ALGEBRAICALLY DETERMINED RINGS OF FUNCTIONS

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Let R be any of the following rings: the smooth functions on $\boldsymbol{R}^{2n}$ with the Poisson bracket, the Hamiltonian vector fields on a symplectic manifold, the Lie algebra of smooth complex vector fields on $\boldsymbol{C}$, or a variety of rings of functions (real or complex valued) over 2nd countable spaces. Then if H is any other Polish ring and $\varphi: H \rightarrow R$ is an algebraic isomorphism, then it is also a topological isomorphism (i.e. a homeomorphism). Moreover, many such isomorphisms between function rings induce a homeomorphism of the underlying spaces. It is also shown that there is no topology in which the ring of real analytic functions on $\boldsymbol{R}$ is a Polish ring.

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CHAPTER 1

INTRODUCTION

A natural question to ask about an algebraic structure is the following: What are the possible Polish topologies which make the structure continuous? There are a number of specialized results for this question. For some groups, there is no Polish topology making the groups into Polish groups. For some rings, there are $2^{\mathfrak{c}}$ many Polish topologies. For a number of groups or rings, there is only one unique topology making them into Polish groups or rings.

Definition 1.1. We call a Polish group [ring, algebra, Poisson algebra, etc] G an algebraically determined Polish group [ring, algebra, Poisson algebra, etc] if, for H a Polish group [ring, algebra, Poisson algebra, etc], $\varphi : H \to G$ is an algebraic group [ring, algebra, Poisson algebra, etc] isomorphism implies that φ is a topological isomorphism (i.e. a homeomorphism).

Many times, when a ring of continuous real or complex valued functions (or related objects like vector fields) on a space is algebraically determined, then a use of a version of "Milnor's Exercise" ([10], pg 11, problem 1-C) gives an underlying homeomorphism between the spaces. Often, if the ring is "nice" in some way, then the homeomorphism is also nice in a related way.

The purpose of this dissertation is to show that some rings of functions are algebraically determined and to give results regarding induced homeomorphisms. Gel'fand and Kolmogorov (originally [3], see [12] for an English translation) showed that isomorphisms of the rings of real valued continuous functions on two first countable spaces induce a homeomorphism between the spaces. Here, we show that various subrings of the ring of continuous real or complex valued functions on a second countable space are algebraically determined (in Chapter 2) and that ismorphisms of these subrings induce a homeomorphism (in Chapters 4 and 5) . We also prove that the group of real analytic functions on $\mathbb{R}$ does not have a topology in which it is a Polish group (Chapter 3). In Chapter 6 we prove that the Lie ring of smooth functions on $\mathbb{R}^{2n}$ with the Poisson bracket is algebraically determined and that for a symplectic manifold M, the Lie ring of Hamiltonian vector fields is algebraically determined. Chapter 7 contains the theorem that the smooth complex vector fields on the complex plane are algebraically determined.

1.1. Basic Definitions and Tools

In the rest of Chapter 1, these preliminary results are all known and thus do not represent original work. Original work will start with Chapter 2.

We call a group G a Polish group if the group is Polish in some topology (i.e. separable and completely metrizable) which makes the group operation and inversion continuous. Here, the term "group" can be replaced with ring, algebra, etc., where any additional algebraic structures must also be continuous.

The following propositions outline very useful cases in which automatic continuity follows. Here, an analytic subset of a topological space is one which is the continuous image of a Polish space. Here, a set A has the Baire property if there is an open set U so that $A \Delta U$ is meager.

Proposition 1.2. *(Mackey, [9]) If X and Y are standard Borel spaces, $\{A_n\}_{n\geq 1}$ is a collection of Borel subsets of Y which sepearates points, and $\varphi : X \to Y$ is so that for all $n \geq 1$, $\varphi^{-1}(A_n)$ is analytic, then φ is measurable with respect to sets with the Baire property.*

Proof:

By Theorem 3.3 of Mackey [9], the $\{A_n\}_{n\geq 1}$ generate the Borel structure of Y and hence φ is measurable with respect to sets with the Baire property.

Proposition 1.3. *(B.J. Pettis, [2]) Let G and H be Polish groups and let $\varphi : G \to H$ be an algebraic isomorphism which is measurable with respect to sets with the Baire property. Then φ is a topological isomorphism. This applies in particular if φ is a Borel mapping.*

Corollary 1.4. *If H is a Polish group, and for any Polish group G and algebraic isomorphism $\varphi : G \to H$, there is a collection $\{A_n\}_{n\geq 1}$ of Borel subsets of H which separate points so that for each $n \geq 1$, $\varphi^{-1}(A_n)$ is analytic, then H an is algebraically determined Polish group.*

This corollary uses a fact proved by Nikodým (which can be found in [8]), that analytic sets are sets with the Baire property.

Another quite useful theorem is the following corollary to the Lusin-Suslin Theorem ([8]). Here, a Borel isomorphism is a bijection for which both it and its inverse are Borel mappings, and standard Borel space is defined to be a Borel isomorphic image of a Polish space.

Theorem 1.5. *Let X and Y be standard Borel spaces and $f : X \to Y$ a Borel mapping. If A is a Borel set so that $f|A$ is injective, then $f(A)$ is a Borel subset of Y and $f|A$ is a Borel isomorphism (in particular f^{-1} is Borel).*

Another useful fact is the following, which will be of use in Chapter 6.

Lemma 1.6. *If R is a Polish Lie ring and $\mathfrak{I}$ is a closed ideal of R, then $R/\mathfrak{I}$ is a Polish Lie ring with bracket $[x+\mathfrak{I}, y+\mathfrak{I}] = [x, y] + \mathfrak{I}$ and the projection map $\pi : R \to R/\mathfrak{I}$ is a continuous Lie ring homomorphism.*

Proof:

Let R be a Polish Lie ring and $\mathfrak{I}$ be an ideal of R. If $x, y \in R$ and $a, b \in \mathfrak{I}$ then $[x+a, y+b] = [x, y]+[x, b]+[a, y]+[a, b] \in [x, y]+\mathfrak{I}$ and $(x+a)+(y+b) = (x+y)+(a+b)$, so the induced bracket $[x + \mathfrak{I}, y + \mathfrak{I}] = [x, y] + \mathfrak{I}$ is a well-defined Lie bracket, the induced addition $(x + \mathfrak{I}) + (y + \mathfrak{I}) = (x + y) + \mathfrak{I}$ is a well-defined addition, so $R/\mathfrak{I}$ is a Lie ring and the quotient map $\pi : R \to R/\mathcal{I}$ is a homomorphism. $R/\mathcal{I}$ with the quotient topology is a metric space since if d is a compatible metric on R, we can define the metric $d_\pi(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ on $R/\mathcal{I}$ which is compatible with the quotient topology on $R/\mathcal{I}$. π is a continuous open mapping from a completely metrizable space onto a metric space and hence by a result of Hausdorff ([4]), $R/\mathcal{I}$ is completely metrizable.

To see that $[\cdot, \cdot] : R/\mathcal{I}^2 \to R/\mathcal{I}$ is continuous, let $x + \mathcal{I}$, $y + \mathcal{I} \in R/\mathcal{I}$ and let U be open about $[x + \mathcal{I}, y + \mathcal{I}]$. Then $[x, y] \in \pi^{-1}(U)$, an open set since π is continuous, and so there are open sets V and W so that $x \in V$, $y \in W$ and $[V, W] \subset \pi^{-1}(U)$. Then $x + \mathcal{I} \in \pi(V)$, $y + \mathcal{I} \in \pi(W)$, and $\pi(V)$ and $\pi(W)$ are open sets so that $[\pi(V), \pi(W)] = \pi([V, W]) \subset U$. Thus $[\cdot, \cdot]$ is continuous. Prove the continuity of $+$ similarly to see that $R/\mathcal{I}$ is a Polish Lie ring. $\square$

The following proposition is contained in a paper to appear at a later date [7], but the proof is included for the convenience of the reader.

Proposition 1.7. . *Let $n \geq 1$. Make $\mathbb{R}^n$ into a commutative Polish ring $(\mathbb{R}^n, \star)$ with the usual topology and vector addition together with the multiplication $(x_1, \ldots, x_n) \star (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$, so $(\mathbb{R}^n, \star) = \prod\limits_{1 \leq \ell \leq n} (\mathbb{R}, \cdot)$ as a product of rings. Let $\mathcal{R}$ be a Polish ring and let $\varphi : \mathcal{R} \to (\mathbb{R}^n, \star)$ be a surjective ring homomorphism such that $\varphi^{-1}(0)$ is an analytic subset of $\mathcal{R}$. Then φ is continuous. In particular $(\mathbb{R}^n, \star)$ is an algebraically determined Polish ring. Similar statements hold for $(\mathbb{R}^\infty, \star) = \prod\limits_{\ell \geq 1} (\mathbb{R}, \cdot)$.*

Proof:

If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define a partial order $\preceq$ by $x \preceq y$ if and only if $x_\ell \leq y_\ell$ for all $1 \leq \ell \leq n$. If $z \in \mathbb{R}^n$, let $U(z) = \{ x \in \mathbb{R}^n \mid z \preceq x \}$ and $L(z) = \{ x \in \mathbb{R}^n \mid x \preceq z \}$. If $S = \{ x \star x \mid x \in \mathbb{R}^n \}$, then $U(z) = z + S$ and $L(z) = z - S$. Therefore if a, $b \in \mathbb{R}^n$ and $a \preceq b$, we have that $B(a, b) = \{ x \in \mathbb{R}^n \mid a \preceq x \preceq b \} = U(a) \cap L(b) = (a + S) \cap (b - S)$.

If $\mathcal{R}$ is a Polish ring and $\varphi : \mathcal{R} \to \mathbb{R}^n$ is a surjective ring homomorphism such that $\varphi^{-1}(0)$ is analytic, then $\varphi^{-1}(x)$ is also an analytic subset of $\mathcal{R}$ for every $x \in \mathbb{R}^n$, for if $w \in \varphi^{-1}(x)$, then $\varphi^{-1}(x) = w + \varphi^{-1}(0)$ is an analytic subset of $\mathcal{R}$. Furthermore, $\varphi^{-1}(B(a,b)) = \varphi^{-1}(a + S) \cap \varphi^{-1}(b - S) = (\varphi^{-1}(a) + \varphi^{-1}(S)) \cap (\varphi^{-1}(b) - \varphi^{-1}(S))$ is an analytic subset of $\mathcal{R}$ since $\varphi^{-1}(S) = \varphi^{-1}(0) + \{ r^2 \mid r \in \mathcal{R} \}$ and therefore both $\varphi^{-1}(a) + \varphi^{-1}(S)$ and $\varphi^{-1}(b) - \varphi^{-1}(S)$ are analytic sets. If $W \subset \mathbb{R}^n$ is open, then W is a countable union of sets of the form $B(a,b)$ and $\varphi^{-1}(W)$ is an analytic subset of $\mathcal{R}$. Hence, Theorem 9.10 in [8] implies that φ, viewed as a homomorphism of additive abelian groups, is continuous since every analytic set is a set with the Baire property. If φ is a bijection, then $\varphi^{-1}(0)$ is a single point and therefore trivially an analytic set, φ is continuous, and finally φ is a topological isomorphism by Proposition 1.3.

For the $(\mathbb{R}^\infty, \star)$ case, let $S = \{ x \star x \mid x \in \mathbb{R}^\infty \}$ as before. For $n \geq 1$ and $x_1, \ldots, x_n \in \mathbb{R}$, let $z \in \mathbb{R}^\infty$ with $z_\ell = 0$ for $1 \leq \ell \leq n$ and $z_\ell = -1$ for $\ell > n$, $z \star S = \{ z \star w \mid w \in S \}$, $U(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots) + S + (z \star S) = \{ y \in \mathbb{R}^\infty \mid x_\ell \leq y_\ell \text{ for } 1 \leq \ell \leq n \text{ and } y_\ell \in \mathbb{R} \text{ for } \ell > n \}$ and $L(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots) - S - (z \star S) = \{ y \in \mathbb{R}^\infty \mid y_\ell \leq x_\ell \text{ for } 1 \leq \ell \leq n \text{ and } y_\ell \in \mathbb{R} \text{ for } \ell > n \}$. If a, $b \in \mathbb{R}^n$ with $a \preceq b$ let $B(a,b) = U(a) \cap L(b) = \{ y \in \mathbb{R}^\infty \mid a_\ell \leq y_\ell \leq b_\ell \text{ for } 1 \leq \ell \leq n \text{ and } y_\ell \in \mathbb{R} \text{ for } \ell > n \}$. If $\mathcal{R}$ is a Polish ring and $\varphi : \mathcal{R} \to \mathbb{R}^\infty$ is a surjective ring homomorphism such that $\varphi^{-1}(0)$ is analytic, then $\varphi^{-1}(x)$ is also an analytic subset of $\mathcal{R}$ for every $x \in \mathbb{R}^\infty$ by the same argument as before. If $a \in \mathbb{R}^n$ then $\varphi^{-1}(U(a)) = \varphi^{-1}((a_1, \ldots, a_n, 0, 0, \ldots) + S + (z \star S)) = \varphi^{-1}((a_1, \ldots, a_n, 0, 0, \ldots)) + \varphi^{-1}(S) + \varphi^{-1}(z)\varphi^{-1}(S)$ is an analytic set, $\varphi^{-1}(S) = \{ r^2 \mid r \in \mathcal{R} \} + \varphi^{-1}(0)$ is an analytic set and $\varphi^{-1}(z \star S) = \varphi^{-1}(z)\varphi^{-1}(S) + \varphi^{-1}(0)$ is an analytic set. Similarly $\varphi^{-1}(L(b))$ is an analytic set for all $b \in \mathbb{R}^n$. Hence, if a, $b \in \mathbb{R}^n$ with $a \preceq b$ then $\varphi^{-1}(B(a,b)) = \varphi^{-1}(U(a)) \cap \varphi^{-1}(L(b))$ is an analytic set. Since every open subset $W \subset \mathbb{R}^\infty$ is a countable union of sets of the form $B(a,b)$, $\varphi^{-1}(W)$ is an analytic subset of $\mathcal{R}$. One finishes this proof in this case as before. $\square$

Also note that in the rest of this document, for topological spaces X and Y, let $C(X)$ denote the collection of continuous real-valued functions on X. For a function $f : X \to Y$, define the support of f, $supp(f) = \overline{\{x \in x : f(x) \neq 0\}}$. Let $C_0(X)$ denote the collection of continuous real valued functions on X with compact support. That is, $f \in C_0(X)$ if and only if $supp(f) = \overline{\{x \in X : f(x) = 0\}}$ is compact. Also if X, Y, and Z are sets, and if $f : X \to Y$ is $1-1$, define the push-forward of f to be f_*, where for each $g : X \to Z$, $f_*(g) = g \circ f^{-1} : Y \to Z$.

1.2. Basic Facts About Analytic Sets

Here are some common facts about analytic sets, with proofs included for the convenience of the reader.

Lemma 1.8. *Let X be a topological space, and let $\{A_n\}_{n\geq 1}$ be a sequence of analytic subsets of X. Then $\bigcap_{n\geq 1} A_n$ is analytic.*

Proof:

Since each A_n is analytic, there each is an image of some continuous function f_n whose domain is X_n, a Polish space. Define $D = \{v \in X^{\mathbb{N}} : \forall i, j \geq 1,\ v(i) = v(j)\}$, a closed subset of $X^{\mathbb{N}}$. Let $F : \prod_{i\geq 1} X_i \to X^{\mathbb{N}}$ be the continuous function defined by $F(x)(i) = f_i(x_i)$. Then $F^{-1}(D)$ is a closed subset of $\prod_{n\geq 1} X_n$ and hence is a Polish space using the subspace topology inherited from $\prod_{n\geq 1} X_n$. Thus $\bigcap_{n\geq 1} A_n = \pi_1 F(F^{-1}(D))$, hence is analytic.

Lemma 1.9. *Let X, Y be standard Borel spaces, $A \subset Y$ analytic, and let $f : X \to Y$ be so that $graph(f)$ is analytic. Then $f^{-1}(A)$ is analytic in X.*

Proof:

A is analytic implies that $X \times A$ is analytic. So, $graph(f) \cap (X \times A)$ is analytic by Lemma 1.8 and hence $\pi_1(graph(f) \cap (X \times A)) = f^{-1}(A)$ is analytic.

1.3. Continuous Functions on Polish Spaces

The following theorem is contained in [8], but a short proof is included using theorems found in many places ([15], for example).

Theorem 1.10. *A Hausdorff, locally compact, second countable space is a Polish space.*

Proof:

If X is a locally compact second countable Hausdorff space, the the one-point compactification $\hat{X}$ of X exists. $\hat{X}$ is a compact second countable Hausdorff space (by using complements of a countable compact basis for X), and so it is completely metrizable by the Urysohn metrization theorem ([15]). Thus the open subspace $X \subset \hat{X}$ is completely metrizable. Since X is a second countable metric space, it is also separable. $\square$

The proofs (though not statements) of the following lemmas can be found in [13] (Lemma 1.9, Theorem 1.11)

Lemma 1.11. *If X is a locally compact, Hausdorff, 2nd countable topological space (and therefore a Polish space), then there is a sequence of compact sets $\{K_i\}_{i\geq 1}$ so that $\bigcup_{i\geq 1} K_i = X$ and for each $i \in \mathbb{N}$, $K_i \subset int(K_{i+1})$.*

Proof:

Let $\{U_i\}_{i\geq 1}$ be an open basis for the topology of X consisting of sets with compact closures. Then, take $K_1 = \overline{U_1}$, and recursively, if $K_i = \overline{U_1 \cup \cdots \cup U_{n_i}}$, then since K_i is a finite union of compact sets and is hence compact, there is an $n_{i+1} > n_i$ so that $K_i \subset U_1 \cup \cdots \cup U_{n_{i+1}}$. Let $K_{i+1} = \overline{U_1 \cup \cdots \cup U_{n_{i+1}}}$. Then each K_i is compact, $K_i = \overline{U_1 \cup \cdots \cup U_{n_i}} \subset U_1 \cup \cdots \cup U_{n_{i+1}} \subset int(K_{i+1})$ and $\bigcup_{i\geq 1} K_i \supset \bigcup_{i\geq 1} U_i = X$.

Definition 1.12. A collection of real valued functions Φ with the same domain, a topological space, X, is called locally finite if for every $x \in X$, there is an open $V \subset X$ containing x so that $\{\phi \in \Phi : \exists y \in V(\phi(y) \neq 0)\}$ is finite.

Theorem 1.13. *If X is a locally compact, Hausdorff, 2nd countable (and hence Polish) space, then $R = C(X)$ can be made into a Polish ring with pointwise multiplication and addition. Moreover $C_0(X)$ is dense in this topology.*

Proof:

Let $\{K_i\}_{i \geq 1}$ be a sequence of compact sets as in Lemma 1.11. Then for $f, g \in R$, define the metric $d(f,g) = \sum_{i \geq 1} 2^{-i} \frac{\sup_{K_i}(|f-g|)}{1+\sup_{K_i}(|f-g|)}$ to define the topology of R. Let $\mathfrak{U} = \{U_i\}_{i \geq 1}$ be a precompact basis for X and for each pair $m, n \in \mathbb{N}$ so that $\overline{U_m} \subset U_n$, let $f_{m,n}$ be a function (given by Urysohn's Lemma) so that $0 \leq f_{m,n} \leq 1$, $f_{m,n}|\overline{U_m} = 1$ and $f_{m,n}|(X - U_n) = 0$. The countable set which will be shown to be dense is the set of finite products & sums of the $f_{m,n}$s and constant rational functions. To see this, one could use the Stone-Weierstrauss theorem, but a direct proof follows. Let $f \in R$, $\epsilon > 0$ and K compact. To approximate f on K, we can assume $f > 0$, since otherwise we could replace f with $f + q$ where q is within $\epsilon/2$ of the minimum of f over K. Define $q_1 \in \mathbb{Q}$ so that $\epsilon/3 < |q_1 - \sup_K(f)| < 2\epsilon/3$. Also, using the compactness of $\{x \in K : f(x) \geq \sup_K(f) - \epsilon/3\}$, let $U_{1,1}, \dots, U_{1,k_1} \subset \mathfrak{U}$ so that for each $U_{1,i}$, there is a $U'_{1,i}$ so that $\overline{U_{1,i}} \subset U'_{1,i} \subset \{x \in K : f(x) > \sup_K(f) - 2\epsilon/3\}$, and also so that $\{x \in K : f(x) \geq \sup_K(f) - \epsilon/3\} \subset \bigcup_{i=1}^{k_1} U_{1,i}$. Now as an abuse of notation, let $f_{1,i}$ be the function above which is 1 on $U_{1,i}$ and 0 on $X - U'_{1,i}$, and let $h_1 = q_1(1 - \prod_{i=1}^{k_1}(1 - f_{1,i}))$. If $|f - h_1| < \epsilon$ on K, then stop, otherwise notice that on $\{x \in K : f(x) \geq \sup_K(f) - \epsilon/3\}$, $|f - h_1| < 2\epsilon/3$, on $\{x \in K : \sup_K(f) - 2\epsilon/3 < f(x) < \sup_K(f) - \epsilon/3\}$, $|f - h_1| < \sup_K(f) - \epsilon/3$, and $f - h_1 = f$ on $\{x \in K : f(x) \leq \sup_K(f) - 2\epsilon/3\}$. Inductively, if $sup_K(f - (h_1 + \cdots + h_n)) \geq \epsilon$, define $q_{n+1} \in \mathbb{Q}$ so that $\epsilon/3 < |q_n - \sup_K(f - (h_1 + \cdots + h_n))| < 2\epsilon/3$. Also, using the compactness of $\{x \in K : f(x) \geq \sup_K(f - (h_1 + \cdots + h_n)) - \epsilon/3\}$, let $U_{n+1,1}, \dots, U_{n+1,k_n} \subset \mathfrak{U}$ so that for each

$U_{n,i}$, there is a $U'_{n,i}$ so that $\overline{U_{n,i}} \subset U'_{n,i} \subset \{x \in K : f(x) > \sup_K(f - (h_1 + \cdots + h_n)) - 2\epsilon/3\}$, and also so that $\{x \in K : (f - (h_1 + \cdots + h_n))(x) \geq \sup_K(f - (h_1 + \cdots + h_n)) - \epsilon/3\} \subset \bigcup_{i=1}^{k_1} U_{1,i}$. Now as an abuse of notation, let $f_{n,i}$ be the function above which is 1 on $U_{n,i}$ and 0 on $X - U'_{n,i}$, and let $h_{n+1} = q_{n+1}(1 - \prod_{i=1}^{k_{n+1}}(1 - f_{n+1,i}))$. If $|f - (h_1 + \cdots + h_{n+1})| < \epsilon$ on K, then stop, otherwise notice that on $\{x \in K : (f - (h_1 + \cdots + h_n))(x) \geq \sup_K((f - (h_1 + \cdots + h_n))) - \epsilon/3\}$, $|f - (h_1 + \cdots + h_{n+1})| < 2\epsilon/3$, on $\{x \in K : \sup_K(f - (h_1 + \cdots + h_n)) - 2\epsilon/3 < (f - (h_1 + \cdots + h_n))(x) < \sup_K(f - (h_1 + \cdots + h_n)) - \epsilon/3\}$, $|f - (h_1 + \cdots + h_{n+1})| < \sup_K(f) - \epsilon/3$, and $f - (h_1 + \cdots + h_{n+1}) = f - (h_1 + \cdots + h_n)$ on $\{x \in K : (f - (h_1 + \cdots + h_n))(x) \leq \sup_K(f - (h_1 + \cdots + h_n)) - 2\epsilon/3\}$. Of course since the maximum is reduced each time by at least $\epsilon/3$, and the minimum stays above $-\epsilon$, eventually there is some n_0 so that $|f - (h_1 + \cdots + h_n)| < \epsilon$.

Now to approximate a function f on all of X, let $0 < \epsilon < 1$ and take $n \in \mathbb{N}$ so that $2^{-n} < \epsilon/2$ and take h a finite sum/product of rational constant functions and functions of the form $f_{m,n}$ as above so that $\sup_{K_n}(|f - h|) < \frac{\epsilon}{2-\epsilon}$. Then $d(f, h) = \sum_{i \geq 1} 2^{-i} \frac{\sup_{K_i}(|f-h|)}{1+\sup_{K_i}(|f-h|)} \leq \frac{\sup_{K_n}(|f-h|)}{1+\sup_{K_n}(|f-h|)} + \sum_{i>n} 2^{-i} < \epsilon$.

To see that (R, d) is complete, let $\{f_k\}_{k \geq 1} \subset R$ be a Cauchy sequence. For each $x \in X$, there is some i_0 so that $x \in K_{i_0}$. Then for each $0 < \epsilon < 2^{-i_0}$, there is an N so that if $n, m \geq N$, $d(f_n, f_m) < \epsilon$. If $n, m \geq N$, then $\epsilon > \sum_{i \geq 1} 2^{-i} \frac{\sup_{K_i}(|f_m - f_n|)}{1+\sup_{K_i}(|f_m - f_n|)} \geq 2^{-i_0} \frac{\sup_{K_{i_0}}(|f_m - f_n|)}{1+\sup_{K_{i_0}}(|f_m - f_n|)}$ and hence $\sup_{K_{i_0}}(|f_m - f_n|) < \frac{2^{i_0}\epsilon}{1-2^{i_0}\epsilon}$ so $|f_m(x) - f_n(x)| < \frac{2^{i_0}\epsilon}{1-2^{i_0}\epsilon}$. So $\{f_m(x)\}_{m \geq 1}$ is a Cauchy sequence and hence converges to some real number $f(x)$. To see that f is continuous, take $x \in X$, $\epsilon > 0$ and $i_0 \in \mathbb{N}$ so that $x \in K_{i_0}$ and hence $x \in int(K_{i_0+1})$. Let $N \in \mathbb{N}$ be so that for $n, m \geq N$, $\sup_{K_{i_0+1}}(|f_m - f_n|) < \epsilon/2$. Then, for each $y \in K_{i_0+1}$ there is an $m_y \geq N$ so that $|f(y) - f_m(y)| < \epsilon/2$, and so if $n \geq N$, $|f(y) - f_n(y)| \leq |f(y) - f_{m_y}(y)| + |f_{m_y}(y) - f_n(y)| < \epsilon$. Hence, f_n converges uniformly on K_{i_0+1} and so if $x_m \to x$, there is some M so that for $m \geq M$, $x_m \in K_{i_0+1}$ and $|f(x) - f(x_m)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_m)| + |f_N(x_m) - f(x_m)| \to 0$ as $m \to \infty$. Thus f is continuous. To see that $f_n \to f$, let $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ be so that $2^{-n_0} < \epsilon$, and let $N \in N$ so that for $n \geq N$, $\sup_{K_{n_0+1}}(|f - f_n|) < \epsilon$. Then $d(f, f_n) = \sum_{i<n_0} 2^{-i} \frac{\sup_{K_i}(|f-f_n|)}{1+\sup_{K_i}(|f-f_n|)} + \sum_{i>n_0} 2^{-i} \frac{\sup_{K_i}(|f-f_n|)}{1+\sup_{K_i}(|f-f_n|)} \leq \frac{\epsilon}{1+\epsilon} + \epsilon$. Thus $f_n \to f$ and hence R is complete.

To see that $C_0(X)$ is dense in R, let $f \in R$ and $\epsilon > 0$. Then take $i_0 \in \mathbb{N}$ so that $2^{-i_0} < \epsilon$. Then, by Urysohn's Lemma there is an $g \in C_0(X)$ so that $g|K_{i_0} = 1$ and $g|(int(K_{i_0+1}))^c = 0$. Then $fg \in C_0(X)$ and $d(fg, f) \leq \sum_{i>i_0} 2^{-i} = 2^{-i_0} < \epsilon$. $\square$

Similarly one can put a Polish topology on $S(X)$, the continuous functions which vanish at infinity (i.e. for every $\epsilon > 0$ there is a compact set K so that $|f|K^c| < \epsilon$) using the supremum metric.

1.4. Basic Facts About Manifolds

A manifold M of dimension n is a topological space which has a basis consisting of open sets which are homeomorphic to $\mathbb{R}^n$. Such pairs of open sets and homeomorphisms are called charts of M. For $1 \leq r \leq \infty$, a C^r manifold is one in which if (U_1, ψ_1) and (U_2, ψ_2) are charts of M, where $U_1 \cap U_2 \neq \emptyset$, then $\psi_1^{-1} \circ \psi_2$ has continuous derivatives of order r. A function f on M is a member of $C^r(M)$ if for any chart (U, ψ), $\psi_*(f) = f \circ \psi^{-1} \in C^r(\psi(U))$, where, for an open set W, $C^r(W)$ is the set of functions on W with continuous derivatives of order r.

Here, a C^r partition of unity subordinate to an open cover $\mathfrak{U}$ is a collection $\mathfrak{F}$ of C^r functions so that each $f \in \mathfrak{F}$, $f : M \to [0, 1]$, $supp(f) \subset U$ for some $U \in \mathfrak{U}$, so that if $m \in M$, there is an open set U containing m so that only finitely many $f \in \mathfrak{F}$ are nonzero on U (called locally finite), and also so that $\sum_{f \in \mathfrak{F}} f = 1$.

Lemma 1.14. *If $0 \leq r \leq \infty$, M is a C^r manifold and $\mathfrak{U}$ is an open cover of M, then there is a C^r partition of unity $\{\phi_i\}_{i \in \mathbb{N}}$ subordinate to $\mathfrak{U}$ so that each ϕ_i has compact support.*

Proof:

First take a sequence $\{K_i\}_{i \geq 1}$ for M as in Lemma 1.11 (and set $K_0 = \emptyset$). Then for

each $p \in M$ let i_p be the largest integer so that $p \in M - K_{i_p}$. Also choose $U_p \in \mathfrak{U}$ so that $p \in U_p$ and let (V_p, τ_p) be a C^r chart centered at p so that $V_p \subset U_p \cap (int(K_{i_p+2}) - K_{i_p})$ and so that $\tau_p(V_p)$ contains the closed cube $[-2,2]^{dim(M)}$. Let $g \in C^\infty(\mathbb{R}^{dim(M)})$ so that $g|[-1,1]^{dim(M)} = 1$, $g|(\mathbb{R}^{dim(M)} - (-2,2)^{dim(M)}) = 0$, and for all $x \in \mathbb{R}^{dim(M)}$, $0 \le g(x) \le 1$. The define $\psi_p : M \to \mathbb{R}$ by $\psi_p(x) = \begin{cases} g \circ \tau_p(x), & \text{if } x \in V_p, \\ 0, & \text{otherwise.} \end{cases}$
Now, ψ_p has compact support which is a subset of V_p and $p \in W_p = \{x \in M : \psi_p(x) = 1$. Since each W_p has nonempty interior, for each $i \ge 1$ there is a finite collection $\{p_{i,1}, \ldots, p_{i,j_i}\} \subset M$ so that $\bigcup_{k=1}^{j_i} \supset K_{i+1} - int(K_i)$. Now the collection $\{\psi_{p_{i,k}}\}_{i \ge 1,\ 1 \le k \le j_i}$ is locally finite and countable, and hence $\Psi = \sum_{i \ge 1,\ 1 \le k \le j_i} \psi_{p_{i,k}}$ is a well-defined member of $C^r(M)$ and moreover $\Psi > 0$ on M. Thus $\{\frac{\psi_{p_{i,k}}}{\Psi}\}_{i \ge 1,\ 1 \le k \le j_i}$ is C^r partition of unity subordinate to $\mathfrak{U}$.

Lemma 1.15. *Let $0 \le r \le \infty$ and let M be a C^r manifold. Then if $F \subset U \subset M$, F is compact, U is open, and $f \in C^r(M)$ so that $f|U > 0$, then there is a function $h \in C_0^r(M)$ with $f|F = h|F$ and $h > 0$ everywhere.*

Proof:

Let d be a metric compatible with the topology on M and let $\epsilon = d(F, U^c) > 0$. Then let $\{\phi_i\}_{i \ge 1}$ be a partition of unity subordinate to $\{B(U^c, 2\epsilon/3), B(F, 2\epsilon/3)\}$ (where for a set A and $\gamma > 0$, $B(A, \gamma) = \{x : d(x, A) < \gamma\}$). Take $h_1 = \sum_{supp(\phi_i) \subset B(F, 2\epsilon/3)} \phi_i$ and $h_2 = \sum_{supp(\phi_i) \subset B(U^c, 2\epsilon/3)} \phi_i$. Then $fh_1 + h_2$ is the desired function since if $x \in F$, $(fh_1 + h_2)(x) = f(x) \cdot 1 + 0 = f(x) > 0$, if $x \in B(U^c, 2\epsilon/3) \cap B(F, 2\epsilon/3)$, $(fh_1 + h_2)(x) \ge min(f(x), 1)(h_1(x) + h_2(x)) = min(f(x), 1) > 0$, and if $x \in B(U^c, 2\epsilon/3) - B(F, 2\epsilon/3)$, then $(fh_1 + h_2)(x) = h_2(x) = 1$. $\square$

Lemma 1.16. *If M is a C^r manifold, $U, V \subset M$ are open in M and $\overline{U} \subset V$, then there is a C^r function h on M which is 0 on V^c and 1 on $\overline{U}$, and for all $x \in M$, $0 \le h(x) \le 1$.*

Proof:

Using Lemma 1.14, let $\{\varphi_i\}_{i\geq 1}$ be a C^r partition of unity subordinate to the cover $\{\overline{U}^c, V\}$. If $h = \sum_{i:\ supp(\varphi_i)\subset V} \varphi_i$, then h is the desired function. $\square$

Lemma 1.17. *If M is a C^r manifold, $U, V \subset M$ are open in M and $\overline{U} \subset V$, then for any $f \in C^\infty(V)$ there is a function $F \in C^\infty(M)$ so that $supp(F) \subset V$ and $F|U = f|U$.*

Proof:

Take $G \in C^r(M)$ which is 1 on $\overline{U}$ so that $supp(G) \subset V$ as in Lemma 1.16. Then define

$$F(m) = \begin{cases} Gf(m), & \text{if } m \in V, \\ 0, & \text{otherwise.} \end{cases}$$

Then $F \in C^r(M)$, $supp(F) \subset supp(G) \subset V$ and $F|U = Gf|U = f|U$. $\square$

CHAPTER 2

THE RING OF CONTINUOUS FUNCTIONS ON A 2ND COUNTABLE SPACE

Unless stated, the following lemmas do not require the space X in question to be Hausdorff.

Lemma 2.1. *Let X be a 2nd countable topological space, and denote by $C(X)$ the continuous real-valued functions on X. Also let R be a subring of $C(X)$ equipped with a topology so that there is a countable dense subset $D \subset X$ such that for all $x \in D$, $E_x : R \to \mathbb{R}$ defined by $E_x(f) = f(x)$ is continuous. Then for $y \in \mathbb{R}$ and open $U \subset X$, $\{f \in R : f(x) \geq y$ for $x \in U\}$ is a closed subset of R.*

Proof:

$\{f \in R : f(x) \geq y \text{ for } x \in U\} = \{f \in R : f(x) \geq y \text{ for } x \in D \cap U\} =$
$\bigcap\limits_{x \in D\cap U} E_x^{-1}([y, \infty))$, which is an intersection of closed sets and hence closed. $\square$

Theorem 2.2. *Let X and R be as in Lemma 2.1, so that*

- *For any $f \in R$ with $f(x) > 0$ for all $x \in X$, $\sqrt{f} \in R$*
- *R is Polish in some topology*
- *There is a collection $\{g_n\}_{n\geq 1} \subset R$ of nonnegative functions so that for any open $U \subset X$, $\bigcup\{int(supp(g_n)) : supp(g_n) \subset U\} = U$*
- *For all $q \in \mathbb{Q}$, $n \in \mathbb{N}$, $qg_n \in R$*
- *There is a collection $\{q_n\}_{n\geq 1} \subset R$ of bounded positive functions so that for each x, $q_n(x) \to 0$ as $n \to \infty$.*

Then R is an algebraically determined Polish ring.

Proof:

Let $q \in \mathbb{Q}$, $U \subset X$ be open, and define $N(U) = \{n : supp(g_n) \subset U\}$, $A_{q,U} = \{f \in R : f \geq q \text{ on } U\}$ and $\mathcal{S} = \{r^2 : r \in R\}$. Note that by Lemma 2.1, $A_{q,U}$ is a closed subset of R. Also notice that $f \geq q$ on U if and only if $fg_n - qg_n \geq 0$ for all $n \in N(U)$ if and only if $fg_n - qg_n + q_\ell > 0$ for all $n \in N(U)$, $\ell \geq 1$. Thus if $f \geq q$ on U, then $fg_n - qg_n + q_\ell \in \mathcal{S}$ for all $n \in N(U)$, $\ell \geq 1$. Conversely, if $fg_n - qg_n + q_\ell \in \mathcal{S}$ for all $n \in N(U)$, $\ell \geq 1$, then $fg_n - qg_n + q_\ell \geq 0$ for all $n \in N(U)$, $\ell \geq 1$, and hence $fg_n - qg_n \geq 0$ for all $n \in N(U)$ and so $f \geq q$ on U. Thus $f \geq q$ on U if and only if $fg_n - qg_n + q_\ell \in \mathcal{S}$ for all $n \in N(U)$, $\ell \geq 1$. $A_{q,U} = \{f \in R : f \geq q \text{ on } U\} = \bigcap_{\ell \geq 1, n \in N(U)} \{f \in R : fg_n - qg_n + q_\ell \in \mathcal{S}\}$.

Let K be a Polish ring, $\varphi : K \to R$ an algebraic isomorphism of Polish rings, $\mathcal{S}' = \{x^2 : x \in K\}$, and $\{U_k\}_{k \geq 1}$ be a neighborhood base of open subsets of X. Then $\varphi^{-1}(A_{q,U_k}) = \bigcap_{\ell \geq 1, n \in N(U_k)} \varphi^{-1}(\{f \in R : fg_n - qg_n + q_\ell \in \mathcal{S}\}) = \bigcap_{\ell \geq 1, n \in N(U_k)} \{r \in K : r\varphi^{-1}(g_n) - \varphi^{-1}(qg_n) + \varphi^{-1}(q_\ell) \in \mathcal{S}'\}$, which is analytic since $\mathcal{S}'$ is analytic, continuous preimages of analytic sets are analytic by Lemma 1.9, and intersections of sequences of analytic sets are analytic by Lemma 1.8. So, $\{A_{q,U_k}\}_{q \in \mathbb{Q}, k \in \mathbb{N}}$ is a collection of Borel subsets of R which separate points and the preimage under φ of each subset is analytic in H. Thus by corollary 1.4, R is an algebraically determined Polish ring. $\square$

It is worth noting that the proof for $R = C(X)$ is slightly simpler since $f \in C(X)$ is nonnegative if and only if $f \in \mathcal{S} = \{g^2 : g \in C(X)\}$. This tells us that $\{f : f \geq q \text{ on } U\} = \bigcap_{n \in N(U)} \{f : (f - q)g_n \in \mathcal{S}\}$. This fact is not necessarily true for other C^r. For example, take something which behaves like x^2 at one point of S^1 but levels off otherwise. Since $|x|$ is not differentiable, for our function to be a square it must behave like x nearby. But this function would have only one sign change, which is impossible in S^1.

Corollary 2.3. *If M is a 2nd countable manifold, $0 \le r \le \infty$, $C^r(M)$ is an algebraically determined Polish ring.*

Proof:

Use $q_n(x) = 1/n$ and if $\{U_n\}_{n \ge 1}$ is a base for the topology of M, define $g_{n,m}$ when $\overline{V_n} \subset V_m$ to be a C^∞ function which is 1 on V_n and 0 outside of V_m. $\square$

Let $S^r(X)$ be the C^r functions on X which vanish at infinity, meaning $f \in S^r(X)$ if $f \in C^r(X)$ and for every $\epsilon > 0$ there is K compact so that $|f|K^c| < \epsilon$. One can put a Polish topology on $S^r(X)$ by using the usual metric on $C^r(X)$ plus the supremum metric.

Corollary 2.4. *If M is a 2nd countable manifold and $0 \le r \le \infty$, then $S^r(M)$ is an algebraically determined Polish ring.*

Proof:

Let $\{K_i\}_{i \ge 1}$ be an increasing sequence of compact sets as in Lemma 1.11 and take $\{f_i\}_{i \ge 1} \subset C_0^r(X)$ with $0 \le f_i \le 1$, $f_i|K_i = 1$ and $f_i|int(K_{i+1})^c = 0$ by Lemma 1.16. Then $f = \sum_{i \ge 1} 2^{-i} f_i$ is a member of $S^r(X)$ (since the sequence of partial sums is a Cauchy sequence) and f is strictly positive. Let $q_n = \frac{1}{n} f$, and apply Theorem 2.2. $\square$

Corollary 2.5. *If X is a 2nd countable locally compact Hausdorff (and hence Polish) space, then $C(X)$, the continuous real valued functions on X, is an algebraically determined Polish ring.*

Proof:

The considerations for Theorem 2.2 are either trivial or implied by Theorem 1.13 or its proof. $\square$

Definition 2.6. A Polish complex star-algebra is a Polish algebra R over $\mathbb{C}$ together with a continuous involution $^* : R \to R$ so that for $r, s \in R$, $c \in \mathbb{C}$, $(r^*)^* = r$, $(r+s)^* = r^* + s^*$, $(rs)^* = s^* r^*$, and $(cr)^* = \overline{c}(r^*)$.

Theorem 2.7. *Let X be a second countable topological space and let R be a subalgebra of $C(X, \mathbb{C})$ (the continuous, complex valued functions on X) or $S(X, \mathbb{C})$ (the continuous, complex valued functions on X which vanish at infinity) so that*

- *There is a countable dense set $D \subset X$ so that if $e \in D$, $r \mapsto r(e)$ is continuous as a map from R to $\mathbb{C}$.*
- *For any $f \in R$ which is real valued with $f(x) > 0$ for all $x \in X$, $\sqrt{f} \in R$*
- *R is Polish in some topology*
- *There is a collection $\{g_n\}_{n \geq 1} \subset R$ of nonnegative real valued functions so that for any open $U \subset X$, $\bigcup \{int(supp(g_n)) : supp(g_n) \subset U\} = U$*
- *There is a collection $\{q_n\}_{n \geq 1} \subset R$ of bounded positive real valued functions so that for each x, $q_n(x) \to 0$ as $n \to \infty$.*
- *If $r \in R$, then $\overline{r} \in R$*

Then R is an algebraically determined Polish complex star-algebra.

Proof:

Let K be a Polish complex star-algebra, and let $\varphi : K \to R$ be an algebraic isomorphism. Let $SA(K) = \{k \in K \ : \ k^* = k\}$. Then $\varphi(SA(K)) = \{f \in R \ : \ f$ is real valued$\}$ since $f \in SA(K)$ if and only if $f^* = f$ if and only if $\overline{\varphi(f)} = \varphi(f^*) = \varphi(f)$ if and only if $\varphi(f)$ is real valued. Now $\varphi | SA(K)$ is a topological isomorphism by Theorem 2.2. Notice that if $k \in K$, $\varphi(k) = \varphi(\frac{k+k^*}{2} + i\frac{k-k^*}{2i}) = \varphi(\frac{k+k^*}{2}) + i\varphi(\frac{k-k^*}{2i})$. $(\frac{x+x^*}{2})^* = \frac{x^*+x}{2} = \frac{x+x^*}{2}$ and $(\frac{x-x^*}{2i})^* = \frac{x^*-x}{\overline{2i}} = \frac{x-x^*}{2i}$, so $\frac{x+x^*}{2}, \frac{x-x^*}{2i} \in SA(K)$. Thus $x \mapsto (\frac{x+x^*}{2}, \frac{x-x^*}{2i}) \mapsto (\varphi(\frac{x+x^*}{2}), \varphi(\frac{x-x^*}{2i})) \mapsto \varphi(\frac{x+x^*}{2}) + i\varphi(\frac{x-x^*}{2i}) = \varphi(x)$ is continuous, hence φ is continuous and hence is a topological isomorphism by Proposition 1.3. $\square$

CHAPTER 3

THE REAL ANALYTIC FUNCTIONS ON $\mathbb{R}$ CANNOT BE MADE INTO A POLISH RING

A function $f : \mathbb{R} \to \mathbb{R}$ is real analytic if for each $a \in \mathbb{R}$, there is an $r > 0$ and a sequence of real numbers $\{a_n\}_{n\geq 0}$ so that if $|x - a| < r$, then $f(x) = \sum_{n\geq 0} a_n(x-a)^n$. In this section, R is the ring of real analytic functions on $\mathbb{R}$ with usual function addition and multiplication.

Lemma 3.1. *If R has a topology in which it is a Polish ring, then the natural injection $\mathfrak{i} : R \to C^\infty(\mathbb{R})$, given by $\mathfrak{i}(f) = f$, is continuous.*

Proof:

Let $a \in \mathbb{R}$, let $\mathcal{I}_a = \{f \in R \ : \ f(a) = 0\}$. Note that $\mathcal{I}_a = (x - a)R$, and hence each $\mathcal{I}_a$ is analytic in R. Also notice that $f \mapsto f(a)$ is a homomorphism onto $\mathbb{R}$, and its kernel is the analytic set $\mathcal{I}_a$, and hence by Proposition 1.7 it is continuous. Thus $g : R \to \mathbb{R}^{\mathbb{Q}}$, $g(f) = \prod_{q\in\mathbb{Q}} f(q)$ is a continuous injection. Also notice that $h : C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{Q}}$, $h(f) = \prod_{q\in\mathbb{Q}} f(q)$ is a continuous injection, hence its image is a Borel set and its inverse is a Borel mapping by Theorem 1.5. So $\mathfrak{i} = h^{-1} \circ g$ is a Borel mapping and hence $\mathfrak{i}$ is continuous by Proposition 1.3. $\square$

Lemma 3.2. *For a fixed $f \in R$, then the map $\mathbb{R} \to R$ given by $\lambda \mapsto \lambda f$ is continuous.*

Proof:

The mapping $\mathbb{R} \to R$ given by $\lambda \mapsto \lambda f$ is a 1-1 group homomorphism (if $f \neq 0$, as otherwise the lemma is trivial) and is Borel by viewing it as a composition of a continuous and a Borel map by $\lambda \in \mathbb{R} \mapsto \lambda\mathfrak{i}(f) = \mathfrak{i}(\lambda f) \in C^\infty(\mathbb{R}) \overset{\mathfrak{i}^{-1}}{\mapsto} \lambda f \in R$, since by Lemma 3.1, $\mathfrak{i}$ is a one-to-one continuous mapping and hence $\mathfrak{i}^{-1}$ is a Borel mapping on the range of $\mathfrak{i}$. Any Borel homomorphism is continuous by [2]. Thus, $\lambda \mapsto \lambda f$ is continuous. $\square$

Theorem 3.3. *R cannot have a topology which makes it a Polish ring.*

Proof:

Assume not, that R has a topology in which it is a Polish ring. For each $f \in R$ there is a $c \in \mathbb{R}$ so that for all $k \in \mathbb{N}$ $|f^{(k)}(0)| \leq k!c^k$ and hence $\bigcup_{c \in \mathbb{N}} \{f \in R \ : \ \forall k \in \mathbb{N},\ |f^{(k)}(0)| \leq k!c^k\} = R$. For each $c \in \mathbb{N}$, $\{f \in R \ : \ \forall k \in \mathbb{N},\ |f^{(k)}(0)| \leq k!c^k\}$ is closed since it is the intersection over $k \in \mathbb{N}$ of sets of the form $\{f \in R \ : \ |f^{(k)}(0)| \leq k!c^k\}$, which are closed since they are the inverses of $[-k!c^k, k!c^k]$ under the composition of the continuous maps $\mathfrak{i} : R \to C^\infty$, differentiation k times ($C^\infty \to C^\infty$), and evaluation at 0 ($C^\infty \to \mathbb{R}$). Thus, by the Baire category theorem, there is some $n_0 \in \mathbb{N}$ and open $U \subset R$ so that $U \subset \{f \in R \ : \ \forall k \in \mathbb{N},\ |f^{(k)}(0)| \leq k!n_0^k\}$. Note that $U - U$ is open and $0 \in U - U \subset \{f \in R \ : \ \forall k \in \mathbb{N},\ |f^{(k)}(0)| \leq k!2n_0^k\}$. Now, for any $f \in R$, $\frac{1}{n}f \to 0$ as $n \to \infty$ by Lemma 3.2, and hence there is some $n \in \mathbb{N}$ so that $\frac{1}{n}f \in U - U$. Hence for any $f \in R$ there is some constant C_f so that for all $k \in \mathbb{N}$, $|f^{(k)}(0)| \leq C_f n_0^k k!$. Take $f(x) = tan^{-1}(2n_0 x)$. Now for all $k \geq 1$, $f^{(2k+1)}(0) = (2n_0)^{2k+1}(2k)!$. By above, for all $k \geq 1$, $|f^{(2k+1)}(0)| \leq C_f n_0^{2k+1}(2k+1)!$, and hence $2^{2k+1} \leq C_f(2k+1)$, a contradiction since $\left\{\frac{2^{2k+1}}{2k+1}\right\}_{k \geq 1}$ is unbounded. $\square$

CHAPTER 4

SUBRINGS OF $C^r(M)$ CONTAINING $C_0^r(X)$

Lemma 4.1. *Let $0 \leq r \leq \infty$, let M be a C^r manifold, and let R be an abstract subring of $C^r(M)$ so that $C_0^r(M) \subset R$ and R has some topology in which it is a Polish ring, then the natural injection $\mathfrak{i} : R \to C^r(M)$ is continuous. Moreover, for each $a \in M$, the associated point evaluation map $f \mapsto f(a)$, $R \to \mathbb{R}$ is continuous.*

Proof:

First it will be shown that for $a \in M$, $\mathcal{I}_a = \{f \in R : f(a) = 0\}$ is an analytic subset of R. Note that $\mathcal{I}_a = \{f \in R : f(a) \geq 0\} \cap -\{f \in R : f(a) \geq 0\}$, and so it will suffice to show that $\{f \in R : f(a) \geq 0\}$ is analytic. First, let (ψ, U) be a chart so that $a \in U$, $\psi(a) = 0$, and $B(0,1) \subset \psi(U)$, and let $g_n \in C_0^\infty(M)$ with $g_n|\psi^{-1}(B(0, \frac{1}{n+1})) = 1$ and $supp(g_n) \subset \psi^{-1}(B(0, \frac{1}{n}))$. Then as will be shown in the following paragraph, $\{f \in R : f(a) \geq 0\} = \bigcap\limits_{q \in \mathbb{Q}^+} \bigcup\limits_{n \geq 1} \{f \in R : fg_n^2 + qg_n^2 \in \mathcal{S}\}$, where $\mathcal{S} = \{s^2 : s \in R\}$. Thus since $\mathcal{S}$ is analytic and $f \mapsto fg_n^2 + qg_n^2$ is continuous, this set is a countable intersection of a countable union of analytic sets, and hence is analytic.

To show the aforementioned set equality, first assume $f \in R$ with $f(a) \geq 0$. Then for each $q \in \mathbb{Q}^+$, $f(a) + q > 0$ and hence $f + q$ is positive in $\psi^{-1}(I_q)$ for some ball I_q about 0. If $n \in \mathbb{N}$ so that $B(0, \frac{1}{n}) \subsetneq I_q$, let $h \in C^\infty(M)$ so that $h|\psi^{-1}(B(0, \frac{1}{n})) = (f+q)|\psi^{-1}(B(0, \frac{1}{n}))$, and $h > 0$ (the existence of h is guaranteed by Lemma 1.15). Then $\sqrt{h} \in C^\infty(M)$, and $(f+q)g_n^2 = hg_n^2$. Thus $fg_n^2 + qg_n^2 = (\sqrt{h}g_n)^2 \in \{k^2 : k \in C_0^\infty(M)\} \subset \mathcal{S}$.

Conversely, suppose that for each $q \in \mathbb{Q}^+$, there is an $n \in \mathbb{N}$ so that $fg_n^2 + qg_n^2 \in \mathcal{S}$. Then $fg_n^2 + qg_n^2 \geq 0$ gives $f(a) + q = (fg_n^2 + qg_n^2)(a) \geq 0$ and so $f(a) \geq -q$ for all $q \in \mathbb{Q}^+$. Thus $f(a) \geq 0$. Hence $\mathcal{I}_a$ is analytic.

For any $a \in M$, the mapping $f \mapsto f(a)$, $R \to \mathbb{R}$ is a ring homomorphism with kernel $\mathcal{I}_a$, an analytic set, and hence it is continuous by Proposition 1.7. Let $\{q_n\}_{n\geq 1} \subset M$ be dense, and then the mapping $g : R \to \mathbb{R}^{\mathbb{N}}$ given by $g(f) = \prod_{n\geq 1} f(q_n)$ is a continuous injection (since any two continuous functions agreeing on a dense set are equal). Also, the map $h : C^\infty(M) \to \mathbb{R}^{\mathbb{N}}$ given by $h(f) = \prod_{n\geq 1} f(q_n)$ is continuous, and hence its inverse mapping is a Borel mapping on the range of h, a Borel set, by Theorem 1.5. Hence, $\mathfrak{i} = h^{-1} \circ g$ is a Borel mapping and hence $\mathfrak{i}$ is continuous by Proposition 1.3. $\square$

Corollary 4.2. *If H is a Polish ring, $0 \leq r \leq \infty$, M is a C^r manifold, and $\psi : H \to C^r(M)$ is a ring isomorphism between H and $R = \psi(H)$ so that $C_0^r(M) \subset R \subset C^r(M)$, then ψ is continuous.*

Proof:

Equip R with the topology inherited from H through ψ. Then $\psi : H \to R$ is a homeomorphism by definition. Moreover, by Lemma 4.1, $\mathfrak{i} : R \to C^r(M)$ is continuous. $\psi : H \to C^r(M)$ is simply $\psi : H \to R$ composed with $\mathfrak{i}$, a composition of continuous maps which is therefore continuous. $\square$

Of course it should be noted that R is not generally closed in $C^r(M)$. For example $\mathfrak{i} : C^\infty(M) \to C^1(M)$ would be continuous but $C^\infty(M)$ is not closed in $C^1(M)$.

Theorem 4.3. *If $0 \leq r \leq \infty$, M is a C^r manifold, and R is an abstract subring of $C^r(M)$ which is Polish in some topology such that $C_0^r(M) \subset R$ then R is an algebraically determined Polish ring.*

Proof:

Let H be a Polish ring and $\varphi : H \to R$ be an algebraic isomorphism. If $\mathfrak{i} : R \to C^r(M)$ is the natural injection, then $\mathfrak{i}$ is continous by Lemma 4.1 and $\varphi \circ \mathfrak{i}$ is continuous by Corollary

4.2. Moreover, $\mathfrak{i}$ is a continuous injection, and hence by Theorem 1.5 its inverse is a Borel mapping between the range of $\mathfrak{i}$, a Borel set, and R. Thus $\varphi = \varphi \circ \mathfrak{i} \circ \mathfrak{i}^{-1}$ is the composition of a continuous and a Borel mapping, and hence is a Borel mapping. Thus by Proposition 1.3, φ is an topological isomorphism. $\square$

Theorem 4.4. *Let $0 \leq r \leq \infty$, let M_1 and M_2 be C^r manifolds, and let R_1 and R_2 be abstract subrings of $C^r(M_1)$ and $C^r(M_2)$, respectively, each with a Polish topology so that each R_i contains $C_0^r(M_i)$ as a dense subset. If $\varphi : R_1 \to R_2$ is an algebraic isomorphism of rings, then there is a C^r diffeomorphism $\alpha : M_1 \to M_2$, so that $\varphi = \alpha_*$ (i.e. $\varphi(g)(x) = g(\alpha^{-1}(x))$ for $g \in R_1$, $x \in M_2$).*

In order to prove the theorem, several lemmas need to be established first. In each of these lemmas, $0 \leq r \leq \infty$ and R is an abstract subring of $C^r(M)$ which is Polish in some topology and contains $C_0^r(M)$ as a dense subset. The following lemma is similar (but more general) than one found in [6] (who in turn references [1] for another similar result).

Lemma 4.5. *Assume $0 \leq r \leq \infty$, M is a C^r manifold, and that R is an abstract subring of $C^r(M)$ which is Polish in some topology and contains $C_0^r(M)$ as a dense subset. If $\mathfrak{J}$ is an ideal of R, then exactly one of the following is true:*

- $\mathfrak{J} \supset C_0^r(M)$
- *there is an $a \in M$ so that for all $f \in \mathfrak{J}$, $f(a) = 0$.*

Proof:

Assume that $\mathfrak{J}$ does not have the second property. Then let $g \in C_0^r(M)$. For each $a \in supp(g)$, there is a function $f_a \in \mathfrak{J}$ so that $f_a(a) \neq 0$. If $U_a = \{m \in M : f_a(m) \neq 0\}$, there are $\{a_i\}_{i=1}^n$ so that $supp(g) \subset \bigcup_{1 \leq i \leq n} U_i$, and thus $f = \sum_{1 \leq i \leq n} f_{a_i}^2$ is strictly positive on $\bigcup_{1 \leq i \leq n} U_i \supset supp(g)$ and $f \in \mathfrak{J}$. Let $h \in C_0^r(M)$ so that $h|supp(g) = \frac{1}{f}$ and $supp(h) \subset \bigcup_{1 \leq i \leq n} U_i$ (which exists by taking a function which is 1 on $supp(g)$ and multiplying it by $1/f$ on U_a). Then $hf \in \mathfrak{J}$ and is 1 on $supp(g)$. Finally notice that $g = hfg \in \mathfrak{J}$. $\square$

Corollary 4.6. *Assume* $0 \le r \le \infty$, *M is a C^r manifold, and that R is an abstract subring of $C^r(M)$ which is Polish in some topology and contains $C_0^r(M)$ as a dense subset. Then the set of ideals of R which are maximal with respect to being both closed and proper is exactly* $\{\mathcal{I}_a : a \in M\}$, *where* $\mathcal{I}_a = \{f \in R : f(a) = 0\}$.

Proof:

For each a, $\mathcal{I}_a$ is an ideal since if $f \in R$ and $g \in \mathcal{I}_a$, $fg(a) = f(a)0 = 0$, and if $h \in \mathcal{I}_a$, $(g+h)(0) = 0 + 0 = 0$. $\mathcal{I}_a$ is proper since there is a member of $C_0^r(M)$ which is nonzero at a. $\mathcal{I}_a$ is closed since point evaluation is continuous by the second proof of Theorem 4.3. To see that $\mathcal{I}_a$ is maximal, let $\mathcal{J} \supset \mathcal{I}_a$ be a closed proper ideal. $\mathcal{J} \not\supset C_0^r(M)$ since otherwise $\mathcal{J} = \overline{\mathcal{J}} \supset \overline{C_0^\infty(M)} = R$, and hence $\mathcal{J}$ is not proper. Thus $\mathcal{J} \subset \mathcal{I}_{a'}$ for some $a' \in M$ by Lemma 4.5. Now, $a' = a$ since otherwise $\mathcal{I}_a \subset \mathcal{I}_{a'}$, which is a contradiction since there is a member of $C_0^r(M)$ which is 0 at a and 1 at a'. Therefore $\mathcal{J} = \mathcal{I}_a$ and hence the $\mathcal{I}_a$'s are maximal with respect to being closed proper ideals.

Now if $\mathcal{J}$ is any maximal closed proper ideal, $\mathcal{J}$ cannot contain $C_0^r(M)$ as above, and thus there must be some $a \in M$ so that $\mathcal{I}_a \supset \mathcal{J}$ by Lemma 4.5. Thus $\mathcal{J} = \mathcal{I}_a$ since $\mathcal{J}$ is maximal and $\mathcal{I}_a$ is a closed proper ideal. $\square$

A Kuratowski closure operation on a set A is a mapping $\bar{\cdot} : \mathcal{P}(A) \to \mathcal{P}(A)$ so that

- $\overline{\emptyset} = \emptyset$
- For $E \in \mathcal{P}(A)$, $E \subset \overline{E}$
- For $E \in \mathcal{P}(A)$, $\overline{(\overline{E})} = \overline{E}$
- For $E, F \in \mathcal{P}(A)$, $\overline{E \cup F} = \overline{E} \cup \overline{F}$

A Kuratowski closure operation defines a topology in which the closed sets are $\{\overline{E} : E \in \mathcal{P}(A)\}$ ([15], Theorem 3.7).

Theorem 4.7. *Assume* $0 \le r \le \infty$, *M is a C^r manifold, and that R is an abstract subring of $C^r(M)$ which is Polish in some topology and contains $C_0^r(M)$ as a dense subset. For the* *space of maximal closed proper ideals of R, $\mathcal{M}$, define an operation $\mathcal{A} \subset \mathcal{M} \mapsto \overline{\mathcal{A}} \subset \mathcal{M}$ by $\overline{\mathcal{A}} = \{B \in \mathcal{M} \ : \ \bigcap_{J \in \mathcal{A}} J \subset B\}$. Then $\bar{\cdot}$ is a Kuratowski closure operator and the map $a \mapsto \mathcal{I}_a$ is a homeomorphism using the topology given by $\bar{\cdot}$ as set closure (sometimes called the hull-kernel topology).*

Proof:

$\mathcal{A} \subset \overline{\mathcal{A}}$ since for $J \in \mathcal{A}$, $J \supset \bigcap_{I \in \mathcal{A}} I$.

If $J \in \overline{\overline{\mathcal{A}}}$ then $J \supset \bigcap_{I \in \overline{\mathcal{A}}} I \supset \bigcap_{I \in \overline{\mathcal{A}}} (\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} A$. So $J \in \overline{\mathcal{A}}$, $\overline{\overline{\mathcal{A}}} \subset \overline{\mathcal{A}} \subset \overline{\overline{\mathcal{A}}}$ and hence they are equal.

$\overline{\emptyset} = \{A \in \mathcal{M} \ : \ A \supset \bigcap_{B \in \emptyset} B = \mathcal{M}\} = \emptyset$.

To see that $\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$, define $\nu : X \to \mathcal{M}$ by $\nu(m) = \mathcal{I}_m$, a bijection. Then we will be done if for $C \subset X$, $\nu(\overline{C}) = \overline{\nu(C)}$, since then $\overline{\mathcal{A} \cup \mathcal{B}} = \nu(\overline{\nu^{-1}(\mathcal{A} \cup \mathcal{B})}) = \nu(\overline{\nu^{-1}(\mathcal{A}) \cup \nu^{-1}(\mathcal{B})}) = \nu(\overline{\nu^{-1}(\mathcal{A})} \cup \overline{\nu^{-1}(\mathcal{B})}) = \nu(\overline{\nu^{-1}(\mathcal{A})}) \cup \nu(\overline{\nu^{-1}(\mathcal{B})}) = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$. If $m \notin \overline{C}$ then there is a function $f \in C_0^r(M)$ so that $f(m) = 1$ and $supp(f) \subset \overline{C}^c$, and hence $\overline{\nu(C)} \subset \nu(\overline{C})$. Also the fact that if $f|C = 0$ then $f|\overline{C} = 0$ will imply that $\nu(\overline{C}) \subset \overline{\nu(C)}$. □

Lemma 4.8. *Let $0 \le r \le \infty$, let M_1 and M_2 be C^r manifolds, and let R_1 and R_2 be abstract subrings of $C^r(M_1)$ and $C^r(M_2)$, respectively, each with a Polish topology so that each R_i contains $C_0^r(M_i)$ as a dense subset, and $\mathcal{M}_i$ is the space of ideals which are maximal with respect to being closed and proper, equipped with the topology from Theorem 4.7. If $\varphi : R_1 \to R_2$ is an algebraic isomorphism, then $\varphi' : \mathcal{M}_1 \to \mathcal{M}_2$ is a homeomorphism, defined by $\varphi'(J) = \{\varphi(f) \ : \ f \in J\}$.*

Proof:

φ is a topological isomorphism by Theorem 4.3, and φ' maps $\mathcal{M}_1$ to $\mathcal{M}_2$ since the image of an element of $\mathcal{M}_1$ is closed because φ is a homeomorphism, and a proper ideal since φ is an isomorphism, and if the image wasn't maximal with respect to these properties, the pullback wouldn't be either. Note that φ' is a bijection since φ is. Also, if $\mathcal{A} \subset \mathcal{M}_1$,

$\varphi'(\overline{\mathcal{A}}) = \varphi'(\{B \in \mathcal{M}_2 \ : \ B \supset \bigcap_{A \in \mathcal{A}} A\}) = \{\varphi(B) \ : \ B \in \mathcal{M}_1$ and $B \supset \bigcap_{A \in \mathcal{A}} A\} = \{\varphi(B) \ : \ B \in \mathcal{M}_1$ and $\varphi(B) \supset \bigcap_{A \in \mathcal{A}} \varphi(A)\} = \{C \in \mathcal{M}_2 \ : \ C \supset \bigcap_{A' \in \varphi'(\mathcal{A})} A'\} = \overline{\varphi'(\mathcal{A})}$, so φ' is a bijection which preserves closures and hence is a homeomorphism. □

Proof of Theorem 4.4:

First note that φ is a topological isomorphism by Theorem 4.3. Let $\phi' : \mathcal{M}_1 \to \mathcal{M}_2$ given by $\varphi'(\mathcal{I}_a) = \{\varphi(f) : f \in \mathcal{I}_a\}$. φ' is a homeomorphism by following the proof of Lemma 4.8. For $i = 1$ or 2, let $\nu_i : M_i \to \mathcal{M}_i$ be defined by $\nu_i(a) = \mathcal{I}_a$. Note that each of ν_1 and ν_2 are homeomorphisms by Theorem 4.7, and thus $\alpha = \nu_2^{-1} \circ \varphi' \circ \nu_1$ is a homeomorphism between M_1 and M_2.

For the second part of the theorem, it is prudent to first observe the behavior of φ on $C_0^r(M_1)$. Two things will be proven: that $\varphi(C_0^r(M_1)) = C_0^r(M_2)$ and that for $g \in C_0^r(M_1)$, $\varphi(g) = \alpha_*(g)$. Notice that since $\varphi|C_0^r(M_1)$ is additive and continuous, it is homogeneous with respect to scalar multiplication. Also notice that if $g \in R_1$, then if $x \in M_1$, $g(x) = 0$ if and only if $\varphi(g)(\alpha(x)) = 0$ since $g(x) = 0$ if and only if $g \in \mathcal{I}_x$ if and only if $\varphi(g) \in \varphi'(\mathcal{I}_x)$ if and only if $\varphi(g) \in \mathcal{I}_{\alpha(x)}$. Thus $supp(\varphi(g)) = \overline{\{y \in M_2 : \varphi(g)(y) \neq 0\}} = \overline{\{y \in M_2 : g(\alpha^{-1}(y)) \neq 0\}} = \overline{\alpha(\{x \in M_1 : g(x) \neq 0\})} = \alpha(\overline{\{x \in M_1 : g(x) \neq 0\}}) = \alpha(supp(g))$ and since α is a homeomorphism, if $g \in R_1$, $supp(g)$ is compact if and only if $\alpha(supp(g)) = supp(\varphi(g))$ is compact. Thus $\varphi(C_0^r(M_1)) = C_0^r(M_2)$. Next, let $M_1 = \bigcup_{j \geq 1} K_j$, where each K_j is compact, $K_j \subset int(K_{j+1})$ for each $j \geq 1$, and for each j, let $g_j \in C_0^r(M_1)$ so that $g_j|K_j = 1$ and $supp(g_j) \subset K_{j+1}$. If $g \in C_0^r(M_2)$, there is a $k_0 \geq 1$ so that $g_k\varphi^{-1}(g) = \varphi^{-1}(g)$ for $k \geq k_0$, and so $\varphi(g_k)g = g$, thus $\varphi(g_k)(x)g(x) = g(x)$ for every $x \in M_2$. Thus for every $x \in M_2$, $\varphi(g_k)(x)$ is eventually 1. So, if $g \in C_0^r(M_1)$ and $x \in M_1$, $(g(x)g_k - g_kg)(x) = 0$ gives that $\varphi(g(x)g_k - g_kg)(\alpha(x)) = 0$ and hence that $g(x)\varphi(g_k)(\alpha(x)) = \varphi(g_k)(\alpha(x))\varphi(g)(\alpha(x))$. The left-hand side of this equation is eventually $g(x)$ and the right-hand is eventually $\varphi(g)(\alpha(x))$. Thus $\varphi(g)(\alpha(x)) = g(x)$ for $g \in C_0^r(M_1)$.

If $x \in M_1$, and $g \in R_1$, then $g = g_k g + (g - g_k g)$ gives $\varphi(g)(\alpha(x)) = \varphi(g_k g)(\alpha(x)) + \varphi(g - g_k g)(\alpha(x))$. Since $g_k g \in C_0^r$, $\varphi(g_k g)(\alpha(x)) = (g_k g)(x)$, which is eventually $g(x)$. Since $(g - g_k g)(x)$ is eventually 0, $\varphi(g - g_k g)(\alpha(x))$ is eventually 0. So the right-hand side is eventually $g(x)$ and hence $\varphi(g)(\alpha(x)) = g(x)$, hence $\varphi = \alpha_*$.

It is worth noting that in the same way that α is the homeomorphism induced by the isomorphism φ, α^{-1} is the homeomorphism induced by φ^{-1}, since $\alpha^{-1} = (\nu_2^{-1} \circ \varphi' \circ \nu_1)^{-1} = \nu_1^{-1} \circ (\varphi')^{-1} \circ \nu_2 = \nu_1^{-1} \circ (\varphi^{-1})' \circ \nu_2$ and so the proof could be repeated for φ^{-1} and α^{-1}, giving that $(\alpha^{-1})_* = \varphi^{-1}$.

To see that α is C^r, it is sufficient to show that in a neighborhood of any point, $\psi \circ \alpha$ is C^r for an arbitrary C^r chart ψ of M_2 about that point. Let $x \in M_1$, and let (ψ, U) be C^r chart of M_2, so that $x \in \alpha^{-1}(U)$. Let $\hat{\psi} \in (C_0^\infty(M_2))^{dim(M_2)}$ so that $\hat{\psi}|V = \psi|V$ for some open $V \subset U$ with $x \in \alpha^{-1}(V)$. Then if $1 \leq n \leq dim(M_2)$, $(\psi \circ \alpha)_n|\alpha^{-1}(V) = (\hat{\psi} \circ \alpha)_n|\alpha^{-1}(V) = \hat{\psi}_n \circ \alpha|\alpha^{-1}(V) = \varphi^{-1}(\hat{\psi}_n)|\alpha^{-1}(V)$. Since each $\varphi^{-1}(\hat{\psi}_n) \in C_0^\infty(M_1)$, each component of $\varphi \circ \alpha$ is C^r in a neighborhood about x, and hence $\varphi \circ \alpha$ is C^r in a neighborhood about x. So α is C^r. α^{-1} is also C^r by the comments in the previous paragraph, and hence α is a C^r diffeomorphism. $\square$

CHAPTER 5

ISOMORPHISMS OF CONTINUOUS FUNCTIONS ON A LOCALLY COMPACT POLISH SPACE GIVE AN UNDERLYING HOMEOMORPHISM

In Corollary 2.5, it was shown that the continuous functions on a locally compact Polish space are algebraically determined. It is also the case that an isomorphism between certain subrings of the continuous functions on two such spaces induces a homeomorphism of the underlying spaces, similar to the results from Chapter 4, and the second half of the proof of Theorem 5.2 and the following corollary are very similar to proofs from Chapter 4 but are included for the sake of clarity.

Theorem 5.1. *Let X_1 and X_2 be locally compact Polish spaces and for $i = 1, 2$, suppose R_i is an abstract subring of $C(X_i)$ which is Polish in some topology, and R_i contains $C_0(X_i)$ as a dense subset. Then each R_i is algebraically determined, and if $\varphi : R_1 \to R_2$ is an algebraic isomporphism, then there is a homeomorphism $\alpha : X_1 \to X_2$ so that $\varphi = \alpha_*$ (i.e. $\varphi(g)(x) = g(\alpha^{-1}(x))$ for all $g \in R_1$, $x \in X_2$).*

As in Chapter 4, the proof of Theorem 5.1 will be broken up into several lemmas.

Theorem 5.2. *Let X be a locally compact Polish space and let R be an abstract subring of $C(X)$ containing $C_0(X)$ which is Polish in some topology. Then the natural injection $\mathfrak{i} : R \to C(X)$ given by $\mathfrak{i}(f) = f$ is continuous. Moreover if $x \in X$, the map $f \mapsto f(a)$, $R \to \mathbb{R}$ is continuous.*

Proof:

Let $\{U_n\}_n \geq 1$ be a precompact basis for X and let $\{f_{m,n}\}_{m,n\in\mathbb{N}}$ be as in the proof of Theorem 1.13. Then if $x \in X$, if $f \in R$ so that $f(x) \geq 0$, then if $q \in \mathbb{Q}^+$, there are

$m, n \in \mathbb{N}$ so that $x \in U_m$, $\overline{U_m} \subset U_n$ and $(f+q)|U_m > 0$ and hence $f_{m,n}(f+q) \geq 0$ and thus $\sqrt{f_{m,n}(f+g)} \in C_0(X)$. This shows that $\{f \in R : f(x) \geq 0\} \subset \bigcap_{q \in \mathbb{Q}^+} \bigcup_{\substack{m,n \geq 1, \\ x \in U_m,\ \overline{U_m} \subset U_n}} \{f \in R : f_{m,n}(f+q) \in \mathcal{S}\}$ where $\mathcal{S} = \{f^2 : f \in R\}$. Conversely if for each $q \in \mathbb{Q}^+$ there are $m, n \geq 1$ with $x \in U_m$ and $\overline{U_m} \subset U_n$ so that $f_{m,n}(f+q)$ is a square, then for each $q \in \mathbb{Q}^+$, $f_{m,n}(x)(f+q)(x) = f(x) + q \geq 0$ and hence $f(x) \geq -q$. Thus $f(x) \geq 0$, showing that $\{f \in R : f(x) \geq 0 = \bigcap_{q \in \mathbb{Q}^+} \bigcup_{\substack{m,n \geq 1, \\ x \in U_m,\ \overline{U_m} \subset U_n}} \{f \in R : f_n(f+q) \in \mathcal{S}\}$, an analytic set since $\mathcal{S}$ is and $f \mapsto f_{m,n}(f+q)$ is continuous. Thus $\mathcal{I}_x = \{f \in R : f(x) = 0\} = \{f \in R : f(x) \geq 0\} \cap \{f \in R : f(x) \geq 0\}$ is analytic.

Now if $x \in X$, the map $f \mapsto f(x)$, $R \to \mathbb{R}$ is continuous since it is a homomorphism onto the reals (since $C_0(X)$ takes all values at each $x \in X$) with analytic kernel $\mathcal{I}_x$ by Theorem 1.7.

For any $x \in X$, the mapping $f \mapsto f(x)$, $R \to \mathbb{R}$ is a ring homomorphism with kernel $\mathcal{I}_x$, an analytic set, and hence it is continuous by Proposition 1.7. Let $\{q_n\}_{n\geq 1} \subset X$ be dense, and then the mapping $g : R \to \mathbb{R}^{\mathbb{N}}$ given by $g(f) = \prod_{n \geq 1} f(q_n)$ is a continuous injection (since any two continuous functions agreeing on a dense set are equal). Also, the map $h : C(X) \to \mathbb{R}^{\mathbb{N}}$ given by $h(f) = \prod_{n \geq 1} f(q_n)$ is continuous, and hence its inverse mapping is a Borel mapping on the range of h, a Borel set, by Theorem 1.5. Hence, $\mathfrak{i} = h^{-1} \circ g$ is a Borel mapping and hence $\mathfrak{i}$ is continuous by Proposition 1.3. $\square$

Corollary 5.3. *If H is a Polish ring, X is a 2nd countable Polish space, and $\psi : H \to C(X)$ is a ring isomorphism between H and $R = \psi(H)$ so that $C_0(X) \subset R \subset C(X)$, then ψ is continuous.*

Proof:

Equip R with the topology inherited from H through ψ. Then $\psi : H \to R$ is a homeomorphism by definition. Moreover, by Lemma 4.1, $\mathfrak{i} : R \to C(X)$ is continuous. $\psi : H \to C(X)$ is simply $\psi : H \to R$ composed with $\mathfrak{i}$, a composition of continuous maps which is therefore continuous. $\square$

Corollary 5.4. *R as defined in Theorem 5.2 is algebraically determined.*

Proof:

Let H be a Polish ring and $\varphi : H \to R$ be an algebraic isomorphism. If $\mathfrak{i} : R \to C(X)$ is the natural injection, then $\mathfrak{i}$ is continous by Theorem 5.2 and $\varphi \circ \mathfrak{i}$ is continuous by Corollary 5.3. Moreover, $\mathfrak{i}$ is a continuous injection, and hence by Theorem 1.5 its inverse is a Borel mapping between the range of $\mathfrak{i}$, a Borel set, and R. Thus $\varphi = \varphi \circ \mathfrak{i} \circ \mathfrak{i}^{-1}$ is the composition of a continuous and a Borel mapping, and hence is a Borel mapping. Thus by Proposition 1.3, φ is an topological isomorphism. $\square$

Lemma 5.5. *Let R be as in Theorem 5.2. If $\mathfrak{J}$ is an ideal of R, then exactly one of the following is true:*

- $\mathfrak{J} \supset C_0(X)$
- *there is an $x \in X$ so that for all $f \in \mathfrak{J}$, $f(x) = 0$.*

Proof:

Assume $\mathfrak{J}$ does not have the second property and let $g \in C_0(X)$. For each $x \in X$ there is $f_x \in \mathfrak{J}$ so that $f_x(x) > 0$ (by squaring if necessary). Then there is some collection $\{x_1, \ldots, x_n\}$ so that $\bigcup_{i=1}^{n} W_{x_i} \supset supp(g)$, where $W_x = \{z \in X : f_x(z) > 0\}$, since the W_x are an open cover of X and $supp(g)$ is compact. Then let $f = \sum_{i=1}^{n} f_i \in \mathfrak{J}$ and note that $f|supp(g) > 0$. Let V be a precompact open set so that $supp(g) \subset V$ and $\overline{V} \subset \bigcup_{i=1}^{n} W_{x_i}$, and let $h_1 \in C_0(X, R)$ so that $h_1|supp(g) = 1$ and $h_1|(X - V) = 0$. Then define $h_2(x) = \begin{cases} \frac{h_1(x)}{f(x)} & \text{for } x \in \overline{V} \\ 0 & x \in X - V \end{cases}$ and notice that h_2 is a continuous function since each piece is continuous and the definitions agree on $\overline{V} \cap (X - V)$. Notice that $h_2 f \in \mathcal{J}$ and $h_2 f | supp(g) = 1$. Thus $g = h_2 f g \in \mathcal{J}$, and hence $C_0(X) \subset \mathcal{J}$. $\square$

Corollary 5.6. *Assume X is a 2nd countable Polish space and that R is an abstract subring of $C^r(M)$ which is Polish in some topology and contains $C_0^r(M)$ as a dense subset. Then the set of ideals of R which are maximal with respect to being both closed and proper is exactly $\{\mathcal{I}_a : a \in X\}$, where $\mathcal{I}_a = \{f \in R : f(a) = 0\}$.*

Proof:

For each $a \in X$, $\mathcal{I}_a$ is an ideal since if $f \in R$ and $g \in \mathcal{I}_a$, $fg(a) = f(a)0 = 0$, and if $h \in \mathcal{I}_a$, $(g+h)(0) = 0+0 = 0$. $\mathcal{I}_a$ is proper since there is a member of $C_0(X)$ which is nonzero at a. $\mathcal{I}_a$ is closed since point evaluation is continuous by Theorem 5.2. To see that $\mathcal{I}_a$ is maximal, let $\mathcal{J} \supset \mathcal{I}_a$ be a closed proper ideal. $\mathcal{J} \not\supset C_0(X)$ since otherwise $\mathcal{J} = \overline{\mathcal{J}} \supset \overline{C_0(X)} = R$, and hence $\mathcal{J}$ is not proper. Thus $\mathcal{J} \subset \mathcal{I}_{a'}$ for some $a' \in X$ by Lemma 5.5. Now, $a' = a$ since otherwise $\mathcal{I}_a \subset \mathcal{I}_{a'}$, which is a contradiction since there is a member of $C_0(X)$ which is 0 at a and 1 at a'. Therefore $\mathcal{J} = \mathcal{I}_a$ and hence the $\mathcal{I}_a$'s are maximal with respect to being closed proper ideals.

Now if $\mathcal{J}$ is any maximal closed proper ideal, $\mathcal{J}$ cannot contain $C_0(X)$ as above, and thus there must be some $a \in X$ so that $\mathcal{I}_a \supset \mathcal{J}$ by Lemma 5.5. Thus $\mathcal{J} = \mathcal{I}_a$ since $\mathcal{J}$ is maximal and $\mathcal{I}_a$ is a closed proper ideal. $\square$

Lemma 5.7. *Let X_1 and X_2 be 2nd countable Polish spaces and let R_1 and R_2 be abstract subrings of $C(X_1)$ and $C(X_2)$, respectively, each with a Polish topology so that each R_i contains $C_0(X_i)$ as a dense subset, and $\mathcal{M}_i$ is the space of ideals which are maximal with respect to being closed and proper, equipped with the topology from Theorem 4.7. If $\varphi : R_1 \to R_2$ is an algebraic isomorphism, then $\varphi' : \mathcal{M}_1 \to \mathcal{M}_2$ is a homeomorphism, defined by $\varphi'(J) = \{\varphi(f) \ : \ f \in J\}$.*

Proof:

φ is a topological isomorphism by Theorem 5.4, and φ' maps $\mathcal{M}_1$ to $\mathcal{M}_2$ since the image of an element of $\mathcal{M}_1$ is closed because φ is a homeomorphism, and a proper ideal since φ is an isomorphism, and if the image wasn't maximal with respect to these properties, the pullback wouldn't be either. Note that φ' is a bijection since φ is. Also, if $\mathcal{A} \subset \mathcal{M}_1$, $\varphi'(\overline{\mathcal{A}}) = \varphi'(\{B \in \mathcal{M}_2 \ : \ B \supset \bigcap\limits_{A \in \mathcal{A}} A\}) = \{\varphi(B) \ : \ B \in \mathcal{M}_1 \text{ and } B \supset \bigcap\limits_{A \in \mathcal{A}} A\} = \{\varphi(B) \ :$ $B \in \mathcal{M}_1 \text{ and } \varphi(B) \supset \bigcap\limits_{A \in \mathcal{A}} \varphi(A)\} = \{C \in \mathcal{M}_2 \ : \ C \supset \bigcap\limits_{A' \in \varphi'(\mathcal{A})} A'\} = \overline{\varphi'(\mathcal{A})}$, so φ' is a bijection which preserves closures and hence is a homeomorphism. □

Proof of Theorem 5.1:

First note that φ is a topological isomorphism by Theorem 5.4. Let $\mathcal{M}_1, \mathcal{M}_2$ be the spaces of maximal closed proper ideals of R_1 and R_2, respectively, and let $\phi' : \mathcal{M}_1 \to \mathcal{M}_2$ given by $\varphi'(\mathcal{I}_a) = \{\varphi(f) \ : \ f \in \mathcal{I}_a\}$. φ' is a homeomorphism by Lemma 5.7. For $i = 1$ or 2, let $\nu_i : M_i \to \mathcal{M}_i$ be defined by $\nu_i(a) = \mathcal{I}_a$. Note that each of ν_1 and ν_2 are homeomorphisms by Theorem 4.7, and thus $\alpha = \nu_2^{-1} \circ \varphi' \circ \nu_1$ is a homeomorphism between M_1 and M_2.

For the second part of the theorem, it is prudent to first observe the behavior of φ on $C_0(X_1)$. Two things will be proven: that $\varphi(C_0(X_1)) = C_0(X_2)$ and that for $g \in C_0(X_1)$, $\varphi(g) = \alpha_*(g)$. Notice that since $\varphi|C_0(X_1)$ is additive and continuous, it is homogeneous with respect to scalar multiplication. Also notice that if $g \in R_1$, then if $x \in X_1$, $g(x) = 0$ if and only if $\varphi(g)(\alpha(x)) = 0$ since $g(x) = 0$ if and only if $g \in \mathcal{I}_x$ if and only if $\varphi(g) \in \varphi'(\mathcal{I}_x)$ if and only if $\varphi(g) \in \mathcal{I}_{\alpha(x)}$. Thus $supp(\varphi(g)) = \overline{\{y \in X_2 : \varphi(g)(y) \neq 0\}} = \overline{\{y \in X_2 : g(\alpha^{-1}(y)) \neq 0\}} = \overline{\alpha(\{x \in X_1 : g(x) \neq 0\})} = \alpha(\overline{\{x \in X_1 : g(x) \neq 0\}}) = \alpha(supp(g))$ and since α is a homeomorphism, if $g \in R_1$, $supp(g)$ is compact if and only if $\alpha(supp(g)) = supp(\varphi(g))$ is compact. Thus $\varphi(C_0(X_1)) = C_0(X_2)$. Next, let $X_1 = \bigcup\limits_{j \geq 1} K_j$, where each K_j is compact, $K_j \subset int(K_{j+1})$ for each $j \geq 1$, and for each j, let $g_j \in C_0(X_1)$ so that $g_j|K_j = 1$ and $supp(g_j) \subset K_{j+1}$. If $g \in C_0(X_2)$, there is a $k_0 \geq 1$ so that $g_k\varphi^{-1}(g) = \varphi^{-1}(g)$ for $k \geq k_0$, and so $\varphi(g_k)g = g$, thus

$\varphi(g_k)(x)g(x) = g(x)$ for every $x \in X_2$. Thus for every $x \in X_2$, $\varphi(g_k)(x)$ is eventually 1. So, if $g \in C_0(X_1)$ and $x \in X_1$, $(g(x)g_k - g_kg)(x) = 0$ gives that $\varphi(g(x)g_k - g_kg)(\alpha(x)) = 0$ and hence that $g(x)\varphi(g_k)(\alpha(x)) = \varphi(g_k)(\alpha(x))\varphi(g)(\alpha(x))$. The left-hand side of this equation is eventually $g(x)$ and the right-hand is eventually $\varphi(g)(\alpha(x))$. Thus $\varphi(g)(\alpha(x)) = g(x)$ for $g \in C_0(X_1)$.

If $x \in X_1$, and $g \in R_1$, then $g = g_kg + (g - g_kg)$ gives $\varphi(g)(\alpha(x)) = \varphi(g_kg)(\alpha(x)) + \varphi(g - g_kg)(\alpha(x))$. Since $g_kg \in C_0(X_1)$, $\varphi(g_kg)(\alpha(x)) = (g_kg)(x)$, which is eventually $g(x)$. Since $(g - g_kg)(x)$ is eventually 0, $\varphi(g - g_kg)(\alpha(x))$ is eventually 0. So the right-hand side is eventually $g(x)$ and hence $\varphi(g)(\alpha(x)) = g(x)$, and thus $\varphi = \alpha_*$ on all of R_1. $\square$

Corollary 5.8. *Let X_1 and X_2 be locally compact Polish spaces and for $i = 1, 2$, suppose R_i is an abstract complex star-subalgebra of $C(X, \mathbb{C})$ which is Polish in some topology, and R_i contains $C_0(X, \mathbb{C})$ as a dense subset. Then each R_i is algebraically determined, and if $\varphi : R_1 \to R_2$ is an algebraic isomorphism, then there is a homeomorphism $\alpha : X_1 \to X_2$ so that $\varphi = \alpha_*$.*

Proof:

The R_i are algebraically determined by Theorem 2.7, and if $SA_i = \{f \in R_i : f^* = f\}$, then $SA_i = \{f \in R_i : f \text{ is real valued}\}$ since $f(x) = f^*(x)$ if and only if $f(x) = \overline{f(x)}$. Also, since $\varphi : SA_1 \to SA_2$ is an algebraic isomorphism, then there is an $\alpha : X_1 \to X_2$ so that $\varphi|SA_1 = \alpha_*$ by Theorem 5.1. If $f \in R_i$, then $\varphi(f) \circ \alpha = \varphi(\frac{f+f^*}{2} + i\frac{f-f^*}{2i}) \circ \alpha = \varphi(\frac{f+f^*}{2}) \circ \alpha + i\varphi(\frac{f-f^*}{2i}) \circ \alpha = \frac{f+f^*}{2} + i\frac{f-f^*}{2i} = f$ and hence $\varphi = \alpha_*$. $\square$

CHAPTER 6

POISSON BRACKETS ON $C^{\infty}(\mathbb{R}^{2n})$ AND THE HAMILTONIAN VECTOR FIELDS

If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are the standard coordinate basis for $\mathbb{R}^{2n}$, then the Poisson bracket operation on $C^{\infty}(\mathbb{R}^{2n})$ is given by $\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$. The goal in this Chapter is to show that the infinitely differentiable functions of $\mathbb{R}^{2n}$ with the Poisson brackets is an algebraically determined Polish Lie ring. When looking at the symplectic structure of R^{2n}, it is natural to divide between "x" and "y", and henceforth the statement $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$ will mean that each of $\vec{a}$ and $\vec{b}$ are real vectors of length n.

Lemma 6.1. *Let $\varphi : H \to (C^{\infty}(\mathbb{R}^{2n}), \{\cdot\}, +)$ be an algebraic isomorphism of Polish Lie rings. Then $\mathcal{C} = \{f : f \text{ is constant}\}$ is closed, $\varphi^{-1}(\mathcal{C})$ is closed and $\varphi|\varphi^{-1}(\mathcal{C})$ is continuous.*

Proof:

First notice that $\mathcal{C}$ is the center of $C^{\infty}(\mathbb{R}^{2n})$ and hence $\varphi^{-1}(\mathcal{C})$ is the center of H and thus both are closed. Next, for each i, define $M_{x,i} = \{f \in C^{\infty}(\mathbb{R}^{2n}) : \exists\lambda \in \mathbb{R}$ such that $f(\vec{x}, \vec{y}) = \lambda x_i\}$, and notice that both $M_{x,i}$ and $\varphi^{-1}(M_{x,i})$ are closed since $M_{x,i} = \bigcap_{1\leq j\leq n} \{f : \{f, x_j\} = 0\} \cap \bigcap_{1\leq j\leq n,\ j\neq i} \{f : \{f, y_j\} = 0\} \cap \{f : \{f, y_i\} \in \mathcal{C}\} \cap \{f : f = \{f, x_i y_i\}\ \}$ and so $\varphi^{-1}(M_{x,i}) = \bigcap_{1\leq j\leq n} \{r \in H : \{r, \varphi^{-1}(x_j)\} = 0\} \cap \bigcap_{1\leq j\leq n,\ j\neq i} \{r \in H : \{r, \varphi^{-1}(y_j)\} = 0\} \cap \{r \in H : \{r, \varphi^{-1}(y_i)\} \in \mathcal{C}\} \cap \{r \in H : r = \{r, \varphi^{-1}(x_i y_i)\}\ \}$. To see why this is true, assume $f \in C^{\infty}(\mathbb{R}^{2n})$ such that for $j \neq i$, $\{f, x_j\} = 0 = \{f, y_j\}$, $\{f, x_j\} = 0$ and $\{f, y_i\} = c$ and $\{f, x_i y_i\} = f$. The first equations give us that $\frac{\partial f}{\partial x_j} = 0 = \frac{\partial f}{\partial y_j}$ for $j \neq i$, $\frac{\partial f}{\partial y_i} = 0$ and $\frac{\partial f}{\partial x_i} = c$, so $f(\vec{x}, \vec{y}) = cx_i + d$. Then $cx_i + d = f = \{f, x_i y_i\} = \frac{\partial f}{\partial x_i} x_i - \frac{\partial f}{\partial y_i} y_i = cx_i$, so $f = cx_i$. Similarly each $M_{y,i} = \{f \in C^{\infty}(\mathbb{R}^{2n}) : \exists\lambda \in \mathbb{R}(f(\vec{x}, \vec{y}) = \lambda y_i)\}$ and $\varphi^{-1}(M_{y,i})$ are closed. Since $\varphi^{-1}(\mathcal{C})$ and $\varphi^{-1}(M_{x,i})$ are closed additive subgroups of H, and since the mapping $\varphi^{-1}(M_{x,i}) \to \varphi^{-1}(\mathcal{C})$ given by $\varphi^{-1}(\lambda x_i) \mapsto \{\varphi^{-1}(\lambda x_i), \varphi^{-1}(y_i)\} = \varphi^{-1}(\lambda)$ is continuous and one-to-one, and hence a homeomorphism by Proposition 1.3. Hence each $\varphi^{-1}(\lambda) \mapsto \varphi^{-1}(\lambda x_i)$ is continuous, and similarly so is each $\varphi^{-1}(\mu) \mapsto \varphi^{-1}(\mu y_i)$. Thus $* : \varphi^{-1}(\mathcal{C})^2 \to \varphi^{-1}(\mathcal{C})$ given by $(\varphi^{-1}(\lambda), \varphi^{-1}(\mu)) \in \varphi^{-1}(\mathcal{C})^2 \mapsto (\varphi^{-1}(\lambda x_1), \varphi^{-1}(\mu y_1)) \in \varphi^{-1}(M_{x,1}) \times \varphi^{-1}(M_{y,1}) \mapsto \{\varphi^{-1}(\lambda x_1), \varphi^{-1}(\mu y_1)\} = \varphi^{-1}(\lambda\mu) \in \mathcal{C}$ is continuous. Thus $(\varphi^{-1}(\mathcal{C}), +, *)$ and $(\mathcal{C}, +, \cdot)$ are each Polish rings which are isomorphic to $(\mathbb{R}, +, \cdot)$. Thus $\varphi|\mathcal{C}$ is a homeomorphism by Proposition 1.7. $\square$

Lemma 6.2. *If $f \in C^\infty(\mathbb{R}^{2n})$ and $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$, then there are $G_1, \ldots, G_n, H_1, \ldots, H_n$ such that $f(\vec{x}, \vec{y}) = f(\vec{a}, \vec{b}) + \sum\limits_{1 \le i \le n} \{\frac{x_i}{2}^2 - a_i x_i, G_i\} + \{H_i, \frac{y_i^2}{2} - b_i y_i\}$.*

Proof:

Expand f in its 2nd order Taylor expansion to get $f(\vec{x}, \vec{y}) = f(\vec{a}, \vec{b}) + \sum\limits_{1 \le i \le n} g_i(\vec{x}, \vec{y})(x_i - a_i) + h_i(\vec{x}, \vec{y})(y_i - b_i)$. Let $H_i(\vec{x}, \vec{y}) = \int\limits_0^{x_i} h_i(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n, \vec{y})dz$ and let $G_i = \int\limits_0^{y_i} g_i(\vec{x}, y_1, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_n)dz$. Then the necessary equality is satisfied since $\{\frac{x_i^2}{2} - a_i x_i, G_i\}(\vec{x}, \vec{y}) = (x_i - a_i)\frac{\partial G_i}{\partial y_i}(\vec{x}, \vec{y}) = (x_i - a_i)g_i(\vec{x}, \vec{y})$ and $\{H_i, \frac{y_i^2}{2} - b_i y_i\}(\vec{x}, \vec{y}) = \frac{\partial H_i}{\partial x_i}(\vec{x}, \vec{y}) \cdot (y_i - b_i) = h_i(\vec{x}, \vec{y})(y_i - b_i)$. $\square$

Corollary 6.3. *Let H be a Polish Lie ring and let $\varphi : H \to C^\infty(\mathbb{R}^{2n})$ be an algebraic isomorphism of Lie rings. For each $r \in H$ and $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$ there is a unique $c \in \varphi^{-1}(\mathcal{C})$ and there are $r_1, \ldots, r_n, s_1, \ldots s_n \in H$ such that $r = c + \sum\limits_{1 \le i \le n} \{\varphi^{-1}(\frac{x_i^2}{2} - a_i x_i), r_i\} + \{s_i, \varphi^{-1}(\frac{y_i^2}{2} - b_i y_i)\}$. Moreover, $c = \varphi^{-1}(\varphi(r)(\vec{a}, \vec{b}))$.*

Proof:

To see existence, apply φ^{-1} to the equation in the statement of Lemma 6.2. To see the uniqueness of c, apply φ to the equation in the statement of this lemma, and plug in $(\vec{a}, \vec{b})$. $\square$

Lemma 6.4. *Using the notation from Corollary 6.3, the mapping $r \mapsto c$ is Borel.*

Proof:

Fix $c_1, c_2 \in \varphi^{-1}(\mathcal{C})$. Define $\mathcal{S}_{\vec{a},\vec{b}} = \{(c, r, r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathcal{C} \times \prod_{i=1}^{2n+1} H : r = c + \sum_{1 \le i \le n} \{\varphi^{-1}(\frac{x_i^2}{2} - a_i x_i), r_i\} + \{s_i, \varphi^{-1}(\frac{y_i^2}{2} - b_i y_i)\}\ \}$, a closed additive subgroup of $\mathcal{C} \times \prod_{i=1}^{2n+1} H$. Notice that by Corollary 6.3, $\pi_2 : \mathcal{S}_{(\vec{a},\vec{b})} \to H$ is onto. If $(c, r, r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathcal{S}_{(\vec{a},\vec{b})}$, then $\pi_2^{-1}(r) = (c, r, r_1, \ldots, r_n, s_1, \ldots, s_n) +$
$\{(0, 0, r_1', \ldots, r_n', s_1', \ldots, s_n') : (0, 0, r_1', \ldots, r_n', s_1', \ldots, s_n') \in \mathcal{S}_{(\vec{a},\vec{b})}\}$, a closed additive subgroup of $\mathcal{S}$. Thus, by Theorem 12.17 in [8], there is a Borel set $U \subset \mathcal{S}$ such that $\pi_2|U$ is a bijection. Since projections are continuous Theorem 1.5 yields that $(\pi_2|U)^{-1}$ is a Borel mapping, and hence $\pi_1 \circ (\pi_2|U)^{-1}$ is a Borel mapping i.e. the mapping $r \mapsto c$ is Borel. $\square$

Theorem 6.5. *Let H be a Polish Lie ring and let $\varphi : H \to (C^\infty(\mathbb{R}^{2n}), +, \{\cdot\})$ an algebraic isomorphism of Lie rings. Then φ is a topological isomorphism.*

Proof:

For each $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$, define $\psi_{(\vec{a},\vec{b})} : H \to \mathbb{R}$ by $\psi_{(\vec{a},\vec{b})}(\varphi^{-1}(f)) = f(\vec{a}, \vec{b})$, which is continuous since is it the composition of the maps $\varphi^{-1}(f) \mapsto \varphi^{-1}(f(\vec{a}, \vec{b})) \mapsto f(\vec{a}, \vec{b})$, which are each continous by Lemmas 6.4 and 6.1. Let $\{(\vec{a_i}, \vec{b_i})\}_{i \ge 1} \subset \mathbb{R}^{2n}$ be dense, and define $\Psi : H \to \prod_{i \ge 1} \mathbb{R}$ by $\Phi = \prod_{i \ge 1} \psi_{(\vec{a_i},\vec{b_i})}$, a continous mapping. Also define $\Phi : C^\infty(\mathbb{R}^{2n}) \to \prod_{i \ge 1} \mathbb{R}$ by $\Phi(f) = \prod_{i \ge 1} f(\vec{a_i}, \vec{b_i})$. Since Φ is continuous and one-to-one, Φ^{-1} is a Borel mapping on the range of Φ by Theorem 1.5, so $\varphi = \Phi^{-1} \circ \Psi$ is a Borel mapping, and hence by Proposition 1.3, φ is a topological isomorphism. $\square$

One might wonder if a similar theorem holds for general symplectic manifolds instead of simply on $\mathbb{R}^{2n}$. It does not, as shown by the following theorem.

Theorem 6.6. *The Lie ring of infinitely differentiable functions on the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the Poisson bracket is not algebraically determined.*

Proof:

Note that if $f \in C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$, f can be written in $C^\infty(T^2, \mathbb{C})$ as a complex Fourier series $\sum\limits_{\ell,k\in\mathbb{Z}} c_{k,\ell} e^{2\pi i(kx+\ell y)}$. Also note that if $m, n, k, \ell \in \mathbb{Z}$, then $\{e^{2\pi i(mx+ny)}, e^{2\pi i(kx+\ell y)}\}$ is only constant if it is 0, since $\{e^{2\pi i(mx+ny)}, e^{2\pi i(kx+\ell y)}\} = (2\pi i m)e^{2\pi i(mx+ny)}(2\pi i \ell)e^{2\pi i(kx+\ell y)} - (2\pi i n)e^{2\pi i(mx+ny)}(2\pi i k)e^{2\pi i(kx+\ell y)} = 4\pi^2(nk - m\ell)e^{2\pi i[(m+k)x+(n+\ell)y]}$, so the bracket is constant if and only if $m = -k$ and $n = -\ell$. But if $m = -k$ and $n = -\ell$, then $nk - m\ell = 0$ and hence the bracket is 0. Thus, if we take a discontinuous group automorphism $\varphi : \mathbb{R} \to \mathbb{R}$ (say, a permutation of a Hamel basis), then the map $\psi : C^\infty(\mathbb{R}^2/\mathbb{Z}^2) \to C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$ defined by $\psi(\sum\limits_{\ell,k\in\mathbb{Z}} c_{k,\ell} e^{2\pi i(kx+\ell y)}) = \varphi(c_{0,0}) + \sum\limits_{\ell,k\in\mathbb{Z},\ (\ell,k)\neq(0,0)} c_{k,\ell} e^{2\pi i(kx+\ell y)}$ is an algebraic isomorphism of Lie rings since the constant terms are eliminated by the bracket and no more constant terms can be introduced. Since ψ is a discontinuous automorphism of $C^\infty(\mathbb{R}^2/\mathbb{Z}^2)$, it cannot be algebraically determined. $\square$

When looking at the symplectic structure of $\mathbb{R}^{2n}$, another important object is the Hamiltonian vector fields. For a symplectic manifold (M, ω), for each $f \in C^\infty(M)$, the Hamiltonian vector field associated with f is the vector field X_f with the property that $-df(Y) = \omega(X_f, Y)$ for any vector field Y (a full discussion can be found in [11]. Also note that in some texts, X_f is defined with the opposite sign). In $\mathbb{R}^2$ they are given by $X_f = f_x \frac{\partial}{\partial y} - f_y \frac{\partial}{\partial x}$. For $f, g \in C^\infty(M)$, $X_f = X_g$ if and only if f and g differ by constants over each connected component. Also, $X_{f+g} = X_f + X_g$ and $[X_f, X_g] = X_{\{f,g\}}$, so the Hamiltonian vector fields on M with the vector field bracket form a Lie ring which is isomorphic with $C^\infty(M)/\mathcal{C}$ with operation induced on the quotient by the Poisson bracket, where $\mathcal{C}$ is the set of locally constant functions (i.e. constant on each connected component). This is the motivation for the following work, to show that the Hamiltonian vector fields on $\mathbb{R}^{2n}$ are algebraically determined by showing that $C^\infty(\mathbb{R}^{2n})/\{f : f \text{ is constant }\}$ is algebraically determined. The methods will be similar to the methods above but the technical details are different.

Lemma 6.7. $\mathcal{C} = \{f \in C^\infty(\mathbb{R}^{2n}) : f$ *is constant* $\}$ *is a closed ideal of* $C^\infty(\mathbb{R}^{2n})$ *(and hence* $C^\infty(\mathbb{R}^{2n})/\mathcal{C}$ *makes sense as a Polish Lie ring).*

Proof:

$\mathcal{C}$ is the center of $C^\infty(\mathbb{R}^{2n})$ and hence is closed. Also, if $f \in \mathcal{C}$, $g \in C^\infty(\mathbb{R}^{2n})$, then $\{f, g\} = 0 \in \mathcal{C}$. $\square$

For Lemma 6.8 through Theorem 6.10, $n \in \mathbb{N}$, and $\mathcal{C} = \{f \in C^\infty(\mathbb{R}^{2n}) : f$ is constant $\}$.

Lemma 6.8. *If* H *is a Polish Lie ring and* $\varphi : H \to C^\infty(\mathbb{R}^{2n})/\mathcal{C}$ *an algebraic isomorphism of Lie rings, then for each* $1 \leq i \leq n$*, the set* $\mathcal{L}_{x,i} = \varphi^{-1}(\{f + \mathcal{C} : \exists c \in \mathbb{R}\ \forall (\vec{x}, \vec{y}) \in \mathbb{R}^{2n},\ f(\vec{x}, \vec{y}) = cx_i + d\})$ *and* $\mathcal{L}_{y,i} = \varphi^{-1}(\{f + \mathcal{C} : \exists c \in \mathbb{R}\ \forall (\vec{x}, \vec{y}) \in \mathbb{R}^{2n},\ f(\vec{x}, \vec{y}) = cy_i + d\})$ *are closed subsets of* H *and* $\varphi|\mathcal{L}_{x,i}$ *and* $\varphi|\mathcal{L}_{y,i}$ *are continuous.*

Proof:

Fix $1 \leq i \leq n$ and notice that $\mathcal{L}_{x,i} = \{h \in H : \{h, \varphi^{-1}(x_i y_i + \mathcal{C})\} = h\} \cap \bigcap_{1 \leq j \leq n} \{h \in H : \{h, \varphi^{-1}(x_j + \mathcal{C})\} = 0 = \{h, \varphi^{-1}(y_j + \mathcal{C})\}\}$ and $\mathcal{L}_{y,i} = \{h \in H : \{\varphi^{-1}(x_i y_i + \mathcal{C}), h\} = h\} \cap \bigcap_{1 \leq j \leq n} \{h \in H : \{h, \varphi^{-1}(x_j + \mathcal{C})\} = 0 = \{h, \varphi^{-1}(y_j + \mathcal{C})\}\}$ and so each $\mathcal{L}_{x,i}$ or $\mathcal{L}_{y,i}$ are closed, additive subgroups of H. To see this, take $h \in H$ such that for $1 \leq j \leq n$, $\{h, \varphi^{-1}(x_j + \mathcal{C})\} = 0 = \{h, \varphi^{-1}(y_j + \mathcal{C})\}$ and such that $\{h, \varphi^{-1}(x_i y_i + \mathcal{C})\} = h$. Then the first equalities give that, if $f \in \varphi(h)$, $-\frac{\partial f}{\partial y_j} \in \mathcal{C}$ and $\frac{\partial f}{\partial x_j} \in \mathcal{C}$, so $f = c + \sum_{1 \leq j \leq n} (a_j x_j + b_j y_j)$, and so the last equality gives that $c + \sum_{1 \leq j \leq n} (a_j x_j + b_j y_j) = f = \{f, x_i y_i\} = a_i x_i - b_i y_i \ (mod\ \mathcal{C})$, so $f \in a_i x_i + \mathcal{C}$. The equality for $\mathcal{L}_{y,i}$ is proved in a similar manner.

In order to show that φ is continuous on the $\mathcal{L}$'s, we will define a multiplication on them which will be continous and will create a ring structure on the $\mathcal{L}$'s which mirrors $\mathbb{R}^{2n}$ with addition and componentwise multiplication, which will then imply that the $\mathcal{L}$'s are homeomorphic with $\mathbb{R}$. In order to do this, first define $Q_{x,i} = \{\varphi^{-1}(f + \mathcal{C}) : \exists c, d \in \mathbb{R}\ \forall (\vec{x}, \vec{y}) \in \mathbb{R}^{2n},\ f(\vec{x}, \vec{y}) = cx_i^2\}$ and $Q_{y,i} = \{\varphi^{-1}(f + \mathcal{C}) : \exists c, d \in \mathbb{R}\ \forall (\vec{x}, \vec{y}) \in \mathbb{R}^{2n},\ f(\vec{x}, \vec{y}) = cy_i^2\}$.

Each $Q_{x,i}$ or $Q_{y,i}$ is closed because $Q_{x,i} = \{h \in H : \{h, \varphi^{-1}(x_i y_i + \mathcal{C})\} = 2h\} \cap \{h \in H : \{h, \varphi^{-1}(y_i + \mathcal{C})\} \in \mathcal{L}_{x,i}\} \cap \bigcap_{1 \le j \le n} \{h \in H : \{h, \varphi^{-1}(x_j + \mathcal{C})\} = 0\} \cap \bigcap_{1 \le j \le n,\ i \neq j} \{h \in H : \{h, \varphi^{-1}(y_j + \mathcal{C})\} = 0\}$ and $Q_{y,i} = \{h \in H : \{\varphi^{-1}(x_i y_i + \mathcal{C}), h\} = 2h\} \cap \{h \in H : \{h, \varphi^{-1}(x_i + \mathcal{C})\} \in \mathcal{L}_{y,i}\} \cap \bigcap_{1 \le j \le n} \{h \in H : \{h, \varphi^{-1}(y_j + \mathcal{C})\} = 0\} \cap \bigcap_{1 \le j \le n,\ i \neq j} \{h \in H : \{h, \varphi^{-1}(x_j + \mathcal{C})\} = 0\}$. To see why this holds, let $h \in H$ such that $\{h, \varphi^{-1}(x_i y_i + \mathcal{C})\} = 2h$, such that for all $j \neq i$, $\{h, \varphi^{-1}(y_j + \mathcal{C})\} = 0 = \{h, \varphi^{-1}(x_j + \mathcal{C})\}$, $\{h, \varphi^{-1}(y_i + \mathcal{C})\} \in \mathcal{L}_{x,i}$ and $\{h, \varphi^{-1}(x_i + \mathcal{C})\} = 0$. Then if $f \in \varphi^{-1}(h)$, this means that for $j \neq i$, $\frac{\partial f}{\partial x_j} = 0 = \frac{\partial f}{\partial y_j}$ $(mod\ \mathcal{C})$, there is some $c \in \mathbb{R}$ so that $\frac{\partial f}{\partial x_i} \in 2cx_i + \mathcal{C}$ and $\frac{\partial f}{\partial y_i} \in \mathcal{C}$. So we have that $f(\vec{x}, \vec{y}) = cx_i^2 + d + \sum_{1 \le j \le n} a_j x_j + b_j y_j$ and hence $2f(x, y) = 2cx_i^2 + 2d + 2 \sum_{1 \le j \le n} (a_j x_j + b_j y_j) = \{f, x_i y_i\} = 2cx_i^2 + a_i x_i - b_i y_i$ $(mod\ \mathcal{C})$. Hence, $a_i x_i + 3b_i y_i + \sum_{j \neq i} (2a_j x_j + 2b_j y_j) \in \mathcal{C}$ and so each $a_j = 0$ and each $b_j = 0$. Thus $f(\vec{x}, \vec{y}) = cx_i^2 + d$ and hence $h = \varphi(f + \mathcal{C}) = \varphi(cx_i^2 + \mathcal{C})$. A similar argument works for the $Q_{y,i}$'s.

So each Q is a closed, additive subgroup of H. Moreover, if $h_1 \in Q_{x,i}$, then for some $b \in \mathbb{R}$, $\varphi(h_1) = bx_i^2 + \mathcal{C}$ and hence $\{h_1, \varphi^{-1}(y_i + \mathcal{C})\} = \varphi^{-1}(2bx_i + \mathcal{C})$, and if $h_2 \in \mathcal{L}_{y,i}$, then for some $c \in \mathbb{R}$, $\varphi(h_2) = cy_i + \mathcal{C}$ and so $\{\varphi^{-1}(\frac{x_i^2}{2} + \mathcal{C}), h_2\} = \varphi^{-1}(cx_i + \mathcal{C})$, and so the maps $h_1 \in Q_{x,i} \mapsto \{h_1, \varphi^{-1}(x_i)\} = \varphi^{-1}(cx_i + \mathcal{C}) \in \mathcal{L}_{x,i}$ and $h_2 \in \mathcal{L}_{y,i} \mapsto \{\varphi^{-1}(\frac{x_i^2}{2}), h_2\} = \varphi^{-1}(cx_i + \mathcal{C}) \in \mathcal{L}_{x,i}$ are continuous isomorphisms of additive Polish groups, and hence their inverses are continuous by Proposition 1.3. These inverses are precisely $\varphi^{-1}(cx_i + \mathcal{C}) \in \mathcal{L}_{x,i} \mapsto \varphi^{-1}(\frac{cx_i^2}{2} + \mathcal{C}) \in Q_{x,i}$ and $\varphi^{-1}(dx_i + \mathcal{C}) \in \mathcal{L}_{x,i} \mapsto \varphi^{-1}(dy_i + \mathcal{C}) \in \mathcal{L}_{y,i}$. So, define $* : \mathcal{L}_{x,i} \times \mathcal{L}_{x,i} \to \mathcal{L}_{x,i}$ by $(\varphi^{-1}(cx_i + \mathcal{C}), \varphi^{-1}(dx_i + \mathcal{C})) \mapsto \varphi^{-1}(cdx_i + \mathcal{C})$. $*$ is continuous since it is the composition of the maps $(\varphi^{-1}(cx_i + \mathcal{C}), \varphi^{-1}(dx_i + \mathcal{C})) \mapsto (\varphi^{-1}(\frac{cx_i^2}{2} + \mathcal{C}), \varphi^{-1}(dy_i + \mathcal{C})) \mapsto \{\varphi^{-1}(\frac{cx_i^2}{2} + \mathcal{C}), \varphi^{-1}(dy_i + \mathcal{C})\} = \varphi^{-1}(cdx_i + \mathcal{C})$, and hence $(\mathcal{L}_{x,i}, +, *)$ is a Polish ring which is isomorphic to $(\mathbb{R}, +, \cdot)$. Thus the isomorphism $\varphi^{-1}(cx_i + \mathcal{C}) \mapsto c$ is continuous by Proposition 1.7 and hence $\varphi|\mathcal{L}_{x,i}$ is continuous since it is the composition of the continuous maps $\varphi^{-1}(cx_i + \mathcal{C}) \in \mathcal{L}_{x,i} \mapsto c \in \mathbb{R} \mapsto cx_i + \mathcal{C} \in C^\infty(\mathbb{R}^{2n})/\mathcal{C}$. The proof is completed by using a similar argument for each $\mathcal{L}_{y,i}$. $\square$

Lemma 6.9. *For each $h \in H$, $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$, $1 \leq i \leq n$, there is a unique $d_{x,i} \in \mathcal{L}_{x,i}$ and a unique $d_{y,i} \in \mathcal{L}_{y,i}$ such that there are $r_1, \ldots, r_n, s_1, \ldots, s_n, t_1, \ldots, t_n, u_1, \ldots, u_n \in H$ such that*

$$h = d_{x,i} + \{\varphi^{-1}(\frac{x_i^3}{3} - a_i x_i^2 + a_i^2 x_i + \mathcal{C}), r_i\} + \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \{\varphi^{-1}(\frac{x_j^2}{2} - a_j x_j + \mathcal{C}), r_j\}$$

$$+ \sum_{1 \leq j \leq n} \{s_j, \varphi^{-1}(\frac{y_j^2}{2} - b_j y_j + \mathcal{C})\}, \text{ and}$$

$$h = d_{y,i} + \{t_i, \varphi^{-1}(\frac{y_i^3}{3} - b_i y_i^2 + b_i^2 y_i + \mathcal{C})\} + \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \{t_j, \varphi^{-1}(\frac{y_j^2}{2} - b_j y_j + \mathcal{C})\}$$

$$+ \sum_{1 \leq j \leq n} \{\varphi^{-1}(\frac{x_j^2}{2} - a_j x_j + \mathcal{C}), u_j\}.$$

Moreover the maps $h \mapsto d_{x,i}$ and $h \mapsto d_{y,i}$ are Borel.

Proof:

The lemma will be proven for $d_{x,1}$. The other $2n-1$ cases are proved in a similar manner. If $f \in C^\infty(\mathbb{R}^{2n})$ and $(\vec{a}, \vec{b}) \in \mathbb{R}^{2n}$, expand f in its 2nd order Taylor expansion about $(\vec{a}, \vec{b})$ and then collect all terms which involve linear factors with variables other than x_1 to see that $f(x, y) = c + ax_1 + u_1(x_1 - a_1)^2 + \sum\limits_{2 \leq j \leq n} u_j(x_j - a_j) + \sum\limits_{1 \leq j \leq n} v_j(y_j - a_j)$, where each u_j and v_j are smooth functions. For each $1 \leq j \leq n$, let $V_j(\vec{x}, \vec{y}) = \int\limits_0^{x_j} v_j(x_1, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_n, \vec{y}) dz$ (an "x_j" antiderivative of v_j) and $U_j = \int\limits_0^{y_j} u_j(\vec{x}, y_1, \ldots, y_{j-1}, z, y_{j+1}, \ldots, y_n) dz$ (a "y_j" antiderivative of u_j). Then $f + \mathcal{C} = ax_1 + \mathcal{C} + \{\frac{x_1^3}{3} - a_1 x_1^2 + a_1^2 x_1 + \mathcal{C}, U_1 + \mathcal{C}\} + \sum\limits_{2 \leq j \leq n} \{\frac{x_j^2}{2} - a_j x_j + \mathcal{C}, U_j + \mathcal{C}\} + \sum\limits_{1 \leq j \leq n} \{V_j + \mathcal{C}, \frac{y_j^2}{2} - b_j y_j + \mathcal{C}\}$. Apply φ^{-1} to get the equation desired in the statement of the lemma. The uniqueness of $\varphi^{-1}(ax_1 + \mathcal{C})$ is obtained by mapping by φ, taking $\frac{\partial}{\partial x_1}$ and then plugging in $(\vec{a}, \vec{b})$ to get that $a = \frac{\partial f}{\partial x_1}(\vec{a}, \vec{b})$.

To see that $h \mapsto d_{x,1}$ is Borel, first define $\mathcal{S}_{\vec{a},\vec{b}} = \{(d, h, r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathcal{L}_{x,1} \times H^{2n+1} : h = d + \{\varphi^{-1}(\frac{x_i^3}{3} - a_i x_i^2 + a_i^2 x_i + \mathcal{C}), r_i\} + \sum\limits_{1 \leq j \leq n,\ j \neq i} \{\varphi^{-1}(\frac{x_j^2}{2} - a_j x_j), r_j\} + \sum\limits_{1 \leq j \leq n} \{s_j, \varphi^{-1}(\frac{y_j^2}{2} - b_j y_j)\}\ \}$, a closed additive subgroup of $\mathcal{L}_{x,1} \times H^{2n+1}$. Define $\pi_1 : \mathcal{S}_{(\vec{a},\vec{b})} \to$

$\mathcal{L}_{x,1}$ and $\pi_2 : \mathcal{S}_{(\vec{a},\vec{b})} \to H$ be the projection from tuples in $\mathcal{S}_{(\vec{a},\vec{b})}$ to the first and second coordinates, and let $\mathcal{Z}_{(\vec{a},\vec{b})} = \pi_1^{-1}(0) \cap \pi_2^{-1}(0)$, a closed additive subgroup of $\mathcal{S}_{(\vec{a},\vec{b})}$. Note that $\pi_2 : \mathcal{S}_{(\vec{a},\vec{b})} \to H$ is surjective, and if $(d, h, r_1, \ldots, r_n, s_1, \ldots, s_n) \in \mathcal{S}_{(\vec{a},\vec{b})}$, then $\pi_2^{-1}(h) = (d, h, r_1, \ldots, r_n, s_1, \ldots, s_n) + \mathcal{Z}_{(\vec{a},\vec{b})}$. Since $Z_{(\vec{a},\vec{b})}$ is a closed subgroup of $\mathcal{S}_{(\vec{a},\vec{b})}$, Theorem 12.17 in [8] gives that there is a Borel subset $U \subset \mathcal{S}_{(\vec{a},\vec{b})}$ such that $\pi_2|U$ is one-to-one. Thus by Theorem 1.5, $(\pi_2|U)^{-1}$ is a Borel mapping on H. Thus the map $\pi_1 \circ (\pi_2|U)^{-1}$, being the composition of a Borel and a continuous mapping, is a Borel mapping. In other words the mapping $h \mapsto d_{x,1}$ is a Borel mapping. $\square$

Theorem 6.10. *If $\varphi : H \to C^\infty(\mathbb{R}^{2n})/\mathcal{C}$ is an algebraic isomorphism of Lie rings, then φ is also a topological isomorphism (i.e. $C^\infty(\mathbb{R}^{2n})/\mathcal{C}$ is algebraically determined)*

Proof:

Using the notation and results from the previous two lemmas, for each $1 \le i \le n$ and each $(\vec{a}, \vec{b})$, $h \mapsto d_{x,i} = \varphi^{-1}(\frac{\partial \varphi(h)}{\partial x_i}|_{(\vec{a},\vec{b})} x_i + \mathcal{C}) \in \mathcal{L}_{x,i}$ is continuous (here, $\frac{\partial \varphi(h)}{\partial x_i}$ means $\frac{\partial f}{\partial x_i}$ for any $f \in \varphi(h)$). Also, $\varphi|\mathcal{L}_{x,i}$ is continuous by Lemma 6.8, and $ax + \mathcal{C} \mapsto a$ is continuous since it is an additive isomorphism of Polish groups and $(\mathbb{R}, +, \cdot)$ are algebraically determined by Proposition 1.7. By composition of these maps we get the continuous map $\varphi^{-1}(f + \mathcal{C}) \mapsto \frac{\partial f}{\partial x_i}(\vec{a}, \vec{b})$. Similarly each $\varphi^{-1}(f + \mathcal{C}) \mapsto \frac{\partial f}{\partial y_i}(\vec{a}, \vec{b})$ is continuous. Now let $\{(\vec{a_j}, \vec{b_j})\}_{j \ge 1} \subset \mathbb{R}^{2n}$ be dense, and let $\Phi : H \to \prod_{j \ge 1} \mathbb{R}^{2n}$ be the continuous map defined by $\Phi(\varphi^{-1}(f + \mathcal{C})) = \prod_{j \ge 1} (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n})(\vec{a_j}, \vec{b_j})$. Define $\Psi : C^\infty/\mathcal{C} \to \prod_{j \ge 1} \mathbb{R}^{2n}$ to be the continuous map given by $\Psi(f + \mathcal{C}) = \prod_{j \ge 1} (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n})(\vec{a_j}, \vec{b_j})$, which is well-defined since each member of a given coset has the same derivatives. Ψ is a one-to-one continuous map, so by Theorem 1.5, Ψ^{-1} is a continuous map on the image of Ψ, a Borel set. Thus $\Psi^{-1} \circ \Phi = \varphi$ is a Borel isomorphism of Polish groups, and hence φ is a topological isomorphism by Proposition 1.3. $\square$

The following theorem uses identical ideas as Theorem 6.10 but is very specific and hence is stated separately. This more specific version will be used to show that the Hamiltonian vector fields on a symplectic manifold are algebraically determined. The following lemmas could be done with any interval replacing $(-1,1)$ (for closed intervals one would need to use the Whitney extension theorem (see [14] for the real analytic version, and see [5] for the C^∞ version))

Theorem 6.11. *Let R be a Lie subring of $(C^\infty((-1,1)^{2n}),\{\cdot\})$ which contains the multinomial functions of orders less than or equal to 3, and so that for each $f \in R$ and $(\vec{a},\vec{b}) \in (-1,1)^{2n}$, then there are $G_1,\dots,G_n,H_1,\dots,H_n \in R$ such that*

$$f(\vec{x},\vec{y}) = f(\vec{a},\vec{b}) + \sum_{1\le i\le n}\{\frac{{x_i}^2}{2} - a_i x_i, G_i\} + \{H_i, \frac{y_i^2}{2} - b_i y_i\}.$$

Then, if R is Polish in some topology, R is algebraically determined. Moreover if $\mathcal{C} = \{f \in R : f \text{ is constant}\}$, and R is so that for each $f \in R$, $(\vec{a},\vec{b}) \in \mathbb{R}^{2n}$ and $1 \le i \le n$, there are $r_1,\dots,r_n,s_1,\dots,s_n,t_1,\dots,t_n,u_1,\dots,u_n \in R$ and $c,d \in \mathbb{R}$ such that

$$f = c + \frac{\partial f}{\partial x_i}(\vec{a},\vec{b})x_i + \{\frac{x_i^3}{3} - a_i x_i^2 + a_i^2 x_i, r_i\} + \sum_{\substack{1\le j\le n\\ j\ne i}}\{\frac{x_j^2}{2} - a_j x_j, r_j\} + \sum_{1\le j\le n}\{s_j, \frac{y_j^2}{2} - b_j y_j\}$$

and

$$f = d + \frac{\partial f}{\partial y_i}(\vec{a},\vec{b})y_i + \{t_i, \frac{y_i^3}{3} - b_i y_i^2 + b_i^2 y_i\} + \sum_{\substack{1\le j\le n\\ j\ne i}}\{t_j, \frac{y_j^2}{2} - b_j y_j\} + \sum_{1\le j\le n}\{\frac{x_j^2}{2} - a_j x_j, u_j\}$$

then $R/\mathcal{C}$ is algebraically determined.

The proof of this theorem is exactly the proof of Theorems 6.5 and 6.10 and the associated lemmas.

Proposition 6.12. *Let (ψ, U) be a symplectic chart of a symplectic manifold M of dimension $2n$ so that $\psi(U) = \mathbb{R}^{2n}$. Define $\mathcal{C} = \{f \in C^\infty(M) : f \text{ is locally constant}\}$ and $\mathcal{C}_\psi =$*

$\{f + \mathcal{C} : f|\psi^{-1}((-1,1)^{2n})$ *is constant*$\}$. $\mathcal{C}$ *is a closed ideal of* $C^\infty(M)$, $\mathcal{C}_\psi$ *is a closed ideal of* $C^\infty(M)/\mathcal{C}$, *and* $(C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi$ *is algebraically determined.*

Proof:

First notice that $\mathcal{C}$ is the center of $C^\infty(M)$ and hence is closed. Also since if $c \in \mathcal{C}$ and $f \in C^\infty(M)$, $\{f, c\} = 0 \in \mathcal{C}$ by Darboux's Theorem (found in [11], this theorem will be used throughout without further mention), $\mathcal{C}$ is an ideal and hence the quotient $C^\infty(M)/\mathcal{C}$ makes sense and is a Polish Lie ring.

$\mathcal{C}_\psi$ is closed since it is the centralizer in $C^\infty(M)/\mathcal{C}$ of $S = \{Z \in C^\infty(M)/\mathcal{C} : \exists f \in Z$ such that $supp(f) \subset \psi^{-1}((-1,1)^{2n})\})$. To see this, let $Z \in C^\infty(M)/\mathcal{C}$ and $f \in Z$ so that $supp(f) \subset \psi^{-1}((-1,1)^{2n})$. Now, if $g \in G \in \mathcal{C}_\varphi$, then if $a \notin supp(f)$, $\{f, g\}(a) = 0$ since all derivatives of f vanish at a. If $b \in supp(f)$, then g is constant in a neighborhood of b and hence all derivatives of g vanish at b, so $\{f, g\}(b) = 0$. Thuse $\{f, g\} = 0$ and hence $\{Z, G\} = 0$ and so $\mathcal{C}_\psi$ is a subset of the centralizer of S. Conversely take G in the centralizer of S and take $a \in (-1,1)^n$ and $1 \leq i \leq n$. Then there is some $f \in C^\infty(M)$ and open $W \subset (-1,1)^{2n}$ so that $a \in \psi^{-1}(W)$, $\phi_*(f)|W = x_i$ and $supp(f) \subset \psi^{-1}((-1,1)^{2n})$. Then if $g \in G$, $\psi_*(\{f, g\}) = \frac{\partial \psi_*(g)}{\partial y_i}$ is constant and so is $\psi_*(\{f^2, g\}) = 2\frac{\partial \psi_*(g)}{\partial y_i} x_i$. Thus $\frac{\partial \psi_*(g)}{\partial y_i} = 0$ on W. Repeating this argument with x_i and y_i interchanged yields that $\psi_*(g)$ is constant on a neighborhood of each point of $(-1,1)^{2n}$ and hence $g|\psi^{-1}((-1,1)^{2n})$ is constant. Hence $G \in \mathcal{C}_\psi$ and so $\mathcal{C}_\psi$ is the centralizer of S.

Note also that if $f \in F \in \mathcal{C}_\psi$ and $g \in C^\infty(M)$, $\{f, g\}|\psi^{-1}((-1,1)^{2n})$ is zero and hence $\{F, g + \mathcal{C}\} \in \mathcal{C}_\psi$, so $\mathcal{C}_\psi$ is an ideal and hence $(C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi$ is a Polish ring.

Take C to be the constant functions in $(-1,1)^{2n}$ and now Theorem 6.11 gives that $(C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi$ is algebraically determined since it embeds isomorphically as a subring of $C^\infty((-1,1)^{2n})/C$ by $(f + \mathcal{C}) + \mathcal{C}_\psi \mapsto \psi_*(f)|(-1,1)^{2n} + C$. To see why $\psi_*((C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi)$ has the desired property from the statement of Theorem 6.11, follow the proof of Lemma 6.9, noting that each $\psi_*(f) \in C^\infty((-2,2)^{2n})$ and so we can apply Lemma 1.17 to the antiderivatives with $U = \psi^{-1}((-1,1)^{2n})$ and $V = \psi^{-1}((-2,2)^{2n})$ to see that those antiderivatives are in $\psi_*((C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi)$. $\square$

Theorem 6.13. *Let M be a smooth symplectic manifold of dimension $2n$. Let $\mathcal{C} = \{f \in C^\infty(M) : f$ is constant on connected components of $M\}$. Then $\mathcal{C}$ is a closed ideal of $(C^\infty(M), \{\cdot\})$ and hence $C^\infty(M)/\mathcal{C}$ is a Polish Lie ring. Moreover, $C^\infty(M)/\mathcal{C}$ is an algebraically determined Polish Lie ring. The Lie ring of Hamiltonian vector fields on a symplectic manifold is also an algebraically determined Polish Lie ring, since it is algebraically isomorphic to $C^\infty(M)/\mathcal{C}$.*

Proof:

By Proposition 6.12, $(C^\infty(M), +, \{\cdot\})$ is a Polish Lie ring. Let $\varphi : H \to C^\infty(M)/\mathcal{C}$ be an algebraic ring isomorphism. Then for any chart (U, ψ) so that $\psi(U) = \mathbb{R}^{2n}$, then as in Proposition 6.12 define $\mathcal{C}_\psi = \{f + \mathcal{C} \in C^\infty(M)/\mathcal{C} : \psi_* f|(-1,1)^{2n}$ is constant$\}$. If S is as in the proof of Proposition 6.12, $\mathcal{C}_\psi$ is the centralizer of S and hence $\varphi^{-1}(\mathcal{C}_\psi)$ is the centralizer of $\varphi^{-1}(S)$ and hence closed. Since $\mathcal{C}_\psi$ is an ideal, so is $\varphi^{-1}(\mathcal{C}_\psi)$, so $H/\varphi^{-1}(C_\psi)$ is a Polish Lie ring. φ induces an isomorphism between the quotients, $\varphi_\psi : H/\varphi^{-1}(\mathcal{C}_\psi) \to (C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi$, by $\varphi_\psi(h + \varphi^{-1}(\mathcal{C}_\psi)) = \varphi(h) + \mathcal{C}_\psi$. Since $(C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi$ is algebraically determined by Proposition 6.12, φ_ψ is a homeomorphism and hence the map $h \mapsto h + \varphi^{-1}(\mathcal{C}_\psi) \mapsto \varphi(h) + \mathcal{C}_\psi$ is continuous.

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & C^\infty(M)/\mathcal{C} \\ \downarrow & & \downarrow \\ H/\varphi^{-1}(\mathcal{C}_\psi) & \xrightarrow{\varphi_\psi} & (C^\infty(M)/\mathcal{C})/\mathcal{C}_\psi \end{array}$$

Now, take $\{(U_i, \psi_i\}_{i \geq 1}$ be a collection of charts of M so that $\{\psi_i^{-1}((-1,1)^{2n})\}_{i \geq 1}$ covers M. For each i, let Φ_i be the continuous map $H \to H/\varphi^{-1}(\mathcal{C}_{\psi_i})$ by $h \mapsto \varphi(h) + \mathcal{C}_{\psi_i}$. Thus $\Phi = \prod_{i \geq 1} \Phi_i$ is a continuous map. Notice also that the map $\Theta : C^\infty(M)/\mathcal{C} \to$

$\prod_{i\geq 1}((C^\infty(M)/\mathcal{C})/\mathcal{C}_{\psi_i})$ given by $\Theta(f+\mathcal{C}) = \prod_{i\geq 1}(f+\mathcal{C})+\mathcal{C}_{\psi_i}$ is continuous, and it is also one-to-one since if $\Theta(f) = \Theta(g)$, then on each factor i, $(\psi_i)_*(f)$ and $(\psi_i)_*(g)$ differ by a constant on $(-1,1)^{2n}$. Since the $\psi_i^{-1}((-1,1)^{2n})$ cover M, $f-g\in\mathcal{C}$ and so $f+\mathcal{C} = g+\mathcal{C}$. Thus, by Theorem 1.5, its inverse is a Borel function on its range, a Borel set. Thus $\varphi = \Theta^{-1}\circ\Phi$ is a Borel mapping and hence a topological isomorphism by Proposition 1.3. $\square$

CHAPTER 7

COMPLEX VECTOR FIELDS

The ring of smooth real vector fields on a smooth manifold was shown to be algebraically determined in [7]. It is unlikely that the ring of smooth complex vector fields on a smooth complex manifold would be, because the ring of complex numbers itself is not algebraically determined. However, if we view it as an algebra instead of just a ring, we get enough structure to determine the Polish topology. Here the case of $\mathcal{L}(\mathbb{C})$, the complex vector fields over the complex plane, is given.

Lemma 7.1. *Let $\mathcal{R}$ be a Polish Lie algebra over $\mathbb{C}$, let $\varphi : \mathcal{R} \to \mathcal{L}(\mathbb{C})$ be an algebraic isomorphism of Lie algebras and $\mathcal{C} = \{ \lambda D \mid \lambda \in \mathbb{C} \}$. Then $\mathcal{C}$ is a closed Lie subalgebra of $\mathcal{L}(\mathbb{C})$ and $\varphi^{-1}(\mathcal{C})$ is closed in $\mathcal{R}$. Continuous binary operations can be defined on both $\mathcal{C}$ and $\varphi^{-1}(\mathcal{C})$ with respect to which they are both algebraically and topologically isomorphic to the Polish algebra $\mathbb{C}$ and such that $\varphi|\varphi^{-1}(\mathcal{C}) : \varphi^{-1}(\mathcal{C}) \to \mathcal{C}$ is a topological isomorphism of Polish fields.*

Proof:

$\mathcal{C}$ is closed in $\mathcal{L}(\mathbb{C})$ and the mapping $\lambda D \to \lambda$, $\mathcal{C} \to \mathbb{C}$ is a topological isomorphism between two additive abelian Polish groups by the definition of the topology on $\mathcal{L}(\mathbb{C})$. Note also that $\mathcal{C} = \{ fD \in \mathcal{L}(\mathbb{C}) \mid [fD, D] = 0 \}$ since $[fD, D] = -f'D$. This observation gives an algebraic proof that $\mathcal{C}$ is a closed commutative Lie subalgebra of $\mathcal{L}(\mathbb{C})$. Hence, $\varphi^{-1}(\mathcal{C}) = \{ r \in \mathcal{R} \mid [r, \varphi^{-1}(D)] = 0 \}$ is closed in $\mathcal{R}$. Notice that the binary operation $(\lambda D, \mu D) \to [\lambda D, [\mu D, \frac{x^2}{2}D]] = [\lambda D, \mu x D] = \lambda\mu D$, $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ makes $\mathcal{C}$ into a commutative Polish algebra which is algebraically isomorphic to the field $\mathbb{C}$. Similarly, notice that the binary operation $(\varphi^{-1}(\lambda D), \varphi^{-1}(\mu D)) \to [\varphi^{-1}(\lambda D), [\varphi^{-1}(\mu D), \varphi^{-1}(\frac{x^2}{2}D)]] =$

$[\varphi^{-1}(\lambda D), \varphi^{-1}(\mu x D)] = \varphi^{-1}(\lambda\mu D)$, $\varphi^{-1}(\mathcal{C}) \times \varphi^{-1}(\mathcal{C}) \to \varphi^{-1}(\mathcal{C})$ makes $\varphi^{-1}(\mathcal{C})$ into a commutative Polish algebra which is algebraically isomorphic via φ to the Polish field $\mathcal{C}$ and therefore to the Polish field $\mathbb{C}$. If $\{\varphi^{-1}(c_n)\}_{n\geq 1} \to \varphi^{-1}(c)$, then $c_n\varphi^{-1}(1) \to c\varphi^{-1}(1)$ and hence $\{c_n\}_{n\geq 1} \to c$, so $\varphi|\varphi^{-1}(\mathcal{C})$ is a topological isomorphism. $\square$

It is important to note that for $g \in C^\infty(\mathbb{C})$, $a \in \mathbb{C}$, there is some $\{a_n\}_{n\geq 0}$ so that $g(x) = \sum\limits_{i\geq 0} a_n(x-a)^n = g(a) + (x-a)\left[\sum\limits_{i\geq 0} \frac{a_{n+1}}{n+1}(x-a)^{n+1}\right]'$. In other words, there is a $G \in C^\infty(\mathbb{C})$ so that $g(x) = g(a) + (x-a)G'(x)$.

Lemma 7.2. *Let $f \in C^\infty(\mathbb{C})$ and $a \in \mathbb{C}$. Then there exist $G \in C^\infty(\mathbb{C})$ and a unique $b \in \mathbb{C}$ such that $fD = bD + GD + [GD, aD] - [GD, xD]$. G is unique up to an additive constant and $b = f(a)$.*

Proof:

$bD + GD + [GD, aD] - [GD, xD] = bD + GD + (G \cdot 0 - G' \cdot a)D - (G \cdot 1 - G' \cdot x)D = (b + (x-a)G')D$. Therefore $fD = bD + GD + [GD, aD] - [GD, xD]$ if and only if $f(x) = b + (x-a)G'(x)$ for all $x \in J$. Therefore $b = f(a)$ and G is uniquely determined up to an additive constant. $\square$

Lemma 7.3. *Fix $c_1 \in \varphi^{-1}(\mathcal{C})$. Then for every $r_1 \in \mathcal{R}$, there is a unique $c_2 \in \varphi^{-1}(\mathcal{C})$ and an $r_2 \in \mathcal{R}$ such that $r_1 = c_2 + r_2 + [r_2, c_1] - [r_2, \varphi^{-1}(xD)\ ]$. The mapping $r_1 \to c_2$, $\mathcal{R} \to \varphi^{-1}(\mathcal{C})$, is a Borel mapping.*

Proof:

$\mathcal{S}_{c_1} = \{\ (r_1, r_2, c_2) \in \mathcal{R}^2 \times \varphi^{-1}(\mathcal{C}) \mid r_1 = c_2 + r_2 + [r_2, c_1] - [r_2, \varphi^{-1}(xD)]\ \}$ is a closed additive subgroup of $\mathcal{R}^2 \times \varphi^{-1}(\mathcal{C})$ since both sides of the equation defining $\mathcal{S}_{c_1}$ are continuous in (r_1, r_2, c_2). If π_ℓ $(1 \leq \ell \leq 3)$ is the projection onto the ℓ-th coordinate in $\mathcal{R}^3$, note that Lemma 7.2 implies that $\pi_1(\mathcal{S}_{c_1}) = \mathcal{R}$ and the uniqueness of c_2 for a given r_1. $\mathcal{Z}_{c_1} = \{\ (0, r, 0) \mid (0, r, 0) \in \mathcal{S}_{c_1}\ \}$ is a closed additive subgroup of $\mathcal{S}_{c_1}$. If $(r_1, r_2, c_2) \in \mathcal{S}_{c_1}$, then $\pi_1^{-1}(r_1) \cap \mathcal{S}_{c_1} = \{\ (r_1, r_2 + r, c_2) \mid (0, r, 0) \in \mathcal{S}_{c_1}\ \} = (r_1, r_2, c_2) + \mathcal{Z}_{c_1}$. Theorem 12.17 of Kechris

([8]) implies that there is a Borel uniformization for the quotient group $\mathcal{S}_{c_1}/\mathcal{Z}_{c_1}$ or, equivalently, there is a Borel uniformization U on $\mathcal{S}_{c_1}$ with respect to π_1, i.e., $U \subset \mathcal{S}_{c_1}$ is a Borel set such that $\pi_1|U : U \to \mathcal{R}$ is a bijection. Theorem 1.5 now implies that $(\pi_1|U)^{-1} : \mathcal{R} \to U \subset \mathcal{S}_{c_1}$ is a Borel mapping and hence that $\pi_3 \circ (\pi_1|U)^{-1} : \mathcal{R} \to \varphi^{-1}(\mathcal{C})$ is a Borel mapping. This is just the statement that the mapping $r_1 \to c_2$, $\mathcal{R} \to \varphi^{-1}(\mathcal{C})$, is a Borel mapping. $\square$

Theorem 7.4. *$\mathcal{L}(\mathbb{C})$ is an algebraically determined Polish Lie algebra.*

Proof:

Let $\mathcal{R}$ be a Polish Lie algebra and let $\varphi : \mathcal{R} \to \mathcal{L}(\mathbb{C})$ be an algebraic isomorphism of Lie algebras. Let $f \in C^\infty(\mathbb{C})$ and $a \in \mathbb{C}$. There is $G \in C^\infty(\mathbb{C})$ such that $f(x) = f(a) + (x-a)G'(x)$ for all $x \in \mathbb{C}$. But then $fD = f(a)D + GD + [GD, aD] - [GD, xD]$ and so $\varphi^{-1}(fD) = \varphi^{-1}(f(a)D) + \varphi^{-1}(GD) + [\varphi^{-1}(GD), \varphi^{-1}(aD)] - [\varphi^{-1}(GD), \varphi^{-1}(xD)]$. Note that both $\varphi^{-1}(f(a)D)$ and $\varphi^{-1}(aD)$ are elements of $\varphi^{-1}(\mathcal{C})$. Lemma 7.3 and Lemma 7.1 imply that the mapping $\psi_a : \varphi^{-1}(fD) \to \varphi^{-1}(f(a)D) \to f(a)D \to f(a)$, $\mathcal{R} \to \varphi^{-1}(\mathcal{C}) \to \mathcal{C} \to \mathbb{C}$, is a Borel mapping for every $a \in \mathbb{C}$.

Let $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ be dense and define $\Psi : \mathcal{R} \to \prod_{n\geq 1} \mathbb{C}$ by $\Psi(\varphi^{-1}(fD)) = \prod_{n\geq 1} \psi_{a_n}(\varphi^{-1}(fD))$ $= \prod_{n\geq 1} f(a_n)$. Then Ψ is a one-to-one Borel mapping and therefore is a Borel isomorphism onto its range, a Borel set, by Theorem 1.5. Similarly the mapping $\Phi : \mathcal{L}(\mathbb{C}) \to \prod_{n\geq 1} \mathbb{C}$ defined by $\Phi(fD) = \prod_{n\geq 1} f(a_n)$ is a continuous one-to-one mapping onto its range. Again, Theorem 1.5 implies that $\Phi(\mathcal{L}(\mathbb{R}))$ is a Borel subset of $\prod_{n\geq 1} \mathbb{C}$ and that $\Phi^{-1} : \Phi(\mathcal{L}(\mathbb{C})) \to \mathcal{L}(\mathbb{C})$ is a Borel mapping. Note that $\Psi(\varphi^{-1}(\mathcal{L}(\mathbb{C}))) = \Phi(\mathcal{L}(\mathbb{C}))$ and therefore $\Phi^{-1} \circ \Psi$ makes sense and is a Borel mapping. But $\Phi^{-1} \circ \Psi : \varphi^{-1}(fD) \to fD$ and therefore coincides with φ. Hence, $\varphi : \mathcal{R} \to \mathcal{L}(\mathbb{C})$ is a Borel isomorphism of additive abelian Polish groups and therefore is a topological isomorphism by Proposition 1.3. $\square$

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