GATEAUX DIFFERENTIABLE POINTS OF SIMPLE TYPE

DISSERTATION

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By

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Every continuous convex function defined on a separable Banach space is Gateaux differentiable on a dense $G_{\delta}$ subset of the space $E$ [Mazur]. Suppose we are given a sequence $(x_n)$ that is dense in $E$. Can we always find a Gateaux differentiable point $x$ such that $x = \sum_{n=1}^{\infty} a_n x_n$ for some sequence $(a_n)$ with infinitely many non-zero terms so that $\sum_{n=1}^{\infty} ||a_n x_n|| < \infty$? According to this paper, such points are called of "simple type," and shown to be dense in $E$. Mazur's theorem follows directly from the result and Rybakov's theorem (A countably additive vector measure $F: E \to X$ on a $\sigma$-field is absolutely continuous with respect to $|x^* F|$ for some $x^* \in X^*$) can be shown without deep measure theoretic involvement.
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CHAPTER I

INTRODUCTION

In 1919, H. Rademacher proved the theorem that for any Lipschitzian mapping of an open set \( G \subseteq \mathbb{R}^m \) into \( \mathbb{R}^n \) the differential exists almost everywhere in \( G \) (6). It was obviously of interest to extend this differentiability property that holds in Euclidean space to infinite dimensional Banach space. The earliest result in this area was Mazur's density theorem (3) which states essentially that the set of Gateaux differentiable points of a continuous convex function on a separable Banach space is a dense \( G_\delta \) set. This type of problem was renewed by E. Asplund (2). He called a Banach space a Weak Differentiability Space (or Strong Differentiability Space) if every continuous convex function is Gateaux (or Frechet) differentiable on a dense \( G_\delta \) subset of its domain of continuity, and he showed that the Banach space whose dual is rotund (or equivalent to rotund) is a Weak Differentiability Space. Since every separable normed linear space can be equivalently normed to have a rotund dual, this is obviously a generalization of Mazur's density theorem. Related topics have been continued and developed by others as E. H. Zarantonello (9) and N. Aronszajn (1). Zarantonello introduced a set valued monotone operator between Banach
spaces, and showed that the set of points where a monotone operator $T: X \to X^*$ from a separable Banach space into its dual is not single valued has an empty interior, and the domain of $T$ is a $F^c_\sigma$ set if it has no empty interior. A subgradient map of a continuous convex function from a Banach space into its dual space is a monotone operator, and a point at which the map is singleton is a Gateaux differentiable point. Thus this theorem, among others, is also an extension of Mazur's theorem. Aronszajn introduced an exceptional set in infinite dimensional Banach space, and discussed the differentiability of a Lipschitzian mapping from an infinite dimensional Banach space into another infinite dimensional Banach space.

At the same time M.A. Rieffel (7) brought the notion of dentable set and began determining geometric equivalents of the Radon-Nikodym property in Banach spaces. In 1975, I. Namioka and R.R. Phelps showed that the differentiability of continuous convex function problem is linked with the geometric aspects of the Radon-Nikodym property. In fact, they showed that if $X$ is an Asplund space then every closed bounded convex subset of $X^*$ is dentable; that is, if $X$ is an Asplund space, then $X^*$ has the Radon-Nikodym property (4). Also the known fact by then was that if $K$ is a closed bounded convex subset of a Banach space with the Radon-Nikodym property, then $K$ is the closed convex hull of its strongly exposed
points (5).

This paper is concerned with the study of Gateaux differentiability of a continuous convex function on a separable Banach space particularly in connection with geometric aspect of the Radon-Nikodym property. And its interest is to find a "simple type" of Gateaux differentiable point of a continuous convex function. By "simple type" is meant a point $x$ of the form $x = \sum_{n=1}^{\infty} a_n x_n$ for some sequence $(x_n)$ in the Banach space and $(a_n) \in \ell_\infty$ so that $\sum_{n=1}^{\infty} \|a_n x_n\| < \infty$.

First we will establish some closed bounded convex subset of the dual space whose exposed points contain every norm gradient of a given Banach space, and by using this we would like to find a "simple type" of smooth point relative to the subspace spanned by the given sequence $(x_n)$.

Secondly, given a continuous convex function defined on a separable Banach space we will define a peak function on some closed bounded convex subset of the dual space and again by using the property of exposed point, we will find a "simple type" of Gateaux differentiable point, which are dense in the space.

Finally, as an obvious result, we will have Mazur's density theorem as a corollary, and will prove Rybakov's theorem (8) (Let $F : \Gamma \rightarrow X$ be a countably additive vector measure. Then there is $x^* \in X^*$ such that $F \ll |x^* F|$) by a very elementary technique without deep measure theoretical involvement.


We will use the same notation used in Asplund's paper (1) for convex functions. Given a function $f$ on a Banach space $X$ with the values in $(-\infty, \infty]$, define the function $f^{\#}$ on $X^*$ by
\[
f^{\#}(\phi) = \sup\{\langle x, \phi \rangle - f(x); \ x \in X\},\]
where $\langle x, \phi \rangle$ denotes the value of the linear functional $\phi$ at $x$. $f^{\#}$ is called the dual function of $f$. Throughout this paper $f$ will be a continuous convex function defined on a real Banach space $X$, $U$ the closed unit ball of $X$, $\partial U$ the boundary of $U$, $U^*$ the norm closed unit ball of $X^*$, and $\partial U^*$ the boundary of $U^*$. We also assume that the effective domain of $f$ defined by $\text{dom} \ f = \{x \in X: f(x) < \infty \}$, is a non-empty, open, and convex subset of $X$.

$f^{\#}$ is lower semi-continuous in the weak* as well as in the norm topology, convex and with values in $(-\infty, \infty]$ if $f \notin (-\infty, \infty)$, which we will assume also. In that case, $(f^{\#})^{\#} = f^{**}$ can be defined on $X$ and $f = f^{**}$ since $f$ is a convex function.

**Definition.** $\phi \in \partial U^*$ is said to be a regularly exposed point of $U^*$ if there exists a unit vector $x \in X$ such that $\hat{x} \in X^{**}(\langle \phi, \hat{x} \rangle = \langle x, \phi \rangle)$ attains its supremum on $U^*$ only at $\phi$.

**Definition.** $f$ is said to be Gateaux differentiable at $x$
if for every \( v \in X \),
\[
\lim_{t \to 0} \frac{1}{t} [f(x + tv) - f(x)]
\]
exists, and converges to \( \langle v, \phi \rangle \) for a unique \( \phi \in X^* \). Such \( \phi \)
is called a Gateaux differential of \( f \) at \( x \). In case \( f \) is a norm function, we call \( x \in \partial U \) a smooth point of \( U \) of \( X \)
\( (x \in \text{sm}(U)) \), and \( \phi \) is called a norm gradient at \( x \).

Definition.
\[ \partial f(x) = \{ \phi \in X^* : \lim_{t \to 0} \frac{1}{t} [f(x+tv) - f(x)] \geq \langle v, \phi \rangle, v \in X \} \]
is said to be a sub-differential of \( f \) at \( x \). The following is an equivalent definition;
\[ \partial f(x) = \{ \phi \in X^*: f(y) \geq f(x) + \langle y-x, \phi \rangle, y \in \text{dom } f \} \]

Lemma 2.1. If \( x \in \text{sm}(U) \), then the norm gradient \( \phi \in X^* \) is a regularly exposed point of \( U^* \), and conversely.

Proof. Let \( \partial ||x|| \) be the sub-differential of the norm function at \( x \). Then
\[ \phi \in \partial ||x|| \text{ if and only if } ||x|| = \langle x, \phi \rangle \text{ and } \phi \in \partial U^* (4, p.30), \]
and
\[ x \in \text{sm}(U) \text{ if and only if } \partial ||x|| \text{ is a singleton set}(4, p.86). \]

Hence the theorem follows.

Definition. A point \( x \) of a closed bounded set \( K \) is strongly exposed if there is an \( x^* \in X^* \) such that
\[ \langle x, x^* \rangle = \sup x^*(K) \], and such that \( \lim_{n \to \infty} ||x_n - x|| = 0 \)
whenever \( (x_n) \) is a sequence in \( K \) with \( \lim_{n \to \infty} \langle x_n, x^* \rangle = \langle x, x^* \rangle \).

In this case, \( x^* \) strongly exposes \( x \) in \( K \).

Theorem 2.2. Let \( (x_n) \) be an arbitrary bounded sequence in \( X \) whose closed span is a subspace \( S \) of \( X \), and \( (r_n) \) be a
sequence of positive real numbers such that \( \sum_{n=1}^{\infty} r_n = 1 \). Then there is a smooth point \( x \) on the boundary of the unit ball of \( S(\text{relative to } S) \) so that \( x = \sum_{n=1}^{\infty} c_n r_n x_n \) for some \( (c_n) \in \ell_\infty \) and \( \sum_{n=1}^{\infty} \|c_n r_n x_n\| < \infty \). Furthermore such smooth points are dense in the boundary of the unit ball of \( S \).

**Proof.** Without loss of generality, we assume \( \|x_n\| = 1 \) for all \( n \). Define \( T: \mathbb{U}^* \rightarrow \ell_1 \) by

\[
T(\phi) = \{r_n \langle x_n, \phi \rangle\}_{n=1}^{\infty},
\]

and let

\[
B = \{T(\phi)\} \phi \in \mathbb{U}^* \quad \text{and} \quad K = \overline{co}(B),
\]

where "\( \overline{co} \)" refers to "closed convex hull of".

\( K \) is a non-empty bounded convex subset of \( \ell_1 \), and \( \ell_1 \) has the Radon-Nikodym property. Hence \( K \) is the closed convex hull of its strongly exposed points (2, p.202). Let \( \xi \in K \) be a strongly exposed point of \( K \), and write \( \xi = \{r_n \xi_n\}_{n=1}^{\infty} \) by using the given sequence \( (r_n) \). Then by the definition of strongly exposed point, there is \( c = (c_n) \in \ell_\infty \) such that

\[
\langle \xi, c \rangle > \langle n, c \rangle \quad \text{for all } n \neq \xi \in K.
\]

We will show that \( \xi = T(\phi) \) for some \( \phi \in \mathbb{U}^* \).

By Krein-Milman's theorem (4, p.74),

\[
\xi \in \{\text{extreme points of } K\} \subseteq B
\]

since \( K \) is compact (3, p.338). Let

\[
\xi = \lim_{k \to \infty} T(\phi_k) \quad \text{or} \quad \xi_n = \lim_{k \to \infty} \langle x_n, \phi_k \rangle \quad \text{for all } n.
\]

Suppose \( \phi \) is a weak* cluster point of \( \phi_k 's \). (Such \( \phi \) exists in \( \mathbb{U}^* \) since \( \mathbb{U}^* \) is weak* compact(4,p.40)). Then we have

\[
\xi_n = \langle x_n, \phi \rangle \quad \text{for all } n, \quad \text{and hence } \xi = T(\phi).
\]
Recapitulating, we obtain 
\[ \langle T(\phi), c \rangle > \langle T(\psi), c \rangle, \psi \in \partial U^* \text{ if } T(\phi) \neq T(\psi) \text{ or } \phi|_S \neq \psi|_S. \]

If we show \( \phi \) is a regularly exposed point of \( U^* \), then by the Lemma 2.1, the proof is complete. By the definition of the mapping \( T \),
\[ \langle T(\phi), c \rangle = \sum_{n=1}^{\infty} c_n r_n \langle x_n, \phi \rangle = \sum_{n=1}^{\infty} c_n r_n \langle x_n, \phi \rangle = \langle x, \phi \rangle \]
if we set \( x = \sum_{n=1}^{\infty} c_n r_n x_n \).

We assume \( \langle x, \phi \rangle = 1 \). Otherwise we may multiply some positive constant to \( c \) to obtain this desired result. Our goal is to show that such \( x = \hat{x} \) obtained is a linear functional defined on \( U^* \) that attains its supremum only at \( \phi \) and \( \phi \in \partial U^* \). We want to show
\[ \phi \in \partial U^* \text{ and } \langle x, \phi \rangle = ||x||. \]

Since \( \phi \in U^* \),
\[ 1 = \langle x, \phi \rangle \leq ||x|| = \sup_{\psi \in U^*} \langle x, \psi \rangle = \sup_{\psi \in U^*} \{ \sum_{n=1}^{\infty} c_n r_n \langle x_n, \psi \rangle \} = \sup_{\psi \in U^*} \langle T(\psi), c \rangle \leq \langle x, \phi \rangle = 1. \]

This implies \( ||x|| = \langle x, \phi \rangle = 1 \), and hence the result follows. Also \( \phi \) is unique in the sense that any two linear functionals agree on the subspace \( S \).

Finally we would like to show that the smooth points obtained by this manner are dense in \( S \).

Let \( L \subseteq \ell_\infty \) be the set of strongly exposing points of \( K \), and define a map \( F: L \rightarrow X \) by \( F(c) = \sum_{n=1}^{\infty} c_n r_n x_n \), where \( (r_n) \) is the given sequence.

\( F \) is norm to norm continuous.

Suppose \( D \), the set of smooth points obtained by this
theorem, is not dense in $S$. Then there exists a $z \in S$ and an open $\varepsilon$-ball (relative to $S$) centered at $z$, $B(z;\varepsilon)$, so that $B(z;\varepsilon) \cap D = \emptyset$. This implies that we can find a $y$ of the form $y = \sum_{n=1}^{\infty} a_n x_n$, a linear combination of $x_n$'s, such that $\| y - z \| < \varepsilon/2$, or equivalently $\| F(a) - z \| < \varepsilon/2$. But $L$ is dense in $\ell_\infty$ and $T$ is continuous. Thus there exists $a' \in L$ such that $\| F(a') - F(a) \| < \varepsilon/2$, which implies

$$\| z - F(a') \| \leq \| z - F(a) \| + \| F(a) - F(a') \| < \varepsilon.$$ 

This contradicts the assumption, $B(z;\varepsilon) \cap D = \emptyset$, since $F(a')$ is dense in $D$. 
CHAPTER BIBLIOGRAPHY


CHAPTER III

PEAK FUNCTIONS AND GATEAUX DIFFERENTIABLE FUNCTIONS

We will find a "simple type" of Gateaux differentiable point of a given continuous convex function $f$ defined on a separable Banach space $X$.

**Definition.** For each $x \in \text{dom } f$, we can define a real valued function $P_x$ on $X^*$ by

$$P_x(\phi) = \langle x, \phi \rangle - f^*(\phi).$$

We call $P_x$ a peak function with respect to $f$ if $P_x$ attains its supremum ($= f(x)$) at a single point $\phi \in X^*$.

**Lemma 3.1.** If $x$ is a Gateaux differentiable point of $f$ with its Gateaux differential $\phi$, then $P_x$ is a peak function with respect to $f$ and assumes its supremum at $\phi$ in $X^*$, and conversely.

**Proof.** We would like to show that

$\phi \in \partial f(x)$ if and only if $f(x) = \langle x, \phi \rangle - f^*(\phi)$.

Suppose $\phi \in \partial f(x)$. Then by the definition,

$$f(y) \geq f(x) + \langle y-x, \phi \rangle, \quad y \in \text{dom } f.$$

This implies for all $y \in \text{dom } f$,

$$f(x) \leq f(y) + \langle x-y, \phi \rangle = \langle x, \phi \rangle - \{ \langle y, \phi \rangle - f(y) \}$$

$$\leq \langle x, \phi \rangle - \sup \{ \langle y, \phi \rangle - f(y) : y \in \text{dom } f \}$$

$$= \langle x, \phi \rangle - f^*(\phi) \leq f(x).$$

Therefore
\[ f(x) = \langle x, \phi \rangle - f^*(\phi) . \]

Conversely, if we assume \( f(x) = \langle x, \phi \rangle - f^*(\phi) \), then
\[
f(x) = \langle x, \phi \rangle - f^*(\phi) \\
= \langle x, \phi \rangle - \sup \{ \langle y, \phi \rangle - f(y) : y \in \text{dom} \ f \} \\
\leq \langle x, \phi \rangle - (\langle y, \phi \rangle - f(y)), \ y \in \text{dom} \ f .
\]

Hence
\[
f(y) \geq f(x) + \langle y-x, \phi \rangle, \ y \in \text{dom} \ f ;
\]
that is \( \phi \in \partial f(x) . \)

This consequence combined with the following fact;
\( \partial f(x) \) is a singleton set if and only if \( x \) is a Gateaux differentiable point of \( f \) (2, p.86), gives the desired result.

**Lemma 3.2.** Let \( h \) be a Lipschitzian mapping defined on an open ball \( G \) in \( X \) with Lipschitz constant \( M \). If there is \( x \) in \( G \) and a sequence \( (\phi_k) \) in \( X^\ast \) so that
\[
\lim_{k \to \infty} \{ \langle x, \phi_k \rangle - h^*(\phi_k) \} = h(x) ,
\]
\( (\phi_k) \) is in \( 2M^* \) = \{ \phi \in X^* : \|\phi\| \leq 2M \} \) eventually, and there is \( \phi \in 2M^* \) such that \( h(x) = \langle x, \phi \rangle - h^*(\phi) . \)

**Proof.** Suppose \( B(x; \varepsilon) = \{ y \in X : \|y-x\| \leq \varepsilon \} \) is in \( G \), and for each \( k \), let \( \varepsilon_k \) be a positive number which tends to zero so that \( h(x) \leq \langle x, \phi_k \rangle - h^*(\phi_k) + \varepsilon_k . \)

This implies
\[
h(x) \leq \langle x, \phi_k \rangle - \langle y, \phi_k \rangle + h(y) + \varepsilon_k
\]
for all \( y \in G \) particularly, or equivalently
\[
\langle y-x, \phi_k \rangle \leq h(y) - h(x) + \varepsilon_k, \ y \in G .
\]

Thus
\[
\sup \{ \langle y-x, \phi_k \rangle, y \in B(x; \varepsilon) \} \leq \sup \{ |h(y)-h(x)|, y \in B(x; \varepsilon) \} + \varepsilon_k .
\]
Therefore \( e \| \phi_k \| \leq M \varepsilon + M \varepsilon \), and \( \| \phi_k \| \leq 2M \) eventually.

Now since \((\phi_k)\) is in \( 2MU^* \) eventually and \( 2MU^* \) is weak*-compact, there exists a weak* cluster point \( \phi \in 2MU^* \). By the lower semi-continuity of \( h^* \)

\[
h(x) \leq <x, \phi> - h^*(\phi).
\]

On the other hand, by the definition of \( h^*(\phi) \),

\[
h(x) \geq <x, \phi> - h^*(\phi).
\]

Hence

\[
h(x) = <x, \phi> - h^*(\phi).
\]

**Definition.** For a continuous convex function \( f \) on a Banach space \( X \), define a restricted dual function \( f^*(\phi | G) \) by

\[
f^*(\phi | G) = \sup \{<x, \phi> - f(x); x \in G\}.
\]

Obviously \( f^*(\phi | G) \leq f^*(\phi) \), and since the function defined by the supremum of any family of continuous functions is lower semi-continuous, \( f^*(\phi | G) \) is, like \( f^*(\phi) \), lower semi-continuous in the weak* as well as in the norm topology and convex.

**Lemma 3.3.** With the same assumption as in the Lemma 3.2 except \( X \) to be separable, define a map \( T: X^* \to \ell_1 \) by

\[
T(\psi) = \{r_n [<x_n, \psi> - h^*(\psi | G)]\}_{n=1}^\infty,
\]

where \((x_n)\) is a dense sequence in \( G \) and \( r_n \) is a positive real number so that \( \sum_{n=1}^\infty r_n = 1 \). Then \( T \) is injective.

**Proof.** Suppose \( T(\psi) = T(\phi) \). Then for each \( n \),

\[
<x_n, \psi> - h^*(\psi | G) = <x_n, \phi> - h^*(\phi | G),
\]

or

\[
<x_n, \psi - \phi> = h^*(\psi | G) - h^*(\phi | G).
\]
\[ \psi - \phi \] is a bounded linear functional on a separable Banach space \( X \), and is constant on \( G \) since \( (x_n) \) is dense in \( G \). Also the value of the constant is 0 as \( G \) contains non-empty interior. Thus \( \psi = \phi \); that is, \( T \) is injective.

**Theorem 3.4.** Let \( f \) be a continuous convex function defined on a separable Banach space \( E \), and \( (x_n) \) be a sequence of points dense in \( E \). Then for every open convex subset \( G \) of \( E \), \( f \) has a Gateaux differentiable point \( x \in G \) such that
\[ x = \sum_{n=1}^{\infty} c_n x_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|c_n r_n x_n\| < \infty , \]
where \( (c_n) \in \ell_\infty \) and \( \sum_{n=1}^{\infty} r_n = 1 \) \( (r_n \geq 0) \). Furthermore such Gateaux differentiable points are dense in \( E \).

**Proof.** Every continuous convex function is locally Lipschitzian (1, p.13), hence we assume that \( f \) is a Lipschitz function on all of \( G \) with Lipschitz constant \( M \). Also we may assume, without loss of generality, that \( G \) is an open ball centered at \( x_0 \); that is, \( G = B(x_0;1) \).

Let \( (r_n^1) \) be a sequence with positive numbers such that \( \sum_{n=1}^{\infty} r_n^1 = 1 \), and define a new sequence \( (r_n) \) by \( r_n = r_n^1 \) if \( x_n \in G \) and \( r_n = 0 \) otherwise.

Define a map \( T : E^* \to \ell_1 \) by
\[ T(\phi) = (r_n (\langle x_n, \phi \rangle - f^*(\phi | G)))_{n=1}^{\infty} . \]

By a slight modification of the Lemma 3.3, we have \( T \) injective. Let
\[ B = \{ T(\phi) \}_{\phi \in 2^M \ell_1^*} , \quad \text{and} \quad K = \overline{\text{co}}(B) . \]

As in the Theorem 2.2, \( K \) is a closed bounded subset of \( \ell_1 \), hence is the closed convex hull of its strongly exposed
points. Also the exposing points of \( K \) are dense in \( \ell_\infty \). Let \( c = (c_n) \in \ell_\infty \) be a strongly exposing point of \( K \) whose components are all positive such that \( \sum_{n=1}^{\infty} c_n r_n = 1 \).

Let \( \xi \in \ell_1 \), a corresponding exposed point of \( K \). We would like to show that
\[
\xi = T(\phi) \quad \text{for some } \phi \in 2MU^*.
\]

By Krein-Milman's theorem,
\[
\xi \in B, \quad \text{or } \xi = \lim_{k \to \infty} T(\phi_k), \quad \{\phi_k\} \subseteq 2MU^*,
\]
and if we write \( \xi = (r_n \xi_n) \), by using the sequence \( (r_n) \) and by setting \( \xi_n = 0 \) if \( r_n = 0 \), then for every \( \xi_n \neq 0 \),
\[
\lim_{k \to \infty} \langle x_n, \phi_k \rangle = \|f(\phi_k | G)\| = \xi_n.
\]

2MU* is weak* compact (1, p.77), hence \( \{\phi_k\} \) has a weak* cluster point, say, \( \phi \in 2MU^* \). For each \( \xi_n \neq 0 \), by the lower semi-continuity of \( f^*(\cdot | G) \),
\[
\langle x_n, \phi \rangle - f^*(\phi | G) \geq \xi_n,
\]
and since \( c_n > 0 \) and \( r_n > 0 \),
\[
\sum_{n=1}^{\infty} c_n r_n \langle x_n, \phi \rangle - f^*(\phi | G) \geq \sum_{n=1}^{\infty} c_n r_n \xi_n,
\]
or equivalently,
\[
\langle T(\phi), x \rangle \geq \langle \xi, x \rangle.
\]

On the other hand, since \( \xi \) is a strongly exposed point of \( K \) and \( T(\phi) \in K \),
\[
\langle T(\phi), x \rangle < \langle \xi, x \rangle, \quad \text{unless } \xi = T(\phi).
\]

Therefore
\[
\xi = T(\phi).
\]

In other words, if we set \( x = \sum_{n=1}^{\infty} c_n r_n x_n \) and take into account that \( T \) is injective and \( \sum_{n=1}^{\infty} c_n r_n = 1 \), then we have
\[ <\xi, c> = <T(\phi), c> = \sum_{n=1}^{\infty} c_n r_n \{<x_n, \phi> - f^*(\phi | G)\} \]
\[ = <\sum_{n=1}^{\infty} c_n r_n x_n, \phi> - \sum_{n=1}^{\infty} c_n r_n f^*(\phi | G) = <x, \phi> - f^*(\phi | G). \]
Consequently,
\[ <x, \phi> - f^*(\phi | G) > <x, \psi> - f^*(\psi | G), \psi \in E^* (\psi \neq \phi). \ldots (1) \]
x \in G = B(x_0, 1) is obvious because
\[ \|x - x_0\| = \|\sum_{n=1}^{\infty} c_n r_n (x_n - x_0)\| \leq \sum_{n=1}^{\infty} c_n r_n \|x_n - x_0\| < 1. \]

Now we would like to claim that such x obtained is a required Gateaux differentiable point.

According to the Lemma 3.1, it suffices to show that \( P_x \) is a peak function with respect to f. We will show that
\[ P_x(\phi) = <x, \phi> - f^*(\phi) = f(x) \]
only at a single point \( \phi \). By the definition of the dual function,
\[ f(x) = f^**(x) = \sup \{<x, \psi> - f^*(\psi) : \psi \in E^*\}. \]
Suppose
\[ f(x) = \frac{L}{\lim_{n \to \infty} \{<x, \psi_n> - f^*(\psi_n)\}}. \]
By the Lemma 3.2, there exists \( \psi \in 2MU^* \) such that
\[ f(x) = <x, \psi> - f^*(\psi), \]
but according to the formula (1) and the fact that
\[ f^*(\psi) \geq f^*(\psi | G), \]
\[ <x, \psi> - f^*(\psi) \leq <x, \psi> - f^*(\psi | G) \]
\[ <x, \phi> - f^*(\phi | G) \quad \text{if } \psi \neq \phi. \]
Now since \( x \in G \) and \( f^*(\phi | G) \geq <x, \phi> - f(x) \) for any \( x \in G \),
\[ <x, \phi> - f^*(\phi | G) \leq <x, \phi> - (<x, \phi> - f(x)) = f(x). \]
Combining all these facts, we see that
\[ f(x) = \langle x, \psi \rangle - f^*(\psi) \]
\[ \langle x, \phi \rangle - f^*(\phi | G) \leq f(x) \text{ unless } \psi = \phi. \]

Hence any such \( \psi \) must be identically equal to \( \phi \), and therefore \( \phi \) is unique. Thus \( x = \sum_{n=1}^{\infty} c_n r_n x_n \) is a required Gateaux differentiable point with its Gateaux differential \( \phi \).

Note that the above proof shows that for any point \( x_0 \) in \( E \) and any \( \varepsilon \)-neighborhood of \( x_0, G \), we can find a Gateaux differentiable point of "simple type" in \( G \). Hence such Gateaux differentiable points are dense in \( E \).
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CHAPTER IV

APPLICATIONS

We are going to apply our result to prove Mazur's density theorem and Rybakov's theorem. It will depend on a very elementary concept.

Lemma 4.1. If $E$ is a separable linear topological space, and $f: E \to \mathbb{R}$ is a continuous real convex function, then $G = \{x \in E : f$ is Gateaux differentiable at $x\}$ is a $G_\delta$ set in $E$.

Proof. Let $(x_n)$ be dense in $E$, and let
$$Df(x; x_n) = \lim_{t \to 0} \frac{1}{t} [f(x + tx_n) - f(x)].$$
Then $x \in G$ if and only if $Df(x; x_n) + Df(x; -x_n) = 0$ because of the convexity of the function $f$. Set
$$G(m; n) = \{x \in E : Df(x; x_n) + Df(x; -x_n) < \frac{1}{m}\}.$$\[
G(m; n) = \{x \in E : \lim_{k \to \infty} [f(x + \frac{1}{k}x_n) + f(x - \frac{1}{k}x_n) - 2f(x)] < \frac{1}{m}\}.
\]
We see that $G(m; n)$ is open for each $n$ and $m$ since
$$G(m; n) = \bigcap_{k=1}^{\infty} \{x \in E : h[f(x + \frac{1}{k}x_n) + f(x - \frac{1}{k}x_n) - 2f(x)] < \frac{1}{m}\}.$$
Hence
$$G = \bigcap_{n, m=1}^{\infty} G(m; n)$$
is a $G_\delta$ set.

Theorem 4.2. [Mazur] In a separable Banach space $E$, every continuous convex function $f$ on an open convex set $G$ of $E$ is Gateaux differentiable on a dense $G_\delta$ subset of $E$.

Proof. Theorem 3.4 together with the Lemma 4.1 give
the result.

In the following theorems, let $\mathcal{E}$ be a $\sigma$-field of subsets of the point set $\Omega$, and $X$ be a Banach space.

**Definition.** Let $\mathcal{E}$ be a field of subsets of $\Omega$, $F: \mathcal{E} \to X$ be a vector measure and $\mu$ be a finite non-negative real-valued measure on $F$. If $\lim_{E \to 0} F(E) = 0$, then $F$ is called $\mu$-continuous and this is signified by $F \ll \mu$.

The following theorem mainly due to Bartle-Dunford-Schwartz is used to prove Rybakov's Theorem. The proof of the last part of forming $\mu$ can be found in the proof of (3, pp. 11-13; Theorem 4).

**Theorem 4.3.** Let $F$ be an $X$-valued countably additive vector measure defined on a $\sigma$-field $\mathcal{E}$. Then there exists a non-negative real valued countably additive measure $\mu$ on $\mathcal{E}$ such that $F \ll \mu$. Moreover $\mu$ can be chosen so that

$$\mu = \sum_{n=1}^{\infty} \beta_n |x_n^* F|$$

for some $(x_n^*) \subseteq U(X^*)$, the unit ball of $X^*$, where $\beta_n \geq 0$, and can be selected such that $\sum_{n=1}^{\infty} \beta_n = 1$.

**Theorem 4.4.** [Rybakov] Let $F: \mathcal{E} \to X$ be a countably additive vector measure. Then there exists $x^* \in X^*$ such that $F \ll |x^* F|$.

**Proof.** By Theorem 4.3, there is a sequence $(x_n^*)$ in the unit ball of $X^*$ such that $F \ll \sum_{n=1}^{\infty} \beta_n |x_n^* F|$, where $\beta_n \geq 0$ and $\sum_{n=1}^{\infty} \beta_n = 1$.

Let $E = \text{ca}(\mathcal{E}, R)$, a Banach space equipped with the total variation norm. Since $F$ is countably additive on $\sigma$-field $\mathcal{E}$,
\((x_n^*F)\) is a bounded sequence of countably additive scalar valued measures (3, p. 9), and hence we can think of the subspace \(S\) of \(E\) spanned by \(\{x_n^*F\}_{n=1}^{\infty}\).

By the Theorem 2.2, there exists a smooth point \(\mu\) on the boundary of the unit ball of \(S\) (relative to \(S\)) so that

\[
\mu = \sum_{n=1}^{\infty} c_n r_n x_n^* F,
\]

If we set \(x^* = \sum_{n=1}^{\infty} c_n r_n x_n^*\), then

\[
x^* F = \sum_{n=1}^{\infty} c_n r_n x_n^* F.
\]

Because of the smoothness of the point \(\mu = x^* F\),

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{ |\mu + \varepsilon \nu_n| - |\mu| \}
\]

exists for all \(n\). This implies that

\[
|\nu_n| = |x^* F| \ll |x^* F| = |\mu| (2, p. 635),
\]

for all \(n\). Therefore

\[
\sum_{n=1}^{\infty} \beta_n |x_n^* F| \ll |x^* F|, \text{ and hence } F \ll |x^* F|.
\]
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