat

CHILI THREE WAYS
ITALIAN SPAGHETTI, CHILI BEANS, AND CHILI MEAT

Extra meat available on above items
PROBLEM TWELVE

The Thompson's dinner bill totaled $28.75. Mr. Thompson wants to leave a tip of about 15%. About how much should he leave for the tip?
PROBLEM THIRTEEN

Which carton has more soda?

6 32-oz. Coca-Cola's

8 16-oz. Pepsi's
Problem Fourteen

Which soda is cheapest?

6 32-oz Coca-Cola's
$1.79

8 16-oz. Pepsi's
$1.29
BIBLIOGRAPHY

Books


Articles


**REPORTS**


Publications of Learned Organizations


Unpublished Materials


