ABILITY ESTIMATION UNDER DIFFERENT ITEM PARAMETERIZATION AND SCORING MODELS

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A Monte Carlo simulation study investigated the effect of scoring format, item parameterization, threshold configuration, and prior ability distribution on the accuracy of ability estimation given various IRT models. Item response data on 30 items from 1,000 examinees was simulated using known item parameters and ability estimates. The item response data sets were submitted to seven dichotomous or polytomous IRT models with different item parameterization to estimate examinee ability. The accuracy of the ability estimation for a given IRT model was assessed by the recovery rate and the root mean square errors. The results indicated that polytomous models produced more accurate ability estimates than the dichotomous models, under all combinations of research conditions, as indicated by higher recovery rates and lower root mean square errors. For the item parameterization models, the one-parameter model out-performed the two-parameter and three-parameter models under all research conditions. Among the polytomous models, the partial credit model had more accurate ability estimation than the other three polytomous models. The nominal categories model performed better than the general partial credit model and the multiple-choice model with the multiple-choice model the least accurate. The results further indicated that certain prior ability distributions had an effect on the accuracy of ability estimation; however, no clear order of accuracy among the four prior distribution groups was identified due to an interaction between prior ability distribution and threshold configuration. The recovery rate was lower when the test items had categories with unequal threshold distances, were close at one end of the ability/difficulty continuum, and were administered to a sample of examinees whose population ability distribution was skewed to the same end of the ability continuum.

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CHAPTER 1

INTRODUCTION

Overview

Testing is essential in education and other social science fields because many decisions, and policies are made according to the results of testing. The purpose of testing is to estimate a person's ability, i.e. latent trait or construct. In a test setting, responses to a set of test items are recorded by each individual. Through a scoring scheme, test scores are assigned to individuals according to their item responses. Test scores provide information from which we infer a person's ability. In educational measurement, Classical Test Theory (CTT) partitions test scores (X) into two components, $X = T + E$, to represent the ability estimate—the true score (T), and error (E). This type of measurement is juxtaposed to measurement in a field like Physics where all factors contained in a model can be accounted for. Error indicates factors that couldn't be accounted for or controlled in the test design, test administration, and/or examinee. Test score reliability and the true score estimates, however, change from one test form to another test form even though the test design and administration are the same. Error in this instance is due to the random sampling of items to form the two tests. The test score reliability and the true score estimates are test-dependent because the properties of items are selected but not controlled in the process of ability estimation in CTT. Item Response Theory (IRT) uses mathematical models to adjust for the item properties making the ability estimates freer from test-dependence.

Different IRT parameterization models adjust for different item properties leading to different ability estimation. 1-parameter (1-PL) IRT adjusts for item difficulty; 2-

parameter (2-PL) IRT accounts for item difficulty and discrimination; and 3-parameter (3-PL) IRT takes into account the effect of item guessing, difficulty and discrimination. If a set of item responses is submitted to a 1-PL, 2-PL, or 3-PL model and item parameters are estimated, a model fit statistic indicates that item parameterization in the models was satisfactorily completed. The three different item parameterization models may yield different ability estimates. It is a known fact that item parameterization will affect the estimation of ability in IRT (Lord & Novick, 1968), however, other factors, e.g. dimensionality of the test, and test-scoring format may also affect ability estimation. The present study deals only with unidimensional IRT models, but will address different test-scoring formats.

It may not be so much an issue of item parameterization, but rather item response format (right/wrong, partial credit, rating scale, etc.), that influences ability estimation. In the first few decades of the development of IRT, research interests were concentrated on dichotomous models, which involve test item responses scored either right or wrong (1, 0). One year after Lord and Novick (1968) established the 1-, 2-, and 3-parameter logistic models for dichotomous items, Samejima (1969) introduced the first polytomous model (Graded Response Model). Although Bock and Samejima (1972) presented a different polytomous model (Nominal Categories Model), it was not until the 1980's that interest in polytomous IRT models began. There have been many polytomous models developed since 1970 (Andrich, 1978, 1982, 1995; Masters, 1982; Muraki, 1990, 1992; Thissen & Steinberg, 1984, Thissen, Steinberg & Fitzpatrick, 1989, etc.). In polytomous models, items in the test are not scored just right or wrong; but instead, each of the categories of response is evaluated and scored according to its degree of correctness or the amount of information provided toward the full answer. The polytomous models are appropriate for multiple-choice items and performance assessments where test items are designed to have steps of difficulty or thresholds. Since choices of categories other than the best answer are given partial credit in polytomous models, instead of no credit as in dichotomous models, ability estimates for individuals are expected to be different depending on whether dichotomous or polytomous models are used in scoring item responses.

Statement of the Problem

If unidimensional tests, e.g. mathematics ability test, are administered to a group of examinees, the ability estimates of the examinees will vary as a function of item parameterization (1-, 2-, 3-PL) and scoring (dichotomous versus polytomous) model applied to the item responses. Since item parameterization and type of scoring model affects ability estimation; one needs to investigate which approach will produce ability estimates closest to the levels of the true ability of examinees, i.e., which combination of the item parameterization and scoring models gives the most accurate ability estimates? The true measure of ability, however, is latent and therefore not known. A comparison can only be made among the ability estimates.

Embretson and Reise (2000) compared the latent trait scores obtained from five different polytomous IRT models and the raw scores. The five models differed in item parameterization. The Partial Credit Model (PCM) and Rating Scale Model (RSM) are Rasch Models, which assume the same slope (1.0) for all items. Graded Response Model (GRM), Modified Graded Response Model (MGRM), and Generalized Partial Credit Model (GPCM) all allow items to differ in slope parameters. They compared the five models on item responses of 350 undergraduates to 12 items on the Neuroticism scale of the Neuroticism Openness Five-Factor Inventory (NEO-FEI) (Costa & McCrae, 1992) and obtained five latent trait scores, i.e. ability estimates. They found that the ability estimates were highly correlated with each other and with the raw scores. The lowest Pearson r was .97. Although the significant correlations indicated that the relative ordering of examinees was basically maintained in all five different polytomous models, they did not give information about the accuracy of the ability estimation of individual models. The information on accuracy of the estimation of individual examinee's ability is as important as their relative ordering in some test settings, e.g. in a criterion-referenced test involving a specific cut-off score. Therefore, it is important to compare the accuracy of ability estimation, not just the correlations of the ability estimates from different models, because the polytomous models with different parameterization may lead to very different ability estimates and thus different variance of the ability estimates. Furthermore, the Embretson and Reise study compared only polytomous models. If dichotomous models with different parameterization were included in the comparison, more diversified ability estimates would be expected. The present study, therefore, investigates different dichotomous and polytomous models to determine which model produces ability estimates closest to the true latent ability of the examinee, using a confidence interval to capture the true latent ability of an examinee, and a root mean square deviation index, for deviation of the ability estimates.

Rationale for the Study

Several factors that affect ability estimation in IRT are of interest in this study. These factors are hypothesized to have an impact on the accuracy of estimation of a person's ability or knowledge. The rationales for several hypotheses are given in the following section.

Item Parameterization and Scoring Models

Studies exist which compared different item parameterization and scoring models, but the emphases were on items, e.g. model fit, recovery of item parameters, and person fit (Wright & Master, 1982; Muraki, 1992). No study specifically compared different models on the ability estimates of examinees. A comparison of dichotomous and polytomous IRT models with regard to the accuracy of ability estimation is therefore needed. Using Monte Carlo methods, item responses of examinees with known ability scores can be simulated, and submitted to different IRT models to compare examinee ability estimation. The sets of examinee ability estimates from the different models can then be compared to the empirically known ability estimates to examine the bias and variance of ability estimation.

Ability Distributions

Different prior ability distributions of the examinees will affect the comparison of the ability estimates from different models. The difference between scoring a test dichotomously and polytomously may not be as prominent in extreme ability groups (low, high) as in medium ability groups. It is reasonable, therefore, to investigate the effect of the prior ability distributions of the examinees on the model comparisons. Four types of distributions will be examined, namely normal, skewed to the right, skewed to the left, and bimodal. The ability estimates of a random sample of examinees are expected to be normally distributed. A sample that has high ability examinees will produce an ability distribution skewed to the left, while that of a sample containing low ability examinees will be skewed to the right. The bimodal distribution of ability represents a sample of examinees with very diversified, even polarized, levels of ability. This study investigated what effect the ability distributions had on the ability estimates under different item parameterization and scoring models.

Threshold Distances

When scored polytomously, the categories of each multiple-choice item can represent different levels of difficulty. The threshold between two adjacent categories is the ability level at which an examinee has equal probability to choose either one of two categories. When polytomous models are applied to multiple-choice items, and partial credits are given to categories other than the best answers, the configuration of the thresholds of an item affects the information function of the item and thus the precision of ability estimation. Configuration of the thresholds includes two aspects, namely the order of the thresholds and the distances between them. Dodd and Koch (1985) found that item information functions for the partial credit model differs as a function of the thresholds. The distance between the first and last thresholds affected the shape of the information function of an item. Items with shorter distances between first and last thresholds had a more peaked information function for a narrower range of ability continuum. In a follow-up study of the issue (Dodd & Koch, 1987), they systematically altered the order of the same set of thresholds to form different items. They concluded that the items with the same set of thresholds yielded the same total amount of information across the entire ability continuum, but different ordering of the thresholds affected the peakedness of the item information curve. They found that the peakedness of the curve increased as the degree of deviation from the sequential order of the thresholds increased. While peaked information function is desired in some testing, e.g. in computerized adaptive testing (CAT), flatter information functions that yield maximum information for a wider range of ability is preferred in tests developed for examinees from all possible ability groups. It is the latter kind of testing under consideration in the present study. Therefore, the thresholds of the items in the test constructed for the present study were in their sequential order, i.e. monotonically increasing from the least to the most difficult. However, distances between the thresholds were varied to investigate its effect on ability estimation. The threshold distances could be equal or unequal. If the threshold distances are unequal and narrower at the lower end, the categories are expected to be less effective in discriminating lower ability groups in the sample, because their responses to each of the two adjacent lower difficulty categories may not be very different. In contrast, narrower threshold distances at the higher end are expected to cause the categories to be less effective in discriminating higher ability groups. With different prior ability distributions, it is meaningful to investigate how different polytomous models perform when the threshold distances are unequal.

Research Questions

The present study hypothesized that the type of item parameterization, scoring model format, prior ability distribution of the examinees, and configuration of category threshold distances were factors affecting the accuracy of ability estimation of examinees. The research questions postulated for this study were as follow:

1. How do dichotomous and polytomous IRT models differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?

2. How do different IRT item parameterization models (i.e. modeling difficulty only; both difficulty & discrimination; and difficulty, discrimination & guessing) differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?
3. How do the polytomous IRT models differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?

Delimitation

The factors identified for examination in this study have fixed levels, which were selected according to the literature review. The generalizability of the findings of this study is limited to the seven IRT models (1-, 2-, and 3-PL dichotomous model, partial credit model, general partial credit model, multiple-choice model, and nominal categories model), four types of prior ability distributions (normal, skewed to the right, skewed to the left and bimodal), and the three types of threshold distance configurations (equal, unequal-close at the lower end, and unequal-close at the higher end). Some factors other than those examined in this study affect ability estimation but they were controlled in the study. The number of examinees in the sample was fixed at 1,000 and the number of items in the test was fixed at 30 according to recommendations in the literature (Dodd and Koch, 1987; Chen, 1996). The number of response categories in each item was four to model typical multiple-choice item tests.

Definition of Terminology

1. Dichotomous item response model—item response model for test with binary items. Examinees taking the test will respond in either one of the two response categories. A test with items scored right or wrong is dichotomous.
2. Polytomous item response model—item response model for items with more than two response categories, e.g. multiple-choice item that allows partial credits for each of the response categories, or constructed-response item with multiple steps.
3. Ability estimate—the estimate of the level of a latent trait of an examinee demonstrated in an observed response pattern to a test.
4. Item response categories—the possible ways pre-assigned by the item writer that an examinee could respond to an item. In the context of multiple-choice items, they are the options provided for the examinee to choose; in constructed-response items, they are the steps or parts of the solution to the item that allow different partial credits to be awarded upon their completion.
5. Item response function (IRF)—the mathematical equation that governs the probability of answering an item correctly as a function of the ability of the examinee attempting the item and the item parameters.
6. Item category response function (ICRF)—the mathematical equation governing the probability of an item category being chosen as a function of the ability of the examinee and item category parameters.
7. Threshold distance—the distance on the ability continuum between two thresholds. The threshold between two adjacent categories is the ability level at which an examinee has equal probability to choose either one of two categories.

8. Item step response function (ISRF)—In polytomous IRT model with ordinal response categories, item step is defined as two adjacent categories. ISRF is the mathematical equation governing the probability of an item step being completed, i.e. an examinee responds in the higher category when the two adjacent categories are given as the condition. ISRF is a function of the ability of the examinee and the step parameters, e.g. thresholds, and slope parameters.
9. Item characteristic curve (ICC)—the curve that demonstrates the relationship between the ability of an examinee and the probability of the examinee answering the item correctly. Sometimes it is referred as a trace line. It is the graph of IRF plotting against the ability parameters.
10. Item category characteristic curve (ICCC)—the curve represents the relationship between the probability of an examinee choosing an item category and the ability of the examinee. ICCCs of all the categories within an item are usually plotted on the same graph.
11. Prior ability distribution—the probability distribution of ability levels in the population of examinees before estimation of parameters. It is usually assumed to be normal. It can also be estimated by the response data in some programs.
12. Posterior ability distribution—the probability distribution of the ability estimate for an examinee across the ability continuum. In marginal maximum likelihood estimation, a probability distribution replaces the point estimate for each examinee in the sample.

13. Item parameterization model—the mathematical model in item response theory through which item properties are calibrated in the measurement of an examinee's ability.
14. Scoring model—the different scoring formats on which the ability estimation and item parameters are modeled.

CHAPTER 2

REVIEW OF RELATED LITERATURE

Overview

A comprehensive review of literature relevant to the present study is provided in this chapter. First, a brief introduction to item parameterization and dichotomous IRT scoring models is presented. Second, a summary of different polytomous scoring models is given. Third, different ability estimation methods are discussed.

Item Parameterization: Basis for Family of IRT Models

Item Response Theory, as its name suggests, models testing at the item level. In contrast to Classical Test Theory (CTT), which depends on the test scores in ability estimation, IRT utilizes mathematical models to estimate the effect of different properties of individual items in the test on ability estimation. IRT item parameterization enables IRT models to be freer from test-dependence and allows estimation of error on an item-by-item basis. The modeling of item properties in IRT also permits individual estimates of standard error of measurement (SEM), instead of assuming an equal SEM for all examinees as in CTT. Therefore, while the primary purpose of IRT is the estimation of ability, item parameterization is very important and distinguishes different IRT models.

Unidimensional IRT models address only one latent trait parameter, but vary in the number of parameters used in item parameterization. The most commonly used dichotomous IRT models are the 1-, 2-, and 3-PL logistic models. Although a 4-PL logistic model was introduced (McDonald, 1967; Barton and Lord, 1981), it is of theoretical interest only, for no practical gain on ability estimation was found by the application of the model (Barton and Lord, 1981). The three item parameters involved in the IRT models are identified by three item properties, i.e. difficulty (or location parameter, usually labeled “b”), item discrimination (or slope parameter, usually labeled “a”), and pseudo-chance (or lower asymptote parameter, usually labeled “c”). These item properties also apply in the polytomous item response models, which will be reviewed in the next section. The development of the three models took decades and bridged across two continents, America, and Europe (Hambleton and Swaminathan, 1985).

One-, Two- and Three-Parameter IRT Models

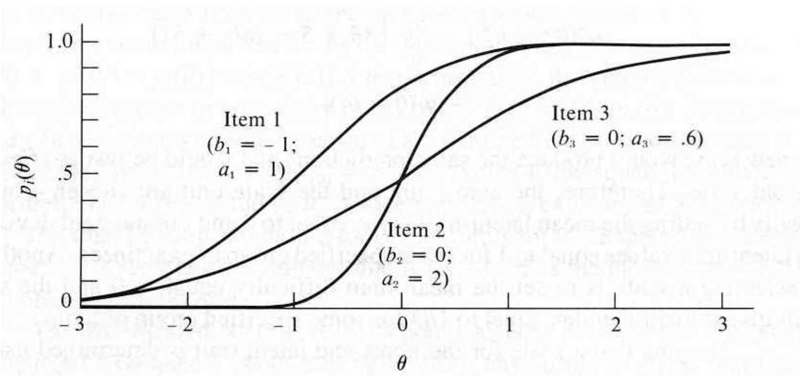
Normal ogive models.

Hambleton and Swaminathan (1985) traced the history of Item Response Theory all the way back to 1916, but credited Frederic M. Lord for providing impetus for the development of the theory in its present form. The model that Lord introduced was the two-parameter normal ogive model (1952). He used the normal ogive curve to model the probabilities of the examinees answering an item correctly as a function of their ability, i.e. the latent trait under measure, and two item parameters. The item response function (IRF) for a two-parameter normal ogive model is:

$$P_i(\theta) = \int_{-\infty}^{a_i(\theta - b_i)} \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) \cdot dz \ ,$$

where $P_i(\theta)$ is the probability of an examinee with ability θ answering item *i* correctly. The two parameters that characterize item *i* are a_i (the item discrimination), and b_i (the item difficulty). The normal ogive curve obtained by plotting $P_i(\theta)$ against θ is called item characteristic curve (ICC) (Figure 1). The b_i value is on the ability continuum where the $P_i(\theta) = 0.5$, and a_i is the slope of the curve at that point.

Figure 1. Item characteristic curves with different item parameters.[1]



The parameter estimation in this model is mathematically complex and computationally involved. The lack of convenient computer programs and high-capacity computers needed for the parameter estimation explained the painstakingly slow pace of the early development of IRT.

Logistic models.

Birnbaum (1957, 1958a, 1958b) made his contribution by introducing the more mathematically tractable logistic model. He used the logistic distribution function to approximate the normal ogive. It had been proved that the two curves differ absolutely by less than .01 for all values of ability θ, if a constant scaler D = 1.7 was applied to the logistic deviate (Haley, 1952). The logistic model not only simplified the computation involved in parameter estimation, but also provided an explicit function for item and ability parameters. The log odds of success in choosing the correct answer over failure is equal to Da_i $(\theta - b_i)$, which is a linear function of the item parameters $a_{i,}$ b_i and ability parameter θ. The endorsement that Lord had given to the logistic model by including Birnbaum's work in his book co-authored with Novick (1968), helped to promote the logistic models replacing normal ogive models in practical use. Birnbaum (1968) added

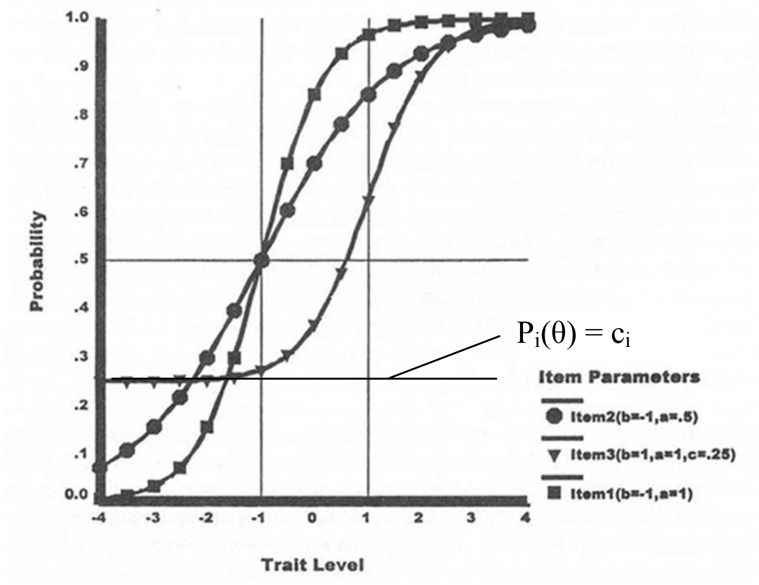
[1] Adopted from Allen and Yen (1979), 255.

one more chapter in his book to introduce the pseudo-guessing parameter, thus creating the 3-PL logistic model. The IRF of the 3-PL logistic model is:

$$P_i(\theta) = c_i + (1 - c_i)\frac{\exp[Da_i(\theta - b_i)]}{1 + \exp[Da_i(\theta - b_i)]}$$

where $P_i(\theta)$, a_i, and b_i, have the same meaning as in the normal ogive model, but c_i is added for the pseudo-guessing, or pseudo-chance parameter, which represents the probability of answering the item correctly even though an examinee does not know the answer. The ICCs of 2- and 3-PL logistic models are shown in Figure 2. The horizontal line $P_i(\theta) = c_i$ is the lower asymptote of the 3-PL logistic ICC.

Figure 2. Item characteristic curves of 2- and 3-PL logistic IRT models.[2]

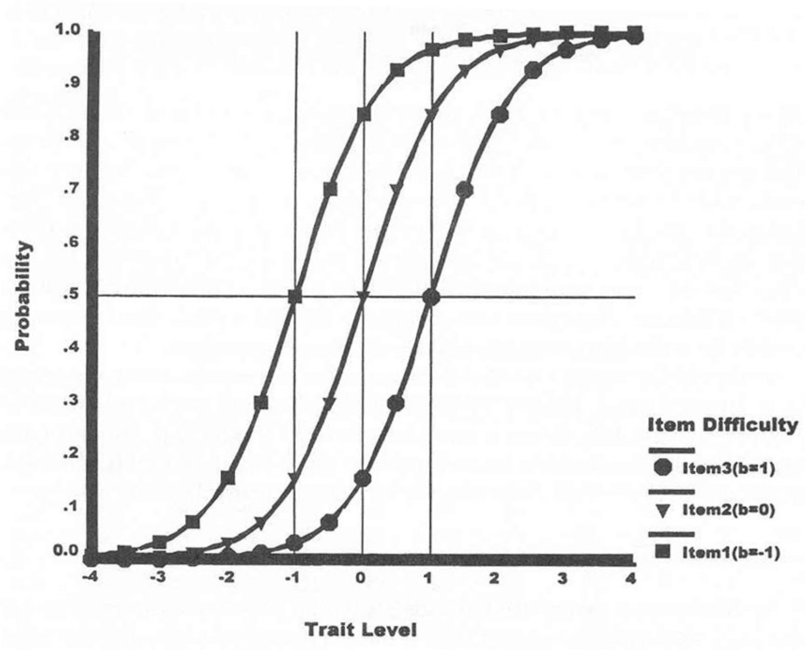


The 3-PL model is the most general among the three logistic models, because the 1- and 2-PL models can be obtained by fixing (i.e. not altering from item to item) one or two of the item parameters in the 3-PL model. The 2-PL logistic model is obtained when c_i is set

[2] Adopted and modified from Embretson and Reise (2000), 47.

to 0. It assumes no guessing on items in the test. When Da_i is fixed (usually set to 1) and c_i is set to 0, a 1-PL logistic model is obtained. The only item parameter in the 1-PL model used to estimate an examinee's ability is item difficulty. The ICCs in a 1-PL model will never intersect each other because they have the same slope at bs (Figure 3).

Figure 3. Item characteristic curves in a 1-PL model.[3]



The 1-PL logistic IRT model included in the comparison shouldn't be mistaken as a Rasch model. While logistic IRT models were developed in America, George Rasch in Denmark introduced a different logistic model. Rasch's model (1960), shares a general form of the logistic model, but was developed prior to and independently from Birnbaum's work. Despite a similar logistic function, the Rasch model has a fundamental difference from the Birnbaum logistic IRT models. The Rasch model forms a common item and person scale that is equal interval and linear, while Birnbaum logistic IRT models are designed to describe the data as close as possible. The Rasch model is more

[3] Adopted from Embretson and Reise (2000), 46.

theory-driven than data-driven (van der Linden & Hambleton, 1997). The Rasch measurement model specifies the conditions that the data must meet. Data that does not fit the model is not good enough to make measures from and thus are questioned or discarded. The existence of sufficient statistics in the Rasch model allows it to separate the item and ability parameter calibrations thus providing sample-free item estimation and item-free ability estimation. The present study compares ability estimation of IRT models under different item parameterization and scoring models. The Rasch model is not included in this study because of its fundamental difference from IRT models, and because the parameter estimation method (MML) applied to the 1-, 2- and 3-PL logistic models is different from that used in the Rasch model.

Summary

When the three IRT item parameterization models are compared on the accuracy of recovering ability estimates in this simulation study, attention should be given to the item properties in the test that generated the item responses of the examinees. With other conditions being equal, the IRT item parameterization model that fits the data better would have better ability estimates. No model will fit perfectly with actual data in an empirical situation. Items in the test may also differ in various item properties. Therefore, a range of item difficulty, item discrimination, and pseudo-chance values should be chosen to simulate item response data to reflect actual practice.

Effects of Scoring: Polytomous IRT Models

Early IRT models used dichotomously scored items, while modern IRT models were developed for polytomous items. When the items in a test have more than two item response categories, IRT models for polytomous items should be used in ability estimation. These polytomous IRT models can be classified into two major types according to the type of response categories the items have (nominal versus ordinal). Nominal response categories do not have a natural order, e.g. questions asked in a personality test, and responses to the questions are classified to reflect different personality types. Ordinal response categories are ordered along the latent trait continuum, e.g. number of steps completed in solving a mathematics problem, or the categories in a Likert scale. Higher ability examinees are more probable to respond in higher categories and lower ability examinees are more probable to respond in lower categories. Bock (1972) developed the Nominal Categories Model for items with nominal response categories, and many polytomous IRT models have been developed for items with ordinal response categories. They were developed for various types of tests and items. In the following section, a summary of Bock's model will be given, followed by a review of the literature on the classifications of the ordinal polytomous models and their relationship to Bock's model. The last part of this section will introduce the polytomous IRT models compared in this study.

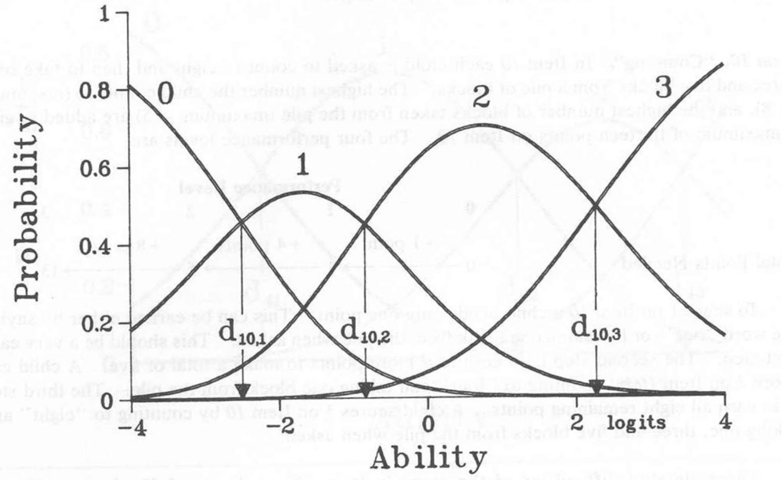
Bock's Nominal Categories Model (NCM)

Bock (1972) proposed the most general form of the polytomous IRT model that can be used to specify the probability of an examinee's response in one of several mutually exclusive and exhaustive categories as a function of person ability and response category characteristics. The response categories are not necessarily ordered. Instead of one IRF in a dichotomous model, the polytomous models have a family of item category response functions (ICRFs) that are derived to portray the probabilities of responses in different categories. The family of ICRFs for Bock's model is:

$$P_{ik}(\theta) = \frac{\exp(a_{ik}\theta + c_{ik})}{\sum_{h=1}^{m} \exp(a_{ih}\theta + c_{ih})}, \qquad k = 1, 2, \ldots, m$$

Where $P_{ik}(\theta)$ is the probability of an examinee with ability θ responding in category k of item *i*; a_{ik} and c_{ik} are the parameters of category k that are analogs to item discrimination and difficulty respectively; m is the number of response categories in item i. Each category has a ICRF. The number of ICRFs in each item is equal to the number of response categories. The model is not fully identified and therefore must be constrained. Bock set the sum of the category parameters within each item to zero, i.e. $\sum_{k=1}^{m} a_{ik} = \sum_{k=1}^{m} c_{ik} = 0$ for item i. With the constraints set, there is only one set of ICRFs for each item.

Figure 4. Item category characteristic curves in a polytomous IRT model.[4]



A category characteristic curve (ICCC) is obtained when an ICRF is plotted against θ. The ICCCs of an item with four response categories are shown in Figure 4. The ICCCs are non-monotonic with the exception of the highest and the lowest response

[4] Adopted from Wright and Masters (1982), 188.

categories. The lowest ICCC represents the response category reflecting lowest ability level, the probability of response in this category decreases along the ability continuum, the ICCC of the category is thus monotonically decreasing. The highest ICCC on the other hand, represents the probability of the response category reflecting highest ability level. The probability of response in this category increases along the ability continuum, and the ICCC of this category is thus monotonically increasing. At each level of θ, the sum of the ICRFs equal to 1, i.e. $\sum_{k=1}^{m} P_{ik}\left(\theta \middle| \theta = \theta_j\right) = 1$, because the categories are mutually exclusive and exhaustive.

Mellenbergh (1995) demonstrated that Bock's model could be reformulated in terms of (m-1) log odds. He conceptually split the nominal response variable with m categories into a series of (m-1) dichotomous response variables. Each one of these dichotomous response variables corresponded to the choices between one of the m categories to a reference category. Because the response categories are nominal, he arbitrarily chose the first category as the reference and set the parameters of that category to zero for convenience. The log odds of choosing a category k over the first category is thus:

$$\ln\left(P_{ik}(\theta) \Big/ P_{i1}(\theta) \right) = \ln\left(\frac{\exp(a_{ik}\theta + c_{ik})}{\exp(a_{i1}\theta + c_{i1})} \right) = (a_{ik} - a_{i1})\theta + (c_{ik} - c_{i1}) = a_{ik}\theta + c_{ik}, \text{ k = 2, 3,...,m.}$$

The above m-1 dichotomous models together are equivalent to the m ICRFs that describes Bock's polytomous model. It is obvious that Birnbaum's 2-PL logistic model is a special case of the Bock's model with m = 2, where the probability of answering the item correctly, $P_i(\theta) = P_{i2}(\theta)$ and the probability of answering the item incorrectly, $Q_i(\theta) = P_{i1}(\theta)$. It follows that:

$$a_i(\theta - b_i) = \ln\left(\frac{P_i(\theta)}{1 - P_i(\theta)}\right) = \ln\left({P_i(\theta)}\Big/{Q_i(\theta)}\right) = \ln\left({P_{i2}(\theta)}\Big/{P_{i1}(\theta)}\right) = a_{i2}\theta + c_{i2},$$

and $a_i = a_{i2}$; $b_i = -c_{i2}/a_i$. Conceptually, Mellenbergh has shown that Bock's model of describing polytomous responses to an item can be viewed as a group of dichotomous response models. The probabilities of responses in these dichotomous models are still governed by logistic distribution functions as in Birnbaum's logistic models, but the sum of the probabilities of the two response categories in those dichotomous models are no longer equal to 1 as in Birnbaum's model. In an up-to-date description of his model, Bock (1997) points out that the NCM is "an elaboration of a primitive, formal model for choice between two alternatives," confirming what Mellenbergh had demonstrated. Mellenbergh went on to show that Bock's model could be used to construct various models for ordinal item responses, because the ordinal polytomous models can be conceptually split into groups of dichotomies. The ordinal models are more restricted since the order of the item responses needs to be preserved. The order is preserved by using contiguous categories or groups of categories in forming the dichotomies.

Polytomous IRT Models with Ordinal Response Categories

Since so many ordinal polytomous IRT models are available, different studies have been conducted to classify them systematically. Three major types were identified out of the many ordinal polytomous models. Bas T. Hemker (2001) credited Molenaar (1983) for being the first person to compare ordinal polytomous models. Thissen and Steinberg (1986) provided a taxonomy of item response models, in which they classified polytomous models by the mathematical form of their ICRFs. Two categories of ordinal polytomous models are identified that way, namely difference models, and divide-by-total models. Their attempt was more an empirical approach in classification of polytomous models. On the other hand, classification was also made according to the theoretical characteristics of the models. Various characteristics have been used to distinguish them. Mellenbergh (1995) used Bock's model as a starting point and distinguished three different order-preserving mechanisms used in splitting the response categories into dichotomies. The three mechanisms led to the three types of models for ordinal polytomous responses. He called them the adjacent-category models, the cumulative probability models and the continuation-ratio models.

In adjacent category models, he split the ordered polytomous item responses into pairs of adjacent categories, i.e. (k^{th} and $(k+1)^{th}$ categories, for k = 1, 2, …, m-1) and applied Bock's model to the log odds of the pairs, as follows:

$$\ln\left(P_{i(k+1)}(\theta) \middle/ P_{ik}(\theta) \right) = \ln\left(\frac{\exp\left(a_{i(k+1)}\theta + c_{i(k+1)}\right)}{\exp\left(a_{ik}\theta + c_{ik}\right)} \right) = \left(a_{i(k+1)} - a_{ik}\right)\theta + \left(c_{i(k+1)} - c_{ik}\right) = a'_{ik}\,\theta + c'_{ik}\,,$$

for k = 1, 2, …, m-1. The m-1 log odds describe an ordinal polytomous model. The order of the categories is preserved in the way that the categories are split into pairs. Some of the ordinal polytomous models in this type are Muraki's (1992) generalized partial credit model (GPCM) (when a'_{ik} = the item discrimination a_i for all k and the $-c'_{ik}$ are step difficulties.), Masters' (1982) partial credit model (PCM) (when a_i equal to 1 for all i and the $-c'_{ik}$ are step difficulties), and other extensions in the partial credit model family, e.g. Andrich's (1978) rating scale model (RSM). In those extensions the step difficulties are further broken down into linear combinations of an item difficulty and a response category parameter.

In cumulative probability models, the ordered polytomous item responses are split into two parts (first k categories and the last m-k categories, for k = 1, 2, …, m-1). The categories within each part are collapsed and a cumulative probability is calculated for each part. The cumulative probability of the first k categories $P^*_{ik}(\theta) = P_{i1}(\theta) + \ldots + P_{ik}(\theta)$ for k = 1, 2, …, m-1, and that of the last m-k categories is equal to 1- $P^*_{ik}(\theta)$. Bock’s model is then applied to the log odds of the pairs of cumulative probabilities, as follows:

$$\ln\left({1 - P^*_{ik}(\theta)} \middle/ {P^*_{ik}(\theta)} \right) = a''_{ik}\,\theta + c''_{ik}, \text{ k = 1, 2, …, m-1.}$$

The m-1 log odds describe another type of ordinal polytomous model. The order of the categories is preserved by using contiguous groups of categories. It is obvious that $P^*_{ik}(\theta)$ is monotonically increasing as k increases, and the log odds associated with $P^*_{ik}(\theta)$ is always larger than or equal to that associated with $P^*_{i(k+1)}(\theta)$ for all k. Two things follow. First, the straight lines represented by the linear functions of θ in the model will not intersect for any value of θ; it is true only when the lines are parallel to each other. Parallel lines imply that the slope parameters a''_{ik} are equal for all k. Second, a straight line associated with higher values of k will always be on the right of the lines associated with lower values of k. This implies that the intercept parameter c''_{ik} changes monotonically as k increases; and thus the category boundaries are in the same order as the categories. An example of this type of ordinal polytomous model is the homogeneous case of Samejima’s (1969) graded response model (GRM).

In the continuation-ratio models, ordinal polytomous item responses are split into continuation ratios. A continuation ratio is the ratio between the probability of a category k to the cumulative probability of categories above k for k = 1, 2, …, m-1. The cumulative probability of categories above k is equal to 1- $P^*_{ik}(\theta)$. Bock’s model is applied to the log odds of the continuation ratio and an ordinal polytomous model is obtained, as follows:

$$\ln\left({1 - P^{*}_{ik}(\theta)} \Big/ {P_{ik}(\theta)} \right) = a'''_{ik}\,\theta + c'''_{ik}, \ \ k = 1, 2, \ldots, m\text{-}1.$$

The order of the categories is preserved by using one contiguous category and a group of categories. One of the examples of this type of model is Tutz' (1990, 1997) sequential model (SM) (assuming the slope parameters are equal across categories and items, i.e. $a'''_{ik} = a'''$ for all k and i).

Although the structure of the three types of models is similar, Mellenbergh concluded that the interpretation of the item parameters was different. He suggested that item features and the cognitive processes involved in answering the item should determine what type of polytomous IRT model should be used. Van Engelenburg (1997), on the other hand, argued that item response models should reflect the task features of the items. He assumed that the process of solving a polytomous item is made up of dichotomous steps, and the task features of the item determine how the steps are linked together. The task features he identified included the step process (simultaneous or sequential); the continuation rule (try-all or try until fail); and the ordering mechanism (fixed or not fixed). A combination of these task features should determine what type of polytomous IRT model should be used. For example, if the step process in the items are sequential with a fixed ordering mechanism, and the examinees are allowed to try the steps until they fail, then the type of polytomous model is the continuation-ratio model.

Akkermans (1998) carried the reasoning one step further. She argued that the interest in IRT is more in scores than in items and that polytomous items should be distinguished by the scoring rule applied to the responses. Which polytomous IRT model is selected should reflect the scoring rule applied. Three different scoring rules were identified in her study, namely graded, parallel, and sequential scoring. Based on the overall judgment of an examinee’s response, graded scoring gives a score within a scale. Parallel scoring gives credit to each feature in the collection to be displayed in the response. The overall score of the item is the sum of the credit points given. Sequential scoring gives credit to a collection of features to be displayed in the response with a fixed order. An overall score will be given as soon as a feature in the collection is not displayed and further features are not considered. From the definition of the three scoring rules, it follows that the three types of ordinal polytomous models should correspond to the rules. The continuation-ratio model should be used for responses scored by sequential scoring; the cumulative probability model for graded scoring, and adjacent category model for parallel scoring. Akkerman gave theoretical and practical reasons for connecting the models to the scoring rules, e.g. GRM and PCM should not be applied to sequentially scored responses, and SM is the preferred model. Her study simulated two score vectors for two completely different item response models and submitted them to a computer for comparison. The computer had to match each score vector to the model that generated it. The results indicated that the sample size needed for the computer to have a 95% rate of correct classification doubled when the two models were from different scoring rules instead of the same scoring rule. Her results indicated that scoring model differences can affect estimation results!

Hemker (2001) summarized the research comparing the three types of models. He classified the polytomous models by three definitions of item step, namely the cumulative, the conditional, and the partial credit. The item steps are the dichotomies that describe an ordinal polytomous response model. The definitions are based on the three different ways of how the polytomous item score is split up by the item steps. He distinguished the three types of item steps by their item step response functions (ISRFs), which is the probability of passing an item step as a function of the ability, θ. The general form of ISRF of a parametric logistic polytomous IRT model is:

$$Y_{ik}(\theta) = \frac{\exp[a_i(\theta - b_{ik})]}{1 + \exp[a_i(\theta - b_{ik})]}, \text{ for k = 1, 2, …, m-1,}$$

where a_i is the item discrimination parameter, and b_{ik} is the location parameter. The interpretation of $b_{ik,}$ however, differs over the three definitions of ISRFs.

The ISRF of a cumulative item step k is given by $Y_{ik}(\theta) = C_{ik}(\theta) = P(X_i \geq k|\theta)$, where X_i is the score of item i. The item steps are cumulative because the item steps have a strict order. If one step is passed, all previous steps are passed; if one step is failed, the following steps are failed. The categories are divided into two parts, one for passed and the other for failed; it is analogous to Mellenbergh's cumulative probability model. The item is scored similar to Akkermans' graded scoring rule. An example of this type of item is a multiple-step mathematics problem. If an examinee fails one step, the rest of the answer will be wrong.

The ISRF of a conditional item step k is given by:

$$Y_{ik}(\theta) = M_{ik}(\theta) = \frac{P(X_i \geq k|\theta)}{P(X_i \geq k-1|\theta)}.$$

The item steps in this type of model have a strict order. They are conditional because an item step will not be tried if the step prior to it was failed. For an item step k, only those examinees who have a score larger than or equal to k-1 will have a chance to try it, because those who scored less than k-1 on the item was either not having a chance to try the k-1 item step or having tried but failed the step, therefore, $X_i \geq k-1$ is the condition for item step k. This type of model is equivalent to Mellenbergh's continuation ratio model. Items in these models are scored using Akkerman's sequential scoring rule. Akkerman gave an example of this kind of scoring for testing psychomotor skills where an action is tried until the first success or repeated until the first failure.

The ISRF of a partial credit item step k is given by:

$$Y_{ik}(\theta) = A_{ik}(\theta) = \frac{P_{ik}(\theta)}{P_{ik}(\theta) + P_{i(k-1)}(\theta)}.$$

This type of model is equivalent to Mellenbergh's adjacent category model. Partial credit is given to each item step being passed. Since the item steps represent a collection of sub-tasks or features displayed by the item response and each of them will be given partial credit, this type of model is scored according to Akkerman's parallel scoring rule. Examples of this type of item are multiple-choice items with partial credit options, e.g. essays using a scoring rubric. A summary of the three types of polytomous IRT models with ordinal responses is given in Table 1.

Table 1

Summary of the Three Types of Ordinal Polytomous IRT Models

Model Type (Mellenbergh)	Scoring Rule (Akkerman)	Item Step (Hemker)	Model Represented
Adjacent category	Parallel	Partial credit	PCM
Cumulative probability	Graded	Cumulative	GRM
Continuation ratio	Sequential	Conditional	SM

Note. PCM = partial credit model; GRM = graded response model; SM = sequential model.

Ordinal polytomous IRT models compared in this study

In this study, three ordinal polytomous IRT models were compared to five other IRT models. They were the partial credit model, the generalized partial credit model and the multiple-choice model (Samejima, 1979, Thissen and Steinberg, 1984, 1997). A brief introduction to the three models is described next.

Partial credit model (PCM).

Masters (1982) extended the Rasch dichotomous model to include polytomous items. He assumed that response categories were ordered by the levels of proficiency they represent. He conceptualized a multiple-step item; in which each step represented the difference in proficiency levels between two adjacent categories. Partial credit was given to each step completed. The resulting PCM is an adjacent category model. If m_i is the number of steps in an item, the response categories of the item can be represented by the partial credit assigned to them, i.e. 0 to m_i. The model is described by m_i log odds:

$$\ln\left(\frac{P_{ik}(\theta)}{P_{i(k-1)}(\theta)}\right) = \theta - b_{ik}\text{, where k = 1, …, m}_i.$$

The ISRF of each step is a response function in the 1-PL logistic model. The one parameter in the model is the item category location parameter b_{ik}. It follows that

$$\frac{P_{ik}(\theta)}{P_{i0}(\theta)} = \frac{P_{ik}(\theta)}{P_{i(k-1)}(\theta)} \cdot \frac{P_{i(k-1)}(\theta)}{P_{i(k-2)}(\theta)} \cdots \frac{P_{i1}(\theta)}{P_{i0}(\theta)} = \prod_{h=1}^{k} \exp(\theta - b_{ih}) = \exp\sum_{h=1}^{k}(\theta - b_{ih})\text{, for all k.}$$

The sum of the quotients for all k is

$$\sum_{k=1}^{m_i} \frac{P_{ik}(\theta)}{P_{i0}(\theta)} = \frac{\sum_{k=1}^{m_i} P_{ik}(\theta)}{P_{i0}(\theta)} = \sum_{k=1}^{m_i} \exp\sum_{h=1}^{k}(\theta - b_{ih}).$$

Since $\sum_{k=0}^{m_i} P_{ik}(\theta) = 1$, it follows that $\sum_{k=1}^{m_i} P_{ik}(\theta) = 1 - P_{i0}(\theta)$.

Therefore, $\frac{1 - P_{i0}(\theta)}{P_{i0}(\theta)} = \sum_{k=1}^{m_i} \exp\sum_{h=1}^{k}(\theta - b_{ih})$. It follows that:

$$P_{i0}(\theta) = \frac{1}{1 + \sum_{k=1}^{m_i} \exp \sum_{h=1}^{k} (\theta - b_{ih})},$$

and the probability of an examinee with ability θ responding in category k can be described by:

$$P_{ik}(\theta) = P_{i0}(\theta) \cdot \exp \sum_{h=1}^{k} (\theta - b_{ih}) = \frac{\exp \sum_{h=1}^{k} (\theta - b_{ih})}{1 + \sum_{k=1}^{m_i} \exp \sum_{h=1}^{k} (\theta - b_{ih})}.$$

For notational convenience, $\sum_{h=0}^{0} (\theta - b_{ih}) \equiv 0$, which implies that $\sum_{h=0}^{k} (\theta - b_{ih}) \equiv \sum_{h=1}^{k} (\theta - b_{ih})$, and $\exp \sum_{h=0}^{0} (\theta - b_{ih}) = 1$, resulting in a simplified PCM:

$$P_{ik}(\theta) = \frac{\exp \sum_{h=0}^{k} (\theta - b_{ih})}{\exp \sum_{h=0}^{0} (\theta - b_{ih}) + \sum_{k=1}^{m_i} \exp \sum_{h=0}^{k} (\theta - b_{ih})} = \frac{\exp \sum_{h=0}^{k} (\theta - b_{ih})}{\sum_{k=0}^{m_i} \exp \sum_{h=0}^{k} (\theta - b_{ih})}, \text{ for k = 0, 1, …, } m_i.$$

The above equation describes the ICRF for responses in category k of a PCM. It is assumed that one credit is given to each step completed. A response in category k will be awarded a partial credit of k out of the possible full credit m_i. The categories are ordered either according to the levels of proficiency demonstrated in the categories or by the sequential order of the item steps needed to be completed. When PCM is applied to multiple-choice items, the former is assumed.

The location parameter b_{ik} can be broken down further to indicate the item location and category threshold, i.e. $b_{ik} = b_i + d_{ik}$. The difference in levels of proficiency between the adjacent k^{th} and $(k+1)^{th}$ categories is called the k^{th} step difficulty or threshold d_{ik}. The thresholds d_{ik} is also equal to a value on the ability continuum $(\theta – b_i)$ where two adjacent ICCCs intersect. An examinee with ability θ = b_{ik} will have an equal probability of choosing either of the adjacent categories, and therefore, the threshold d_{ik} is just like the boundary between those two categories. In PCM, thresholds need not be ordered. Harder steps could be followed by easier steps or vice versa. The configuration of the thresholds, however, could have an effect on the discrimination of the item since the same amount of credit is given to each item step completed despite its difficulty.

Generalized partial credit model (GPCM).

Muraki (1992) generalized the partial credit model by adding a slope parameter to the model. The slope parameter is analogous to item discrimination in the 2-PL dichotomous model. In GPCM, the slope parameter is constant across the categories within each item, but could be different between items. In dichotomous 2-PL, or 3-PL IRT models, the slope parameter alone is fully responsible for providing item discrimination. In GPCM, the slope parameter in combination with the configuration of item thresholds determines the discrimination of an item. The ICRFs of GPCM are very similar to those of PCM, given as:

$$P_{ik}(\theta) = \frac{\exp \sum_{h=0}^{k} Da_i(\theta - b_{ih})}{\sum_{k=0}^{m_i} \exp \sum_{h=0}^{k} Da_i(\theta - b_{ih})}, \text{ for k = 0, 1, …, m}_\text{i},$$

where a_i is the slope parameter, i.e. item discrimination, of item i, and D = 1.7 is the scalar constant that transforms the ability scale into the same metric as the normal ogive model. The ICRFs of GPCM can be derived with similar derivation in the last section starting with the log odds:

$$\ln\left(\frac{P_{ik}(\theta)}{P_{i(k-1)}(\theta)}\right) = a_i(\theta - b_{ik}), \text{ where k = 1, …, m}_\text{i}.$$

Without presumption of order in the categories, the left hand side of the equation can be:

$$\ln\left(P_{ik}(\theta) \Big/ P_{i(k-1)}(\theta) \right) = \ln\left(\frac{\exp(a_{ik}\theta + c_{ik})}{\exp(a_{i(k-1)}\theta + c_{i(k-1)})} \right) = \left(a_{ik} - a_{i(k-1)}\right)\theta + \left(c_{ik} - c_{i(k-1)}\right).$$

Therefore, for a nominal categories model to be constrained to a more restricted general partial credit model, it follows that $\left(a_{ik} - a_{i(k-1)}\right)\theta + \left(c_{ik} - c_{i(k-1)}\right) = a_i\left(\theta - b_{ik}\right)$, and the relationship between the parameters in the two models can be expressed as:

$$a_i = a_{ik} - a_{i(k-1)}, \text{ and } b_{ik} = -\frac{c_{ik} - c_{i(k-1)}}{a_{ik} - a_{i(k-1)}}.$$

It confirms the observations of Thissen and Steinberg (1984) and Bock (1997).

Multiple-choice model.

In Bock's nominal category model, the category with lowest a_{ik} (most negative) will have an ICCC that decreases monotonically from the left tail value of 1 to a right tail value of 0, while the other ICCCs are with a left tail decreasing to 0. When it is applied to multiple choice items, the ICCCs imply that all examinees with low ability will select the same incorrect category (the one with lowest a_{ik}). Empirical studies have shown that often this is not the case (Levine and Drasgow, 1983). The non-modeled discrepancy is caused by examinees selecting different categories as their answer by purely guessing.

Samejima (1979) introduced a latent category to allow nonzero left tails for all ICCCs. Thissen and Steinberg (1984) called that latent category the "don't know" (DK) category. The DK category that Samejima suggested is described by a response distribution function of the ability parameter θ:

$$P_{DK}(\theta) = \frac{\exp(a_{i0}\theta + c_{i0})}{\sum_{k=0}^{m} \exp(a_{ik}\theta + c_{ik})},$$

where DK is treated as one extra nominal response category to the item, and a_{i0}, c_{i0} are the parameter for that category. It is assumed that examinees in the DK category select one of the item categories at random. Therefore, the probability of a specific item category being selected by a DK examinee is 1/m, where m is the number of item categories. With this assumption, Samejima suggested a multiple-choice model described by the following ICRFs:

$$P_{ik}(\theta)=\frac{\exp(a_{ik}\theta+c_{ik})+{}^{1}\!/_{m}\exp(a_{i0}\theta+c_{i0})}{\sum_{h=0}^{m}\exp(a_{ih}\theta+c_{ih})}, \text{ for k = 1, 2, …, m.}$$

Thissen and Steinberg (1984) argued that it is implausible to assume the DK examinees would select their answer randomly. They believed that the DK examinees would be drawn to different options at differential rates. They introduced another parameter d_{ik}, and used it instead of the constant 1/m to represent the probability of a DK examinee choosing category k. It is obvious that d_{ik} for all ks lie in the interval (0,1) and the sum of d_{ik} within an item is 1, i.e. $\sum_{k=1}^{m} d_{ik}=1$. The dual constraint on d_{ik} is imposed by expressing them in terms of a set of psuedo-parameters d^{*}_{ik}, where $d_{ik}=\frac{\exp(d^{*}_{ik})}{\sum_{h=1}^{m}\exp(d^{*}_{ih})}$.

The psuedo-parameters are undetermined and the constraint $\sum_{k=1}^{m} d^{*}_{ik}=0$ is imposed to make them identifiable. The ICRFs of the multiple-choice model become:

$$P_{ik}(\theta)=\frac{\exp(a_{ik}\theta+c_{ik})+d_{ik}\exp(a_{i0}\theta+c_{i0})}{\sum_{h=0}^{m}\exp(a_{ih}\theta+c_{ih})}, \text{ where k = 1, 2, …, m.}$$

It is this multiple-choice model that is compared to other models in the present study.

Summary

In the literature, it has been shown that Bock's nominal category model is the most general polytomous IRT model, and its relationships to the dichotomous models and different types of ordinal polytomous IRT models have been investigated. As shown in Akkerman's (1998) study, model differences in polytomous IRT models have a theoretical basis and are practical. How items are scored determines what IRT model should be used for ability estimation. When the wrong type of IRT model is applied, specification error is made and bias is introduced in examinee ability estimation. Mellenbergh (1995) has shown that differences between the three types of models disappear when the items are scored dichotomously.

In this study, tests with multiple-choice items scored dichotomously and polytomously are compared. It is assumed that all the items are scored according to the parallel scoring rule, therefore, only the partial credit type of ordinal polytomous IRT models will be used for comparison. Thissen & Steinberg's multiple-choice model was included for modeling the effect of guessing in a polytomous model. Bock's model was included to investigate how much bias is present when the ordinal nature of the response categories is not specified.

Ability Estimation

Different approaches and techniques are applied in item response theory to estimate ability. The ability parameter can be estimated jointly with the item parameters or estimated with known item parameters, which have been previously estimated. If the ability parameter is not estimated jointly with the item parameters, the item parameters are first estimated from the item responses with the influence of the ability parameter taken away; the ability parameter is either eliminated through conditioning or integrated out through marginalization. Techniques used in parameter estimation include the maximum likelihood procedure (Baker, 1992); logistic regression (Reynolds, Perkins and Brutten, 1994); minimum chi-quadrant (Zwinderman and van der Wollenberg, 1990), and Bayesian modal estimation procedure (Mislevy, 1986; Baker, 1992). The maximum likelihood procedure with Bayesian estimates (MAP, EAP) was used in this study; therefore, a review of maximum likelihood and Bayesian estimation methods in the literature will be given in the following section.

Maximum Likelihood method

Likelihood is a probabilistic function of modeled observations; a specific item response vector in the case of IRT models. When local independence is assumed, the likelihood function is the product of the probabilities associated with individual item responses in a vector. Since the probability of an item response is a function of ability and item parameters, the likelihood function is also a function of those parameters. For example, the likelihood function of examinee *j*'s response to n items using Birnbaum's 2-PL logistic model is $L(\mathbf{x}|\theta_{\mathrm{j}}, \mathbf{a}, \mathbf{b}) = \prod_{i=1}^{n} f(x_i|\theta_j, a_i, b_i)$, where **x** = ($x_1$, x_2, …, x_n) is the response vector of examinee j to the n items (x_i = 1 or 0 for all i), **a** = (a_1, a_2, …, a_n) and **b** = (b_1, b_2, …, b_n) are the item parameter vectors and *f* is the item response function of the 2-PL logistic model. Therefore, *L* is the probability of obtaining a response vector **x** given the parameters θ_j, **a**, and **b**. Maximum likelihood estimation is used to find the value of the parameters that maximize the value of *L*.

Since the likelihood function *L* involves a product of functions, it is easier to work with a logarithm of *L* instead of *L* itself. A logarithm is a monotonic function; therefore, ln *L* is maximized when *L* reaches its maximum. A logarithm of the likelihood function is used to find a solution in practice. Estimators of the parameters are obtained from the solution of the first derivative equation $\frac{\partial \ln L}{\partial \theta} = 0$ (likelihood equation), which is the condition for a local maximum to occur. If there is more than one local maximum, the largest should be chosen. For the entire response data set of N examinees tested on n dichotomous scored items, the likelihood function is given by:

$$L(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N | \theta_1, \theta_2, ..., \theta_N) = \prod_{j=1}^{N} L(\mathbf{x}_j | \theta_j) = \prod_{j=1}^{N} \prod_{i=1}^{n} L(x_{ij} | \theta_j) = \prod_{j=1}^{N} \prod_{i=1}^{n} P_{ij}^{\ x_{ij}} Q_{ij}^{\ 1-x_{ij}} .$$

And after taking logarithms on both sides of the equation,

$$\ln L(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N | \theta_1, \theta_2, ..., \theta_N) = \sum_{j=1}^{N} \sum_{i=1}^{n} \left[x_{ij} \ln P_{ij} + (1 - x_{ij}) \ln (1 - P_{ij}) \right].$$

The maximum likelihood estimates of the ability parameters $\theta_1, \theta_2, ..., \theta_N$ are obtained by solving the simultaneous equations:

$$\frac{\partial}{\partial \theta_j} \ln L(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N | \theta_1, \theta_2, ..., \theta_N) = 0, \text{ for } j = 1, 2, \ldots, \text{N}.$$

A solution for the likelihood equation is possible using the Newton-Raphson algorithm, when the likelihood function is twice differentiable, i.e. the second derivative of the likelihood function is available. The Newton-Raphson algorithm starts with an initial value for the estimate of the parameter in the model. The number of items correct is usually used for the ability estimates, and CTT item statistics, e.g. proportion correct and biserial correlation are used for item estimates. In each iteration, a new estimate for the parameters is generated based on the estimate obtained from the previous iteration.

For example, if $\left[\hat{\theta}_j\right]_t$ is the ability estimate of the examinee *j* at the t$^{\text{th}}$ iteration, the ability estimate for the (t+1)$^{\text{th}}$ iteration is:

$$\left[\hat{\theta}_j\right]_{t+1} = \left[\hat{\theta}_j\right]_t - \left[\frac{\partial \ln L(\mathbf{x}|\theta)}{\partial \theta_j}\right]_t \Bigg/ \left[\frac{\partial^2 \ln L(\mathbf{x}|\theta)}{\partial {\theta_j}^2}\right]_t .$$

The differences between the new and old estimates ($\left[\hat{\theta}_j\right]_{t+1} - \left[\hat{\theta}_j\right]_t$) are calculated for each iteration. The iterations continue until the difference is smaller than a pre-set minimal value, then the estimate has converged and is the maximum likelihood estimate of the parameter. Baker (1992) gave very detailed derivations for estimation equations with the Newton-Raphson algorithm applied to different dichotomous models. Three types of maximum likelihood estimation are often used to estimate parameters in IRT, namely Joint Maximum Likehood (JML), Conditional Maximum Likehood (CML), and Marginal Maximum Likelihood (MML).

Joint Maximum Likelihood Estimation (JML).

The JML estimation method was developed by Birnbaum (1968). He used an iterative two-stage procedure for jointly estimating item and ability parameters. Each iteration was carried out in two stages. Iterations started by estimating the ability parameters with the initial values of the item parameters known; then the final values in the estimation were treated as known ability parameters to estimate the item parameters. This two-stage procedure was repeated until both the estimates of the ability and item parameters converged.

The JML method was straightforward, but several problems associated with the method limited its use. First of all, parameter estimation in JML is inconsistent. When the item and ability parameters are estimated jointly, the item parameters are structural parameters, which are fixed by the length of the test, and the ability parameters are incidental parameters, because the number of ability parameters increases as the sample size increases. Neyman and Scott (1948) have shown that large numbers of incidental parameters adversely affects the consistency of the estimation of structural parameters. This implies that the estimates of the item parameters will not converge to their true values when the sample size of examinees increases to a large number. Moreover, the item parameter estimates in JML are biased. De Gruijter (1990) found that the bias of parameter estimates, i.e. the difference between the true value of the parameter and the estimate, depended on the expected total score distribution. As the test increases in length, the bias will become less important, because the ability parameter can be estimated more precisely. Large sample sizes and lengthy tests are required to minimize bias in parameter estimates in the JML procedure, thus making it less popular, especially in small-scale studies. The JML method cannot apply to items and examinees with zero or perfect scores. An examinee with a zero score will have an ability estimate of $-\infty$, while an examinee with a perfect score will have an estimate of $+\infty$. Similarly, an item that all examinees fail will have an item difficulty estimate of $+\infty$, while an item that everybody answered correctly will have an item difficulty estimate of $-\infty$.

Conditional Maximum Likelihood Estimation (CML).

In JML, the item parameter estimates could be inconsistent and biased because they are estimated jointly with the ability parameter. The CML estimation technique (Andersen, 1972), in contrast, provided consistent and efficient parameter estimates by factoring out the unknown ability parameters from the likelihood equations. It required sufficient statistics for the ability and item parameters, which were only available in the

1-PL logistic model. The number of items correct (count) is a sufficient statistic for the ability parameter and the number of correct responses to an item is a sufficient statistic for the item difficulty parameter. The likelihood function $L(\mathbf{x} \mid \boldsymbol{\theta})$ is replaced by $L(\mathbf{x} \mid \mathbf{r})$ in CML, where **x** is a response vector containing the response patterns of each examinee in the sample, and **r** is a vector containing the number correct of each examinee. It can be shown that $L(\mathbf{x} \mid \mathbf{r}) = L(\mathbf{x} \mid \boldsymbol{\theta})/ L(\mathbf{r} \mid \boldsymbol{\theta})$, which is independent of θ because the terms in the numerator and denominator cancel out each other.

While CML has the advantage of separately estimating the ability and item parameters, it has some limitations. First, no parameter estimates can be obtained for zero or perfect scores. Second, examinees that have the same number of items correct but different response patterns will be given the same ability estimate. Third, CML has problems in estimating parameters for a long test, complicated patterns of missing data, and polytomous items with many response categories.

Marginal Maximum Likelihood Estimation (MML).

While CML permits item parameter estimation free from the condition of ability parameters, it requires a sufficient statistic for the ability parameter. That condition limits its application to the 1-PL logistic models. Bock & Lieberman (1970) introduced an approach to handle the unknown ability parameters. Instead of using the likelihood function conditioned on an examinee's ability θ, i.e. $L(\mathbf{x} \mid \theta)$, they suggested the unconditional likelihood function $L(\mathbf{x})$, which is the probability of observing the pattern **x** from an examinee of unknown ability drawn at random from a population. The observed response data is regarded as a random sample from a population. If the population has an ability distribution described by a continuous density function g(θ), the unconditional likelihood function is given by the definite integral,

$$L(\mathbf{x}) = \int_{-\infty}^{\infty} L(\mathbf{x}|\theta) \cdot g(\theta) d\theta .$$

$L(\mathbf{x})$ is a function of the item parameters only because the ability parameter θ has been integrated out. The definite integral generally cannot be expressed in closed form, thus a procedure called Gaussian quadrature is used to find the value of the integral.

Since the unconditional likelihood function is an expected value over a population rather than an observed value, estimation procedures are different from that used in the JML estimation method. The original approach that Bock and Lieberman (1970) introduced posed a formidable computational task, and thus was not practical for lengthy tests. In a subsequent reformulation of the MML approach, Bock and Aitken (1981) introduced the EM algorithm as a procedure for MML estimation. The EM algorithm has two stages, namely the expectation and the maximization stage. In the expectation stage, expected values of the frequencies at quadrature points and expected frequencies of examinees passing the items are computed. These expected values are then submitted to the estimation equations for maximum likelihood estimation in the maximization stage. The E and M stages repeat until the estimates converge. The Newton-Gauss method is used to solve the maximum likelihood equation.

MML has some advantages over other maximum likelihood methods. It is applicable to all IRT models and efficient for any test length. It provides estimates for perfect scores and thus no loss of information by trimming items and examinees with perfect scores. The estimates of item standard error in MML are good approximations of expected sampling variance of the estimates. Despite these advantages, MML has some limitations. First, MML estimation is computationally involved and sophisticated. Second, an ability distribution must be assumed. It is assumed to be normal if the prior ability distribution is not known. The effect of departure from normality, however, seems minimal and the population ability distribution actually can be estimated from data. Third, Baker (1992) pointed out that MML with EM algorithm resolved the problem of inconsistent item parameter estimates in JML, but the problem of deviant ability estimates in some data sets remained unsolved. There is simply no way to estimate parameters for aberrant response patterns. It is this last limitation that led to the use of Bayesian estimates in MML (Bock and Aitkin, 1981; Mislevy, 1986).

Bayesian estimation

Bayesian parameter estimation procedures are based on Bayes' theorem, which states the relationship between the conditional and the unconditional probabilities of the occurrence of an event. Specifically, Bayes' theorem can be expressed as:

$$P(A|B) \propto P(B|A) \cdot P(A),$$

where *P(A|B)* and *P(B|A)* are conditional probabilities and P(A) is the unconditional probability of occurrence of event A. The likelihood function $L(\mathbf{x}|\ \theta)$ is a conditional probability. It follows from the Bayes' theorem that:

$$P(\theta|\mathbf{x}) \propto L(\mathbf{x}|\theta) \cdot P(\theta).$$

$P(\theta|\mathbf{x})$, the probability of θ under the condition of item response vector **x**, can be understood as the distribution of ability estimates according to the item responses, i.e. the density function of the posterior ability distribution. The unconditional probability $P(\theta)$ is the distribution of the ability prior to estimation, i.e. density function of the prior ability distribution. The posterior ability distribution is therefore proportional to the product of the likelihood function and the prior ability distribution.

Bayesian estimation procedure makes use of this relationship and the information from the prior ability distribution in the estimation of the ability parameters. The prior ability distribution is informative if the variance of the distribution is small. A large variance, however, would make a prior distribution uninformative. An uninformative prior distribution would have less impact on ability parameter estimation. Informative prior distributions, on the other hand, will shrink the ability estimates toward the mean of the prior distribution and thus prevent the ability estimates from diverging to unreasonable values in the estimation process. This characteristic of Bayesian estimation helps to solve the problem of deviant estimates in maximum likelihood estimation, and provides a means for estimating aberrant response patterns, e.g. all zero or all perfect responses to items in a test. Two Bayesian estimates are commonly applied in MML, i.e. maximum a posteriori (MAP) and expected a posteriori (EAP).

Maximum a posteriori (MAP) estimates.

MAP is the value of ability estimate θ that maximizes the logarithm of the density function of the posterior ability distribution:

$$\ln P(\theta|\mathbf{x}) = \ln[L(\mathbf{x}|\theta)\cdot P(\theta)] = \ln L(\mathbf{x}|\theta) + \ln P(\theta).$$

MAP estimate is also called Bayesian modal estimate. It is obtained by the solution of the likelihood equation using the Newton-Raphson procedure:

$$\frac{\partial \ln P(\theta|\mathbf{x})}{\partial\theta} = \frac{\partial \ln L(\mathbf{x}|\theta)}{\partial\theta} + \frac{\partial \ln P(\theta)}{\partial\theta} = 0\,.$$

Since marginal Bayesian modal estimation can be considered an extension of MML, the estimation procedure utilizes the EM algorithm as described before. The two-stage procedure is repeated until a convergence criterion is met. The criterion is typically specified as a number of EM cycles. The Bayesian modal estimation procedure always converges and thus ability estimates are available, even for examinees with zero total score or a perfect score.

Expected a posteriori (EAP) estimates.

EAP is the mean of the posterior ability distribution of an examinee, i.e. the expected value of θ in the posterior ability distribution. It is given by:

$$E(\theta|\mathbf{x}) = \frac{\int_{-\infty}^{\infty} L(\mathbf{x}|\theta)P(\theta)\theta d\theta}{\int_{-\infty}^{\infty} L(\mathbf{x}|\theta)P(\theta)d\theta}.$$

The above equation involves integrations. In practice, the integrations can be approximated by the Gauss-Hermite quadrature. The approximation is obtained by

$$E(\theta|\mathbf{x}) = \frac{\sum_{k=1}^{q} L(\mathbf{x}|X_k)A(X_k)X_k}{\sum_{k=1}^{q} L(\mathbf{x}|X_k)A(X_k)}.$$

The equation is not iterative and EAP estimates are obtained directly. Bock and Aitkin (1981) introduced EAP as part of the MML estimation procedure.

Summary

In the present study, ability estimates are obtained using different item parameterization and scoring models. The ability estimates are compared to see which combination of item parameterization and scoring model provides the best estimation of the examinees' ability. It is important for the same estimation approach be used in all the models under comparison. In the literature, MML was recommended among the three maximum likelihood methods. Caution was also taken in choosing the IRT computer software that used MML for parameter estimation. The software used for parameter estimation in this study is MULTILOG (Thissen, 1991). MML in combination with Bayesian MAP and EAP estimation were used for parameter estimation in the item calibration and scoring runs. MULTILOG provides analysis for all the models, dichotomous and polytomous, covered in the present study. The dichotomous models are treated as special cases of some polytomous models covered by the software.

CHAPTER 3

METHODS AND PROCEDURES

Data Simulation and Design

This study employed a $7 \times 4 \times 3$ factorial design with seven item parameterization and scoring models, four prior ability distributions (normal, skewed to the right, skewed to the left, bimodal), and three kinds of threshold distances (equal, unequal-low, unequal-high). The seven models being compared were the 1-, 2-, 3-PL dichotomous logistic model, the Generalized Partial Credit Model with item discrimination a_i set to a constant (GPCM-1), the Generalized Partial Credit Model (GPCM), the Multiple Choice Model (MCM), and the Nominal Categories Model (NCM). The 84 combinations for the three factors are listed in Table 2.

Table 2

Combinations for Study Design

	Normal			Skewed to right			Skewed to left			Bimodal		
	ET	UL	UH	ET	UL	UH	ET	UL	UH	ET	UL	UH
1-PL	C_{ne1}	C_{nl1}	C_{nh1}	C_{re1}	C_{rl1}	C_{rh1}	C_{le1}	C_{ll1}	C_{lh1}	C_{be1}	C_{bl1}	C_{bh1}
2-PL	C_{ne2}	C_{nl2}	C_{nh2}	C_{re2}	C_{rl2}	C_{rh2}	C_{le2}	C_{ll2}	C_{lh2}	C_{be2}	C_{bl2}	C_{bh2}
3-PL	C_{ne3}	C_{nl3}	C_{nh3}	C_{re3}	C_{rl3}	C_{rh3}	C_{le3}	C_{ll3}	C_{lh3}	C_{be3}	C_{bl3}	C_{bh3}
GPCM-1	C_{ne4}	C_{nl4}	C_{nh4}	C_{re4}	C_{rl4}	C_{rh4}	C_{le4}	C_{ll4}	C_{lh4}	C_{be4}	C_{bl4}	C_{bh4}
GPCM	C_{ne5}	C_{nl5}	C_{nh5}	C_{re5}	C_{rl5}	C_{rh5}	C_{le1}	C_{ll5}	C_{lh5}	C_{be5}	C_{bl5}	C_{bh5}
MCM	C_{ne6}	C_{nl6}	C_{nh6}	C_{re6}	C_{rl6}	C_{rh6}	C_{le6}	C_{ll6}	C_{lh6}	C_{be6}	C_{bl6}	C_{bh6}
NCM	C_{ne7}	C_{nl7}	C_{nh7}	C_{re7}	C_{rl7}	C_{rh7}	C_{le7}	C_{ll7}	C_{lh7}	C_{be7}	C_{bl7}	C_{bh7}

Note. ET= equal threshold distances, UL= unequal threshold-low, UH= unequal threshold-high.

With each combination of prior ability distribution and threshold distances, a set of polytomous item responses of 1,000 subjects to 30 items was simulated using a computer program described in later section. Therefore, twelve different sets of polytomous item responses were generated. Those sets of item response data were then analyzed given the seven IRT parameterization and scoring models using computer programs described in the next section for estimation of item parameters and ability estimates. For each combination of the three factors, ability estimates were computed. A total of 84 different sets of ability estimates were computed, one set for each combination in Table 2.

Number of Replications in Monte Carlo Estimation

In order to obtain stable ability estimates on each individual in the sample of 1,000 examinees, 50 replications of the data simulation and ability estimation described in the above paragraph were completed to produce a total of 4,200 (84×50) different sets of ability estimates. The number of replications used in simulation studies of IRT models in the past varied from as few as 5 to as many as 100 replications (Kamata, 1998; Boughton, Klinger and Gierl, 2001). Kamata, who varied the number of replications in his study from 3 to 100 across six levels, and Yang (1995), have both shown that estimates of parameters were stable after 50 replications. Therefore, this study employed 50 replications for each combination of conditions based on suggestions from these Monte Carlo studies. In each of the 50 replications of data simulation, the random seed (seed1 in the SAS program in Appendix A) that was used to create the random sample of original ability estimates for the 1,000 examinees was kept constant and the random seed (seed3 in the same program in Appendix A) that was used in generating the item response data was changed in each replication, so that the 50 item response data sets were different but with the same sample of examinees. For each combination of conditions, the mean of the 50 ability estimates of each examinee generated in the 50 replications was calculated and served as the estimate of the ability parameter for that person under those specific conditions. The sampling error of the mean was also calculated and a 95% confidence interval was computed around the mean. Therefore, for each of the 84 combinations of conditions, 1,000 confidence intervals were formed.

Criteria for Evaluation

Two criteria were used to determine how close the estimated ability estimates were to the original ability estimates. First, each of the 1,000 original ability estimates was examined to determine how many of them fell in the 95% C.I.s of the corresponding bootstrap ability estimates. The number of original ability estimates that fell in the C.I.s for each combination of conditions was recorded. The rate of recovery, i.e. the number of recovered original ability estimates divided by 1,000, served as an indicator of how well the different models recovered an examinee’s ability estimate under the different research design combinations. Therefore, 84 of such recovery rates, one for each study design combination, were generated in each data simulation and analysis cycle.

Second, a root mean square error (RMSE) of ability estimation was calculated across samples for each of the study design combination. The RMSE indicated the bias and variance of ability estimation, and thus served as an indicator of precision. The RMSE was calculated as:

$$\text{RMSE}(\text{C}_{\text{xyz}}) = \sqrt{\sum_{j=1}^{n} \frac{\left(\hat{\theta}_j - \theta_j\right)^2}{n}}$$

For n = number of examinees;

$\theta_j \equiv$ original ability score of the jth examinee

$\hat{\theta}_j \equiv$ mean of the ability estimates of the jth examinee from 50 replications

Where C_{xyz} is the research design combination of “x prior distribution”, “y threshold distance configuration” and “z IRT model”, and therefore 84 RMSE were calculated in each data simulation and analysis cycle. A flowchart illustrating one cycle of data simulation and analysis is listed in Appendix B followed by the computer programs used in each phase of the cycle.

Sample Size and Power Analysis for Hypothesis Testing

In order to answer the research questions, six three-way ANOVAs were run to test the stated hypotheses. Eighteen cycles of the data simulation and analysis process described in the previous section were conducted to generate 1,552 (84×18) recovery rates and RMSE. The number of cycles was chosen in reference to the F test for the hypothesis with most levels involved, which was the three-way interaction effect of the hypothesis comparing the four different polytomous models under the combinations of four levels of prior distributions, and three levels of threshold configurations ($4\times 4\times 3$), because it involved the largest number of cells. According to Cohen’s (1988) power table for F test at level of significance, $\alpha = .05$, and degree of freedom for the numerator, df = 15, and a medium effect size (f = .25), a cell size of 18 will give the F test a power approximately equal to .76; but if df = 24 and the other conditions remain unchanged, the power of the test increases to .89. The F test was chosen with reference df = 18, and therefore the power of the F test will be around .80. From one cycle to the other, the random number seed (seed1), which was kept constant within each cycle, was changed, so that the samples of 1,000 examinees were different from those in the other cycles.

Construction of Test Items

The parameters of the test items used in the present study were chosen to reflect typical item discrimination and difficulty of real test data. The item parameters were generated to model partial credit items. Thirty discrimination parameters ranged from .75 to 2 were randomly assigned to the thirty items representing adequate but varied discriminations. Different levels of difficulty were implemented in the test by having 10 easy, 10 moderate and 10 difficult items. The thresholds of the easy items were all negative, those of the difficult items were all positive, and those of the moderate items were symmetric around values close to zero. The values of the thresholds in the 30 items were ranging from –2.25 to 2.25 to ensure enough range for measuring all the ability levels in the simulated samples. Since the number of response categories in each item was set to four, there were three thresholds in each item, namely b_1, b_2, and b_3. Three different configurations of thresholds were generated by varying the distances between the thresholds, i.e. equally distributed, shorter on the lower end, or shorter on the higher end. The three different configurations of thresholds were acquired by changing the value of the middle threshold. For the condition of equal threshold distances, the value of the middle threshold, b_2 was set to the mean of b_1 and b_3. For threshold distances shorter at the lower end, the value of the middle threshold, $b^{'}_2$ was set to the mean of b_1 and b_2. For the third condition, the value of the middle threshold, $b^{''}_2$ was set to the mean of b_2 and b_3. The item parameters of the 30 items are listed in Appendix C.

Data Simulation Computer Program

The SAS data simulation program was adopted and modified from the unpublished dissertation of Susan Chen (1999). The first part of the program used the random number function RANOR (for the other three ability distributions, other random number generating functions are used, see Appendix A) to generate a set of 1,000 random numbers to represent the ability parameters for the different IRT item parameterization and scoring models. The ability parameters generated by RANOR were normally distributed with the mean equal to zero and standard deviation equal to one. The 1,000 random numbers were then submitted to the second part of the program as the known ability estimates of 1,000 examinees to generate response patterns for 30 items. The SAS program did two things. First, it generated the probability of response in each of the four item categories of an item for each examinee. The probabilities were generated according to the ICRFs of the general partial credit model, the most general form of partial credit models. The probability of each item category was conceptually represented by a line segment with the length equal to the magnitude of the probability. The four probabilities were concatenated to form a 0 to 1 interval, (the sum of the probabilities equal to 1). Second, another 1,000 uniformly distributed random numbers between 0 and 1 were used to generate the item responses. For each examinee, a random number between 0 and 1 was generated and compared to the interval formed by the four probabilities. The number must fall into one of the four line segments, and the category represented by that line segment was the item response of that examinee for that item. Since an examinee with higher ability estimate will have a higher probability in choosing higher categories and the random number chosen is from a uniform distribution, a higher ability examinee has higher chance to get a proportionally higher response categories in this transformation. The SAS sub-routines were repeated for each item and each examinee. A dichotomous response data set was obtained by setting to 1 the category in the polytomous response data set whose ICCC was monotonic increasing, and the other three categories to 0.

Item Parameterization and Ability Estimation Programs

The commercial computer software package MULTILOG developed by David Thissen (1991) was used in the present study to estimate parameters for different item parameterization and scoring models. The package employs MML in the item parameter estimation phase, maximum likelihood and Bayesian estimation in the scoring phase. It provides estimation for the three normal and logistic dichotomous models and many polytomous models. The program files of individual models are listed in Table 3 and some examples of the programs are listed in Appendix D.

Table 3

Computer Programs Used for Different Item Parameterization and Scoring Models

	Scoring format		
Item Parameterization	Dichotomous	Ordinal Polytomous	Nominal Polytomous
1-parameter	L1.cmd L1s.cmd	Pc.cmd Pcs.cmd	
2-parameter	L2.cmd L2s.cmd	Gp.cmd Gps.cmd	Nc.cmd Ncs.cmd
3-parameter	L3.cmd L3s.cmd	Mc.cmd Mcs.cmd	

CHAPTER 4

RESULTS

Overview

The recovery rate of original ability estimates (Thetarec) and the root mean squared error (RMSE) for the ability estimates were tabulated for each study design combination (a portion of the table from one data simulation and analysis cycle is listed in Appendix E). Five grouping variables were created to facilitate comparisons posed by the research questions. The five grouping variables were the scoring format (ScorFor) [1 = dichomtomous, 2 = polytomous], the item parameterization (ItemPar) [1 = 1-PL, 2 = 2-PL, 3 = 3PL], the polytomous IRT models compared in the study (PolyMod) [1 = GPCM-1, 2 = GPCM, 3 = MCM, 4 = NCM], the prior ability distribution (Thetadis) [1 = normal, 2 = skewed to the right, 3 = skewed to the left, 4 = bimodal], and the item threshold configuration (Threconf) [1 = evenly distributed, 2 = closer at the low end, 3 = closer at the high end]. A total of 1,512 recovery rates and RMSEs respectively were acquired through the analysis of the simulated item response data sets using seven different IRT models. The minimum recovery rate was .049, and the maximum was .945; where a zero recovery rate represented non-recovery and 1 represented prefect recovery. The RMSE recorded a minimum value of .029384 and a maximum value of .372022, where lower RMSE represented higher accuracy in ability estimation. In view of the range of the original ability estimates, mostly (over 95%) within the range of –4 to 4, the differences in the RMSEs across the models were not as dramatic as what the recovery rates indicated shown. Descriptive information for these two variables in different groups by the five grouping variables was summarized in Table 4 and Table 5.

Table 4

Descriptive Information of the Recovery Rates across Different Groups

	Group Size	Mean	Standard Deviation	Minimum	Maximum
Scoring Format	1,512	.50500	.22121	.049	.945
Dichotomous	648	.34240	.12795	.049	.650
Polytomous	864	.62695	.19661	.086	.945
Item Parameterization	1,296	.47747	.22449	.049	.945
One type[a] (1-PL)	432	.63704	.23816	.271	.945
Two types[b] (2-PL)	432	.45351	.14767	.134	.806
Three types[c] (3-PL)	432	.34186	.16843	.049	.894
Polytomous Models	864	.62695	.19661	.086	.945
GPCM-1	216	.86056	.08908	.339	.945
GPCM	216	.52787	.12414	.291	.806
MCM	216	.44919	.15859	.086	.894
NCM	216	.67018	.09141	.417	.913
Prior Ability Distribution	1,512	.50500	.22121	.049	.945
Bimodal	378	.50215	.20854	.251	.945
Skewed to the left	378	.48772	.22980	.106	.898
Normal	378	.48484	.24397	.049	.925
Skewed to the right	378	.54529	.19482	.209	.923
Thresholds Configuration	1,512	.50500	.22121	.049	.945
Equal Distances	504	.51631	.22376	.082	.945
Unequal-High end	504	.49305	.22704	.049	.931
Unequal-Low end	504	.50564	.21237	.086	.925

[a] item difficulty; [b] item difficulty and discrimination, [c] item difficulty, discrimination and guessing

Table 5

Descriptive Information of the RMSE across Different Groups

	Group Size	Mean	Standard Deviation	Minimum	Maximum
Scoring Format	1,512	.134138	.0837995	.029384	.372022
Dichotomous	648	.212244	.0697662	.111041	.372022
Polytomous	864	.075560	.0250993	.029384	.211636
Item Parameterization	1,296	.144302	.0860949	.029384	.372022
One type[a]	432	.094320	.0497234	.029384	.196774
Two types[b]	432	.142517	.0619508	.047906	.265009
Three types[c]	432	.196066	.1037707	.043347	.372022
Polytomous Models	864	.075560	.0250993	.029384	.211636
GPCM-1	216	.047900	.0118312	.029384	.090294
GPCM	216	.083270	.0110293	.047906	.123693
MCM	216	.097910	.0241639	.043347	.211636
NCM	216	.073160	.0186354	.036222	.103875
Prior Ability Distribution	1,512	.134138	.0837995	.029384	.372022
Bimodal	378	.120133	.0828093	.029384	.307099
Skewed to the left	378	.135304	.0621354	.042988	.268961
Normal	378	.136334	.0917573	.034395	.372022
Skewed to the right	378	.144781	.0932271	.040336	.360555
Thresholds Configuration	1,512	.134138	.0837995	.026384	.372022
Equal Distances	504	.132676	.0841862	.026384	.352136
Unequal-High end	504	.138665	.0894819	.031623	.372022
Unequal-Low end	504	.131073	.0772566	.032879	.342053

[a] item difficulty; [b] item difficulty and discrimination, [c] item difficulty, discrimination and guessing

The two criteria, recovery rate and root mean squared error, were expected to be correlated to a certain extent. While both criteria are measuring the accuracy of the models' ability estimation, they refer to different aspects. They were correlated to examine their strength of association. A Pearson correlation coefficient of -.730 was obtained for the two criteria. The negative sign indicates that the two criteria vary in opposite directions. High recovery rate or low RMSE indicates high accuracy in ability estimation. One criterion explains about 50% of the variance in the other. Despite their shared variance, each criterion offers unique information on the accuracy of the ability estimation.

It is apparent from the wide range of recovery rates and RMSEs that the different IRT models vary greatly in their ability to recover the original ability estimates. The group means and standard deviations alone do not reveal how the two criteria varied by different combinations of grouping variables. Three-way ANOVAs were performed to examine the effects of the five factors, namely scoring format, item parameterization, different polytomous models, prior ability distribution, and thresholds configurations, on the two criteria of ability estimation accuracy. The results of the analyses are summarized in the following section by research question. ANOVA was chosen instead of MANOVA because the primary interest of the present study was on how the factors affected the two criteria individually, not on their combined measure of accuracy.

Research Question 1

How do dichotomous and polytomous IRT models differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?

Two three-way ANOVAs were conducted with Scorfor (S), Thetadis (P), Threconf (T) as independent variables, and Thetarec and RMSE as dependent variables, respectively. The results for these analyses are summarized in Tables 6 and 7, and the discussion addressed the research question.

Table 6

Three-Way Fixed Effects ANOVA on Recovery Rate (S, P, T)

Source*	SS	Df	MS	F	p	η^2	Power†
S	29.983	1	29.983	59.211	.000	.406	1.000
P	1.349	3	.450	17.326	.000	.018	1.000
T	.161	2	.08066	3.109	.045	.002	.599
S×P	2.680	3	.893	34.431	.000	.036	1.000
S×T	.512	2	.256	9.867	.000	.007	.984
P×T	.337	6	.05619	2.166	.044	.005	.775
S×P×T	.653	6	.109	4.193	.000	.009	.980
Error	38.605	1488	.02594				
Total	73.937	1511					

*S = Scoring Format, P = Prior Ability Distribution, T = Threshold Configuration

† Computed using level of significance = .05

The independent variables together explained about 48% (model R square was .478) of the variance in the recovery rate in this ANOVA. From Table 6, the three-way interaction effect is statistically significant at the level of .0005 with a small effect size (η^2 = .009, f = .10) according to Cohen (1988)[5]. Since the three-way interaction was

[5] Cohen (1988) defined f as the ratio between variance of group means and pooled variance of the dependence variable. The relationship between f and η^2 is given by $f^2 = \eta^2 / (1 - \eta^2)$. Cohen gave f = .1, .25, and .4 as general reference for small, medium and large effect for F tests in ANOVA.

statistically significant, the other effects could not be interpreted unambiguously. Simple interaction tests of scoring format (2) × prior ability distribution (4) at the three levels of threshold configuration were used to explain results. The two-way interaction S×P was statistically significant at the level of .05 at all levels of the threshold configuration, (F(3, 496) = 11.68, p < .0005; F(3, 496) = 3.25, p = .022; and F(3, 496) = 26.53, p < .0005; for "equal threshold distances", "unequal-close at the lower end", and "unequal-close at the higher end" respectively). Therefore, simple-simple main effects of scoring format were examined within each combination of threshold configurations and prior ability distributions. They were all statistically significant (p < .0005) except within the combination of an "unequal-close at the lower end" threshold configuration and "skewed to the right" prior ability distribution (F(1, 124) = .578, p = .448). Comparisons of the individual cell means for the statistically significant effects showed that the cell means of the polytomous models (scorfor = 2) were much higher than those of the dichotomous models (scorfor = 1). The effect sizes ranged from η^2 = .178 to .689, or f = .47 to 1.49, and they were large effects by Cohen's standard. This result indicated that the polytomous models had much higher recovery rates than the dichotomous models. The only exception was when a group of predominantly lower ability examinees were tested on items with categories other than the most completed answer being close to their levels of difficulty at the lower end of the ability continuum.

The simple-simple main effects of prior ability distribution were also tested within each combination of threshold configurations and scoring formats. With each combination of threshold configurations and dichotomous scoring, the tests were all statistically significant (with p-value < .0005 in each case) with large effect sizes (ranging from η^2 = .294 to .375 or f = .65 to .77). Post hoc tests revealed that the effects came from the difference between "skewed to the right" prior and the other prior distributions, and between the bimodal prior to the other priors. The "skewed to the right" prior group had the highest recovery rates, the bimodal prior group in the middle, and the other two prior groups had the lowest. With polytomous scoring, the test was statistically significant (F(3, 284) = 7.698, p < .0005) only when the threshold configuration of the items were "unequal-closer at the lower end" and the effect size was much smaller (η^2 = .075, or f = .28). Post hoc tests showed that the "skewed to the left" prior group had statistically significantly (at the level of .05) higher mean recovery rate than the other three prior groups.

Table 7

Three-Way Fixed Effects ANOVA on RMSE (S, P, T)

Source*	SS	Df	MS	F	p	η^2	Power†
S	6.918	1	6.918	3009.755	.000	.652	1.000
P	.121	3	.04046	17.605	.000	.011	1.000
T	.02261	2	.01131	4.919	.007	.002	.808
S×P	.101	3	.03355	14.598	.000	.010	1.000
S×T	.03495	2	.01748	7.603	.001	.003	.946
P×T	.0009136	6	.0001523	.066	.999	.000	.066
S×P×T	.0005335	6	.00008892	.039	1.000	.000	.059
Error	3.420	1488	.002299				
Total	10.611	1511					

*S = Scoring Format, P = Prior Ability Distribution, T = Threshold Configuration

† Computed using level of significance = .05

The same group of independent variables together explained about 68% (model $R^2 = .678$) of the variance in RMSE in this ANOVA. The major contributor to the explained variance was the main effect of scoring format. However, two of the three two-way interaction effects were statistically significant at the level of .001, with small effect sizes ($\eta^2 = .010$ and .003). Therefore, the main effects could not be interpreted unambiguously. Tests were performed for the simple main effects of scoring format within each level of prior ability distribution and threshold configuration. The tests were all statistically significant (with $p < 0.0005$ in all tests) and had very large effect sizes (ranging from $\eta^2 = .574$ to .773 or $f = 1.16$ to 1.85). Comparison of the cell means indicated that polytomous models had much smaller RMSE in ability estimation than the dichotomous models at all levels of the other two independent variables.

The simple main effect of the prior ability distribution was also tested within the two levels of scoring format. For the dichotomous models, the simple main effect was statistically significant ($F(3, 644) = 8.308$, $p < 0.0005$) with a small effect size ($\eta^2 = .037$, or $f = .20$). Comparison of cell means revealed two homogenous subsets in the four prior distribution groups. The “skewed to the left” and bimodal prior groups had smaller RMSEs compared to the normal and “skewed to the right” prior groups. For the polytomous models, the simple main effect was statistically significant ($F(3, 860) = 66.7$, $p < .0005$) with a large effect size ($\eta^2 = .187$, or $f = .48$). Comparison of cell means indicated that the bimodal prior group had the lowest RMSE, with the normal prior group in the middle, and the two skewed prior groups with the highest RMSE.

The simple main effect of the threshold configuration was tested for dichotomous and polytomous models. It was not significant when the scoring format was polytomous.

With dichotomous scoring, it was statistically significant (F(2, 645) = 5.011, p = .007) with a small effect size (η^2 = .015, or f = .12). The effect came from the difference between the “unequal-close at the lower end” group and the “unequal-close at the higher end” group. The former had a lower RMSE. The result indicated that polytomous scoring out-performed the dichotomous scoring in the accuracy of ability estimation, indicated by lower RMSE under all combinations of the research conditions. The prior distribution of the ability estimates had effect on the RMSE. The bimodal prior had the lowest RMSE.

To answer the first research question, the analysis indicated that polytomous scoring provided more accurate ability estimation, both in terms of higher recovery rate and lower RMSE, than the dichotomous scoring under all combinations of prior ability distribution and threshold configuration. Further tests were conducted to investigate the effect of the combinations within dichotomous and polytomous scoring models. Figures 5 to 8 visualized the effects that the combinations of the two factors had on the recovery rate and RMSE.

For dichotomous models, the threshold configuration did not have much effect on the accuracy of ability estimation, as expected, but the different prior ability distributions did affect the recovery rate and RMSE differently. Figures 5 and 6 show how the four prior groups differed from each other. The normal prior group performed almost identically for both criteria with the lowest recovery rate and highest RMSE. The bimodal and the “skewed to the right” prior group did better on both criteria, but the latter did much better in recovery rate.

Figure 5. Marginal means of recovery rate by prior distribution (dichotomous model).

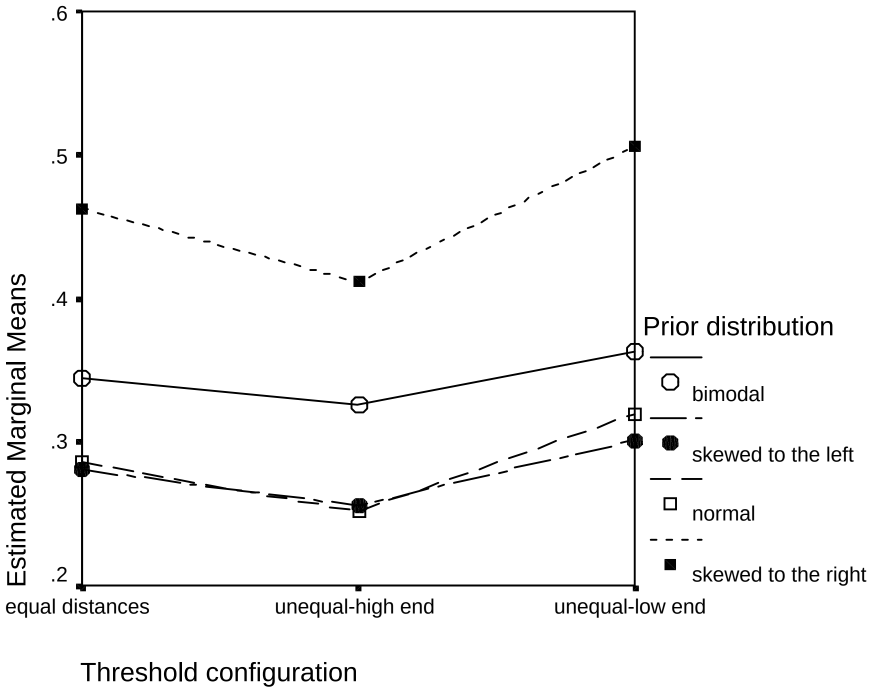
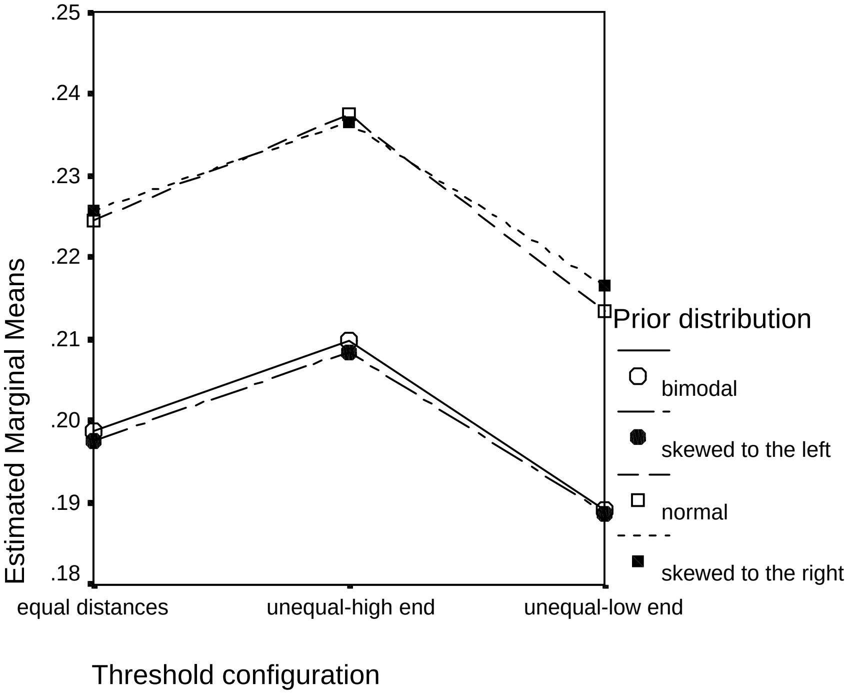


Figure 6. Marginal means of RMSE by prior distribution (dichotomous model).



For the polytomous models, the threshold configuration did not have much effect on the RMSE, but affected the recovery rate and had statistically significant interaction (F(6, 852) = 4.829, p < .0005) with a small to medium effect size (η^2 = .033, or f = .18) with prior ability distribution. It is apparent from Figure 7 that the four prior groups had very similar recovery rates when the threshold distances were equal, but their recovery rates diverged when the threshold distances were unequal. The dispersion of the group means was larger when the threshold distances were closer at the lower end (about .2) than when they were closer at the higher end (about .1). Furthermore, the ascending order of the recovery rates of the four prior groups (skewed to the left, normal, bimodal and skewed to the right) was the same as that in the dichotomous models only when the threshold distances were closer at the higher end. The order reversed when the threshold distances were equal or closer at the lower end. In the case of the RMSE, the threshold configuration had no effect, and the four prior groups differed slightly.

Figure 7. Marginal means of recovery rate by prior distribution (polytomous model).

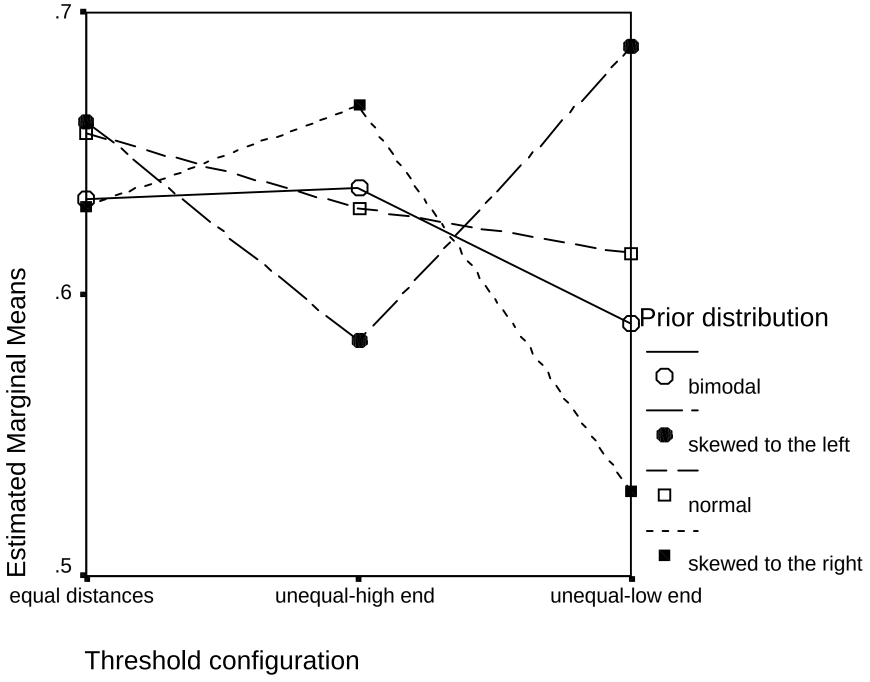
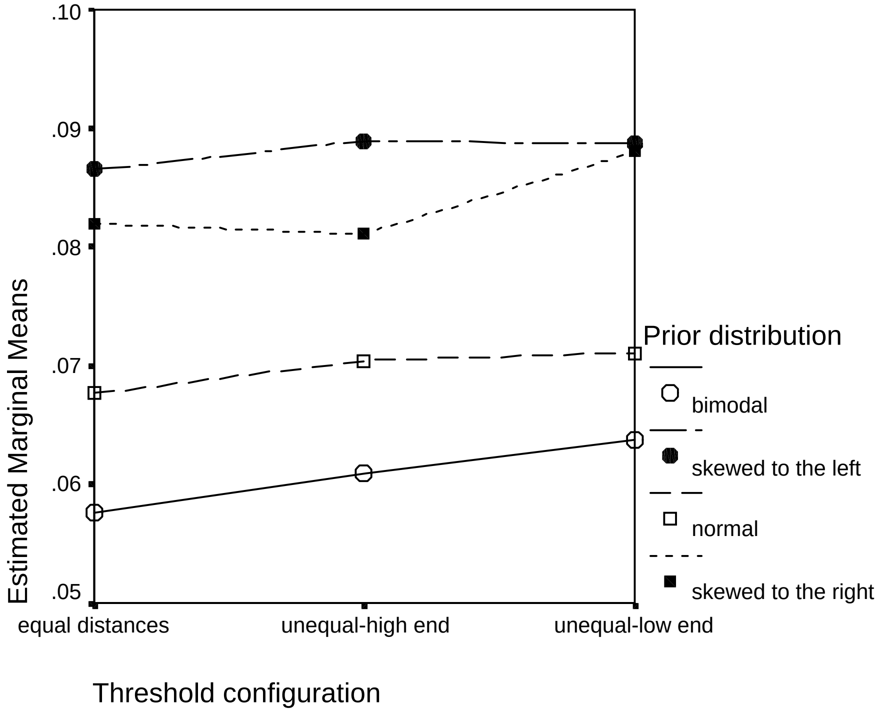


Figure 8. Marginal means of RMSE by prior distribution (polytomous model).



Research Question 2

How do different IRT item parameterization models (i.e. modeling difficulty only; both difficulty & discrimination; and difficulty, discrimination & guessing) differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?

Two three-way ANOVAs were conducted with ItemPar (I), Thetadis (P), Threconf (T) as independent variables, and Thetarec and RMSE as dependent variables, respectively. The results for these analyses are summarized in Tables 8 and 9.

Table 8

Three-Way Fixed Effects ANOVA on Recovery Rate (I, P, T)

Source[*]	SS	Df	MS	F	p	η^2	Power[†]
I	19.192	2	9.596	292.289	.000	.294	1.000
P	1.151	3	.384	11.690	.000	.018	1.000
T	.07475	2	.03737	1.138	.321	.001	.252
I×P	1.861	6	.310	9.449	.000	.029	1.000
I×T	.747	4	.187	5.691	.000	.011	.981
P×T	.273	6	.04557	1.388	.216	.004	.548
I×P×T	.596	12	.04970	1.514	.112	.009	.819
Error	41.367	1260	.03283				
Total	65.264	1295					

*I = Item Parameterization, P = Prior Ability Distribution, T = Threshold Configuration

† Computed using level of significance = .05

The independent variables explained about 37% (model $R^2 = 3.66$) of the variance in the recovery rates. Most of the variance explained came from the main effect of the item parameterization. However, in Table 8, the F tests on two of the two-way interaction effects were statistically significant with a small effect size. Therefore, tests on the simple main effects of item parameterization within different levels of the other two grouping variables were conducted. The tests were statistically significant ($p < .0005$) for all three threshold configurations and four prior ability distributions with larger effect sizes (ranging from $\eta^2 = .208$ to $.572$ or $f = .51$ to 1.16). The recovery rates of the models with different item parameterizations differed greatly. The models with only item difficulty (includes 1-PL dichotomous model and the GPCM-1 model) had the highest recovery rate, the models with item difficulty and discrimination had the medium recovery rate, and the models with item difficulty, discrimination and guessing had the lowest recovery rate. The same order was maintained under all levels of the other two variables.

The simple main effect of the prior ability distribution was not significant for the three-parameter models. It was statistically significant at the level of .05 ($F(3, 428) = 2.949$, $p = 0.033$) with a small effect size ($\eta^2 = .020$, or $f = .14$) for the one-parameter models. Comparison of the cell means revealed that the "skewed to the right" prior group had the highest recovery rate, but the difference between the four prior groups were small. For the two-parameter models, the test for simple main effect was statistically significant ($F(3, 428) = 47.606$, $p < .0005$) with a large effect size ($\eta^2 = .250$, or $f = .58$). Comparison of cell means revealed that the "skewed to the right" prior had a much higher recovery rate than the other three prior groups, whose recovery rates were similar. The magnitude of the recovery rates for the three prior groups reversed its order from that of the three groups within the one-parameter models. Although the simple main effects of threshold configuration were statistically significant at the level of .05 within the two-parameter and three-parameter models, the effect sizes were small. Comparison of cell means revealed that the differences among them were small and the recovery rate of "unequal-close at the high end" group was slightly lower than the other two.

Table 9

Three-Way Fixed Effects ANOVA on RMSE (I, P, T)

Source*	SS	Df	MS	F	p	η^2	Power†
I	2.238	2	1.119	197.722	.000	.233	1.000
P	.08907	3	.02969	5.246	.001	.009	1.000
T	.01727	2	.0086356	1.526	.218	.002	.326
I×P	.117	6	.01954	.0453	.002	.012	.947
I×T	.004854	4	.001214	.214	.930	.001	.097
P×T	.0009978	6	.0001663	.029	1.000	.000	.057
I×P×T	.0003369	12	.00002808	.005	1.000	.000	.052
Error	7.131	1260	.00566				
Total	9.599	1295					

*I = Item Parameterization, P = Prior Ability Distribution, T = Threshold Configuration

† Computed using level of significance = .05

The model R^2 of the ANOVA was .257, and most of the effect came from item parameterization (η^2 = .233, or f = .55). The interaction effect of item parameterization and prior ability distribution was statistically significant with a small effect size. The tests of simple main effects showed that the three levels of item parameterization statistically significantly (with $p < .0005$ in each test) differed from each other in RMSE for all prior groups. They differed most within the “skewed to the right” prior groups (η^2 = .330, or f = .70), but less within the normal prior groups (η^2 = .160, or f = .35). In all cases the one-parameter models had the lowest RMSE, and the three-parameter models had the highest RMSE. The simple main effects of prior ability distribution within different parameterization models were statistically significant at the level of .05 with small effect sizes (ranging from η^2 = .021 to .056 or f = .15 to .24). The bimodal prior group gave the lowest RMSE within one- and two-parameter models, and a close second within the three-parameter models. The "skewed to the right" prior group had the highest RMSE within the two- and three-parameter models. However, the differences in RMSEs between the four different prior groups were small. Although the main effects and all the interaction effects involving the threshold configuration were not significant at the level of .05, examination of cell means revealed that the "unequal-close at the high end" group always had a RMSE slightly higher than the other threshold configuration groups.

To answer the second research question, the ANOVA indicated that under all combinations of prior ability distributions and threshold configurations, the 1-PL models (with only item difficulty) had the most accurate ability estimation, and the 3-PL models (with three types of parameters) were less accurate in ability estimation among the three different item parameterization models. Within each category of item parameterization, the effect of prior ability distribution and threshold configuration was assessed. Figures 9 to 14 show how the recovery rates and RMSEs differed across these two variables within each category of item parameterization. The effect of threshold configuration was similar in all three categories of item parameterization. The ability estimation using items with "unequal-close at high end" configuration were less accurate, indicated by lower recovery rates and higher RMSEs, than the other two kinds of configurations. The effect of prior ability distribution was not clear. The "skewed to the right" prior group had the highest recovery rate within 1-PL and 2-PL models, lowest within the 3-PL models; but had the highest RMSE with 2-PL and 3-PL models, second lowest within the 1-PL models. On the other hand, the bimodal prior group provided the lowest RMSE all the time, and also had the highest recovery rate within the 3-PL models. A similar pattern for the effect of prior ability distribution was observed within the dichotomous and polytomous models.

Figure 9. Marginal means of recovery rate by prior distribution (1-PL model).

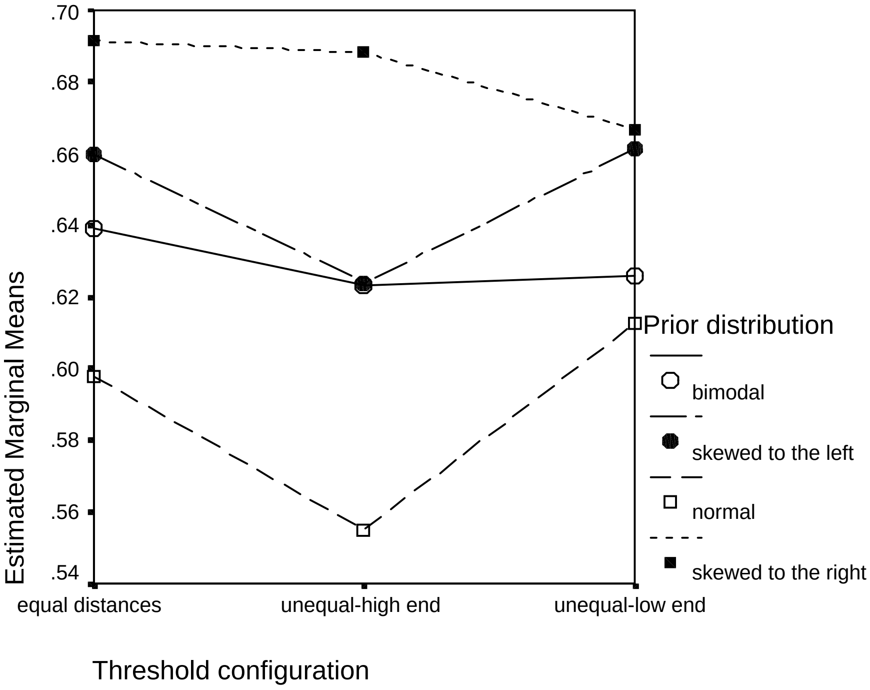


Figure 10. Marginal means of RMSE by prior distribution (1-PL model).

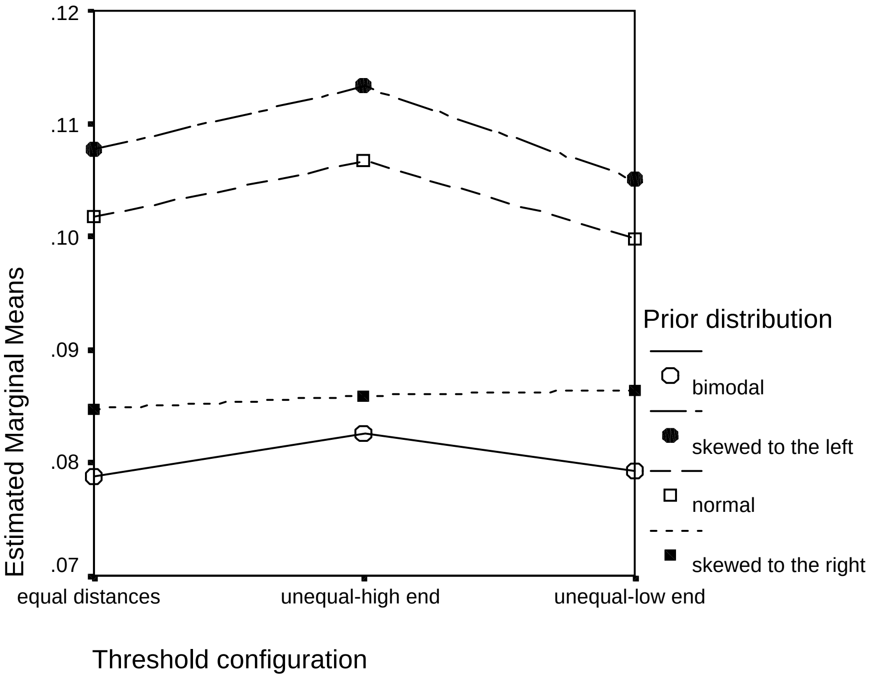


Figure 11. Marginal means of recovery rate by prior distribution (2-PL model).

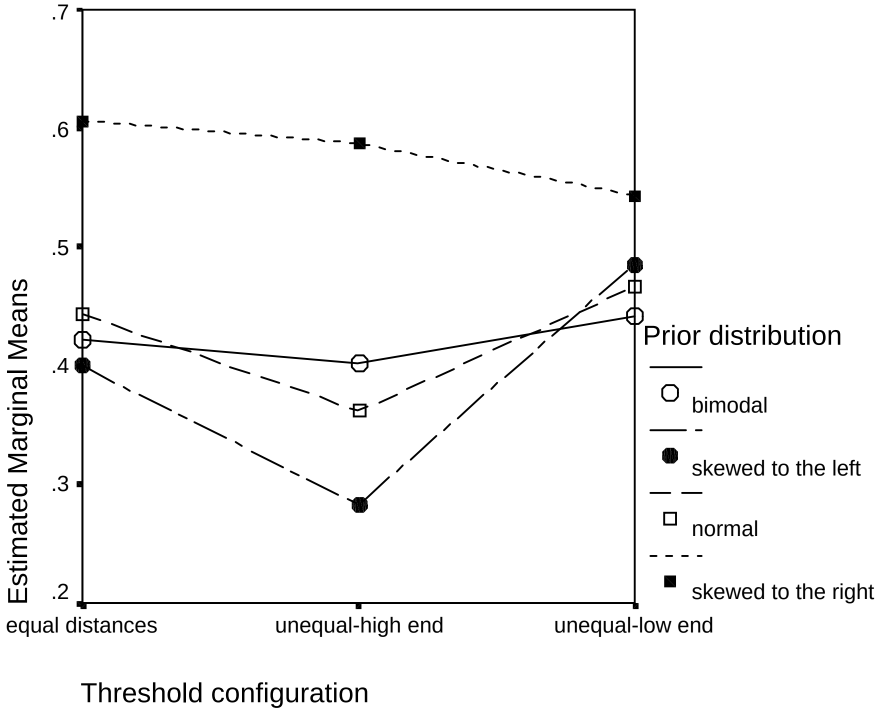


Figure 12. Marginal means of RMSE by prior distribution (2-PL model).

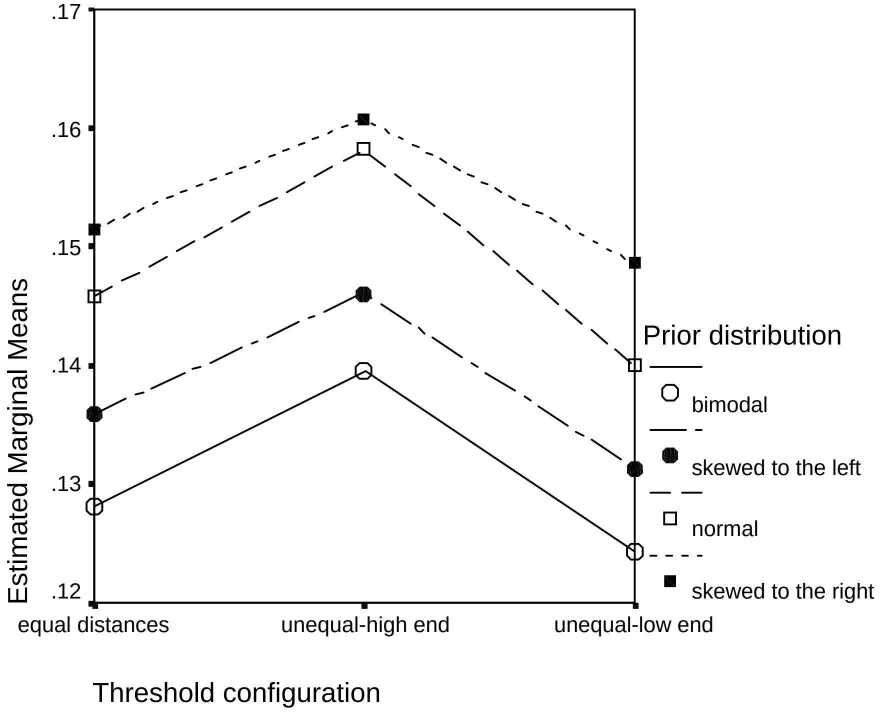


Figure 13. Marginal means of recovery rate by prior distribution (3-PL model).

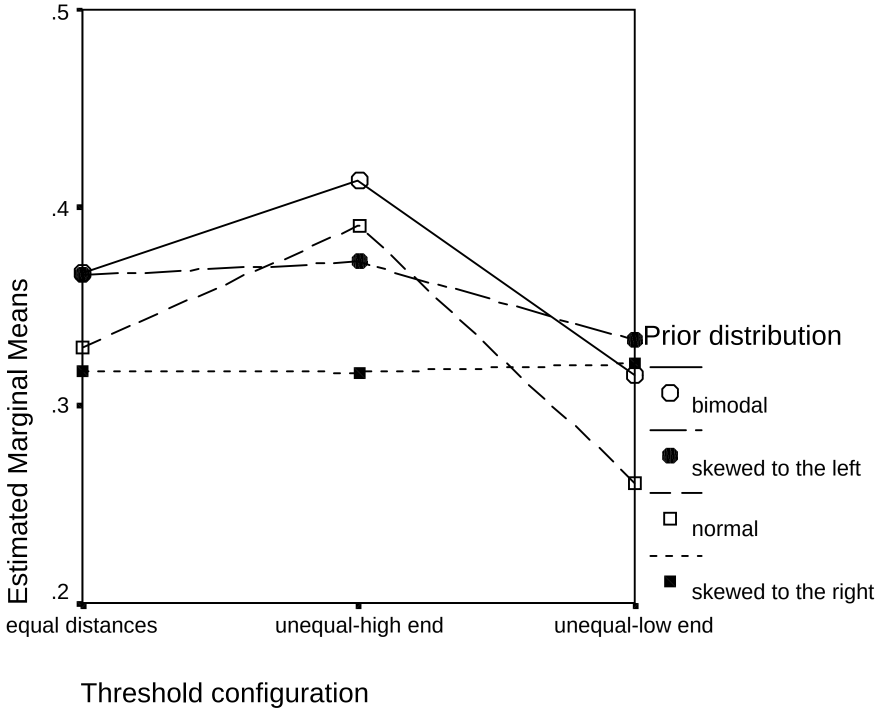
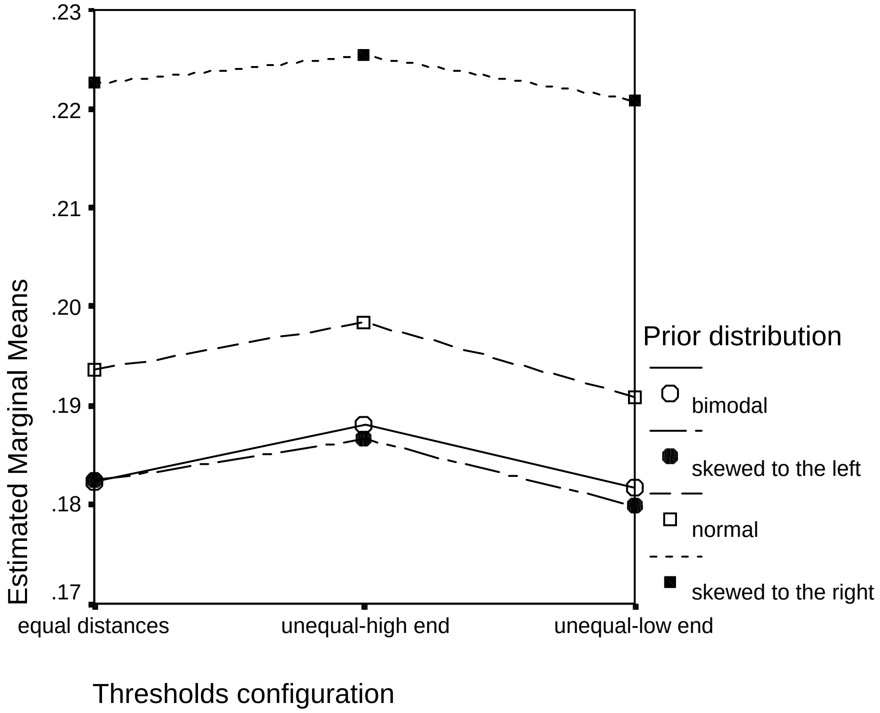


Figure 14. Marginal means of RMSE by prior distribution (3-PL model).



Research Question 3

How do the polytomous IRT models differ in accuracy of recovering ability estimates in different combinations of prior ability distributions and item category threshold distance configurations?

Two 3-way ANOVAs were conducted with PolyMod (Po), Thetadis (P), Threconf (T) as independent variables, and with Thetarec and RMSE as dependent variables. The results for these analyses are summarized in Tables 10 and 12.

Table 10

Three-Way Fixed Effects ANOVA on Recovery Rate (Po, P,T)

Source*	SS	Df	MS	F	p	η^2	Power†
Po	21.137	3	7.046	1259.247	.000	.634	1.000
P	.153	3	.05094	9.104	.000	.005	.996
T	.238	2	.119	21.284	.000	.007	1.000
Po×P	3.368	9	.374	66.884	.000	.101	1.000
Po×T	1.505	6	.251	44.827	.000	.045	1.000
P×T	1.084	6	.181	32.301	.000	.033	1.000
Po×P×T	1.309	18	.07272	12.997	.000	.039	1.000
Error	4.566	816	.005595				
Total	33.361	863					

*Po = Polytomous Models, P = Prior Ability Distribution, T = Threshold Configuration
† Computed using level of significance = .05

The model effect of this ANOVA was high (R^2 = .863). About 86% of the variance of the recovery rate could be explained by the three independent variables. Most of the model effect came from the differences among the four polytomous models. Since the three-way interaction effect was statistically significant (F(18, 816) = 12.997, p <

.0005), tests for the simple interaction effects were conducted. The simple interaction effects Po×P were statistically significant (($F(9, 272) = 26.268$, $p < .0005$; $F(9, 272) = 41.954.25$, $p < .0005$; and $F(9, 272) = 19.044$, $p < .0005$) with large effect sizes (ranging from $\eta^2 = .387$ to .581 or $f = .79$ to 1.18) within all three categories of threshold configuration. The interaction effect was strongest within the "unequal-close at the high end" group, and weakest within the "unequal-close to the low end" group.

Simple-simple main effects of PolyMod were tested within each combination of prior ability distribution and threshold configuration. All combinations were statistically significant ($p < .0005$) with large effect sizes (ranging from $\eta^2 = .299$ to .994 or $f = .65$ to 12.87). The large range of effect sizes indicated that the four different polytomous models differed on accuracy of ability estimation, and the differences changed from one combination of threshold configuration and prior abiltiy distribution to the other. Table 11 summarized the effect sizes of the simple-simple main effects of PolyMod. It is apparent that the effect sizes were large; most of them were larger than .95, i.e. the recovery rates of the four polytomous models were different from each other in most cases. Post hoc analyses were conducted to compare the four cell means in each of the twelve combinations of conditions. Cell means that were not statistically significantly different at the level of .05 were put together in a homogeneous subset. Therefore, cell means of the models between homogenous subsets were statistically significantly different from each other at the level of .05. In the last column of Table 11, polytomous models in a homogenous subset were put together inside a bracket, and commas separate the homogenous subsets of models. The cell means of the models within and between homogenous subsets are listed in descending order. A clear pattern (1, 4, 2, 3) emerged when the homogenous subsets were compared across the twelve combinations of conditions. GPCM-1 had the highest recovery rate in all cases, most of the time statistically significantly higher than the other three models. NCM had the second best recovery rate except in three cases, and in two of those cases it was very close to the second best. MCM had the lowest recovery rate most of the time except when the threshold configuration was “unequal-close at the high end.” The same order (GPCM-1, NCM, GPCM, MCM) was maintained when the threshold configuration was “equal distances” or “unequal-close at the low end”, with only one pair of models reversing their

Table 11

Simple-Simple Main effect (PolyMod) on Recovery Rate and Homogeneous Subsets

Threconf	Thetadis	Effect size (η^2)	Homogeneous subsets[*]
Equal distances	bimodal	.993	1, 4, (2, 3) [†]
Equal distances	skewed (left)	.952	1, 4, (3, 2)
Equal distances	normal	.413	(1, 4), (2, 3)
Equal distances	skewed (right)	.982	1, 4, 2, 3
Unequal-high	bimodal	.990	1, 4, 3, 2
Unequal-high	skewed (left)	.967	1, 3, 4, 2
Unequal-high	normal	.299	(1, 3, 4), 2
Unequal-high	skewed (right)	.985	1, 4, 2, 3
Unequal-low	bimodal	.994	1, 4, 2, 3
Unequal-low	skewed (left)	.968	1, (2, 4), 3
Unequal-low	normal	.715	(1, 4), 2, 3
Unequal-low	skewed (right)	.979	1, 4, 2, 3

[*] the means of the models in a homogenous subset are not significantly different at the level of .05.

[†] 1 = GPCM-1, 2 = GPCM, 3 = MCM, 4 = NCM; models inside a bracket are in same homogenous group.

order when the prior distribution was “skewed to the left”. The models with reversed order were in the same homogeneous subset. It seems that the “unequal-close to the high end” threshold configuration enhanced the recovery rate of MCM and jeopardized that of the GPCM. With the prior ability distributions, it seems that three of the prior groups did not affect the order, but “skewed to left” prior group did change the order.

Table 12

Three-Way Fixed Effects ANOVA on RMSE (Po, P, T)

Source[*]	SS	Df	MS	F	p	η^2	Power[†]
Po	.287	3	.09574	975.535	.000	.528	1.000
P	.103	3	.03421	348.553	.000	.189	1.000
T	.002908	2	.001454	14.813	.000	.005	.999
Po×P	.05696	9	6.329	64.493	.000	.105	1.000
Po×T	.01182	6	.001970	20.078	.000	.022	1.000
P×T	.001277	6	.0002128	2.168	.044	.002	.774
Po×P×T	.0007763	18	.00004313	.439	.979	.001	.321
Error	.08008	816	.00009814				
Total	.544	863					

[*] Po = Polytomous Models, P = Prior Ability Distribution, T = Threshold Configuration

[†] Computed using level of significance = .05

The model effect of the ANOVA on RMSE was high ($R^2 = .853$) and most of the effect came from the main effect of the independent variable PolyMod. The main effects can not be interpreted unambiguously because two-way interaction effects were statistically significant at the level of .05 in the analysis. Tests were therefore conducted to investigate simple main effects. The simple main effects of PolyMod were statistically significant ($p < .0005$) with effect sizes $\eta^2 = .111$, .129, and .082 or f = .35, .38 and .30.

Post hoc comparisons indicated that the order of the four models by magnitude of their RMSE remained unchanged across the three levels of threshold configuration, specifically the order was GPCM-1, NCM, GPCM, MCM with RMSE ascending. It also showed that the order of the four prior ability distribution groups by the magnitude of RMSE remained the same across the levels of threshold configuration. The order was bimodal, normal, skewed to the right, and skewed to the left with ascending RMSE.

To answer the third research question, the analysis indicated that the four polytomous IRT models used in the present study differed in accuracy of ability estimation. The tests of simple effects and post hoc tests revealed that the four models basically maintained the same order of accuracy in terms of higher recovery rate and lower RMSE, with different combinations of threshold configurations and prior distributions, although the magnitudes of the differences between models varied from one combination of conditions to the other. Figures 15 to 18 show the recovery rates and the RMSEs of the four models across the categories of threshold configuration and across different prior groups.

Figure 15. Marginal means of recovery rate by polytomous model (within Threconf).

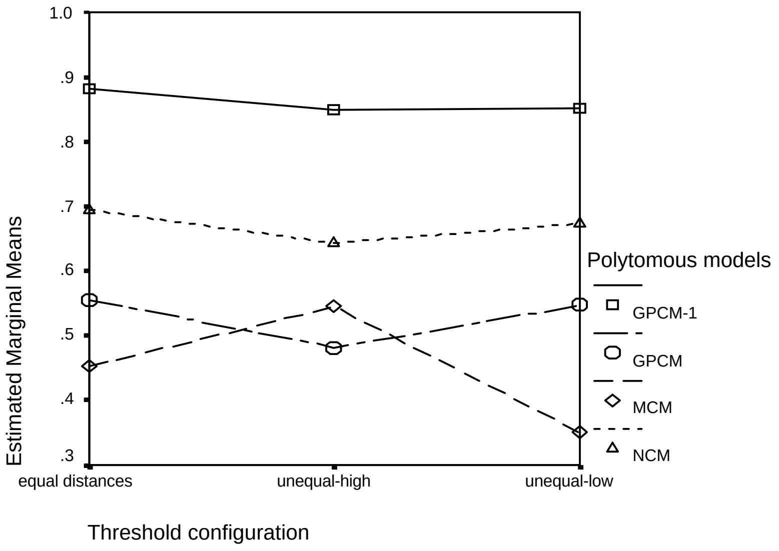


Figure 16. Marginal means of recovery rate by polytomous model (within Thetadis).

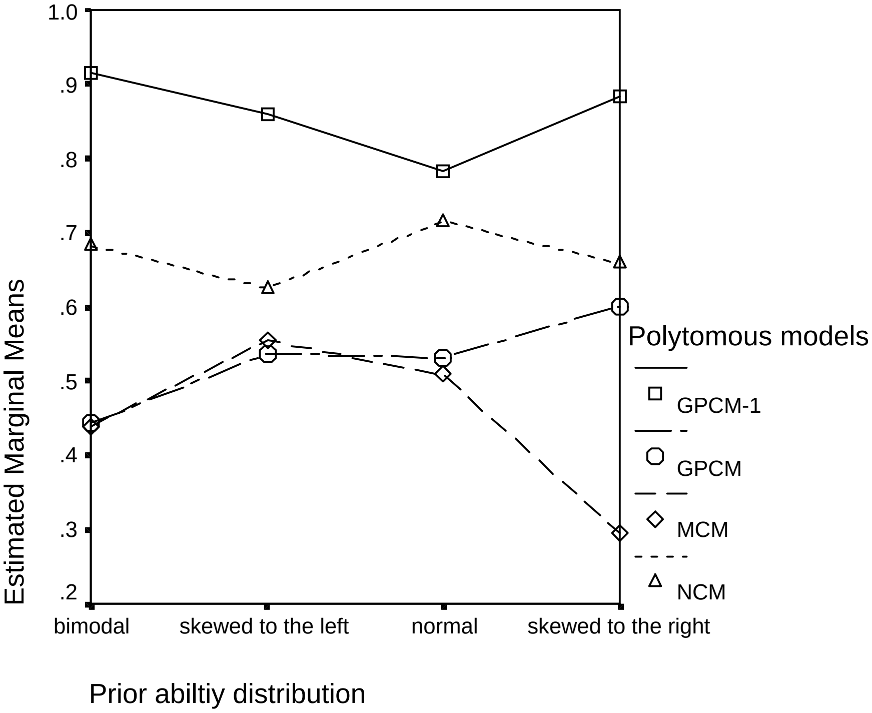


Figure 17. Marginal means of RMSE by polytomous model (within Threconf).

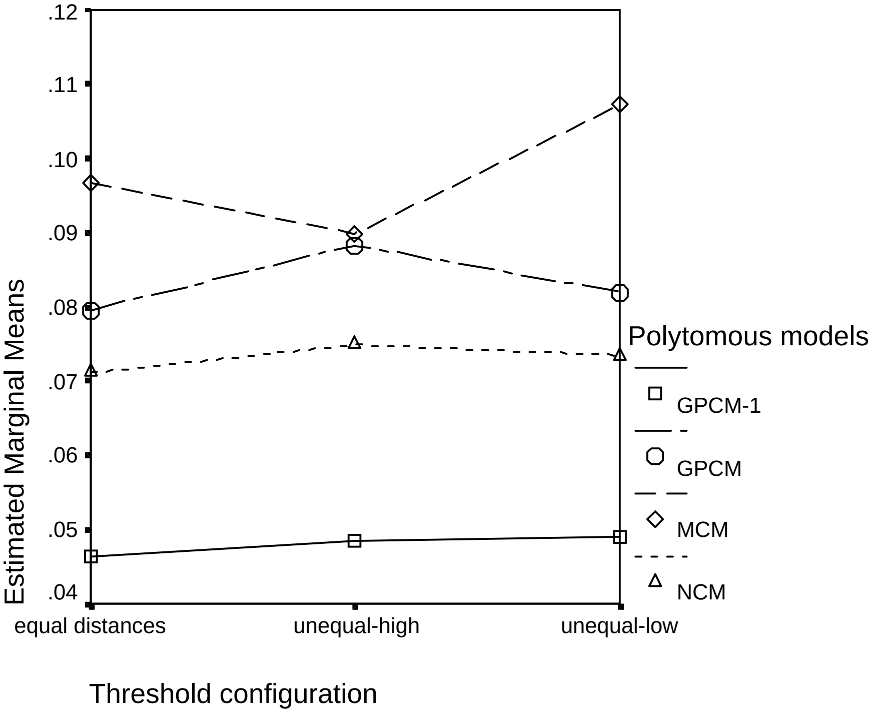
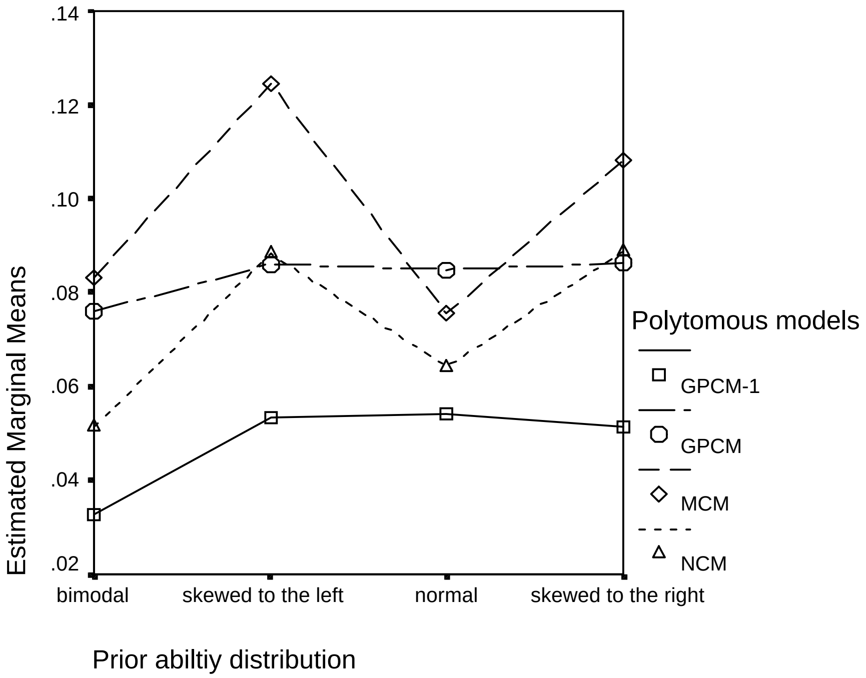


Figure 18. Marginal means of RMSE by polytomous model (within Thetadis).



Examination of the interaction effects in the four figures revealed that they were mostly ordinal. Ordinal interactions indicate that the order of the groups within a grouping variable remains unchanged, while the magnitudes of the differences between the group means varied from one pair of groups to the other. The order of the four models with ascending RMSEs, and descending recovery rates was GPCM-1, NCM, GPCM, and MCM. The order only changed when the threshold configuration was "unequal-close at the high end," or when the prior ability distribution was normal.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

The findings concerning the effects of scoring format and item parameterization, in combination with prior ability distribution and item threshold configuration, on the accuracy of ability estimation will be discussed in the first part of this chapter. Some conclusions will be drawn from the discussion and the practical educational importance of the findings will be addressed. Recommendations for future study will be made in the second part of this chapter.

Conclusions

Findings of the Present Study

1. The two criteria of ability estimation accuracy, recovery rate and RMSE, were negatively correlated as expected. In most of the hypotheses tested in this study, the effect sizes on recovery rate were larger than that of the RMSE.
2. Polytomous scoring models provided more accurate ability estimation, both in terms of higher recovery rate and lower RMSE, than the dichotomous scoring models under all combinations of prior ability distribution and threshold configuration.
3. Under all combinations of prior ability distributions and threshold configurations, the 1-PL models (with only item difficulty) had the most accurate ability estimation, and the 3-PL models (with three types of parameters) were less accurate in ability estimation among the three different types of item parameterization models.

4. Four polytomous IRT models used in the present study differed in accuracy of ability estimation. The four models basically maintained the same order of accuracy under all the combinations of threshold configuration and prior distribution. The order was GPCM-1, NCM, GPCM, and MCM with ascending RMSEs and descending recovery rates.
5. Threshold configuration indicated small effect on RMSE. When the threshold configuration was “unequal-close at the high end”, RMSE was slightly lower than the RMSE obtained under the other two threshold configurations. Threshold configuration had a much larger effect on recovery rates. For dichotomous models, the “unequal-close at the high end” group had recovery rates slightly lower than the other two groups. For polytomous models, different threshold configurations had different effects on different prior groups. For the bimodal and the “skewed to the right” prior group, the “unequal-close at the low end” had the lowest recovery rate, and the “unequal-close at the high end” had the highest.” For the “skewed to the left” prior group, the order was reversed. The threshold configuration had little effect when the prior distribution was normal.
6. Prior ability distribution affected recovery rates and RMSEs in all combinations of the research conditions. However, no conclusive pattern of its effect was identified. The “skewed to the right” prior had the highest recovery rate and highest RMSE most of the time, while bimodal prior had the lowest RMSE most of the time.

Recovery Rate and Root Mean Squared Error

In the present study, the accuracy of IRT models in ability estimation was compared using the recovery rate of original ability estimates and RMSE. Both criteria were obtained by comparing two sets of ability estimates. One set was the original ability estimates of 1,000 examinees that were used to generate the simulated item response data, and the other set was the means of fifty ability estimates for each of the 1,000 examinees obtained as the output when applying an IRT model to fifty simulated item response data sets from the same 1,000 examinees. The first set of estimates was used as known ability estimates and the second set as the bootstrap estimators of the known ability estimates in the present study. RMSE was a measure of the average deviation of the 1,000 bootstrap estimators from the 1,000 known ability estimates. The recovery rate, was the percentage of the 1,000 known ability estimates captured by the 95% confident interval constructed around their respective estimators, which was obtained by checking each of the 1,000 pairs of known ability estimates and their respective estimators. Therefore, while both criteria assessed the overall accuracy of ability estimates, the recovery rate was more sensitive to individual deviations. This explains why the recovery rate had larger effect sizes than the RMSE. Since individual deviations are important in the overall accuracy of an IRT model's ability estimation, the recovery rates were more informative than the RMSEs on the accuracy of the IRT model's ability estimation.

Scoring Format

This study has demonstrated that polytomous models have better accuracy in ability estimation, both in terms of higher recovery rates and lower RMSEs, in all combinations of prior distribution and threshold configuration. Dichotomous models had less information about examinees' ability by ignoring their differences in choosing different categories other than the most completed answer. It has been discussed in the literature review that a polytomous model can be viewed as a sequence of dichotomous models and thus is more refined as a measuring tool of the latent construct under investigation. This study not only pointed out that two scoring formats will give different ability estimates when applied to multiple-choice items, but also gave reference to the magnitude of the difference of the ability estimates. The recovery rate for polytomous scoring almost doubled that of dichotomous scoring (63% versus 34%) when averaged over all research conditions. The average RMSE for polytomous scoring was about one third of that for dichotomous scoring (.075 versus .212). These differences are not small and should not be ignored. This finding has important implications for educational testing.

Large-scale assessments and achievement tests are predominantly constructed of multiple-choice items scored dichotomously (correct, incorrect). In view of the large difference in ability estimation that this study has found between dichotomous scoring and polytomous scoring models, the scoring format of tests should be changed to improve the accuracy of ability estimation. Tests are used to guide educational decision-making and the accuracy of their outcomes should be a priority. Moreover, the use of multiple-choice items versus constructed response items (e.g. Bennett, R. E. & Ward, W.C. Eds., 1993) has been an ongoing debate on for a long time. Many issues were discussed, but the objectivity and reliability of the scoring rubrics is paramount. Scoring multiple-choice items is relatively inexpensive and highly reliable in comparison to scoring constructed response items. However, concerns have been raised about the inability of multiple-choice items to test students on multi-level or multi-step questions. Multiple-choice items with well-constructed response categories, scored polytomously with partial credits, would be a legitimate option to consider. The ordinal nature of the response categories could be used to test multi-level or multi-step tasks.

Item parameterization

The three categories of item parameterization were expected to give different ability estimates, but it was unexpected to have the IRT models with one type of parameter (1-PL) having the highest recovery rates and lowest RMSEs across all combinations of research conditions. Because the item response data used in the study was simulated with known item parameters, in which the item discrimination parameters varied from .8 to 2 (see Appendix C), the 2-PL models should fit the data better than 1-PL models. Two factors may be possible explanations for the unexpected outcome. First, Multilog software analyzes 1-PL dichotomous models as a special case of the graded response model in which item discrimination is not set free as a parameter to be estimated. In the item calibration stage of the program, the item discrimination is not set to 1 but to a constant value estimated from all 30 items in the test. The constant value will change from test to test, but remains unchanged across the items within the same test. The constant value probably served as the "average item discrimination" for all 30 items in the test, and may have enhanced the fit of the IRT model to the item response data. Second, the 2-PL dichotomous model has 30 extra parameters to be estimated and therefore has more chances to accumulate error.

For polytomous scoring models, GPCM-1 out-performed GPCM. This finding seems to contradict Muraki' study (1992), in which he compared the partial credit model and general partial credit model using simulated and real data. His results demonstrated that the general partial credit model yielded a better fit to the data than the partial credit model. Present study indicated that the partial credit model (GPCM-1) yielded more accurate ability estimates than the general partial credit model (GPCM). However, GPCM-1 only yielded more accurate ability estimates, not better model fit than GPCM. Similar factors working in the dichotomous 1-PL model are present in GPCM-1. Multilog software analyzes the partial credit model as a constrained case of the nominal categories model. The only difference between the GPCM-1 and GPCM was that an "average item discrimination" was estimated for all 30 items in GPCM-1 with item discrimination free to vary from item to item in GPCM. The "average item discrimination" may have enhanced the fit of the GPCM-1 model. GPCM had a slightly better fit than GPCM-1 in this study; but it was not enough to compensate for the increase in the number of parameters estimated.

The 3-PL models were expected to be less accurate in ability estimation in this study because the guessing factor was not modeled in the simulated item response data (a limitation of this study). The Multilog software analyzed the dichotomous 3-PL model as a special case of the multiple-choice model with the number of categories equal to two. The multiple-choice model in this study used .25 as an initial value in the calibration run to estimate the pseudo-chance parameter for each item, however, there was no guessing effect in the item response data. The 3-PL models had more parameters to be estimated and yield a worse fit to the item response data. These findings demonstrated that an IRT model with a better model fit to the data may give less accurate ability estimation, especially when the IRT model can not compensate for the increased number of parameters to be estimated.

Comparison of Polytomous Models

In the present study, polytomous models were not only compared to the dichotomous models but also among themselves. Three of the four polytomous models, GPCM-1, GPCM, MCM had been compared as part of the 1-PL, 2-PL, and 3-PL models under the item parameterization study. When only the polytomous models were being compared, the order of accuracy of those three models remained the same as it was in the item parameterization study. The fourth model, NCM, was included in the study to compare the ability estimation of nominal and ordinal polytomous models. It was discussed in the literature review that NCM is the most general polytomous model, and the other ordinal polytomous models can be derived from NCM by putting certain constraints on the item parameters to preserve the order of the categories. In Multilog, GPCM-1 and GPCM are obtained by imposing constraints on the parameters a_k and c_k of the NCM through the application of T-matrixes (polynomial and triangle). The two constrained versions of NCM were expected to be more accurate in their ability estimation than the general NCM, because the item categories of the test used to simulate item response data were ordinal. However, the findings of this study indicated that for most of the combinations of threshold configuration and prior ability distribution, NCM out-performed GPCM, although both were out-performed by GPCM-1. This finding indicated that constraints on parameters may not help to improve the model fit. It also indicated that polytomous scoring models would give better ability estimation even for multiple-choice items with nominal response categories, because NCM provided ability estimation as good as, if not better than, ordinal polytomous models.

Prior Ability Distribution

The present study indicated that prior ability distribution had an effect on the accuracy of ability estimation. It also revealed that prior ability distribution interacted with other factors in the study, e.g. scoring format, item parameterization, and threshold configuration. The differences in group means for the four prior groups varied greatly from one combination of research conditions to the other. No clear overall pattern for the variation was indicated, however some patterns within some combinations were identified. Furthermore, caution needs to be taken in the interpretation of this finding, because prior ability distribution was assumed to be normal by Multilog software in the item calibration and scoring programs. According to Thissen (1991), there is little or no theory available to evaluate a population-distribution fitting procedure, so it is difficult to assess the effect of prior distribution on the item calibration and test scoring in IRT. Although Thissen introduced the Johnson family of distributions as a possible tool to characterize the population density in MML estimation, he warned against routine use of it because it was a "relatively untested scheme in an area fraught with difficulties." In Multilog software, all prior ability distributions are assumed to be normal, and Gaussian quadratures are used to approximate the distribution in MML estimation. Therefore, the normal prior group should be used as a reference point in the interpretation of the mean differences of the four prior groups in different combinations of the research conditions.

Although a clear overall pattern cannot be identified for the effect of prior ability distribution on the accuracy of ability estimation, the differences of group means for the prior groups in the present study have practical and educational importance. In education testing, the population distribution of the latent construct is often unknown and assumed to be normal. In practice, however, the sample of examinees is more often from a population with an ability distribution different from normal. Three of the four prior groups represented in this study simulated such kinds of populations; specifically they represented predominantly high ability groups, predominantly low ability groups and groups with diversified or even polarized ability. The present study indicated the effect on the accuracy of ability estimation (i.e. enhanced or reduced in comparison to the normally distributed group) when the sample of examinees represented one of the three kinds of populations described above.

Threshold Configuration

When the items are scored dichotomously, the unequal distance and equal distance threshold configurations will not affect the response data because all categories other than the complete answer will be scored as zero. The small differences in the findings were due to random error from the simulation process. For the polytomous models, when threshold configuration categories were examined across item parameterization and individual models, the same pattern emerged. The three categories were in the same order ("unequal-close at the low end", "equal threshold", and "unequal-at the high end") with descending recovery rates and ascending RMSE, if the mean differences were statistically significant. However, the effect sizes for those effects were small. The mean differences amounted to less than 10% in recovery rate and hundredths difference in RMSE. The values for threshold are usually unknown before item calibration, but it is important to realize that the distance between thresholds has an effect on the accuracy of ability estimation. If a partial credit is given to an easy step as well as a hard step, it impacts how different levels of ability in the examinees are estimated.

When the effects of threshold configurations were examined across the four prior distribution groups, the RMSEs of the three categories showed no significant difference. In contrast, the recovery rates of the three categories differed significantly across the four prior groups. When the prior was normal, there was no significant difference among the three group means. When the prior was “skewed to the left”, the three categories followed the same order as in other combinations of research conditions, however, the “unequal-high end” category had a much lower recovery rate than the other two. When the prior was “skewed to the right” and “bimodal”, the order of the three categories reversed, i.e. “unequal-at the high end”, “equal threshold”, and “unequal-at the low end” with descending recovery rates. The effect sizes of the mean differences were large. The reversal of order happened because of the presence of a higher density of low ability examinees in the “bimodal” and “skewed to the right” prior groups. The presence of a larger low ability group reduced the recovery rate when the item thresholds were close at the low end, which decreased the power of the lower categories in discriminating low ability examinees. The fact that “skewed to the left” prior group had a lowest recovery when the threshold configuration was “unequal-close at the high end” could be explained by similar reason. In summary, the effect of threshold configuration indicated that the accuracy of ability estimation would reduce if multiple-choice items with categories close to each other at one end of the ability continuum were administrated to a group with higher population density at the same end of the continuum.

Recommendations

Future studies are recommended in three areas. First, the effect of prior ability distribution on the accuracy of ability estimation cannot be assessed unambiguously until a new approach for appropriately characterizing population density is developed. Second, the effect of threshold configuration has not been fully assessed. Threshold configurations other than the three examined in this study should be applied and varied systematically throughout a set of items, and the effects of those variations should be examined. Third, while most large-scale assessments and achievement tests in the past were constructed of multiple-choice items scored dichotomously, tests with multiple item formats have become popular in recent years. These tests often include multiple-choice items mixed with a variety of constructed-response items (e.g. short answer, matching, multiple-steps, short essay, etc.). Examples of mixed item format tests are the National Assessment of Educational Progress (NAEP) (Calderone, King & Horkay, 1997), the Test of English as a Foreign Language (TOEFL) (Tang & Eignor, 1997), and state assessments administered in states such as Massachusetts, North Carolina, and Wisconsin. If tests that combine multiple-choice items and constructed-response items are unidimensional, a single ability estimate per examinee is generated according to all item responses. There are different approaches to handling mixed model estimates for a test in order to produce a common score scale. Weighted combinations of the two parts (testlets) are widely applied. The selection of weights, however, poses a problem. Lukhele, Thissen, and Wainer (1994) pointed out that the composite ability scores may have lower reliability than the component ability scores; if the weights applied to create the composite score are ill chosen. Rosa, Swygert, Nelson, and Thissen (2001) developed a technique based on IRT to use the item response data to determine the relative weights of the components.

Multiple item formats call for mixed scoring IRT models. The approach typically taken calibrates all of the items jointly using suitable IRT models for each item. Dichotomous models are used for the multiple-choice items and polytomous models for constructed-response items. A likelihood function of response patterns for each combination of summed scores for the two components is then constructed. A single ability score for each combination of summed scores is estimated using a likelihood function. A scale-scoring table is then constructed to list the ability score for all the possible combinations of summed scores for the two components. Examinees obtain their ability estimates from the scale-scoring table according to the summed scores, which came from the two parts of the test. It is meaningful to compare the ability estimates obtained from a mixed scoring model to those from dichotomous and polytomous models that have only one item type, e.g. all multiple choice items.

APPENDIX A

SAS Program for Data Simulation

```
/*******************************************************************/
/*      This program is to generate response patterns             */
/*      based on Muraki's generalized partial credit model.       */
/*      The possible responses for item i are 0 to (ncat-1).      */
/*******************************************************************/


options ps=52 ls=72;
%let ne=1000;  /*no. of examinees*/
%let ni=30;   /*no. of items*/
%let thlist=th(i,1) th(i,2) th(i,3);/*threshold input format*/
%let ncb=3;/*no. of thresholds*/
%let ncat=4;/*no. of categories*/
%let ipmfl='h:\diss_sim\etip.dat'; /* item paramenter input path*/
%let ipmfmt=a(i) 8.2 (&thlist) (&ncb*8.2);/*item parameter input
format*/
%let outfl='h:\diss_sim\ner.dat';  /*generated data output path*/
%let seed1=6734;                /* seed number for normal dist.. */
%let seed3=3422;                /*seed number for r*/
%let putfmt=@1 id 4.0 @6 z 7.3 @14 (r1-r&ni) (&ni*1.0);
/*******************************************************************/


data know (keep=z);           /* created normally distributed sample */
     do i=1 to &ne;           /* of the examinees’ ability estimates */
       z=rannor(&seed1);
       output;
     end;

filename ipm &ipmfl;          /* read in the input parameter file */
data iparm;
     array th(&ni, &ncb);
     array a(&ni);
     do i=1 to &ni;
          infile ipm;
          input &ipmfmt;
     end;

filename resp &outfl;         /* set up the output file format */
data respdat (keep= z r1-r&ni e1 den pbc1-pbc&ncat r tot);
     if _n_=1 then set iparm; set know;
     array sd(&ni, &ncb);
     array a(&ni);
     array pbc(&ncat);
     array rr(*) r1-r&ni;
         do i=1 to &ni;
            rr(i)=0.0;
         end;
         do i=1 to &ni;       /* calculate the probabilities of an   */
            den=1.0;          /* examinee choosing each of the four  */
            do j=1 to &ncb;   /* categories for each items.          */
```

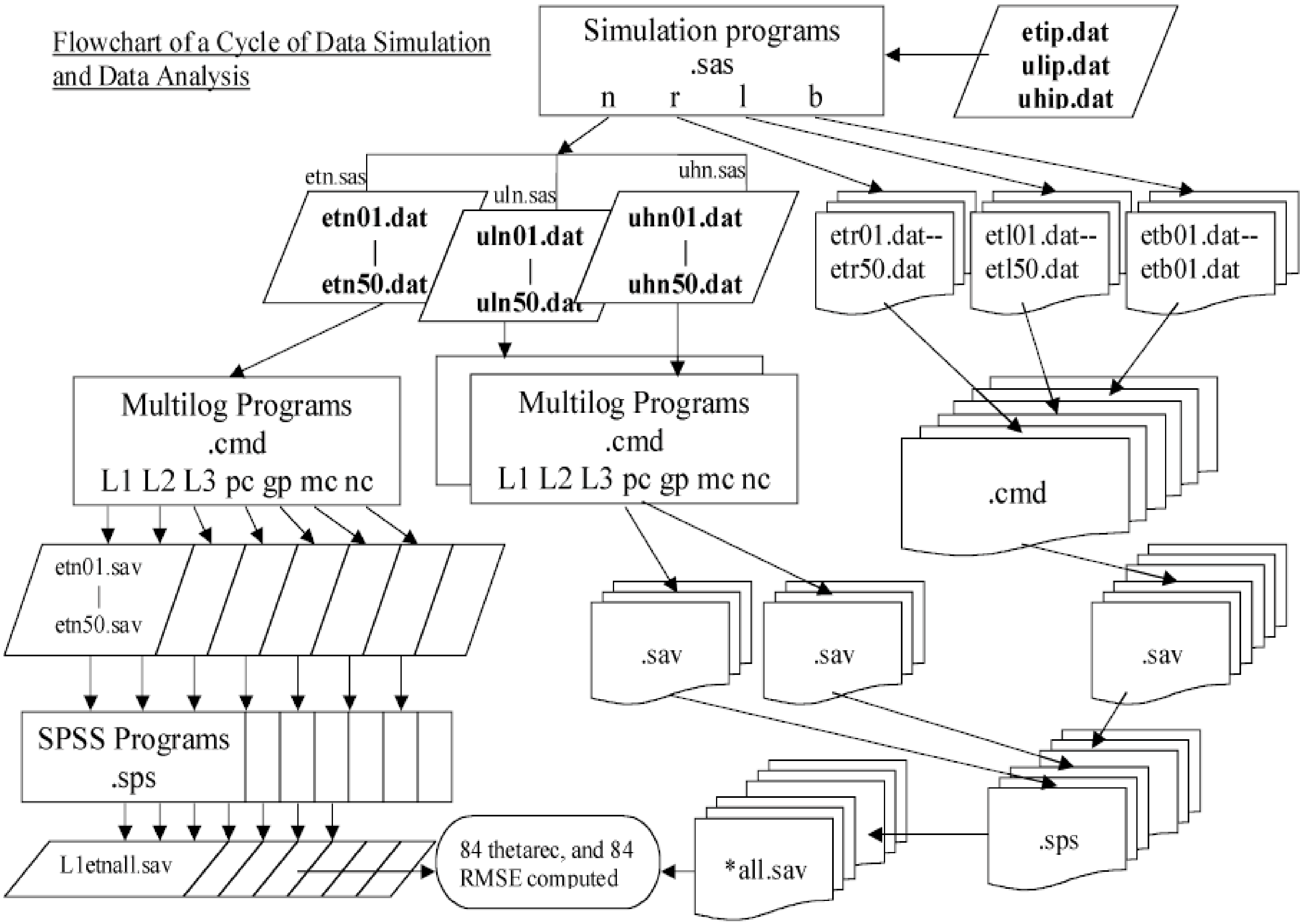
```
               e1=0.0;
               do k=1 to j;
                  e1=e1+a(i)*(z-sd(i,k));

               end;
               pbc(j+1)=exp(e1);
               den=den+exp(e1);
            end;
            pbc(1)=1/den;
            do j=2 to &ncat;
                pbc(j)=pbc(j)/den;
            end;
            tot=0.0;
            r=ranuni(&seed3);       /* converts the probabilities to */
            do k=1 to &ncat;        /* item responses using a sample */
              tot=tot+pbc(k);       /* of random numbers within (0,1)*/
              if r>tot then rr(i)=k;
         end;

        end;
        id=_n_;                     /* set up for next examinee */
        file resp;
        put &putfmt;
proc univariate normal vardef=n; var z;
proc chart;
  hbar z;
run;
```

SAS Program for Generating Prior Ability Distribution

```
/******************************************************************/
/*       This program is to generate random numbers with          */
/*       with different density functions of distribution         */
/*       to be used as the prior distribution of ability in       */
/*       a simulated item responses data set.                     */
/******************************************************************/

options ps=52 ls=72;
%let seed1=6734;

/* generate a bimodal prior distribution  */
data theta1 (keep=x);
     do i=1 to 500;           /* created a normally distributed half*/
       x1=rannor(&seed1);     /* sample shifted 1.5 sd to the left. */
       x=x1-1.5;
       output;
     end;
data theta2 (keep=x);         /* created a normally distributed half*/
     do i=1 to 500;           /* sample shifted 1.5 sd to the right.*/
       x1=rannor(&seed1);
       x=x1+1.5;
       output;
     end;
data theta  (keep=x x_std);   /* two half samples combined to form a*/
    set theta1 theta2;        /* bimodal sample centered at zero.   */
      x_std=x;
run;
proc standard data=theta mean=0 std=1 out=thetaz;  /* standardization*/
 var x_std;
run;
proc univariate normal vardef=n; var x_std;
proc chart;
  hbar x_std;
run;

/* generate a positively skewed prior distribution using
the density function of beta distribution. (beta distribution has two
parameter α and β, where α>1 and β>1. The shape of the distribution is
obtained by manipulating the values of the two parameters (Novick &
Jackson, 1974, 112). */

data thetas (keep=y y_std);
     do i=1 to 1000;
        x1=RANGAM(&seed1,1.8);      /* α = 1.8 */
        x2=RANGAM(&seed1,5);        /* β = 5   */
        y=x1/(x1+x2);               /* the beta density function */
        y_std=y;
        output;
     end;
run;
proc standard data=thetas mean=0 std=1 out=thetasz; /*standardization*/
 var y_std;
run;
proc univariate normal vardef=n; var y_std;
proc chart;
  hbar y_std;
run;


/* generate a negatively skewed prior distribution using
the density function of beta distribution  */

data thetas1 (keep=y y_std);
     do i=1 to 1000;
        x1=RANGAM(&seed1,5);          /* α = 5   */
        x2=RANGAM(&seed1,1.8);        /* β = 1.8 */
        y=x1/(x1+x2);
        y_std=y;
        output;
     end;
run;
proc standard data=thetas1 mean=0 std=1 out=thetas1z;
 var y_std;
run;
proc univariate normal vardef=n; var y_std;
proc chart;
  hbar y_std;
run;
```

APPENDIX B

Flowchart of a Cycle of Data Simulation and Data Analysis



APPENDIX C

Parameters of Items in the Constructed Tests for Data Simulation

Item	a_i	b_{i1}	b'_{i2}	b_{i2}	b''_{i2}	b_{i3}
1	1.60	-2.25	-2.000	-1.75	-1.500	-1.25
2	.80	-2.25	-1.900	-1.55	-1.200	-.85
3	2.00	-2.20	-1.975	-1.75	-1.525	-1.30
4	1.50	-2.00	-1.800	-1.60	-1.400	-1.20
5	1.70	-1.65	-1.425	-1.20	-0.975	-.75
6	1.80	-2.10	-1.850	-1.60	-1.350	-1.10
7	1.20	-1.90	-1.600	-1.30	-1.000	-.70
8	.90	-1.55	-1.325	-1.10	-.925	-.75
9	1.70	-1.80	-1.600	-1.40	-1.200	-1.00
10	1.20	-2.05	-1.775	-1.50	-1.225	-.95
11	.90	-.65	-.325	.00	.325	.65
12	1.90	-.60	-.250	.10	.450	.80
13	1.40	-.55	-.275	.00	.275	.55
14	1.80	-.45	-.250	-.05	.150	.35
15	1.30	-.50	-.250	.00	.250	.50
16	1.60	-.50	-.225	.05	.325	.60
17	1.10	-.45	-.225	.00	.225	.45
18	1.80	-.40	-.200	.00	.200	.40
19	1.20	-.30	-.050	.20	.450	.70
20	1.80	-.35	-.175	.00	.175	.35
21	1.70	1.30	1.525	1.75	1.975	2.20
22	1.30	.85	1.200	1.55	1.900	2.25
23	.90	.50	.700	.90	1.100	1.30
24	1.00	1.15	1.400	1.65	1.900	2.15
25	1.60	.80	1.100	1.40	1.700	2.00
26	1.60	1.00	1.225	1.45	1.675	1.90
27	1.90	.75	.975	1.20	1.425	1.65
28	1.10	1.05	1.275	1.50	1.725	1.95
29	2.00	.75	1.025	1.30	1.575	1.85
30	1.80	.95	1.225	1.50	1.775	2.05

APPENDIX D

Ability Estimation Programs

Multilog Command Programs For Different IRT Models

1-PL logistic model (L1.cmd & L1s.cmd).

```
MML PARAMETER ESTIMATION FOR THE 1PL MODEL
>PROBLEM RANDOME IN NITEMS=30 NGROUPS=1 NE=1000, NCHAR=8;
>TEST ALL L1;
>SAVE;
>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)

ESTIMATION OF ABILITY  1PL MODEL
>PROBLEM SCORE IND NITEMS=30 NGROUPS=1 NE=1000 NCHAR=8;
>TEST ALL L1;
>SAVE;
>START ALL;
Y

>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)
```

2-PL logistic model (L2.cmd & L2s.cmd).

```
MML PARAMETER ESTIMATION FOR THE 2PL MODEL
>PROBLEM RANDOME IN NITEMS=30 NGROUPS=1 NE=1000, NCHAR=8;
>TEST ALL L2;
>SAVE;
>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)


ESTIMATION OF ABILITY 2PL MODEL
>PROBLEM SCORE IND NITEMS=30 NGROUPS=1 NE=1000 NCHAR=8;
>TEST ALL L2;
>SAVE;
>START ALL;
Y
```

```
>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)
```

3-PL logistic model (L3.cmd & L3s.cmd).

```
MML PARAMETER ESTIMATION, 3PL logistic
>PROBLEM RANDOM IN NITEMS=30 NGROUPS=1 NE=1000 NC=8;
>TEST ALL L3;
>SAVE;
>PRIORS ALL DK=1 PA=(-1.4,1.0);
>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)

ESTIMATION OF ABILITY, 3PL logistic
>PROBLEM SCORE IND NITEMS=30 NGROUPS=1 NE=1000 NC=8;
>TEST ALL L3;
>SAVE;
>PRIORS ALL DK=1 PA=(-1.4,1.0);
>START ALL;
Y

>END;
           2
01
111111111111111111111111111111
N
(4X,8A1,T45,30A1)
```

Partial credit model (pc.cmd & pcs.cmd).

```
MML PARAMETER ESTIMATION, General PARTIAL CREDIT MODEL
>PRO RA IN NI=30 NG=1 NE=1000 NCHARS=8;
>TEST ALL NO NC=(4(0)30) HI=(4(0)30);
>SAVE;
>TMATRIX ALL AK POLY;
>EQUAL ALL AK=1;
>FIX ALL AK=(2,3) VALUE=0.0;
>TMATRIX ALL CK TRIANGLE;
>END;
           4
1234
111111111111111111111111111111
222222222222222222222222222222
333333333333333333333333333333
444444444444444444444444444444
(4X,8A1,1X,30A1)
```

```
ESTIMATION OF ABILITY partial credit model
>PRO SCORE IND NE=1000 NG=1 NI=30 NCHARS=8;
>TEST ALL NO NC=(4(0)30)HI=(4(0)30);
>TMATRIX ALL AK POLY;
>EQUAL ALL AK=1;
>FIX ALL AK=(2,3) VALUE=0.0;
>TMATRIX ALL CK TRIANGLE;
>SAVE;
>START ALL;
Y

>END;

           4
1234
1111111111111111111111111111111111
2222222222222222222222222222222222
3333333333333333333333333333333333
4444444444444444444444444444444444
(4X,8A1,1X,30A1)
```

General Partial credit model (gp.cmd & gps.cmd).

```
MML PARAMETER ESTIMATION, General PARTIAL CREDIT MODEL
>PRO RA IN NI=30 NG=1 NE=1000 NCHARS=8;
>TEST ALL NO NC=(4(0)30) HI=(4(0)30);
>SAVE;
>TMATRIX ALL AK POLY;
>FIX ALL AK=(2,3) VALUE=0.0;
>TMATRIX ALL CK TRIANGLE;
>END;
           4
1234
111111111111111111111111111111
222222222222222222222222222222
333333333333333333333333333333
444444444444444444444444444444
(4X,8A1,1X,30A1)

ESTIMATION OF ABILITY general partial credit model
>PRO SCORE IND NE=1000 NG=1 NI=30 NCHARS=8;
>TEST ALL NO NC=(4(0)30)HI=(4(0)30);
>TMATRIX ALL AK POLY;
>FIX ALL AK=(2,3) VALUE=0.0;
>TMATRIX ALL CK TRIANGLE;
>SAVE;
>START ALL;
Y

>END;

           4
1234
1111111111111111111111111111111111
2222222222222222222222222222222222
```

```
3333333333333333333333333333333333
4444444444444444444444444444444444
(4X,8A1,1X,30A1)
```

Multiple choice model (mc.cmd & mcs.cmd).

```
MML PARAMETER ESTIMATION multiple choice model
>PRO RA IN NI=30 NG=1 NE=1000 NC=8;
>TEST ALL BS NC=(5(0)30) HI=(5(0)30);
>EQUAL ALL DK=(1,2,3);
>SAVE;
>END;
           4
1234
222222222222222222222222222222
333333333333333333333333333333
444444444444444444444444444444
555555555555555555555555555555
(4X,8A1,1X,30A1)

Estimation of ability multiple choice model
>PRO SCORE IND NI=30 NG=1 NE=1000 NC=8;
>TEST ALL BS NC=(5(0)30) HI=(5(0)30);
>EQUAL ALL DK=(1,2,3);
>SAVE;
>START ALL;
Y


>END;
           4
1234
222222222222222222222222222222
333333333333333333333333333333
444444444444444444444444444444
555555555555555555555555555555
(4X,8A1,1X,30A1)
```

Nominal categories model (nc.cmd & ncs.cmd).

```
MML PARAMETER ESTIMATION, Nominal categories model
>PRO RA IN NI=30 NG=1 NE=1000 NC=8;
>TEST ALL NO NC=(4(0)30) HI=(4(0)30);
>SAVE;
>END;
           4
1234
111111111111111111111111111111
222222222222222222222222222222
333333333333333333333333333333
444444444444444444444444444444
(4X,8A1,1X,30A1)

ESTIMATION OF ABILITY NOMINAL CATEGORIES MODEL
```

```
>PRO SCORE IND NE=1000 NG=1 NI=30 NCHARS=8;
>TEST ALL NO NC=(4(0)30)HI=(4(0)30);
>SAVE;
>START ALL;
Y

>END;

           4
1234
1111111111111111111111111111111111
2222222222222222222222222222222222
3333333333333333333333333333333333
4444444444444444444444444444444444
(4X,8A1,1X,30A1)
```

APPENDIX E

Ability Estimation Result

C_{xyz}	ScorFor	ItemPar	PolyMod	Thetadis	Threconf	Thetarec	RMSE
l1eb6734	1	1		1	1	.374	.120083
l1el6734	1	1		2	1	.445	.134239
l1en6734	1	1		3	1	.282	.168582
l1er6734	1	1		4	1	.483	.196774
l1hb6734	1	1		1	2	.357	.123329
l1hl6734	1	1		2	2	.408	.125419
l1hn6734	1	1		3	2	.281	.154952
l1hr6734	1	1		4	2	.475	.185338
l1lb6734	1	1		1	3	.386	.118448
l1ll6734	1	1		2	3	.452	.184065
l1ln6734	1	1		3	3	.271	.185284
l1lr6734	1	1		4	3	.511	.250619
l2eb6734	1	2		1	1	.406	.215963
l2el6734	1	2		2	1	.231	.197889
l2en6734	1	2		3	1	.181	.197282
l2er6734	1	2		4	1	.575	.265009
l2hb6734	1	2		1	2	.379	.229717
l2hl6734	1	2		2	2	.206	.173263
l2hn6734	1	2		3	2	.134	.175585
l2hr6734	1	2		4	2	.496	.238663
l2lb6734	1	2		1	3	.427	.207196
l2ll6734	1	2		2	3	.247	.282400
l2ln6734	1	2		3	3	.216	.232357
l2lr6734	1	2		4	3	.625	.352136
l3eb6734	1	3		1	1	.303	.338231
l3el6734	1	3		2	1	.159	.298060
l3en6734	1	3		3	1	.082	.244724
l3er6734	1	3		4	1	.347	.372022
l3hb6734	1	3		1	2	.264	.352136
l3hl6734	1	3		2	2	.130	.269314
l3hn6734	1	3		3	2	.049	.221427
l3hr6734	1	3		4	2	.305	.334664
l3lb6734	1	3		1	3	.316	.322800
l3ll6734	1	3		2	3	.170	.030671
l3ln6734	1	3		3	3	.095	.050289
l3lr6734	1	3		4	3	.406	.082000
pceb6734	2	1	1	1	1	.942	.052792
pcel6734	2	1	1	2	1	.868	.032619
pcen6734	2	1	1	3	1	.481	.053787
pcer6734	2	1	1	4	1	.907	.090294

pchb6734	2	1	1	1	2	.926	.050398
pchl6734	2	1	1	2	2	.832	.035270
pchn6734	2	1	1	3	2	.339	.051565
pchr6734	2	1	1	4	2	.914	.072581
pclb6734	2	1	1	1	3	.899	.058129
pcll6734	2	1	1	2	3	.842	.069556
pcln6734	2	1	1	3	3	.664	.082171
pclr6734	2	1	1	4	3	.856	.103344
gpeb6734	2	2	2	1	1	.473	.082583
gpel6734	2	2	2	2	1	.570	.082207
gpen6734	2	2	2	3	1	.417	.087601
gper6734	2	2	2	4	1	.682	.118743
gphb6734	2	2	2	1	2	.450	.090956
gphl6734	2	2	2	2	2	.374	.071708
gphn6734	2	2	2	3	2	.327	.084054
gphr6734	2	2	2	4	2	.690	.095300
gplb6734	2	2	2	1	3	.482	.085206
gpll6734	2	2	2	2	3	.714	.079391
gpln6734	2	2	2	3	3	.532	.131681
gplr6734	2	2	2	4	3	.520	.070314
mceb6734	2	3	3	1	1	.405	.107517
mcel6734	2	3	3	2	1	.569	.069065
mcen6734	2	3	3	3	1	.823	.126293
mcer6734	2	3	3	4	1	.300	.069101
mchb6734	2	3	3	1	2	.522	.095802
mchl6734	2	3	3	2	2	.598	.091203
mchn6734	2	3	3	3	2	.843	.136272
mchr6734	2	3	3	4	2	.376	.076864
mclb6734	2	3	3	1	3	.307	.117686
mcll6734	2	3	3	2	3	.510	.044744
mcln6734	2	3	3	3	3	.687	.089566
mclr6734	2	3	3	4	3	.250	.085475
nceb6734	2		4	1	1	.757	.088034
ncel6734	2		4	2	1	.608	.048177
ncen6734	2		4	3	1	.561	.091684
ncer6734	2		4	4	1	.685	.094715
nchb6734	2		4	1	2	.722	.089878
nchl6734	2		4	2	2	.518	.050794
nchn6734	2		4	3	2	.417	.091635
nchr6734	2		4	4	2	.716	.078492
nclb6734	2		4	1	3	.717	.089247
ncll6734	2		4	2	3	.671	.120083
ncln6734	2		4	3	3	.703	.134239
nclr6734	2		4	4	3	.599	.168582

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