ANALYSIS OF SEQUENTIAL BARYCENTER RANDOM PROBABILITY MEASURES VIA DISCRETE CONSTRUCTIONS

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Hill and Monticino (1998) introduced a constructive method for generating random probability measures with a prescribed mean or distribution on the mean. The method involves sequentially generating an array of barycenters that uniquely defines a probability measure. This work analyzes statistical properties of the measures generated by sequential barycenter array constructions. Specifically, this work addresses how changing the base measures of the construction affects the statististics of measures generated by the SBA construction. A relationship between statistics associated with a finite level version of the SBA construction and the full construction is developed. Monte Carlo statistical experiments are used to simulate the effect changing base measures has on the statistics associated with the finite level construction.

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Chapter 1

Introduction

The notion of a sequential barycenter array (SBA) random probability measure (rpm) construction was introduced by Hill and Monticino [19]. This work extends the characterizations of the support of this construction. In particular, this work investigates the relationships between the generating measure used in the SBA rpm construction and the distributions of the standard deviation, skewness, third central moment, kurtosis and fourth central moment of the measures in the support of the SBA rpm. Because of the analytical intractability of this relationship, much of the investigation is accomplished through numerical simulation. Among other results, the simulations imply that base measures which concentrate their mass near 1 are more likely to produce measures with a larger standard deviations than those which concentrate their mass near 0. Also, base measures which concentrate their mass near 1 are more likely to produce measures that are more symmetric than those with mass near 0.

Random probability measure constructions are techniques for specifying a prior on the space of probability measures. Priors are an important component of Bayesian statistical analysis. Priors based upon parametric classes have been extensively studied. Due, in part, to improvements in computability, interest in nonparametric Bayesian techniques has increased. Several nonparametric constructions have been given in the literature. These methods include Ferguson's Dirichlet processes [15] and [16], the Dubins-Freedman scheme for generating random distribution functions

[11], Graf, Mauldin and Williams' random rescaling scheme [17], Mauldin, Sudderth and Williams' Polya tree priors [24], Monticino and Mauldin's generalization of the random rescaling scheme [23], and Bloomer's variance split arrays [5]. In varying degree, all these techniques comply with Ferguson's [15, 16] criteria that nonparametric constructions be analytically manageable and have large support. However, while general properties of measures in the support of the constructions are known, more detailed properties are not. For example, if the base measure used in the SBA construction has full support on the interval [0,1], then the associated prior has full support on the set of probability measures on [0,1]. On the other hand, it is not known how the selection of the base measure affects the standard deviations of the generated measures.

All the constructions mentioned above share the common feature that they involve a recursive process which makes them analytically manageable and relatively straightforward to simulate. In this work, the SBA construction is simulated and the relationship between the base measure and the types of measures in the support is investigated. Chapter 2 reviews the SBA construction given by Hill and Monticino [19] and develops relationships between statistics associated with a finite level version of the SBA construction and those from the full construction. Chapter 3 gives some partial results on a random splitting problem which was motivated by the relationship between the finite and full SBA constructions. Chapter 4 presents results of the Monte Carlo simulations.

Chapter 2

Background

2.1 The Sequential Barycenter Array Construction

The sequential barycenter array (SBA) construction gives a general and natural method for randomly generating probability measures with a prescribed mean. Before describing the construction, some definitions are in order.

Throughout this work, let X be a real-valued random variable with distribution function F, such that $E[|X|] < \infty$.

Definition 2.1 *The* ***F–barycenter of the interval*** $(a, c]$*,* $b_F(a, c]$*, is given by*

$$b_F(a,c] = \begin{cases} E[X|X \in (a,c]] = \frac{\int_{(a,c]} x dF(x)}{F(c)-F(a)}, & \text{if } F(c)>F(a) \\ a & \text{if } F(c)=F(a). \end{cases}$$

That is, the F–barycenter of $(a, c]$ is the conditional expectation of X over the interval $(a, c]$. The following lemma characterizes some elementary properties of F–barycenters.

Lemma 2.2 (Hill and Monticino [19]) *Fix* $a < c$ *such that* $P[X \in (a, c]] > 0$*, and let* $b = b_F(a, c]$*. Then* *i.* $F(c) > F(a)$ *if and only if* $b > a$,

ii. $(F(c) - F(a))b = (F(b) - F(a))b_F(a,b] + (F(c) - F(b))b_F(b,c]$,

iii. $b_F(a,b] = b$ *if and only if* $b_F(b,c] = b$,

iv. $b \geq b_F(a,x]$, *for all* $x \in (a,c]$.

Definition 2.3 (Hill and Monticino [19]) *For a random variable* X *with distribution function* F*, the* ***sequential barycenter array*** *(SBA) of* F *is the triangular array* $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1} = \{m_{n,k}(F)\}$ *defined inductively by*

i. $m_{1,1} = E[X]$,

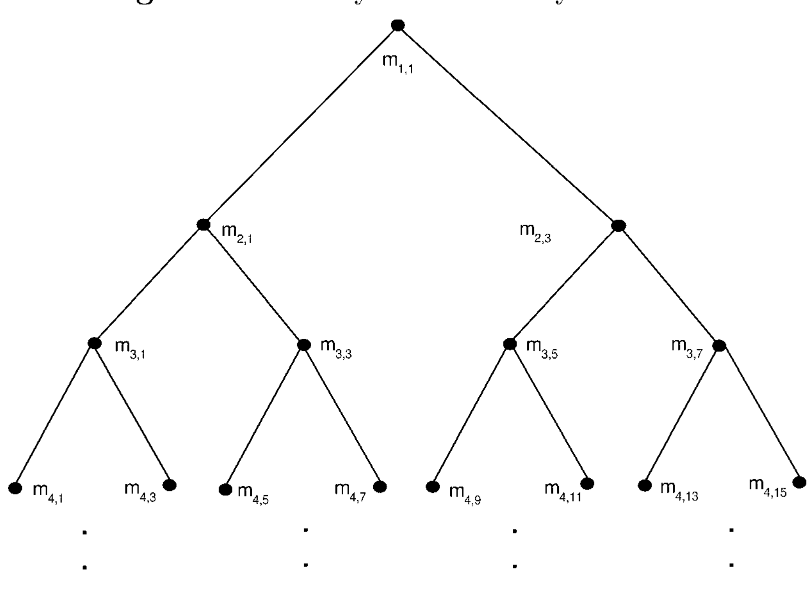
ii. $m_{n,2j} = m_{n-1,j}$, *for* $n \geq 1$ *and* $j = 1, \ldots, 2^{n-1} - 1$,

iii. $m_{n,2j-1} = b_F(m_{n-1,j-1}, m_{n-1,j}]$, *for* $j = 1, \ldots, 2^{n-1}$,

with the convention that $m_{n,0} = -\infty$ *and* $m_{n,2^n} = \infty$.

Note that (iii) of Definition 2.3, defines $m_{n,2j-1}$ as the *conditional barycenter* of X over $(m_{n-1,j-1}, m_{n-1,j}]$, for $n \geq 1$. These conditional barycenters can be viewed as an infinite binary tree, as shown in Figure 2.4.

Figure 2.4 Binary tree of F-barycenters



Example 2.5 Suppose that the random variable X is uniformly distributed over $[0,1]$. Then the sequential barycenter array of X is given by

$$\{m_{n,k}\}_{n=1,k=1}^{\infty,\,2^n-1} = \left\{\frac{k}{2^n}\right\}_{n=1,\,k=1}^{\infty,\,2^n-1}. \tag{2.1}$$

Example 2.6 Suppose X is a geometric random variable with parameter $p = 1/2$. Then

$$\begin{aligned}
m_{1,1} &= E(X) = 2,\\
m_{2,1} &= b_F(-\infty, 2] = \frac{\frac{1}{2} + 2(\frac{1}{2})(\frac{1}{2})}{F(2) - F(-\infty)} = \frac{4}{3},\\
&\textit{and}\\
m_{2,3} &= b_F(2,\infty) = \frac{2 - [1/2 + 2(1/2)^2]}{F(\infty) - F(2)} = \frac{1}{1 - \frac{3}{4}} = 4.
\end{aligned}$$

The next level of barycenters are

$$\begin{aligned}
m_{3,1} &= b_F(-\infty, 4/3] = \frac{\frac{1}{2}}{F(4/3) - F(-\infty)} = \frac{\frac{1}{2}}{\frac{1}{2} - 0} = 1,\\
m_{3,3} &= b_F(4/3, 2] = 2,\\
m_{3,5} &= b_F(2,4] = \frac{3(1/2)^3 + 4(1/2)^4}{F(4) - F(2)} = \frac{\frac{5}{8}}{\frac{3}{16}} = \frac{10}{3},\\
&\textit{and}\\
m_{3,7} &= b_F(4,\infty] = \frac{2 - [1/2 + 2(1/2)^2 + 3(1/2)^3 + 4(1/2)^4]}{F(\infty) - F(4)} = \frac{\frac{3}{8}}{1 - \frac{15}{16}} = 6.
\end{aligned}$$

The fourth level barycenters are

$$\begin{aligned}
m_{4,1} &= b_F(-\infty, 1] = 1,\\
m_{4,3} &= b_F(1, 4/3] = 1,\\
m_{4,5} &= b_F(4/3, 2] = 2,\\
m_{4,7} &= 2,
\end{aligned}$$

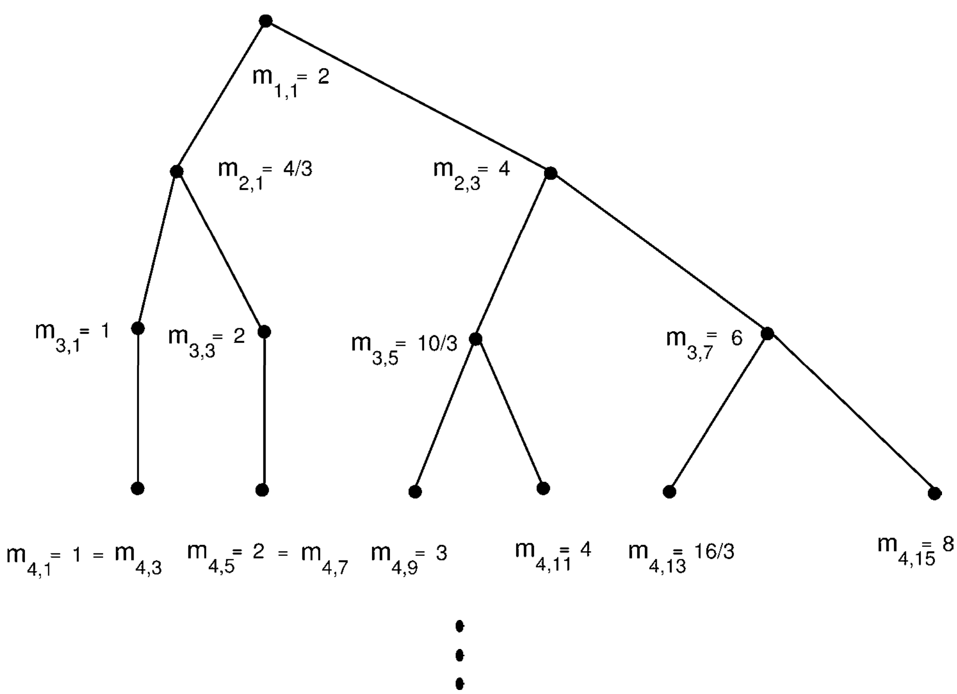
$$
\begin{aligned}
m_{4,9} &= b_F(2, 10/3] = \frac{\frac{3}{8}}{\frac{7}{8} - \frac{3}{4}} = 3, \\
m_{4,11} &= b_F(10/3, 4] = 4, \\
m_{4,13} &= b_F(4, 6] = \frac{5(1/2)^5 + 6(1/2)^6}{F(6) - F(4)} = \frac{16}{3},
\end{aligned}
$$

and

$$
\begin{aligned}
m_{4,15} &= b_F(6, \infty) \\
&= \frac{2 - [1/2 + 2(1/2)^2 + 3(1/2)^3 + 4(1/2)^4 + 5(1/2)^5 + 6(1/2)^6]}{1 - \frac{63}{64}} = 8.
\end{aligned}
$$

Figure 2.7 represents the barycenters as a binary tree.

Figure 2.7 The binary tree of barycenters for Example 2.6



Hill and Monticino [19] show that a probability measure is completely determined by its sequential barycenter array and they give an inversion formula (Theorem 2.8) for recovering the distribution function of the probability measure from its SBA.

Theorem 2.8 (Hill and Monticino, [19]) *F is completely determined by the values* $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1}$. *In particular,* $F(m_{n,k})$ *is given inductively by* $F(m_{n,0}) = 0, F(m_{n,2^n}) = 1$; *by Definition 2.3(ii) for even k; and, for* $k = 2j - 1$,

$$F(m_{n,2j-1}) = F(m_{n-1,j-1}) + (F(m_{n-1,j}) - F(m_{n-1,j-1}))\frac{m_{n+1,4j-1} - m_{n+1,4j-2}}{m_{n+1,4j-1} - m_{n+1,4j-3}},$$

with the convention that $\frac{0}{0} = 1$.

The following theorem gives conditions so that a given array is a sequential barycenter array.

Theorem 2.9 (Hill and Monticino, [19]) *A triangular array* $M = \{m_{n,k}\}_{n=1,k=1}^{\infty,2^n-1}$ *is an SBA for some distribution function F of X if and only if M satisfies ii, of Definition 2.3,*

i. $m_{n,k-1} \leq m_{n,k}$, *for all* $n \geq 1$ *and* $k = 1, 2, \ldots, 2^n$, *and*

ii. $m_{n,4k-3} = m_{n,4k-2}$, *if and only if* $m_{n,4k-1} = m_{n,4k-2}$, *for all* $n \geq 2$ *and* $k = 1, 2, \ldots, 2^{n-2}$.

Hill and Monticino [19] introduce a method for generating probability measures with support on [0,1] by randomly generating arrays which are almost surely sequential barycenter arrays for probability measures. The construction proceeds as follows.

Let μ and μ_0 be probability measures with support on [0,1] and [0,1) respectively. Denote by $\mathcal{P}([0,1])$ the set of all Borel probability measures on [0,1]. Let $\{X_{n,2j-1}\}_{n=1,j=1}^{\infty,2^n-1}$ be an array of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $X_{1,1}$ has distribution μ_0 and for $n \geq 2$, each $X_{n,k}$ has distribution μ.

Define a random array $M = \{m_{n,k}\}_{n=1,k=1}^{\infty,2^n-1}$ by

i. $m_{1,1} = X_{1,1}$,

ii. $m_{n,2j} = m_{n-1,j}$, for $n > 1$ and $j = 1, 2, \ldots, 2^{n-1} - 1$, iii.

$$m_{n,4j-3} = \begin{cases} m_{n-1,2j-1}, & \text{if } m_{n-1,2j-1} = m_{n-1,2j}; \\ & m_{n-1,2j-2} = m_{n-1,2j-1}; \\ & X_{n,4j-3} = 0, \text{ or } X_{n,4j-1} = 0, \\ m_{n-1,2j-1} - & \\ X_{n,4j-3}(m_{n-1,2j-1} - m_{n-1,2j-2}) & \text{otherwise,} \end{cases}$$

and

iv.

$$m_{n,4j-1} = \begin{cases} m_{n-1,2j-1}, & \text{if } m_{n,4j-3} = m_{n-1,2j-1}; \\ m_{n-1,2j-1} + & \\ X_{n,4j-1}(m_{n-1,2j} - m_{n-1,2j-1}) & \text{otherwise.} \end{cases}$$

Endow the set of triangular arrays $\mathcal{A} = [0,1] \times [0,1]^3 \times \cdots \times [0,1]^{2^n-1} \times \cdots$ with the standard product topology. Let $A \in \mathcal{A}$ be the Borel subset of arrays which satisfy (ii) of Definition 2.3 and conditions (i) and (ii) of Theorem 2.9. Clearly, $M(\omega) \in A$ for all $\omega \in \Omega$. Let $\mathcal{Q}_{(\mu_0,\mu)}$ be the distribution of M on $\mathcal{A}$. It is shown by Hill and Monticino [19] that the mapping T, induced by (ii) of Definition 2.3 and Theorem 2.8, which sends an array $\{m_{n,k}\} \in A$ to its associated distribution, $T(\{m_{n,k}\})$, is Borel from A to $\mathcal{P}([0,1])$ given the weak* topology.

Definition 2.10 (Hill and Monticino, [19]) *The sequential barycenter array random probability measure (SBA rpm) $B_{\mu_0,\mu}$is the Borel measure $\mathcal{Q}_{(\mu_0,\mu)}T^{-1}$ on $\mathcal{P}([0,1])$.*

What kind of measures are in the support of $B_{\mu_0,\mu}$? Hill and Monticino [19] give some general properties of the support of $B_{\mu_0,\mu}$ — for instance, Theorem 2.11 states conditions on μ and μ_0 so that $B_{\mu_0,\mu}$ has full support on $\mathcal{P}([0,1])$. Theorem 2.12 specifies when all measures in the support are continuous, discrete or have finite support.

A probability measure ν defined on a compact Hausdorff space $\mathcal{H}$ has *full support* if $\mathcal{H}$ is the smallest compact set whose complement has ν-measure zero.

Theorem 2.11 (Hill and Monticino, [19])
If μ_0 *and* μ *have full support on [0,1], then* $B_{\mu_0,\mu}$*has full support on* $\mathcal{P}([0,1])$.

Theorem 2.12 (Hill and Monticino, [19])

i. $B_{\mu_0,\mu}$*-almost all SBA measures are continuous on [0,1] if and only if* $\mu_0(\{0,1\}) = 0 = \mu(\{0\})$.

ii. If $\mu_0(\{0\}) > 0$, *then almost all SBA measures are discrete.*

iii. If $\mu_0(\{0\}) \geq 1 - \sqrt{2}$, *then* $B_{\mu_0,\mu}$*-almost all measures have finite support.*

One of the objectives of this work is to extend the characterization of the support of $B_{\mu_0,\mu}$. For instance, what is the distribution of the standard deviation (SD) of the measures generated by an SBA construction? What is the distribution on the third or fourth moments? The shape and form of a distribution of a random variable X with mean m can be described by its departure from symmetry about the mean and from its degree of peakedness or flatness (Wilks [31]). The statistics that serve as descriptors are calculated using the third and fourth central moments of X.

The *ith central moment* of a random variable X is defined by

$$\eta_i = E\left[(X-m)^i\right]. \tag{2.2}$$

The SD, $\sigma = \sqrt{\eta_2}$, gives a measure of dispersion of the distribution about the mean. For a random variable on [0,1], with mean m between 0 and 1, the SD has values between 0 and $\sqrt{m - m^2}$ (Corollary 2.26). The third central moment, η_3, gives a measure of how symmetric the distribution of mass is relative to the mean. If a distribution is symmetric about the mean, then η_3 is zero. A positive value for η_3 implies that most of the mass of a distribution lies to the left of the mean and a negative value for η_3 means that most of the mass lies to the right of the mean.

The third moment of the standardized random variable

$$X^* = \frac{X-m}{\sigma} \tag{2.3}$$

is referred to as the skewness of the distribution of X and is denoted by $\alpha_3 = \eta_3/\sigma^3$. If $\alpha_3 < 0$, then most of the mass of the distribution is to the left of the mean and the distribution is said to be negatively skewed. If $\alpha_3 > 0$, then the distribution is positively skewed and most of the mass is to the right of the mean. If $\alpha_3 = 0$, then the distribution of X is symmetric about m. The range for skewness of a random variable on [0,1] is $(-\infty, \infty)$.

Kurtosis is defined by $\alpha_4 = E(X^{*^4}) - 3$. If $\alpha_4 = 0$, the distribution is neither excessively peaked nor flat. If $\alpha_4 > 0$, the distribution is sharply peaked (leptokurtic). If $\alpha_4 < 0$ the distribution is flat (platykurtic). For a detailed discussion about central moments, the SD, skewness and kurtosis, see Wilks [31]. The kurtosis of the random variable on [0,1] takes on values in $[-2, \infty)$ [27].

Unfortunately, deriving the distributions of the moments of measures in the support of $B_{\mu_0,\mu}$ appears to be analytically intractable. Therefore, simulation will be used to empirically characterize them. One question addressed here is how well do the statistics derived from simulation represent the actual statistics associated with $B_{\mu_0,\mu}$? To address this question, the notion of an SBA approximation of a random probability measure is developed.

2.2 Approximations Of SBA Random Probability Measure Constructions

This section introduces a method for approximating a probability measure using the measure's SBA.

Definition 2.13 *Let X be a real-valued random variable with cumulative distribution function (cdf) F and SBA $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1}$. For $n \geq 1$, the* ***nth level SBA approximation*** *of X is the discrete random variable, $X^{(n)}$, with support on $\{m_{n,2i-1}\}_{i=1}^{2^{n-1}}$ such that*

$$P\left(X^{(n)} = m_{n,2i-1}\right) = F(m_{n,2i}) - F(m_{n,2i-2}),$$

with the convention $F(-\infty) = 0$ and $F(\infty) = 1$.

The cdf, $F^{(n)}$, of $X^{(n)}$, is given by

$$F^{(n)}(x) = \begin{cases} 0 & \text{if } x < m_{n,1} \\ F(m_{n,2i}) & \text{if } m_{n,2i-1} \leq x < m_{n,2i+1}, \text{ for all } 1 \leq i \leq 2^{n-1}-2 \\ 1 & \text{if } x \geq m_{n,2^n-1}. \end{cases} \tag{2.4}$$

The distribution, $\mu^{(n)}$, of $X^{(n)}$ is given by

$$\mu^{(n)} = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \delta_{m_{n,2i-1}}, \tag{2.5}$$

where $p_i^{[n]} = F(m_{n,2i}) - F(m_{n,2i-2})$.

Example 2.14 Let X be random variable such that $P\left(X = \frac{1}{6}\right) = \frac{1}{8} = P\left(X = \frac{1}{3}\right)$, and $P\left(X = \frac{2}{3}\right) = \frac{3}{4}$. Then $m_{1,1} = E[X] = 9/16$ and the first level approximation $X^{(1)}$ of X has measure $\delta_{m_{1,1}}$. Now, $m_{2,1} = b_F(0, 9/16] = 1/4$ and $m_{2,3} = b_F(9/16, 1] = 2/3$. Thus, the second level approximation $X^{(2)}$ of X has cdf

$$F^{(2)}(x) = \begin{cases} 0 & x < 1/4 \\ 1/8 & 1/4 \leq x < 2/3 \\ 1 & x \geq 2/3. \end{cases}$$

Continuing with this process, we get $m_{3,1} = b_F(0, 1/4] = 1/6$, $m_{3,3} = b_F(1/4, 1/2] = 1/3$ and $m_{3,5} = m_{3,7} = 2/3$; $m_{4,1} = b_F(0, 1/6] = 1/6 = m_{4,3} = b_F(1/6, 1/4] = m_{3,1}$, $m_{4,5} = b_F(1/4, 1/3] = m_{4,7} = 1/3 = m_{3,3}$ and $\{m_{4,i}\}_{i=9}^{15} = 2/3$. By applying Lemma 2.2 (ii) and (iii) repeatedly, it is straightforward to see that all the elements of the SBA of X after the first three rows are equal to 1/6, 1/3 or 2/3. Moreover, for all $n \geq 3$, the nth level SBA approximation of X has distribution function

$$F^{(n)}(x) = \begin{cases} 0 & x < 1/6 \\ 1/8 & 1/6 \leq x < 1/3 \\ 1/4 & 1/3 \leq x < 2/3 \\ 1 & x \geq 2/3, \end{cases}$$

which is equal to F.

This example illustrates the general case which will be shown in Theorem 2.20: If X has support on a set of k points, then $X^{(n)} = X$, for all $n \geq k$. Note that due to the binary nature of the SBA approximation, it might seem that $X^{(n)}$ should equal X for any n that $2^n \geq k$. However, Example 2.14 shows that it may be necessary to have $n \geq k$ before $X^{(n)} \stackrel{\mathcal{L}}{=} X$.

A few preliminary definitions are needed before proving Theorem 2.20. Denote by $\mathcal{B}(\mathbb{R})$ the Borel subsets of $\mathbb{R}$. Suppose $B \in \mathcal{B}(\mathbb{R})$ and $P(B) \neq 0$, then the **conditional random variable** X_B of X is the random variable such that

$$P[X_B \in A] = \frac{P[X \in A \cap B]}{P[X \in B]},$$

for $A \in \mathcal{B}(\mathbb{R})$. Denote the cdf of X_B by F_B.

The notion of a *nth level SBA decomposition* of the measure of a random variable is given next.

Definition 2.15 *Let X be a random variable with cdf F, distribution μ, and SBA $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1}$. For $n \geq 1$ and $1 \leq i \leq 2^{n-1}$ let*

$$X_i^{[n]} = X_{(m_{n,2i-2}, m_{n,2i}]}, \tag{2.6}$$

and denote the distribution of $X_i^{[n]}$ by $\mu_i^{[n]}$. The nth-level SBA decomposition of μ is given by

$$\mu = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \mu_i^{[n]}. \tag{2.7}$$

Note that, for $A \in \mathcal{B}(\mathbb{R})$,

$$\mu(A) = \left(\sum_{i=1}^{2^{n-1}} p_i^{[n]} \mu_i^{[n]} \right)(A) = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \mu_i^{[n]}(A). \tag{2.8}$$

Using the inversion formula given in Theorem 2.8, the values $\{p_i^{[n]}\}$ can be calculated from the SBA of X without explicitly knowing μ. Also, note that $\mu_i^{[n]}$ has density

$\frac{1}{P(X \in (m_{n,2i-2}, m_{2i}])} I_{(m_{n,2i-2}, m_{n,2i}]}$ with respect to μ. (For $A \subseteq \Omega$, the indicator function, I_A, is the function on Ω that assumes the value 1 on A and 0 on the complement of A.)

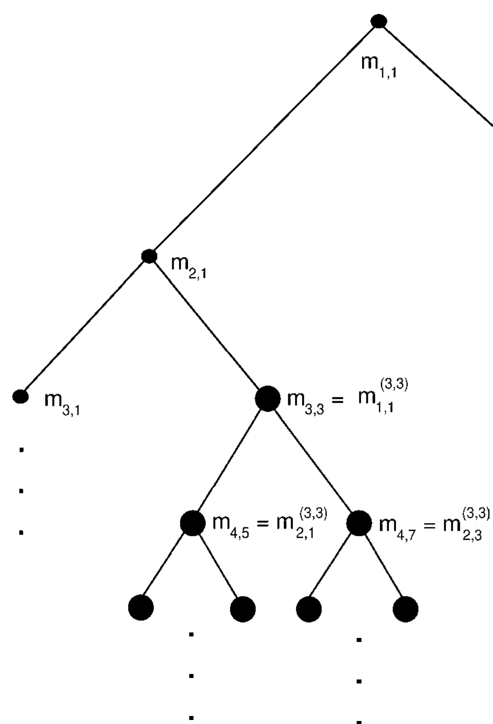
Definition 2.16 *Let M be the SBA for the random variable X. For each* **odd** *integer i between 1 and* $2^n - 1$*, the* **(n, i) conditional SBA** $M_i^{[n]} = \{m_{k,j}^{(n,i)}\}_{k \geq 1}^{j \leq 2^k - 1}$ *of X is defined by*

$$m_{k,j}^{(n,i)} = m_{n+k-1,(i-1)2^{k-1}+j}, \tag{2.9}$$

for $k \geq 2$ *and* $j = 1, 2, \ldots, 2^k - 1$.

For example, the (3, 3) conditional SBA, $M_3^{[3]}$, is the triangular subtree of M with apex $m_{3,3}$, as shown in the following figure.

Figure 2.17 Subtree comprising $M_3^{[3]}$



Proposition 2.18 *For a random variable X with SBA* $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1}$, *the* $(n, 2i-1)$ *conditional SBA,* $M_{2i-1}^{[n]}$, *is the SBA for the random variable* $X_i^{[n]}$ *with distribution* $\mu_i^{[n]}$.

Proof: First, by Definition 2.15,

$$\begin{aligned} E(X_i^{[n]}) &= \int x\, d\mu_i^{[n]}(x) \\ &= \int x I_{(m_{n,2i-2}, m_{n,2i}]} \frac{1}{F(m_{n,2i}) - F(m_{n,2i-2})}\, d\mu(x) && (2.10) \\ &= \frac{\int_{(m_{n,2i-2}, m_{n,2i}]} x\, d\mu(x)}{F(m_{n,2i}) - F(m_{n,2i-2})} && (2.11) \\ &= m_{n,2i-1} && (2.12) \\ &= m_{1,1}^{(n,i)}. && (2.13) \end{aligned}$$

Equality (2.10) follows from Billingsley [4] (Theorem 16.10) and the fact that $I_{(m_{n,2i-2}, m_{n,2i}]} \frac{1}{F(m_{n,2i}) - F(m_{n,2i-2})}$ is the density of $\mu_i^{(n)}$ with respect to μ. Equality (2.12) follows from Definition 2.1 and Equality (2.13) from Definition 2.16.

Let $k \geq 1$, $1 \leq j \leq 2^{k-1} - 1$. By Definition 2.16, and Definition 2.3 (ii),

$$\begin{aligned} m_{k,2j}^{(n,2i-1)} &= m_{n+k-1,((2i-1)-1)2^{k-1}+2j} \\ &= m_{(n+k-1)-1,(2i-2)2^{k-2}+j} \\ &= m_{k-1,j}^{(n,2i-1)}. \end{aligned}$$

Thus, $M_{2i-1}^{[n]}$ satisfies the second condition of Definition 2.3 for the SBA of $X_i^{[n]}$. Lastly,

$$\begin{aligned} m_{k,2j-1}^{(n,2i-1)} &= m_{n+k-1,(2i-2)2^{k-1}+2j-1} && (2.14) \\ &= m_{n+k-1,2[(2i-2)2^{k-2}+j]-1} && (2.15) \\ &= b_F(m_{n+k-2,(2i-2)2^{k-2}+j-1}, m_{n+k-2,(2i-2)2^{k-2}+j}] && (2.16) \\ &= \frac{\int_{(m_{n+k-2,(2i-2)2^{k-2}+j-1}, m_{n+k-2,(2i-2)2^{k-2}+j}]} x\, d\mu(x)}{F(m_{n+k-2,(2i-2)2^{k-2}+j}) - F(m_{n+k-2,(2i-2)2^{k-2}+j-1})} && (2.17) \end{aligned}$$

Equality (2.14) follows from Definition 2.16. Equality (2.15) follows from Definition 2.3 and (2.16) from Definition 2.1. Let $a = m_{n+k-2,(2i-2)2^{k-2}+j-1}$ and $b = m_{n+k-2,(2i-2)2^{k-2}+j}$. Since $\mu_i^{[n]}$ has density $I_{(m_{n,2i-2},m_{n,2i}]}\frac{1}{F(m_{n,2i})-F(m_{n,2i-2})}$ with respect to μ and by Definition 2.16, (2.17) equals

$$\frac{F(m_{n,2i})-F(m_{n,2i-2})}{F(b)-F(a)}\int_{(a,b]} x\, I_{(m_{n-2i-2},m_{n,2i}]}\frac{1}{F(m_{n,2i})-F(m_{n,2i-2})}\,d\mu(x) \tag{2.18}$$

$$= \frac{1}{F_i^{[n]}(b)-F_i^{[n]}(a)}\int_{(a,b]} x\, d\mu_i^{[n]}(x) \tag{2.19}$$

$$= \frac{1}{F_i^{[n]}(m_{k-1,j}^{(n,2i-1)})-F_i^{[n]}(m_{k-1,j-1}^{(n,2i-1)})}\int_{(m_{k-1,j-1}^{(n,2i-1)},m_{k-1,j}^{(n,2i-1)}]} x\, d\mu_i^{[n]}(x) \tag{2.20}$$

$$= b_{F_i^{[n]}}(m_{k-1,j-1}^{(n,2i-1)}, m_{k-1,j}^{(n,2i-1)}]. \tag{2.21}$$

Hence $M_{2i-1}^{[n]}$ satisfies the three conditions of Definition 2.3 for the SBA of $X_i^{[n]}$. □

Proposition 2.19 *Let $n, k \geq 1$ and let X be a random variable with distribution μ. Let $\sum_{i=1}^{2^{n-1}} p_i^{[n]}\mu_i^{[n]}$ be the nth level SBA decomposition of μ. For each i, let $(\mu_i^{[n]})^{(k)} = \sum_{j=1}^{2^{k-1}} q_{i,j}^{[k]}\delta_{m_{k,2j-1}^{(n,2i-1)}}$ be the kth level SBA approximation of $\mu_i^{[n]}$. Then*

$$\mu^{(n+k-1)} = \sum_{l=1}^{2^{(n+k-1)-1}} p_l^{[n+k-1]}\delta_{m_{n+k-1,2l-1}} \tag{2.22}$$

$$= \sum_{i=1}^{2^{n-1}} p_i^{[n]}(\mu_i^{[n]})^{(k)}. \tag{2.23}$$

Proof: Note that (2.22) can be written as

$$\mu^{(n+k-1)} = \sum_{i=1}^{2^{n-1}}\left(\sum_{l=(i-1)2^{k-1}+1}^{i2^{k-1}} p_l^{[n+k-1]}\delta_{m_{n+k-1,2l-1}}\right). \tag{2.24}$$

To prove the proposition it will suffice to show that for each $1 \leq i \leq 2^{n-1}$,

$$p_i^{[n]}(\mu_i^{[n]})^k = \sum_{l=(i-1)2^{k-1}+1}^{i2^{k-1}} p_l^{[n+k-1]} \delta_{m_{n+k-1,2l-1}}. \tag{2.25}$$

Re-indexing the sum on the right hand side of (2.25), this is equivalent to showing that for each $1 \leq j \leq 2^{k-1}$,

$$p_i^{[n]} q_{i,j}^{[k]} \, \delta_{m_{k,2j-1}}^{(n,2i-1)} = p_{(i-1)2^{k-1}+j}^{[n+k-1]} \, \delta_{m_{n+k-1,\, 2((i-1)2^{k-1}+j)-1}}. \tag{2.26}$$

But, by Definition 2.16,

$$\begin{aligned} m_{k,\,2j-1}^{(n,\,2i-1)} &= m_{n+k-1,\,(2i-2)2^{k-1}+2j-1} \\ &= m_{n+k-1,\,2((i-1)2^{k-1}+j)-1} \end{aligned} \tag{2.27}$$

By Definition 2.15 and Proposition 2.18,

$$\begin{aligned} p_i^{[n]} q_{i,j}^{[k]} &= (F(m_{n,2i}) - F(m_{n,2i-2})) \left(F_i^{[n]}(m_{k,2j}^{(n,\,2i-1)}) - F_i^{[n]}(m_{k,2j-2}^{(n,\,2i-1)}) \right) && (2.28) \\ &= (F(m_{n,2i}) - F(m_{n,2i-2})) \frac{F(m_{k,2j}^{(n,\,2i-1)}) - F(m_{k,2j-2}^{(n,\,2i-1)})}{F(m_{n,2i}) - F(m_{n,2i-2})}. && (2.29) \\ &= F(m_{k,2j}^{(n,\,2i-1)}) - F(m_{k,2j-2}^{(n,\,2i-1)}) && (2.30) \\ &= F(m_{n+k-1,2((i-1)2^{k-1}+j)}) - F(m_{n+k-1,2((i-1)2^{k-1}+j)-2}) && (2.31) \\ &= p_{(i-1)2^{k-1}+j}^{[n+k-1]}. && (2.32) \end{aligned}$$

Equality (2.28) follows from Definition 2.15, and equality (2.29) follows from the definition of a conditional random variable. Equality (2.31) follows from (2.27), and equality (2.32) from Definition 2.15. □

The next theorem shows that a random variable with support on a finite set of k elements will be equal to its nth level SBA approximation, for all $n \geq k$.

Theorem 2.20 *Let X be a random variable with measure μ having support on the finite set $\{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{R}$. Let F be the cdf of X. Then $X^{(n)} \stackrel{\mathcal{L}}{=} X$, for all $n \geq k$.*

Proof: The proof proceeds by induction on the cardinality of the support of X, $|supp(X)|$. First, suppose X gives unit mass to $\{a\}$, i.e., $\mu = \delta_a$. By Definition (2.1),

$$\begin{aligned} m_{1,1} &= b_F(-\infty, \infty) \\ &= \int_{\mathbb{R}} x dF(x) \\ &= a. \end{aligned} \tag{2.33}$$

Thus,

$$m_{2,1} = b_F(-\infty, m_{1,1}] \tag{2.34}$$

$$= \frac{\int_{(-\infty, m_{1,1}]} x\, dF(x)}{F(m_{1,1}) - F(-\infty)} \tag{2.35}$$

$$= \frac{\int_{(-\infty, a]} x\, dF(x)}{F(a) - F(-\infty)} \tag{2.36}$$

$$= \frac{a}{1-0}. \tag{2.37}$$

By Lemma 2.2 (iii), it follows that $m_{2,3} = b_F(m_{1,1}, \infty) = a$. Moreover, by repeated application of Definition 2.1, Lemma 2.2 (iii) and Definition 2.3, it follows that $m_{n,2i-1} = a$, for all $n \geq 1$ and $1 \leq i \leq 2^{n-1}$. Thus,

$$\mu^{(n)} = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \delta_{m,\, 2i-1} = \delta_a = \mu, \tag{2.38}$$

for all $n \geq 1$. Assume that if μ has support on a set of $k-1$ elements, then $X^{(n)} \stackrel{\mathcal{L}}{=} X$ for all $n \geq k-1$.

Suppose X has support on a set of k distinct elements, $\{a_1, a_2, \ldots, a_k\}$. Let μ be the distribution of X. Write μ as $\mu = p_1^{[2]}\mu_1^{[2]} + p_2^{[2]}\mu_2^{[2]}$. Recall $\mu_1^{[2]}$ is the conditional measure of μ over $(-\infty, m_{1,1}]$; $\mu_2^{[2]}$ is the conditional measure over $(m_{1,1}, \infty)$; $p_1^{[2]} = P(X \leq m_{1,1})$, and $p_2^{[2]} = 1 - p_1^{[2]}$. There can be at most $k-1$ a_i's less than or equal to $m_{1,1} = E(X)$. Similarly, there can be at most $k-1$ a_i's greater than or equal to $m_{1,1}$. Thus $\mu_1^{[2]}$ and $\mu_2^{[2]}$ have support on at most $k-1$ elements. By the induction assumption, $(\mu_1^{[2]})^{(n)} = \mu_1^{[2]}$ and $(\mu_2^{[2]})^{(n)} = \mu_2^{[2]}$, for all $n \geq k-1$. Thus, by Proposition 2.19, for all $n \geq k$,

$$\begin{aligned} \mu &= p_1^{[2]}\mu_1^{[2]} + p_2^{[2]}\mu_2^{[2]} && (2.39) \\ &= p_1^{[2]}(\mu_1^{[2]})^{(n-1)} + p_2^{[2]}(\mu_2^{[2]})^{(n-1)} && (2.40) \\ &= \mu^{(n)}. && (2.41) \end{aligned}$$

□

For a random variable with support on an infinite set, some more background is needed in order to continue the discussion of how well a random variable is approximated by its SBA approximation. The following discussion can be found in Billingsley [3] for a general metric space S. Here S is often assumed to be [0,1] with the *Euclidean* metric β. Let P be a probability measure on the class of Borel subsets, $\mathcal{B}(S)$ of S. Let $\mathcal{P}(S)$ denote the space of probability measures on $(S, \mathcal{B}(S))$. Make $\mathcal{P}(S)$ into a Hausdorff space by taking basic neighborhoods $B = \left\{Q : \left|\int f_i\, dQ - \int f_i\, dP\right| < \epsilon,\ i = 1, \ldots, k\right\}$, where ϵ is positive and $f_1, f_2, \ldots, f_k$ are in the class of bounded, continuous real functions on S. The *topology of weak convergence* is the resulting topology, and it is metrizable by the following metric.

Definition 2.21 *The Prohorov distance, ρ, between two measures P and Q on $(S, \mathcal{B}(S))$, denoted by $\rho(P, Q)$, is the infimum of all $\epsilon > 0$ so that*

$$P(A) \leq Q(A_\epsilon) + \epsilon \tag{2.42}$$

and

$$Q(A) \leq P(A_\epsilon) + \epsilon \tag{2.43}$$

for all sets $A \in \mathcal{B}(S)$, where $A_\epsilon = \{x : \beta(x, A) < \epsilon\}$.

Note that if $\rho(P, Q) = 0$, then P and Q agree on closed sets and therefore are identical, since every probability measure on $(S, \mathcal{B}(S))$ is regular. Compact subsets of a metric space are closed, hence one can characterize ρ on compact sets only.

The notation $\rho(X, Y)$, for random variables X and Y, denotes the Prohorov distance $\rho(\mathcal{L}(X), \mathcal{L}(Y))$ between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, the laws of X and Y, respectively.

Lemma 2.22 places a bound on the Prohorov distance between a probability measure μ, with $m = \int x \, d\mu(x)$, and the Dirac measure δ_m.

Lemma 2.22 (Bloomer, [5]) *Suppose X is a random variable with $m = E(X)$ and $V = Var(X)$. Then,*

$$\rho(X, \delta_m) \leq \sqrt[3]{V}. \tag{2.44}$$

Using Lemma 2.22 and the following notion of the balayage random variable, a bound on the Prohorov distance between a random variable and its nth level SBA approximation is obtained in Theorem 2.28.

Definition 2.23 (Hill and Kertz, [18]) *Let Y be an integrable random variable and let a,b be constants such that $-\infty < a < b < \infty$. The (balayage) random variable, Y_a^b, is equal to Y if $Y \notin [a, b]$, is equal to a with probability $(b-a)^{-1} \int_{Y \in [a,b]} (b - Y) \, dP$, and is equal to b with probability $(b-a)^{-1} \int_{Y \in [a,b]} (Y - a) \, dP$.*

Note that $(b-a)^{-1} \int_{Y \in [a,b]} (b - Y) \, dP + (b-a)^{-1} \int_{Y \in [a,b]} (Y - a) \, dP = P(Y \in [a, b])$.

Hill and Kertz [18] note that Y_a^b is the random variable with maximum variance which coincides with Y off $[a, b]$, and which has expectation $E(Y)$. This statement can be extended to the following: If Y takes only positive values, Y_a^b is the random variable with maximum kth moment, $k \geq 1$, which coincides with Y off $[a, b]$, and which has expectation $E(Y)$. Denote the kth moment of Y_a^b by $E({Y_a^b}^k)$.

Lemma 2.24 *Let Y be a nonnegative integrable random variable and $0 < a < b < \infty$. Then $E(Y^k) \leq E({Y_a^b}^k)$, for $k \geq 1$, with equality when $k = 1$.*

The proof follows directly from the proof given by Hill and Kertz [18] (Lemma 2.2).

Proof:

$$
\begin{aligned}
E(Y_a^b) &= \int Y_a^b \, dP \\
&= \int_{Y \notin [a,b]} Y_a^b \, dP + \int_{Y \in [a,b]} Y_a^b \, dP \\
&= \int_{Y \notin [a,b]} Y_a^b \, dP + \left[\left(\frac{1}{b-a} \int_{Y \in [a,b]} (b-Y) \, dP \right) a + \left(\frac{1}{b-a} \int_{Y \in [a,b]} (Y-a) \, dP \right) b \right] \\
&= \int_{Y \notin [a,b]} Y \, dP + \frac{1}{b-a} \int_{Y \in [a,b]} (b-a) Y \, dP \\
&= \int_{Y \notin [a,b]} Y \, dP + \int_{Y \in [a,b]} Y \, dP \\
&= \int Y \, dP \\
&= E(Y).
\end{aligned}
$$

Let $\psi_k(y) = y^k$. Then,

$$
\begin{aligned}
\int_{Y \in [a,b]} \psi_k(Y) \, dP &= \int_{Y \in [a,b]} \psi_k \left(\frac{b-Y}{b-a} a + \frac{Y-a}{b-a} b \right) \, dP \\
&\leq \int_{Y \in [a,b]} \frac{b-Y}{b-a} \psi_k(a) + \frac{Y-a}{b-a} \psi_k(b) \, dP \qquad (2.45) \\
&= \psi_k(a) \frac{1}{b-a} \int_{Y \in [a,b]} (b-Y) \, dP + \quad \psi_k(b) \frac{1}{b-a} \int_{Y \in [a,b]} (Y-a) \, dP \\
&= \int_{Y \in [a,b]} \psi_k(Y_a^b) \, dP.
\end{aligned}
$$

Equality (2.45) follows from the convexity of $\psi_k(y)$ over $[0,\infty)$. Thus,

$$
\begin{aligned}
E(\psi_k(Y)) &= \int_{Y \notin [a,b]} \psi_k(Y) \, dP + \int_{Y \in [a,b]} \psi_k(Y) \, dP \\
&\leq \int_{Y \notin [a,b]} \psi_k(Y_a^b) \, dP + \int_{Y \in [a,b]} \psi_k(Y_a^b) \, dP \\
&= E\left(\psi_k(Y_a^b) \right). \qquad \square
\end{aligned}
$$

Note that the condition that Y take nonnegative values in Lemma 2.24 is not necessary for k even.

Corollary 2.25 *Suppose* $0 < a < b < \infty$*, Y has support on [a,b], and let* $m = E(Y)$*. Then* $E(Y) = E(Y_a^b)$ *and, for all* $k \geq 1$*,* $E(Y^k) \leq E({Y_a^b}^k) = a^k\left(\frac{b-m}{b-a}\right) + b^k\left(\frac{m-a}{b-a}\right)$*.*

Corollary 2.26 gives a bound for the variance of a random variable with support on $[a, b]$ in terms of $E(X) = m$, a and b.

Corollary 2.26 *Suppose X is a random variable with support on [a,b] and let* $m = E(X)$*. Then* $Var(X) \leq m(b+a) - m^2 - ab \leq \dfrac{(b-a)^2}{4}$*.*

Proof: By Corollary 2.25,

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &\leq E((X_a^b)^2) - E(X_a^b)^2 \\ &= a^2\left(\frac{b-m}{b-a}\right) + b^2\left(\frac{m-a}{b-a}\right) - m^2 \\ &= m(b+a) - m^2 - ab. \end{aligned} \tag{2.46}$$

Equality (2.46), as a function of m, is maximized over $[a, b]$ at $m = \frac{b+a}{2}$ with maximum value of $\frac{(b-a)^2}{4}$. Thus

$$Var(X) \leq \frac{(b-a)^2}{4}. \tag{2.47}$$

□

The following result is given in Bloomer [5].

Lemma 2.27 (Bloomer [5]) *If* $P = pQ_1 + (1-p)Q_2$ *and* $P' = pQ_1' + (1-p)Q_2'$ *are probability measures, with* $0 \leq p \leq 1$*, and* $\rho(Q_1, Q_1') = \epsilon_1$ *and* $\rho(Q_2, Q_2') = \epsilon_2$*, then* $\rho(P, P') \leq \max\{\epsilon_1, \epsilon_2\}$*.*

A bound for the Prohorov distance between a random variable X and its nth level SBA approximation $X^{(n)}$ can now be given.

Theorem 2.28 *Suppose X is a random variable with support on the interval [0,1]. Let $\{m_{n,k}\}_{n=1}^{\infty}{}_{k=1}^{2^n-1}$ be the SBA of X and let $X^{(n)}$ be the nth level SBA approximation of X. Then*

$$\rho(X^{(n)}, X) \leq \max_{1\leq i\leq 2^{n-1}} \sqrt[3]{\frac{(m_{n,2i}-m_{n,2i-2})^2}{4}} \tag{2.48}$$

for each $n \geq 1$.

Proof: Fix $n \geq 1$. Let μ be the distribution of X and let $\mu^{(n)}$ be the distribution of $X^{(n)}$. Then, by Definitions 2.13 and 2.15,

$$\mu = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \mu_i^{[n]} \tag{2.49}$$

and

$$\mu^{(n)} = \sum_{i=1}^{2^{n-1}} p_i^{[n]} \delta_{m_{n,2i-1}}, \tag{2.50}$$

where $p_i^{[n]} = F(m_{n,2i}) - F(m_{n,2i-2})$, $\mu_i^{[n]}$ is the distribution of $X_i^{[n]}$ with support on $[m_{n,2i-2}, m_{n,2i}]$, and $m_{n,2i-1} = E(X_i^{[n]})$. By Lemmas 2.22, 2.27 and Corollary 2.26,

$$\begin{aligned}
\rho(\mu, \mu^{(n)}) &\leq \max_{1\leq i\leq 2^{n-1}} \rho(\mu_i^{[n]}, \delta_{m_{n,2i-1}}) \\
&\leq \max_{1\leq i\leq 2^{n-1}} \sqrt[3]{\frac{(m_{n,2i}-m_{n,2i-2})^2}{4}}.
\end{aligned}$$

□

Example 2.29 Let X be U[0,1]. Then

$$\rho(X, X^{(n)}) \leq \sqrt[3]{\frac{\left(\frac{1}{2^{n-1}}\right)^2}{4}} = \sqrt[3]{\frac{1}{2^{2n}}}, \text{ for each } n \geq 1. \tag{2.51}$$

Example 2.30 Suppose $X \sim Beta(1,2)$. Then

$$m_{1,1} = b_F(0,1] = \int_0^1 2x(1-x)\,dx = \frac{1}{3}$$

$$
\begin{aligned}
m_{2,1} &= b_F(0, \frac{1}{3}] \\
&= \frac{\int_0^{1/3} 2x(1-x)\,dx}{\int_0^{1/3} 2(1-x)\,dx - 0} = \frac{7}{45} \\
m_{2,3} &= \frac{\int_{1/3}^{1} 2x(1-x)\,dx}{1 - \int_0^{1/3} 2(1-x)\,dx} = \frac{5}{9} \\
m_{3,1} &= \frac{\int_0^{7/45} 2x(1-x)\,dx}{\int_0^{7/45} 2(1-x)\,dx} = \frac{847}{11205} \\
m_{3,3} &= \frac{\int_{7/45}^{1/3} 2x(1-x)\,dx}{\int_0^{1/3} 2(1-x)\,dx - \int_0^{7/45} 2(1-x)\,dx} = \frac{553}{2295} \\
m_{3,5} &= \frac{\int_{1/3}^{5/9} 2x(1-x)\,dx}{\int_0^{5/9} 2(1-x)\,dx - \int_0^{1/3} 2(1-x)\,dx} = \frac{59}{135} \\
m_{3,7} &= \frac{\int_{5/9}^{1} 2x(1-x)\,dx}{1 - \int_0^{5/9} 2(1-x)\,dx} = \frac{19}{27}.
\end{aligned}
$$

Using Theorem 2.28, $\rho(X^{(3)}, X) \leq$

$$
\max\left\{ \sqrt[3]{\left(\frac{847}{11205}\right)^2}, \sqrt[3]{\left(\frac{7}{45} - \frac{847}{11205}\right)^2}, \sqrt[3]{\left(\frac{553}{2295} - \frac{7}{45}\right)^2}, \sqrt[3]{\left(\frac{1}{3} - \frac{553}{2295}\right)^2}, \right.
$$

$$
\left. \sqrt[3]{\left(\frac{59}{135} - \frac{1}{3}\right)^2}, \sqrt[3]{\left(\frac{5}{9} - \frac{59}{135}\right)^2}, \sqrt[3]{\left(\frac{19}{27} - \frac{5}{9}\right)^2}, \sqrt[3]{\left(1 - \frac{19}{27}\right)^2} \right\} / \sqrt[3]{4}
$$

$$
= \frac{\max\{0.17878, 0.43082, 0.19393, 0.20435, 0.22073, 0.24128, 0.27998, 0.44444\}}{\sqrt[3]{4}} \approx 0.28.
$$

2.3 The Distribution Of Moments In The Support Of $B_{\mu_0,\mu}$

The goal of this section is to specify two methods of assessing the accuracy of using an nth level simulation of the SBA construction to estimate the distribution of the moments of measures in the support of $B_{\mu_0,\mu}$. The first method gives upper and lower bounds on $G_k^{\mu_0,\mu}(y) = B_{\mu_0,\mu}\{X : E(X^k) \leq y\}$, the distribution function of $E(X^k)$, with respect to $B_{\mu_0,\mu}$ in terms of the distribution of the Prohorov distance between X and $X^{(n)}$. The second method is less analytically tractable but allows for a sharper bound for simulations.

For the SBA rpm $B_{\mu_0,\mu}$, let

$$G_k^{(n)^{\mu_0,\mu}}(y) = B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y\} \tag{2.52}$$

denote the distrubution function of the kth moment of the nth level SBA approximation, $X^{(n)}$. Let $F_n^{\mu_0,\mu}$ be the distribution function of the length of the largest (barycenter) subinterval induced at the nth level of the $B_{\mu_0,\mu}$ construction. That is,

$$F_n^{\mu_0,\mu}(y) = B_{\mu_0,\mu}\left\{X : \max_{1\leq i\leq 2^n-1}\{m_{n,i}(X) - m_{n,i-1}(X)\} \leq y\right\}. \tag{2.53}$$

Proposition 2.31 *For the SBA rpm $B_{\mu_0,\mu}$,*

$$\max_{\delta>0}\left\{G_k^{(n)^{\mu_0,\mu}}(y-\delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)\right]\right\} \;\leq\; G_k^{\mu_0,\mu}(y) \leq$$

$$\min_{\delta>0}\left\{G_k^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)\right\}. \tag{2.54}$$

For the proof of Proposition 2.31, two lemmas will be needed. Lemma 2.32 gives a lower bound for $B_{\mu_0,\mu}\left\{X : \rho(X, X^{(n)}) \leq \epsilon\right\}$ in terms of $F_{n-1}^{\mu_0,\mu}$ and Lemma 2.33 bounds the difference between moments of two random variables in terms of the Prohorov distance.

Lemma 2.32 *Let $\epsilon > 0$, then*

$$B_{\mu_0,\mu}\left\{X : \rho(X, X^{(n)}) \leq \epsilon\right\} \geq F_{n-1}^{\mu_0,\mu}(2\epsilon^{3/2}). \tag{2.55}$$

Proof: By Theorem 2.28, $\rho(X, X^{(n)}) \leq \max_{1\leq i\leq 2^{n-1}} \sqrt[3]{\frac{(m_{n,2i}-m_{n,2i-2})^2}{4}}$. Hence,

$$\begin{aligned} F_{n-1}^{\mu_0,\mu}(2\epsilon^{3/2}) &= B_{\mu_0,\mu}\left\{X : \max_{1\leq i\leq 2^{n-1}} \{m_{n-1,i}(x) - m_{n-1,i-1}(x)\} \leq 2\epsilon^{3/2}\right\} \\ &= B_{\mu_0,\mu}\left\{X : \max_{1\leq i\leq 2^{n-1}} \sqrt[3]{\frac{(m_{n,2i}(x) - m_{n,2i-2}(x))^2}{4}} \leq \epsilon\right\} \qquad (2.56) \\ &\leq B_{\mu_0,\mu}\left\{X : \rho(X, X^{(n)}) \leq \epsilon\right\}. \end{aligned}$$

Equality 2.56 follows from Definition 2.3. □

For a metric space (S, d), the *Lipschitz* semi-norm for a real-valued function f on S is defined by $||f||_L = \sup_{x\neq y} |f(x) - f(y)|/d(x, y)$. The supremum norm on f is given by $||f||_\infty = \sup_x |f(x)|$. Lemma 2.33 is essentially Corollary 11.6.5 (to Strassen's Theorem) of Dudley [12].

Lemma 2.33 *For any separable metric space (S, d) and random variables X and Y on S, $|E(X^k) - E(Y^k)| \leq \left(||x^k||_L + 2||x^k||_\infty\right) \rho(X, Y)$, for any $k \geq 1$. In particular, for $S = [0, 1]$, $|E(X^k) - E(Y^k)| \leq (k + 2)\rho(X, Y)$.*

Proof: Let $\epsilon > 0$. By Corollary 11.6.4 (Dudley [12]), there exists a probability space (Ω, P) and random variables $\tilde{X}$ and $\tilde{Y}$ such that $\mathcal{L}(\tilde{X}) = \mathcal{L}(X)$, $\mathcal{L}(\tilde{Y}) = \mathcal{L}(Y)$, and $P(d(\tilde{X}, \tilde{Y}) > \rho(\mathcal{L}(X), \mathcal{L}(Y)) + \epsilon) < \rho(\mathcal{L}(X), \mathcal{L}(Y)) + \epsilon$. Let $A = \{(x, y) \in S \times S : d(x, y) \leq \rho(X, Y) + \epsilon\}$, and let R be the measure of $(\tilde{X}, \tilde{Y})$ on $S \times S$. Following the proof of Corollary 11.6.5 (Dudley [12]),

$$\begin{aligned} |E(X^k) - E(Y^k)| &\leq E|X^k - Y^k| \\ &\leq \int_A |x^k - y^k| dR(x, y) + \int_{A^c} |x^k - y^k| dR(x, y) \\ &\leq ||x^k||_L \int_A d(x, y) dR(x, y) + 2||x^k||_\infty(\rho(X, Y) + \epsilon) \\ &\leq ||x^k||_L \int_A (\rho(X, Y) + \epsilon) dR(x, y) + 2||x^k||_\infty(\rho(X, Y) + \epsilon) \\ &\leq ||x^k||_L(\rho(P, Q) + \epsilon) + 2||x^k||_\infty(\rho(X, Y) + \epsilon) \\ &= (||x^k||_L + 2||x^k||_\infty)(\rho(X, Y) + \epsilon). \qquad (2.57) \end{aligned}$$

The first portion of the proof is complete since $\epsilon > 0$ was arbitrary.

For the second part of the lemma, note that for $S = [0,1]$ and $k \geq 1$, $||x^k||_\infty = 1$ and $||x^k||_L = \sup_{x \neq y} \dfrac{|x^k - y^k|}{|x-y|} = k$. $\square$

Proof of Proposition 2.31: Define $D_k^n(X) = |E(X^k) - E(X^{(n)^k})|$. Then, for all $\delta > 0$,

$$\begin{aligned}
G_k^{\mu_0,\mu}(y) &= B_{\mu_0,\mu}\{X : E(X^k) \leq y\} \\
&= B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y + E(X^{(n)^k}) - E(X^k)\} \\
&\leq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y + D_k^n(X)\} \\
&= B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y + D_k^n(X), D_k^n(X) \leq \delta\} + \\
&\quad\; B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y + D_k^n(X), D_k^n(X)) > \delta\} \\
&\leq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y + \delta\} + B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\} \\
&= G_k^{(n)^{\mu_0,\mu}}(y+\delta) + B_{\mu_0,\mu}\left\{X : D_k^n(X) > \delta\right\} \qquad (2.58) \\
&\leq G_k^{(n)^{\mu_0,\mu}}(y+\delta) + B_{\mu_0,\mu}\{X : (k+2)\rho(X, X^{(n)}) > \delta\} \qquad (2.59) \\
&= G_k^{(n)^{\mu_0,\mu}}(y+\delta) + B_{\mu_0,\mu}\left\{X : \rho(X, X^{(n)}) > \frac{\delta}{k+2}\right\} \\
&= G_k^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - B_{\mu_0,\mu}\left\{X : \rho(X, X^{(n)}) \leq \frac{\delta}{k+2}\right\} \\
&\leq G_k^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right). \qquad (2.60)
\end{aligned}$$

Inequality (2.60) follows from Lemma 2.32 and Lemma 2.33. Thus, $G_k^{\mu_0,\mu}(y) \leq \min_{\delta>0}\left\{G_k^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)\right\}$.

Similarly, for all $\delta > 0$,

$$\begin{aligned}
G_k^{\mu_0,\mu}(y) &= B_{\mu_0,\mu}\left\{X : E(X^k) \leq y\right\} \\
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y - D_k^n(X)\} \\
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y - D_k^n(X),\ D_k^n(X) \leq \delta\}
\end{aligned}$$

$$
\begin{aligned}
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y-\delta,\ D_k^n(X) \leq \delta\} \\
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)^k}) \leq y-\delta\} \\
&\quad + B_{\mu_0,\mu}\{X : D_k^n(X) \leq \delta\} - 1 && (2.61) \\
&\geq G_k^{(n)\mu_0,\mu}(y-\delta) + F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right) - 1 && (2.62) \\
&= G_k^{(n)\mu_0,\mu}(y-\delta) - \left(1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)\right) && (2.63)
\end{aligned}
$$

Inequality (2.61) follows from the Bonferroni inequality and (2.62) follows from Lemma 2.33. Thus, $G_k^{\mu_0,\mu}(y) \geq \max_{\delta>0}\left\{G_k^{(n)\mu_0,\mu}(y-\delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)\right]\right\}$. □

Corollary 2.34 places bounds on the distribution function of the standard deviation of the measures in the support of $B_{\mu_0,\mu}$in terms of the distribution function of the second central moment and $F_{n-1}^{\mu_0,\mu}$. Let $m = E(X)$, let

$$C_k^{\mu_0,\mu}(y) = B_{\mu_0,\mu}\{X : E(X - E(X))^k \leq y\}. \qquad (2.64)$$

and let

$$C_k^{(n)\mu_0,\mu}(y) = B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^k \leq y\}. \qquad (2.65)$$

Note that $B_{\mu_0,\mu}\{X : \sqrt{E(X^2) - E(X)^2} \leq y\} = C_2^{\mu_0,\mu}(y^2)$. Also, note that $E(X) = E(X^{(n)})$. The proofs for Corollaries 2.34, 2.35, and 2.36 are similar to the proof of Proposition 2.31.

Corollary 2.34 (Corollary to Proposition 2.31.) *For the SBA rpm $B_{\mu_0,\mu}$,*

$$
\begin{aligned}
&\max_{\delta>0}\left\{C_2^{(n)\mu_0,\mu}(y^2-\delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(\frac{\delta^{3/2}}{4}\right)\right]\right\} \leq \\
&B_{\mu_0,\mu}\{X : \sqrt{E(X^2) - E(X)^2} \leq y\} \leq \\
&\min_{\delta>0}\left\{C_2^{(n)\mu_0,\mu}(y^2+\delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(\frac{\delta^{3/2}}{4}\right)\right\}. \qquad (2.66)
\end{aligned}
$$

Proof: For $\delta > 0$,

$$B_{\mu_0,\mu}\{X : \sqrt{E(X - E(X))^2} \le y\}$$

$$\begin{aligned}
&= B_{\mu_0,\mu}\{X : E(X - E(X))^2 \le y^2\} \\
&\le B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 + |E(X^{(n)^2}) - E(X^2)|\} \qquad (2.67) \\
&= B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 + D_2^n(X), D_2^n(X) \le \delta\} + \\
&\quad B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 + D_2^n(X), D_2^n(X) > \delta\} \\
&\le B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 + \delta\} + B_{\mu_0,\mu}\{X : D_2^n(X) > \delta\} \\
&\le C_2^{(n)\mu_0,\mu}(y^2 + \delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(\frac{\delta^{\frac{3}{2}}}{4}\right). \qquad (2.68)
\end{aligned}$$

Inequality 2.67 follows because $E(X) = E(X^{(n)})$. Inequality 2.68 follows from Lemma 2.32. Also, for $\delta > 0$,

$$B_{\mu_0,\mu}\{X : E(X - E(X))^2 \le y^2\}$$

$$\begin{aligned}
&\ge B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 - D_2^n(X)\} \\
&\ge B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 - D_2^n(X), D_2^n(X) \le \delta\} \\
&\ge B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^2 \le y^2 - \delta\} + \\
&\quad B_{\mu_0,\mu}\{X : D_2^n(X) \le \delta\} - 1 \qquad (2.69) \\
&\ge C_2^{(n)\mu_0,\mu}(y^2 - \delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(\frac{\delta^{\frac{3}{2}}}{4}\right)\right].
\end{aligned}$$

Inequality (2.69) follows from the Bonferroni inequality. Since (2.68) and (2.69) hold for all $\delta > 0$, the result follows. □

Corollary 2.35 places similar bounds on the third central moment. The proof is similar to the proof for Corollary 2.34.

Corollary 2.35 (Corollary to Proposition 2.31.) *For the SBA rpm $B_{\mu_0,\mu}$,*

$$\max_{\delta>0}\left\{C_3^{(n)\mu_0,\mu}(y - \delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(\left(\frac{\delta}{17}\right)^{\frac{3}{2}}\right)\right]\right\} \le C_3^{\mu_0,\mu}(y) \quad \le$$

$$\min_{\delta>0}\left\{C_3^{(n)\mu_0,\mu}(y+\delta)+1-F_{n-1}^{\mu_0,\mu}\left(\left(\frac{\delta}{17}\right)^{\frac{3}{2}}\right)\right\}. \tag{2.70}$$

Proof: Recall that $E(X) = E(X^{(n)})$ and that $E(X) \leq 1$. Let $SD_3^n(X) = |E(X^{(n)^3}) - E(X^3)| + 3|E(X^{(n)^2}) - E(X^2)|$. Then for $\delta > 0$,

$$\begin{aligned}
& B_{\mu_0,\mu}\{X : E(X-E(X))^3 \leq y\} \\
\leq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y + |E(X^{(n)}-E(X^{(n)}))^3 - E(X-E(X))^3|\} \\
\leq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y + SD_3^n(X)\} \\
\leq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y + \delta\} + B_{\mu_0,\mu}\{X : SD_3^n(X) > \delta\} \\
\leq \quad & C_3^{(n)\mu_0,\mu}(y+\delta) + B_{\mu_0,\mu}\{X : 17\rho(X,X^{(n)}) > \delta\} && (2.71) \\
\leq \quad & C_3^{(n)\mu_0,\mu}(y+\delta) + 1 - B_{\mu_0,\mu}\{X : 17\rho(X,X^{(n)}) \leq \delta\} \\
\leq \quad & C_3^{(n)\mu_0,\mu}(y+\delta) + 1 - F_{n-1}^{\mu_0,\mu}\left(\left(\frac{\delta}{17}\right)^{\frac{3}{2}}\right). && (2.72)
\end{aligned}$$

Inequality (2.71) follows from Lemma 2.33. Inequality (2.72) follows from Lemma 2.32.

Also, for $\delta > 0$,

$$\begin{aligned}
& B_{\mu_0,\mu}\{X : E(X-E(X))^3 \leq y\} \\
\geq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y - SD_3^n(X)\} \\
\geq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y - SD_3^n(X), SD_3^n(X) \leq \delta\} \\
\geq \quad & B_{\mu_0,\mu}\{X : E(X^{(n)}-E(X^{(n)}))^3 \leq y - \delta\} + \\
& \quad B_{\mu_0,\mu}\{X : SD_3^n(X) \leq \delta\} - 1 && (2.73) \\
\leq \quad & C_3^{\mu_0,\mu}(y-\delta) - [1 - B_{\mu_0,\mu}\{X : 17\rho(X,X^{(n)}) \leq \delta\}] \\
= \quad & C_3^{\mu_0,\mu}(y-\delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(\left(\frac{\delta}{17}\right)^{\frac{3}{2}}\right)\right]. && (2.74)
\end{aligned}$$

Inequality (2.73) follows from the Bonferroni inequality. Since (2.72) and (2.74) hold true for all $\delta > 0$, the result is true. □

Corollary 2.36 places bounds on the fourth central moment. The proof is similar to Corollaries 2.34 and 2.35.

Corollary 2.36 (Corollary to Proposition 2.31.) *For the SBA rpm $B_{\mu_0,\mu}$,*

$$\max_{\delta>0}\left\{C_4^{(n)\mu_0,\mu}(y-\delta)-\left[1-F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{50}\right)^{\frac{3}{2}}\right)\right]\right\}\leq C_4^{\mu_0,\mu}(y)\leq$$

$$\min_{\delta>0}\left\{C_4^{(n)\mu_0,\mu}(y+\delta)+1-F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{50}\right)^{\frac{3}{2}}\right)\right\}. \tag{2.75}$$

Proof: Let $SD_4^n(X)=|E(X^{(n)^4})-E(X^4)|+4|E(X^{(n)^3})-E(X^3)|+6|E(X^{(n)^2})-E(X^2)|$. Recall that $E(X)=E(X^{(n)})$ and that $E(X)\leq 1$. Then for all $\delta>0$,

$$\begin{aligned}
&B_{\mu_0,\mu}\{X:E(X-E(X))^4\leq y\}\\
&\leq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y+|E(X^{(n)}-E(X^{(n)}))^4-E(X-E(X))^4|\}\\
&=B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y+SD_4^n(X),SD_4^n(X)\leq\delta\}+\\
&\quad B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y+SD_4^n(X),SD_4^n(X)>\delta\}\\
&\leq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y+\delta\}+B_{\mu_0,\mu}\{X:SD_4^n(X)>\delta\}\\
&\leq C_4^{(n)\mu_0,\mu}(y+\delta)+1-B_{\mu_0,\mu}\left\{X:50\rho(X,X^{(n)})\leq\delta\right\}. \qquad (2.76)\\
&\leq C_4^{(n)\mu_0,\mu}(y+\delta)+1-F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{50}\right)^{\frac{3}{2}}\right).
\end{aligned}$$

Inequality (2.76) follows from Lemma 2.32. Also,

$$\begin{aligned}
&B_{\mu_0,\mu}\{X:E(X-E(X))^4\leq y\}\\
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y-SD_4^n(X)\}\\
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y-SD_4^n(X),SD_4^n(X)\leq\delta\}\\
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^4\leq y-\delta\}+\\
&\qquad B_{\mu_0,\mu}\{X:SD_4^n(X)\leq\delta\}-1 \qquad (2.77)\\
&\geq C_4^{\mu_0,\mu}(y-\delta)-[1-B_{\mu_0,\mu}\{X:50\rho(X,X^{(n)})\leq\delta\}].
\end{aligned}$$

$$\geq C_4^{\mu_0,\mu}(y-\delta) - \left[1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{50}\right)^{\frac{3}{2}}\right)\right].$$

Inequality (2.77) follows from the Bonferroni inequality. Since each inequality in (2.75) holds for all $\delta > 0$, the result is true. □

Note that in the proof of Proposition 2.31, $B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\}$ is bounded by $1 - F_{n-1}^{\mu_0,\mu}\left(2\left(\frac{\delta}{k+2}\right)^{3/2}\right)$ which is likely not to be a sharp bound (Refer to Chapter 4). In numerical simulation, a better approximation for $B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\}$ may be obtained from Proposition 2.38.

Proposition 2.37 *For a random variable X, and its nth level SBA approximation $X^{(n)}$,*

$$|E(X^k) - E((X^{(n)})^k)| \leq \sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| m_{n,2i-2}^k \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}}\right) + m_{n,2i}^k \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}}\right) - m_{n,2i-1}^k \right|. \tag{2.78}$$

Proof:

$$\begin{aligned}
|E(X^k) - E((X^{(n)})^k)| &= \left| \int_{(-\infty,\infty)} x^k d\mu(x) - \int_{(-\infty,\infty)} x^k d\mu^{(n)}(x) \right| \\
&= \left| \sum_{i=1}^{2^{n-1}} \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k d\mu(x) - \sum_{i=1}^{2^{n-1}} \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k d\mu^{(n)}(x) \right| \\
&\leq \sum_{i=1}^{2^{n-1}} \left| \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k d\mu(x) - \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k d\mu^{(n)}(x) \right| \\
&= \sum_{i=1}^{2^{n-1}} \left| \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k p_i^{[n]} d\mu_i^{[n]}(x) - \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k p_i^{[n]} d\delta_{m_{n,2i-1}} \right| \\
&\leq \sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| \int_{(m_{n,2i-2}\,-\,m_{n,2i}]} x^k d\mu_i^{[n]}(x) - (m_{n,2i-1})^k \right| \\
&\leq
\end{aligned}$$

$$\sum_{i=1}^{2^{n-1}} p_i^{[n]} | m_{n,2i-2}^k \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}} \right) + m_{n,2i}^k \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}} \right) - m_{n,2i-1}^k |. \quad (2.79)$$

Inequality (2.79) follows from Corollary 2.25. □

The distribution $B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\}$ on the right hand side of equality 2.58 (Proposition 2.31) is bounded by the distribution function of the longest (barycenter) subinterval (2.72). This bound can be replaced by the distribution of the bound given in Proposition 2.37. This leads to Proposition 2.38.

Proposition 2.38 *For the SBA rpm $B_{\mu_0,\mu}$, let*

$$\begin{aligned} R_{n,k}(\delta) \quad = \quad B_{\mu_0,\mu} \Bigg\{ X : \sum_{i=1}^{2^{n-1}} p_i^{[n]} \Bigg| m_{n,2i-2}^k \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}} \right) \\ + m_{n,2i}^k \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}} \right) - m_{n,2i-1}^k \Bigg| \leq \delta \Bigg\}. \quad (2.80) \end{aligned}$$

Then

$$\max_{\delta>0} \left\{ G_k^{(n)\mu_0,\mu}(y-\delta) - \left[1 - R_{n,k}^{\mu_0,\mu}(\delta)\right] \right\} \leq G_k^{\mu_0,\mu}(y) \leq \min_{\delta>0} \left\{ G_k^{(n)\mu_0,\mu}(y+\delta) + 1 - R_{n,k}^{\mu_0,\mu}(\delta) \right\}.$$

Proof: By (2.58) and (2.61) in the proof of Proposition 2.31,

$$\begin{aligned} \max_{\delta>0} \left\{ G_k^{(n)\mu_0,\mu}(y-\delta) - B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\} \right\} \leq G_k^{\mu_0,\mu}(y) \leq \\ \min_{\delta>0} \left\{ G_k^{(n)\mu_0,\mu}(y+\delta) + B_{\mu_0,\mu}\{X : D_k^n(X) > \delta\} \right\}. \end{aligned}$$

The result now follows from Proposition 2.37. □

Corollary 2.39 places bounds in terms of $R_{n,k}^{\mu_0,\mu}$ on the distribution function of the standard deviation.

Corollary 2.39 (Corollary to Proposition 2.38.) *For the SBA rpm $B_{\mu_0,\mu}$,*

$$\max_{\delta>0} \left\{ C_2^{(n)\mu_0,\mu}(y^2-\delta) - \left[1 - R_{n,2}^{\mu_0,\mu}(\delta)\right] \right\} \quad \leq \quad B_{\mu_0,\mu}\{X : \sqrt{E(X^2) - E(X)^2} \leq y\}$$

$$\leq \min_{\delta>0}\left\{C_2^{(n)\mu_0,\mu}(y^2+\delta)+1-R_{n,2}^{\mu_0,\mu}(\delta)\right\}. \tag{2.81}$$

Proof: Let $\delta>0$. Then $B_{\mu_0,\mu}\{X:\sqrt{E(X-E(X))^2}\leq y\}$

$$\begin{aligned}
&= B_{\mu_0,\mu}\{X:E(X-E(X))^2\leq y^2\}\\
&\leq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2+|E(X^{(n)^2})-E(X^2)|\}\\
&= B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2+D_2^n(X),D_2^n(X)\leq\delta\}+\\
&\quad B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2+D_2^n(X),D_2^n(X)>\delta\}\\
&\leq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2+\delta\}+B_{\mu_0,\mu}\{X:D_2^n(X)>\delta\}\\
&\leq C_2^{(n)\mu_0,\mu}(y^2+\delta)+1-R_{n,2}^{\mu_0,\mu}(\delta). \qquad (2.82)
\end{aligned}$$

Inequality (2.82) follows from Lemma 2.37. Also, for $\delta>0$,

$B_{\mu_0,\mu}\{X:E(X-E(X))^2\leq y^2\}$

$$\begin{aligned}
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2-D_2^n(X)\}\\
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2-D_2^n(X),D_2^n(X)\leq\delta\}\\
&\geq B_{\mu_0,\mu}\{X:E(X^{(n)}-E(X^{(n)}))^2\leq y^2-\delta\}+\\
&\quad B_{\mu_0,\mu}\{X:D_2^n(X)\leq\delta\}-1 \qquad (2.83)\\
&\geq C_2^{\mu_0,\mu}(y^2-\delta)-[1-R_{n,2}^{\mu_0,\mu}].
\end{aligned}$$

Inequality (2.83) follows from the Bonferroni inequality. Since inequalities (2.82) and (2.83) hold for all $\delta>0$, then (2.81) follows. □

Corollary 2.40 places similar bounds on the third central moment.

Corollary 2.40 (Corollary to Proposition 2.38.) *For the SBA rpm $B_{\mu_0,\mu}$, let*

$RS_{n,3}^{\mu_0,\mu}(\delta)=$

$$B_{\mu_0,\mu}\left\{X:\sum_{i=1}^{2^{n-1}}p_i^{[n]}\left|m_{n,2i-2}^3\left(\frac{m_{n,2i}-m_{n,2i-1}}{m_{n,2i}-m_{n,2i-2}}\right)+m_{n,2i}^3\left(\frac{m_{n,2i-1}-m_{n,2i-2}}{m_{n,2i}-m_{n,2i-2}}\right)-m_{n,2i-1}^3\right|\right.$$

$$+3\sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| m_{n,2i-2}^2 \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}} \right) + m_{n,2i}^2 \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}} \right) - m_{n,2i-1}^2 \right| \leq \delta \Bigg\}. \quad (2.84)$$

Then

$$\max_{\delta>0} \left\{ C_3^{(n)^{\mu_0,\mu}}(y-\delta) - \left[1 - RS_{n,3}^{\mu_0,\mu}(\delta) \right] \right\} \leq C_3^{\mu_0,\mu}(y) \leq \max_{\delta>0} \left\{ C_3^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - RS_{n,3}^{\mu_0,\mu}(\delta) \right\}. \quad (2.85)$$

Proof: Let $\delta > 0$ and let $SD_3^n = |E(X^{(n)^3}) - E(X^3)| + 3|E(X^{(n)^2}) - E(X^2)|$. Recall that $E(X) = E(X^{(n)})$ and $E(X) \leq 1$. Then

$$\begin{aligned}
& B_{\mu_0,\mu}\{X : E(X - E(X))^3 \leq y\} \\
& \leq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + |E(X^{(n)} - E(X^{(n)}))^3 - E(X - E(X))^3|\} \\
& \leq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + SD_3^n(X), SD_3^n(X) \leq \delta\} + \\
& \quad B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + SD_3^n(X), SD_3^n(X) > \delta\} \\
& \leq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + \delta\} + B_{\mu_0,\mu}\{X : SD_3^n(X) > \delta\} \\
& \leq C_3^{\mu_0,\mu}(y+\delta) + 1 - RS_{n,3}^{\mu_0,\mu}(\delta). \qquad (2.86)
\end{aligned}$$

Inequality (2.86) follows from Lemma 2.37. Also, for $\delta > 0$,

$$\begin{aligned}
& B_{\mu_0,\mu}\{X : E(X - E(X))^3 \leq y\} \\
& \geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y - SD_3^n(X)\} \\
& \geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y - SD_3^n(X), SD_3^n(X) \leq \delta\} \\
& \geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y - \delta\} + \\
& \quad B_{\mu_0,\mu}\{X : SD_3^n(X) \leq \delta\} - 1 \qquad (2.87) \\
& \geq C_3^{\mu_0,\mu}(y-\delta) - [1 - RS_{n,3}^{\mu_0,\mu}(\delta)].
\end{aligned}$$

Inequality (2.87) follows from the Bonferroni inequality. Since inequalities (2.86) and (2.87) each holds for all $\delta > 0$, then (2.85) holds true. □

Corollary 2.41 places bounds on the fourth central moment.

Corollary 2.41 (Corollary to Proposition 2.38.) *For the SBA rpm $B_{\mu_0,\mu}$, let* $RS_{n,4}^{\mu_0,\mu}(\delta) =$

$$B_{\mu_0,\mu}\left\{X : \sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| m_{n,2i-2}^4 \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}}\right) + m_{n,2i}^4 \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}}\right) - m_{n,2i-1}^4 \right| \right.$$
$$+4\sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| m_{n,2i-2}^3 \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}}\right) + m_{n,2i}^3 \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}}\right) - m_{n,2i-1}^3 \right|$$
$$\left. +6\sum_{i=1}^{2^{n-1}} p_i^{[n]} \left| m_{n,2i-2}^2 \left(\frac{m_{n,2i} - m_{n,2i-1}}{m_{n,2i} - m_{n,2i-2}}\right) + m_{n,2i}^2 \left(\frac{m_{n,2i-1} - m_{n,2i-2}}{m_{n,2i} - m_{n,2i-2}}\right) - m_{n,2i-1}^2 \right| \leq \delta \right\}.$$

Then,

$$\max_{\delta>0}\left\{C_4^{(n)^{\mu_0,\mu}}(y-\delta) - \left[1 - RS_{n,4}^{\mu_0,\mu}(\delta)\right]\right\} \leq C_4^{\mu_0,\mu}(y) \leq \min_{\delta>0}\left\{C_4^{(n)^{\mu_0,\mu}}(y+\delta) + 1 - RS_{n,4}^{\mu_0,\mu}(\delta)\right\}. \tag{2.88}$$

Proof: Let $\delta > 0$ and let $SD_4^n = |E(X^{(n)^4}) - E(X^4)| + 4|E(X^{(n)^3}) - E(X^3)| + 6|E(X^{(n)^2} - E(X^2)|$.

$$\begin{aligned}
&B_{\mu_0,\mu}\{X : E(X - E(X))^4 \leq y\} \\
&\leq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^4 \leq y + |E(X^{(n)} - m)^4 - E(X - E(X))^4|\} \\
&= B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^4 \leq y + SD_4^n(X), SD_4^n(X) \leq \delta\} + \\
&\quad B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + SD_4^n(X), SD_4^n(X) > \delta\} \\
&\leq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^3 \leq y + \delta\} + B_{\mu_0,\mu}\{X : SD_4^n(X) > \delta\} \\
&\leq C_4^{\mu_0,\mu}(y+\delta) + 1 - RS_{n,4}^{\mu_0,\mu}(\delta).
\end{aligned} \tag{2.89}$$

Inequality (2.89) follows from Lemma 2.37. Also, for $\delta > 0$,
$B_{\mu_0,\mu}\{X : E(X - E(X))^3 \leq y\}$

$$\begin{aligned}
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^4 \leq y - SD_4^n(X)\} \\
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^4 \leq y - SD_4^n(X), SD_4^n(X) \leq \delta\} \\
&\geq B_{\mu_0,\mu}\{X : E(X^{(n)} - E(X^{(n)}))^4 \leq y - \delta\} +
\end{aligned}$$

$$
\begin{aligned}
& B_{\mu_0,\mu}\{X : SD_4^n(X) \leq \delta\} - 1 \\
\geq \;\; & C_4^{\mu_0,\mu}(y-\delta) - [1 - RS_{n,4}^{\mu_0,\mu}(\delta)].
\end{aligned} \tag{2.90}
$$

Inequality (2.90) follows from the Bonferroni inequality. Since each of (2.89) and (2.90) hold true for all $\delta > 0$, the result stated in inequality (2.88) follows. □

In the next chapter, some partial results on the closed form of $F_n^{\mu_0,\mu}$ are given.

Chapter 3

Random Splittings Of An Interval

This chapter develops a model and numerical procedure for approximating the distribution function of the longest (barycenter) subinterval at level n of the $B_{\mu_0,\mu}$ construction. Denote this distribution function by $F_n^{\mu_0,\mu}$. Although $F_n^{\mu_0,\mu}$ can be defined recursively (in terms of $F_{n-1}^{\mu_0,\mu}$), the form of $F_n^{\mu_0,\mu}$ becomes analytically unmanageable as n gets large, except for trivial distributions of μ_0 and μ. After introducing a "random splitting" model for $F_n^{\mu_0,\mu}$, examples for special cases of $\mu_0 = \mu$ are given along with some partial results on the general form of $F_n^{\mu_0,\mu}$. Then the numerical procedure for approximating $F_n^{\mu_0,\mu}$ is specified.

Note that many random splitting models appear in the literature (see, for instance Feller [14], Lloyd and Williams [21], Lloyd [22] and Holst [20]). As Feller [14] indicates, these models are important in physics, chemistry and statistics. However, it does not appear that any of the papers in the literature address the problem of calculating $F_n^{\mu_0,\mu}$.

3.1 A Splitting Model

Let $\mu = \mu_0$ be a distribution on [0,1] with density $f(x)$. (It is straightforward to generalize the results if $\mu \neq \mu_0$.) The first step of the process is to split the interval [0,1] into two pieces, where the splitting point is chosen according to $\mu(= \mu_0)$. Next, split each of the two subintervals into two pieces. The splitting points are chosen independently according to μ scaled to the length of each subinterval respectively. Each of the four resulting subintervals is split in two pieces, and so on. For $n \geq 1$, let $L_n^{\mu_0,\mu}$ denote the length of the longest subinterval after n steps of the splitting process. It is straightforward to see that

$$P(L_n^{\mu_0,\mu} \leq y) = F_n^{\mu_0,\mu}(y) = B_{\mu_0,\mu}\{X : \max_{1\leq i\leq 2^n} |m_{n,i}(X) - m_{n,i-1}(X)| \leq y\}. \qquad (3.1)$$

Note,

$$P(L_1^{\mu_0,\mu} \leq y) = \mu\{x : \max\{x, 1-x\} \leq y\} = \begin{cases} 0 & y \leq \frac{1}{2} \\ \int_{1-y}^{y} f(x)\,dx & \frac{1}{2} < y \leq 1 \\ 1 & y > 1, \end{cases} \qquad (3.2)$$

and, for $n \geq 1$, the following recursive relation holds.

$$F_n^{\mu_0,\mu}(y) = P(L_n^{\mu_0,\mu} \leq y) = \int_0^1 F_{n-1}(\frac{y}{x})F_{n-1}(\frac{y}{1-x})f(x)\,dx. \qquad (3.3)$$

Thus, in theory, $F_n^{\mu_0,\mu}$ can be determined recursively for any $n \geq 1$ and density $f(x)$.

Example 3.1 Suppose $\mu_0 = \mu$ is uniformly distributed over the interval [0,1]. (This is often the assumption made for splitting processes analyzed in the literature. See Feller [14], as well as Lloyd and Williams [21], Holst [20] and Lloyd [20]). Then, by (3.2),

$$F_1^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{2} \\ 2y-1 & \text{if } \frac{1}{2} < y \leq 1 \\ 1 & \text{if } y > 1. \end{cases} \qquad (3.4)$$

By (3.3), $F_2^{\mu_0,\mu}$ is given by

$$\begin{aligned} F_2^{\mu_0,\mu}(y) &= \int_0^1 F_1^{\mu_0,\mu}(\frac{y}{x})F_1^{\mu_0,\mu}(\frac{y}{1-x})f(x)\,dx \\ &= \int_0^1 F_1^{\mu_0,\mu}(\frac{y}{x})F_1^{\mu_0,\mu}(\frac{y}{1-x})\,dx \end{aligned}$$

$$= \quad 2\int_0^{1/2} F_1^{\mu_0,\mu}(\frac{y}{x}) \times F_1^{\mu_0,\mu}(\frac{y}{1-x})\,dx, \tag{3.5}$$

where

$$F_1^{\mu_0,\mu}(y/x) = \begin{cases} 1 & \text{if } x \le y \\ 2\dfrac{y}{x} - 1 & \text{if } y < x \le 2y \\ 0 & \text{if } x > 2y, \end{cases} \tag{3.6}$$

and

$$F_1^{\mu_0,\mu}(\frac{y}{1-x}) = \begin{cases} 0 & \text{if } x \le 1-2y \\ 2\dfrac{y}{1-x} - 1 & \text{if } 1-2y < x \le 1-y \\ 1 & \text{if } x > 1-y. \end{cases} \tag{3.7}$$

Thus, through straightforward, but tedious, calculus,

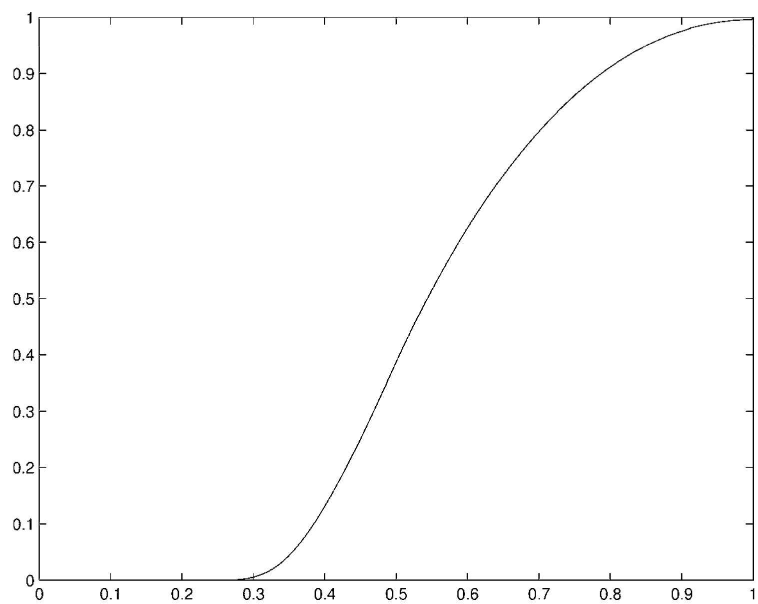
$$F_2^{\mu_0,\mu}(y) \quad = \quad \begin{cases} 2\left[\int_0^y F_1(y/x)F_1(\frac{y}{1-x})\,dx + \int_y^{2y} F_1(y/x)F_1(\frac{y}{1-x})\,dx \right. \\ \quad \left. + \int_{2y}^{1/2} F_1(y/x)F_1(\frac{y}{1-x})\,dx\right] & \text{if } 0 \le y \le \frac{1}{4}, \\ 2\left[\int_0^{1-3y} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx + \int_{1-3y}^{y} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx \right. \\ \quad \left. + \int_y^{1-2y} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx + \int_{1-2y}^{1/2} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx\right] & \text{if } \frac{1}{4} \le y \le \frac{1}{3}, \\ 2\left[\int_0^{1-2y} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx + \int_{1-2y}^{y} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx \right. \\ \quad \left. + \int_y^{1/2} F_1(\frac{y}{x})F_1(\frac{y}{1-x})\,dx\right] & \text{if } \frac{1}{3} \le y \le \frac{1}{2}, \\ 2\left[\int_0^{1-y} F_1(\frac{y}{1-x})\,dx + \int_{1-y}^{1/2}\,dx\right] & \text{if } \frac{1}{2} \le y \le 1 \end{cases}$$

$$= \quad \begin{cases} 0 & \text{if } 0 \le y \le \frac{1}{4}, \\ 2\int_{1-2y}^{1/2} \left(\frac{2y}{1-x} - 1\right)\left(2\frac{y}{x} - 1\right)\,dx & \text{if } \frac{1}{4} \le y \le \frac{1}{3}, \\ 2\left[\int_{1-2y}^{y} \left(\frac{2y}{1-x} - 1\right)\,dx + \int_y^{1/2} \left(\frac{2y}{x} - 1\right)\left(\frac{2y}{1-x} - 1\right)\right] & \text{if } \frac{1}{3} \le y \le \frac{1}{2}, \\ 2\left[\int_0^{1-y} \left(\frac{2y}{1-x} - 1\right)\,dx\right] + 2y - 1 & \text{if } \frac{1}{2} \le y \le 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } y \leq 1/4, \\ 8y^2 \ln\left(\frac{2y}{1-2y}\right) + 4y \ln\left(\frac{1-2y}{2y}\right) + 4y - 1 & \text{if } 1/4 < y \leq 1/3, \\ 8y^2 \ln\left(\frac{1-y}{y}\right) + 8y \ln\left(\frac{y}{1-y}\right) + 4y \ln 2 - 8y + 3 & \text{if } 1/3 < y \leq 1/2, \\ 4y\left(1 - \ln(y)\right) - 3 & \text{if } 1/2 < y \leq 1, \\ 1 & \text{if } y > 1. \end{cases} \tag{3.8}$$

The graph of $F_2^{\mu_0,\mu}$ is given in Figure 3.2.

Figure 3.2 $F_2^{\mu_0,\mu}$ for $\mu_0 = \mu$ uniform over [0,1]



Example 3.3 Suppose $\mu_0 = \mu$ is beta(1,2). Then

$$F_1^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{2} \\ \int_{1-y}^{y} 2(1-x)\,dx = 2y - 1 & \text{if } \frac{1}{2} < y \leq 1 \\ 1 & \text{if } y > 1. \end{cases} \tag{3.9}$$

If μ is beta(2,1), then

$$F_1^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq \frac{1}{2} \\ \int_{1-y}^{y} 2x\,dx = 2y-1 & \text{if } \frac{1}{2} < y \leq 1 \\ 1 & \text{if } y > 1. \end{cases} \tag{3.10}$$

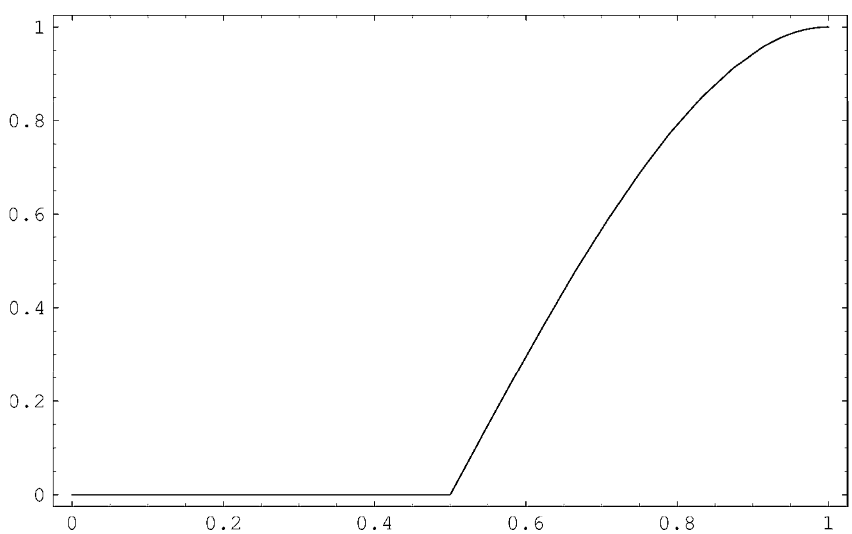
Thus, by (3.3) and (3.4), $F_n^{\mu_0,\mu}$ for $\mu_0 = \mu$ equal to beta(1,2) or beta (2,1) is the same as when $\mu_0 = \mu$ is equal to the uniform distribution over [0,1].

Example 3.4 Suppose $\mu_0 = \mu$ is beta(2,2), then

$$F_1^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } y \leq \frac{1}{2}, \\ -4y^3 + 6y^2 - 1 & \text{if } \frac{1}{2} < y \leq 1, \\ 1 & \text{if } y > 1. \end{cases} \tag{3.11}$$

The graph of $F_1^{\mu_0,\mu}$, in this case, is given in Figure 3.5.

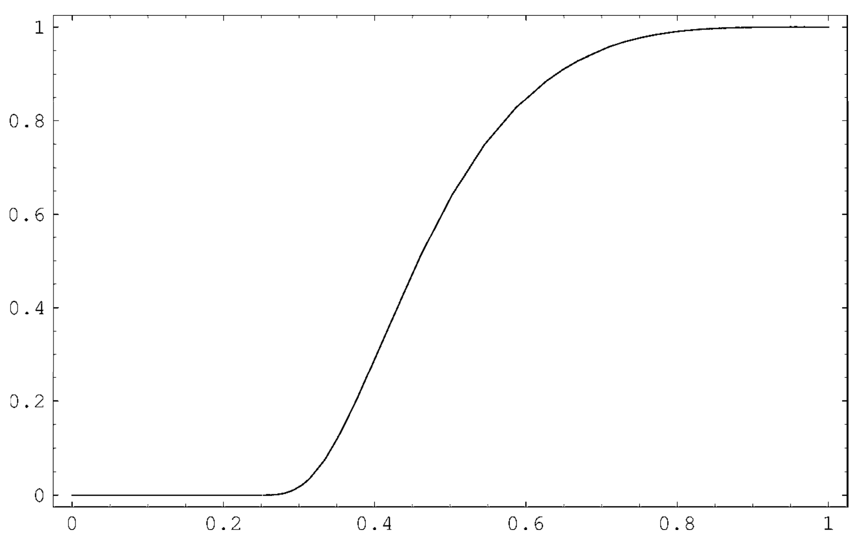
Figure 3.5 $F_1^{\mu_0,\mu}$ for $\mu_0 = \mu =$ beta(2,2).



$F_2^{\mu_0,\mu}$ is given by

$$F_2^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } y \le 1/4, \\ 12y^2(-3-2y+18y^2-24y^3+16y^4)\ln(\frac{2y}{2y-1}) & \\ \quad +(9+6y-54y^2+72y^3-48y^4)\ln(\frac{2y-1}{2y}) & \\ \qquad -192y^5+240y^4+304y^3-72y^2-1 & \text{if } 1/4 < y \le 1/3, \\ \frac{1}{y-1}[-2+2y-66y^2+182y^3-68y^4 & \\ \qquad -24y^2(-3+y+2y^2)\ln(1-y)] & \\ +\frac{1}{y-1}(-1+y-114y^2+302y^3+148y^4-768y^5+384y^6) & \\ \quad +\frac{24y^2}{y-1}\ln(\frac{y-1}{y})(3-y-20y^2+42y^3-40y^4+16y^5) & \\ \qquad -112y^3+(72y^2+48y^3)\ln(2y) & \text{if } 1/3 < y \le 1/2, \\ -3-108y^2+112y^3-24y^2(3+2y)\ln(y) & \text{if } 1/2 < y \le 1, \\ 1 & \text{if } y > 1. \end{cases} \tag{3.12}$$

Figure 3.6 $F_2^{\mu_0,\mu}$, $\mu_0 = \mu =$beta(2,2)



As indicated by Example 3.4, tractable analytical expressions for $F_n^{\mu_0,\mu}$ do not appear possible except in simple cases. Nonetheless, some partial results on the form of $F_n^{\mu_0,\mu}$ are given in the next two propositions.

Proposition 3.7 *For the SBA construction $B_{\mu_0,\mu}$, $F_n^{\mu_0,\mu}(y) = 0$, for all $0 \le y < \frac{1}{2^n}$ and $n \ge 1$.*

Proof: At the nth stage of the splitting process, the interval [0,1] is split into 2^n subintervals. Obviously, at least one subinterval must be of length greater than or equal $\frac{1}{2^n}$. Thus, $B_{\mu_0,\mu}\{L_n^{\mu_0,\mu} \le y\} = F_n^{\mu_0,\mu} = 0$, for $0 \le y < \frac{1}{2^n}$. □

Lemma 3.8 is used in Proposition 3.9 to give a closed form solution to $F_n^{\mu_0,\mu}(y)$, for $y \in [\frac{1}{2}, 1]$ and $\mu_0 = \mu$ uniform over [0,1].

Lemma 3.8 *If $\mu_0 = \mu$, $n \ge 2$ and $\frac{1}{2} < y \le 1$ then*

$$\int_{1-y}^{y} F_{n-1}^{\mu_0,\mu}(\frac{y}{x}) F_{n-1}^{\mu_0,\mu}(\frac{y}{1-x}) \times f(x)\, dx = \int_{1-y}^{y} f(x)\, dx. \tag{3.13}$$

If $n = 1$, then $F_1^{\mu_0,\mu}(y) = \int_{1-y}^{y} f(x)\, dx$ as in (3.2).

Proof: For $\frac{1}{2} < y \le 1$ and $1 - y \le x \le y$, both $\frac{y}{x}$ and $\frac{y}{1-x}$ are greater than or equal to 1. Thus, each of $F_{n-1}^{\mu_0,\mu}(\frac{y}{x})$ and $F_n^{\mu_0,\mu}(\frac{y}{1-x})$ are equal to 1, and so (3.13) holds. □

Proposition 3.9 *For $\mu_0 = \mu$ uniform over $[0,1]$, $n \ge 1$ and $\frac{1}{2} \le y \le 1$,*

$$F_n^{\mu_0,\mu}(y) = 2^n y \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} [\ln(y)]^k - (2^n - 1). \tag{3.14}$$

Proof: By(3.4) of Example 3.1, (3.14) holds for $n = 1$.

Assume (3.14) holds for all $j \le n$. Then, for $y \ge \frac{1}{2}$,

$$\begin{aligned}
F_{n+1}^{\mu_0,\mu}(y) &= \int_0^1 F_n^{\mu_0,\mu}(\frac{y}{x}) F_n^{\mu_0,\mu}(\frac{y}{1-x}) dx \\
&= 2\left[\int_0^{\frac{1}{2}} F_n^{\mu_0,\mu}(\frac{y}{x}) F_n^{\mu_0,\mu}(\frac{y}{1-x}) dx\right] \\
&= 2\left[\int_0^{1-y} F_n^{\mu_0,\mu}(\frac{y}{x}) F_n^{\mu_0,\mu}(\frac{y}{1-x})\, dx + \int_{1-y}^{\frac{1}{2}} dx\right]. \qquad (3.15) \\
&= 2\left[\int_0^{1-y} F_n^{\mu_0,\mu}(\frac{y}{1-x})\, dx + \left(y - \frac{1}{2}\right)\right]. \qquad (3.16)
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{1-y}\left[2^n\frac{y}{1-x}\left(\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}[\ln(\frac{y}{1-x})]^k\right)-(2^n-1)\right]dx \\
&\quad +2y-1 \qquad (3.17)\\
&= 2^{n+1}\int_0^{1-y}\frac{y}{1-x}\left(\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}[\ln(\frac{y}{1-x})]^k\right)dx+\left(-2^{n+1}+2\right)\int_0^{1-y}dx \\
&\quad +2y-1 \\
&= 2^{n+1}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\int_0^{1-y}\frac{y}{1-x}\left([\ln(\frac{y}{1-x})]^k\right)dx+\left(-2^{n+1}+2\right)(1-y) \\
&\quad +2y-1 \\
&= 2^{n+1}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\int_0^{1-y}\left(\frac{y}{1-x}\right)[\ln(\frac{y}{1-x})]^k\,dx+2^{n+1}y-(2^{n+1}-1) \\
&= 2^{n+1}\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\left(\frac{-y}{k+1}\right)[\ln(y)]^{k+1}+2^{n+1}y-(2^{n+1}-1) \qquad (3.18)\\
&= 2^{n+1}y\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(k+1)!}[\ln(y)]^{k+1}+2^{n+1}y-(2^{n+1}-1) \\
&= 2^{n+1}y\sum_{k=1}^{n}\frac{(-1)^k}{k!}[\ln(y)]^k+2^{n+1}y-(2^{n+1}-1) \qquad (3.19)\\
&= 2^{n+1}y\sum_{k=0}^{n}\frac{(-1)^k}{k!}[\ln(y)]^k-(2^{n+1}-1). \qquad (3.20)
\end{aligned}
$$

Equality (3.15) follows from Lemma 3.8. Equality (3.16) holds since $y \geq \frac{1}{2}$ and $0 \leq x \leq 1-y$ implies $\frac{y}{x} \geq 1$. Equality (3.17) follows from the induction assumption. Equality (3.18) follows from the Fundamental Theorem of Calculus and equality (3.19) holds by re-indexing. Thus, by induction, (3.20) implies (3.14) holds for all $n \geq 1$. □

Finding a closed form for $\frac{1}{2} \leq y \leq 1$ when μ is equal to another base measure such as beta(2,2) or beta(1,10) is not as easy. However, $F_n^{\mu_0,\mu}$ over $[\frac{1}{2},1]$ must follow the form given in Proposition 3.10 regardless of the generating distribution.

Proposition 3.10 *Let $\mu_0 = \mu$, $n \geq 1$, $\frac{1}{2} < y \leq 1$, and H_{n-1}, G_{n-1} and f be differentiable functions on [0,1], with $H'_{n-1}(x) = F_{n-1}^{\mu_0,\mu}(\frac{y}{1-x})$ and $G'_{n-1}(x) = F_{n-1}^{\mu_0,\mu}(\frac{y}{x})$.*

Then

$$F_n^{\mu_0,\mu}(y) = H_{n-1}(1-y)f(1-y) - H_{n-1}(0)f(0) - \int_0^{1-y} f'(x)H_{n-1}(x)\,dx$$
$$+G_{n-1}(1)f(1) - G_{n-1}(y)f(y) - \int_y^1 f'(x)G_{n-1}(x)\,dx + F_1^{\mu_0,\mu}(y) \quad (3.21)$$

Proof: By integration by parts, and because $F_1^{\mu_0,\mu}(y) = \int_{1-y}^{y} f(x)\,dx$ over (1/2,1],

$$F_n^{\mu_0,\mu}(y) = \int_0^{1-y} F_{n-1}^{\mu_0,\mu}(\frac{y}{1-x})f(x)\,dx + \int_{1-y}^{y} f(x)\,dx + \int_y^1 F_{n-1}^{\mu_0,\mu}(\frac{y}{x})f(x)\,dx \quad (3.22)$$
$$= H_{n-1}(1-y)f(1-y) - H_{n-1}(0)f(0) - \int_0^{1-y} f'(x)H_{n-1}(x)\,dx$$
$$+G_{n-1}(1)f(1) - G_{n-1}(y)f(y) - \int_y^1 f'(x)G_{n-1}(x)\,dx + F_1^{\mu_0,\mu}(y). \quad (3.23)$$

□

Example 3.11 In the case of a uniform generating measure, as in Example 3.1, $f(x) = 1$ and for $n = 2$, $F_1^{\mu_0,\mu}(y) = 2y - 1$, $H_1(x) = -2y\ln(1-x) - x$ and $G_1(x) = 2y\ln x - x$. Hence, over $[\frac{1}{2}, 1]$

$$F_2^{\mu_0,\mu}(y) = -2y\ln y - 1 + y - 1 - 2y\ln y + y + 2y - 1 = 4y(1-\ln y) - 3 \quad (3.24)$$

which agrees with Example 3.1.

Example 3.12 For a beta(2,2) generating measure, as in Example 3.4,

$$F_1^{\mu_0,\mu}(\frac{y}{x}) = -1 + 6\frac{y^2}{x^2} - 4\frac{y^3}{x^3},\ G_1(x) = -x - 6\frac{y^2}{x} + 2\frac{y^3}{x^2}, \quad (3.25)$$

and,

$$F_1^{\mu_0,\mu}(\frac{y}{1-x}) = -1+6\frac{y^2}{(1-x)^2} - 4\frac{y^3}{(1-x)^3},\ H_1(x) = -x+6\frac{y^2}{1-x} - 2\frac{y^3}{(1-x)^2}. \quad (3.26)$$

Hence, over $[\frac{1}{2}, 1]$,

$$\begin{aligned}
F_2^{\mu_0,\mu}(y) &= 36y^2 - 30y^3 - 6y - (-3(1-y)^2 + 4(1-y)^3 + 12y^2 + 72y^2(1-y) \\
&\quad -12y^3 + 36y^2 \ln y + 24y^3 \ln y) + 30 * y^2(1-y) \\
&\quad - \int_y^1 \left(-6x - 36\frac{y^2}{x} + 12(\frac{y^3}{x^2}) + 12x^2 + 72y^2 - 24(\frac{y^3}{x}) \right) dx \\
&\quad -1 + 6y^2 - 4y^3 \\
&= -3 - 108y^2 + 112y^3 - 24y^2(3+2y)\ln y.
\end{aligned}$$

3.2 Numerical Methods

This section specifies a procedure for numerically approximating $F_n^{\mu_0,\mu}$ and gives an error bound on the approximation. Note that this section is developed for the case $\mu_0 = \mu$.

3.2.1 An Algorithm For Approximating F_n

The approximation proceeds inductively. Let $\tilde{F}_n^{\mu_0,\mu}$ denote the approximation of $F_n^{\mu_0,\mu}$ to be determined. Fix N large. For an integral $\int_0^1 g(x)\,dx$, let $Simp_N(\int_0^1 g(x)\,dx)$ denote the approximation of the integral using Simpson's rule with respect to the partition $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$.

Step 1. Assume $\tilde{F}_{n-1}^{\mu_0,\mu}$ has been calculated. It is straightforward to calculate $F_1^{\mu_0,\mu}(y)$ and often $F_2^{\mu_0,\mu}(y)$, so in general the algorithm can be initialized with $\tilde{F}_1^{\mu_0,\mu}(y) = F_1^{\mu_0,\mu}(y)$ or $\tilde{F}_2^{\mu_0,\mu}(y) = F_2^{\mu_0,\mu}(y)$.

Step 2. For $y_i = \frac{i}{N}$, $1 \le i \le N$, set

$$\hat{F}_n^{\mu_0,\mu}(y_i) = Simp_N\left(\int_0^1 \tilde{F}_{n-1}^{\mu_0,\mu}\left(\frac{y_i}{x}\right) \tilde{F}_{n-1}^{\mu_0,\mu}\left(\frac{y_i}{1-x}\right) f(x)\,dx \right). \tag{3.27}$$

Step 3. Set

$$\tilde{F}_n^{\mu_0,\mu}(y) = \begin{cases} 0 & \text{if } y \leq \frac{1}{2^n}, \\ \tilde{F}_n^{\mu_0,\mu}(y_i)\frac{y-y_{i-1}}{y_i-y_{i-1}} + \tilde{F}_n^{\mu_0,\mu}(y_{i-1})\frac{y_i-y}{y_i-y_{i-1}} & \text{if } \frac{1}{2^n} < y \leq 1 \text{ and } y_{i-1} \leq y \leq y_i, \\ & \text{ for } i = 1, \ldots, N, \\ 1 & \text{if } y > 1. \end{cases} \tag{3.28}$$

Step 4. Iterate with respect to n.

Example 3.13 (Continuation of Example 3.1). Let $\mu_0 = \mu$ be the uniform distribution over [0,1]. Figures 3.14 through 3.18 give the numerical approximation for $F_3^{\mu_0,\mu}$, $F_4^{\mu_0,\mu}$, $F_6^{\mu_0,\mu}$, $F_7^{\mu_0,\mu}$, $F_9^{\mu_0,\mu}$. The value of N in Simpson's rule is equal to 500.

Figure 3.14 $F_3^{\mu_0,\mu}$

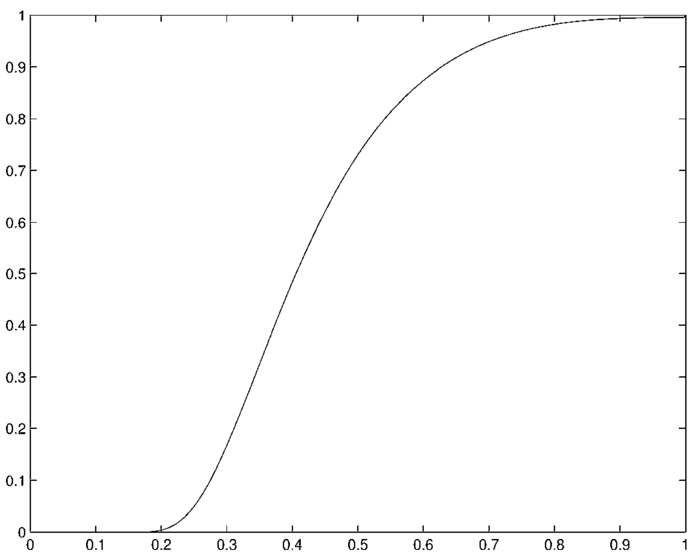


Figure 3.15 $F_4^{\mu_0,\mu}$

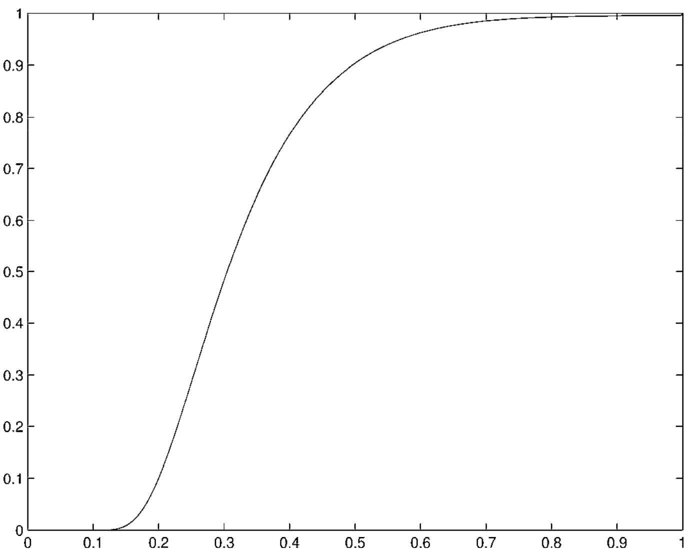


Figure 3.16 $F_6^{\mu_0,\mu}$

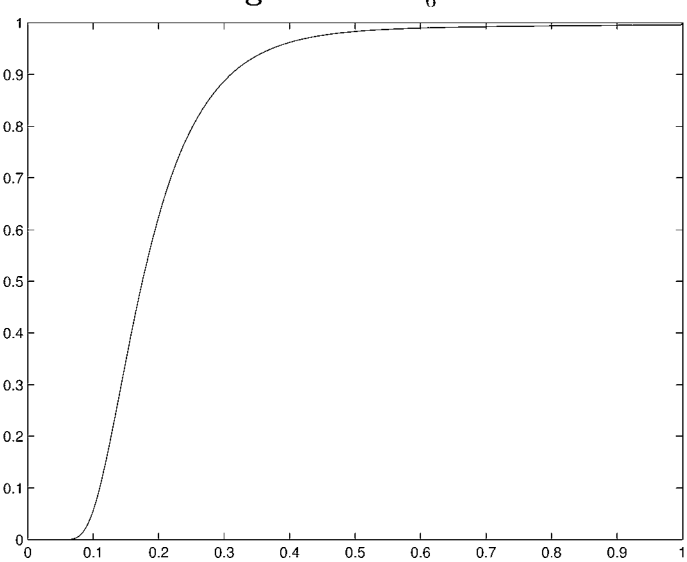


Figure 3.17 $F_7^{\mu_0,\mu}$

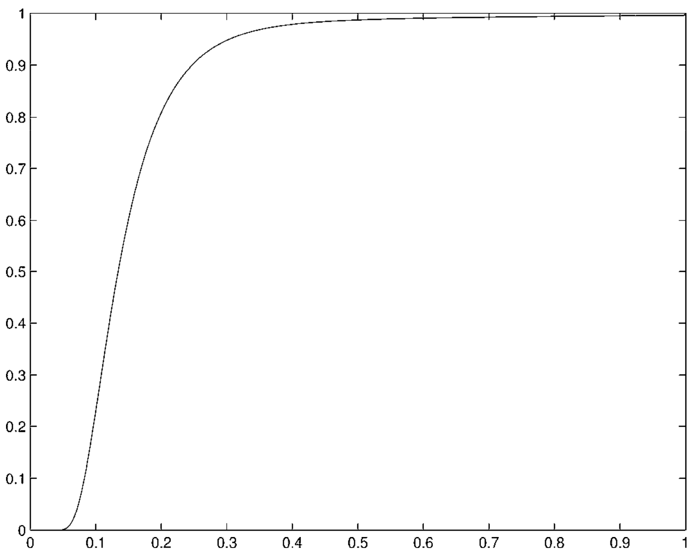
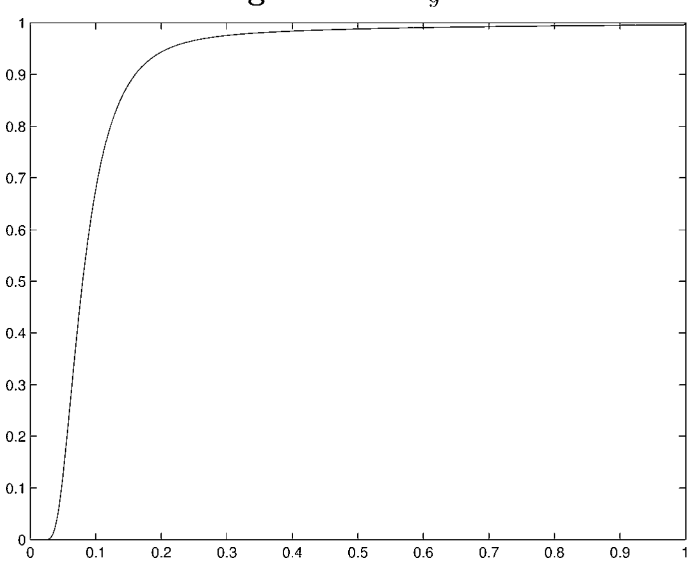
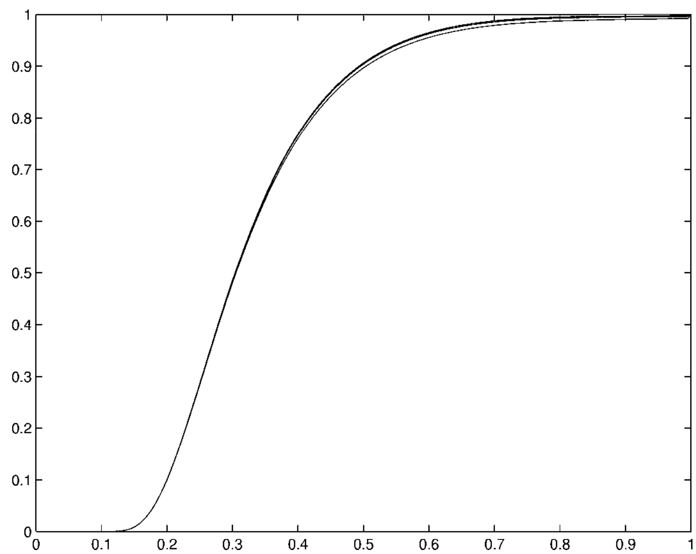


Figure 3.18 $F_9^{\mu_0,\mu}$



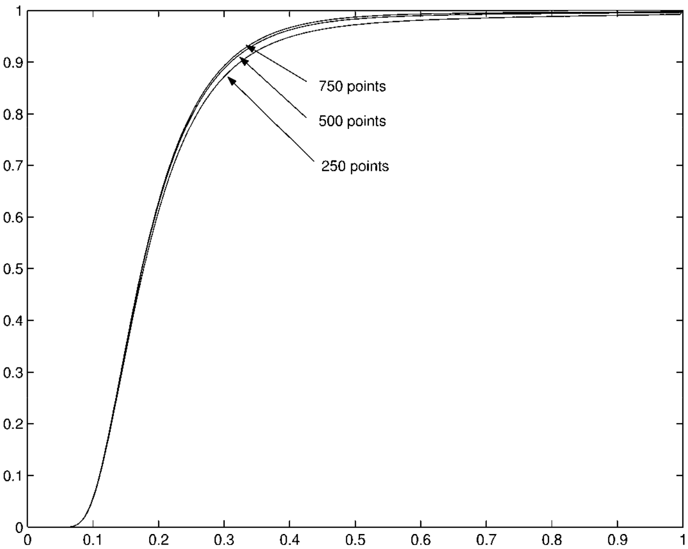
Error Estimates: Figures 3.19 through 3.21 represent the plots of $F_4^{\mu_0,\mu}$, $F_6^{\mu_0,\mu}$ and $F_9^{\mu_0,\mu}$, with $N = 250$, 500 and 750. The figures show that changing the number, N, has a greater effect on the accuracy of the graph of $F_n^{\mu_0,\mu}$ for larger n than it does for smaller n. This is due to the recursive process of determining $F_n^{\mu_0,\mu}$, specifically for $n \geq 3$. Error is introduced in the estimation of $F_3^{\mu_0,\mu}$. This error is added to the error from the estimation of $F_4^{\mu_0,\mu}$, $F_5^{\mu_0,\mu}$ and so on. This is why the plots for $F_4^{\mu_0,\mu}$ in Figure 3.19 are much closer together than the plots for $F_9^{\mu_0,\mu}$ in Figure 3.21. Comparing the empirical distributions for $F_4^{\mu_0,\mu}$, where $N = 250$ and 500 shows that the maximum difference between the two graphs over all y is 0.0073, while the maximum difference between the graphs for $F_4^{\mu_0,\mu}$ over all y, when comparing the graphs for $N = 500$ and 750, is 0.0024.

Figure 3.19 $F_4^{\mu_0,\mu}, N = 250, 500$ and 750



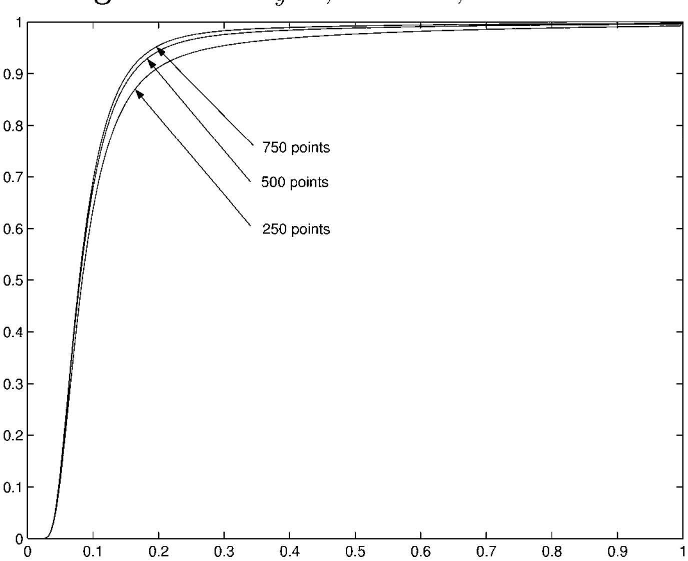
Comparing the empirical distributions for $F_6^{\mu_0,\mu}$, where $N = 250$ and 500 shows that the maximum difference between the two graphs over all y is 0.0158, while the maximum difference between the graphs for $F_6^{\mu_0,\mu}$, where $N = 500$ and 750 is 0.0053. Note that the difference of the graphs of $F_6^{\mu_0,\mu}$ in Figure 3.20 is more apparent than in Figure 3.19.

Figure 3.20 $F_6^{\mu_0,\mu}, N = 250, 500$ and 750



When comparing the empirical distributions for $F_9^{\mu_0,\mu}$, for the same values of N the maximum difference between the two graphs over all y is 0.041, for $N = 250$ and 500, respectively. The maximum difference between the graphs for $F_9^{\mu_0,\mu}$ over all y, where $N = 500$ and 750 is 0.0143.

Figure 3.21 $F_9^{\mu_0,\mu}, N = 250, 500$ and 750



Since there is a closed form for $F_n^{\mu_0,\mu}$ for all $y \geq 1/2$, the graphs of the empirical distribution functions can be compared against the true distribution function given in Proposition 3.9 to see how well the empirical distribution function approximates the true distribution function. The results of the comparison show that as N increases, the difference between the empirical distribution function and the true distribution function decreases substantially. For the case where $n = 4$, the maximum difference over all y between the empirical distribution for $F_4^{\mu_0,\mu}$ and the closed form for $F_4^{\mu_0,\mu}$, for $N = 250$, is 0.0146. For $N = 500$, the maximum difference over all y is 0.0073, and for $N = 750$, the maximum difference over all y is 0.0049. For the case where $n = 6$, the maximum difference over all y between the empirical distribution for $F_6^{\mu_0,\mu}$ and the closed form for $F_6^{\mu_0,\mu}$, for $N = 250$, is 0.0224. For $N = 500$, the maximum difference over all y is 0.0113, and for $N = 750$, the maximum difference over all y is 0.0076. Finally, for the case where $n = 9$, the maximum difference over all y between the empirical distribution for $F_9^{\mu_0,\mu}$ and the closed form for $F_9^{\mu_0,\mu}$, for $N = 250$, is 0.0184. For $N = 500$, the maximum difference over all y is 0.0083, and for $N = 750$, the maximum difference over all y is 0.0052.

Chapter 4

Simulations Of Statistics For SBA Constructions

4.1 Introduction And Simulation Implementation

The goal of this chapter is to determine the distribution of the moments of SBA rpms and how these distributions are affected by the base measure of the construction. Since it does not appear that the distribution on the moments generated with the SBA construction can be obtained analytically, the idea is to approximate the distribution functions $G_k^{\mu_0,\mu}$and $C_k^{\mu_0,\mu}$ by $G_k^{(n)^{\mu_0,\mu}}$ and $C_k^{(n)^{\mu_0,\mu}}$, respectively, which are in turn estimated through simulation. Specifically, empirical bounds for $G_k^{\mu_0,\mu}$ are obtained using Proposition 2.31 and Proposition 2.38 and empirical bounds for $C_k^{\mu_0,\mu}$ are obtained from Corollaries 2.39, 2.40, and 2.41. How the distribution of the standard deviation (SD), skewness, kurtosis, second third and fourth moments, and the third and fourth central moments are affected by the base measure is also investigated.

In this chapter, differences between statistics associated with five base measures, uniform [0,1], beta (2,2), beta (10,1), beta (1,10) and beta (.5,.5) will be discussed. These base measures were used to simulate SBA constructions in order to answer some basic questions about the support of $B_{\mu_0,\mu}$, such as what base measures should be chosen to be more likely to yield a small or large SD, skewness or kurtosis; or which base measures are more likely to yield large or small central moments. For example, from the base measures used in this study, it appears that base measures which concentrate their mass near 1 are more likely to produce measures with a large SD. Base measures which concentrate their mass near zero are more likely to produce measures with a larger skewness and kurtosis, but smaller SD.

Statistics associated with each the five generating measures are compared against one another in the following six sections. Section 4.2 is devoted to the second moment, section 4.3 to the SD, section 4.4 to the third moment, section 4.5 to the third central moment and to skewness, section 4.6 to the fourth moment, and section 4.7 to kurtosis and the fourth central moment.

The rest of this section is devoted to the discussion of implementation issues of the simulation – the number of Monte Carlo points and the number of SBA levels – and confidence bands for the empirical results obtained from the simulations. Then a discussion will follow about the choice of base measures and the determination of the smallest nonrandom bounds on the distribution functions.

Recall that if H_N is an empirical distribution obtained from i.i.d. samples from a distribution, H, then the distribution of the random variable

$$D_N = \sup_y |H_N(y) - H(y)|. \tag{4.1}$$

is independent of the distribution of H (Feller [14]). Asymptotic percentiles are computed using the following limiting distribution

$$\lim_{N\to\infty} P\left(D_N < \frac{z}{N}\right) = 1 - 2\sum_{j=1}^{\infty}(-1)^{j-1}\exp(-2j^2z^2) \tag{4.2}$$

$$\approx 1 - 2\exp\left(-2z^2\right). \tag{4.3}$$

Thus for large N,

$$P\left(D_N < x\right) \approx 1 - 2\exp\left(-2(xN)^2\right). \tag{4.4}$$

For instance, for large N,

$$P\left(|H_N(y) - H(y)| \le \frac{\sqrt{-\frac{1}{2}\ln\left(\frac{1-.995}{2}\right)}}{N}\right) \approx 0.995. \tag{4.5}$$

The value

$$\frac{\sqrt{-\frac{1}{2}\ln\left(\frac{1-.995}{2}\right)}}{N} \tag{4.6}$$

will be called **the 99.5% confidence band width for N.** Table 4.1 gives confidence band widths for different confidence levels with $N = 65,000$ points.

Table 4.1: Confidence Band Widths, N=65,000

Confidence Level	Confidence Band Width
.999	0.00003
.995	0.000028
.99	0.000025
.95	0.00002
.9	0.00001

For each base measure in this section, a quasi-Monte Carlo method [8] was used to generate 65,000 finite SBAs (n levels). Then the distributions of those SBAs were calculated.

Proposition 2.38 and (4.6) were used to determine 99.5% confidence bands on $G_2^{\mu_0,\mu}$, $G_3^{\mu_0,\mu}$, and $G_4^{\mu_0,\mu}$ and Corollaries 2.39 through 2.41 and (4.6) were used to calculate 99.5% confidence bands on the SD, $C_3^{\mu_0,\mu}$and $C_4^{\mu_0,\mu}$. (Appendix B gives details of the simulation procedure.) The rest of this section provides a discussion on the parameters used in the simulations.

The base measures: To investigate the support of $B_{\mu_0,\mu}$, the base measure μ_0 is chosen so that $\mu_0\{\frac{1}{2}\} = 1$. For μ, five base measures were chosen because of the different ways that they distribute their mass: beta (2,2), beta (10,1), beta (1,10), beta (.5,.5) and uniform$[0,1]$. The beta (2,2) distribution has most of its mass near the mean ($m = 0.5$) while beta (.5,.5) has most of its mass concentrated near the endpoints 0 and 1 of its support. The base measures beta (1,10) and beta (10,1) have most of their mass near one of the endpoints – 0 for the former and 1 for the latter. The uniform $[0,1]$ generating measure has its mass evenly distributed over $[0,1]$.

The choice of δ: The value of δ is used to give nonrandom bounds on the distribution function of the kth moment, $G_k^{\mu_0,\mu}$, discussed in Proposition 2.38 and the distribution function of the kth central moment $C_k^{\mu_0,\mu}$, discussed in Corollaries 2.39, 2.40 and 2.41. For each experiment, a δ^* value was determined. The value of δ^* was that δ which minimized the value

$$\max_{0\le y\le 1}\left\{\left(G_k^{(n)\mu_0,\mu}(y+\delta^*)+1-R_{n,k}^{\mu_0,\mu}(\delta^*)\right)-\left(G_k^{(n)\mu_0,\mu}(y-\delta^*)-\left[1-R_{n,k}^{\mu_0,\mu}(\delta^*)\right]\right)\right\}, \tag{4.7}$$

over all $y\in[0,1]$. The bounds on $G_k^{\mu_0,\mu}$ discussed in Proposition 2.31 and the bounds on $C_k^{\mu_0,\mu}$ given in Corollaries 2.34, 2.35, 2.36 were found to be fairly large and not adequate for this study.

Table 4.2 gives the values of δ^* for the second, third and fourth moments with uniform $[0,1]$, beta (2,2), beta (10,1) and beta (1,10) generating measures. Table 4.3 gives the values of δ^* for the SD, $C_3^{\mu_0,\mu}$, $C_4^{\mu_0,\mu}$, for each of the five base measures. The last column gives the range of the δ's tested. Note that the bounds given by the δ^* chosen are conservative in the sense that the bounds given by Proposition 2.38 and Corollaries 2.39, 2.40 and 2.41 are guaranteed to be better than that associated with any particular choice of δ.

The number of levels in the SBA approximation and the number of Monte Carlo points: For each base measure, a collection of 9-level SBA approximations was generated. Sixty-five thousand distributions were generated for each base measure. For each base measure, the second, third and fourth moments, the SD, and the third and fourth central moments of each SBA approximation were calculated; and the distribution of these statistics was determined.

The number of Monte Carlo points, 65 thousand, and the number of levels in the SBA construction were at the upper limits of available computing power. In particular, the total number of values generated for each simulation was $2^9\times 65,000 = 33.28$ million. The confidence bands presented in this chapter show that the number

Table 4.2: This table lists the values of δ^* which give smallest confidence band widths for $G_2^{\mu_0,\mu}$, $G_3^{\mu_0,\mu}$, $G_4^{\mu_0,\mu}$ and the range of δ^*'s tested for each base measure.

Base Measures	Moments	δ^*	Range of δs in experiment
Uniform [0,1]			
	Second moment	0.00006	[0.000001, 0.01]
	Third moment	0.00009	[0.00001, 0.001]
	Fourth moment	0.0001	[0.0001, 0.01]
Beta (2,2)			
	Second moment	0.00006	[0.000001, 0.1]
	Third moment	0.0002	[0.000001, 0.01]
	Fourth moment	0.0002	[0.00001, 0.001]
Beta (10,1)			
	Second moment	0.002	[0.000005, 0.005]
	Third moment	0.003	[0.0005,0.005]
	Fourth moment	0.003	[0.0005, 0.005]
Beta (1,10)			
	Second moment	0.00004	[0.000001, 0.001]
	Third moment	0.0001	[0.00005, 0.005]
	Fourth moment	0.00009	[0.00001, 0.005]
Beta (.5,.5)			
	Second moment	0.00009	[0.00001, 0.0001]
	Third moment	0.0001	[0.0001, 0.001]
	Fourth moment	0.0001	[0.0001, 0.001]

of levels in the SBA construction and the number of Monte Carlo points used provide a good estimate for the distribution functions of the moments, the central moments, and the SD.

Table 4.3: This table lists the values of δ^* which give smallest confidence band widths for $C_2^{\mu_0,\mu}$, $C_3^{\mu_0,\mu}$, $C_4^{\mu_0,\mu}$ and the range of the δ^*'s tested for each base measure.

Base Measures	Moments	δ^*	Range of δs
Uniform [0,1]			
	SD	0.000009	[0.000001, 0.005]
	Third central moment	0.001	[0.00001, 0.005]
	Fourth central moment	0.012	[0.00001, 0.005]
Beta (2,2)			
	SD	0.0001	[0.0009, 0.005]
	Third central moment	0.0001	[0.0009, 0.005]
	Fourth central moment	0.00005	[0.00001, 0.005]
Beta (10,1)			
	SD	0.0006	[0.0003, 0.003]
	Third central moment	0.0001	[0.00005,0.0003]
	Fourth central moment	0.00002	[0.00001, 0.03]
Beta (1,10)			
	SD	0.000009	[0.000001, 0.01]
	Third central moment	0.0001	[0.00001, 0.005]
	Fourth central moment	0.0001	[0.000001, 0.005]
Beta (.5,.5)			
	SD	0.0006	[0.00001, 0.005]
	Third central moment	0.0002	[0.00001, 0.005]
	Fourth central moment	0.0002	[0.00001, 0.005]

4.2 The Second Moment

Figures 4.1 through 4.5 show confidence bands about the distribution function of the second moment, $G_2^{\mu_0,\mu}$, associated with the base measures uniform $[0,1]$, beta (2,2), beta (10,1), beta (1,10) and beta (.5,.5). Recall the upper and lower bounds for $G_k^{\mu_0,\mu}$ are given in Proposition 2.38. Shown in the following graphs are the upper half of the 99.5% confidence band about the upper bound on $G_k^{\mu_0,\mu}$,

$$G_2^{(9)\mu_0,\mu}(y-\delta^*) - \left[1 - R_{9,2}^{\mu_0\mu}(\delta^*)\right],$$

and the lower half of the 99.5% confidence band about the lower bound on $G_k^{\mu_0,\mu}$,

$$G_2^{(9)^{\mu_0,\mu}}(y - \delta^*) + 1 - R_{9,2}^{\mu_0\mu}(\delta^*).$$

Hence, 99% of the distribution functions of the second moment of the random variable X, in the support of $B_{\mu_0,\mu}$, are within this band. The resulting graphs are of the 99% confidence band about the distribution function of the $2nd$ moment, $G_2^{\mu_0,\mu}$. The 99% confidence bands are displayed for each of the base measures – uniform $[0, 1]$, beta (2,2), beta (10,1), beta (1,10) and beta (.5,.5).

For most cases, the separation between the upper and lower functions is small. The exception is beta (10,1) (see Figure 4.3) which presents a clearer distinction between the upper and lower bands. It is not known why the confidence bands have a greater width than the other confidence bands. The relatively large width of the confidence band might be caused by the inefficient algorthim used to generate random numbers according to the beta (10,1) distribution. (Refer to [8] and Appendix B).

In Figure 4.1, the bound (2.54) given by Proposition 2.31, is plotted along with the 99% confidence band for $G_2^{(9)^{\mu_0,\mu}}$. It is clear from this figure that the bound from Proposition 2.31, which uses $\tilde{F}_{n-1}^{\mu_0,\mu}$, is not useful since the corresponding bounds are too large. Many more levels of the SBA approximation would be needed for $\tilde{F}_{n-1}^{\mu_0,\mu}$ to provide a more precise bound. Bounds on selected values of the distribution functions accompany Figures 4.1 through 4.5 in tabular form.

Figures 4.1 through 4.5 show that the beta (10,1) measure is more likely to generate measures with large second moment than those associated with the base measures. Intuitively, this happens because at each level of the construction beta (10,1), scaled to the appropriate interval, behaves similarly to the balayage random variable over that interval. Just as the balayage random variable maximizes moments over an interval (Corollary 2.25), the beta (10,1) base measure is more likely to generate measures with large moments. The beta (1,10) base measure, scaled to the appropriate interval at each level of the construction, behaves in an manner opposite to the balyage. Thus, the conditional barycenters are more likely to be close to previous level barycenters, generating measures with smaller moments. With the beta (.5,.5) case, at each level of the construction, conditional barycenters are equally likely to be close to previous level barycenters as they are to be pushed away from the previous level's barycenter. Hence the beta (.5,.5) base measure is more likely to generate distributions with larger second moment than the beta (1,10) case, but not as likely as the beta (10,1) case.

A summary of results from analysis of the simulated data follows: 50% of the distributions associated with beta (10,1) have second moment between 0.45 and 0.48. For the uniform $[0, 1]$ case, 50% of the distributions have second moment between 0.28 and 0.36 and 50% of the distributions associated with beta (2,2) have second moment between 0.37 and 0.43. For the beta (1,10) case, 50% of the distributions associated have second moment between 0.264 and 0.288 while 50% of the distributions associated with beta (.5,.5) have second moment between 0.26 and 0.39. Thus the beta (10,1) base measure is most likely to produce measures with large second moment than the other base measures and is least likely to produce measures with small second moment. Also, the base measure beta (1,10) is most likely to produce measures with small second moments. Figures 4.6 through 4.10 support this with density plots.

Figure 4.1 The 99% confidence band about G_2, $\delta^* = 0.00006$, with a uniform $[0, 1]$ generating measure. The strips near zero and one are the bounds given by Proposition 2.31. The narrow band represents the bound given by Proposition 2.38. The upper band represents the upper bound of the 99.5% confidence band around $G_2^{(9)}(y + 0.00006) + R_{9,2}^{\mu_0,\mu}(0.00006)$ and the lower band represents the lower bound of the 99.5% confidence band about $G_2^{(9)}(y - 0.00006) - R_{9,2}^{\mu_0,\mu}(0.00006)$.

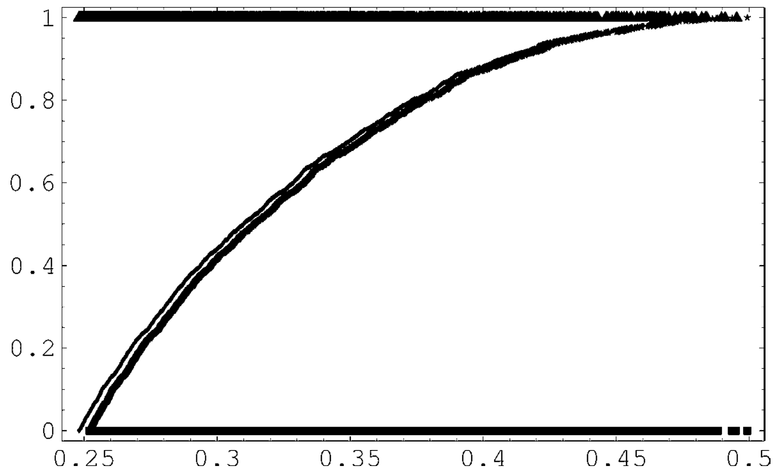


Table 4.1.1: The 99% confidence bands associated with selected values of the distribution function of the second moment in the uniform case.

	99% Confidence Interval
$G_2^{\mu_0,\mu}(.25)$	[0,0.001]
$G_2^{\mu_0,\mu}(.277)$	[0.248,0.25]
$G_2^{\mu_0,\mu}(.3)$	[0.429,0.431]
$G_2^{\mu_0,\mu}(.312)$	[0.499,0.5]
$G_2^{\mu_0,\mu}(.363)$	[0.748,0.75]
$G_2^{\mu_0,\mu}(.4)$	[0.877,0.879]
$G_2^{\mu_0,\mu}(.45)$	[0.966,0.968]

Figure 4.2 The 99% confidence band about G_2, $\delta^* = 0.00006$, with a beta (2,2) generating measure. The narrow band represents the bound given by Proposition 2.38. The upper band represents the upper bound of the 99.5% confidence band around $G_2^{(9)}(y+0.00006)+R_{9,2}^{\mu_0,\mu}(0.00006)$ and the lower band represents the lower bound of the 99.5% confidence band about $G_2^{(9)}(y-0.00006)-R_{9,2}^{\mu_0,\mu}(0.00006)$.

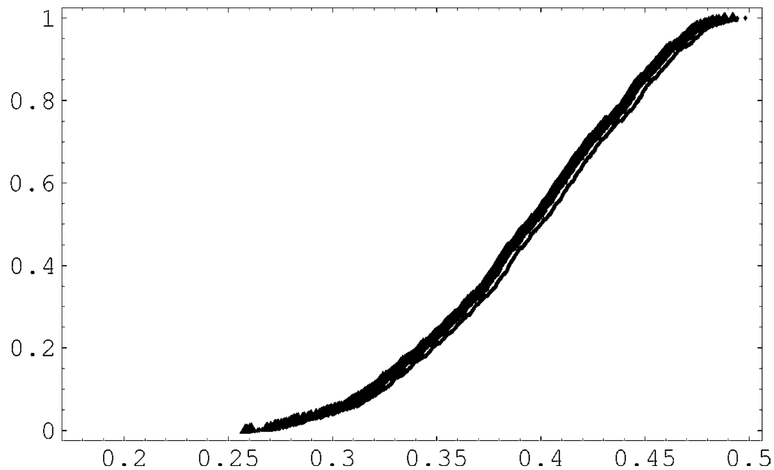


Table 4.2.1: The 99% confidence bands associated with selected values of the distribution function of the second moment in the beta (2,2) case are given below.

	99% Confidence Interval
$G_2^{\mu_0,\mu}(.3)$	[0.045,0.049]
$G_2^{\mu_0,\mu}(.37)$	[0.248,0.25]
$G_2^{\mu_0,\mu}(.397)$	[0.498,0.5]
$G_2^{\mu_0,\mu}(.434)$	[0.749,0.75]
$G_2^{\mu_0,\mu}(.45)$	[0.844,0.848]
$G_2^{\mu_0,\mu}(.49)$	[0.993,0.997]

Figure 4.3 The 99% confidence band about G_2, $\delta^* = 0.002$, with a beta (10,1) generating measure. The upper band represents the upper bound of the 99.5% confidence band around $G_2^{(9)}(y+0.002)+R_{9,2}^{\mu_0,\mu}(0.002)$ and the lower band represents the lower bound of the 99.5% confidence band about $G_2^{(9)}(y-0.002)-R_{9,2}^{\mu_0,\mu}(0.002)$.

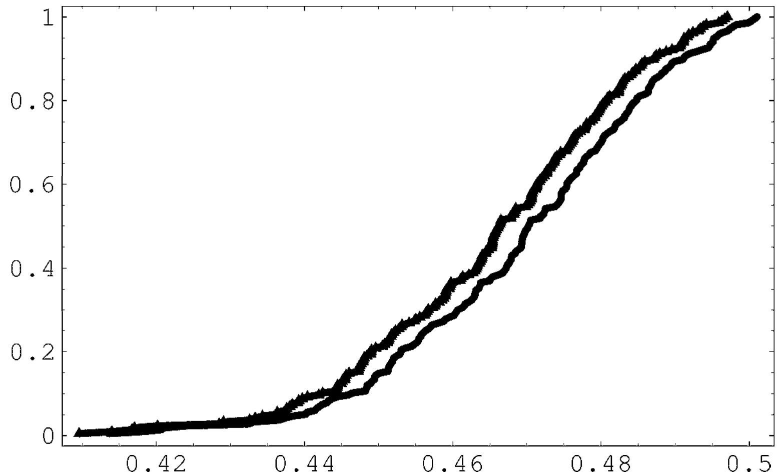


Table 4.3.1: The 99% confidence bands associated with selected values of the distribution function of the second moment in the beta (10,1) case are given below.

	99% Confidence Interval
$G_2^{\mu_0,\mu}(.44)$	[0.046,0.087]
$G_2^{\mu_0,\mu}(.45)$	[0.22,0.25]
$G_2^{\mu_0,\mu}(.46)$	[0.28,0.366]
$G_2^{\mu_0,\mu}(.466)$	[0.38,0.5]
$G_2^{\mu_0,\mu}(.479)$	[0.68,0.75]
$G_2^{\mu_0,\mu}(.49)$	[0.89,0.92]

Figure 4.4 The 99% confidence band about G_2, $\delta^* = 0.00004$, with a beta (1,10) generating measure. The narrow band represents the bound given by Proposition 2.38. The upper band represents the upper bound of the 99.5% confidence band around $G_2^{(9)}(y+0.00004)+R_{9,2}^{\mu_0,\mu}(0.00004)$ and the lower band represents the lower bound of the 99.5% confidence band about $G_2^{(9)}(y-0.00004)-R_{9,2}^{\mu_0,\mu}(0.00004)$.

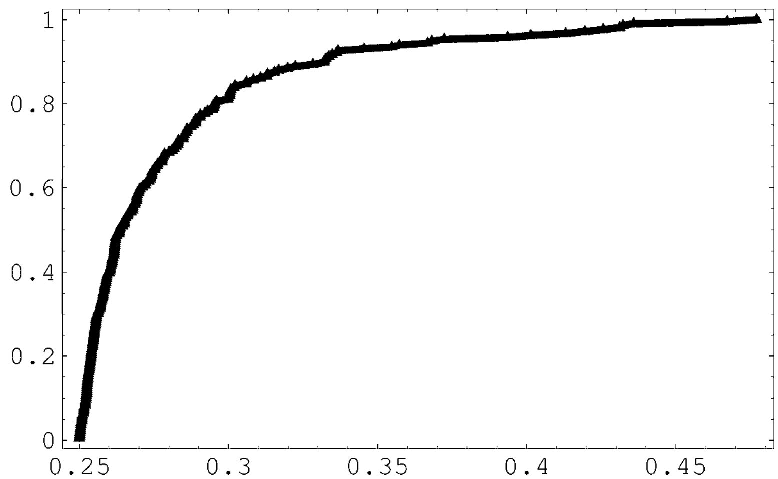


Table 4.4.1: The 99% confidence bands associated with selected values of the distribution function of the second moment in the beta (1,10) case are given below.

	99% Confidence Interval
$G_2^{\mu_0,\mu}(.25)$	[0.046,0.047]
$G_2^{\mu_0,\mu}(.255)$	[0.245,0.25]
$G_2^{\mu_0,\mu}(.26)$	[0.393,0.398]
$G_2^{\mu_0,\mu}(.264)$	[0.495,0.5]
$G_2^{\mu_0,\mu}(.28)$	[0.68,0.685]
$G_2^{\mu_0,\mu}(.288)$	[0.745,0.75]
$G_2^{\mu_0,\mu}(.3)$	[0.81,0.82]
$G_2^{\mu_0,\mu}(.4)$	[0.958,0.962]

Figure 4.5 The 99% confidence band about G_2, $\delta^* = 0.004$, with a beta (.5,.5) generating measure. The narrow band represents the bound given by Proposition 2.38. The upper band represents the upper bound of the 99.5% confidence band around $G_2^{(8)}(y + 0.00009) + R_{9,2}^{\mu_0,\mu}(0.00009)$ and the lower band represents the lower bound of the 99.5% confidence band about $G_2^{(8)}(y - 0.00009) - R_{9,2}^{\mu_0,\mu}(0.00009)$.

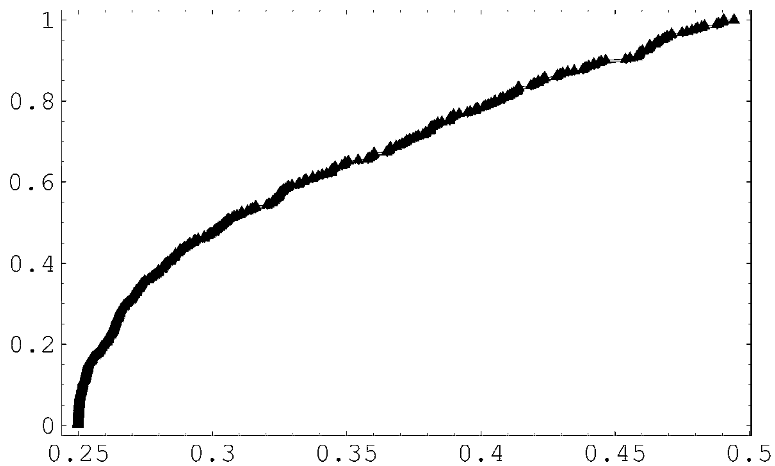


Table 4.5.1: The 99% confidence bands associated with selected values of the distribution function of the second moment in the beta (.5,.5) case are given below.

	99% Confidence Interval
$G_2^{\mu_0,\mu}(.253)$	[0.13,0.14]
$G_2^{\mu_0,\mu}(.259)$	[0.19,0.20]
$G_2^{\mu_0,\mu}(.263)$	[0.24,0.25]
$G_2^{\mu_0,\mu}(.283)$	[0.39,0.4]
$G_2^{\mu_0,\mu}(.305)$	[0.49,0.5]
$G_2^{\mu_0,\mu}(.388)$	[0.74,0.75]
$G_2^{\mu_0,\mu}(.41)$	[0.79,0.80]

Figures 4.6 through 4.10 give the frequency distributions of the second moment for each base measure. While these figures do not provide confidence bands, they do provide visual support for the the analysis given above. The empirical average for the second moment for each base measure are: 0.47 for beta (10,1), 0.33 for beta (.5,.5), 0.32 for uniform, 0.35 for beta (2,2), and 0.27 for beta (1,10).

Figure 4.6 The frequency distribution of the second moment associated with generating uniform [0, 1] generating measure. The average of the distribution is 0.32.

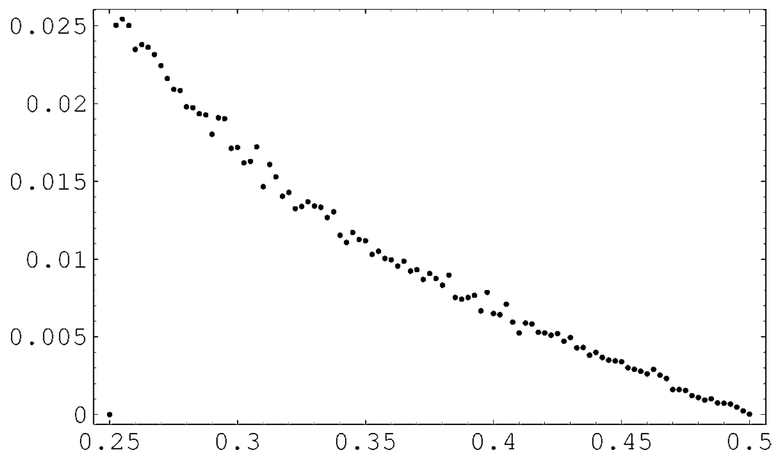


Figure 4.7 The frequency distribution of the second moment associated with generating beta (2,2) generating measure. The average of the distribution is 0.35.

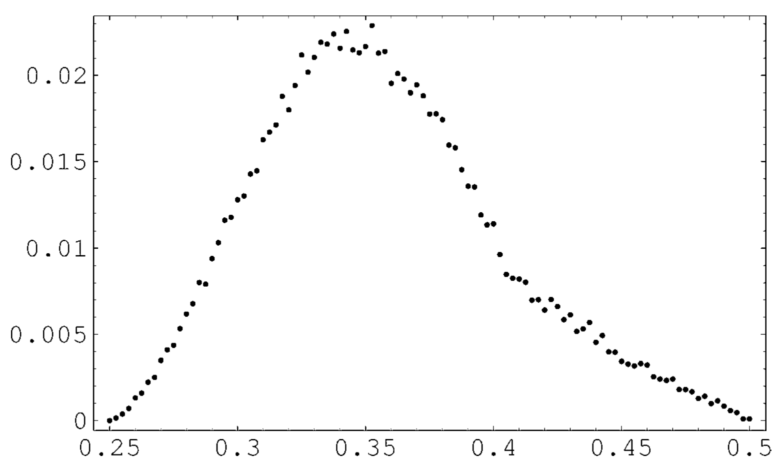


Figure 4.8 The frequency distribution of the second moment associated with generating beta (10,1) generating measure. The average of the distribution is 0.47.

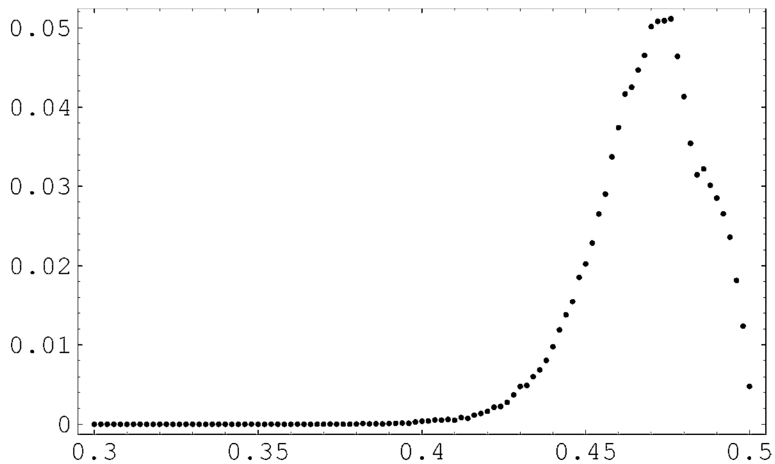


Figure 4.9 The frequency distribution of the second moment, $G_2^{(9)^{\mu_0,\mu}}$ associated with generating beta (1,10) generating measure. The average of the distribution is 0.27.

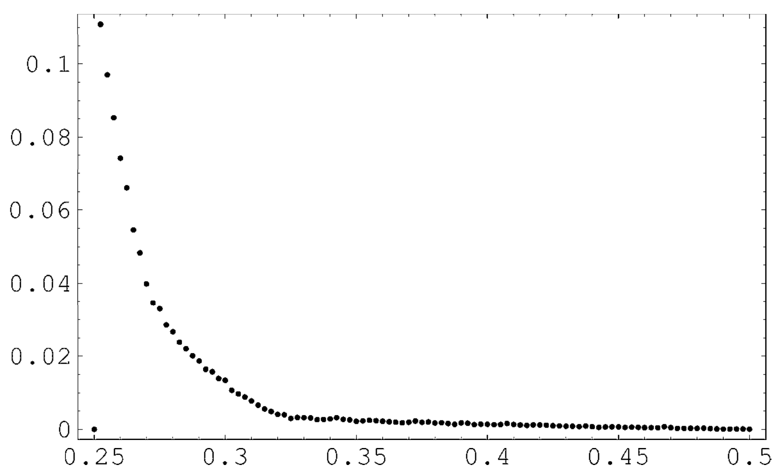
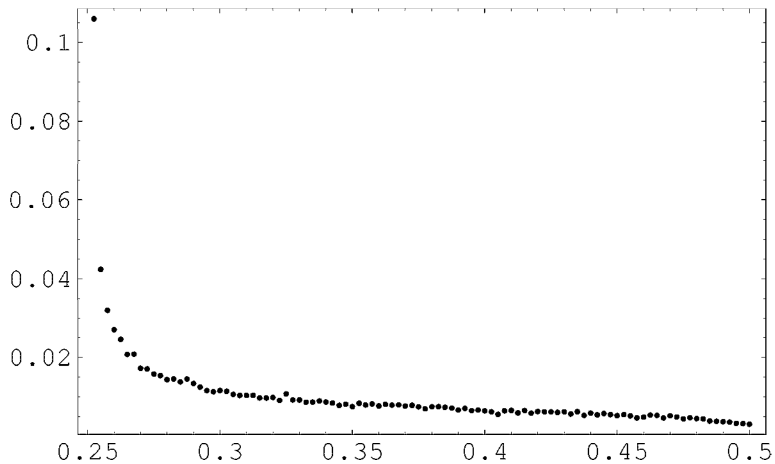


Figure 4.10 The frequency distribution of the second moment, $G_2^{(9)^{\mu_0,\mu}}$ associated with beta (.5,.5) generating measure. The average of the distribution is 0.33.



4.3 The Standard Deviation

Figures 4.11 through 4.15 give the 99% confidence bands about the standard deviation associated with each of the generating measures. For the beta (10,1) case, 50% of the generated distributions have SD between 0.45 and 0.48. For the beta (.5,.5) case, 50% of the distributions have SD between 0.12 and 0.37 while 50% of the distributions generated from the base measure beta (1,10) have SD between 0.09 and 0.19. For the uniform base measure, 50% of the distributions have SD between 0.15 and 0.325. For the beta (2,2) base measure, 50% of the distributions have SD between 0.27 and 0.35. This implies that the beta (10,1) base measure is most likely to produce measures with large SD. The beta (1,10) base measure is most likely to produce measures with small SD. Figures 4.16 through 4.20 support this with density plots. The beta (10,1) base measure is most likely to produce measures with large SD. Again, intuitively, this follows since the beta (10,1) base measure behaves like the balayage random variable at each step in the construction and the mean is fixed at 1/2. Conversely, the beta (10,1) more likely concentrates mass near the barycenters at each stage of the construction (behaving in an opposite fashion to the balayage random variable at each step), and so is more likely to generate measures with small SD.

Figure 4.11 The 99% confidence band about the SD, with a uniform $[0,1]$ generating measure and $\delta^* = 0.000009$. The upper band represents the upper bound of the 99.5% confidence band around $C_2^{(9)}(y^2+0.000009)+R_{9,2}^{\mu_0,\mu}(0.000009)$ and the lower band represents the lower bound of the 99.5% confidence band about $C_2^{(9)}(y^2-0.000009)-R_{9,2}^{\mu_0,\mu}(0.000009)$.

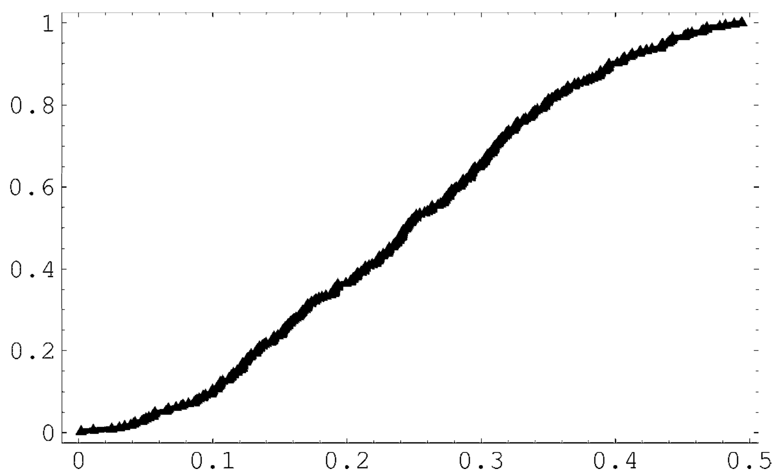


Table 4.11.1: The following are 99% confidence bands for selected values of the the distribution function of the SD.

	99% Confidence Interval
$B_{\mu_0,\mu}(\sqrt{X^2-\frac{1}{4}} \leq .151)$	[0.23,0.25]
$B_{\mu_0,\mu}(\sqrt{X^2-\frac{1}{4}} \leq .24)$	[0.485,0.5]
$B_{\mu_0,\mu}(\sqrt{X^2-\frac{1}{4}} \leq .31)$	[0.718,0.73]
$B_{\mu_0,\mu}(\sqrt{X^2-\frac{1}{4}} \leq .325)$	[0.742,0.75]
$B_{\mu_0,\mu}(\sqrt{X^2-\frac{1}{4}} \leq .376)$	[0.847,0.859]

Figure 4.12 The 99% confidence band for the SD, with a beta (2,2) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around

$C_2^{(9)}(y^2 + 0.0001) + R_{9,2}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_2^{(9)}(y^2 - 0.0001) - R_{9,2}^{\mu_0,\mu}(0.0001)$.

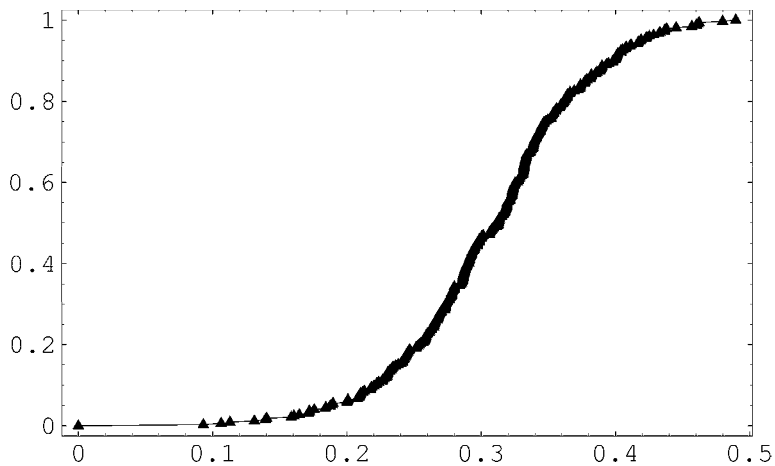


Table 4.12.1: For the distribution function of the standard deviation associated with the beta (2,2) base measure, the following are 99% confidence bands for selected values of the distribution function of the SD.

	99% Confidence Interval
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .177)$	[0.037,0.04]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .265)$	[0.239,0.25]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .312)$	[0.488,0.5]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .350)$	[0.748,0.75]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .37)$	[0.83,0.84]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .416)$	[0.936,0.94]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .447)$	[0.979,0.98]

Figure 4.13 The 99% confidence band for the SD, with a beta (10,1) generating measure and $\delta^* = 0.0006$. The upper band represents the upper bound of the 99.5% confidence band around $C_2^{(9)}(y^2 + 0.0006) + R_{9,2}^{\mu_0,\mu}(0.0006)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_2^{(9)}(y^2 - 0.0006) - R_{9,2}^{\mu_0,\mu}(0.0006)$.

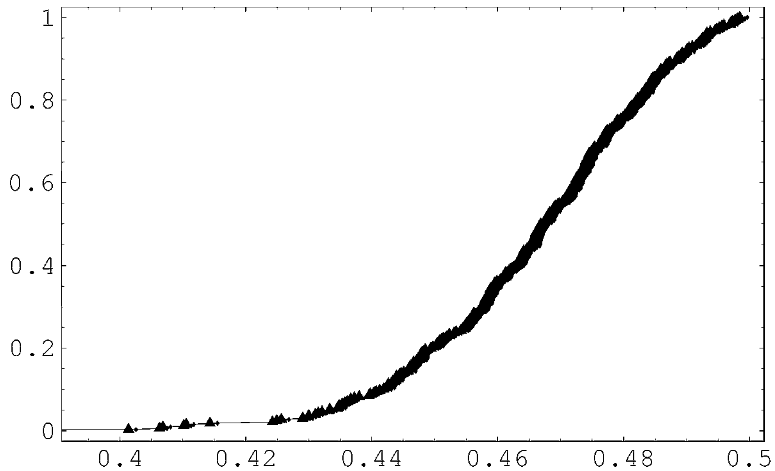


Table 4.13.1: For the distribution function of the standard deviation associated with the beta (10,1) base measure, the following are 99% confidence bands for selected values of the distribution function of the SD.

	99% Confidence Interval
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .416)$	[.015,.018]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .447)$	[0.134,0.196]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .453)$	[0.22,0.25]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .461)$	[0.322,0.387]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .466)$	[0.41,0.5]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .473)$	[0.555,0.64]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .477)$	[0.69,0.752]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .484)$	[0.797,0.86]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .491)$	[0.90,0.94]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .495)$	[0.95,0.98]

Figure 4.14 The 99% confidence band for the SD, with a beta (1,10) generating measure and $\delta^* = 0.000009$. The upper band represents the upper bound of the 99.5% confidence band around

$C_2^{(9)}(y^2 + 0.000009) + R_{9,2}^{\mu_0,\mu}(0.000009)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_2^{(9)}(y^2 - 0.000009) - R_{9,2}^{\mu_0,\mu}(0.000009)$.

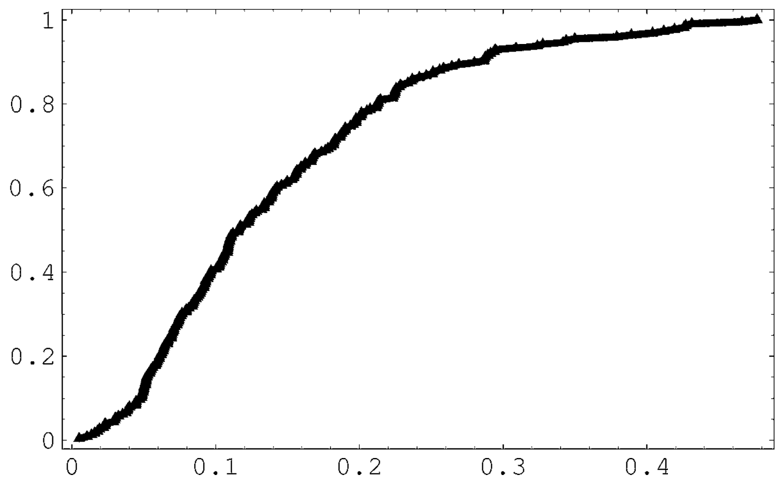


Table 4.14.1: For the distribution function of the standard deviation associated with the beta (1,10) base measure, the following are 99% confidence bands for selected values of the distribution function of the SD.

	99% Confidence Interval
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq 0)$	[0.01,0.11]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .089)$	[0.22,0.257]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .1)$	[0.317,0.347]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .126)$	[0.46,0.5]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .132)$	[0.525,0.535]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .141)$	[0.609,0.624]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .190)$	[0.743,0.753]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .32)$	[0.95,0.96]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .39)$	[0.999,1]

Figure 4.15 The 99% confidence band for the SD, with a beta (.5,.5) generating measure and $\delta^* = 0.0006$. The upper band represents the upper bound of the 99.5% confidence band around $C_2^{(9)}(y^2 + 0.0006) + R_{9,2}^{\mu_0,\mu}(0.0006)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_2^{(9)}(y^2 - 0.0006) - R_{9,2}^{\mu_0,\mu}(0.0006)$.

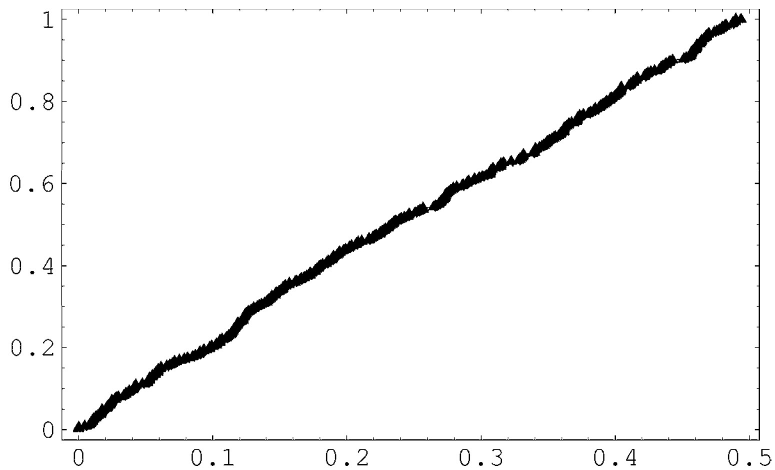


Table 4.15.1: For the distribution function of the standard deviation associated with the beta (.5,.5) base measure, the following are 99% confidence bands for selected values of the distribution function of the SD.

	99% Confidence Interval
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .041)$	[0.09,0.1]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .12)$	[0.24,0.25]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .234)$	[0.49,0.5]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .371)$	[0.74,0.75]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .42)$	[0.84,0.85]
$B_{\mu_0,\mu}(\sqrt{X^2 - \frac{1}{4}} \leq .495)$	[0.945,0.988]

Figures 4.16 through 4.20 give the density plots of the the standard deviation associated with each base measure. The empirical average of the SD for each case are as

follows: 0.25 for uniform, 0.32 for beta (2,2), 0.45 for beta (10,1), 0.247 for and beta (.5,.5), and 0.145 for beta (1,10).

Figure 4.16 The frequency distribution of the SD associated with generating uniform $[0,1]$ generating measure. The average of the distribution is 0.25.

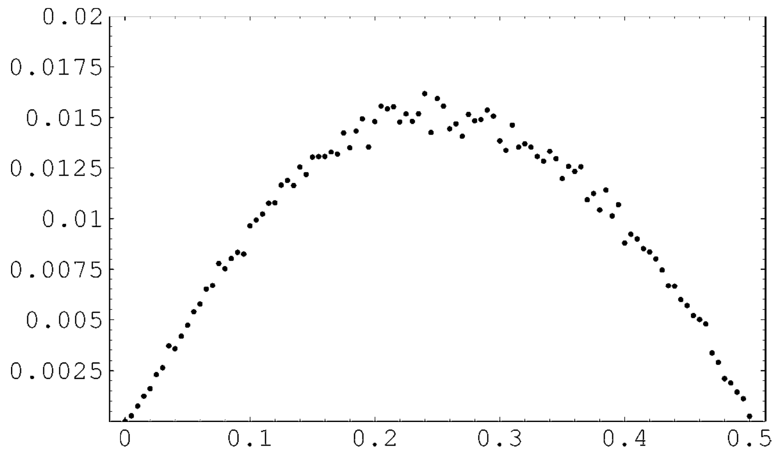


Figure 4.17 The frequency distribution of the SD with a beta (2,2) generating measure. The distribution on the SD has an average of 0.32.

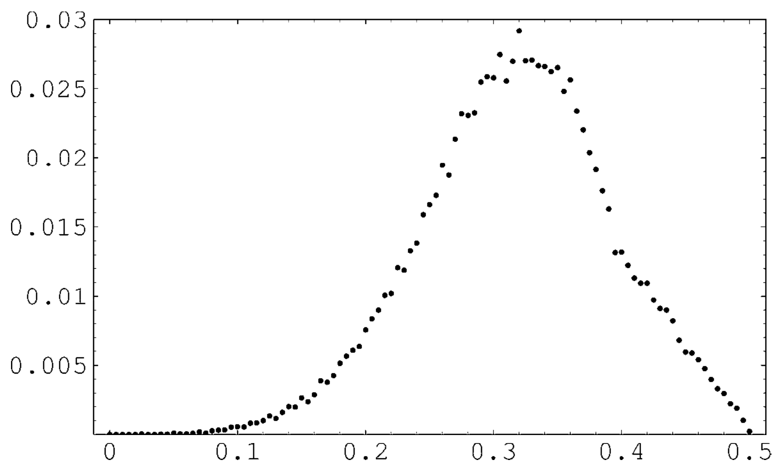


Figure 4.18 The frequency distribution of the SD associated with the beta (10,1) base measure. The mean of the distribution is 0.45.

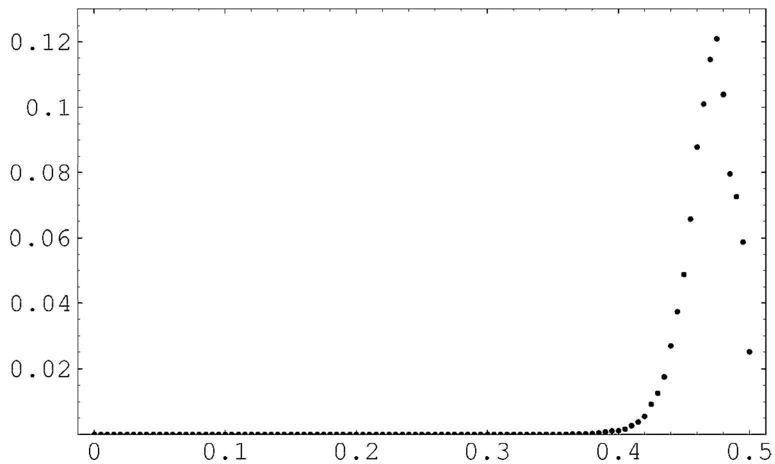


Figure 4.19 The frequency distribution of the SD associated with the beta (1,10) base measure. The average of the distribution is 0.145.

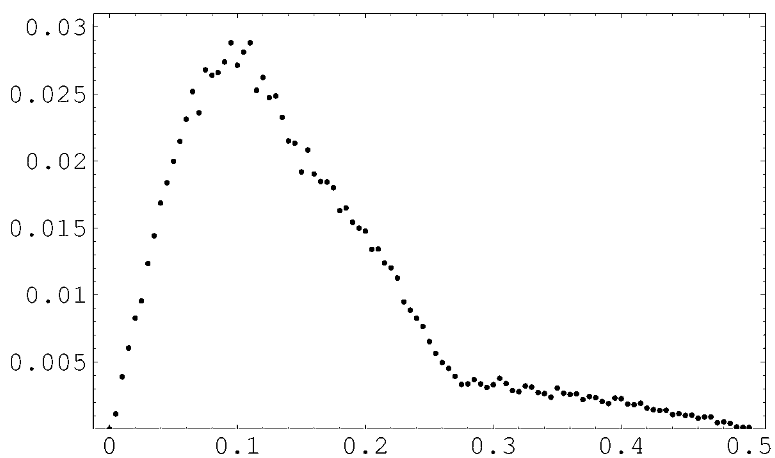
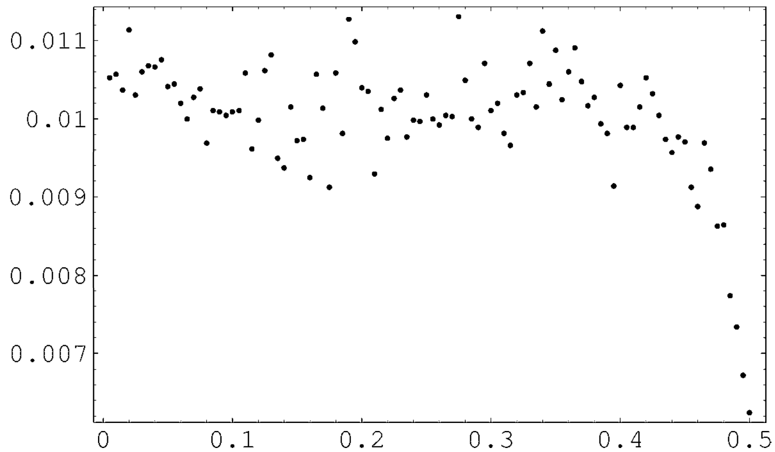


Figure 4.20 The frequency distribution of the SD with a generating measure beta (.5,.5). The mean of the distribution is 0.247.



4.4 The Third Moment

Figures 4.21 through 4.25 show 99% confidence bands around the distribution function of the third moment associated with each of the base measures. The results for the third moment are similar to the results for the second moment and the standard deviation in that the beta (10,1) base measure is more likely to generate distributions with a large third moment than the the other base measures.

For the beta (10,1) case, 50% of the distributions generated have third moment between 0.43 and 0.47, while for the beta (.5,.5) case, 50% for the distrubutions generated have third moment between 0.15 and 0.33. For the beta (1,10) case, 50% of the distributions generated have third moment between 0.132 and 0.183 and 50% of the distributions have third moment between 0.16 and 0.297, for the uniform base measure. For the beta (2,2) measure, 50% of the distributions have third moment between 0.285 and 0.398. Thus beta (10,1) base measure is more likely to produce measures with a large third moment. The beta (1,10) base measure is more likely to generate measures with small third moment. Figures 4.26 through 4.30 give density plots that support this characterization.

Figure 4.21 The 99% confidence band for $G_3^{\mu_0,\mu}$ with $\delta^* = 0.00009$ with a uniform $[0,1]$ generating measure. The upper strip is the upper bound of the 99.5% confidence band around $G_3^{(9)^{\mu_0,\mu}}(y + 0.00009) + R_{9,3}^{\mu_0,\mu}(0.00009)$ and the lower strip is the lower bound of the 99.5% confidence band for $G_3^{(9)^{\mu_0,\mu}}(y - 0.00009) - R_{9,3}^{\mu_0,\mu}(0.00009)$

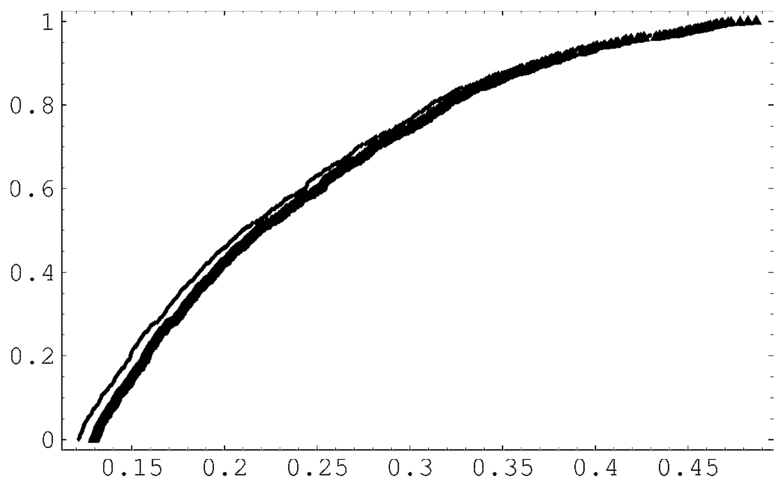


Table 4.21.1: The following are 99% confidence bands for selected values of the distribution function of the third moment associated with the uniform $[0,1]$ base measure.

	99% Confidence Interval
$G_3^{\mu_0,\mu}(.125)$	[0.001,0.002]
$G_3^{\mu_0,\mu}(.15)$	[0.174,0.176]
$G_3^{\mu_0,\mu}(.16)$	[0.247,0.25]
$G_3^{\mu_0,\mu}(.213)$	[0.495,0.5]
$G_3^{\mu_0,\mu}(.25)$	[0.619,0.621]
$G_3^{\mu_0,\mu}(.297)$	[0.747,0.75]
$G_3^{\mu_0,\mu}(.455)$	[0.978,0.981]
$G_3^{\mu_0,\mu}(.483)$	[0.999,1]

Figure 4.22 The 99% confidence band for $G_3^{\mu_0,\mu}$ with a beta (2,2) generating measure and $\delta^* = 0.0002$. The upper band represents the upper bound of the 99.5% confidence band around

$G_3^{(9)^{\mu_0,\mu}}(y+0.0002)+R_{9,3}^{\mu_0,\mu}(0.0002)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_3^{(9)^{\mu_0,\mu}}(y-0.0002)-R_{9,3}^{\mu_0,\mu}(0.0002)$.

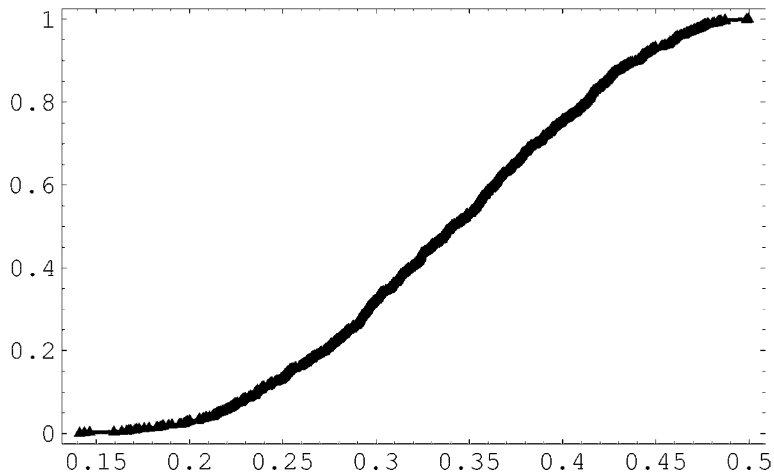


Table 4.22.1: The following are 99% confidence bands for selected values of the distribution function of the third moment associated with the beta (2,2) base measure.

	99% Confidence Interval
$G_3^{\mu_0,\mu}(.2)$	[0.0309,0.031]
$G_3^{\mu_0,\mu}(.285)$	[0.249,0.25]
$G_3^{\mu_0,\mu}(.3)$	[0.3259,0.326]
$G_3^{\mu_0,\mu}(.34)$	[0.498,0.5]
$G_3^{\mu_0,\mu}(.398)$	[0.749,0.75]
$G_3^{\mu_0,\mu}(.45)$	[0.9319,0.932]

Figure 4.23 The 99% confidence band for $G_3^{\mu_0,\mu}$ with a beta (10,1) generating measure and $\delta^*=0.003$. The upper band represents the upper bound of the 99.5% confidence band around $G_3^{(9)^{\mu_0,\mu}}(y+0.003)+R_{9,3}^{\mu_0,\mu}(0.003)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_3^{(9)^{\mu_0,\mu}}(y-0.003)-R_{9,3}^{\mu_0,\mu}(0.003)$.

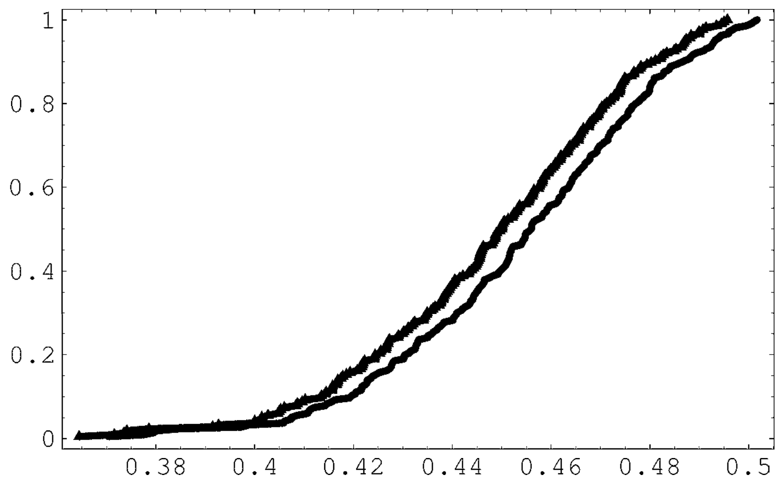


Table 4.23.1: The following are 99% confidence bands for selected values of the distribution function of the third moment associated with the beta (10,1) base measure.

	99% Confidence Interval
$G_3^{\mu_0,\mu}(.4)$	[0.032,0.037]
$G_3^{\mu_0,\mu}(.42)$	[0.102,0.157]
$G_3^{\mu_0,\mu}(.43)$	[0.19,0.25]
$G_3^{\mu_0,\mu}(.44)$	[0.278,0.366]
$G_3^{\mu_0,\mu}(.45)$	[0.41,0.5]
$G_3^{\mu_0,\mu}(.46)$	[0.56,0.64]
$G_3^{\mu_0,\mu}(.468)$	[0.68,0.75]
$G_3^{\mu_0,\mu}(.48)$	[0.84,0.89]
$G_3^{\mu_0,\mu}(.49)$	[0.92,0.97]

Figure 4.24 The 99% confidence band for $G_3^{\mu_0,\mu}$ with a beta (1,10) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $G_3^{(9)^{\mu_0,\mu}}(y+0.0001) + R_{9,3}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_3^{(9)^{\mu_0,\mu}}(y-0.0001) - R_{9,3}^{\mu_0,\mu}(0.0001)$.

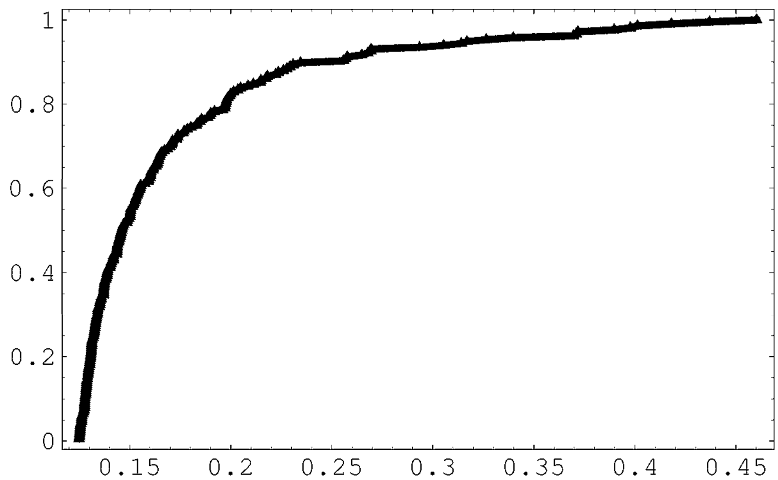


Table 4.24.1: The following are 99% confidence bands for selected values of the distribution function of the third moment associated with the beta (1,10) base measure.

	99% Confidence Interval
$G_3^{\mu_0,\mu}(.126)$	[0.042,0.046]
$G_3^{\mu_0,\mu}(.132)$	[0.245,0.25]
$G_3^{\mu_0,\mu}(.146)$	[0.495,0.5]
$G_3^{\mu_0,\mu}(.183)$	[0.745,0.75]
$G_3^{\mu_0,\mu}(.2)$	[0.819,0.828]
$G_3^{\mu_0,\mu}(.25)$	[0.898,0.902]
$G_3^{\mu_0,\mu}(.3)$	[0.935,0.940]
$G_3^{\mu_0,\mu}(.4)$	[0.99,0.99]

Figure 4.25 The 99% confidence band for $G_3^{\mu_0,\mu}$ with a beta (.5,.5) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $G_3^{(9)^{\mu_0,\mu}}(y+0.0001) + R_{9,3}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_3^{(9)^{\mu_0,\mu}}(y-0.0001) - R_{9,3}^{\mu_0,\mu}(0.0001)$.

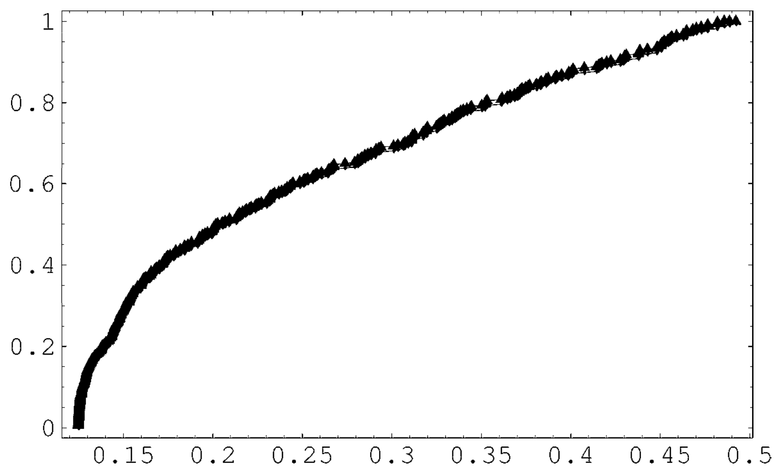


Table 4.25.1: The following are 99% confidence bands for selected values of the distribution function of the third moment associated with the beta (.5,.5) base measure.

	99% Confidence Interval
$G_3^{\mu_0,\mu}(.125)$	[0.020,0.043]
$G_3^{\mu_0,\mu}(.138)$	[0.18,0.2]
$G_3^{\mu_0,\mu}(.145)$	[0.235,0.25]
$G_3^{\mu_0,\mu}(.20)$	[0.49,0.5]
$G_3^{\mu_0,\mu}(.328)$	[0.739,0.75]
$G_3^{\mu_0,\mu}(.384)$	[0.84,0.85]
$G_3^{\mu_0,\mu}(.42)$	[0.89,0.9]
$G_3^{\mu_0,\mu}(.45)$	[0.94,0.95]

Figures 4.26 through 4.30 give the density plots of the third moment associated with each of the generating measures. The average was 0.45 for the beta (10,1) case. For the beta (.5,.5) case, the average was 0.25. The average associated with the beta (1,10) case is 0.17. The uniform case has an average of 0.23 while the beta (2,2) case has an average of 0.28. These averages indicate that the beta (10,1) base measure is most likely to generate distributions with large third moment than the other base

measures and the beta (1,10) base measure is most likely to generate measures with small third moment.

Figure 4.26 The frequency distribution of the third moment associated with the uniform [0, 1] base measure. It has an average of 0.23.

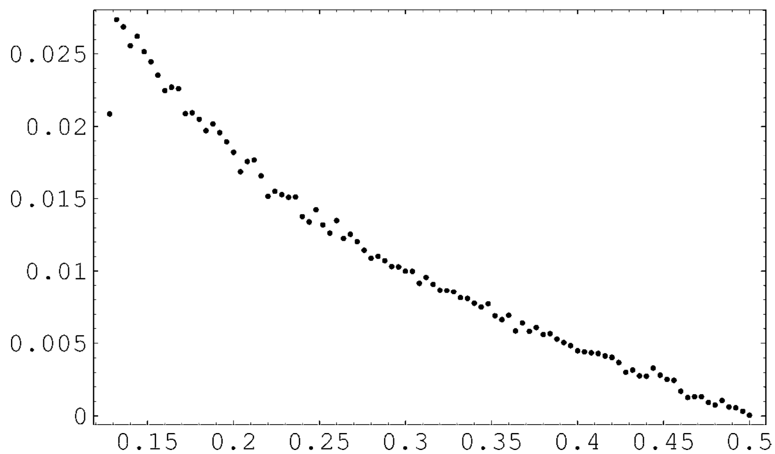


Figure 4.27 The frequency distribution of the third moment associated with a beta (2,2) base measure. The mean is 0.28.

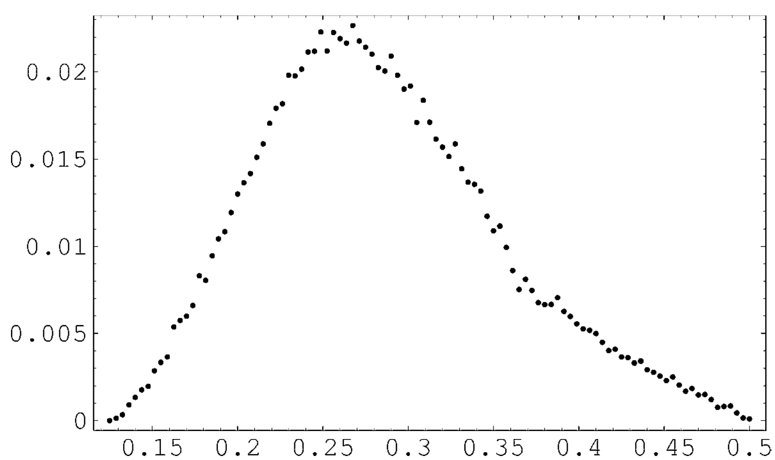


Figure 4.28 The frequency distribution of the third moment associated with a beta (10,1) base measure. The average of the distribution is 0.45.

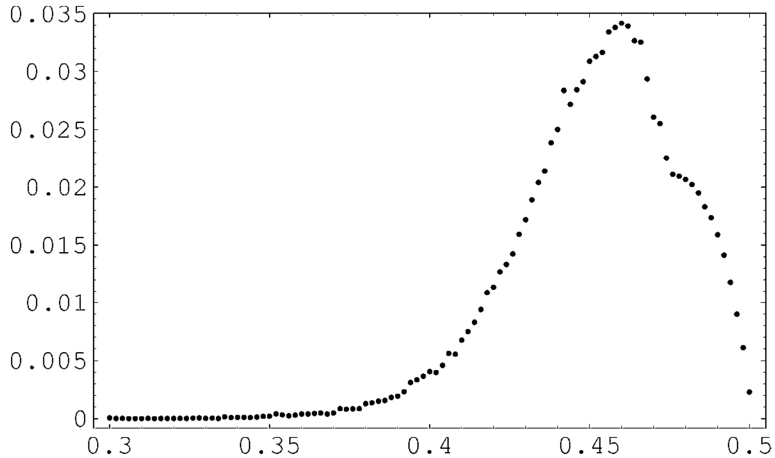


Figure 4.29 The distribution on the third moment associated with a beta (1,10) generating measure. The average of the distribution is 0.17.

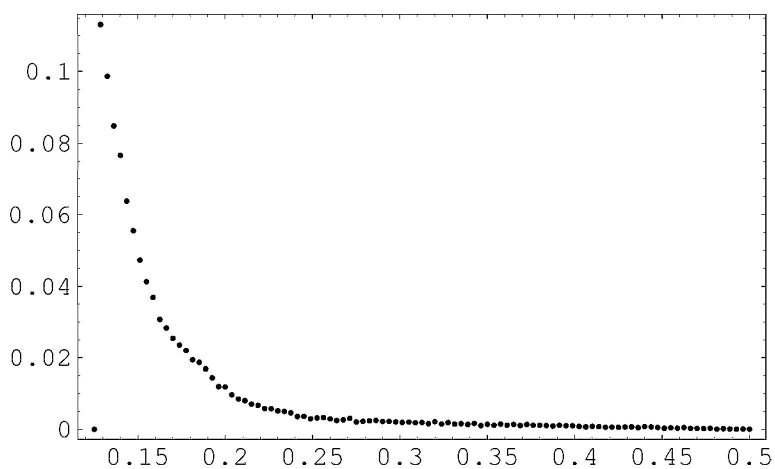
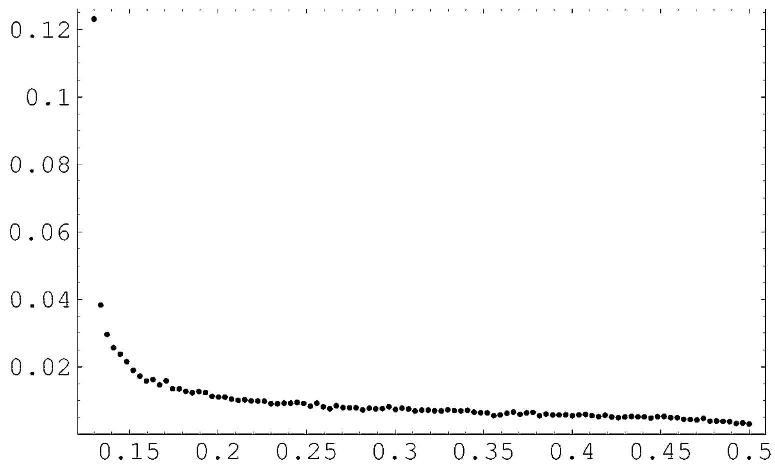


Figure 4.30 The frequency distribution of the third moment associated with a beta (.5,.5) base measure. The mean is 0.25.



4.5 The Third Central Moment and Skewness

The distribution functions and density plots below imply that beta (10,1) measure is more likely to produce a third central moment with a value near zero, which implies that these base measures are most likely to produce approximately symmetric measures. The graphs show that most of the distributions occur over a very small interval centered about zero making it difficult to infer sufficient information from the data. The distribution functions for the third central moment associated with each of the base measures is given in Figures 4.31 through 4.35. For the beta (10,1) base measure (Figures 4.33 and 4.38), 50% of the distributions have third central moment between -0.002 and 0.0006. For beta (1,10), 50% of the distributions have third central moment between -0.0017 and 0.0019. Note that this interval is nearly twice as large as the interval in the beta (10,1) case, but still very close to having length zero. (See Figures 4.34 and 4.39). The beta (1,10) base measure is most likely to produce measures with smallest third central moments. For the beta (.5,.5) base measure (Figures 4.35 and 4.40), 50% of the distributions have third central moment between -0.006 and 0.004. For beta (2,2), 50% of the distributions have third central moment between -0.008 and 0.006. (Refer to Figures 4.32 and 4.37). For the uniform case, 50% of the distributions have third central moment between -0.007 and 0.004. (See Figures 4.31 and 4.36). The beta (2,2) base measure is most likely to produce measures with largest third central moments.

Figure 4.31 The 99% confidence band for the third central moment, with a uniform $[0,1]$ generating measure and $\delta^* = 0.001$. The upper band represents the upper bound of the 99.5% confidence band around $C_3^{(9)^{\mu_0,\mu}}(y+0.001) + RS_{9,3}^{\mu_0,\mu}(0.001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_3^{(9)^{\mu_0,\mu}}(y-0.001) - RS_{9,3}^{\mu_0,\mu}(0.001)$.

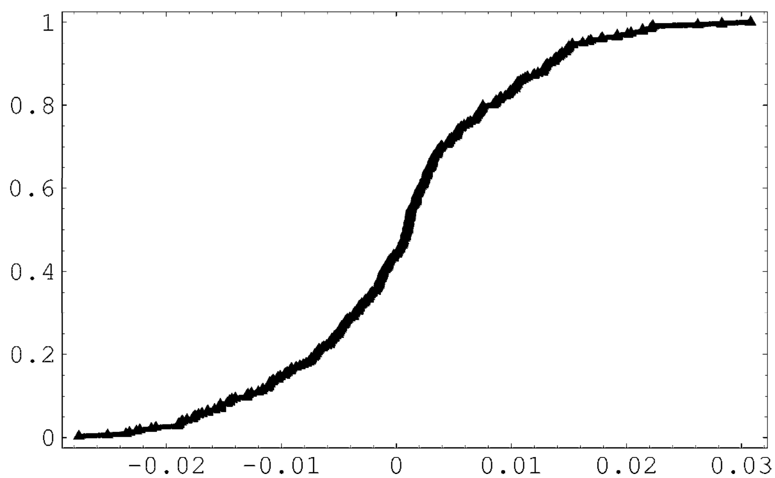


Table 4.31.1: The following are 99% confidence bands for selected values of the distribution function of the third central moment associated with the uniform base measure.

	99% Confidence Interval
$C_3^{\mu_0,\mu}(-.027)$	[0.0031,0.006]
$C_3^{\mu_0,\mu}(-.01)$	[0.15,0.178]
$C_3^{\mu_0,\mu}(-.007)$	[0.202,0.25]
$C_3^{\mu_0,\mu}(-.001)$	[0.41,0.5]
$C_3^{\mu_0,\mu}(.0039)$	[0.702,0.75]
$C_3^{\mu_0,\mu}(.01)$	[0.834,0.874]
$C_3^{\mu_0,\mu}(.02)$	[0.972,0.985]

Figure 4.32 The 99% confidence band for the third central moment, with a beta (2,2) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $C_3^{(9)^{\mu_0,\mu}}(y+0.0001) + RS_{9,3}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_3^{(9)^{\mu_0,\mu}}(y-0.0001) - RS_{9,3}^{\mu_0,\mu}(0.0001)$.

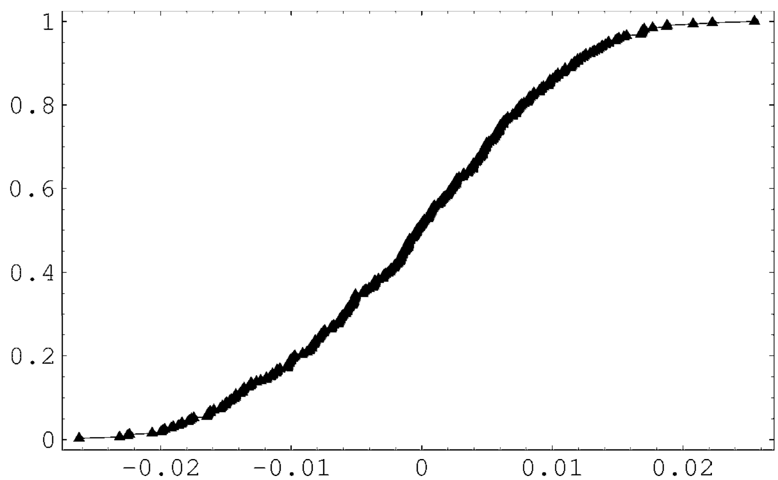


Table 4.32.1: The following are 99% confidence band for selected values of the distribution function of the third central moment associated with the beta (2,2) base measure.

	99% Confidence Interval
$C_3^{\mu_0,\mu}(-.02)$	[0.0183,0.0184]
$C_3^{\mu_0,\mu}(-.01)$	[0.17,0.19]
$C_3^{\mu_0,\mu}(-.0077)$	[0.245,0.252]
$C_3^{\mu_0,\mu}(-.00037)$	[0.485,0.5]
$C_3^{\mu_0,\mu}(.006)$	[0.733,0.752]
$C_3^{\mu_0,\mu}(.01)$	[0.856,0.859]
$C_3^{\mu_0,\mu}(.02)$	[0.988,0.991]

Figure 4.33 The 99% confidence band for the third central moment, with a beta (10,1) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $C_3^{(9)^{\mu_0,\mu}}(y + 0.0001) + RS_{9,3}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_3^{(9)^{\mu_0,\mu}}(y - 0.0009) - RS_{9,3}^{\mu_0,\mu}(0.0001)$.

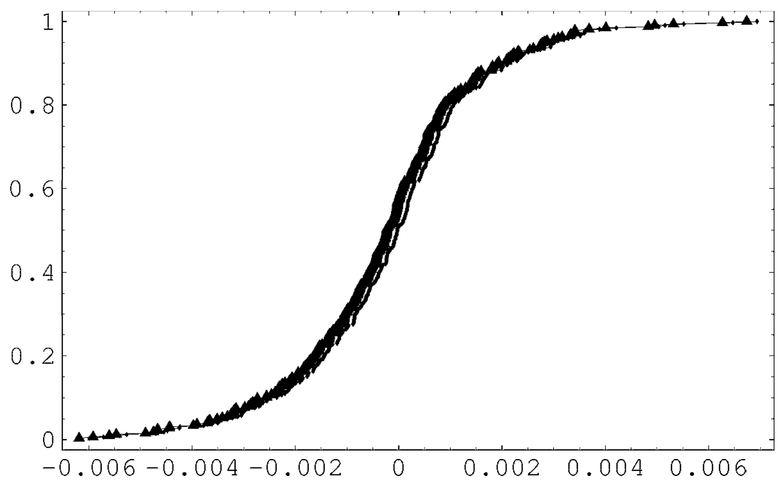


Table 4.33.1: The following are 99% confidence bands for selected values of the distribution function of the third central moment associated with the beta (10,1) base measure.

	99% Confidence Interval
$C_3^{\mu_0,\mu}(-.005)$	[0.004,0.012]
$C_3^{\mu_0,\mu}(-.002)$	[0.131,0.147]
$C_3^{\mu_0,\mu}(-.001)$	[0.22,0.25]
$C_3^{\mu_0,\mu}(-.0002)$	[0.439,0.5]
$C_3^{\mu_0,\mu}(.0003)$	[0.60,0.65]
$C_3^{\mu_0,\mu}(.0006)$	[0.71,0.75]
$C_3^{\mu_0,\mu}(.002)$	[0.89,0.90]
$C_3^{\mu_0,\mu}(.004)$	[0.98,0.98]
$C_3^{\mu_0,\mu}(.0067)$	[0.997,1]

Figure 4.34 The 99% confidence band for the third central moment, with a beta (1,10) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $C_3^{(9)^{\mu_0,\mu}}(y + 0.0001) + RS_{9,3}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_3^{(9)^{\mu_0,\mu}}(y - 0.0009) - RS_{9,3}^{\mu_0,\mu}(0.0001)$.

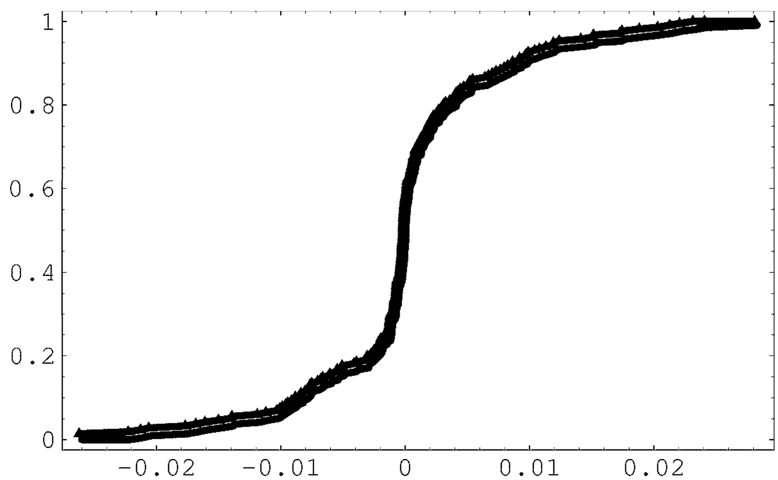


Table 4.34.1: The following are 99% confidence bands for selected values of the distribution function of the third central moment associated with the beta (1,10) base measure.

	99% Confidence Interval
$C_3^{\mu_0,\mu}(-.01)$	[0.0051,.007]
$C_3^{\mu_0,\mu}(-.0017)$	[0.22,0.25]
$C_3^{\mu_0,\mu}(-.00015)$	[0.417,0.5]
$C_3^{\mu_0,\mu}(.00022)$	[0.57,0.625]
$C_3^{\mu_0,\mu}(.0019)$	[0.72,0.75]
$C_3^{\mu_0,\mu}(.005)$	[0.829,0.843]
$C_3^{\mu_0,\mu}(.01)$	[0.907,0.926]
$C_3^{\mu_0,\mu}(.02)$	[0.967,0.982]

Figure 4.35 The 99% confidence band for the third central moment, with a beta (.5,.5) generating measure and $\delta^* = 0.0002$. The upper band represents the upper bound of the 99.5% confidence band around $C_3^{(9)^{\mu_0,\mu}}(y + 0.0002) + RS_{9,3}^{\mu_0,\mu}(0.0002)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_3^{(9)^{\mu_0,\mu}}(y - 0.0002) - RS_{9,3}^{\mu_0,\mu}(0.0002)$.

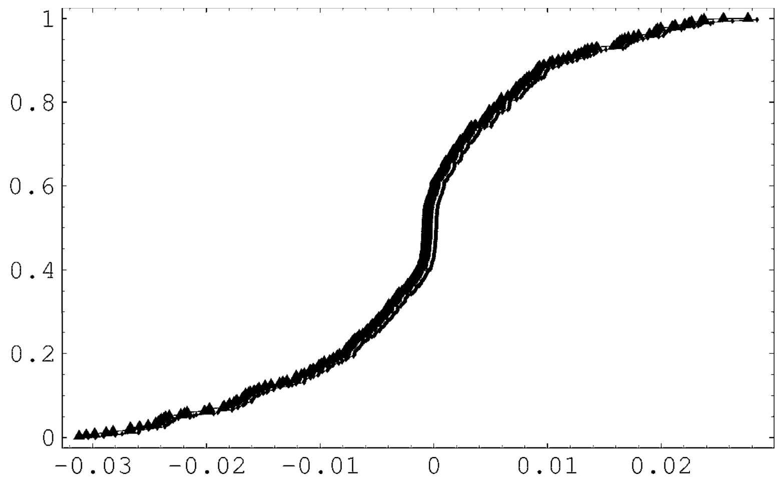


Table 4.35.1: The following are 99% confidence bands for selected values of the distribution function of the third central moment associated with the beta (.5,.5) base measure.

	99% Confidence Interval
$C_3^{\mu_0,\mu}(-.006)$	[0.24,0.25]
$C_3^{\mu_0,\mu}(-.0006)$	[0.29,0.5]
$C_3^{\mu_0,\mu}(.004)$	[0.74,0.75]
$C_3^{\mu_0,\mu}(.008)$	[0.84,0.85]
$C_3^{\mu_0,\mu}(.010)$	[0.89,0.90]

Figures 4.36 through 4.40 show the average of each distribution is near 0. The base measures beta (10,1) produces third central moments that are likely to be nearest to zero. The average of the distribution associated with beta (10,1) is 4.5×10^{-5}. The distribution for the beta (.5,.5) case has a mean of .0002. The averages for the other cases are as follows: 0.00011 for beta (1,10), 0.00015 for beta (2,2), and 5.1×10^{-5} for uniform. Although the averages for each of the cases are very close to each other, the following density plots show that the beta (10,1) case concentrates its mass on a smaller interval than do the other cases, although the difference in interval size is very small. Thus, the beta (10,1) case is most likely to generate a zero valued third central moment than the other cases, while the beta (2,2) and beta (.5,.5) cases are most likely produces measures with larger third central moment.

Figure 4.36 The frequency distribution of the third central moment associated with the uniform $[0, 1]$ base measure. It has an average of 5.1×10^{-5}.

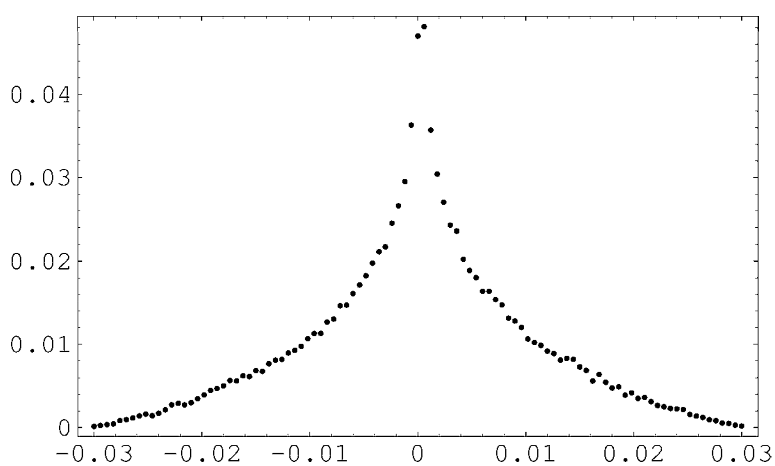


Figure 4.37 The frequency distribution of the third central moment associated with a beta (2,2) base measure. The mean is 0.00015.

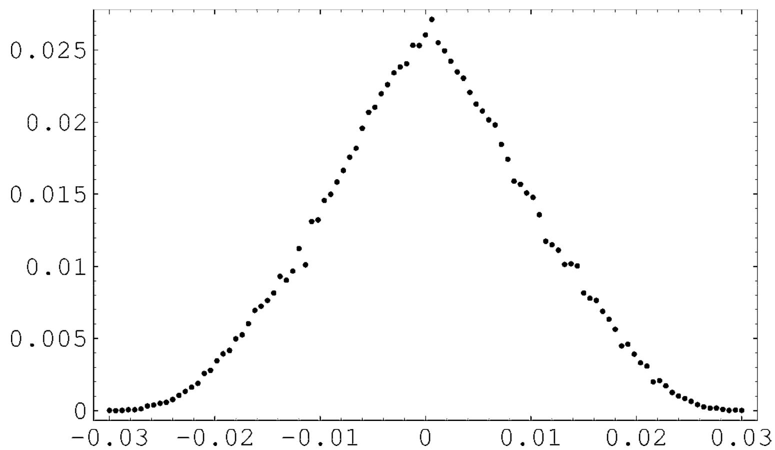


Figure 4.38 The frequency distribution of the third central moment associated with a beta (10,1) base measure. The average of the distribution is 4.5×10^{-5}.

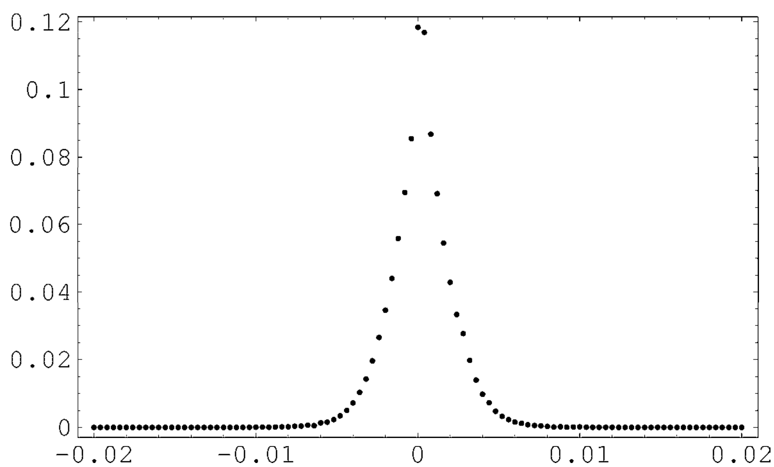


Figure 4.39 The distribution on the third central moment associated with a beta (1,10) generating measure. The average of the distribution is 0.00011.

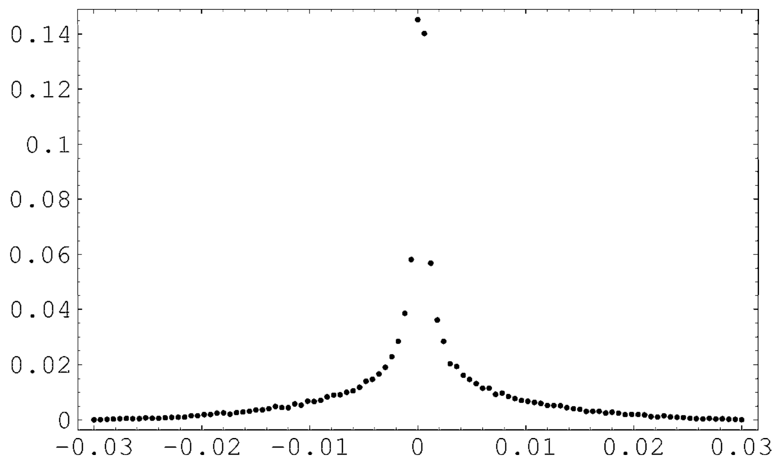
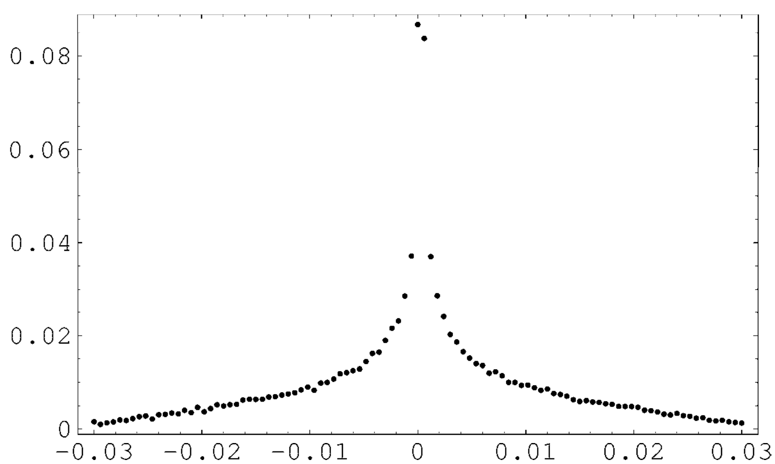


Figure 4.40 The frequency distribution of the third central moment associated with a beta (.5,.5) base measure. The mean is 0.0002.



Because the density plots for the third central moment occur over small intervals, the plots are not nearly as informative as the density plots for the skewness (Figures 4.41 through 4.45). These figures imply that the beta (10,1) measure generates distributions most likely to take on values for skewness that are closer than the distribution associated with the other cases. For the beta (10,1) case, 50% of the measures have skewness between -0.01 and 0.01, while 50% of the measures generated have by the beta (1,10) base measure have skewness between -1 and 1. For the uniform case, 50% of the measures generated have skewness from -0.46 to 0.45 while 50% of the measures generated by the beta (2,2) base measure have skewness on the between -0.22 and 0.26. For the beta (.5,.5) case, 50% of the measures generated have skewness between -0.7 and 0.7. Therefore, the beta (10,1) base measure is most likely to produce measures with symmetric shape. However, the beta (1,10) base measure is not only most likely to produce measures with positive skewness (tails to the right), but it is also most likely to produce measures with negative skewness (tails to the left). That is, it is most likely to produce a skewed distribution. This is because the beta (1,10) base measure generates distributions that are most likely to have the smallest SD. Since all the base measures generate distributions with very similar third central moments (see Figures 4.36 through 4.40), the relatively small SD in the beta (1,10) case makes a large skewness most likely.

Figure 4.41 The frequency distribution of the skewness associated with the uniform [0, 1] base measure. It has an average of 0.047.

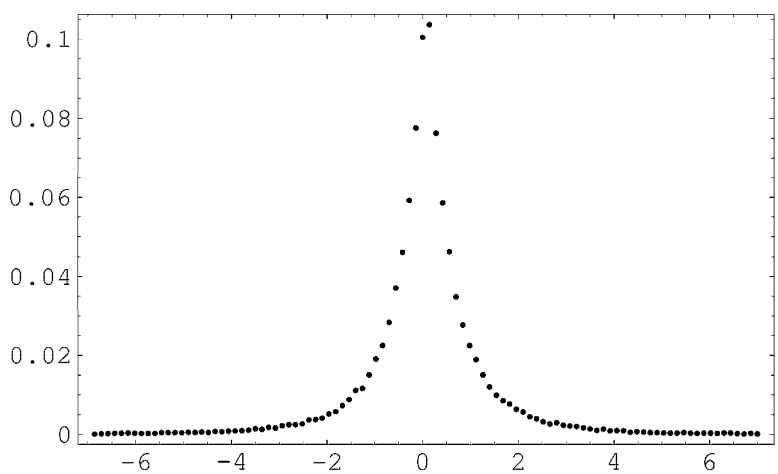


Figure 4.42 The frequency distribution of the skewness associated with a beta (2,2) base measure. The mean is 0.032.

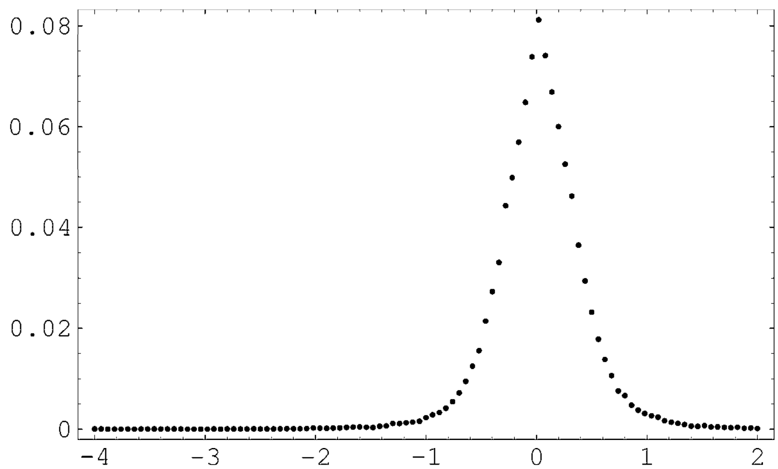


Figure 4.43 The frequency distribution of the skewness associated with a beta (10,1) base measure. The average of the distribution is 0.0008

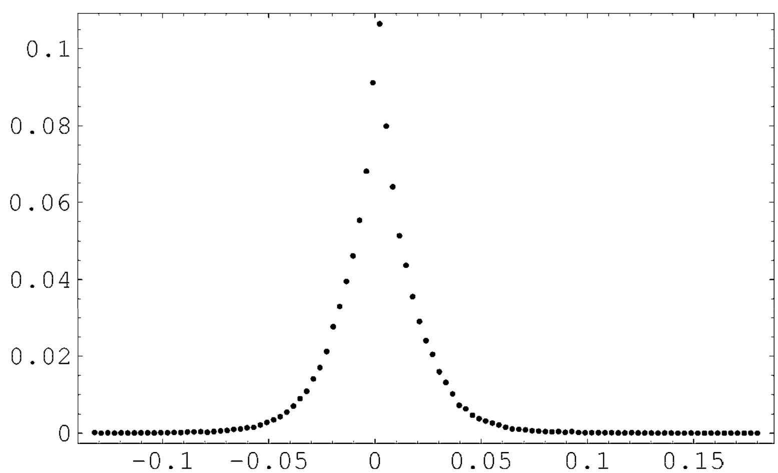


Figure 4.44 The distribution on the skewness associated with a beta (1,10) generating measure. The average of the distribution is 0.48.

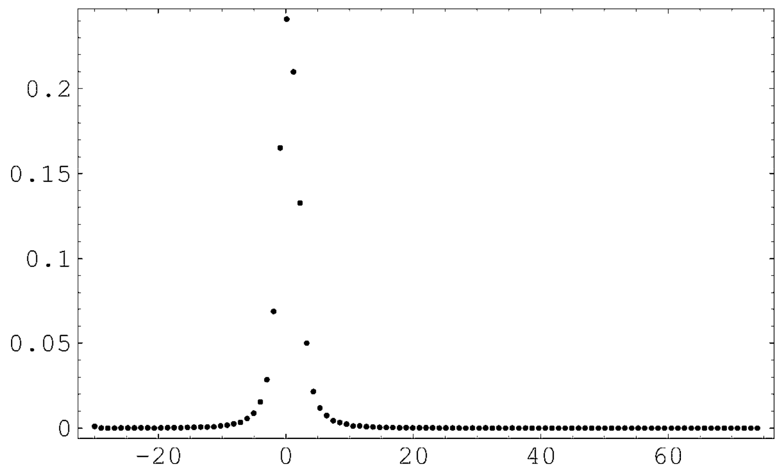
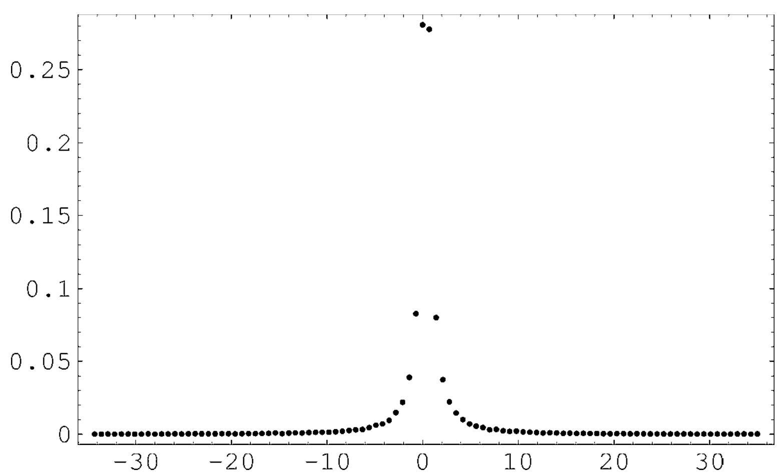


Figure 4.45 The frequency distribution of the skewness associated with a beta (.5,.5) base measure. The mean is 0.106.



4.6 The Fourth Moment

The distribution functions of the fourth moments associated with each base measure are given in Figures 4.46 through 4.50. For the beta (10,1) measure, 50% of the distributions have fourth moment between 0.417 and 0.460 while for the beta (.5,.5) measure, 50% of the distributions have fourth moment between 0.09 and 0.29. For the beta (1,10) case, 50% of the distributions have fourth moment between 0.071 and 0.123 while 50% of the distributions have fourth moment between 0.103 and 0.248 for the uniform case. For the beta (2,2) measure, 50% of the distributions have fourth moment between 0.241 and 0.382. Note that the intervals over which 50% of the distributions have fourth moment is about three times longer for the beta (10,1) case. The following distribution functions and density plots show that the beta (10,1) measure is most likely to produce measures with larger fourth moment than the other cases while the beta (1,10) measure is most likely to produce measures with small fourth moments. Note that the beta (.5,.5) measure is almost as likely to produce measures with small fourth moments as the beta (1,10) base measure.

Figure 4.46 The 99% confidence band for $G_4^{\mu_0,\mu}$ with $\delta^* = 0.0001$ with a uniform $[0,1]$ generating measure. The narrow band represents the bound given by Proposition 2.38. The upper strip represents the upper bound of the 99.5% confidence band around $G_4^{(9)^{\mu_0,\mu}}(y+0.0001)+R_{9,4}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_4^{(9)^{\mu_0,\mu}}(y-0.0001)-R_{9,4}^{\mu_0,\mu}(0.0001)$

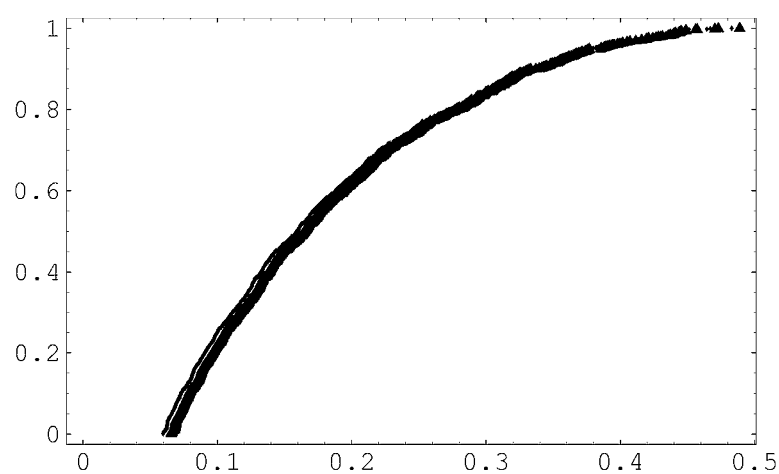


Table 4.46.1: The following are 99% confidence bands for selected values of the distribution function of the fourth moment associated with the uniform [0, 1] base measure.

	99% Confidence Interval
$G_4^{\mu_0,\mu}(.063)$	[0,0.002]
$G_4^{\mu_0,\mu}(.1)$	[0.231,0.233]
$G_4^{\mu_0,\mu}(.103)$	[0.247,0.25]
$G_4^{\mu_0,\mu}(.162)$	[0.497,0.5]
$G_4^{\mu_0,\mu}(.2)$	[0.626,0.628]
$G_4^{\mu_0,\mu}(.248)$	[0.746,0.75]
$G_4^{\mu_0,\mu}(.3)$	[0.841,0.843]
$G_4^{\mu_0,\mu}(.4)$	[0.962,0.965]

Figure 4.47 The 99% confidence band for $G_4^{\mu_0,\mu}$ with $\delta^* = 0.0002$ associated with a beta (2,2) generating measure. The upper strip represents the upper bound of the 99.5% confidence band around $G_4^{(9)^{\mu_0,\mu}}(y + 0.0002) + R_{9,4}^{\mu_0,\mu}(0.0002)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_4^{(9)^{\mu_0,\mu}}(y - 0.0002) - R_{9,4}^{\mu_0,\mu}(0.0002)$

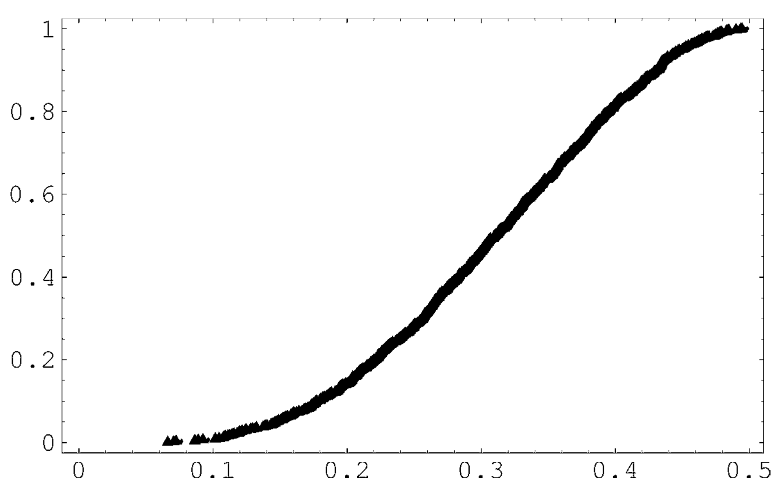


Table 4.47.1: The following are 99% confidence bands for selected values of the distribution function of the fourth moment associated with the beta (2,2) base measure.

	99% Confidence Interval
$G_4^{\mu_0,\mu}(.1)$	[0.007,0.008]
$G_4^{\mu_0,\mu}(.2)$	[0.1389,0.139]
$G_4^{\mu_0,\mu}(.241)$	[0.249,0.25]
$G_4^{\mu_0,\mu}(.3)$	[0.4549,0.455]
$G_4^{\mu_0,\mu}(.31)$	[0.498,0.5]
$G_4^{\mu_0,\mu}(.382)$	[0.748,0.75]
$G_4^{\mu_0,\mu}(.4)$	[0.8089,0.809]
$G_4^{\mu_0,\mu}(.5)$	[0.9959,0.996]

Figure 4.48 The 99% confidence band for $G_4^{\mu_0,\mu}$ with $\delta^* = 0.003$ associated with a beta (10,1) generating measure. The upper strip represents the upper bound of the 99.5% confidence band around $G_4^{(9)^{\mu_0,\mu}}(y + 0.003) + R_{9,4}^{\mu_0,\mu}(0.003)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_4^{(9)^{\mu_0,\mu}}(y - 0.003) - R_{9,4}^{\mu_0,\mu}(0.003)$

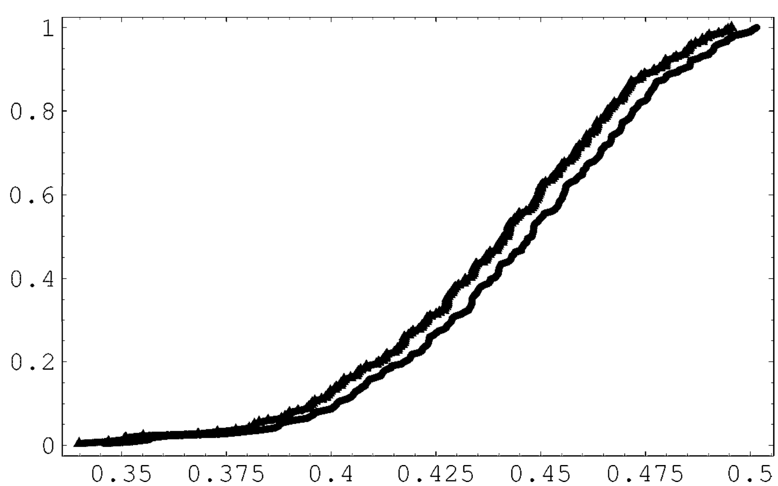


Table 4.48.1: The following are 99% confidence bands for selected values of the distribution function of the fourth moment associated with the beta (10,1) base measure.

	99% Confidence Interval
$G_4^{\mu_0,\mu}(.375)$	[0.028,0.032]
$G_4^{\mu_0,\mu}(.4)$	[0.083,0.129]
$G_4^{\mu_0,\mu}(.417)$	[0.20,0.25]
$G_4^{\mu_0,\mu}(.425)$	[0.26,0.31]
$G_4^{\mu_0,\mu}(.442)$	[0.44,0.5]
$G_4^{\mu_0,\mu}(.45)$	[0.54,0.61]
$G_4^{\mu_0,\mu}(.465)$	[0.69,0.75]
$G_4^{\mu_0,\mu}(.475)$	[0.82,0.89]
$G_4^{\mu_0,\mu}(.485)$	[0.894,0.926]

Figure 4.49 The 99% confidence band for $G_4^{\mu_0,\mu}$ with $\delta^* = 0.00009$ associated with a beta (1,10) generating measure. The upper strip represents the upper bound of the 99.5% confidence band around $G_4^{(9)^{\mu_0,\mu}}(y + 0.00009) + R_{9,4}^{\mu_0,\mu}(0.00009)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_4^{(9)^{\mu_0,\mu}}(y - 0.00009) - R_{9,4}^{\mu_0,\mu}(0.00009)$

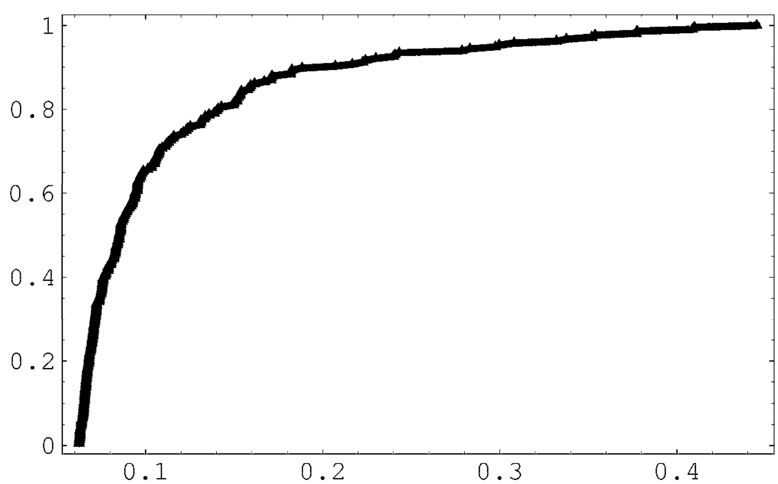


Table 4.49.1: The following are 99% confidence bands for selected values of the distribution function of the fourth moment associated with the beta (1,10) base measure.

	99% Confidence Interval
$G_4^{\mu_0,\mu}(.06)$	[0.0185,0.032]
$G_4^{\mu_0,\mu}(.07)$	[0.24,0.25]
$G_4^{\mu_0,\mu}(.08)$	[0.426,0.43]
$G_4^{\mu_0,\mu}(.085)$	[0.49,0.5]
$G_4^{\mu_0,\mu}(.1)$	[0.653,0.657]
$G_4^{\mu_0,\mu}(.123)$	[0.75,0.75]
$G_4^{\mu_0,\mu}(.2)$	[0.898,0.903]
$G_4^{\mu_0,\mu}(.4)$	[0.986,0.991]

Figure 4.50 The 99% confidence band for $G_4^{\mu_0,\mu}$ with $\delta^* = 0.0001$ associated with a beta (.5,.5) generating measure. The upper strip represents the upper bound of the 99.5% confidence band around $G_4^{(9)^{\mu_0,\mu}}(y + 0.0001) + R_{9,4}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $G_4^{(9)^{\mu_0,\mu}}(y - 0.0001) - R_{9,4}^{\mu_0,\mu}(0.0001)$

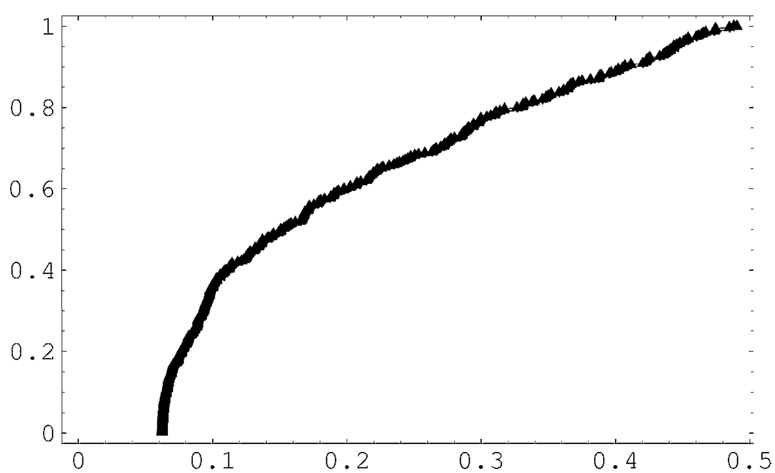


Table 4.50.1: The following are 99% confidence bands for selected values of the distribution function of the fourth moment associated with the beta (.5,.5) base measure.

	99% Confidence Interval
$G_4^{\mu_0,\mu}(.35)$	[0.03,0.014]
$G_4^{\mu_0,\mu}(.4)$	[0.055,0.159]
$G_4^{\mu_0,\mu}(.41)$	[0.10,0.25]
$G_4^{\mu_0,\mu}(.434)$	[0.3,0.5]
$G_4^{\mu_0,\mu}(.45)$	[0.45,0.69]
$G_4^{\mu_0,\mu}(.454)$	[0.49,0.75]
$G_4^{\mu_0,\mu}(.475)$	[0.77,0.92]

Figures 4.51 through 4.55 give the frequency plots of the fourth moment for each of the different base measures used in the experiments. The beta (10,1) case has a mean is 0.44 while the distribution of the fouth moment in the beta (.5,.5) case has a mean of 0.20. The mean of the distribution of the fourth moment in the beta (1,10) case has a mean of 0.11. The distribution of the fourth moment in the uniform case has a mean of 0.19 while the beta (2,2) case has a mean of 0.31.

Figure 4.51 The frequency distribution of the fourth moment associated with the uniform $[0, 1]$ base measure. The average is 0.19.

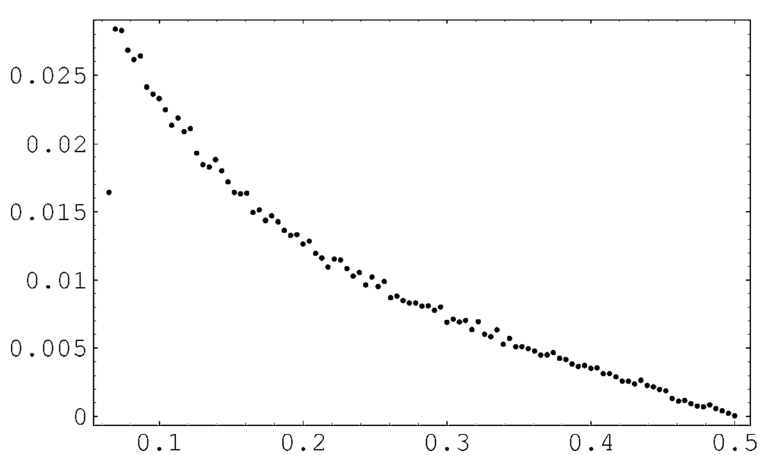


Figure 4.52 The frequency distribution of the fourth moment associated with the beta (2,2) base measure. The mean is 0.31.

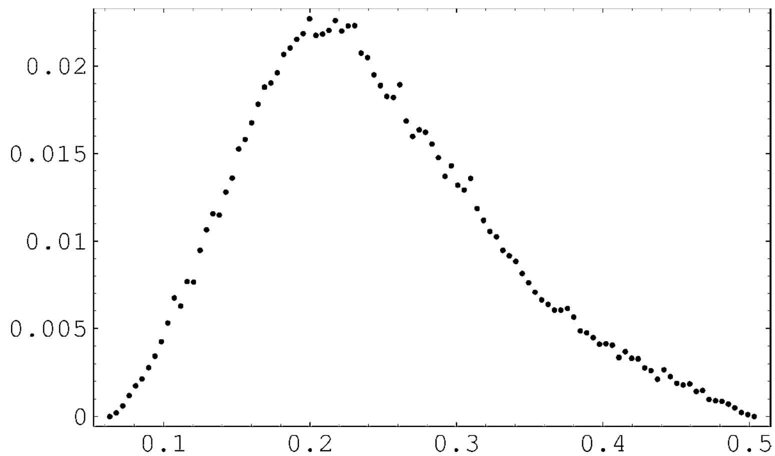


Figure 4.53 The frequency distribution of the fourth moment associated with the beta (10,1) generating measure. The mean of the distribution is 0.44.

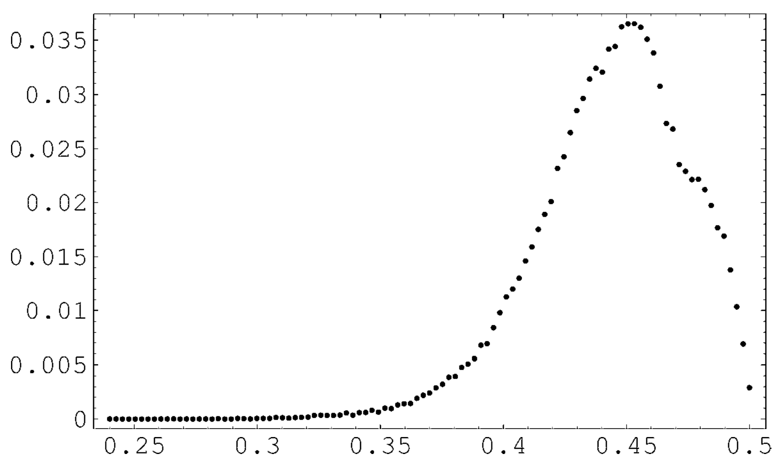


Figure 4.54 The frequency distribution of the fourth moment associated with the beta (1,10) base measure. The mean of the distribution is 0.11.

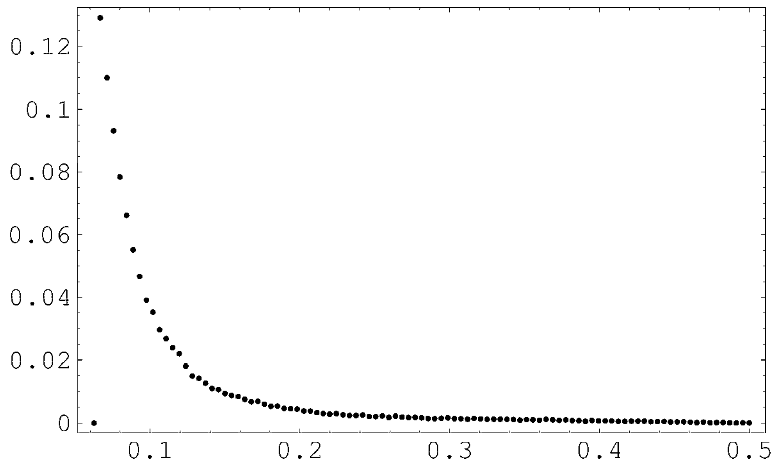
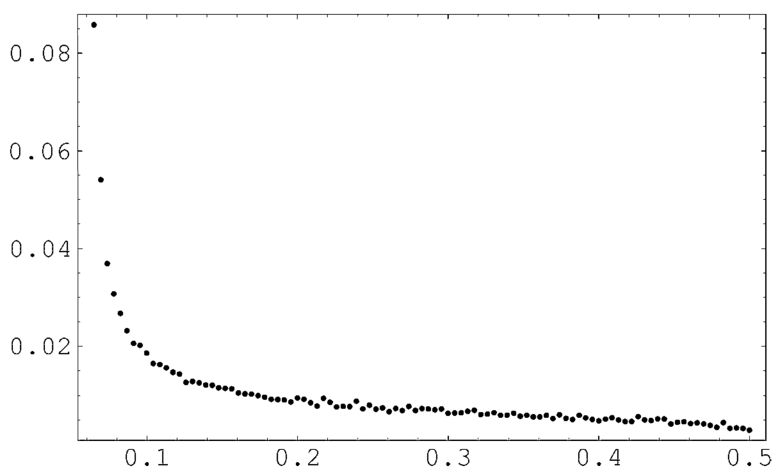


Figure 4.55 The frequency distribution of the fourth moment associated with the beta (.5,.5) base measure. The average is 0.20.



4.7 The Fourth Central Moment and Kurtosis

Figures 4.56 through 4.60 give the confidence bands about the distribution function of the fourth central moment associated with each of the base measures involved in the experiments. Figures 4.61 through 4.65 give the density plots. Figures 4.66 through 4.70 give the density plots of the distribution of the kurtosis associated with each of the five base measures used in the experiments. The analysis of the fourth central moment and kurtosis associated with each base measure is included for the sake of completeness. However, little can be inferred from these summary measures about the shape of measures generated by the different base measures. The summary statistics for the fourth central moment associated with the five base measures will be compared in order to note which measure or measures generate the largest and smallest fourth central moment.

The analysis of kurtosis does not provide enough information to infer characteristics about the shape of a measure associated with any particular base measure. This is because kurtosis is a summary of the extent to which the peak of a unimodal probability distribution departs from the shape of the normal distribution [13]. Since $\mu(\{1/2\}) \neq 0$, then almost all distribution functions generated by the construction are stricly singular (Hill and Monticino [19]). In particular, almost all distributions are multimodal and kurtosis values are not informative for multimodal distributions. Also, the kurtosis does not seem to give a good indication of the shape of a distribution over $[0, 1]$ and in fact it can even be misleading. Kendall, Stuart and Ord [27] suggest that for many symmetrical or asymmetrical distributions, the terms flat and peaked are better used to describe the sign of the α_4 rather than the shape of the distribution.

The beta (10,1) base is more likely to generate measures with larger fourth central moments. For the beta (10,1) base measure, 50% of the distributions have fourth central moment between 0.05 and 0.057 while 50% of the distributions associated with the beta (1,10) base measure have fourth central moment between 0.0019 and 0.0051. For the uniform measure, 50% of the distributions have fourth central moment between 0.0033 and 0.019 while for the beta (2,2) base measure, 50% of the distributions have fourth central moment between 0.012 and 0.024. For the beta (.5,.5) base measure, 50% of the distributions have fourth central moment between 0.002 and 0.026. Thus, from the distribution functions and the density plots for the fourth central moments, the beta (10,1) measure is most likely to generate measures with largest fourth central moments, while the beta (1,10) measure is most likely to produce measures with smallest fourth central moment; although the difference between smallest and largest is small. Note that the uniform measure is also likely to produce measures with small fourth central moment.

For the beta (10,1) base measure, 50% of the distributions have kurtosis between -1.92 and -1.82 with the median kurtosis at -1.9. For the beta (1,10) base measure, 50% of the distributions generated have kurtosis between -0.29 and 8.71, with the median at 2.01. For the uniform case, 50% of the distributions have kurtosis between -1.44 and 2.41 with the median at -0.5. In the beta (2,2) case, 50% of the generated measures have kurtosis between -1.5 and -0.59 with the median at -1.2. 50% of the distributions have kurtosis between -1.98 and 8.5 in the beta (.5,.5) case with a median of 4.5. Hence, the beta (1,10) base measure is most likely to produce measures that are peaked while the beta (10,1) base measure is most likely to produce measures that are flat.

Figure 4.56 The 99% confidence band for the fourth central moment, $C_4^{\mu_0,\mu}$, with a uniform $[0,1]$ generating measure and $\delta^* = 0.012$. The upper band represents the upper bound of the 99.5% confidence band around $C_4^{(9)^{\mu_0,\mu}}(y+0.012) + RS_{9,4}^{\mu_0,\mu}(0.012)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_4^{(9)^{\mu_0,\mu}}(y-0.012) - RS_{9,4}^{\mu_0,\mu}(0.012)$.

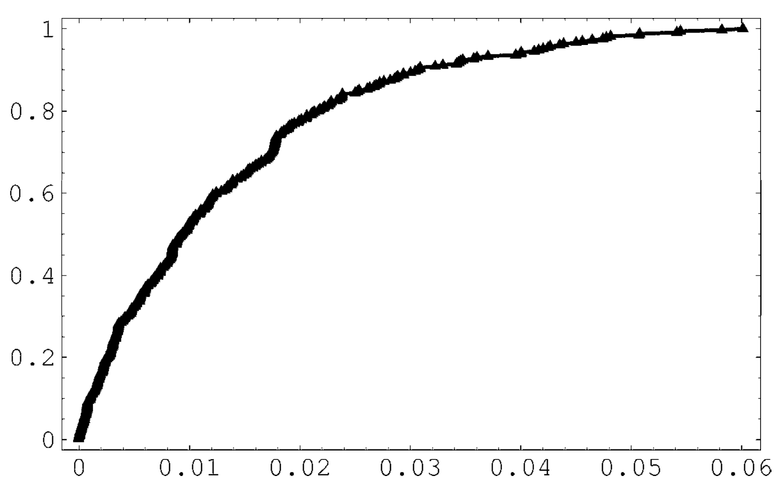


Table 4.56.1: The following are 99% confidence bands for selected values of the distribution function of the fourth central moment associated with the uniform [0, 1] base measure.

	99% Confidence Interval
$C_4^{\mu_0,\mu}(.002)$	[0.0021,0.027]
$C_4^{\mu_0,\mu}(.0033)$	[0.242,0.252]
$C_4^{\mu_0,\mu}(.0094)$	[0.491,0.5]
$C_4^{\mu_0,\mu}(.01)$	[0.518,0.524]
$C_4^{\mu_0,\mu}(.019)$	[0.749,0.752]
$C_4^{\mu_0,\mu}(.03)$	[0.890,0.893]
$C_4^{\mu_0,\mu}(.05)$	[0.981,0.984]

Figure 4.57 The 99% confidence band for the fourth central moment, $C_4^{\mu_0,\mu}$, with a beta (2,2) generating measure and $\delta^* = 0.00005$. The upper band represents the upper bound of the 99.5% confidence band around $C_4^{(9)^{\mu_0,\mu}}(y + 0.00005) + RS_{9,4}^{\mu_0,\mu}(0.00005)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_4^{(9)^{\mu_0,\mu}}(y - 0.00005) - RS_{9,4}^{\mu_0,\mu}(0.00005)$.

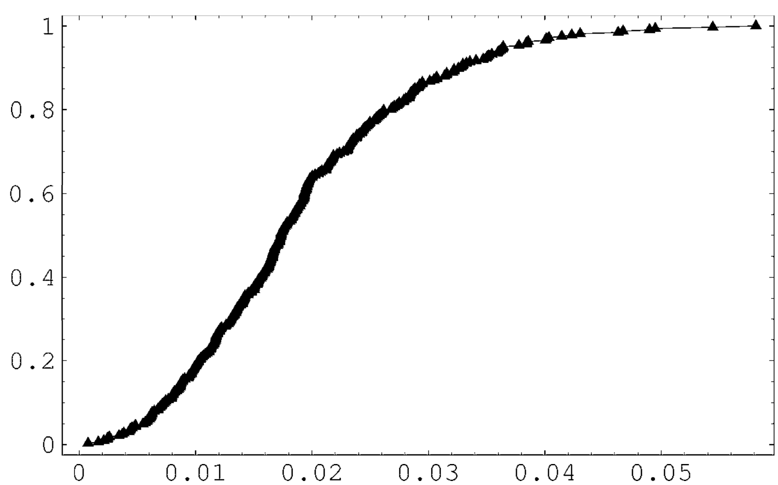


Table 4.57.1: The following are 99% confidence bands for selected values of the distribution function of the fourth central moment associated with the beta (2,2) base measure.

	99% Confidence Interval
$C_4^{\mu_0,\mu}(.01)$	[0.178,0.184]
$C_4^{\mu_0,\mu}(.012)$	[0.248,0.25]
$C_4^{\mu_0,\mu}(.0174)$	[0.491,0.5]
$C_4^{\mu_0,\mu}(.02)$	[0.631,0.638]
$C_4^{\mu_0,\mu}(.024)$	[0.745,0.75]
$C_4^{\mu_0,\mu}(.03)$	[0.859,0.865]
$C_4^{\mu_0,\mu}(.05)$	[0.991,0.993]

Figure 4.58 The 99% confidence band for the fourth central moment, $C_4^{\mu_0,\mu}$, with a beta (10,1) generating measure and $\delta^* = 0.00002$. The upper band represents the upper bound of the 99.5% confidence band around $C_4^{(9)^{\mu_0,\mu}}(y + 0.00002) + RS_{9,4}^{\mu_0,\mu}(0.00002)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_4^{(9)^{\mu_0,\mu}}(y - 0.00002) - RS_{9,4}^{\mu_0,\mu}(0.00002)$.

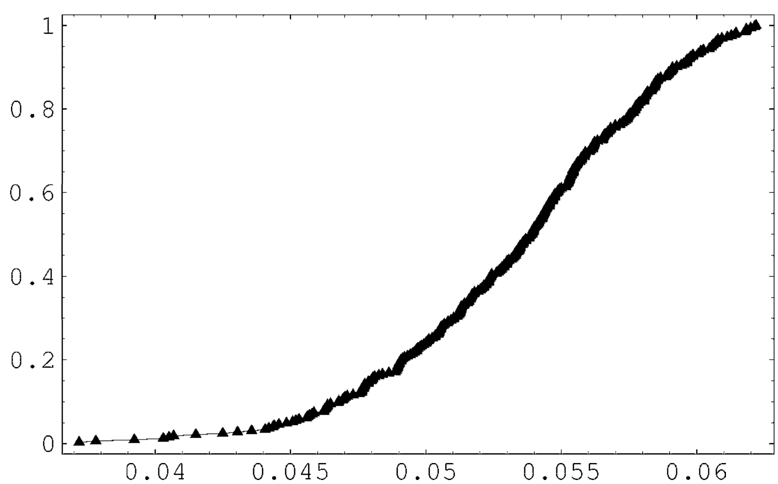


Table 4.58.1: The following are 99% confidence bands for selected values of the distribution function of the fourth central moment associated with the beta (10,1) base measure.

	99% Confidence Interval
$C_4^{\mu_0,\mu}(.045)$	[0.049,0.049]
$C_4^{\mu_0,\mu}(.05)$	[0.248,0.252]
$C_4^{\mu_0,\mu}(.054)$	[0.497,0.5]
$C_4^{\mu_0,\mu}(.055)$	[0.607,0.61]
$C_4^{\mu_0,\mu}(.057)$	[0.745,0.752]
$C_4^{\mu_0,\mu}(.058)$	[0.816,0.819]
$C_4^{\mu_0,\mu}(.06)$	[0.929,0.932]

Figure 4.59 The 99% confidence band for the fourth central moment, $C_4^{\mu_0,\mu}$, with a beta (1,10) generating measure and $\delta^* = 0.0001$. The upper band represents the upper bound of the 99.5% confidence band around $C_4^{(9)^{\mu_0,\mu}}(y + 0.0001) + RS_{9,4}^{\mu_0,\mu}(0.0001)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_4^{(9)^{\mu_0,\mu}}(y - 0.0001) - RS_{9,4}^{\mu_0,\mu}(0.0001)$.

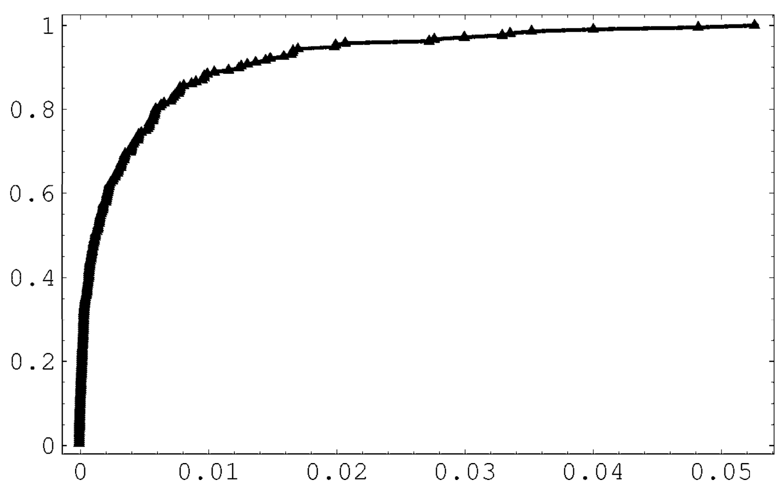


Table 4.59.1: The following are 99% confidence bands for selected values of the distribution function of the fourth central moment associated with the beta (1,10) base measure.

	99% Confidence Interval
$C_4^{\mu_0,\mu}(.0001)$	[0.004,0.18]
$C_4^{\mu_0,\mu}(.0019)$	[0.12,0.25]
$C_4^{\mu_0,\mu}(.005)$	[0.32,0.36]
$C_4^{\mu_0,\mu}(.0012)$	[0.468,0.5]
$C_4^{\mu_0,\mu}(.0051)$	[0.745,0.75]
$C_4^{\mu_0,\mu}(.01)$	[0.879,0.88]
$C_4^{\mu_0,\mu}(.02)$	[0.94,0.954]
$C_4^{\mu_0,\mu}(.04)$	[0.986,0.99]

Figure 4.60 The 99% confidence band for the fourth central moment, $C_4^{\mu_0,\mu}$, with a beta (.5,.5) generating measure and $\delta^* = 0.0002$. The upper band represents the upper bound of the 99.5% confidence band around $C_4^{(9)^{\mu_0,\mu}}(y + 0.0002) + RS_{9,4}^{\mu_0,\mu}(0.0002)$ and the lower band represents the lower bound of the 99.5% confidence band for $C_4^{(9)^{\mu_0,\mu}}(y - 0.0002) - RS_{9,4}^{\mu_0,\mu}(0.0002)$.

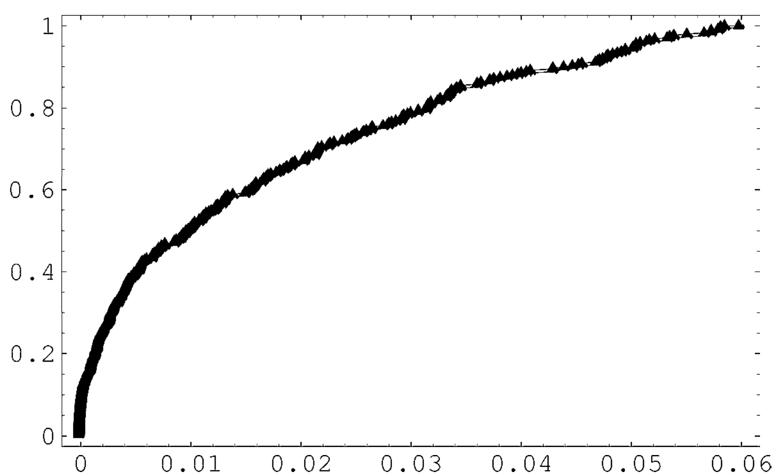


Table 4.60.1: The following are 99% confidence band for selected values of the distribution function of the fourth central moment associated with the beta (.5,.5) base measure.

	99% Confidence Interval
$C_4^{\mu_0,\mu}(.002)$	[0.23,0.25]
$C_4^{\mu_0,\mu}(.01)$	[0.48,0.5]
$C_4^{\mu_0,\mu}(.026)$	[0.74,0.75]
$C_4^{\mu_0,\mu}(.044)$	[0.89,0.90]

Figures 4.61 through 4.65 give the density plot of the fourth central moment for for each of the five base measures used in the experiments. The average of each of the distributions is as follows: 0.053 is the mean for the beta (10,1) case, 0.0042 for the beta (1,10) case, 0.014 for the uniform case, 0.020 for the beta (2,2) case, and 0.016 is the average for the beta (.5,.5) case.

Figure 4.61 The frequency distribution of the fourth central moment associated with the uniform $[0, 1]$ base measure. It has an average of 0.00012.

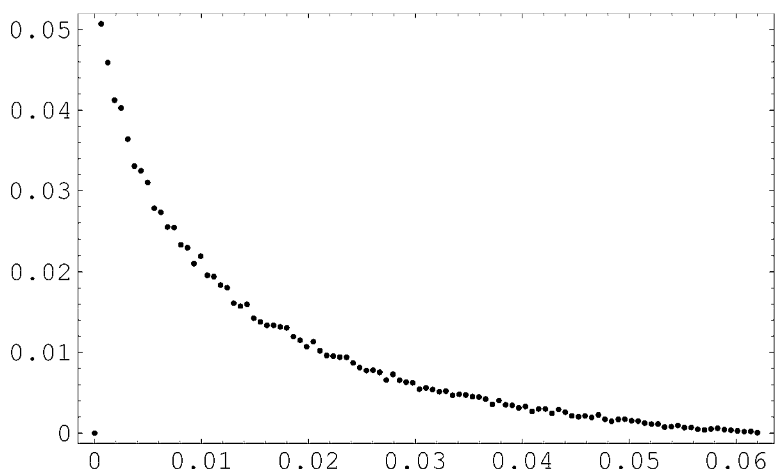


Figure 4.62 The frequency distribution of the fourth central moment associated with a beta (2,2) base measure. The mean is 0.020.

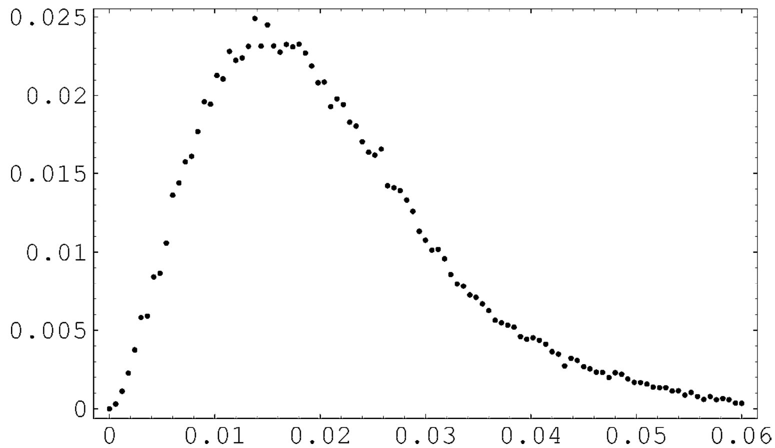


Figure 4.63 The frequency distribution of the fourth central moment associated with a beta (10,1) base measure. The average of the distribution is 0.053.

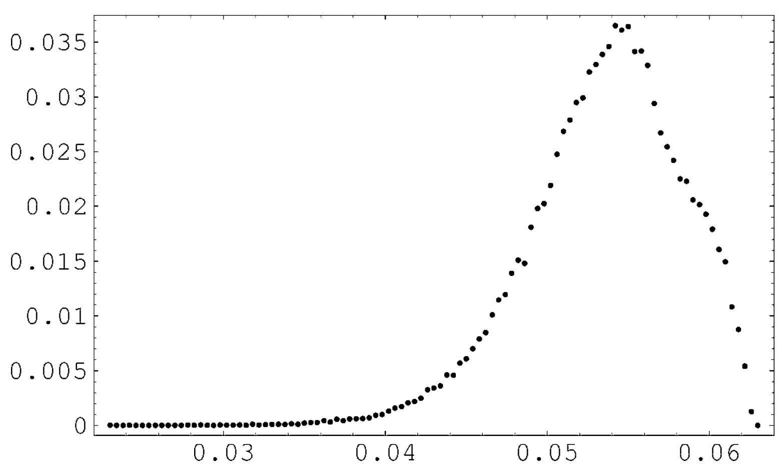


Figure 4.64 The distribution on the fourth central moment associated with a beta (1,10) generating measure. The average of the distribution is 0.0042.

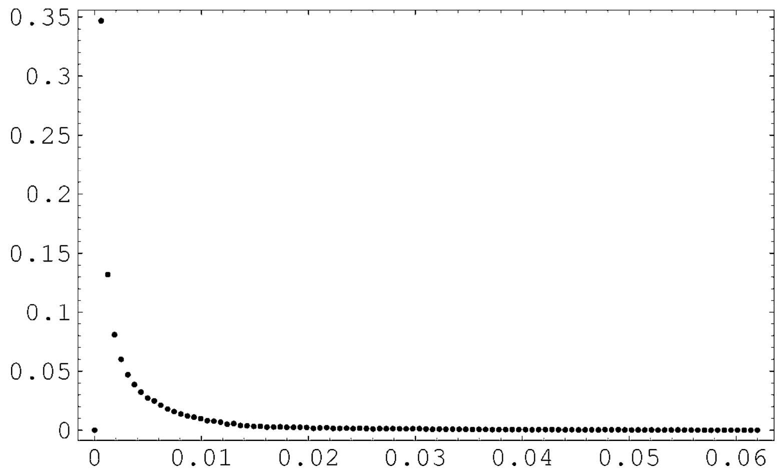
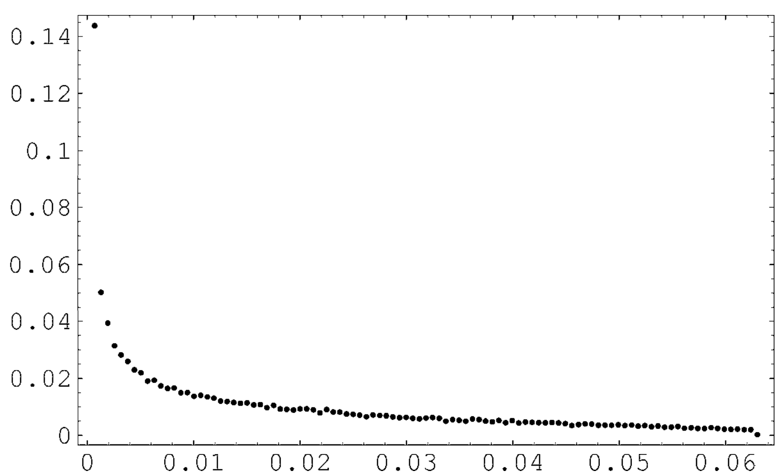


Figure 4.65 The frequency distribution of the fourth central moment associated with a beta (.5,.5) base measure. The mean is 0.016.



Figures 4.66 through 4.70 give the density plots of the distribution of the kurtosis assoicated with each of the five base measures used in the experiments.

Figure 4.66 The frequency distribution of the kurtosis associated with the uniform [0, 1] base measure. The average is 4.8.

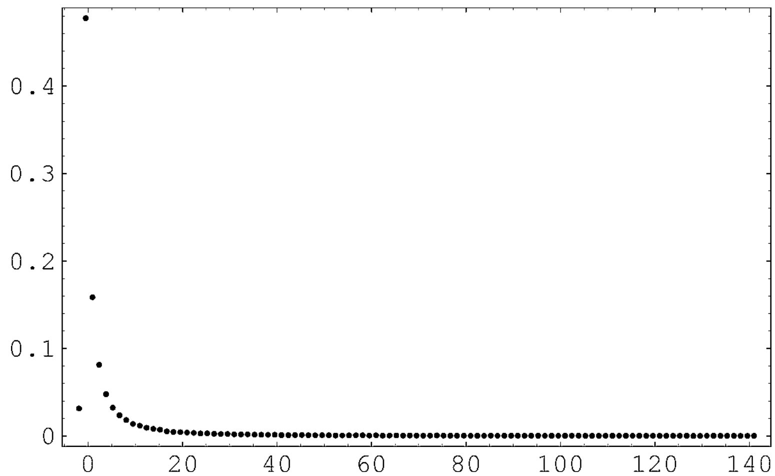


Figure 4.67 The frequency distribution of the kurtosis associated with the beta (2,2) base measure. The mean is -0.76

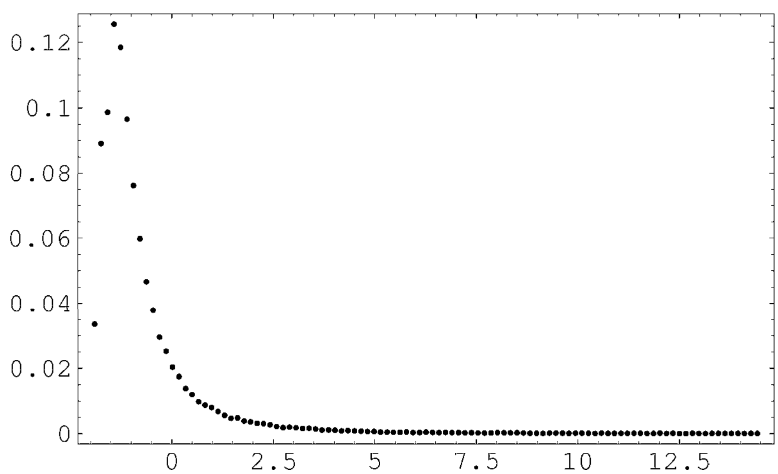


Figure 4.68 The frequency distribution of the kurtosis associated with the beta (10,1) generating measure. The mean of the distribution is -1.87.

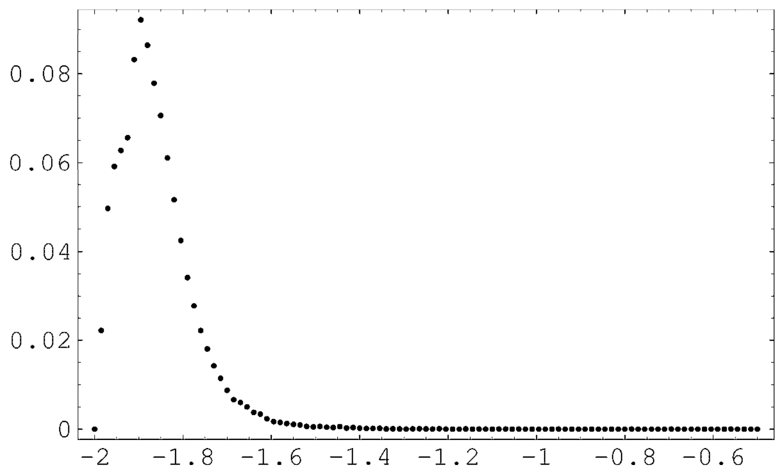


Figure 4.69 The frequency distribution of the kurtosis associated with the beta (1,10) base measure. The mean of the distribution is 10.5.

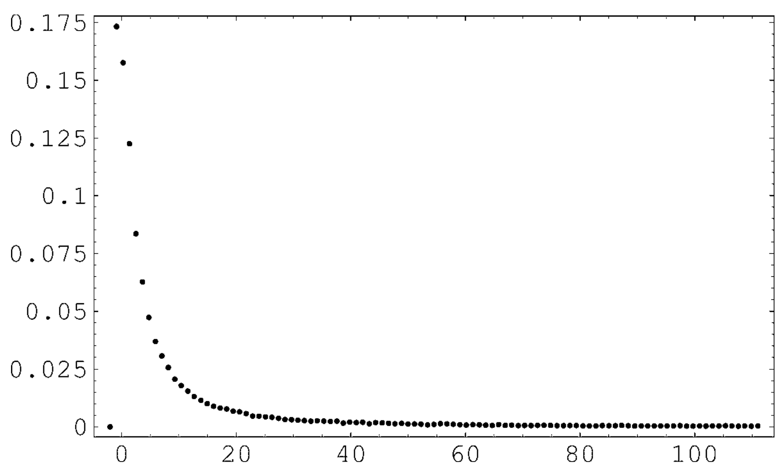
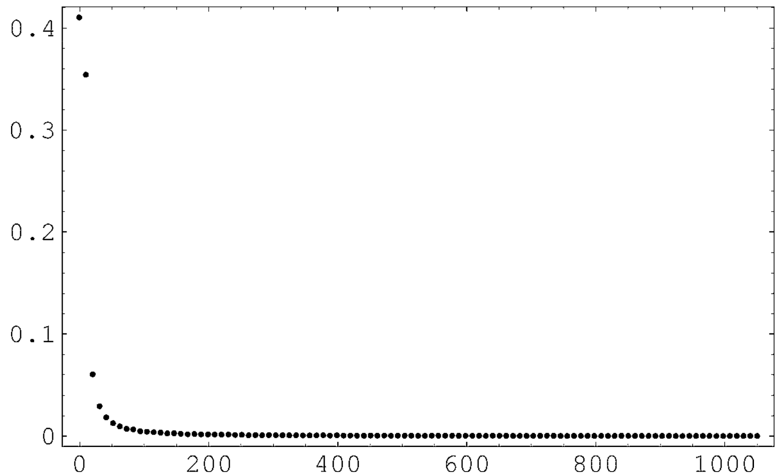


Figure 4.70 The frequency distribution of the kurtosis associated with the beta (.5,.5) base measure. The average is 28.



4.8 Conclusions And Open Questions

From the generated measures examined:

- SBA rpms are most likely to produce measures with small SD if the generating measure concentrates its mass near 0.

- SBA rpms are most likely to produce measures with large SD if the generating measure concentrates its mass near 1.

- SBA rpms that concentrate their mass near 1 are also most likely to produce measures with large second, third and fourth moments.

This is because in the construction given on pages 7 - 8, the child barycenter is pushed out away from the parent barycenter (such as with the balayage random variable). From the simulations, the beta (10,1) base measure produced measures with large SD, and second through fourth moments.

Of the base measures examined, the beta (1,10) measure most likely produced measures with small SD. This is because child barycenters are not pushed out very far from parent barycenters in the construction. The beta (1,10) base measure also is more likely to produce measures with small second, third and fourth moments.

Measures that concentrate their mass near 1 are also most likely to produce measures with symmetric shape. The experiments showed that beta (10,1) base measure is more likely to produce measures with third central moment and skewness near zero. The beta (1,10), beta (2,2), beta (.5,.5), and uniform base measures each are also likely to produce measures that are symmetric. However, since there is a likelihood that each base measure will produce base measures that are nonsymmetric, it is difficult to determine which base measures should be used to generated measures that are positively or negatively skewed.

Measures that concentrate their mass near zero should be used to generate measures that are peaked ($\alpha_4 > 0$). The beta (1,10) base measure generated base measures that were most likely to have kurtosis be not only be greater than zero, but to also be quite large. Measures that concentrate their mass near one should be used to generate measures that generate flat measures ($\alpha_4 < 0$). The beta (10,1) base measures is most likely to produce measures with negative values for kurtosis.

Some questions remain such as:

- Why are the confidence band widths for beta (10,1) wide for the moments, but relatively thin for the SD and central moments?
- What is the correlation between the SD and the skewness and kurtosis of the generated measures and how does symmetry of the generated measures affect the correlation?
- What is the correlation between moments of the generated measures?

Further analysis directed to answer these open questions will help provide a complete understanding of measures in the support of $B_{\mu_0,\mu}$.

Appendix A

A Discussion Of The Simulation Procedure

The simulation procedure discussed in Chapter 4 is discussed in detail in this appendix. The first step in the simulation is to construct an n–level SBA generated from a prescribed distribution (any of the five base measures – uniform [0,1], beta (2,2), beta (10,1), beta (1,10) or beta (.5,.5)). Let $\{r_i\}$, $0 \leq i \leq 2^n - 2$, denote an array of random numbers generated according to one of the five base measures used for μ. The n-level SBA is constructed as follows: The first element is 0 and the last element is 1. The mean is 1/2. That is, μ_0 is such that $\mu_0\{1/2\} = 1$. The next element to the left in the array is found as follows. The left conditional barycenter is $m_{2,1} = m_{1,1} - s_1$ and $m_{2,3} = m_{1,1} + s_2$, where $s_1 = r_1$ according to μ scaled to $[0, m_1, 1]$ and $s_2 = r_2$ according to μ scaled to $[m_1, 1, 1]$. This process is continued down to the nth level. Applying the definition of the nth level approximation (Definition 2.13) to the nth level SBA, an nth level approximation to the distribution function is generated.

The second, third and fourth moments of the approximation are then calculated along with the the bound on the Prohorov distance between the true random variable and the nth level SBA approximation given in Theorem 2.28. A representation of $R_{n,k}^{\mu_0,\mu}$, for $k = 2, 3, 4$ is also calculated, using Proposition 2.37. The above construction is repeated 65,000 times for each base measure to obtain empirical estimates for

$G_k^{(n)^{\mu_0,\mu}}$, and $C_k^{(n)^{\mu_0,\mu}}$, a distribution on the bound on $F_n^{\mu_0,\mu}$ (given in Lemma 2.32), and an empirical estimate for $R_{n,k}^{\mu_0,\mu}$. Appendix B gives the annotated source code used in the simulation procedure.

Appendix B

Annotated Source Code

This appendix contains selected annotated C++ source code of algorithms used in the simulation of the distribution functions of the moments, standard deviation, and central moments associated with each of the generating measures beta (2,2), beta (10,1), beta (1,10), beta (.5,.5) and uniform [0,1]. Included in the algorithms are the construction of the barycenter array, the construction of the moments, and the nth level SBA approximation.

In the DistGen class, the baryarray method generates the array of barycenters used in MomentsMain.cc. The baryarray method takes the array of random numbers and scales them to generate a Sequential Barycenter Array. The function values are for graphing later on and the weights are for calculating moments.

DistGen.h

```
#ifndef DistGenH
#define DistGenH

class DistGen()
{

private:
//Data members
long double* baryarr, *barylist;
long int numpts;
```

```
int _depth   ;
long double* DF_vals ;
long double* dist_vals;

public: //Methods
//Default constructor
DistGen();

//Destructor
~DistGen();

//Constructor
DistGen(long double* ba,int depth );

// Calculates the $n$th level barycenters.
long double* baryarray();

//Calculate the values of the distribution of the $n$th level SBA.
void ClcDistVal ();

// Produces a bound on the distribution function
// of the moments, central moments, etc.
long double* momentBound();

}  ;  //end of DistGen Class
#endif
```

DistGen.cc

```
#include "DistGen.h"

// Constructor
DistGen::DistGen(long double* ba,int depth )
{
  barylist = ba;
  numpts=(long int) pow (2.,int(depth))+1;
  _depth = depth ;
```

```
} ;

// Destructor
DistGen::~DistGen (){delete[] baryarr;
            delete[] DF_vals;
            };

long double* DistGen::baryarray()
{
  int wtpts= (numpts-1)/2 +1;
  long double min,temp ;
  int minind;
  baryarr = new long double[numpts];
  DF_vals = new long double[wtpts];
  baryarr[0] =  *barylist;
  baryarr[1] = *(barylist+1);
  baryarr[2]=1.0;
  DF_vals[0]=0;
  DF_vals[2]=1;
  long int mark=1;
  long int place=2;
  long int itv=0;
  long int bpt =3;
  int d=1;

  long double numleft=0.0;
  long double numright =0.0;

  //Start iterative algorithm to calculate SBA and the distribution
  for (int ptr=2;ptr<= _depth; ptr++)
   {
    //Sets up an array of scales at each level

    long int numer=(long int) pow(2.,int(ptr-1));
    long double*scalearr = new long double[numer];
    for (int i=0;i<numer;i++)
    scalearr[i]=baryarr[i+1]-baryarr[i];
```

```
mark=1;
itv=0;
d=1;
//Generates each parent's child and the distribution value of each parent
for (int ctr=1;ctr<=(numer/2);ctr++)
{
 numleft =baryarr[mark]-(*(barylist+place)*scalearr[itv]);
 numright=baryarr[mark]+(*(barylist+place+1)*scalearr[itv+1]);
 place=place+2;
 itv=itv+2;
 baryarr[bpt]=numleft;
 baryarr[bpt+1]=numright;

 // Perform a check to prevent dividing by 0.
 long double denom = numright - numleft;
 if(denom == 0)
 {
   denom = 0.0000001;
 }

 DF_vals[d] = DF_vals[d-1]+
            (DF_vals[d+1]-DF_vals[d-1])*(numright-baryarr[mark])/denom;

 mark=mark+2;
 bpt=bpt+2;
 d=d+2;
}

//Here, the weights are renamed to set the next level of weights
if (ptr< _depth){
for (int i=numer;i>=0;i--)
 DF_vals[2*i]=DF_vals[i];


//The existing part of the SBA is sorted with a selection sort...
for (int i=0; i<place+1;++i)
{
```

```
      min = baryarr[i];
      minind = i;
      for (int j=i+1;j<place+1;++j)
      if (baryarr[j] < min)
      {
         min = baryarr[j];
         minind = j;
      }

      //...perform the switch...

      if(min < baryarr[i])
      {
         temp = baryarr[i];
         baryarr[i]= baryarr[minind];
         baryarr[minind] = temp;
      }

     }
     delete[] scalearr;
   }  //...go back to the beginning of the iterative calculation of SBA and distribution
   return (baryarr);
} ;  //end of baryarray function.
```

The method calcDistVal finds the distribution (density) values and it is used in momentBound, the class member that uses the local algorithm to place bounds on the moments.

```
void DistGen::calcDistVal ()
{
 long int Bdepth = pow (2.0,int(_depth -1))+1;
 dist_vals = new long double[Bdepth];
 dist_vals[0]=0;
 for (int i=1;i<Bdepth;i++)
 dist_vals[i]=DF_vals[i]-DF_vals[i-1];
 return ;
}
```

The momentBound method involves finding error in distance between kth moments of rv X and its nth level SBA approximation over barycenter subintervals formed by the construction.

```
long double* DistGen::momentBound( )
{
 long int Epoints=powl(2.0, int(_depth-1));
 int k =2;
 long double* Diff_Moms = new long double [3];
 long double diff_moms=0.0;
 for (int i=0;i<3;i++){
 diff_moms = dist_vals[1]*fabsl((powl(baryarr[2],int(k))*(baryarr[1]/baryarr[2])
                   -powl(baryarr[1],int(k))));

 for (int j=1; j< Epoints; j++)
 {
  long double denom =  baryarr[2*j+2]-baryarr[2*j];
  // Perform another check to make sure compiler is not dividing by 0.
  if( denom == 0 )
  {
     denom = 0.0000001;
  }

  long double quot=(baryarr[2*j+2]-baryarr[2*j+1])/denom;
  long double int_case_l=(powl(baryarr[2*j],int(k))*(quot));
  long double int_case_r=(powl(baryarr[2*j+2],int(k))*(1-quot));
  long double int_case=dist_vals[j]*
                 fabsl(int_case_l+int_case_r- powl(baryarr[2*j+1],int(k)));
  diff_moms = int_case + diff_moms ;
  } //end of inner for loop
  Diff_Moms[i] = diff_moms;
  //diff_moms = 0.0;
  k++;
 }  //end of outer for loop
 return Diff_Moms;
}

...
```

```
}//end of DistGen class
```

The next class is the RandNum class which contains the random number generators used in the simulations. The uniform generators ran0, ran1 and oldran1 follow very closely to the uniform generators found in Chapter 7 of [28]. The beta random number generator follows from a psuedo algorithm given in [8].

RandNum.h

```
#ifndef RandNumH
#define RandNumH
#include "RandNum.h"

class RandNum ()
{
private:
long int * idum;

protected:
// RandNum cannot be instantiated but is accessed as a singleton.
// It is not abstract.
RandNum();

public:
float alpha;
float beta;

//Class destructor
~RandNum();

//The two random number generators that follow are from adapted Numerical Recipes
//in C, by Press, et. al.

//ran0 has a much smaller period than ran1
float ran0 (long * idum);

//Returns a uniform random deviate between 0.0 and
//1.0.  Set idum to any negative value to initialize
```

```
//or reinitialize the sequence.
static long double ran1(long& idum);

static long double beta_rand(
  float& alpha, float& beta,long double& uni_1, long double& uni_2);

//Old random number generator.
//Returns a uniform random deviate between 0.0 and
//1.0.  Set idum to any negative value to initialize
//or reinitialize the sequence
float oldran1(int idum);

 }; //end of RandNum class
```

Randnum.cc

```
#include "RandNum.h"

//default constructor
RandNum::RandNum(){};

//destructor
RandNum::~RandNum(){};

...

long double RandNum::beta_rand(float& alph, float& bet, long double & uni_1,
                                long double& uni_2)
{
 float alpha( alph );
 float beta( bet );

 long double exp1= (long double)1/alpha;
 long double exp2=(long double) 1/beta;
 long double denom= min(powl(uni_1,exp1)+powl(uni_2,exp2),(long double)1.0) ;
 long double numerator =  powl(uni_1,exp1);
 long double betaRand =  numerator/denom;
```

```
 return betaRand;
}
```

The MomentsMain.cc class contains methods that calculate the moments of the nth level SBA approximation. The generation of the unscaled random array is also implemented here. These methods are included in this source code but could be methods in separate classes to follow good object oriented programming practice.

MomentsMain.cc

```
#include <math.h>
#include "Moments.h"
#include "RandNum.h"
#include "DistGen.h"

main()
{
  //This first part converts the user's string entries to integer values.
  float alpha = StrToFloat(alpha_val->Text);
  float beta = StrToFloat(beta_val->Text);
  long seed = StrToInt(Seed->Text);
  float M = StrToFloat(Mean->Text);
  float Lev = StrToFloat(Levels->Text);
  int K = StrToInt(Monte->Text);

  //This is necessary to scale the window and set the origin for the display.
  long int num = (long int) pow(2.,int(Lev));
  float r=0.0;
  if(M==1) r =  RandNum::ran1(seed);
  else r=M;

  long double EV=0.0;
  long int numwts = num / 2;

  //This big loop is the Monte Carlo Simulation to estimate an empirical
  //distribution of the bound on the moments of a random variable generated
  //by the SBA process.
  for (int MCPnts=1; MCPnts<= K;++MCPnts){
```

```
  long double* randarray = new long double[num+1];

  //Next we initialize randarray.
  randarray[0]=0;
  randarray[1]=r;
  randarray[num]=1;

  //If r=0, then the array of unscaled points is generated at random.
  if (r == 0)
   {
    r=.5;
    for (int i=1; i<=num-1;i++)
      randarray[i]= RandNum::ran1(seed);
   }

  // Put result of Pull down here to toggle between beta and uni.
  if(alpha == 1 && beta == 1)
   {
    for(int i=2; i<=num-1;i++)
      {
       randarray[i]= RandNum::ran1(seed);
      }
   }
  else
    {
     for(int i=2; i<=num-1;i++)
       {
        long double uni_1= RandNum::ran1(seed);
        long double uni_2= RandNum::ran1(seed);
        randarray[i]= RandNum::beta_rand(alpha,beta,uni_1,uni_2);
       }
    }

//Call baryarray function in Genstruct class.
  DistGen gen_info(randarray, Lev) ;
  long double* SBA = gen_info.baryarray();
  long double* DF=gen_info.CalcDist();
```

```
gen_info.ClcDist_val ();
for (int i=0; i<numwts;i++)
  DFun.listarray(DF[2*i],DF[2*i+1]);

//The next part calculates the moment of the approximation down to Lev
//levels and store them in the  array Exp.  The bound on the error is also
//stored in he last three memory locations of Exp and are written to the file
//MomArray.txt.

long double* Exp = new long double[7];
Exp[0]=r;

//This calls the moment bound function that is based on the global worst
//case bound.
//The max_int_n method gives back the bound on the Prohorov distance between the
//true random variable and its $n$thlevel SBA approximation.
long double* imprv_bnd = gen_info.Bet_MomBnd();
long double max_int = gen_info.max_int_n();

//This next part creates the array with moments for the current monte carlo
//realization of the SBA generated approximation and writes them to a
//file.

long double** Mom_pts = new long double*[2];
for (int m=0;m<2;m++)
 Mom_pts[m]=new long double[numwts];

//The first for loop initializes the barycenter part and stores in the
//first row of the Mom_pts array.
for (int p=0; p< numwts  ;p++)
   Mom_pts[0][p]= *(SBA+2*p+1);

//This for loop calculates the probability and puts it in the second row of
//the Mom_pts array.
for (int p=0;p<numwts;p++)
  Mom_pts[1][p]= *(DF+2*p+3)-*(DF+2*p+1);
```

```
  //This nested for loop calculates the second, third and fourth moments.
  //Ev is the array consisting of these three moments.
  for (int i=1;i<=3;i++)
    {
    EV=0.0;
    for (int p=0;p<numwts;p++)
     {
     EV=EV+ (powl(Mom_pts[0][p],i+1) * Mom_pts[1][p]);
     }
      *(Exp+i)=EV;
  }

  //This part writes the array of moments and error bounds to a text file
  //based on improved method to a file.

  for(int i=0;i<=2;i++)
   *(Exp+(i+4))=*(imprv_bnd+i);
  for(int i=0;i<=6;i++)
   BetMom.listarray1(*(Exp+i)) ;
  BetMom.listarray2(max_int);

  for (int p=0;p<2;p++)
  delete[] Mom_pts[p];  //step 1: Delete the columns
  delete[] Mom_pts;    //step 2: Delete the rows
  delete[] Exp;
  delete[] randarray;
  } //End of MCPoints loop.

}
```

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