DUELLY SEMIMODULAR CONSISTENT LATTICES

DISSERTATION

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By

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A lattice $L$ is said to be dually semimodular if for all elements $a$ and $b$ in $L$, $a \lor b$ covers $b$ implies that $a$ covers $a \land b$. $L$ is consistent if for every join-irreducible $j$ and every element $x$ in $L$, the element $x \lor j$ is a join-irreducible in the upper interval $[x,1]$. In this paper, finite dually semimodular consistent lattices are investigated. Examples of these lattices are the lattices of subnormal subgroups of a finite group.

In 1954, R. P. Dilworth proved that in a finite modular lattice, the number of elements covering exactly $k$ elements is equal to the number of elements covered by exactly $k$ elements. Here, it is established that if a finite dually semimodular consistent lattice has the same number of join-irreducibles as meet-irreducibles, then it is modular. Hence, a converse of Dilworth's theorem, in the case when $k$ equals 1, is obtained for finite dually semimodular consistent lattices.

Several combinatorial results are shown for finite consistent lattices similar to those already established for finite geometric lattices. The reach of an element $x$ in a lattice $L$ is the difference between the rank of $x^*$, the join
of $x$ and all the elements covering $x$, and the rank of $x$; the maximum reach of all elements in $L$ is the reach of $L$. Sharp lower bounds for the total number of elements and the number of elements of a given reach in a semimodular consistent lattice given the rank, the reach, and the number of join-irreducibles are found. Extremal lattices attaining these bounds are described. Similar results are then obtained for finite dually semimodular consistent lattices.
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CHAPTER I

INTRODUCTION

Basic Definitions

We first present some basic definitions and results in lattice theory. We follow the terminology found in Birkhoff [2] and Aigner [1].

A partially ordered set or poset is a set \( P \) in which a binary relation \( x \leq y \) is defined satisfying the following conditions for all \( x, y, \) and \( z \):

\[
\begin{align*}
\text{P1. } & x \leq x \quad \text{(reflexive)} \\
\text{P2. } & \text{If } x \leq y \text{ and } y \leq x, \text{ then } x = y. \quad \text{(anti-symmetry)} \\
\text{P3. } & \text{If } x \leq y \text{ and } y \leq z, \text{ then } x \leq z. \quad \text{(transitivity)}
\end{align*}
\]

If \( x \leq y \) in a poset \( P \), the interval \([x,y]\) is the set of elements \( z \) between \( x \) and \( y \), that is,

\[
[x,y] = \{ z \mid x \leq z \leq y \}.
\]

An element \( p \) covers an element \( q \) if \( p \geq q \) and the interval \([q,p]\) contains two elements. If \( x \leq y \) or \( y \leq x \) for all \( x \) and \( y \) in \( P \), then \( P \) is totally ordered and is called a chain.

The dual of a poset \( P \) is the poset obtained from \( P \) by inverting the order relation.

A lattice is a poset \( P \) in which any two elements \( a \) and \( b \) have a greatest lower bound or meet, denoted by \( a \wedge b \), and a least upper bound or join, denoted by \( a \vee b \).
Throughout this paper, all lattices are assumed to be finite. Thus, \( L \) has a unique minimum 0 and a unique maximum 1. Elements which cover 0 are called atoms and elements covered by 1 are called coatoms.

A lattice is semimodular if \( a \lor b \) covers \( b \) whenever \( a \) covers \( a \land b \), and is dually semimodular if \( b \) covers \( a \land b \) whenever \( a \lor b \) covers \( a \). Any interval of a semimodular [respectively, dually semimodular] lattice is also semimodular [respectively, dually semimodular]. A lattice is modular if \( L \) is both semimodular and dually semimodular. A lattice in which every element is the join of atoms is called an atomic lattice. A coatomic lattice is a lattice in which every element is the meet of coatoms. A geometric lattice is a semimodular atomic lattice.

A lattice \( L \) satisfies the Jordan-Dedekind chain condition if all maximal chains between the same endpoints have the same length. The Jordan-Dedekind chain condition holds in any semimodular or dually semimodular lattice. Hence, a rank function \( r: L \to \mathbb{N} \) is defined on \( L \) by the properties: \( r(0) = 0 \) and \( r(a) = r(b) + 1 \) if \( a \) covers \( b \) in \( L \). The rank \( r(L) \) of a lattice is the rank of the maximum 1. In a lattice \( L \) satisfying the Jordan-Dedekind chain condition, \( W_k \) denotes the number of elements in \( L \) of rank \( k \) and is called the kth Whitney number of \( L \).

An element \( j \) in a lattice \( L \) is a join-irreducible if \( j = a \lor b \) implies \( j = a \) or \( j = b \). A meet-irreducible is a
join-irreducible in the order dual. A lattice $L$ is consistent if for every join-irreducible $j$ and every element $x$ in $L$, the element $x \lor j$ is a join-irreducible in the upper interval $[x, 1]$. We shall denote by $J$ the set of join-irreducibles in a lattice and $M$ denotes the set of meet-irreducibles. The following lemma will be useful.

**Lemma I.1.** Let $L$ be a finite consistent lattice. If $x$ is in $L$, then the upper interval $[x, 1]$ is consistent.

Let $x_*$ be the meet of $x$ and all elements covered by $x$ and let $x^*$ be the join of $x$ and all elements covering $x$.

The coreach of $x$ is the difference between the rank of $x$ and the rank of $x_*$: $\text{coreach}(x) = r(x) - r(x_*)$. Similarly, the reach of $x$ equals $r(x^*) - r(x)$. The coreach of $L$ is the maximum coreach of all the elements in $L$ and the reach of $L$ is the maximum reach of all the elements in $L$.

The Möbius function of a lattice $L$ is the function $\mu : L \times L \rightarrow \mathbb{Z}$ defined by:

$$\mu(x, y) = 0 \quad \text{if } x \nleq y,$$

$$\mu(x, x) = 1,$$

and

$$\mu(x, y) = - \sum_{x \leq z \leq y} \mu(x, z) = - \sum_{x \leq z < y} \mu(x, z) \quad \text{if } x < y.$$

A useful fact concerning the Möbius function of a geometric lattice is Rota's Theorem [13].
Theorem 1.2. Let \( x \leq y \) in a finite geometric lattice \( L \). Then \( \mu(x, y) \) is non-zero and the sign of \( \mu(x, y) \) is \((-1)^{r(y)-r(x)}\).

If \( A \) and \( B \) are subsets of a lattice \( L \), \( A \) is concordant with \( B \) if, for every element \( x \) in \( L \), either \( x \) is in \( B \) or there exists an element \( x^+ \) such that

(CS1) the Möbius function \( \mu(x, x^+) \neq 0 \) and

(CS2) for every element \( a \) in \( A \), \( x \vee a \neq x^+ \).

In the remaining sections of this introduction, we discuss other results necessary for establishing the results in this dissertation.

Subnormal Subgroups

We summarize results used in Chapter II in this section.

A series of a group \( G \) is a finite sequence of subgroups of \( G \) such that

\[
1 = G_0 \leq G_1 \leq \ldots \leq G_{n-1} \leq G_n = G
\]

and \( G_{i-1} \) is normal in \( G_i \) for \( 1 \leq i \leq n \). A normal series of \( G \) is a series of \( G \) in which every subgroup \( G_i \) is also normal in \( G \). A composition series of \( G \) is a chain of subgroups

\[
1 = G_0 \leq G_1 \leq G_2 \leq \ldots \leq G_{n-1} \leq G_n = G
\]
such that \( G_{i-1} \) is normal in \( G_i \) and the factors \( G_i / G_{i-1} \) are simple for \( i = 1, 2, \ldots, n \). A subgroup \( H \) of \( G \) is subnormal if and only if \( H \) is a term of some composition series of \( G \). Only finite groups are considered in this study. Finite
groups have composition series.

The theory of subnormal subgroups was initiated by H. Wielandt [14]. We first state, without proof, two important theorems needed for forming the lattice of subnormal subgroups due to Wielandt.

Theorem 1.3. Let \( H \) and \( K \) be subnormal subgroups of a group \( G \). Then \( H \cap K \) is subnormal in \( G \).

If \( H \) and \( K \) are subnormal subgroups of \( G \), \( HK \) need not necessarily be a subgroup of \( G \); an example is given in Rose [12] where \( G \) is the dihedral group of order \( 2^n \) with \( n \geq 3 \). Let \( \langle H, K \rangle \) be the subgroup generated by \( H \) and \( K \); that is, the intersection of all subgroups in \( G \) containing both \( H \) and \( K \).

Theorem 1.4. Suppose that \( G \) has a composition series. If \( H \) and \( K \) are subnormal subgroups of \( G \), then \( \langle H, K \rangle \) is subnormal in \( G \).

One well-known result of group theory we will use is the third isomorphism theorem.

Theorem 1.5. Let \( H \) be a subgroup of a group \( G \) and \( K \) a normal subgroup of \( G \). Then \( H \cap K \) is a normal subgroup of \( H \) and

\[
\frac{H}{H \cap K} \cong \frac{HK}{K}.
\]
The last subject studied in Chapter II deals with the relationship between a group G and its lattice L(G) of subgroups. The center Z(G) is the subgroup
\[ \{ y \mid xy = yx \text{ for all } x \in G \}. \]
A factor H/K of a series of G is a central factor of G if K is normal in G and H/K ≤ Z(G/K). The group G is nilpotent if it has a series all of whose factors are central factors of G. The group G is soluble if it has a series all of whose factors are abelian. A group G is said to be supersoluble if it has a normal series all of whose factors are cyclic.

**Theorem I.6.** Let G be a nilpotent group G. Then every subgroup of G is subnormal in G.

The following result is due to Iwasawa [7].

**Theorem I.7.** The lattice of subgroups of a finite group G satisfies the Jordan-Dedekind chain condition if and only if G is supersoluble.

The Radon Transform and the Upper Complement Transform

In Chapter III, a converse, in the case of dually semimodular consistent lattices, is found for the following theorem due to R. P. Dilworth [3].
Theorem 1.8. Let \( L \) be a finite modular lattice and \( k \) a positive integer. The number of elements covering exactly \( k \) elements equals the number of elements covered by exactly \( k \) elements.

In particular, in a modular lattice, the number of join-irreducibles equals the number of meet-irreducibles. The main theorem of this chapter establishes that a dually semimodular consistent lattice with the same number of join-irreducibles as meet-irreducibles is modular. The results needed to prove this are summarized here.

Let \( L \) be a finite lattice and \( f: L \to \mathbb{Q} \) a function from \( L \) to the rational numbers \( \mathbb{Q} \). The Radon transform of \( f \) is the function \( T_f: L \to \mathbb{Q} \) defined by

\[
T_f(x) = \sum_{y \leq x} f(y).
\]

The upper complement transform of \( f \) is the function \( C_f: L \times L \to \mathbb{Q} \) defined by

\[
C_f(x, y) = \sum_{z \leq y \leq x} f(z).
\]

An important property of the Möbius function is the Möbius inversion formula defined as follows. Let \( f, g: L \to \mathbb{Q} \) be functions defined from \( L \) to the rationals. Then

\[
g(x) = \sum_{y \leq x} f(y).
\]
if and only if
\[ f(x) = \sum_{y \leq x} \mu(y, x)g(y). \]

Using this inversion formula, Dowling and Wilson [6] obtained the following formulas:
\[ f(x) = \sum_{y \leq x} \mu(y, x)Tf(y) \]
and
\[ Cf(x, y) = \sum_{x \leq z \leq y} \mu(z, y)Tf(z). \]

An explicit inversion formula for a function derived from the upper complement transform was established by Race in [11].

**Theorem 1.9.** Let \( L \) be a finite lattice and \( f \) a function from \( L \) to the rational numbers. Let \( Cf \) be the upper complement transform of \( f \). Then
\[
f(x) = \sum_{k \in L} \left[ \sum_{y \leq k \wedge x} \mu(y, x)\Delta(y, k) \right]Cf(k, k^+) \]
\[
= \sum_{k \in L} \lambda(x, k)Cf(k, k^+),
\]
where \( x \rightarrow x^+ \) is a function from \( L \) into \( L \) such that
\( \mu(x, x^+) \neq 0 \) for all \( x \). For a fixed function \( x \to x^+ \), we say a finite sequence \((p_i)_{i=0}^n\) of elements \( p_i \) in \( L \) is a path from \( x \) to \( y \) if

1) \( x = p_0 < p_1 < \ldots < p_{n-1} < p_n = y \)
and \( p_i \in [p_{i-1}, p_{i-1}^+] \) if \( x < y \); or

2) \( x = p_0 = y \) if \( x = y \).

Now if \((p_i)_{i=0}^n\) is a path from \( x \) to \( y \), we define

\[
\Pi(x, y, (p_i)) = \sum_{i=1}^{n} \frac{\mu(p_i, p_{i-1}^+)}{\mu(p_{i-1}, p_{i-1}^+)} \quad \text{if } x < y
\]

\[
\Pi(x, y, (p_i)) = 1 \quad \text{if } x = y.
\]

Also, \( \Delta(x, y) \) is defined by

\[
\Delta(x, y) = \frac{1}{\mu(y, y^+)} \sum_{(p_i)} (-1)^n \Pi(x, y, (p_i)),
\]

where the summation is over all paths from \( x \) to \( y \).

The Radon transform was used by Kung \([8,9]\) to find matchings in a consistent lattice.

**Theorem I.10.** Let \( L \) be a finite consistent lattice. Then the incidence matrix \( I(\mathcal{M}|\mathcal{I}) \) has rank \( |\mathcal{I}| \), and \( |\mathcal{I}| \leq |\mathcal{M}| \).
Theorem 1.11. Let $L$ be a finite consistent lattice. Then $J$ is concordant with $M$. In particular, $|J| \leq |M|$.

Briefly, we examine the use of the Radon transform in this setting, see [10]. Let $f: J \to \mathbb{Q}$ be a function defined from the join-irreducibles to the rational numbers. The Radon transform $Tf$ of the function $f$ is the function $Tf: L \to \mathbb{Q}$ defined by

$$Tf(x) = \sum \{ f(j) \mid j \in J \text{ and } j \leq x \}.$$  

This transform induces a linear transformation $T: f \mapsto Tf$ from the vector space $J^*$ of functions from $J$ to $\mathbb{Q}$ to the space $L^*$ of functions from $L$ to $\mathbb{Q}$. The partial Radon transforms, $T^*$ and $T^*_*$, are defined by restricting the domain of $Tf$ to $M$, obtaining the linear transformation $T^*: J^* \to M^*$ and similarly for $T^*_*: J^* \to J^*$. It is shown that if $L$ is consistent, then there exists a linear transformation $R$ from $M^*$ to $J^*$ such that the following diagram commutes:

\[ \begin{array}{ccc}
J^* & \xymatrix{ & M^*} \\
T^* \ar[ru] & \ar[d] R & T^*_* \\
J^* & \ar[ru] & \ar[d] R \end{array} \]

Therefore, $T^*_* = R \circ T^*$ and the matrix $T^*$ relative to the standard bases is $I(M|J)$ which has rank $|J|$, and thus, for a consistent lattice, $|J| \leq |M|$. In Chapter III, we conclude
that if the lattice is also dually semimodular, then
equality holds only if the lattice is modular.

Combinatorial Inequalities

Results inspired by the following theorem due to Dowling
and Wilson [5] describing the "slimmest" geometric lattice
are established in Chapters IV and V.

**Theorem I.12.** Let \( L \) be a finite geometric lattice of rank \( n \)
with \( p \) atoms. Then

\[
\hat{w}_k \geq \binom{n-2}{k-1}(p-n) + \binom{n}{k}, \quad 0 \leq k \leq n.
\]

When \( r \geq 4 \), equality holds for some \( k \) if and only if \( L \) is
isomorphic to the direct product of a modular plane and a
free geometry.

In our study of these two types of lattices, we find
sharp lower bounds for the total number of elements in these
lattices dependent on the rank, the number of
join-irreducibles, and either the reach or coreach of the
lattice. Bounds on the number of elements of a particular
reach in semimodular lattices are given, while lower bounds
for the number of elements of a given coreach in dually
semimodular lattices are found. The following fact
concerning consistent lattices will be important in proving
Theorem 1.13. Let $L$ be a finite consistent lattice and $x \in L$ such that $x \neq 0$. If $y$ is a join-irreducible in $[x,1]$, then there exists a $j \in J$ so that $j \lor x = y$.

Finally, extremal lattices attaining these bounds are described using the following geometrical definitions and a lemma of Dilworth and Hall [4].

The free geometry or boolean algebra with $j$ atoms is the geometric lattice of rank $k$ isomorphic to the lattice of all subsets of its point set under the inclusion order. A $j$-point line is a geometric lattice of rank two with $j$ atoms. A $j$-point projective plane is a connected, modular geometric lattice of rank three with $j$ atoms. Each of these three types of lattices is modular, hence their direct product is also modular. A modular plane is either projective, or if not connected, the direct product of a line and a one-point free geometry.

Lemma 1.14. Let $L_1$ and $L_2$ be lattices with maximums $u_1$ and $u_2$ and minimums $z_1$ and $z_2$ respectively. Let the quotient lattice $u_1/a_1$ of $L_1$ be isomorphic to the quotient lattice $a_2/z_2$ of $L_2$. If isomorphic elements are identified, then the union $L$ of $L_1$ and $L_2$ forms a lattice which contains $L_1$ and $L_2$ as sublattices and is modular if and only if $L_1$ and
L_2 are modular.

The lattice L described in the preceding lemma will be referred to as the **Dilworth-Hall sum** of L_1 and L_2, and be denoted as L_1 \oplus L_2. With restrictions upon the isomorphic quotient lattices of L_1 and L_2, we find that if L_1 and L_2 are [dually] semimodular, then L_1 \oplus L_2 is also [dually] semimodular. Also, we prove that the Dilworth-Hall sum of L_1 and L_2 is consistent if and only if L_1 and L_2 are both consistent.


CHAPTER II

LATITUDES OF SUBNORMAL SUBGROUPS OF A GROUP

We now consider the lattice of subnormal subgroups of a finite group and first prove that this lattice is dually semimodular and consistent. In fact, more general consistent properties hold for such lattices. Next, several concordant sets in this lattice are described. Several examples are given in seeking to characterize finite groups whose lattice of subgroups are dually semimodular and consistent. We end this chapter with the Hasse diagrams of the lattice of subgroups of several finite p-groups.

Dually Semimodular Consistent Lattices

In this section, we examine an example to motivate our analysis of dually semimodular consistent lattices. Kung shows in [4] that the lattice L(G) of all subgroups of a finite group G is not necessarily consistent, as in L(H20) shown in Figure 2. Thus, we first consider those groups in which the lattice of subgroups is consistent. Since all subgroups are normal in any abelian group, the lattice of subgroups is modular and thus consistent. To obtain examples of dually semimodular lattices which are not necessarily modular, we note that the lattices of subgroups
of both finite p-groups and finite nilpotent groups are dually semimodular, see [1]. Since in these types of groups every subgroup is subnormal, we are led to study the broader class of lattices formed by the subnormal subgroups of a group. Before we show they are dually semimodular and consistent, some preliminary definitions are needed.

Let $H$ and $K$ be subnormal subgroups of $G$. We define $H \wedge K$ to be the intersection of $H$ and $K$, $H \cap K$, and define $H \vee K$ to be the subgroup $\langle H, K \rangle$ generated by $H$ and $K$: that is, $H \vee K$ is the intersection of all the subgroups containing both $H$ and $K$. As discussed in the introduction, Wielandt [9] proved that $H \cap K$ and $\langle H, K \rangle$ are both subnormal subgroups of a finite group $G$. We shall form a meet and join sublattice of the lattice of subgroups of $G$ and denote this lattice by $W(G)$.

**Theorem II.1.** Let $G$ be a finite group and $W(G)$ be the lattice of subnormal subgroups of $G$. Then $W(G)$ is dually semimodular.

Proof: To show that $W(G)$ is dually semimodular, it suffices to show that if $X$ and $Y$ are elements in $W(G)$ such that $X \vee Y$ covers $X$, then $Y$ covers $Y \wedge X$. Suppose $X \vee Y$ covers $X$. Since $X$ is a normal subgroup of $\langle X, Y \rangle$ and $Y$ is a subgroup of $\langle X, Y \rangle$, $XY$ is a subgroup of $\langle X, Y \rangle$ and thus $\langle X, Y \rangle = XY$. Using the third isomorphism theorem from group theory, $X \cap Y$ is
normal in $Y$ and

$$XY/X \cong Y/X \cap Y.$$ 

Thus, since $X \cap Y = X \land Y$, $XY = X \lor Y$, and $X \lor Y$ covers $X$, $Y$ covers $X \land Y$ in $W(G)$. Hence, $W(G)$ is dually semimodular. □

Recall that an element $j$ is a **join-irreducible** if $j = a \lor b$ implies $j = a$ or $j = b$; equivalently, $j = 0$ or $j$ covers exactly one element. A subgroup $G$ is a **join-irreducible** in $W(G)$ if and only if $G$ contains a unique maximal normal subgroup. More generally, let $k$ be a positive integer. An element $j$ is said to be a **$k$-covering** if it covers at most $k$ elements. Thus the $1$-covering elements are precisely the join-irreducibles. Dually, an element $m$ is **$k$-covered** if it is covered by at most $k$ elements. A lattice $L$ is **$k$-consistent** if for every $k$-covering $j$ and every element $x$ in $L$, $x \lor j$ is a $k$-covering in the upper interval $[x,1]$. If $k = 1$, we say the lattice is consistent, as defined earlier in the introduction.

**Theorem II.2.** Let $G$ be a finite group and $W(G)$ be the lattice of subnormal subgroups of $G$. Then $W(G)$ is $k$-consistent.

**Proof:** Let $J$ be a $k$-covering in $W(G)$ and $X$ any element in $W(G)$ such that $J \not\leq X$. We may assume without loss of generality that $\langle X, J \rangle = G$. 
Let \( L = G \downarrow [X] \) be defined to be the meet of all elements \( M \) covered by \( G \) such that \( X \leq M \). Since maximal subnormal subgroups are normal in \( G \), \( L \) is the intersection of normal subgroups and therefore, \( L \) is normal in \( G \).

If \( G \) covers \( M \) in \([L,G]\), since \( X \leq L \), \( G \) covers \( M \) in \([X,G]\). Now if \( G \) covers \( M \) in \([X,G]\), then \( X \leq M \) and by definition, \( L \leq M \) so that \( G \) covers \( M \) in \([L,G]\). Hence, \( G \) covers the same elements in \([L,G]\) as in \([X,G]\).

Note that \( X \leq L \) and \( G = \langle X,J \rangle = \langle L,J \rangle \). Since \( L \) is a normal subgroup in \( G \) and \( J \) is a subgroup of \( G \), \( LJ = \langle L,J \rangle = G \). By the third isomorphism theorem, \( L \cap J \) is a normal subgroup of \( J \) and further

\[
LJ/L \cong J/L \cap J.
\]

\( J \) is a \( k \)-covering in \( W(G) \) and thus covers at most \( k \) elements in the upper interval \([L \wedge J,J]\). Also,

\[
W(J/L \cap J) \cong W(LJ/L).
\]

Thus, \( G \) is a \( k \)-covering in \([L,G]\). Since \( G \) covers the same elements in \([L,G]\) as in \([X,G]\), \( G \) is a \( k \)-covering in \([X,G]\). Thus, \( W(G) \) is \( k \)-consistent.

\[\square\]

A General Consistency Theorem

A more general version of consistency in the lattice of subnormal subgroups can be stated using the idea of excluding subintervals. From this, we see that \( W(G) \) is not only \( k \)-consistent, but also consistent relative to the coreach of an element.
The coreach of an element \( x \) is the difference:
\[
\text{rank}(x) - \text{rank}(x*)
\]
where \( x* \) denotes the meet of \( x \) and all the elements covered by \( x \). We generalize and define the set \( C(k) \) to be the set of elements with coreach at most \( k \), that is, \( C(k) = \{ z \mid \text{coreach}(z) \leq k \} \).

Let \( x \) and \( z \) be elements of \( W(G) \) such that \( x \leq z \). Define \( Z[x] \) to be that element which is the meet of all the elements covered by \( z \) containing \( x \). Recall that the elements covered by \( z \) are normal subgroups of \( z \) and the intersection of normal subgroups is also normal in \( z \). Hence \( Z[x] \) is a normal subgroup of \( z \) containing \( x \) and \( [Z[x], z] \) is an upper interval of \([x, z] \).

Let \( \mathcal{M} = M_\delta \) be a collection of lattices. \( P(\mathcal{M}) \) is the property that \([x*, x] \) does not contain \( M_\delta \) as an upper subinterval. An element \( j \) in a lattice is consistent relative to property \( P(\mathcal{M}) \) if \( j \) satisfies property \( P(\mathcal{M}) \) and for every \( x \) in \( L \), \( x \lor j \) satisfies property \( P(\mathcal{M}) \) in the upper interval \([x, 1] \).

**Theorem II.3.** The lattice \( W(G) \) is consistent relative to property \( P(\mathcal{M}) \).

**Proof:** Let \( J \) satisfy \( P(\mathcal{M}) \) and suppose that \( W(G) \) is not consistent. Then there exists an element \( Z \) in \( W(G) \) such that \( J \not\leq Z \) but \([J \lor Z]_Z, J \lor Z \) does contain some \( M_\delta \) as an upper subinterval. Define \( F = (J \lor Z)_Z \). \( F \) is a normal...
subgroup of $\langle J, Z \rangle$ such that $Z \leq F$. Thus $JF = \langle J, F \rangle = \langle J, Z \rangle$.

By the third isomorphism theorem, $J \cap F$ is a normal subgroup in $J$ and we conclude

$$JF/F \cong J/J \cap F \quad \text{and} \quad W(JF/F) \cong W(J/J \cap F).$$

Therefore, $J \cap F$ is the meet of elements covered by $J$ and thus $J_* \leq J \cap F$. Hence, $[J_*, J]$ contains an $M_a$ in $\mathcal{M}$ as an upper subinterval, contradicting the hypothesis that $J$ satisfies $P(\mathcal{M})$.

\[ \Box \]

Corollary II.4. The elements of $\mathcal{C}(k)$ in $W(G)$ are consistent.

Proof: Note that $\mathcal{C}(k) = P(\mathcal{M}_{k+1})$, where

$$\mathcal{M}_{k+1} = \{ M_a \mid M_a \text{ is a lattice of subnormal subgroups of rank at least } k + 1 \}.$$

Let $\mathcal{J}(k) = \{ j \mid j \text{ covers at most } k \text{ elements in } W(G) \}$, the set of $k$-covering elements. The following corollary states that $W(G)$ is $k$-consistent.

Corollary II.5. The elements of $\mathcal{J}(k)$ in $W(G)$ are $k$-consistent.

Proof: Define $\mathcal{M}_{k+1} = \{ M_a \mid M_a \text{ is a lattice of subnormal subgroups with at least } k+1 \text{ coatoms } \}$. $\mathcal{J}(k)$ is by definition $P(\mathcal{M}_{k+1})$. $\Box$
Concordant Sets

We continue the study of the lattice of subnormal subgroups of a group by exploring further the covering relations in the lattice. Using concordant sets, we discover matchings in the lattice.

Let $J$ and $M$ be subsets of a finite lattice $L$. $J$ is said to be concordant with $M$ if, for every element $x$ in $L$, either $x$ is in $M$ or there exists an element $x^+$ such that

(CS1) the Möbius function $\mu(x,x^+) \neq 0$ and

(CS2) for every element $j$ in $J$, $x \lor j \neq x^+$.

Let $x \in L$. Denote the join of $x$ and all elements covering $x$ by $x^*$, and the meet of $x$ and all elements covered by $x$ as $x_*$. Then the reach of $x$ is defined to be $r(x^*) - r(x)$.

Further, let

$$C(k) = \{ x \mid \text{coreach}(x) \leq k \}$$

and

$$R(k) = \{ x \mid \text{reach}(x) \leq k \}$$

be subsets of $W(G)$. Before we proceed to show that $C(k)$ is concordant with the set $R(k)$ in $W(G)$, some preliminary information about the lattice of subnormal subgroups of a group is needed. The following lemma will be useful.

**Lemma II.6.** If the lattice $W(G)$ of subnormal subgroups of a group $G$ is coatomic, then it is modular and the Möbius function $\mu(0,1)$ is non-zero with sign $(-1)^n$, where $n$ is the rank of $W(G)$.
Proof: Recall that intersections of normal subgroups are normal subgroups. In $W(G)$, the coatoms are all normal subgroups of $G$ and therefore if $W(G)$ is coatomic, every element in $W(G)$ is a normal subgroup of $G$ and hence, $W(G)$ is the lattice of normal subgroups of $G$. By a theorem of Dedekind [2], the lattice of normal subgroups is modular.

Since $W(G)$ is coatomic, it is atomic. Thus $W(G)$ is a geometric lattice and by Rota's Theorem I.2 (see [7]) the Möbius function $\mu(0,1)$ is non-zero with sign $(-1)^n$, where $n$ is the rank of $W(G)$.

Let $\mathcal{M}$ be any collection of finite coatomic lattices; that is, let

$$\mathcal{M} = \{ M_\alpha \mid \text{$M_\alpha$ is a coatomic lattice having property P} \}.$$ 

Recall from Lemma II.6 that if the lattice $W(G)$ of subnormal subgroups is coatomic, it is actually modular and geometric. Thus, in this setting, we are using the stronger condition that $\mathcal{M}$ is a collection of finite atomic modular lattices, not just coatomic.

Define $E^*_\ast(\mathcal{M})$ and $E^\ast(\mathcal{M})$ as follows:

$$E^\ast(\mathcal{M}) = \{ x \mid [x, x^*] \text{ does not contain any of the } M_\alpha \text{ in } \mathcal{M} \text{ as a lower subinterval} \}.$$ 

$$E^*_\ast(\mathcal{M}) = \{ x \mid [x^*, x] \text{ does not contain any of the } M_\alpha \text{ in } \mathcal{M} \text{ as an upper subinterval} \}.$$ 

The relationship between these two sets lead to the following general theorem.
Theorem II.7. Let $G$ be a finite group and $W(G)$ be the lattice of subnormal subgroups of $G$. Suppose $\mathcal{M}$ is a collection of finite coatomic lattices. Then $E^*_*(\mathcal{M})$ is concordant with $E^*(\mathcal{M})$.

Proof: Let $X$ be an element of $W(G)$ and suppose that $X \notin E^*_*(\mathcal{M})$. Then $[X,X^*]$ does contain an $M_\sigma$, for some $M_\sigma$ in $\mathcal{M}$, as a lower subinterval. This means that there exists an element $Y \in [X,X^*]$ such that $[X,Y]$ is a coatomic lattice belonging to $\mathcal{M}$. Since $[X,Y]$ is coatomic, the preceding lemma shows that $p(X,Y) \neq 0$, and CS1 holds.

It remains to show that for every $J \in E^*_*(\mathcal{M})$, $X \lor J \neq Y$. Suppose not, then there exists a $J \in E^*_*(\mathcal{M})$ such that $X \lor J = Y$. Now since $X$ is the meet of coatoms of $Y$ in $[X,Y]$, $X$ is normal in $Y$ and $<X,J> = XJ = Y$. By the third isomorphism theorem, $[X \land J,J] \cong [X,Y]$. Therefore, $[X \land J,J]$ is coatomic and it follows that $J_* \leq X \land J$. So $[X \land J,J]$ is an upper subinterval of $[J_*,J]$ isomorphic to $[X,Y]$ and thus in $\mathcal{M}$. This implies that $J \notin E^*_*(\mathcal{M})$, contradicting the hypothesis that $J \in E^*_*(\mathcal{M})$. Thus for every $J \in E^*_*(\mathcal{M})$, $X \lor J \neq Y$, and CS2 holds. By definition then, $E^*_*(\mathcal{M})$ is concordant with $E^*(\mathcal{M})$. $\square$

We now establish the following concordant pair of subsets.
Corollary II.8. Let $G$ be a finite group and $W(G)$ the lattice of subnormal subgroups of $G$. Then $C(k)$ is concordant with $\mathcal{R}(k)$.

Proof: Let $\mathcal{M}_{k+1}$ be the collection of coatomic lattices of rank $k+1$. Then $C(k)$ is defined to be $\mathcal{E}^*(\mathcal{M}_{k+1})$ and $\mathcal{R}(k) \subseteq \mathcal{E}^*(\mathcal{M}_{k+1})$ since every element $X$ with reach at most $k$ implies the rank of $[X,X^*]$ is at most $k$. Since $\mathcal{E}^*(\mathcal{M}_{k+1})$ is concordant with $\mathcal{E}^*(\mathcal{M}_{k+1})$, it follows that $C(k)$ is concordant with $\mathcal{R}(k)$. □

Since $C(k)$ is concordant with $\mathcal{R}(k)$, there exists a matching from $C(k)$ to $\mathcal{R}(k)$, see [5]. Thus there is an injection $\sigma: C(k) \rightarrow \mathcal{R}(k)$ such that $x \leq \sigma(x)$ for all $x$ in $C(k)$. Note that join-irreducibles are those elements with coreach 1 and meet-irreducibles are those of reach 1. Hence the set of join-irreducibles is concordant with the set of meet-irreducibles in $W(G)$, and we have established that the number of join-irreducibles is less than or equal to the number of meet-irreducibles in $W(G)$.

Examples of the Lattice of Subgroups

As stated in Theorem I.6, if $G$ is a finite nilpotent group, then every subgroup of $G$ is subnormal in $G$, see [6]. Thus the lattice $L(G)$ of subgroups is precisely the lattice $W(G)$ of subnormal subgroups. From this, we conclude the
Corollary II.9. Let $G$ be a finite nilpotent group. Then $L(G)$ is dually semimodular and consistent.

Iwasawa [3] proved that a finite group is supersoluble if and only if its lattice of subgroups satisfies the Jordan-Dedekind chain condition, as stated in Theorem I.7. Thus, a natural problem is to characterize finite groups whose lattice of subgroups is dually semimodular and consistent. We first give an example of a non-nilpotent group whose lattice of subgroups is dually semimodular and consistent.

Example II.10. $S_3$, the symmetric group on three letters is not nilpotent but has a dually semimodular and consistent lattice of subgroups. $S_3$ is not nilpotent since the center is the identity. $S_3$ has three proper subgroups of order two and one proper normal subgroup of order three. The lattice $L(S_3)$ of subgroups and the lattice $W(S_3)$ of subnormal subgroups are dually semimodular and consistent lattices. Their Hasse diagram is shown in Figure 1.

We next cite two examples of the lattice of subgroups of a group to show that the ideas of consistency, solubility, and supersolubility are logically independent.
Example II.11. Let $H_{20}$ be the group with defining relation given by $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^2 \rangle$. $H_{20}$ is metacyclic, supersoluble, and thus soluble. $L(H_{20})$ is illustrated in Figure 2. This lattice is neither consistent nor dually semimodular.

Example II.12. Let $A_4$ be the alternating group on four letters. $A_4$ is soluble but not supersoluble. The subgroup lattice $L(A_4)$ does not satisfy the Jordan-Dedekind chain condition and hence is not dually semimodular, but it is consistent. $L(A_4)$ is shown in Figure 3.
$L(H_{20})$

Figure 2

$L(A_4)$

Figure 3
Before studying dually semimodular consistent lattices in general, we present several examples of the lattice of subgroups of p-groups of small order. In these cases, the lattice of subgroups is dually semimodular and consistent with coreach two. These lattices are illustrated in Figures 4 through 12.

First, two groups of order 8 are shown. The dihedral group $D_4$ of order 8 is defined by the presentation
\[ \langle x, y \mid x^4 = 1, y^2 = 1, xy = yx^{-1} \rangle. \]
Its lattice of subgroups is given in Figure 4. The quaternion group $Q$ is defined by
\[ \langle x, y \mid x^4 = 1, x^2 = y^2, xy = yx^{-1} \rangle, \]
and its lattice of subgroups is shown in Figure 5.

Next, five non-abelian groups of order 16 are given:

Figure 6. $G = \langle x, y \mid x^4 = 1, y^4 = 1, xy = yx^{-1} \rangle$, a semidirect product of a cyclic group of order 4 with a cyclic group of order 4. All the proper subgroups of $G$ are abelian. This group is isomorphic to a subgroup of the direct product of the quaternion group and a cyclic group of order 4.

Figure 7. $G = \langle x, y \mid x^8 = 1, y^2 = 1, xy = yx^5 \rangle$, a semidirect product of a cyclic group of order 8 with a cyclic group of order 2. All the proper subgroups of $G$ are abelian.

Figure 8. $G = \langle x, y \mid x^8 = 1, y^2 = 1, xy = yx^{-1} \rangle$, the dihedral group of order 16.
Figure 9. \( G = \langle x, y \mid x^8 = 1, y^2 = 1, xy = yx^3 \rangle \), a semidirect product of a cyclic group of order 8 with a cyclic group of order 2. This is the unique group of order 16 with five elements of order 2.

Figure 10. \( G = \langle x, y \mid x^8 = 1, y^2 = x^4, xy = yx^{-1} \rangle \), the dicyclic group of order 16, denoted by \( Q_8 \).

Finally, the two non-abelian subgroups of order 27 are illustrated and defined as follows:

Figure 11. \( G \) is the group of 3x3 unipotent matrices over \( \mathbb{Z}_3 \) with the defining relation given by
\[
\langle x, y \mid x^3 = 1, y^3 = 1, (xy)^3 = 1, (x^{-1}y)^3 = 1 \rangle.
\]

Figure 12. \( G = \langle x, y \mid x^9 = 1, y^3 = 1, xy = yx^4 \rangle \), a semidirect product of a cyclic group of order 9 with a cyclic group of order 3.
Figure 4

L(D₄)

Figure 5

L(Q)

Figure 6

L(G), G = \langle x, y \mid x^4 = 1, y^4 = 1, xy = yx^{-1} \rangle
\[ L(G), \ G = \langle \ x, y \mid x^8 = 1, \ y^2 = 1, \ xy = yx^5 \rangle \]

Figure 7

\[ L(G), \ G = \langle \ x, y \mid x^6 = 1, \ y^2 = 1, \ xy = yx^{-1} \rangle \]

Figure 8

\[ L(G), \ G = \langle \ x, y \mid x^8 = 1, \ y^2 = 1, \ xy = yx^3 \rangle \]

Figure 9
\[ L(G), \ G = \langle x, y \mid x^8 = 1, \ y^2 = x^4, \ xy = yx^{-1} \rangle \]

Figure 10

\[ L(G), \ G = \langle x, y \mid x^3 = 1, \ y^3 = 1, \ (xy)^3 = 1, \ (x^{-1}y)^3 = 1 \rangle \]

Figure 11

\[ L(G), \ G = \langle x, y \mid x^9 = 1, \ y^3 = 1, \ xy = yx^4 \rangle \]

Figure 12
CHAPTER BIBLIOGRAPHY


CHAPTER III

A CONVERSE TO DILWORTH'S COVERING THEOREM

Preliminary Results

The main aim in this chapter is to show that any consistent dually semimodular lattice with the same number of join-irreducibles as meet-irreducibles is modular. This gives a partial converse to Dilworth's theorem [1] for this type of lattice.

We shall prove this theorem by induction on the rank of the lattice. Several preliminary lemmas are required.

Lemma III.1. Let L be a finite consistent lattice such that \( |J| = |M| \). Then for any \( x \in L \), the upper interval \([x,1]\) contains the same number of join-irreducibles and meet-irreducibles.

Proof: Let \( x \) be an element of \( L \) such that \( x \neq 0 \). Let \( J[x,1] \) be the set of join-irreducibles in the upper interval \([x,1]\) and \( M[x,1] \) the set of meet-irreducibles in \([x,1]\). Note that \( M = M[x,1] \cup M[x,1] \), where \( M[x,1] \) is the set of meet-irreducibles in \( L \) but not in \([x,1]\).

Since \( L \) is consistent, \([x,1]\) is consistent by Lemma I.1. Thus
Therefore we have

$$|\mathcal{J}| = |M[x,1]| + |M\setminus[x,1]| \geq |\mathcal{J}[x,1]| + |M\setminus[x,1]| = |S|,$$

where $S$ is defined to be the union $\mathcal{J}[x,1] \cup M\setminus[x,1]$. We shall show using Radon transforms that $|\mathcal{J}| \leq |S|$. From this, we conclude

$$|\mathcal{J}| = |M[x,1]| + |M\setminus[x,1]| \geq |S| \geq |\mathcal{J}|$$

and hence

$$|M[x,1]| = |\mathcal{J}[x,1]|.$$

It remains to show that $|\mathcal{J}| \leq |S|$. Let $S^*$ be the vector space of functions from $S$ to $Q$. By restricting the domain of $T_f$ to $S$, we obtain the linear transformation $T^*_S: J^* \to S^*$, $f \mapsto T_f|_S$. Now we have the following diagram of linear transformations:

\[ \begin{array}{ccc} J^* & \xrightarrow{T^*_S} & S^* \\ \downarrow{T^\#} & & \downarrow{T^*_X} \\ M^* & \xrightarrow{T^\#} & S^* \end{array} \]

In [3], Kung proved that there exists a linear transformation $R$ such that $T^\# = R \circ T^*$. We will show that there exists a linear transformation $T^\#_X$ from $S^*$ to $M^*$ such that $T^\# = T^\#_X \circ T^*_S$. Note however that $T^\#_X$ is simply the
linear transformation induced by the Radon transform on 
[\mathcal{I},\mathcal{J}], denoted by \( T_{\mathcal{J}} \). If \( f: \mathcal{J} \to \mathcal{Q} \) is a function which is non-zero only on \( \mathcal{J} \), we define the function \( f_{\mathcal{J}}: \mathcal{J}[x,1] \to \mathcal{Q} \) by

\[
f_{\mathcal{J}}(j) = \sum_{h \in \mathcal{J}} (f(h) \mid h \in \mathcal{J} \text{ and } h \uparrow x = j).
\]

It is clear that \( f_{\mathcal{J}} \) is non-zero only on \( \mathcal{J}[x,1] \) and further, for \( y \in [x,1] \), \( T_{\mathcal{J}}f_{\mathcal{J}}(y) = Tf(y) \). Since \( m \in \mathcal{M}[x,1] \) if and only if \( m \in \mathcal{M} \) and \( x \leq m \), \( T_{\mathcal{J}}f_{\mathcal{J}}(m) = Tf(m) \). Therefore, if the partial Radon transform \( T_{\mathcal{J}}^f: \mathcal{S} \to \mathcal{Q} \) is known, the partial Radon transform \( T_{\mathcal{J}}^f: \mathcal{M} \to \mathcal{Q} \) can be reconstructed using the Radon transform \( T_{\mathcal{J}} \) on \( [x,1] \).

Since \( T_{\mathcal{J}}^f \) is nonsingular and \( T_{\mathcal{J}}^f \) is a nonsingular square matrix of rank \(|\mathcal{J}|\), \( T_{\mathcal{J}}^f \) and \( T_{\mathcal{S}}^f \) have rank at least \( |\mathcal{J}| \).

Hence, the matrix of \( T_{\mathcal{S}}^f \) relative to the standard bases is \( I(\mathcal{S}|\mathcal{J}) \) and has rank \(|\mathcal{J}| \). Thus, \(|\mathcal{J}| \leq |\mathcal{S}| \).

Lemma III.2. Let \( \mathcal{L} \) be a dually semimodular lattice such that for \( a \neq 0 \), the upper interval \([a,1]\) is modular. If \( \mathcal{L} \) is not modular, there exists a pair of distinct atoms \( x \) and \( y \) such that \( x \uparrow y \) has rank 3.

Proof: Suppose \( \mathcal{L} \) is not modular. Then there exist atoms \( a \) and \( b \) such that the rank of \( a \uparrow b \) is strictly greater than two. Now if \( r(a \uparrow b) = 3 \), then the lemma is true. So suppose \( r(a \uparrow b) = 4 \). Let \( a < a_1 < a_2 < a \uparrow b \) and \( b < b_1 < b_2 < a \uparrow b \) be chains from \( a \) and \( b \) to \( a \uparrow b \). Since
L is dually semimodular, there exists an element c such that \(a_2 \land b_2 = c\). Now \(a_2\) covers \(a_1\) and \(c\) and thus \(a_1\) and \(c\) cover \(a_1 \land c\), call this element x. Note that \(x\) has rank 1.

Similarly, \(c\) and \(b_1\) cover \(b_1 \land c\).

Now if \(x = b_1 \land c = a_1 \land b_1\), then \(a_1 \lor b_1\) has rank 3 since \([x,1]\) is modular. But \(a_1 \lor b_1 = a \lor b\) having rank 4, and thus \(x \neq b_1 \land c\).

Note that \(b \leq x \lor b \leq b_2\). Suppose there exists an element \(d\) such that \(b < d < b_2\) and \(x \lor b = d\). Then \(x = a_1 \land d\) and thus \(a_1 \lor d\) has rank 3 by the modularity of \([x,1]\). But \(a_1 \lor d = a \lor b\) and \(a \lor b\) has rank 4. Therefore, \(x \lor b = b_2\), and \(r(b_2) = 3\). Hence, \(x\) and \(b\) are distinct atoms such that the rank of \(x \lor b\) is 3.

Assume the lemma holds if \(a \lor b\) has rank less than \(k\).

We consider a lattice such that \(a \lor b\) has rank \(k\), and use an argument analogous to that with \(r(a \lor b) = 4\). Let

\[a < a_1 < a_2 < \ldots < a_{k-3} < a_{k-2} < a \lor b\]

and

\[b < b_1 < b_2 < \ldots < b_{k-3} < b_{k-2} < a \lor b\]

be chains from \(a\) and \(b\) to \(a \lor b\). Since the lattice is
dually semimodular, $a_{k-2} \wedge b_{k-2}$ is covered by both $a_{k-2}$ and $b_{k-2}$. Let $c = a_{k-2} \wedge b_{k-2}$. Now $a \not\leq c$ and $b \not\leq c$ and also $a_i \not\leq c$ and $b_j \not\leq c$ for every $i,j$, such that $1 \leq i,j \leq k-2$. Thus, there exists the following chain:

$$a_{k-3} \wedge c > a_{k-4} \wedge a_{k-3} \wedge c > \ldots > a \wedge a_1 \wedge \ldots \wedge c$$

or more simply:

$$a_{k-3} \wedge c > a_{k-4} \wedge a \wedge c > \ldots > a_1 \wedge a \wedge c = 0.$$  

There exists a similar chain for the $b_i$'s. Now let $a_1 \wedge c = x$ and $b_1 \wedge c = y$, and note that both $x$ and $y$ have rank 1.

We now consider two cases. Note that if $x = y$, then since $[x,1]$ is modular, $a_1 \vee b_1$ covers $a_1$ and $b_1$ and thus the rank of $a_1 \vee b_1$ is 3. But $a \vee b = a_1 \vee b_1$, and the rank of $a \vee b$ is $k$. Thus if $k > 3$, $x \neq y$. Since $x \neq y$, then either $r(x \vee y) = 2$ or $r(x \vee y) > 2$.

1) Suppose the rank of $x \vee y$ is 2. Since $[y,1]$ is modular,
(x V y) V b_1 covers both x V y and b_1. Now
b < x V b ≤ (x V y) V b. Suppose r(x V b) = 2, and let
x V b = z. Then x is covered by a_1 and z and thus a_1 V z
has rank 3, but a_1 V z = a V b since a ≤ a_1 and b ≤ z. This
is not possible since the rank of a V b is k. Therefore,
x V b = (x V y) V b which has rank 3. Hence, x and b are
distinct atoms such that r(x V b) = 3.

2) If x V y has rank greater than 2, then since x ≤ c and
y ≤ c with [0,c] being a dually semimodular lattice of rank
k - 2, there exist atoms x_1 and y_1 such that the rank of
x_1 V y_1 is 3 by the inductive hypothesis.

Thus in each case, there exist distinct atoms with their
join having rank 3.

Lemma III.3. Let L be a finite lattice and a an element in
L. Then a is equal to the meet of all the meet-irreducibles
containing a.

Proof: If a = 1 or is covered by 1, then a is a
meet-irreducible and the lemma holds for a.

Assume the lemma is true for elements greater than a.
If x is an element in L, let
\[ M_x = \{ m \mid m \in M \text{ and } x \leq m \} \]
where M is the set of meet-irreducibles in L. Let b = \bigwedge_{m \in M_a} m.

Since a ≤ m for every m ∈ M_a, a ≤ b.
If \( a \) is a meet-irreducible, then \( a \in M_a \), \( b \leq a \), and \( a = b \). Hence we can suppose that \( a \) is not a meet-irreducible. Then there exist two elements covering \( a \). Moreover, since \( b \geq a \), \( b \geq d \) for some element \( d \). Thus, there exist elements \( c \) and \( d \) such that \( c \) and \( d \) cover \( a \) and \( b \geq d \geq a \). By induction, \( c = \bigwedge_{m \in M_c} m \) and \( d = \bigwedge_{m \in M_d} m \). Since \( c \) and \( d \) are not equal, there exists an \( m \in M_c \) such that \( m \not\in M_d \). Therefore, \( b \wedge m = a \). Since \( m \in M_a \), \( b \leq m \) and we conclude that \( b \wedge m = b = a \).

The Main Theorem

We shall now use the upper complement transform and the inversion formula described in Theorem 1.9 to prove the following theorem, see [2,4,5].

**Theorem III.4.** Let \( L \) be a finite consistent dually semimodular lattice. Then the number of join-irreducibles equals the number of meet-irreducibles if and only if \( L \) is modular.

**Proof:** We proceed by induction on the rank of \( L \). If \( r(L) \leq 2 \), the theorem holds trivially. We assume then that the theorem holds for all dually semimodular consistent lattices of rank less than \( n \).

Suppose \( r(L) = n \). Let \( a \) and \( b \) be elements in \( L \) covering
a \wedge b. Now if a \wedge b > 0, then there exists an atom x such that x \leq a \wedge b. Since the rank of [x,1] is n - 1 and [x,1] is dually semimodular and consistent, J[x,1] = M[x,1] by Lemma III.1. Hence, by the inductive hypothesis, [x,1] is modular. Therefore, a V b covers a and b in [x,1] and in L.

Now suppose that a \wedge b = 0 and suppose that L is not modular. By Lemma III.2, there exist atoms x and y such that x V y has rank 3. Let x' and y' be elements covered by x V y such that x' covers x and y' covers y. Since L is dually semimodular, x' \wedge y' is covered by x' and y'. Let x' \wedge y' = z. Note that x' is not a join-irreducible as it covers both x and z.

Now consider a function f: L \rightarrow \mathbb{Q}, where f(x) is zero unless x is a join-irreducible. The upper complement transform Cf: L x L \rightarrow \mathbb{Q} is defined by

$$Cf(x, x^+) = \sum_{z \vee x = x^+} f(z).$$

Let x \mapsto x^+, L \rightarrow L be a function satisfying the following property:
if $x$ is a meet-irreducible,
\[ x^+ = x^* \]
and,

if $x$ is not a meet-irreducible,
\[ x^+ = y \]
such that $r(y) = r(x) + 2$ and $y$ is not a join-irreducible in $[x, 1]$. Note that by definition, $\mu(x, x^+) \neq 0$ and $Cf(x, x^+)$ is zero if $x$ is not a meet-irreducible.

The function $f$ can be reconstructed from the function $Cf$ defined on the meet-irreducibles by the inversion formula
\[ f(x) = \sum_{z \in M} \lambda(x, z)Cf(z, z^+). \]

Because the number of join-irreducibles equals the number of meet-irreducibles, the dimension of the space $J^*$ is the same as the space $M^*$. The subspace spanned by $Cf$ equals the space $M^*$ of all functions from $M$ to $\mathbb{Q}$. Therefore, if
\[ 0 = f(x) = \sum_{z \in M} \lambda(x, z)Cf(z, z^+) , \]
then $\lambda(x, z) = 0$ for every meet-irreducible $z$. Thus, to finish the proof, it suffices to find an element $x$ in $L$ which is not a join-irreducible (and so, $f(x) = 0$) and a meet-irreducible $m$ where $\lambda(x, m)$ is non-zero, thus obtaining a contradiction.

Recall from Lemma III.3 that for $a \in L$,
\[ a = \Lambda \{ m \mid a \leq m, \ m \in M \} . \]
Since $y \neq y'$, there exists a meet-irreducible $m$ such that $y' \notin m$ but $y \leq m$. Now since $y \leq x \lor y$ and $y \leq m,$
$y \leq (x \lor y) \land m \leq x \lor y$. $(x \lor y) \land m$ is not equal to $x \lor y$ since if that is the case, $y' \leq m$. Suppose $(x \lor y) \land m = w$
with $x \lor y$ covering $w$ and $w$ covering $y$. Then $x \lor y$ covers $y'$ and $w$ and thus is not a join-irreducible in $[y,1]$. Since $L$ is consistent and $x$ a join-irreducible in $L$, $x \lor y$ must be
a join-irreducible in $[y,1]$. Thus $(x \lor y) \land m = y$.

Now $x' \notin m$. If $x' \notin m$, since $y \leq m$, $x' \lor y \leq M$, but $x' \lor y = x \lor y$ and $x \lor y \notin m$. Then
$x' \land m \leq (x \lor y) \land m = y$. If $x' \land m = y$, then $y \leq x'$ and
$x' = x \lor y$ which contradicts $x'$ being covered by $x \lor y$.
Hence $x' \land m$ must be zero.

The element $x'$ is not a join-irreducible and hence
$f(x') = 0$. The meet-irreducible $m$ satisfies $x' \land m = 0$.
Therefore,
$$O = f(x') = \sum_{z \in \mathcal{M}} \lambda(x',z)Cf(z,z^+)$$

By the linear independence of the function defined on the
meet-irreducibles, $\lambda(x',z) = 0$ whenever $z \in \mathcal{M}$. But
$$\lambda(x',m) = \sum_{a \leq 0} \eta(a,x')A(a,m) = \eta(0,x')A(0,m)$$
where $\eta(0,x')$ is non-zero since $x'$ covers at least $x$ and $z$.

We now proceed to choose the function $x \to x^+$ in such a way
that there exists only one path from $0$ to $m$ and hence $A(0,m)$
is non-zero for this choice of $x \to x^+$.

There exists at least one chain from $y$ to $m$. Let $y_1', y_2', \ldots, y_k'$ be one such chain with

$$0 < y < y_1 < y_2 < \ldots < y_{k-1} < y_k < m.$$ 

Now define a function $x \to x^+$ using the following choices:

$$0^+ = y'$$

$$y^+ = y' \lor y_1$$

$$y_1^+ = y' \lor y_1 \lor y_2$$

$$y_2^+ = y' \lor y_1 \lor y_2 \lor y_3$$

$$\vdots$$

$$y_{k-1}^+ = y' \lor y_1 \lor y_2 \lor \ldots \lor y_k$$

$$y_k^+ = y' \lor y_1 \lor y_2 \lor \ldots \lor y_k \lor m = m^*.$$ 

Note that by the modularity of $[y, 1]$, $0^+$ covers $z$ and $y$, $y^+$ covers $0^+$ and $y_1$, $y_1^+$ covers $y^+$ and $y_2$, ..., and $y_k^+$ covers $y_{k-1}^+$ and $m$. Thus, $\mu(0, 0^+)$, $\mu(y, y^+)$, and $\mu(y_i, y_i^+)$ are non-zero for all $i$ such that $1 \leq i \leq k$.

We now show that the chain $0, y, y_1, y_2, \ldots, y_k, m$ is the only path from $0$ to $m$ given our choice for $x^+$. Suppose the contrary. Then there exists an element $y_i^+$ which covers two elements, say $y_{i+1}$ and $z_{i+1}$, with both contained in $m$. Then $y_{i+1} \lor z_{i+1} = y_i^+$ which implies $y_i^+ \leq m$. But $y' = 0^+ < y^+ < y_1^+ < \ldots < y_i^+$. Therefore $y' \leq y_i^+ \leq m$ which contradicts $y' \notin m$. We have
\[ \Lambda(0, m) = \frac{(-1)^{k+2} \mu(y, 0^+) \mu(y_1^+, y^+) \cdots \mu(y_{k-1}^+, y_{k-1}^+) \mu(m, y_k^+)}{\mu(m, m^*) \mu(0, 0^+) \mu(y, y^+) \cdots \mu(y_{k-1}^+, y_{k-1}^+) \mu(y_k^+, y_k^+)} \]

and simplifying,

\[ \Lambda(0, m) = \frac{-1}{\mu(0, 0^+) \mu(y, y^+) \mu(y_1^+, y_1^+) \cdots \mu(y_{k-1}^+, y_{k-1}^+) \mu(y_k^+, y_k^+)} \neq 0. \]

Hence if \( x' \wedge m = 0 \), \( \lambda(x', m) \neq 0 \), contradicting \( \lambda(x', m) = 0 \). \( \Box \)


CHAPTER IV

COMBINATORIAL INEQUALITIES IN SEMIMODULAR CONSISTENT LATTICES

We present several combinatorial inequalities for semimodular consistent lattices in this chapter. First, we establish lower bounds for the total number of elements dependent upon the rank, reach, and number of join-irreducibles in the lattice. Then we describe extremal lattices which attain these bounds. Finally, we find bounds for the number of elements of a given reach.

The Total Number of Elements in a Consistent Semimodular Lattice

Given the number of join-irreducibles in a semimodular consistent lattice, we want to find the minimum number of elements in the lattice. Several cases will be considered dependent upon the relationship between the rank and the reach of the lattice. First, a preliminary lemma is needed.

Lemma IV.1. Let L be a finite semimodular consistent lattice. Then for $x \in L$, $[x,x^*]$ is geometric.

Proof: First, we consider $L = [0,0^*]$. If $L$ has rank 2,
then the lemma is true. Let \( n \) be the rank of \( L \) and assume \( n \geq 3 \). We will proceed by induction on \( n \). Thus, if \( L \) is a finite semimodular consistent lattice such that \( 0^* = 1 \) with rank less than \( n \), then \( L \) is geometric.

Let \( p \) be an atom in \( L \) and consider \([p,1]\). Since \( 0^* = 1 \), \( p^* = 1 \) also, or else for every atom \( a \) in \( L \), \( a \vee p \) covers \( p \). Therefore \( a \vee p \leq p^* \) which implies \( 0^* \leq p^* \), but \( 0^* = 1 \).
Thus \([p,1]\) has rank \( n - 1 \) and is consistent and semimodular with \( p^* = 1 \). Hence, \([p,1]\) is geometric.

Suppose the lemma is false. More specifically, suppose there exists an element \( x \) of rank 2 which is not the join of atoms. Then \( x \) is a join-irreducible. Since \( 0^* = 1 \), there exists an atom \( a \) such that \( a \not\leq x \). With \( L \) being semimodular, \( a \vee x \) covers \( x \). Since \( L \) is consistent, \( a \vee x \) is a join-irreducible in \([a,1]\). Since \( x \) has rank 2, \( a \vee x \) has rank 3 in \( L \) and in \([a,1]\) has rank 2. But \([a,1]\) is geometric and the join-irreducibles in \([a,1]\) have rank 0 or 1. Thus, we conclude that every element of rank 2 is the join of atoms.

Now for \( y \) in \( L \) such that \( r(y) > 2 \), there exists an atom \( p \) such that \( p \leq y \). Thus \( y \) is the join of atoms in \([p,1]\) and is the join of the join of atoms in \( L \). Hence \( L \) is geometric, if \( 0^* = 1 \).

Finally, since intervals of semimodular lattices are semimodular and by Lemma I.1 also consistent, \([x,x^*]\) is a semimodular consistent lattice. Thus \([x,x^*]\) is geometric. \( \Box \)
Theorem IV.2. Let $L$ be a finite semimodular consistent lattice of rank $n$ and reach $k$ with $m$ join-irreducibles. Then if $n = k$, or $m = n + 1$, or $k = 2$,

$$|L| \geq 2^{k-2}(m - 1 - n) + 2^k + n - k,$$

and if $k \geq 3$, $m > n + 1$, and $n > k$,

$$|L| \geq 2^k + m - k.$$

Proof: Suppose $n = k$. Then $L = [0,0^*]$ and using Lemma IV.1, $L$ is geometric. As stated in Theorem I.12, Dowling and Wilson [2] established the following lower bounds on the size of Whitney numbers for geometric lattices:

$$w_1 \geq \left[ \begin{array}{c} n - 2 \\ i - 1 \end{array} \right] (w_1 - n) + \left[ \begin{array}{c} n \\ i \end{array} \right]$$

for $0 \leq i \leq n$,

where $w_1$ is the number of atoms in $L$. Since $|L| = \sum_{i=0}^{n} w_i$,

$$|L| \geq 2^{n-2}(w_1 - n) + 2^n.$$

Since $L$ is geometric, a join-irreducible is either an atom or $0$, and hence $w_1$ equals $m - 1$. We conclude that

$$|L| \geq 2^{k-2}(m - 1 - n) + 2^k.$$

Next, let $n > k$ and assume $m = n + 1$. We are left to show that $|L| \geq 2^k + m - 1 - k$. There exists an element $x$ in $L$ having reach $k$ and thus

$$|L| \geq |[x,x^*]| + |L\setminus[x,x^*]|.$$
Since $L$ has rank $n > k$, there exist at least $n - k$ elements in $L \setminus [x, x^*]$. With $[x, x^*]$ having rank $k$, there exist at least $k$ elements covering $x$. As before, because $[x, x^*]$ is geometric containing at least $k$ atoms, $|[x, x^*]| \geq 2^k$.

Therefore,

$$|L| \geq 2^k + n - k = 2^k + m - 1 - k.$$ 

Note that $2^k = n - k$ is a lower bound for any lattice we are considering if $n > k$. However, in the case with $m > n + 1$, equality will never hold for $k \geq 3$, that is, no extremal lattices exist. We shall find a precise lower bound for the number of elements in $L$, with extremal lattices existing, in terms of the reach and the number of join-irreducibles.

Now suppose $n > k$ and $m > n + 1$. Since there exists an element $x$ of reach $k$ in $L$, we first assume no non-zero element has reach $k$. Thus $0$ has reach $k$ and

$$|L| \geq |[0, 0^*]| + |L \setminus [0, 0^*]|.$$ 

Let $r$ denote the number of atoms in $L$. Then since $[0, 0^*]$ is geometric of rank $k$, there are $r + 1$ join-irreducibles in $[0, 0^*]$.

We consider two cases, the first with $k = 2$. There exist at least $m - (r + 1)$ elements in $L \setminus [0, 0^*]$ such that $|[0, 0^*]| \geq 2^{k-2}y + 2^k$, where $r - k = y$. Then

$$|L| \geq 2^{k-2}y + 2^k + m - r - 1$$
$$\geq y + 2^k + m - y - k - 1$$
$$\geq 2^k + m - k - 1.$$
or
\[ |L| \geq m + 1. \]

Second, if we consider \( k \geq 3 \),
\[ |L| \geq |[0,0^*]| + |L\setminus[0,0^*]|. \]

If \( r - k = y > 0 \), then as above
\[ |L| \geq 2^{k-2}y + 2^k + m - k - y - 1. \]
Since \( y > 0 \),
\[ |L| \geq 2^k + m - k - 1 + y \]
\[ \geq 2^k + m - k. \]

If \( y = 0 \), then there exist exactly \( k + 1 \) join-irreducibles in \([0,0^*]\) and \( m - k - 1 \) join-irreducibles in \( L\setminus[0,0^*] \).

Since \( m > n + 1 \), \( m - k - 1 > n - k \). There exists a join-irreducible of rank \( k \) or two join-irreducibles of rank greater than \( k \) of the same rank. Choose the pair with minimal rank and denote them \( z_1 \) and \( z_2 \). (Note that \( z_2 \) could be \( 0^* \).) By semimodularity, \( z_1 \lor z_2 \) covers both \( z_1 \) and \( z_2 \).
Therefore, there is an element, \( z_1 \lor z_2 \), in \( L\setminus[0,0^*] \) such that \( z_1 \lor z_2 \) is not a join-irreducible. Then
\[ |L| \geq 2^{k-2}y + 2^k + m - k - 1 + 1. \]

With \( y = 0 \),
\[ |L| \geq 2^k + m - k. \]

We next consider the case in which \( L \) contains a non-zero element, \( x \), of reach \( k \). There exists an atom \( p \) such that \( p \leq x \) and \([p,1]\) has reach \( k \) and rank \( n - 1 \). We proceed by induction on the rank of \( L \). Thus if \( L \) is a finite semimodular consistent lattice of rank \( n' < n \) with reach \( k' \)
and m' join-irreducibles, then either
\[ |L| \geq 2^{k'} + m' - k' \]
if \( n' > k' \), \( m' > n' + 1 \), and \( k' > 3 \),
or
\[ |L| \geq 2^{k-2}(m' - 1 - n') + 2^k + n' - k' \]
if \( n' = k' \) or \( m' = n' + 1 \) or \( k' = 2 \).

Now we have
\[ |L| \geq |[p,1]| + |\{0\}| + |L\setminus([p,1]U\{0\})|. \]
Since \( L \) is consistent, for each join-irreducible \( z \) in \([p,1]\), denoted \( z \in J_{[p,1]} \), there exists a join-irreducible \( j \) in \( L \) such that \( j \lor p = z \), as shown by Theorem 1.13, see [4]. The map \( j \mapsto j \lor p \) is a surjection from \( J \) to \( J_{[p,1]} \) and partitions \( J \) into inverse images of \( J_{[p,1]} \). Now suppose there are \( s - 1 \) atoms \( z_1, z_2, \ldots, z_{s-1} \) in \([p,1]\). Let
\[ r_i = |\{ j \mid j \in J \text{ and } j \lor p = z_i \}| \]
for \( 1 \leq i \leq s - 1 \), and note that \( r_i \geq 1 \). Note also that if \( z_i \notin J \), then \( r_i \geq 1 \). There exist element(s) \( j \in J \) such that \( j \notin [p,1] \) and \( j \lor p = z_i \). Therefore, let
\[ b_i = \begin{cases} 0 & \text{if } z_i \in J \\ 1 & \text{if } z_i \notin J \end{cases} \]
such that \( \sum_{i=1}^{s-1} b_i \)
is equal to the number of join-irreducibles in \([p,1]\) but not in \( L \). Let
\[ B = \sum_{i=1}^{s-1} b_i = |\{ z_i \mid z_i \notin J \}|. \]
Then
\[ |L\setminus([p,1] \cup \{0\})| \geq | \text{join-irreducibles in } L \text{ not in } [p,1]|, \]
or specifically
\[
|L| \geq |[p,1]| + 1 + \sum_{i=1}^{s-1} (r_i - 1) + \sum_{i=1}^{s-1} b_i,
\]
where
\[
\sum_{i=1}^{s-1} (r_i - 1) + \sum_{i=1}^{s-1} b_i = \left| \{ j \mid j \lor p = z_i \text{ and } j \notin [p,1] \} \right|.
\]
Now
\[
\sum_{i=1}^{s-1} (r_i - 1) = r_1 + r_2 + \ldots + r_{s-1} - (1 + 1 + \ldots + 1)
\]
\[
= m - 2 - (s - 1)
\]
\[
= m - s - 1
\]
since \(r_1 + r_2 + \ldots + r_{s-1} = m - |\{0\}|.\)
Thus
\[
|L| \geq |[p,1]| + 1 + m - s - 1 + B.
\]
Note that if \(m - s - 1 > 0\), then \(B > 0\). If \(m - s - 1 = 0\), \(B = 0\) only if every \(z_i \in J\). This implies that every
join-irreducible in \([p,1]\) is also a join-irreducible in \(L\).

Again, we suppose \(k = 2\). By the inductive hypothesis
with \([p,1]\) having reach 2,
\[
|L| \geq 2^k + s - k - 1 + 1 + m - s - 1 + B
\]
\[
\geq 2^k + m - k - 1
\]
\[
\geq m + 1.
\]
We suppose \(k \geq 3\) and consider several cases pertaining to
the lattice \([p,1]\). In each case however, we show that
$|L| \geq 2^k + m - k$. We shall prove in the next section that equality does hold for certain extremal lattices.

We first consider the case with $k = n - 1$. Then $[p,1]$ has the property that $p^* = 1$. Thus, $[p,1]$ is geometric and

$$|[p,1]| \geq 2^{k-2}y + 2^k,$$

where $y = s - 1 - (n - 1) = s - n$. Now if $s = n$, then $y = 0$. Since $m > n + 1 = s + 1$, $m - s - 1 > 0$. This implies $B > 0$. Therefore

$$|[p,1]| \geq 2^k$$ and $$|L| \geq 2^k + 1 + m - s - 1 + B.$$

With $B \geq 1$ and $s = k + 1$, then

$$|L| \geq 2^k + m - k - 1 + B \geq 2^k + m - k.$$

Now suppose $s > n$. Then $y = s - n > 0$ and

$$|L| \geq 2^{k-2}y + 2^k + 1 + m - s - 1 + B.$$

Since $s = y + k + 1$,

$$|L| \geq 2^{k-2}y + 2^k + m - y - k - 1 + B \geq y + 2^k + m - k - 1 + B \geq 2^k + m - k.$$

Finally, we suppose $n - 1 > k$. Again, we consider the two cases in which $s = n$ or $s > n$. Assume $s = n$, then $m > n + 1 = s + 1$ and $m - s - 1 > 0$ and $B \geq 1$. In this case, we obtain $|[p,1]| \geq 2^k + s - k - 1$ from the inductive step. With $B \geq 1$,

$$|L| \geq 2^k + s - k - 1 + 1 + m - s - 1 + B \geq 2^k + m - k - 1 + B.$$

Thus,

$$|L| \geq 2^k + m - k.$$
We consider \( s > n \). From the induction hypothesis,
\[
|L| \geq 2^k + s - k + 1 + m - s - 1 + B.
\]
Thus,
\[
|L| \geq 2^k + m - k.
\]
Hence in all cases if \( k \geq 3 \), \( m > n + 1 \), and \( n > k \),
\[
|L| \geq 2^k + m - k.
\]

Slimmest Semimodular Consistent Lattices

In this section, the extremal lattices are described in detail. We establish that, in general, the slimmest semimodular consistent lattice is a Dilworth-Hall sum [1] of a free geometry dependent upon the reach with a \( j \)-point line where \( j \) is in terms of the rank and number of join-irreducibles. Before stating the main theorem, several preliminary lemmas regarding the Dilworth-Hall sum of two lattices are needed.

Lemma IV.3. Let \( L_1 \) and \( L_2 \) be lattices with unit elements \( u_1 \), \( u_2 \) and null elements \( z_1 \), \( z_2 \) respectively. Let the quotient lattice \( u_1/a_1 \) of \( L_1 \) be isomorphic to the quotient lattice \( a_2/z_2 \) of \( L_2 \) and further let this quotient lattice be modular. Let \( L \) be the Dilworth-Hall sum of \( L_1 \) and \( L_2 \). Then \( L \) is semimodular if and only if \( L_1 \) and \( L_2 \) are semimodular.

Proof: If \( L \) is semimodular, then since \( L_1 \) and \( L_2 \) are intervals of \( L \), \( L_1 \) and \( L_2 \) are also semimodular. Conversely,
let $L_1$ and $L_2$ be semimodular.

Let $a \in L_1$ and $b \in L_2$ and suppose that $a$ covers $a \wedge b$. Since $a \in L_1$, $a \wedge b = a \wedge (b \wedge u_1)$, an element in $L_1$. Therefore $a \vee (b \wedge u_1)$ covers $b \wedge u_1$ by the semimodularity of $L_1$.

Note that $z_2 \leq a \vee (b \wedge u_1)$ and thus
\[ a \vee (b \wedge u_1) = (z_2 \vee a) \vee (b \wedge u_1) \text{ and similarly} \]
\[ b \wedge (a \vee z_2) \leq u_1 \text{ so that} \]
\[ b \wedge (a \vee z_2) = (u_1 \wedge b) \wedge (a \vee z_2). \]

Since these two elements are in $[z_2, a_2] = [a_1, u_1]$, with this quotient lattice being modular, we have $(b \wedge u_1) \vee (a \vee z_2)$ covering $b \wedge u_1$.

This implies $a \vee z_2$ covers $(b \wedge u_1) \wedge (a \vee z_2)$. Since $L_2$ is semimodular, $(a \vee z_2) \vee b$ covers $b$. Note $a \vee b = (a \vee z_2) \vee b$ such that $a \vee b$ covers $b$. Hence, $L$ is semimodular.

\[ \square \]

**Lemma IV.4.** Let $L_1$ and $L_2$ be lattices with unit elements
u_1, u_2 and null elements z_1, z_2 respectively, and let the quotient lattice u_1/a_1 of L_1 be isomorphic to the quotient lattice a_2/z_2 of L_2. Let L be the Dilworth-Hall sum of L_1 and L_2. Then L is consistent if and only if L_1 and L_2 are consistent.

Proof: If L is consistent, since L_1 and L_2 are intervals of L, Lemma I.1 shows that L_1 and L_2 are consistent. Thus we shall show L is consistent given that L_1 and L_2 are both consistent. Suppose j ∈ J_L_1 and x ∈ L_2 \ L_1.

Note that x ∨ j = x ∨ (z_2 ∨ j) since x ≥ z_2. L_1 is consistent and j ∨ z_2 = j ∨ a_1 in L_1. Therefore j ∨ z_2 is a join-irreducible in [a_1, u_1]. Also, j ∨ z_2 is a join-irreducible in [z_2, u_1] ⊆ J[z_2, u_2] = J_L_2.

Since L_2 is consistent, (j ∨ z_2) ∨ x is a join-irreducible in [x, 1] = [x, u_2]. Since (j ∨ z_2) ∨ x = j ∨ x,

j ∨ x is an element of J[x, 1]. Hence, L is consistent. □

**Theorem IV.5.** Let L be a finite semimodular consistent lattice of rank n ≥ 3 and reach k with m join-irreducibles. Then

|L| = 2^{k-2}(m - 1 - n) + 2^k + n - k

i) for n = k if and only if L is isomorphic to the direct product of a modular plane and a free geometry.
ii) for \( n = m - 1 \) if and only if \( L \) is isomorphic to a chain attached to a free geometry with \( k \) atoms,

iii) for \( k = 2 \) if and only if \( L \) is isomorphic to a chain attached to a \((m - n + 1)\)-point line.

Otherwise, if \( k \geq 3, m \geq n + 1, \) and \( n > k \),

\[
|L| = 2^k + m - k
\]

if and only if \( L \) is isomorphic to a chain attached to the Dilworth-Hall sum of a free geometry with \( k \) atoms and a \((m - n + 1)\)-point line, with the identified quotient lattice consisting of a two-element chain.

Proof: If \( n = k \), then \( L \) is a geometric lattice of rank \( n \) with \( m - 1 \) atoms. With \( |L| = 2^{k-2}(m - n - 1) + 2^k \),

\[
W_i = \left[ \begin{array}{c} n - 2 \\ i - 1 \end{array} \right] (m - n - 1) + \left[ \begin{array}{c} n \\ i \end{array} \right] \quad \text{for } 0 \leq i \leq n.
\]

Thus for \( n \geq 4 \), equality holds if and only if \( L \) is isomorphic to the direct product of a modular plane and a free geometry. It also holds true for \( n = 3 \) when the free geometry is trivial.

Suppose \( n > k \) and \( n = m + 1 \). Then

\[
|L| = 2^{k-2}(m - n - 1) + 2^k + n - k = 2^k + n - k.
\]

Since \( L \) has at least one element \( x \) of reach \( k \), then

\[
|[x,x^*]| \geq 2^k.
\]

Also, there exist at least \( n - k \) elements in \( L\setminus[x,x^*] \). Then \( |[x,x^*]| = 2^k \) and

\[
|L\setminus[x,x^*]| = n - k.
\]

Hence \([x,x^*]\) is a geometric lattice of
rank $k$ with $k$ atoms and so isomorphic to a free geometry with $k$ atoms. Each element in $L\{x,x^*\}$ is both a join-irreducible and a meet-irreducible since there are only $n - k$ elements. Thus $L$ is isomorphic to a chain attached to a free geometry on $k$ atoms.

Conversely, suppose $L$ is isomorphic to a chain attached to a free geometry on $k$ atoms. There exists an element $x$ in $L$ such that $x$ is covered by $k$ atoms and $[x,x^*]$ has rank $k$. Then $[x,x^*]$ has $2^k$ elements. $L\{x,x^*\}$ consists of a chain in which every element is both a join-irreducible and a meet-irreducible. Therefore, $|L| = 2^k + n - k$. Since there are $k + 1$ join-irreducibles of $L$ in $[x,x^*]$ and $n - k$ join-irreducibles in $L\{x,x^*\}$, $m = n + 1$. Then

$$|L| = 2^k + m - 1 - k = 2^k + n - k,$$

and in general,

$$|L| = 2^{k-2}(m - n - 1) + 2^k + n - k.$$ 

Now suppose $k = 2$, and

$$|L| = 2^{k-2}(m - n - 1) + 2^k + n - k = m + 1.$$ 

First note that $L$ has at least one element $x$ of reach 2. Then at least 2 distinct elements $a$ and $b$ cover $x$. By the semimodularity of $L$, $a \lor b$ covers $a$ and $b$. Thus $a \lor b$ is not a join-irreducible. But all elements except $a \lor b$ are join-irreducibles since $|L| = m + 1$. Note also that all elements except $x$ are meet-irreducibles. Otherwise if an element $y$ has reach 2, there exists an element of $L$ which is not a join-irreducible. Thus there are $n - 2$ elements in $L$
which are both join and meet-irreducibles. Since \( x \) is a join-irreducible, \( m = 1 + n - 2 + z \) where \( z \) is the number of elements which are both a join-irreducible and a meet-irreducible covering \( x \) and covered by \( a \lor b \). Now \( x \) covers \( m - n + 1 \) elements. Therefore \([x,x^*]\) is isomorphic to a \((m - n + 1)\)-point line with a chain attached.

Conversely, suppose \( L \) is isomorphic to a chain attached to a \((m - n + 1)\)-point line. There exists only one element covering more than one element and all other elements are join-irreducibles. Hence \(|L| = m + 1\).

Finally, suppose \( k \geq 3, m > n + 1 \), and \( n > k \). Suppose further that \(|L| = 2^k + m - k\). Since there exists at least one element \( x \) of reach \( k \), \([x,x^*]\) has at least \( 2^k \) elements. \( L \setminus [x,x^*] \) has at least \( n - k \) elements. If \(|L| = 2^k + n - k\), then there are only \( n + 1 \) join-irreducibles in \( L \). However, since \( m > n + 1 \), and

\[
2^k + n - k < 2^k + m - 1 - k < 2^k + m - k = |L|,
\]

there exist more join-irreducibles in \( L \). Either \([x,x^*]\) or \( L \setminus [x,x^*] \) contain additional join-irreducibles. We shall add these elements to this minimal lattice in such a manner as to minimize the number of elements in \( L \) while remaining semimodular and consistent. Suppose \( m = n + 1 + z \) such that there are \( z \) join-irreducibles to be added to \( L \).

First, consider an additional join-irreducible in \([x,x^*]\). Then, with \([x,x^*]\) being geometric,

\[
|[x,x^*]| \geq 2^{k-2}(k + 1 - k) + 2^k = 2^{k-2} + 2^k.
\]
An addition of one join-irreducible contributes at the least to an additional \(2^{k-2}\) elements in \(L\).

Next, consider the addition of join-irreducibles in \(L \setminus [x,x^*]\) and its influence on the size of \(L\). Since \(n > k\), either there exists a meet-irreducible \(b\) such that \(b\) is covered by \(x\) and \(x\) is covered by a join-irreducible \(y\), or \(b\) is covered by \(x^*\) and \(x^*\) is covered by a join-irreducible \(y\).

In either case, let \(a_1, a_2, \ldots, a_z, a_{z+1}\) be \(z + 1\) additional join-irreducibles covering \(b\) and covered by \(y\). Since \(y\) is no longer a join-irreducible in \(L\), we have gained \(z\) additional join-irreducibles while adding a total of \(z + 1\) elements to \(L\). Note that \([b,y]\) is a \((z + 2)\)-point line and therefore modular and consistent, while \([x,x^*]\) is geometric of rank \(k\) with \(k\) atoms. Then since \([b,x]\) or \([x^*,y]\) is common to both \([x,x^*]\) and \([b,y]\), we form the Dilworth-Hall sum of \([x,x^*]\) and \([b,y]\). As shown in Lemma IV.3 and Lemma IV.4, this sum is semimodular and consistent.

Since \(2^{k-2} \geq z + 1\) for every \(z \geq 1\), if we form the Dilworth-Hall sum we have minimized \(L\). Then \(|[x,x^*]| = 2^k\) and \(|L \setminus [x,x^*]| = n - k + z + 1\). But since \(m = n + 1 + z\),
\[
|L \setminus [x,x^*]| = n - k + m - n - 1 + 1 = m - k,
\]
and
\[
|L| = 2^k + m - k.
\]

Then \(L\) consists of the Dilworth-Hall sum of \([x,x^*]\) and \([b,y]\) with \(n - k - 1\) elements in \(L \setminus ([x,x^*] \uparrow [b,y])\) in which every element is a join-irreducible and a meet-irreducible.
Hence, $L$ is isomorphic to a chain attached to the Dilworth-Hall sum of a free geometry with $k$ atoms and a $(m - n + 1)$-point line.

To prove the converse, let $L$ be isomorphic to a chain attached to the Dilworth-Hall sum of a free geometry with $k$ atoms and a $(m - n + 1)$-point line, with the isomorphic quotient lattice equal to a two-element chain. Observe that

$$|L| = |L_1 \uparrow L_2| + |L \setminus (L_1 \uparrow L_2)|$$

in which $L_1$ is a $(m - n + 1)$-point line and $L_2$ is a free geometry on $k$ atoms. Since the rank of $L_1 \uparrow L_2$ is $k + 1$, there exist at least $n - (k + 1)$ elements in $L \setminus (L_1 \uparrow L_2)$ in which every element is both a join-irreducible and a meet-irreducible. Now

$$|L_1 \uparrow L_2| = |L_1| + |L_2| - |L_1 \cap L_2|.$$ 

Since $L_1 \cap L_2$ is a two-element chain,

$$|L_1 \uparrow L_2| = |[x,x^*]| + |[b,y]| - 2 = 2^k + m - n + 1 + 2 - 2,$$

and

$$|L| = 2^k + m - n + 1 + n - k - 1 = 2^k + m - k. \quad \square$$

Lower Bounds on the Number of Elements of Reach $i$

Let $R_i$ be the number of elements with reach $i$. Lower bounds for $R_i$ are given dependent upon the rank, reach, and number of join-irreducibles in a consistent semimodular lattice.
Theorem IV.6. Let \( L \) be a finite semimodular consistent lattice of rank \( n \) and reach \( k \) with \( m \) join-irreducibles. Then

\[
R_0 = 1 \quad \text{and} \quad R_1 \geq m - 1 \quad \text{and}
\]

i) if \( n = k \), \( m = m + 1 \), or \( k = 2 \), then

\[
R_i \geq \left[ \frac{k - 2}{k - i - 1} \right] (m - 1 - n) + \left[ \frac{k}{k - i} \right] \quad \text{for} \quad 2 \leq i \leq k,
\]

ii) if \( n > k \), \( m > n + 1 \), and \( k \geq 3 \), then

\[
R_2 \geq \left[ \frac{k}{k - 2} \right] + 1 \quad \text{and} \quad R_i \geq \left[ \frac{k}{k - 2} \right] \quad \text{for} \quad 3 \leq i \leq k.
\]

Proof: If \( n = k \), then \( L = [0,0^*] \). \( L \) is geometric and all the join-irreducibles are atoms, except 0. Thus \( L \) has \( m - 1 \) elements of rank 1.

Note that in any geometric lattice, if \( y \) has rank \( n - i \), then \( y \) has reach \( i \). For any atom \( a \subseteq y \), \( a \lor y \) covers \( y \) by semimodularity. Then

\[
y^* = \lor \{ b \mid b \text{ covers } y \} \geq \lor \{ a \mid a \text{ covers } 0 \}
\]

since either \( a \subseteq y \) or \( a \subseteq b \) for some \( b \). Since \( 0^* = 1 \), \( y^* = 1 \), and

\[
\text{reach}(y) = r(y^*) - r(y) = n - (n - i) = i.
\]

Therefore, \( R_i \geq W_{n-i} \) for \( 0 \leq i \leq n \), where \( W_{n-i} \) represents the Whitney number of rank \( n - i \). Lower bounds for these
numbers were established in Theorem I.12.

\[ R_i \geq W_{k-i} \geq \left(\begin{array}{c} k-2 \\ k-i-1 \end{array}\right)(m-1-n) + \left(\begin{array}{c} k \\ k-i \end{array}\right), \quad \text{for } 0 \leq i \leq n. \]

Suppose \( n > k \). \( L \) has reach \( k \) and there exists at least one element of reach \( k \), \( x \). Then \([x,x^*]\) is a geometric lattice of rank \( k \) containing at least \( k \) atoms. Again, note that all join-irreducibles in \([x,x^*]\) have rank \( \leq 1 \) in \([x,x^*]\).

Since \( L \) is consistent, \( m = |\mathcal{J}| \leq |\mathcal{M}| = R_0 + R_1 \). Thus, \( R_1 \geq m - 1 \) for consistent lattices [3]. Note that \( m \) is equal to the number of join-irreducibles of \( L \) in \([x,x^*]\) plus the number of join-irreducibles of \( L \) in \( L\setminus[x,x^*] \). Also, the number of join-irreducibles in \( L\setminus[x,x^*]\) \( \geq n - k \).

If \( m = n + 1 \), then

\[ m = n + 1 \geq |\{ j \mid j \in \mathcal{J} \cap [x,x^*]\}| + n - k. \]

The number of join-irreducibles of \( L \) in \([x,x^*]\) \( \leq k + 1 \) and there exist \( k \) atoms in \([x,x^*]\). Let \( W_{k-i}^x \) be the number of elements of rank \( k - i \) in \([x,x^*]\). Then

\[ R_i \geq W_{k-i}^x \geq \left(\begin{array}{c} k \\ k-i \end{array}\right), \quad \text{for } 2 \leq i \leq k. \]

If \( m > n + 1 \), then either the number of atoms in \([x,x^*]\) = \( k + y \) with \( y > 0 \) or the number of join-irreducibles of \( L \) in \( L\setminus[x,x^*] \) = \( n - k + z \) with \( z > 0 \). If \( y > 0 \), then
noting that $R_i \geq W_{k-i}^k$, we have

$$R_i \geq \left[\frac{k - i - 1}{k - i}\right]y + \left[\frac{k}{k - 1}\right]$$
for $2 \leq i \leq k$.

If $k = 2$, then $R_2 \geq \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 1$.

If $k \geq 3$, then $R_2 \geq \left[\frac{k - 2}{k - 3}\right]y + \left[\frac{k}{k - 2}\right]$ and since $y > 0$,

$$R_2 \geq 1 + \left[\frac{k}{k - 2}\right].$$

Otherwise,

$$R_i \geq \left[\frac{k}{k - i}\right]$$
for $3 \leq i \leq k$.

Now if $y = 0$, then $W_{i}^k \geq \left[\frac{k}{i}\right]$ for $0 \leq i \leq k - 2$. Let $r(x) = g$ and $r(x^*) = g + k$. Since $z > 0$, there exist at least two elements $a$ and $b$, one may be $x$ or $x^*$, of rank less than or equal to $g$ or of rank greater than or equal to $g + k$. There exists a minimal such pair in terms of rank such that $a$ and $b$ cover $a \wedge b$. Then, $a \vee b$ covers $a$ and $b$. Thus $a \wedge b$ has reach at least 2. If $a \wedge b$ is an element in $L\setminus\{x, x^*\}$, then

$$R_2 \geq W_{k-2}^k + 1 \geq \left[\frac{k}{k - 2}\right] + 1,$$

and

$$R_i \geq \left[\frac{k}{k - i}\right]$$
for $3 \leq i \leq k$. 
The same inequalities hold if $a \wedge b$ is an element in $[x,x^*]$. In this case, $a \wedge b$ is a coatom of $[x,x^*]$ and instead of having reach 1, it has reach 2. □
CHAPTER BIBLIOGRAPHY


CHAPTER V

COMBINATORIAL INEQUALITIES IN DUALLY SEMIMODULAR
CONSISTENT LATTICES

In this chapter, we prove combinatorial inequalities for dually semimodular consistent lattices similar to those shown in Chapter IV. We present lower bounds for the total number of elements in such lattices in terms of the rank, coreach, and number of join-irreducibles, and describe the extremal lattices attaining these bounds. Lower bounds for the number of elements of a given coreach are also presented.

Preliminary Results

The main result in this section is to show that in a dually semimodular consistent lattice, the interval \([x_*, x]\) is not only dually geometric, but modular. We first establish several preliminary lemmas.

**Lemma V.1.** Let \(L\) be a finite dually semimodular lattice, \(x\) a join-irreducible in \(L\), and \(y\) an element in \(L\). If \(x \leq y\), then \(x_* \leq y_*\).

**Proof:** Let \(x\) be a join-irreducible in \(L\) and \(y\) an element in
L such that $x \leq y$. Then if $x = y$, then $x_* \leq y_*$. 

Suppose $y$ covers $x$ and let $z$ be an element covered by $y$. Since $L$ is dually semimodular, $x \land z$ is covered by both $x$ and $z$. However, $x$ is a join-irreducible and $x \land z$ must be $x_*$. Thus, for every $z$ covered by $y$, $x_* = x \land z \leq z$, and $x_* \leq y_*$. 

We now assume that if $r(y) - r(x) < n$, with $x \leq y$, then $x_* \leq y_*$. Let $y \in L$ with $x \leq y$ such that $r(y) - r(x) = n$. There exists an element $y_1$ such that $x \leq y_1$ and $y_1$ is covered by $y$. Let $z$ be an element covered by $y$. Since $y$ covers $y_1$ and $z$, $y_1 \land z$ is covered by $y_1$ and $z$. Now $r(y_1) - r(x) = n - 1$ and from the inductive hypothesis, $x_* \leq (y_1)_*$, and $(y_1)_* \leq y_1 \land z$. Then, $x_* \leq y_1 \land z \leq z$. Therefore for every $z$ covered by $y$, $x_* \leq z$ and $x_* \leq y_*$. □

Lemma V.2. Let $L$ be a finite dually semimodular consistent lattice. If $L$ is coatomic, then it is modular and atomic.

Proof: If $L$ is a dually semimodular coatomic lattice, then it is dually geometric. Thus, $W_1 \geq W_{n-1}$ and $L$ is atomic. Since $L$ is consistent [3], $|J| \leq |M|$. Since $L$ is atomic and coatomic, $|J| = W_0 + W_1$ and $|M| = W_{n-1} + W_n$. Hence $W_1 \leq W_{n-1}$. Thus $W_1 = W_{n-1}$ and $|J| = |M|$. By Theorem III.4, $L$ is modular. □

Lemma V.3. Let $L$ be a modular atomic lattice of rank $n$. 

Suppose there exist \( n \) coatoms \( a_1, a_2, \ldots, a_n \) such that
\[
\bigwedge_{i=1}^{n} a_i = 0
\]
and a coatom \( x \) such that \( x \neq a_i \) for \( 1 \leq i \leq n \).

Then there exist two coatoms, \( c_1 \) and \( c_2 \), such that
\[
c_1 \land c_2 = c_1 \land c_2 \land x = b.
\]
Further, there exists a join-irreducible \( j \) such that \( b \lor j = x \).

**Proof:** We begin by induction on \( n \), the rank of the lattice.
The lemma is true if \( n = 2 \) since \( x \) is itself a join-irreducible. We assume the lemma is true for lattices of rank less than \( n \).

The lemma is true if there exist coatoms \( a_1 \) and \( a_2 \) such that the rank of \( a_1 \land a_2 \land x \) is \( n - 2 \). Suppose then, that
\[
\bigwedge_{i=1}^{n} a_i \land a_j \land x
\]
has rank less than or equal to \( n - 3 \).

Let \( y = \bigwedge_{i=2}^{n} a_i \) and observe that \( \bigvee_{i=2}^{n} a_i = 1 \). Since \( a_1 \) is covered by 1, \( y \) covers \( \bigwedge_{i=1}^{n} a_i = 0 \) by the modularity of \( L \).

Thus, \( r(y) = 1 \).

Consider the element \( (a_1 \land x) \lor y \). Note that since
\[
\bigwedge_{i=1}^{n} a_i = 0, \ y \nsubseteq a_1.
\]
We first assume \( y \nsubseteq x \).

Using the modular identity,
\[
r((a_1 \land x) \lor y) = r(a_1 \land x) + r(y) - r(a_1 \land x \land y)
\]
\[
r((a_1 \land x) \lor y) = n - 2 + 1 - r(x \land (\bigwedge_{i=1}^{n} a_i))
\]
\[
= n - 1 - r(0)
\]
Letting \( c_1 = a_1 \) and \( c_2 = (a_1 \land x) \lor y \), \( c_1 \) and \( c_2 \) are coatoms in \( L \) covering \( a_1 \land x \). Since \( L \) is atomic, there exists a join-irreducible \( j \) such that \((a_1 \land x) \lor j = x\).

Suppose \( y \leq x \). Then \([y,1]\) is a modular atomic lattice of rank \( n - 1 \) and there exist coatoms \( a_2, a_3, \ldots, a_n \) such that \( \bigwedge_{i=2}^{n} a_i = y \). By the inductive hypothesis, there exist two coatoms \( c_1 \) and \( c_2 \) such that \( c_1 \land c_2 = c_1 \land c_2 \land x = b \) in \([y,1]\). \( c_1 \) and \( c_2 \) are also coatoms in \( L \). Again, since \( L \) is atomic, there exists a join-irreducible \( j \) such that \( b \lor j = x \). 

**Theorem V.4.** Let \( L \) be a finite dually semimodular consistent lattice such that \( 1_* = 0 \). Then \( L \) is coatomic.

**Proof:** We begin by induction on \( n \), the rank of \( L \). The theorem is certainly true for \( n = 2 \). We assume \( L \) is coatomic if it is dually semimodular, consistent and \( 1_* = 0 \) with rank less than \( n \).

Let \( L \) be as in the hypothesis with rank \( n \), and let \( m \) be a coatom of \( L \). Since \( L \) is dually semimodular, \( m \) covers \( m \land a \) for every coatom \( a \neq m \). Thus, \( m_* \leq 1_* = 0 \), such that \([0,m]\) is a dually semimodular consistent lattice with \( m_* = 0 \) and rank \( n - 1 \). By the inductive hypothesis, \([0,m]\) is coatomic. It suffices to show that any element of rank
n - 2 is the meet of coatoms.

Let \( x \in L \) such that \( x \) is covered by \( m \) and no other coatom. Now since \( L \) has rank \( n \) with \( 1^* = 0 \), there exist at least \( n \) distinct coatoms, \( a_1, a_2, \ldots, a_n \), such that \( \Lambda a_i = 0 \) for \( 1 \leq i \leq n \). Also, \([0,m]\) has rank \( n - 1 \) and \( m^* = 0 \). Therefore \( m \) covers \( m \Lambda a_i \) or \( m = m \Lambda a_i \) for some \( i \).

In any case, there are at least \( n - 1 \) distinct coatoms of the form \( m \Lambda a_i \) in \([0,m]\) such that \( \Lambda (m \Lambda a_i) = 0 \). Choose \( n - 1 \) of these. Using Lemma V.2, \([0,m]\) is modular and geometric and since \( x \notin m \Lambda a_i \) for every \( i \) such that \( 1 \leq i \leq n \), the hypothesis for Lemma V.3 is satisfied. Then, there exist coatoms \( m \Lambda a_k \) and \( m \Lambda a_p \) such that \((m \Lambda a_k) \Lambda (m \Lambda a_p) = a_k \Lambda a_p \Lambda x\). Denote this element by \( z \). Using Lemma V.3, there exists a join-irreducible \( j \) such that \( z \vee j = x \) since \( x \) covers \( z \). Both \( m \Lambda a_k \) and \( m \Lambda a_p \) join \( j \) equals \( m \) since \( j \notin m \Lambda a_k \) and \( j \notin m \Lambda a_p \). Thus \( j \notin a_k \) and \( j \notin a_p \). Since \( a_k \Lambda a_p \) covers \( z \), then \( a_k \Lambda a_p \notin m \). It remains to show that \((a_k \Lambda a_p) \vee j = 1\).

Suppose \((a_k \Lambda a_p) \vee j = b\) for some coatom \( b \) covering \( a_k \) and \( a_p \). Then \( b \) covers \( b \Lambda m \). By dual semimodularity, \( a_k \Lambda a_p \) covers \( a_k \Lambda a_p \Lambda b \Lambda m \), which is \( z \Lambda b \). \( a_k \Lambda a_p \) covers \( z \) and \( z \Lambda b \) which implies \( z = z \Lambda b \) since \( z \Lambda b \leq z \).
Since \( j \vee z = x \), \( j \nleq b \wedge m \). Then \( j \nleq b \), and
\[
(a_k \wedge a_p) \vee j = 1.
\]
But \( 1 \) covers \( a_k \) and \( a_p \) in \([a_k \wedge a_p, 1]\)
and therefore contradicts the fact that \( L \) is consistent.
Hence, \( L \) is coatomic.

**Theorem V.5.** Let \( L \) be a finite dually semimodular
consistent lattice. Then for \( x \in L \), the interval \([x^x, x]\) is
modular and geometric.

**Proof:** If \( x \in L \), the interval \([x^x, x]\) is a dually
semimodular consistent lattice. It follows from **Theorem V.4**
that \([x^x, x]\) is coatomic and from **Lemma V.3** that it is
modular and geometric.

The Total Number of Elements in a Consistent
Dually Semimodular Lattice

We give lower bounds for the number of elements in a
dually semimodular consistent lattice.

**Theorem V.6.** Let \( L \) be a finite dually semimodular
consistent lattice of rank \( n \) and coreach \( k \) with \( m \)
join-irreducibles. Then:
if \( n = k \),
\[
|L| = 2^{n-2} (m - 1 - n) + 2^k;
\]
if \( m = n + 1 \) or \( k = 2 \),
\[ |L| \geq 2^{k-2}(m - 1 - n) + 2^k + n - k; \]

if \( k \geq 3, m > n + 1, \) and \( n > k, \)
\[ |L| \geq 2^k + m - k. \]

Proof: Suppose \( n = k. \) Then \( L = [1_k, 1] \) and by Theorem V.5, \( L \) is modular and geometric. As stated in Theorem I.12 [2]

\[ W_i \geq \left\lfloor \frac{n - 2}{i - 1} \right\rfloor (W_1 - n) + \left( \begin{array}{c} n \\ i \end{array} \right) \text{ for } 0 \leq i \leq n, \]

where \( W_i \) is the number of atoms in \( L, \) which equals \( m - 1. \)

Since \( |L| = \sum_{i=0}^{n} W_i, \)
\[ |L| = 2^{n-2}(W_1 - n) + 2^n. \]

We conclude that
\[ |L| = 2^{k-2}(m - 1 - n) + 2^k. \]

Next, let \( n > k \) and assume \( m = n + 1. \) There exists an element \( x \) in \( L \) having coreach \( k \) and
\[ |L| \geq |[x_*, x]| + |L \setminus [x_*, x]|. \]

Since \( L \) has rank \( n > k, \) there exist at least \( n - k \) elements in \( L \setminus [x_*, x] \) and, with \( [x_*, x] \) having rank \( k, \) there exist at least \( k \) elements covering \( x. \) As before, because \( [x_*, x] \) is modular and geometric containing at least \( k \) atoms,
\[ |[x_*, x]| \geq 2^k. \] Thus,
\[ |L| \geq 2^k + n - k \]
\[ \geq 2^k + m - 1 - k. \]

Let \( n > k \) and \( m > n + 1, \) and suppose \( k = 2. \) Again,
\(|L\backslash[x_*,x]\| \geq n - 2\), and \(|[x_*,x]| = w + 2\), where \(w\) represents the number of atoms in \([x_*,x]\). Therefore we have
\[|L| \geq w + 2 + n - 2 = w + n.\]
If \( |L\backslash[x_*,x]| = n - 2\), then each of these elements is a join-irreducible and thus \(w = m - n + 1\). Then
\[|L| \geq w + n = m + 1.\]
If \( |L\backslash[x_*,x]| > n - 2\), then there exist two elements \(a\) and \(b\) of a given rank such that \(a \lor b\) is not a join-irreducible. Thus,
\[|L| \geq m + 2.\]
Hence if \( |L\backslash[x_*,x]| = n - 2\), the number of elements in \(L\) is minimized and
\[|L| \geq m + 1.\]

Now suppose \(n > k, m > n + 1,\) and \(k \geq 3\). Since there exists an element \(x\) of coreach \(k\) in \(L\), we first assume there exists an element \(x\) in \(L\) with coreach \(k\) such that \(x^* = 0\). Then,
\[|L| \geq |[0,x]| + |L\backslash[0,x]|.\]
Let \(k + y\) denote the number of atoms in \([0,x]\). Since \([0,x]\) is modular and geometric of rank \(k\), there are \(k + y + 1\) join-irreducibles in \([0,x]\).

There exist at least \(m - (k + y + 1)\) elements in \(L\backslash[0,x]\), and \(|[0,x]| \geq 2^{k-2}y + 2^k\). Then
\[|L| \geq 2^{k-2}y + 2^k + m - k - y - 1.\]
If \(y > 0,\)
\[|L| \geq 2^k + m - k - 1 + y\]
\[ \geq 2^k + m - k. \]

If \( y = 0 \), there exist exactly \( k + 1 \) join-irreducibles in \([0, x]\) and \( m - k - 1 \) join-irreducibles in \( L \setminus [0, x] \). Since \( m > n + 1 \), \( m - k - 1 > n - k \), and there exists a join-irreducible of rank \( k \) or two join-irreducibles of rank greater than \( k \) of the same rank. Choose the pair with maximal rank, \( z_1 \) and \( z_2 \). (Note that \( z_2 \) could be \( x \).) Then, there is an element, \( z_1 \lor z_2 \), in \( L \setminus [0, x] \) such that this element is not a join-irreducible. Therefore

\[
|L| \geq 2^{k-2}y + 2^k + m - k - 1 + 1 \\
\geq 2^k + m - k.
\]

We next consider the case in which \( L \) contains an element \( x \) of coreach \( k \) such that \( x_* > 0 \). There exists an atom \( p \) with \( p \leq x_* \) such that \([p, 1]\) has coreach \( k \) and rank \( n - 1 \). We proceed by induction on the rank of \( L \), similarly as in Theorem IV.2. Let \( s \) be the number of join-irreducibles of \([p, 1]\). Note that

\[
|L| \geq |[p, 1]| + |(0)| + |L\setminus([p, 1] \cup (0))|.
\]

Since \( L \) is consistent [4], we may write this inequality as before as

\[
|L| \geq |[p, 1]| + 1 + m - s - 1 + B,
\]

where \( B \) is equal to the number of join-irreducibles of \([p, 1]\) which are not join-irreducibles in \( L \).

If \( m - s - 1 > 0 \), then \( B > 0 \), and if \( m - s - 1 = 0 \), \( B = 0 \) only if every join-irreducible in \([p, 1]\) is also a join-irreducible in \( L \).
We next consider several cases depending on the lattice \([p,1] \). 

We first consider the case with \( k = n - 1 \). Then, \([p,1]\) is modular and geometric and
\[
| [p,1] | = 2^{k-2}y + 2^k,
\]
where \( y = s - 1 - (n - 1) = s - n \). If \( y = 0 \), then
\[
m - s - 1 > 0 \text{ since } m > n + 1 = s + 1. \text{ This implies } B > 0.
\]
Hence,
\[
| [p,1] | = 2^k \text{ and } |L| \geq 2^k + 1 + m - s - 1 + B.
\]
Since \( B \geq 1 \) and \( s = k + 1 \),
\[
|L| \geq 2^k + m - k - 1 + B \geq 2^k + m - k.
\]
If \( y > 0 \), then \( s > n \) and \( s = y + k + 1 \). Therefore,
\[
|L| \geq 2^{k-2}y + 2^k + 1 + m - s - 1 + B \geq y + 2^k + m - k - 1 + B.
\]
Since \( B \geq 0 \) and \( y \geq 1 \),
\[
|L| \geq 2^k + m - k.
\]

Finally, we suppose \( n - 1 > k \). Again, we consider the two cases where \( s = n \) or \( s > n \). Assume \( s = n \), then
\[
m > n + 1 = s + 1 \text{ such that } m - s - 1 > 0 \text{ and } B \geq 1. \text{ In this case, } | [p,1] | \geq 2^k + s - k - 1 \text{ by the inductive step.}
\]
With \( B \geq 1 \),
\[
|L| \geq 2^k + s - k - 1 + 1 + m - s - 1 + B \geq 2^k + m - k - 1 + B \geq 2^k + m - k.
\]
Now consider \( s > n \). By the induction hypothesis,
\[(p,1) \geq 2^k + s - k. \text{ Then}
\]
\[|L| \geq 2^k + s - k + 1 + m - s - 1 + B.\]

With \(B \geq 0,\)
\[|L| \geq 2^k + m - k.\]

Hence, with \(k \geq 3, m > n + 1,\) and \(n > k,\) we have
\[|L| \geq 2^k + m - k.\]

\[\square\]

**Slimmest Dually Semimodular Consistent Lattices**

In this section, we describe the dually semimodular consistent lattices which attain the bounds given in Theorem V.6. We find that, as in the case of semimodular consistent lattices, the slimmest of these is a Dilworth-Hall sum \([1]\) of a free geometry with a \(j\)-point line.

**Theorem V.7.** Let \(L\) be a finite dually semimodular consistent lattice of rank \(n \geq 3\) and coreach \(k\) with \(m\) join-irreducibles. Then
\[|L| = 2^{k-2}(m - 1 - n) + 2^k + n - k\]
i) for \(n = k\) if and only if \(L\) is isomorphic to the direct product of a modular plane and a free geometry,

ii) for \(n = m - 1\) if and only if \(L\) is isomorphic to a chain attached to a free geometry with \(k\) atoms,

iii) for \(k = 2\) if and only if \(L\) is isomorphic to a chain attached to a \((m - n + 1)\)-point line.

Otherwise, if \(k \geq 3, m \geq n + 1,\) and \(n > k,\)
\[ |L| = 2^k + m - k \]

if and only if \( L \) is isomorphic to a chain attached to the Dilworth-Hall sum of a free geometry with \( k \) atoms and a \((m - n + 1)\)-point line, with the identified quotient lattice consisting of a two-element chain.

**Proof:** If \( n = k \), then \( L \) is a modular and geometric lattice of rank \( n \) with \( m - 1 \) atoms. From Theorem 1.12,
\[ |L| = 2^{k-2}(m - n - 1) + 2^k, \]
and
\[ W_i = \begin{bmatrix} n - 2 \\ i - 1 \end{bmatrix}(m - n - 1) + \begin{bmatrix} n \\ i \end{bmatrix} \text{ for } 0 \leq i \leq n. \]

Thus for \( n \geq 4 \), equality holds if and only if \( L \) is isomorphic to the direct product of a modular plane and a free geometry. It also holds true for \( n = 3 \) when the free geometry is trivial.

Suppose \( n > k \) and \( n = m + 1 \) and
\[ |L| = 2^{k-2}(m - n - 1) + 2^k + n - k = 2^k + n - k. \]
Since \( L \) has at least one element of coreach \( k, x \),
\[ |[x^*,x]| \geq 2^k. \]
Also, there exist at least \( n - k \) elements in \( L \setminus [x^*,x] \). Then \( |[x^*,x]| = 2^k \) and
\[ |L \setminus [x^*,x]| = n - k. \] Thus, \([x^*,x]\) is a geometric lattice of rank \( k \) with \( k \) atoms and hence isomorphic to a free geometry with \( k \) atoms. Every element in \( L \setminus [x^*,x] \) is both a join-irreducible and a meet-irreducible since there are only \( n - k \) elements. Thus \( L \) is isomorphic to a chain attached to a free geometry on \( k \) atoms.
Conversely, suppose \( L \) is isomorphic to a chain attached to a free geometry on \( k \) atoms. There exists an element \( x \) in \( L \) such that \( x \) covers \( k \) coatoms and \([x_*, x]\) has rank \( k \). Then \([x_*, x]\) has \( 2^k \) elements. \( L \setminus [x_*, x] \) consists of a chain in which every element is both a join-irreducible and a meet-irreducible. Therefore, \(|L| = 2^k + n - k\). There are \( k + 1 \) join-irreducibles of \( L \) in \([x_*, x]\) and \( n - k \) join-irreducibles in \( L \setminus [x_*, x] \). With \( m = n + 1 \),

\[
|L| = 2^k + m - 1 - k
= 2^k + n - k.
\]

If \( k = 2 \), then

\[
|L| = 2^{k-2}(m - n - 1) + 2^k + n - k
= m + 1.
\]

First note that \( L \) has only one element \( x \) of coreach 2. Then at least 2 distinct elements \( a \) and \( b \) are covered by \( x \). By the dually semimodularity of \( L \), \( a \land b \) must be covered by \( a \) and \( b \), and \( a \land b = x_* \). Then \([x_*, x]\) has rank 2. Since \( L \) has rank \( n \), there exist \( n - 2 \) join-irreducibles in \( L \setminus [x_*, x] \) and therefore there exist \( m - (n - 2) - 1 \) atoms in \([x_*, x]\).

Hence \([x_*, x]\) is isomorphic to a \((m - n + 1)\)-point line with a chain attached.

Now suppose \( L \) is isomorphic to a chain attached to a \((m - n + 1)\)-point line. There exists only one element covering more than one element and all other elements are join-irreducibles. Hence \(|L| = m + 1\).

Finally, suppose \( k \geq 3 \), \( m > n + 1 \), and \( n > k \). Suppose
further that \(|L| = 2^k + m - k\). Since there exists at least one element \(x\) of coreach \(k\), \([x^*, x]\) has at least \(2^k\) elements and \(L \setminus [x^*, x]\) has at least \(n - k\) elements. If \(|L| = 2^k + n - k\), then there are only \(n + 1\) join-irreducibles in \(L\). However, since \(m > n + 1\), and

\[
2^k + n - k < 2^k + m - 1 - k < 2^k + m - k = |L|
\]

there exist more join-irreducibles in \(L\). Then, either \([x^*, x]\) or \(L \setminus [x^*, x]\) contain additional join-irreducibles. We shall add these elements to this minimal lattice in such a manner as to minimize the number of elements in \(L\). Suppose \(m = n + 1 + z\) such that there are \(z\) join-irreducibles to be added to \(L\).

First, consider an additional join-irreducible in \([x^*, x]\). Then, with \([x^*, x]\) being modular and geometric,

\[
|x^*, x| = 2^{k-2}(k + 1 - k) + 2^k
= 2^{k-2} + 2^k.
\]

Thus, an addition of one join-irreducible contributes at the least to an additional \(2^{k-2}\) elements in \(L\).

Next, consider the addition of join-irreducibles in \(L \setminus [x^*, x]\) and its influence on the size of \(L\). Since \(n > k\), either there exists a join-irreducible \(y\) such that \(y\) covers \(x\) and \(x\) covers a meet-irreducible \(b\), or there exists a join-irreducible \(y\) covering \(x^*\) where \(x^*\) covers a meet-irreducible \(b\). In either case, let \(a_1, a_2, ..., a_z, a_{z+1}\) be \(z + 1\) additional join-irreducibles covering \(b\) and covered by \(y\). Since \(y\) is no longer a join-irreducible in \(L\),
we have gained $z$ additional join-irreducibles while adding a
total of $z + 1$ elements to $L$. Note that $[b,y]$ is a
$(z + 2)$-point line and therefore modular and consistent,
while $[x_s,x]$ is modular and geometric of rank $k$ with $k$
atoms. Since $[b,x]$ or $[x_s,y]$ is common to both $[x_s,x]$ and
$[b,y]$, we form the Dilworth-Hall sum of $[x_s,x]$ and $[b,y]$.
Since both intervals are modular, by Lemma 1.14, this sum is
modular and thus consistent.

Since $2^{k-2} > z + 1$ for every $z \geq 1$, if we form the
Dilworth-Hall sum we have minimized $L$. Then $|[x_s,x]| = 2^k$
and $|L\backslash[x_s,x]| = n - k + z + 1$. Since $m = n + 1 + z$,

$$\left|L\backslash[x_s,x]\right| = n - k + m - n - 1 + 1 = m - k,$$

such that

$$\left|L\right| = 2^k + m - k.$$

Then $L$ consists of the Dilworth-Hall sum of $[x_s,x]$ and
$[b,y]$, and $n - k - 1$ elements in $L\backslash([x_s,x] \uparrow [b,y])$ in which
every element is both a join-irreducible and
meet-irreducible. Hence, $L$ is isomorphic to a chain
attached to the Dilworth-Hall sum of a free geometry with $k$
atoms and a $(m - n + 1)$-point line.

To prove the converse, let $L$ be isomorphic to a chain
attached to the Dilworth-Hall sum of a free geometry with $k$
atoms and a $(m - n + 1)$-point line, with the isomorphic
quotient lattice equal to a two-element chain. Note that

$$\left|L\right| = \left|L_1 \uparrow L_2\right| + \left|L\backslash(L_1 \uparrow L_2)\right|$$

where $L_1$ is a $(m - n + 1)$-point line and $L_2$ is a free
geometry on \( k \) atoms. Since the rank of \( L_1 \uparrow L_2 \) is \( k + 1 \), there exist at least \( n - (k + 1) \) elements in \( L \setminus (L_1 \uparrow L_2) \) in which every element is both a join-irreducible and a meet-irreducible. Now

\[
|L_1 \uparrow L_2| = |L_1| + |L_2| - |L_1 \cap L_2|.
\]

Since \( L_1 \cap L_2 \) is a two-element chain,

\[
|L_1 \uparrow L_2| = |[x*,x]| + |[b,y]| - 2 = 2^k + m - n + 1 + 2 - 2,
\]

and

\[
|L_1| = 2^k + m - n + 1 + n - k - 1 = 2^k + m - k.
\]

\[\square\]

Lower Bounds on the Number of Elements of Coreach \( i \)

We present lower bounds for \( C_i \), the number of elements with coreach \( i \), in a consistent dually semimodular lattice.

**Theorem V.8.** Let \( L \) be a finite dually semimodular consistent lattice of rank \( n \) and coreach \( k \) with \( m \) join-irreducibles. Then

\[
C_0 = 1 \quad \text{and} \quad C_1 = m - 1 \quad \text{and}
\]

i) if \( n = k \), then

\[
C_i = \left[ \frac{k - 2}{i - 1} \right](m - 1 - n) + \left[ \frac{k}{i} \right] \quad \text{for} \ 2 \leq i \leq k.
\]

ii) if \( m = n + 1 \) or \( k = 2 \), then
\[
C_i \geq \left[ \binom{k}{i} \right]
\quad \text{for } 2 \leq i \leq k
\]

iii) if \( n > k \), \( m > n + 1 \), and \( k \geq 3 \), then

\[
C_2 \geq \left[ \binom{k}{2} \right] + 1 \quad \text{and} \quad C_i \geq \left[ \binom{k}{i} \right] \quad \text{for } 3 \leq i \leq k.
\]

Proof: In any case, there is exactly one element of coreach \( 0 \) and \( C_1 \) is by definition \( m - 1 \).

If \( n = k \), then \( L = [1,1] \). By Theorem V.5, \( L \) is modular and geometric. Note that the coreach of an element is the rank of that element in a geometric lattice. Recalling Theorem I.12,

\[
C_i = w_i \geq \left[ \binom{k-2}{i-1} \right] (m - 1 - n) + \left[ \binom{k}{i} \right], \quad \text{for } 2 \leq i \leq k.
\]

Now suppose \( n > k \). \( L \) has coreach \( k \) and there exists at least one element of coreach \( k \), \( x \). Then \([x,x] \) is a modular and geometric lattice of rank \( k \) containing at least \( k \) atoms. Again, all join-irreducibles in \([x,x] \) have rank \( \leq 1 \) in \([x,x] \).

Note that \( m \) is equal to the number of join-irreducibles of \( L \) in \([x,x] \) plus the number of join-irreducibles of \( L \) in \( L \setminus [x,x] \). Also, we observe that the number of join-irreducibles in \( L \setminus [x,x] \) is greater than or equal to \( n - k \).
If $m = n + 1$, then

$$m = n + 1 \geq |\{ j \mid j \in \mathcal{J} \cap [x_{*}, x]\}| + n - k.$$ 

Thus, the number of join-irreducibles of $L$ in $[x_{*}, x] \leq k + 1$ and there exist $k$ atoms in $[x_{*}, x]$. Let $W^x_i$ be the number of elements of rank $i$ in $[x_{*}, x]$. Then

$$C_i \geq W^x_i \geq \left[ \begin{array}{c} k \\ i \end{array} \right] \quad \text{for } 2 \leq i \leq k.$$

If $k = 2$, then there exists at least one element of coreach 2 and the above inequality holds since $\left[ \begin{array}{c} 2 \\ 2 \end{array} \right] = 1$.

If $m > n + 1$, then either the number of atoms in $[x_{*}, x] = k + y$ with $y > 0$ or the number of join-irreducibles of $L$ in $L \setminus [x_{*}, x] = n - k + z$ with $z > 0$. If $y > 0$, then with $C_i \geq W^x_i$,

$$C_i \geq \left[ \begin{array}{c} k - 2 \\ i - 1 \end{array} \right]y + \left[ \begin{array}{c} k \\ i \end{array} \right] \quad \text{for } 2 \leq i \leq k.$$

If $k \geq 3$, then

$$C_2 \geq \left[ \begin{array}{c} k - 2 \\ 1 \end{array} \right]y + \left[ \begin{array}{c} k \\ 2 \end{array} \right]$$

and since $y > 0$,

$$C_2 \geq 1 + \left[ \begin{array}{c} k \\ 2 \end{array} \right].$$

Otherwise,

$$C_i \geq \left[ \begin{array}{c} k \\ i \end{array} \right] \quad \text{for } 3 \leq i \leq k.$$
If \( y = 0 \), then \( W_i^x \geq \begin{bmatrix} k \\ i \end{bmatrix} \) for \( 2 \leq i \leq k \). Let \( r(x_*) = p \) and \( r(x) = p + k \). Since \( z > 0 \), there exist at least two elements \( a \) and \( b \), one may be \( x \) or \( x_* \), of rank less than or equal to \( p \) or of rank greater than or equal to \( p + k \). There exists a maximal such pair in terms of rank such that \( a \) and \( b \) are covered by \( a \lor b \). Then, \( a \) and \( b \) cover \( a \land b \). Thus \( a \lor b \) has coreach at least 2. If \( a \lor b \) is an element in \( L \setminus [x_*, x] \),

\[
C_2 \geq W_2^x + 1 \geq \begin{bmatrix} k \\ 2 \end{bmatrix} + 1,
\]

and

\[
C_i \geq \begin{bmatrix} k \\ i \end{bmatrix} \quad \text{for } 3 \leq i \leq k.
\]

Otherwise, if \( a \lor b \) is in \( [x_*, x] \), then \( a \lor b \) is an atom of \( [x_*, x] \) and does not have coreach 1 but has coreach at least 2. Hence, the inequalities given above hold. \( \Box \)
CHAPTER BIBLIOGRAPHY


